STRUCTURAL ASPECTS OF PLANAR BRAID GROUPS

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Declaration

The work presented in this thesis has been carried out by me under the guidance of Dr Mahender Singh at the Indian Institute of Science Education and Research Mohali. This work has not been submitted in part or in full for a degree, a diploma, or a fellowship to any other university or institute. Whenever contributions of others are involved, every effort is made to indicate this clearly, with due acknowledgement of collaborative research and discussions. This thesis is a bonafide record of original work done by me and all sources listed within have been detailed in the bibliography.

Neha Nanda

In my capacity as the supervisor of the candidate's thesis work, I certify that the above statements by the candidate are true to the best of my knowledge.

Dr Mahender Singh
(Supervisor)

Dedicated to Maa, Paa and Bhai

By the Grace of God, I am what I am
1 Corinthians 5:10

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Abstract

Artin braid groups are celebrated objects which appear in and affix several areas of mathematics and theoretical physics. A geometric interpretation given by Artin in his pioneering work in the 1920s, which captures the behaviour of intertwined strings in the Euclidean 3-space, has led to a deeply rooted connection with links in the 3-space. Since then the theory has been ramified by topologists and algebraists both. This naturally leads to a question of how the strings would intertwine if considered on a plane, and how it can be signified algebraically. The thesis explores this direction and presents a detailed investigation of structural aspects of planar braid groups and their (higher genus) virtual analogues.

Study of certain isotopy classes of a finite collection of immersed circles (called doodles on surfaces) without triple or higher intersections on closed oriented surfaces is considered as a planar analogue of virtual knot theory with the genus zero case corresponding to the classical knot theory. In the case of doodles on the 2-sphere, the role of groups is played by a class of right-angled Coxeter groups called twin groups. For the higher genus case in the virtual setting, the role of groups is played by a new class of groups called virtual twin groups.

We give a topological description of virtual twin groups and establish Alexander and Markov theorems for oriented virtual doodles. This paves a way for constructing invariants for doodles on surfaces. We investigate structural aspects of (pure) virtual twin groups in detail. More precisely, we obtain a presentation of the pure virtual twin group and deduce that it is an irreducible right-angled Artin group. We then prove that pure virtual twin groups can be written as iterated semidirect products of infinite rank free groups. Consequently, it follows that pure virtual twin groups have trivial centers, which confirms a well-known conjecture about triviality of centers of irreducible non-spherical Artin groups. We also compute the automorphism group of pure virtual twin groups in full generality and give applications to twisted conjugacy.

We investigate the conjugacy problem in twin groups and derive a formula for the number of conjugacy classes of involutions, which, quite interestingly, is related to the well-known Fibonacci sequence. We also investigate z-classes in twin groups and derive a recursive formula for the number of z-classes of involutions. Finally, we determine automorphism groups of twin groups and give applications to twisted conjugacy.

List of notations

$\mathbb{Z}_n$	Cyclic group of order n
S_n	Symmetric group on n symbols
A_n	Alternating group on n symbols
K_4	Klein four group
D_n	Dihedral group of order n
F_n	Free group of rank n
B_n	Braid group on n strands
P_n	Pure braid group on n strands
VB_n	Virtual braid group on n strands
T_n	Twin group on n strands
PT_n	Pure twin group on n strands
VT_n	Virtual twin group on n strands
$\mathcal{VT}_n$	Diagrammatic group of virtual twins on n strands
PVT_n	Pure virtual twin group on n strands
$G*H$	Free product of groups G and H
$[a,b]$	$a^{-1}b^{-1}ab$
x^g	$g^{-1}xg$
$\hat{g}$	Inner automorphism induced by g
$\overline{g}$	Right coset representative of coset Hg
$\ell(w)$	Length of an expression representing a word w
$C_G(S)$	Centraliser of a subset S of group G
$\gamma_2(G)$	Commutator subgroup of group G
$\mathrm{Z}(G)$	Center of group G
$R(\phi)$	Reidemeister number of an automorphism ϕ
$A \sqcup B$	Disjoint union of sets A and B
$\|A\|$	Number of elements in set A
$\mathbb{R}$	Real line
$\mathbb{R}^2$	Real plane
$\mathbb{D}$	Unit disk centred at the origin of $\mathbb{R}^2$

$\partial\mathbb{D}$	Boundary of disk $\mathbb{D}$
$\mathbb{R}^2 \setminus \mathbb{D}^\circ$	Complement of interior of unit disk $\mathbb{D}$
$\mathbb{S}^1$	Unit circle centred at the origin of $\mathbb{R}^2$
$\sqcup_n \mathbb{S}^1$	Disjoint union of n copies of $\mathbb{S}^1$
Σ	Closed oriented surface
$\mathrm{cl}(\beta)$	Closure of a twin on the 2-sphere or closure of a virtual twin on the plane
$m \otimes \beta$	(Virtual) twin obtained by adding m strands on the left of β
$V(D)$	Set of all crossings of a diagram D
$V_R(D)$	Set of all real crossings of a diagram D
A_Γ	Right-angled Artin group associated to graph Γ
$lk(v)$	Link of a vertex v
$st(v)$	Star of a vertex v
$\mathrm{Ker}(\phi)$	Kernel of a homomorphism ϕ
$\mathrm{Aut}(G)$	Group of automorphisms of a group G
$\mathrm{Inn}(G)$	Group of inner automorphisms of a group G
$\mathrm{Aut}_{gr}(G)$	Group of graph automorphisms of a right-angled Artin group G
$\mathrm{Aut}_{inv}(G)$	Group of inversions of a right-angled Artin group G
$\mathrm{Aut}_{tr}(G)$	Group of transvections of a right-angled Artin group G
$\mathrm{Aut}_{pc}(G)$	Group of partial conjugations of a right-angled Artin group G

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Chapter 1

Introduction

Artin braid groups are celebrated objects which appear in and affix various areas of mathematics and theoretical physics. One of the notable features of these groups is their deeply rooted connection with classical links which are defined as embeddings of 1-dimensional closed manifolds (disjoint union of circles) into the 3-sphere. An explicit geometric account revolving around the behaviour of interlinked strings with crossing information in the 3-space was given by Artin in the 1920s. Since then it has been ramified by topologists and algebraists both. This naturally leads to a question of how the strings would intertwine if considered on a plane, and that how can it be signified algebraically. In this direction, Fenn and Taylor [27] introduced doodles on the 2-sphere as finite collections of Jordan curves lying on the 2-sphere which are required not to have triple intersections. They focused on three components doodles and studied relations between doodles and commutator identities in free groups using a certain type of intersection number of components. Two doodles are equivalent if one can be obtained from the other by a finite sequence of the move called a Whitney move. Khovanov [52] then considered a finite collection of immersed circles on any fixed closed oriented surface in his definition of a doodle. This further allowed him to consider a move which adds or removes a kink in a doodle, and therefore the equivalence relation on a set of doodles with a fixed number of components was refined accordingly. By considering the definition given by Khovanov [52] and extending the idea of Fenn and Taylor [27], it has been proved recently by Bartholomew-Fenn-Kamada-Kamada [8] that there is a bijection between cobordism classes of coloured doodles and weak equivalence classes of elementary commutator identities. More interestingly, these objects can be viewed as planar analogues of links in the Euclidean 3-space. Consequently, the role of groups in the theory of doodles can be contemplated. As a matter of course, Khovanov considered abstract groups T_n which he called twin groups, and gave a geometric interpretation similar to the one for Artin braid groups. He considered configurations of n arcs in the infinite strip $\mathbb{R} \times [0,1]$ connecting

n marked points on each of the parallel lines $\mathbb{R} \times \{1\}$ and $\mathbb{R} \times \{0\}$ such that each arc is monotonic and no three arcs have a point in common. Two such configurations are equivalent if one can be deformed into the other by a homotopy of such configurations in $\mathbb{R} \times [0,1]$ keeping the end points of arcs fixed. An equivalence class under this equivalence is called a *twin*. The product of two twins can be defined by the juxtaposition of one twin on top of the other and shrinking the interval back to $[0,1]$. The collection of all twins with n arcs under this operation forms a group isomorphic to the twin group T_n. Taking the one-point compactification of the plane, one can define the closure of a twin on the 2-sphere, in resemblance to the operation defined for geometric braids in the 3-space. Evidently, closure of a twin gives a doodle on the 2-sphere. In fact, Khovanov [52] proved that every oriented doodle on the 2-sphere is the closure of some twin. Recently, Gotin [37] established the relation between twins with equivalent closures and doodles on the 2-sphere, that is, the Markov description of oriented doodles on the 2-sphere in terms of Markov equivalence classes of twins. To classify doodles on the 2-sphere, following the work of Burau to construct link invariants via representations, an Alexander type invariant for oriented doodles has been constructed in a recent work [18]. Twin groups, being right-angled Coxeter groups, are known to be linear groups via the well-known Tits representation [15, p.96]. Using a deformation of this faithful representation, a polynomial invariant has been constructed which vanishes on unlinked doodles with more than one component.

Even before this interpretation, twin groups appeared in the work of Shabat and Voevodsky [81] in the context of curves over number fields, who referred them as Grothendieck cartographical groups. Later, these groups appeared in the work of Khovanov [51] on real $K(\pi,1)$ arrangements. These groups are also referred as traid groups or planar braid groups in the literature [36, 39, 63, 64].

There is a natural surjection of T_n onto the symmetric group on n symbols trailing how the end points of the strands are connected, whose kernel is known as the pure twin group and is denoted by PT_n. It is well-known that the pure Artin braid group P_n is the fundamental group of the configuration space of ordered n-tuples of distinct points in $\mathbb{R}^2$. Björner and Welker [12, 14] considered a more general class of manifolds to which the space

$$X_n = \mathbb{R}^n \setminus \{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \mid x_i = x_j = x_k = x_i,\ i \neq j \neq k \neq i\}$$

belongs. They studied the cohomology of these spaces and proved that $H^i(X_n, \mathbb{Z})$ is free for all i. Through their works, a lower bound of an exponential order to the rank of pure twin groups has been estimated. Further, in [12] it was conjectured whether the space X_n is a $K(\pi,1)$, which was later dealt by Khovanov [51] in his work on real $K(\pi,1)$ arrangements.

He proved that the conjecture holds and that the fundamental group of the space X_n is isomorphic to the pure twin group on n strands.
Recently, the algebraic perspective of these groups has gained much attention. Bardakov-Singh-Vesnin [4] gave an upper bound to the rank of PT_n and proved that PT_n is free for $n = 3, 4$ and not free for $n \geq 6$. It was also proved that PT_n is torsion-free for every $n \geq 3$. Further, it was conjectured that PT_5 is also a free group of rank 31, and the same has been established recently by González-León-Medina-Roque [36]. It has been proven in [63] that PT_6 is a free product of the free group F_{71} and 20 copies of the free abelian group $\mathbb{Z} \oplus \mathbb{Z}$. A description of a presentation of PT_n for all $n \geq 3$ has been given in a recent work by Mostovoy [64]. González-León-Medina-Roque [36] also computed the Lusternik-Schnirelmann category and the higher topological complexity of PT_n. These groups also belong to the class of so called diagram groups [30, 38].
Even with a slightly simpler presentation than classical Artin braid groups and a resembling geometric interpretation, it is interesting to understand how the twin groups differ algebraically. A part of this thesis concentrate on algebraic aspects of these groups. In particular, we explore conjugacy classes of involutions, centralisers, automorphisms, representations, R_∞-property and (co)-Hopfianity of twin groups.
From ramifications of links and braids, arose virtual knots which were later shown to be links in closed oriented thickened surfaces with a slightly more robust notion of equivalence. One can think of the study of isotopy classes of immersed circles without triple or higher intersection points on closed oriented surfaces as a planar analogue of virtual knot theory. Bartholomew-Fenn-Kamada-Kamada [9] extended the study of doodles to immersed circles on closed oriented surfaces of any genus and called them doodles on surfaces. Then, they introduced the notion of a virtual doodle which is a generic immersion of a closed one-dimensional manifold (disjoint union of circles) on the plane with finitely many real or virtual crossings such that there are no triple or higher real intersection points. With the aim of classifying these geometric objects, coloring of diagrams using a special type of algebra called doodle switch, has been established to construct an invariant for virtual doodles [7]. They also discussed Gauss codes for virtual doodles and defined left canonical Gauss codes which turns out to be a complete invariant for oriented virtual doodles. More precisely, they proved that virtual doodles are uniquely represented by left canonical Gauss codes [10]. Through their works [9], classes of doodles on surfaces can uniquely be captured into diagrammatic study of classes of virtual doodles on the plane. In this thesis, we characterise Alexander-Markov description of doodles on surfaces, which captures the information about these objects into the algebraic perspective. In other words, we give a one to one correspondence between equivalence classes of doodles on surfaces and Markov equivalence classes of virtual twins

(possibly with different number of strands). From a wider perspective, a recent work of Bartholomew and Fenn [5] look at which Alexander and Markov theories can be defined for generalised knot theories.

For constructing algebraic counterparts for doodles on surfaces, we examine an abstract generalisation of twin groups called virtual twin groups defined in [4] and denoted by VT_n. In this thesis, we give a topological interpretation of the group VT_n as group of classes of configurations called virtual twins of n arcs with real or virtual crossings satisfying suitable conditions. Once this is established, we prove Alexander and Markov theorems for oriented virtual doodles on the plane which completely classify them in terms of virtual twins with possibly different number of strands. This opens up the possibility of constructing algebraic invariants to classify virtual doodles on the plane. There is a natural surjection of VT_n onto the symmetric group on n symbols which traces the end points of n strands and whose kernel PVT_n is called the pure virtual twin group. We then focus on examining the structural properties of virtual twin groups and pure virtual twin groups.

The following subsections give a brief outline of the thesis.

1.1 Structural properties of twin groups

For an integer $n \geq 2$, the *twin group* T_n is defined as the group with a presentation

$$\left\langle s_1, s_2, \ldots, s_{n-1} \mid s_i^2 = 1 \text{ for } 1 \leq i \leq n-1 \text{ and } s_i s_j = s_j s_i \text{ for } |i-j| \geq 2 \right\rangle.$$

Let S_n be the symmetric group on n symbols. Then there is a natural homomorphism from T_n onto S_n, which maps each generator s_i to the transposition $(i,\ i+1)$. The kernel of this map is defined as the *pure twin group* and is denoted by PT_n.

We derive a formula for the number of conjugacy classes of involutions in T_n. Quite interestingly, it is closely related to the well-known Fibonacci sequence.

Theorem 1.1.1. *Let ρ_n denote the number of conjugacy classes of involutions in T_n. Then*

$$\rho_n = 1 + \rho_{n-1} + \rho_{n-2}$$

for all $n \geq 4$, where $\rho_2 = 1$ and $\rho_3 = 2$.

Corollary 1.1.2. *For each $n \geq 2$, $\rho_n + 1 = \boldsymbol{F}_{n+1}$, where $(\boldsymbol{F}_n)_{n\geq 1}$ is the well-known Fibonacci sequence with $\boldsymbol{F}_1 = \boldsymbol{F}_2 = 1$. In particular,*

$$\rho_n = \sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} \binom{n-k}{k}.$$

Two elements x, y of a group G are said to be *z-equivalent* if their centralisers $C_G(x)$ and $C_G(y)$ are conjugates in G. A z-equivalence class is called a *z-class*. We compute the number of z-classes of involutions in T_n for $n \geq 2$, and we have the following results.

Proposition 1.1.3. *T_n has finitely many z-classes if and only if $n = 2$ or 3.*

Theorem 1.1.4. *Let λ_n denote the number of z-classes of involutions in T_n, $n \geq 2$. Then, for $n \geq 7$,*

$$\lambda_n = \Big(\sum_{i=3}^{n-2} \lambda_i\Big) - \lambda_{n-4} + n - 2,$$

where $\lambda_2 = 1$, $\lambda_3 = 2$, $\lambda_4 = 2$, $\lambda_5 = 5$ and $\lambda_6 = 8$.

We compute the group of automorphisms for T_n in full generality. Note that the automorphism group of $T_3 \cong \mathbb{Z}_2 * \mathbb{Z}_2$ is well-known.

Theorem 1.1.5. *Let T_n be the twin group with $n \geq 3$. Then the following hold:*

(1) $\mathrm{Aut}(T_3) \cong T_3 \rtimes \mathbb{Z}_2$.

(2) $\mathrm{Aut}(T_4) \cong T_4 \rtimes S_3$.

(3) $\mathrm{Aut}(T_n) \cong T_n \rtimes D_8$ *for $n \geq 5$, where D_8 is the dihedral group of order 8.*

Using the structure of the group $\mathrm{Aut}(T_n)$, we study the R_∞-property of T_n. Let G be a group and ϕ an automorphism of G. Two elements $x, y \in G$ are said to be (ϕ-twisted conjugate) ϕ-conjugate if there exists an element $g \in G$ such that $x = gy\phi(g)^{-1}$. The relation of ϕ-conjugation is an equivalence relation and divides the group into ϕ-conjugacy classes. Taking ϕ to be the identity automorphism gives the usual conjugacy classes. The number of ϕ-conjugacy classes $R(\phi) \in \mathbb{N} \cup \{\infty\}$ is called the *Reidemeister number* of the automorphism ϕ. We say that a group G has R_∞-property if $R(\phi) = \infty$ for each $\phi \in \mathrm{Aut}(G)$.

Theorem 1.1.6. *T_n satisfy R_∞-property for all $n \geq 3$.*

Residual properties of groups are of great interest to combinatorial group theorists. Twin groups belong to the special class of right-angled Coxeter groups which are known to be linear, and hence residually finite and Hopfian. A group is said to be co-Hopfian (respectively Hopfian) if every injective (respectively surjective) endomorphism is an automorphism. For twin groups we have the following result.

Theorem 1.1.7. *T_n is not co-Hopfian for $n \geq 3$.*

1.2 Virtual twin groups and doodles on surfaces

The virtual twin group VT_n is presented by generators $\{s_1, s_2, \ldots, s_{n-1}, \rho_1, \rho_2, \ldots, \rho_{n-1}\}$ and following defining relations

- relations of the twin group:

$$\begin{aligned} s_i^2 &= 1 \text{ for } i = 1, 2, \ldots, n-1, \\ s_i s_j &= s_j s_i \text{ for } |i-j| \geq 2, \end{aligned}$$

- relations of the symmetric group:

$$\begin{aligned} \rho_i^2 &= 1 \text{ for } i = 1, 2, \ldots, n-1, \\ \rho_i \rho_j &= \rho_j \rho_i \text{ for } |i-j| \geq 2, \\ \rho_i \rho_{i+1} \rho_i &= \rho_{i+1} \rho_i \rho_{i+1} \text{ for } i = 1, 2, \ldots, n-2, \end{aligned}$$

- mixed relations:

$$\begin{aligned} \rho_i s_j &= s_j \rho_i \text{ for } |i-j| \geq 2, \\ \rho_i \rho_{i+1} s_i &= s_{i+1} \rho_i \rho_{i+1} \text{ for } i = 1, 2, \ldots, n-2. \end{aligned}$$

In this thesis, we give a topological interpretation of these groups. Consider a set Q_n of n points in $\mathbb{R}$. A *virtual twin diagram* on n strands is a subset D of $\mathbb{R} \times [0,1]$ which consists of n intervals called *strands* such that $\partial D = Q_n \times \{0,1\}$ and the following conditions are satisfied:

1. the natural projection $\mathbb{R} \times [0,1] \to [0,1]$ maps each strand homeomorphically onto the unit interval $[0,1]$,
2. the set $V(D)$ of all crossings of the diagram D consists of transverse double points of D where each crossing has the pre-assigned information of being a real or a virtual

crossing as depicted in Figure 1.1. A virtual crossing is depicted by a crossing encircled with a small circle.

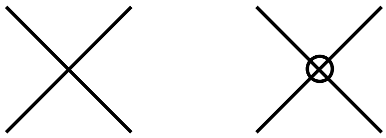

Fig. 1.1. Real and virtual crossing

We say that the two virtual twin diagrams on n strands are *equivalent* if one can be obtained from the other by a finite sequence of isotopies of the plane and the moves as in Figure 1.2. Such an equivalence class is called a *virtual twin*. We prove that the set of virtual twins on n strands forms a group under the operation of concatenation, isomorphic to the group VT_n.

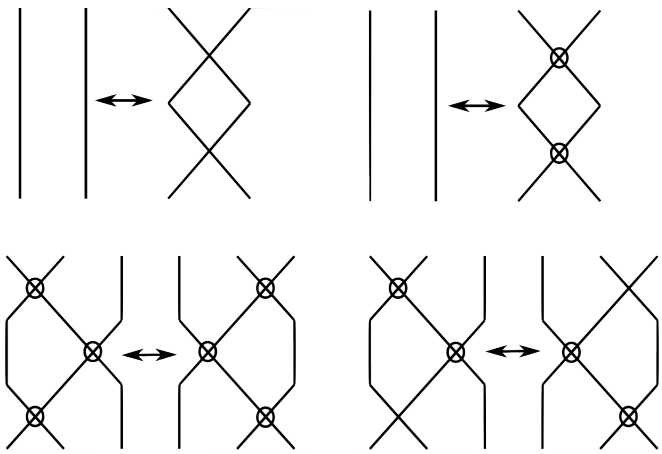

Fig. 1.2. Reidemeister moves for virtual twin diagrams

A *virtual doodle diagram* is a generic immersion of disjoint union of finitely many circles on the plane $\mathbb{R}^2$ with finite number of real and virtual crossings such that there are no triple or higher real intersection points. Two virtual doodle diagrams are *equivalent* if they are related by a finite sequence of R_1, R_2, VR_1, VR_2, VR_3, M moves as shown in Figure 1.3 and isotopies of the plane.

With the preceding setup, we have the following results.

Theorem 1.2.1. *Every oriented virtual doodle on the plane is equivalent to closure of a virtual twin diagram.*

We now consider the following moves:

$(M0)$ Defining relations of VT_n,

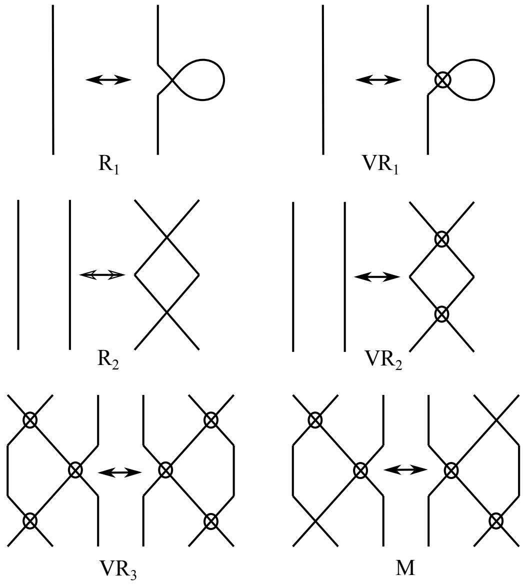

Fig. 1.3. Reidemeister moves for virtual doodle diagrams

$(M1)$ Conjugation: $\alpha^{-1}\beta\alpha \sim \beta$,

$(M2)$ Right stabilisation of real or virtual type: $\beta s_n \sim \beta$ or $\beta\rho_n \sim \beta$,

$(M3)$ Left stabilisation of real type: $(1 \otimes \beta)s_1 \sim \beta$,

$(M4)$ Right exchange: $\beta_1 s_n \beta_2 s_n \sim \beta_1 \rho_n \beta_2 \rho_n$,

$(M5)$ Left exchange: $s_1(1 \otimes \beta_1)s_1(1 \otimes \beta_2) \sim \rho_1(1 \otimes \beta_1)\rho_1(1 \otimes \beta_2)$,

for $\alpha, \beta, \beta_1, \beta_2 \in VT_n$, $n \geq 2$ and $1 \otimes \beta \in VT_{n+1}$ the virtual twin obtained by putting a trivial strand on the left of β.

Theorem 1.2.2. *Two virtual twin diagrams on the plane (possibly on different number of strands) have equivalent closures if and only if they are related by a finite sequence of moves* $(M0)-(M5)$.

1.3 Structural properties of pure virtual twin groups

The kernel of the natural surjection from VT_n onto S_n is called the *pure virtual twin group* and is denoted by PVT_n. We determine the presentation of PVT_n which, quite interestingly, turns out to be an irreducible right-angled Artin group.

Theorem 1.3.1. *The pure virtual twin group PVT_n on $n \geq 2$ strands has the presentation*

$$\left\langle \lambda_{i,j},\ 1 \leq i < j \leq n \mid \lambda_{i,j}\lambda_{k,l} = \lambda_{k,l}\lambda_{i,j} \text{ for distinct integers } i, j, k, l \right\rangle,$$

where $\lambda_{i,j}$ is shown in Figure 1.4.

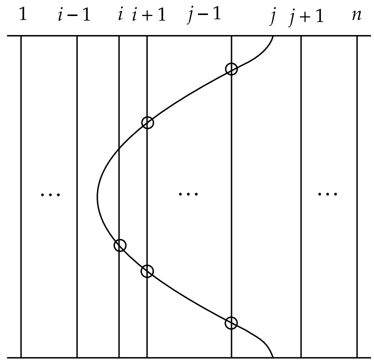

Fig. 1.4. The generator $\lambda_{i,j}$ of pure virtual twin group

As a result, we get significant information about the groups VT_n and PVT_n.

Corollary 1.3.2. *The virtual twin group VT_n is residually finite and Hopfian for each $n \geq 2$.*

We also show that PVT_n is isomorphic to an iterated semidirect product of infinite rank free groups, through which we deduce that the center of VT_n and PVT_n is trivial for $n \geq 2$ and $n \geq 3$, respectively. Next, in the direction of examining the group of automorphisms of PVT_n, we have the following results.

Theorem 1.3.3. *Let $n \geq 5$. Then*

$$\mathrm{Aut}(PVT_n) \cong PVT_n \rtimes (\mathbb{Z}_2^{n(n-1)/2} \rtimes S_n).$$

Since $PVT_2 \cong \mathbb{Z}$ and $PVT_3 \cong F_3$, their automorphism groups are well-known. The case $n = 4$ is exotic and the following result describes the structure of the group of automorphisms in this case.

Theorem 1.3.4. *Let PVT_4 be the pure virtual twin group on 4 strands. Then*

$$\mathrm{Aut}(PVT_4) \cong ((\mathbb{Z}^2 * \mathbb{Z}^2 * \mathbb{Z}^2) \rtimes (\mathbb{Z}^2 * \mathbb{Z}^2 * \mathbb{Z}^2)) \rtimes \big((\mathrm{GL}_2(\mathbb{Z}) \times \mathrm{GL}_2(\mathbb{Z}) \times \mathrm{GL}_2(\mathbb{Z})) \rtimes S_3\big).$$

As an application, we deduce the following result.

Theorem 1.3.5. *PVT_n has R_∞-property if and only if $n \geq 3$.*

Throughout the thesis, we consider doodles on 2-sphere as defined by Khovanov, unless specified. Only for the purpose of illustration, the diagrams are shown as piecewise linear.

The thesis is organised as follows. In Chapter 2, we develop necessary background required for the subsequent chapters. In Chapter 3, we give a topological description of virtual twin groups. We state and prove Alexander and Markov theorems for oriented virtual doodles on the plane. In Chapter 4, we investigate the conjugacy problem in twin groups and derive a formula for the number of conjugacy classes of involutions in T_n. We also investigate z-classes (conjugacy classes of centralisers of elements) in twin groups and derive a recursive formula for the number of z-classes of involutions. We conclude the chapter by addressing algebraic link problem for doodles on the 2-sphere. In Chapter 5, we determine the group of automorphisms $\mathrm{Aut}(T_n)$ for all $n \geq 3$ and give applications of the same. In particular, we show that twin groups satisfy the R_∞-property. Furthermore, we construct a representation of T_n into $\mathrm{Aut}(F_n)$. We also prove that T_n is not co-Hopfian for $n \geq 3$. In Chapter 6, we investigate structural aspects of (pure) virtual twin groups in detail. More precisely, we obtain a presentation of the pure virtual twin group PVT_n, which proves that it is an irreducible right-angled Artin group. We then prove that PVT_n can be written as an iterated semidirect product of infinite rank free groups, as a consequence of which it follows that both PVT_n and VT_n have trivial center. In Chapter 7, we compute the automorphism group of PVT_n in full generality. Finally, in Chapter 8, we give a reduced presentation of VT_n and use it to compute the commutator subgroup of VT_n. We also prove that VT_n is residually nilpotent if and only if $n = 2$.

Chapter 2

Preliminaries

In this chapter, we establish preliminaries which will be used in subsequent chapters. The results can be found in [9, 52, 60].

2.1 Twin and pure twin groups

Let us consider the infinite strip $\mathbb{R} \times [0,1]$ and n marked points on lines $\mathbb{R} \times \{0\}$ and $\mathbb{R} \times \{1\}$, respectively. Let us fix the points say $\{(1,0),(2,0),\dots,(n,0),(1,1),(2,1),\dots,(n,1)\}$, for convenience. We consider configurations of n strands connecting points $(1,0),\dots,(n,0)$ and $(1,1),\dots,(n,1)$ in some permutation such that the following conditions hold.

(i) Each strand maps homeomorphically onto the interval $[0,1]$. In other words, the strands are monotonic.

(ii) No three or more arcs have a common intersection point.

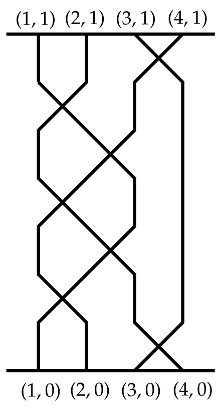

Fig. 2.1. Example of a twin on four strands

Figure 2.1 gives an example of such a configuration of 4 strands. We say that two configurations are *equivalent* if one can be obtained from the other by a homotopy of arcs in $\mathbb{R} \times [0,1]$ such that conditions (i) and (ii) hold and endpoints are fixed throughout the homotopy. It is not difficult to see that this is an equivalence relation on the set of configurations of n strands.

Definition 2.1.1. *A twin is an equivalence class of configurations of n strands.*

Consider the set of twins on n strands. The product C_1C_2 of two configurations C_1 and C_2 is defined by placing C_1 on top of C_2 and then shrinking the interval to $[0,1]$. It is clear that if C_1 is equivalent to C_1' and C_2 is equivalent to C_2' , then C_1C_2 is equivalent to $C_1'C_2'$. Thus, there is a well-defined binary operation on the set of twins with fixed number of strands. It is easy to see that this operation is indeed associative. Note that the twin represented by a configuration of n strands with no crossings is the identity element with respect to the binary operation.

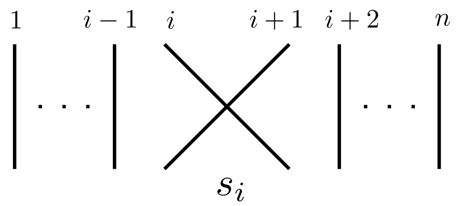

Fig. 2.2. The twin s_i

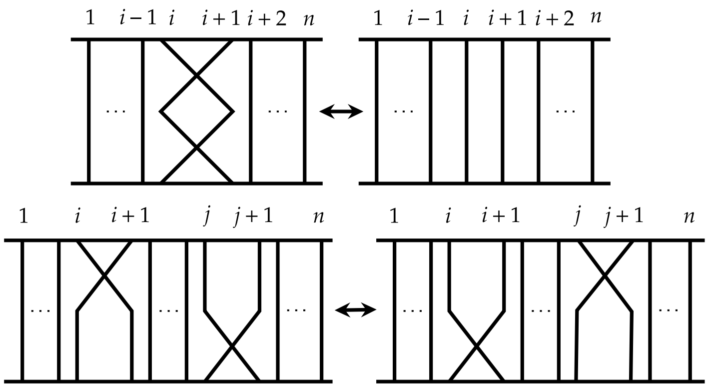

Fig. 2.3. Equivalence of configurations of n arcs

Also, any twin can be represented by a configuration such that each intersection point is at a distinct level, when mapped onto the interval $[0,1]$. Thus, any twin is a composition of some basic twins $s_1, s_2, \ldots, s_{n-1}$, where s_i is depicted in Figure 2.2. By the definition of equivalence of configurations (see Figure 2.3), we see that

$$\begin{aligned} s_i^2 &= 1, \ \ i = 1,2,\ldots,n-1, \\ s_i s_j &= s_j s_i, \ \ |i-j| > 1. \end{aligned}$$

We also note that due to the restriction that no three or more arcs have a common intersection point, the move shown in Figure 2.4 is forbidden. Khovanov [52] showed that the set of twins on a fixed number of strands with the operation of concatenation forms a group. The following result gives a presentation for the same.

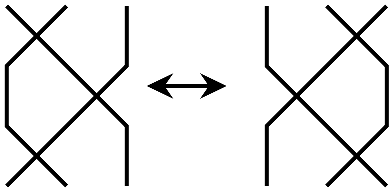

Fig. 2.4. Forbidden move for twins

Proposition 2.1.2 ([52], Proposition 1.1)**.** *The group of twins on n strands is isomorphic to the group T_n with presentation*

$$\big\langle s_1, s_2, \ldots, s_{n-1} \mid s_i^2 = 1 \textit{ for } 1 \le i \le n-1 \textit{ and } s_i s_j = s_j s_i \textit{ for } |i-j| \ge 2 \big\rangle.$$

From now on, the notation T_n is interchangeably used for diagrammatic group of twins on n strands and abstract group defined in Proposition 2.1.2.

Definition 2.1.3. *The pure twin group PT_n is the subgroup of T_n consisting of twins with strands connecting pairs $(i,0)$ with $(i,1)$ for all $i = 1,2,\ldots,n-1$.*

In other words, there is a natural surjection of T_n onto the symmetric group S_n on n symbols by sending each generator s_i to the transposition $(i,i+1)$. The kernel of this map is the pure twin group PT_n. Figure 2.5 depicts an example of a pure twin on four strands.
Consider the following space

$$X_n = \mathbb{R}^n \setminus \{(x_1,x_2,\ldots,x_n) \in \mathbb{R}^n \mid x_i = x_j = x_k = x_i, \ i \ne j \ne k \ne i\}.$$

Björner and Welker [12, 14] studied the cohomology of these spaces. They proved that $H^i(X_n,\mathbb{Z})$ is free for all i, $H^i(X_n,\mathbb{Z}) \ne 1$ if and only if $1 \le i \le n/3$ and that the rank of $H^i(X_n,\mathbb{Z})$ is $\sum_{i=3}^{n} \binom{n}{i} \binom{i-1}{2}$. It was also conjectured in [12] whether the space X_n is Eilenberg-MacLane space $K(\pi,1)$. Khovanov [51] proved that the conjecture holds and that PT_n is the fundamental group of X_n.

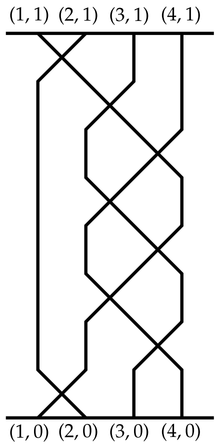

Fig. 2.5. Example of a pure twin on four strands

It is shown in [4] that PT_3 is the infinite cyclic group generated by $(s_1s_2)^3$, whereas PT_4 is a free group generated by

$$(s_1s_2)^3, ((s_1s_2)^3)^{s_3}, ((s_1s_2)^3)^{s_3s_2}, ((s_1s_2)^3)^{s_3s_2s_1},$$

$$(s_2s_3)^3, ((s_2s_3)^3)^{s_1}, ((s_2s_3)^3)^{s_1s_2}.$$

It was conjectured in [4] that PT_5 is a free group of rank 31 which has been recently proved in [36]. The group PT_6 is not free as $(s_1s_2)^3$ and $(s_4s_5)^3$ commutes in PT_6, and it is proved in [63] that PT_6 is isomorphic to the free product of the free group F_{71} and 20 copies of the free abelian group $\mathbb{Z} \oplus \mathbb{Z}$. A minimal presentation of PT_n is described in a recent work [64].

2.2 Doodles on 2-sphere

Definition 2.2.1. *A doodle is a finite collection of immersed circles on a fixed closed oriented surface without any triple intersections.*

A doodle is *oriented* if the underlying 1-dimensional manifold has an orientation. Two (oriented) doodles $D_0 = (D_0^1, \dots, D_0^n)$ and $D_1 = (D_1^1, \dots, D_1^n)$ are said to be *equivalent* if there exists (an orientation preserving) a continuous one parameter family of doodles $\{D_t = (D_t^1, \dots, D_n^t)\}$ joining D_0 and D_1. Two doodles are *equivalent* if and only if they are related to each other by a finite sequence of local moves shown in Figure 2.6.

It should be noted that by picking a point ∞ disjoint from a doodle on 2-sphere, the doodle can be represented in the Euclidean plane $\mathbb{R}^2$. For the sake of convenience, we draw doodles on a plane. Also, we assume that all the components meet transversely so that each component

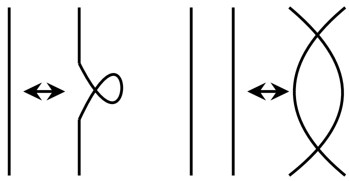

Fig. 2.6. Reidemeister moves for doodles

is well distinguished. Figure 2.7 shows some examples of doodles with different number of components.

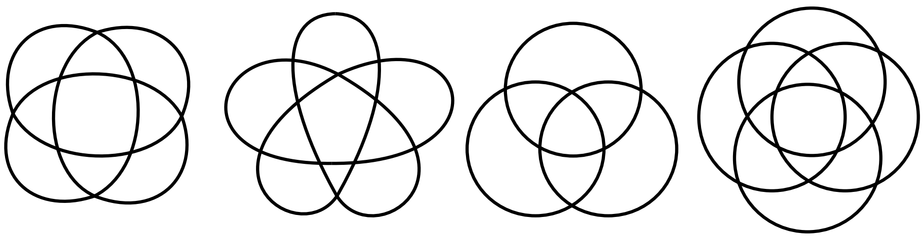

Fig. 2.7. Doodles on different number of components

We now define an operation crucial for building a bridge between twins and doodles. Consider a twin $\beta \in T_n$ represented by a configuration C of n strands. Let $\mathrm{cl}(C)$ be the doodle obtained by joining each pairs $(i,0)$ and $(i,1)$ on the plane, disjoint from the configuration, with non-intersecting arcs shown in Figure 3.7. We note that if C' is another configuration representing β, then $\mathrm{cl}(C)$ is equivalent to $\mathrm{cl}(C')$. Consequently, the equivalence class of $\mathrm{cl}(C)$ depends only on the equivalence class of β.
This equivalence class of doodle is called the *closure* of twin β and is denoted by $\mathrm{cl}(\beta)$. The orientation of β induces an orientation on its closure. The following result is an analogue of the classical Alexander theorem for oriented links [47, Chapter 2, Theorem 2.3].

Theorem 2.2.2 ([51], Theorem 2.1)**.** *Every oriented doodle on* 2*-sphere is the closure of a twin.*

The above theorem gives us the existence of a twin for each doodle on 2-sphere. The natural question that arises here is that if two twins have equivalent closures on the 2-sphere, then how are they related? For instance, it is not difficult to check that for twins $\alpha, \beta \in T_n$, the doodles $\mathrm{cl}(\alpha^{-1}\beta\alpha)$ and $\mathrm{cl}(\beta)$ are equivalent. This was answered by Gotin [37] who proved an analogue of Markov theorem [47] for oriented doodles on 2-sphere.
Let $m \otimes \beta$ (similarly, $\beta \otimes m$) be the twin obtained by adding m strands to the left (right) of the diagram of β, for a twin β with possibly different number of strands. For any positive integer n and $\alpha, \beta \in T_n$, define the following moves:

$$\mathrm{M}_1 \; : \beta \otimes 1 \to 1 \otimes \; \beta,$$

$$\mathrm{M}_2 \; : \beta \to \alpha^{-1} \beta \alpha,$$

$$\mathrm{M}_3 \; : \beta \to (\beta \otimes 1) s_n s_{n-1} \dots s_{i+1} s_i s_{i+1} \dots s_{n-1} s_n,$$

$$\mathrm{M}_4 \; : \beta \to (1 \otimes \beta) s_1 s_2 \dots s_{i-1} s_i s_{i-1} \dots s_2 s_1,$$

where $s_i \in T_{n+1}$.

Definition 2.2.3. *Two twins are said to be* M*-equivalent if one can be obtained from the other by a finite sequence of moves* $\mathrm{M}_1 - \mathrm{M}_4$ *and their inverses.*

If $\mathrm{M}_i(\beta)$ is the twin obtained from β by applying the M_i-move, then it is not difficult to prove that $\mathrm{cl}(\mathrm{M}_i(\beta))$ is equivalent to $\mathrm{cl}(\beta)$. For example, the closure of $(s_1 s_2)^3 \in T_3$ and the closure of $(s_2 s_3)^3 s_1 s_2 s_1 \in T_4$ are equivalent by M_4-move as shown in Figure 2.8. We have the following result.

Theorem 2.2.4 ([37], Theorem 4.1)**.** *Any two twins with equivalent closures are* M*-equivalent.*

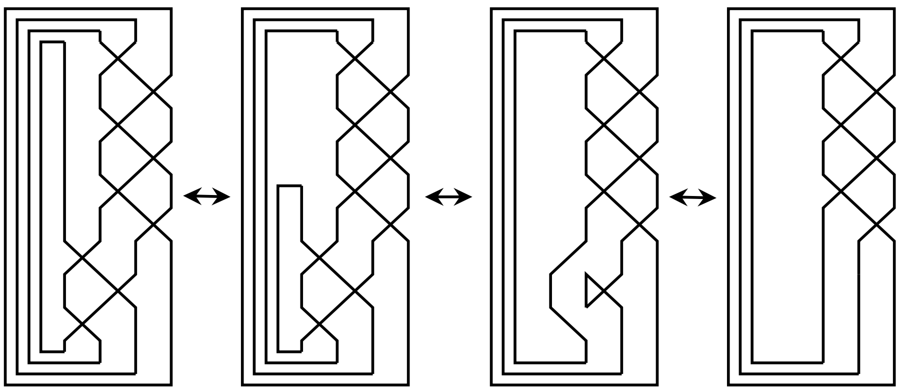

Fig. 2.8. The closures of $(s_1 s_2)^3$ and $(s_2 s_3)^3 s_1 s_2 s_1$ being equivalent as doodles.

2.3 Doodles on surfaces

In the previous section, we considered doodles on a fixed closed oriented surface, in particular on the 2-sphere. We now extend the range of doodles to immersed circles on closed oriented surfaces of any genus defined in [9].

A representative of a doodle is defined by a pair (f, Σ) consisting of a smooth map

$$f : \sqcup_n \mathbb{S}^1 \to \Sigma$$

from n disjoint circles to a closed oriented surface Σ such that $|f^{-1}f(x)| < 3$ for every $x \in \sqcup_n \mathbb{S}^1$. The condition $|f^{-1}f(x)| < 3$ means that there are no triple or higher intersection points. The image of f on Σ is called a *doodle diagram* on the surface Σ. We assume that the doodle diagram intersects with each connected component of Σ. By *orientation* of a doodle diagram, we mean an orientation of underlying disjoint n circles. Two representatives are said to be *equivalent* if one can be obtained from the other by equivalence generated by $(1)-(3)$ defined as follows.

(1) **Homeomorphic equivalence.** Two representatives (f,Σ) and (f',Σ') are *homeomorphic equivalent* if there exists homeomorphisms $s : \sqcup_n \mathbb{S}^1 \to \sqcup_n \mathbb{S}^1$ and $S : \Sigma \to \Sigma'$ such that the following diagram commutes.

$$\begin{array}{ccc} \sqcup_n \mathbb{S}^1 & \xrightarrow{f} & \Sigma \\ {\scriptstyle s}\downarrow & & \downarrow{\scriptstyle S} \\ \sqcup_n \mathbb{S}^1 & \xrightarrow[f']{} & \Sigma' \end{array}$$

If we consider oriented doodle diagrams, then the maps s and S respect orientations of circles and Σ, respectively.

(2) **Homotopic equivalence.** For a fixed surface Σ, two representatives (f,Σ) and (f',Σ) with same number of components are *homotopic equivalent* if their images are related to each by a finite sequence of moves shown in Figure 2.9, that is the moves which generate and delete curls and bigons.

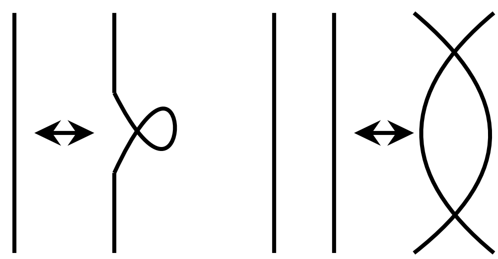

Fig. 2.9. Moves for homotopy equivalence of representatives of doodles

(3) **Surface surgery.** Surface surgery involves a finite sequence of addition and removal of handles disjoint from doodle diagrams. Consider two closed discs on the surface disjoint from the diagram. First we remove the interior of the two discs, and replace it with annulus $\mathbb{S}^1 \times [0,1]$ by glueing the boundary of annulus to the boundary of two discs (see Figure 2.10). This procedure is known as *handle addition*. The removal of

the handle is the reversal of handle addition. Two representatives are (f,Σ) and (f,Σ') are *equivalent* if Σ' is obtained from Σ by a sequence of handle additions and removal of handles disjoint from the doodle diagram.

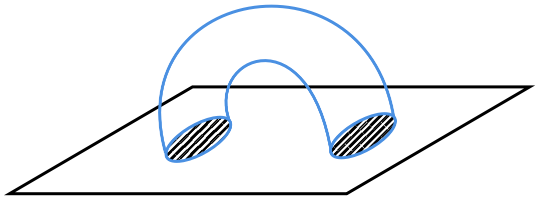

Fig. 2.10. Handle addition on surface Σ

Definition 2.3.1. *An equivalence class of a doodle diagram is called doodle on a surface.*

Figure 2.11 shows an example of a doodle diagram on the surface of genus two.

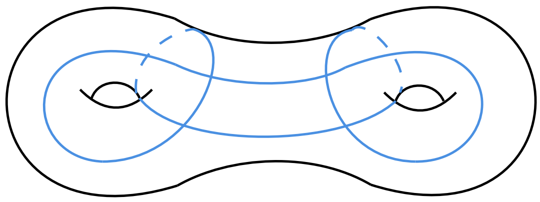

Fig. 2.11. Kishino doodle on the surface of genus two

Example 2.3.2. *Figure 2.12 illustrates the equivalence of doodle diagrams through an example. The first homeomorphism follows from the fact that if α and β are any two nonseperating simple closed curves in a surface Σ, then there exists a homeomorphism of the surface Σ sending α to β. In other words, upto homeomorphism, there exists a unique nonseperating simple closed curve in a fixed surface [23, Section 1.3.1]. The second equivalence involves removal of handle disjoint from the curve, whereas the third equivalence is due to the homeomorphism between the 2-sphere and a closed cylinder.*

2.4 Virtual doodles

In this section, we define virtual doodles which have resemblance with doodles on the 2-sphere having additional crossings. The role of virtual doodles is crucial in capturing the information of doodles on surfaces in a diagrammatic manner. The results of this section can be found in [9].

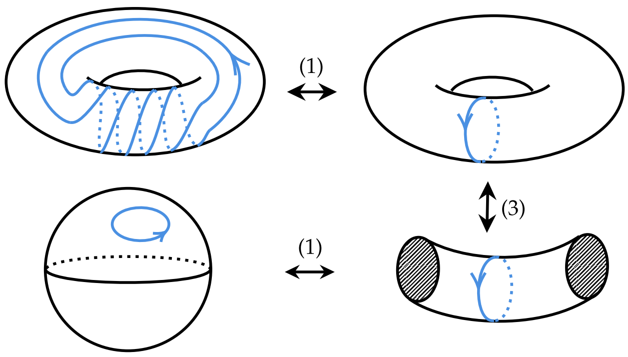

Fig. 2.12. Doodle on torus being equivalent to trivial doodle on the 2-sphere

Definition 2.4.1. *A virtual doodle diagram is a generic immersion of a closed one-dimensional manifold (disjoint union of circles) on the plane* $\mathbb{R}^2$ *with finitely many real or virtual crossings (as in Figure 2.13) such that there are no triple or higher real intersection points.*

By the term generic, we mean that all the crossings are transversal.

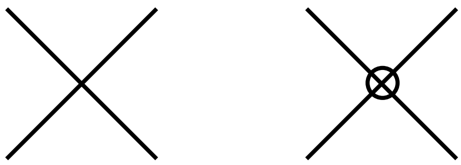

Fig. 2.13. Real and virtual crossings

Example 2.4.2. *An example of a virtual doodle is shown in Figure 2.14. The figure represents a flat virtual knot called the flat Kishino knot which was proved to be non-trivial as a flat virtual knot in [28, 44]. Thus, the flat Kishino knot is also non-trivial as a virtual doodle. The nomenclature is motivated by the Kishino knot diagram which is a diagram of a virtual knot whose non-triviality as a virtual knot is proven, for example, in [6, 53].*

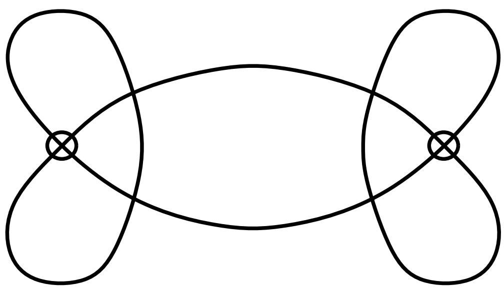

Fig. 2.14. Flat Kishino knot as virtual doodle

Two virtual doodle diagrams are said to be *equivalent* if they are related by a finite sequence of isotopies of the plane and R_1, R_2, VR_1, VR_2, VR_3, M moves as shown in Figure 2.15. Note that VR_1, VR_2, VR_3 and M are flat versions of virtual Reidemeister moves in virtual knot theory [48]. The moves R_1 and R_2 are also referred as flat versions of Reidemeister moves for classical knots.

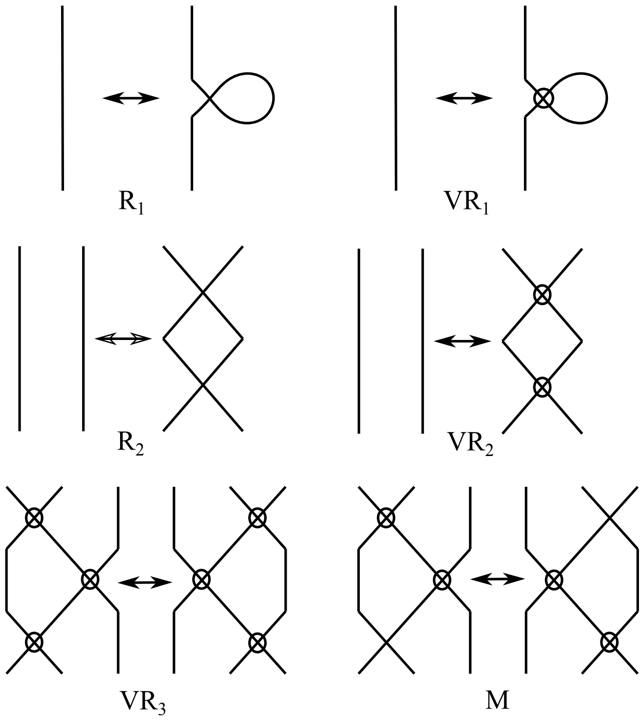

Fig. 2.15. Moves for virtual doodle diagrams

An *oriented virtual doodle diagram* is a doodle diagram with an orientation on each component of the underlying immersion. It is easy to see that there are a total of 28 moves for oriented virtual doodle diagrams. Further, any oriented move can be obtained as a composition of moves in Figure 2.16 and planar isotopies. From now on, by a virtual doodle diagram we mean an oriented virtual doodle diagram unless stated otherwise.

Definition 2.4.3. *A virtual doodle on the plane $\mathbb{R}^2$ is an equivalence class of a virtual doodle diagram.*

Remark 2.4.4. *Every classical link diagram can be regarded as an immersion of circles in the plane with an extra structure (of over/under crossing) at double points. If we take a diagram without this extra structure, then it is simply a shadow of some link in $\mathbb{R}^3$ and such crossings are called flat crossings in the literature [48]. An easy check shows that if one is allowed to apply the classical Reidemeister moves to such a diagram, then the diagram*

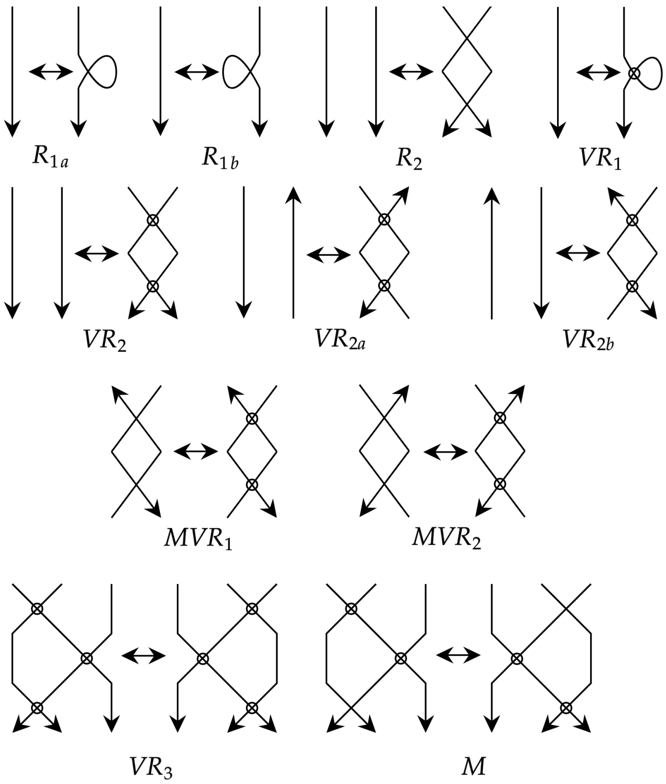

Fig. 2.16. Moves for oriented virtual doodle diagrams

can be reduced to a disjoint union of circles. However, this does not happen in flat virtual diagrams, that is, diagrams which have both flat and virtual crossings. It is worth noting that if we include the first forbidden move in the moves for virtual doodle diagrams, then we get precisely the theory of flat virtual knots initiated in [48].

The following result relates doodles on surfaces with virtual doodles on the plane.

Theorem 2.4.5 ([9], Theorem 6.4)**.** *There is a bijection from the family of oriented (or unoriented) virtual doodles to the family of oriented (or unoriented) doodles on surfaces.*

2.5 Gauss data

The Gauss data plays a major role in the proof of Theorem 2.4.5. Let K be a virtual doodle diagram on the plane with n real crossings. Let $N_1, N_2, \ldots, N_n$ be closed 2-disks each enclosing exactly one real crossing of the diagram K and $W(K)$ the closure of $\mathbb{R}^2 \setminus \cup_{i=1}^n N_i$ in the plane. Note that $W(K)$ consists of immersed arcs and loops in the plane where the intersection points are precisely the virtual crossings. Let $V_R(K)$ be the set of real crossings of K. Since we are considering oriented virtual doodle diagrams, for each real crossing c_i, the set $\partial N_i \cap c_i$ consists of four points and are assigned symbols as in Figure 2.17.

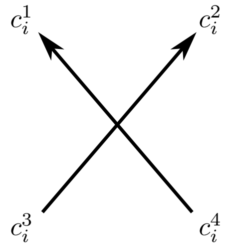

Fig. 2.17. Labelling at real crossing

Define

$$V_\partial(K) = \left\{c_i^j \mid i = 1,2,\ldots,n \text{ and } j = 1,2,3,4\right\}$$

and

$$\begin{aligned} X(K) \quad = \quad & \big\{(a,b) \in V_\partial(K) \times V_\partial(K) \mid \text{there is an arc in } K \cap W(K) \text{ starting} \\ & \text{at } a \text{ and ending at } b\big\}. \end{aligned}$$

Definition 2.5.1. *The pair $\big(V_R(K), X(K)\big)$ is called the Gauss data of a virtual doodle diagram K.*

Let K and K' be two virtual doodle diagrams each with n real crossings. We say that K and K' have the *same Gauss data* if there is a bijection $\sigma : V_R(K) \to V_R(K')$ such that whenever $(a,b) \in X(K)$, then $\big(\bar{\sigma}(a), \bar{\sigma}(b)\big) \in X(K')$, where $\bar{\sigma} : V_\partial(K) \to V_\partial(K')$ is defined as

$$\bar{\sigma}(c_i^j) = \sigma(c_i)^j.$$

The following result is proved in [9, Lemma 6.1].

Lemma 2.5.2. *Let K and K' be virtual doodle diagrams with the same number of real crossings. Then the following are equivalent:*

(i) K and K' have the same Gauss data with respect to a bijection between their real crossings,

(ii) K and K' are related by a finite sequence of moves VR_1, VR_2, VR_3 and M modulo isotopies of the plane,

(iii) K and K' are related by a finite sequence of Kauffman's detour moves (shown in Figure 2.18) modulo isotopies of the plane.

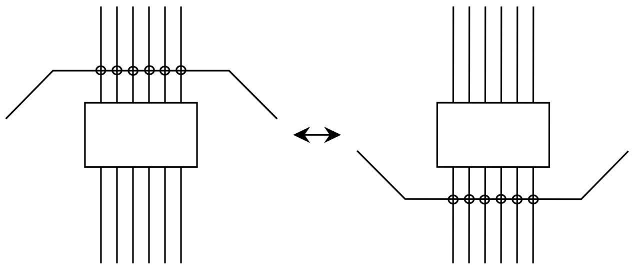

Fig. 2.18. Kauffman's detour move

The Gauss data will be crucial in establishing Alexander and Markov theorems for virtual doodles which we prove in the subsequent chapter.

2.6 Reidemeister-Schreier theorem

Reidemeister-Schreier method is one of the most applicable algorithms in combinatorial group theory, named after Kurt Reidemeister and Otto Schreier, which is used to compute presentations of subgroups of a group. This algorithm has been used at a few instances in the succeeding chapters. We refer the reader to [60, Theorem 2.6] for more details. The algorithm works even for groups with infinite presentations and subgroups of infinite index. However, we assume that the groups are finitely presented and that the index of the subgroup is finite. Let G be a group with presentation

$$\langle x_1, x_2, \dots, x_n \mid R_1, R_2, \dots, R_m \rangle,$$

and that the subgroup H of G is of finite index say t.
In this section, we recall the required terminologies and results which are fruitful in determining an explicit presentation of a subgroup H of G.
We begin by considering a full set of right coset representatives of H in G in which all the words are written in letters x_i's. Given any element $g \in G$, there exists a right coset representative which corresponds to the coset Hg. Let $\Lambda = \{\lambda_1, \lambda_2, \dots, \lambda_t\}$ be a complete set of right coset representatives of H in G. Then we know that every λ_i can be expressed as

$$\lambda_i = x_{i_1}^{\varepsilon_1} x_{i_2}^{\varepsilon_2} \dots x_{i_k}^{\varepsilon_k},$$

for $\varepsilon_i = \pm 1$ and some positive integer k.

Definition 2.6.1. *The set $\Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_t\}$ is said to be a Schreier system if for each $\lambda_i = x_{i_1}^{\varepsilon_1} x_{i_2}^{\varepsilon_2} \ldots x_{i_k}^{\varepsilon_k}$, the initial expressions*

$$x_{i_1}^{\varepsilon_1},\ x_{i_1}^{\varepsilon_1} x_{i_2}^{\varepsilon_2},\ \ldots,\ x_{i_1}^{\varepsilon_1} x_{i_2}^{\varepsilon_2} \ldots x_{i_{k-1}}^{\varepsilon_{k-1}}$$

also belong to the set Λ.

For every element $g \in G$, let $\overline{g}$ be the right coset representative of Hg in Λ. The following result gives the generators of the subgroup H in terms of words in letters x_i's.

Theorem 2.6.2 ([69], Proposition 6.2). *(1) For $g \in G$ and $\lambda_i \in \Lambda$, the element*

$$\gamma(\lambda_i, g) = (\lambda_i g)(\overline{\lambda_i g})^{-1}$$

belongs to the subgroup H.

(2) The subgroup H of G is generated by

$$S = \{\gamma(\lambda_i, x_j) = (\lambda_i x_j)(\overline{\lambda_i x_j})^{-1} \mid i = 1, 2, \ldots, t \text{ and } j = 1, 2, \ldots, n\}.$$

Next, to find the defining relations for the presentation of H with generating set S, we consider a word $x_{i_1}^{\varepsilon_1} x_{i_2}^{\varepsilon_2} \ldots x_{i_k}^{\varepsilon_k}$ in H. Note that $\overline{x_{i_1}^{\varepsilon_1} x_{i_2}^{\varepsilon_2} \ldots x_{i_k}^{\varepsilon_k}} = 1$. We consider the following transformation which expresses an element of H written as a word in generators x_i's into a word in generators of H mentioned in the preceding theorem.

$$\tau(x_{i_1}^{\varepsilon_1} x_{i_2}^{\varepsilon_2} \ldots x_{i_k}^{\varepsilon_k}) = \gamma(1, x_{i_1}^{\varepsilon_1})\gamma(\overline{x_{i_1}^{\varepsilon_1}}, x_{i_2}^{\varepsilon_2}) \cdots \gamma(\overline{x_{i_1}^{\varepsilon_1} x_{i_2}^{\varepsilon_2} \ldots x_{i_{k-1}}^{\varepsilon_{k-1}}}, x_{i_k}^{\varepsilon_k}).$$

This transformation τ is commonly known as a *rewriting process*.
The following result gives the full set of generators and relations for the subgroup H of group G.

Theorem 2.6.3 ([69], Theorem, 6.3). *Let $\Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_t\}$ be a Schreier system. Then the subgroup H of G has a presentation $H = \langle S \mid R \rangle$, where*

$$R = \{\tau(\lambda_i R_l \lambda_i^{-1}) \mid i = 1, 2, \ldots, t \text{ and } l = 1, 2, \ldots, m\rangle.$$

Chapter 3

Alexander and Markov theorems for doodles on surfaces

The virtual twin group VT_n was introduced in [4, Section 5] as an abstract generalisation of the twin group T_n. The group VT_n has generators $\{s_1, s_2, \ldots, s_{n-1}, \rho_1, \rho_2, \ldots, \rho_{n-1}\}$ and defining relations

$$\begin{aligned}
s_i^2 &= 1 \text{ for } i = 1, 2, \ldots, n-1, \\
s_i s_j &= s_j s_i \text{ for } |i-j| \geq 2, \\
\rho_i^2 &= 1 \text{ for } i = 1, 2, \ldots, n-1, \\
\rho_i \rho_j &= \rho_j \rho_i \text{ for } |i-j| \geq 2, \\
\rho_i \rho_{i+1} \rho_i &= \rho_{i+1} \rho_i \rho_{i+1} \text{ for } i = 1, 2, \ldots, n-2, \\
\rho_i s_j &= s_j \rho_i \text{ for } |i-j| \geq 2, \\
\rho_i \rho_{i+1} s_i &= s_{i+1} \rho_i \rho_{i+1} \text{ for } i = 1, 2, \ldots, n-2.
\end{aligned} \tag{3.1}$$

The kernel of the natural surjection from VT_n onto S_n, which sends each generator s_i and ρ_i to the transposition $(i, i+1)$, is called the *pure virtual twin group* and is denoted by VPT_n. In this chapter, we show that virtual twin groups play the role of groups in the theory of virtual doodles. We then establish the Alexander-Markov relation of oriented doodles on surfaces with certain classes of virtual twins on the plane.

The chapter is organised as follows. In Section 3.1, we give a topological interpretation of virtual twin groups. In Section 3.2, we define closed virtual twin diagrams of fixed degree and prove Alexander theorem for oriented virtual doodles. Lastly, in Section 3.3, we prove Markov theorem for oriented virtual doodles. The results are from the work [72].

3.1 Topological interpretation of virtual twin groups

Consider a set Q_n of n points in $\mathbb{R}$. A *virtual twin diagram* on n strands is a subset D of $\mathbb{R} \times [0,1]$ consisting of n intervals called *strands* with $\partial D = Q_n \times \{0,1\}$ and satisfying the following conditions.

(1) The natural projection $\mathbb{R} \times [0,1] \to [0,1]$ maps each strand homeomorphically onto the unit interval $[0,1]$, that is, each strand is monotonic.

(2) The set $V(D)$ of all crossings of the diagram D consists of transverse double points of D, where each crossing has the pre-assigned information of being a real or a virtual crossing as depicted in Figure 3.1. A virtual crossing is depicted by a crossing encircled with a small circle.

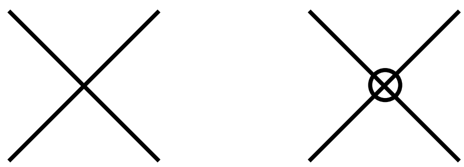

Fig. 3.1. Real and virtual crossings

Two virtual twin diagrams D_1 and D_2 on n strands are said to be *equivalent* if one can be obtained from the other by a finite sequence of moves as shown in Figure 3.2 and isotopies of the plane. We define a *virtual twin* to be an equivalence class of such virtual twin diagrams. Figure 3.3 shows an example of a virtual twin diagram on four strands.
Let $\mathcal{VT}_n$ denote the set of all virtual twins on n strands. The product D_1D_2 of two virtual twin diagrams D_1 and D_2 is defined by placing D_1 on top of D_2 and then shrinking the interval to $[0,1]$. It is clear that if D_1 is equivalent to D_1' and D_2 is equivalent to D_2' , then D_1D_2 is equivalent to $D_1'D_2'$. Thus, the binary operation makes $\mathcal{VT}_n$ a semigroup.

Remark 3.1.1. *It is worth noting that the moves in Figure 3.4 are forbidden and cannot be obtained from moves in Figure 3.2 (see Proposition 5.4.4).*

Lemma 3.1.2. *For each $n \geq 2$, the set $\mathcal{VT}_n$ of virtual twins forms a group under the operation defined above.*

Proof. We begin by noting that the virtual twin represented by a diagram of n strands with no crossings is the identity element with respect to the binary operation on the set $\mathcal{VT}_n$ of virtual twins. Let us define $\tilde{s}_i$ and $\tilde{\rho}_i$, $i = 1,2,\ldots,n-1$, to be the virtual twins represented

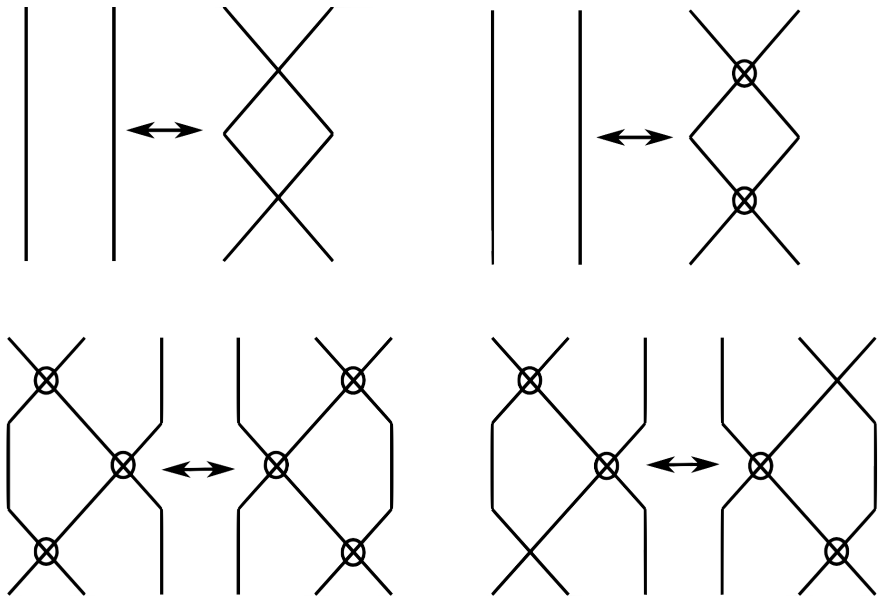

Fig. 3.2. Moves for virtual twin diagrams

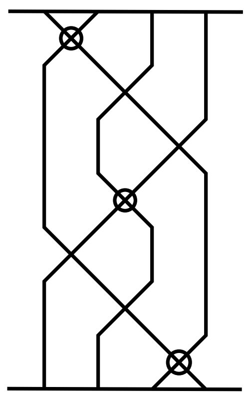

Fig. 3.3. Example of a virtual twin diagram on four strands

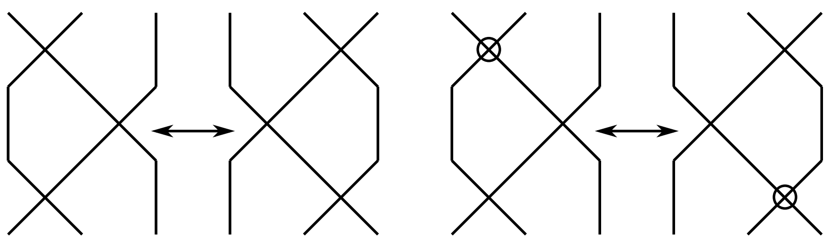

Fig. 3.4. Forbidden Moves

by diagrams as in Figure 3.5. Let β be any arbitrary element in $\mathcal{VT}_n$. Then after applying isotopies of the plane, β can be represented by a diagram $D \subset \mathbb{R} \times [0,1]$ such that the projection $\mathbb{R} \times [0,1] \to [0,1]$ restricted to the set $V(D)$ of all crossings is injective, that is, each crossing is at a distinct level. Further, it follows from the moves given in Figure 3.2 that $\tilde{s}_i^2 = 1$ and $\tilde{\rho}_i^2 = 1$ for all $i = 1,2,\dots,n-1$. Thus, we can write $\beta = \tilde{s}_{i_1}^{\varepsilon_1} \tilde{\rho}_{i_2}^{\varepsilon_2} \dots \tilde{s}_{i_k}^{\varepsilon_k}$ for some k, where $\varepsilon_i \in \{0,1\}$. Since $\tilde{s}_i$ and $\tilde{\rho}_i$ are self inverses, the element β has the inverse $\tilde{s}_{i_k}^{\varepsilon_k} \dots \tilde{\rho}_{i_2}^{\varepsilon_2} \tilde{s}_{i_1}^{\varepsilon_1}$. $\square$

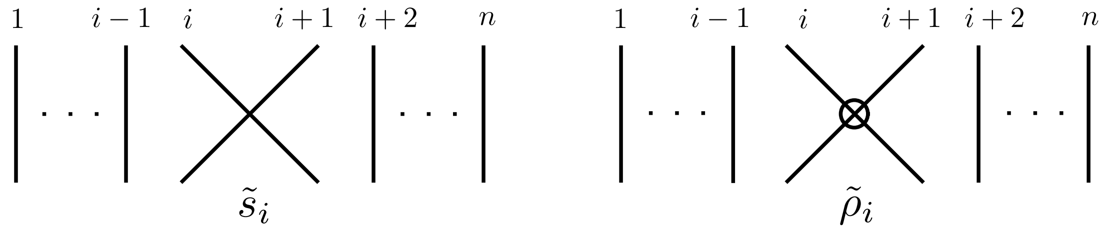

Fig. 3.5. Generators $\tilde{s}_i$ and $\tilde{\rho}_i$

Proposition 3.1.3. *The diagrammatic group $\mathcal{VT}_n$ and the abstract group VT_n are isomorphic for all $n \geq 2$.*

Proof. It follows from the definition of equivalence of two virtual twin diagrams on n strands that the generators $\tilde{s}_i$ and $\tilde{\rho}_i$ satisfy the following relations.

$$\begin{aligned}
\tilde{s}_i^2 &= 1 \text{ for } i = 1,2,\dots,n-1,\\
\tilde{s}_i\tilde{s}_j &= \tilde{s}_j\tilde{s}_i \text{ for } |i-j| \geq 2,\\
\tilde{\rho}_i^2 &= 1 \text{ for } i = 1,2,\dots,n-1,\\
\tilde{\rho}_i\tilde{\rho}_j &= \tilde{\rho}_j\tilde{\rho}_i \text{ for } |i-j| \geq 2,\\
\tilde{\rho}_i\tilde{\rho}_{i+1}\tilde{\rho}_i &= \tilde{\rho}_{i+1}\tilde{\rho}_i\tilde{\rho}_{i+1} \text{ for } i = 1,2,\dots,n-2,\\
\tilde{\rho}_i\tilde{s}_j &= \tilde{s}_j\tilde{\rho}_i \text{ for } |i-j| \geq 2,\\
\tilde{\rho}_i\tilde{\rho}_{i+1}\tilde{s}_i &= \tilde{s}_{i+1}\tilde{\rho}_i\tilde{\rho}_{i+1} \text{ for } i = 1,2,\dots,n-2.
\end{aligned}$$

Thus, there exists a unique group homomorphism

$$f_n : VT_n \to \mathcal{VT}_n$$

given by $f_n(s_i) = \tilde{s}_i$ and $f_n(\rho_i) = \tilde{\rho}_i$ for $i = 1,2,\dots,n-1$. Since every $\beta \in \mathcal{VT}_n$ can be written as a product of $\tilde{s}_i$ and $\tilde{\rho}_i$, the map f_n is surjective. For an element $\tilde{s}_{i_1}^{\varepsilon_1} \tilde{\rho}_{i_2}^{\varepsilon_2} \dots \tilde{s}_{i_k}^{\varepsilon_k} \in \mathcal{VT}_n$, where $\varepsilon_i \in \{0,1\}$, define

$$g_n : \mathcal{VT}_n \to VT_n$$

by $g_n\big(\tilde{s}_{i_1}^{\varepsilon_1}\tilde{\rho}_{i_2}^{\varepsilon_2}\ldots\tilde{s}_{i_k}^{\varepsilon_k}\big) = s_{i_1}^{\varepsilon_1}\rho_{i_2}^{\varepsilon_2}\ldots s_{i_k}^{\varepsilon_k}$. We prove that g_n is well-defined. Let D be a virtual twin diagram representing the element $\tilde{s}_{i_1}^{\varepsilon_1}\tilde{\rho}_{i_2}^{\varepsilon_2}\ldots\tilde{s}_{i_k}^{\varepsilon_k}$. A diagram obtained by a planar isotopy on D that does not change the order of the image of $V(D)$ in $[0,1]$ under the projection map $\mathbb{R}\times[0,1]\to[0,1]$ is again represented by the element $\tilde{s}_{i_1}^{\varepsilon_1}\tilde{\rho}_{i_2}^{\varepsilon_2}\ldots\tilde{s}_{i_k}^{\varepsilon_k}$. Any move that interchanges two points in the image of $V(D)$ under the projection $\mathbb{R}\times[0,1]\to[0,1]$ exchanges the subwords $\tilde{s}_i\tilde{s}_j$ and $\tilde{s}_j\tilde{s}_i$, $\tilde{s}_i\tilde{\rho}_j$ and $\tilde{\rho}_j\tilde{s}_i$ or $\tilde{\rho}_i\tilde{\rho}_j$ and $\tilde{\rho}_j\tilde{\rho}_i$ in the word $\tilde{s}_{i_1}^{\varepsilon_1}\tilde{\rho}_{i_2}^{\varepsilon_2}\ldots\tilde{s}_{i_k}^{\varepsilon_k}$ for some $|i-j|\geq 2$. Under each of these cases, the images of the corresponding words under g_n are the same element in VT_n. The move that adds (respectively, removes) two points in $V(D)$ adds (respectively, removes) subwords of the form $\tilde{s}_i\tilde{s}_i$ or $\tilde{\rho}_i\tilde{\rho}_i$ in the word $\tilde{s}_{i_1}^{\varepsilon_1}\tilde{\rho}_{i_2}^{\varepsilon_2}\ldots\tilde{s}_{i_k}^{\varepsilon_k}$. But $s_i^2 = 1 = \rho_i^2$ in VT_n, and hence both the words are mapped to same element under g_n. The third move interchanges the subwords $\tilde{\rho}_i\tilde{\rho}_{i+1}\tilde{\rho}_i$ and $\tilde{\rho}_{i+1}\tilde{\rho}_i\tilde{\rho}_{i+1}$ in the word $\tilde{s}_{i_1}^{\varepsilon_1}\tilde{\rho}_{i_2}^{\varepsilon_2}\ldots\tilde{s}_{i_k}^{\varepsilon_k}$. But VT_n has the relation $\rho_i\rho_{i+1}\rho_i = \rho_{i+1}\rho_i\rho_{i+1}$. Finally, the last move replaces the subwords $\tilde{\rho}_i\tilde{\rho}_{i+1}\tilde{s}_i$ and $\tilde{s}_{i+1}\tilde{\rho}_i\tilde{\rho}_{i+1}$, but VT_n has the relation $\rho_i\rho_{i+1}s_i = s_{i+1}\rho_i\rho_{i+1}$, and hence g_n is well-defined. Since $g_n\circ f_n = \mathrm{id}$, f_n is injective and the proof is complete. □

Since the diagrammatic group $\mathcal{VT}_n$ and the abstract group VT_n have been identified, from now onwards, the generators s_i and ρ_i will be represented geometrically as in Figure 3.5.

3.2 Alexander Theorem for virtual doodles

Consider the space $\mathbb{R}^2\setminus\mathbb{D}^\circ$, where $\mathbb{D}^\circ$ is the interior of the closed unit 2-disk $\mathbb{D}$ centred at the origin. Let K be an oriented virtual doodle diagram on the plane satisfying the following:

(1) K is contained in $\mathbb{R}^2\setminus\mathbb{D}^\circ$.

(2) If $\pi : \mathbb{R}^2\setminus\mathbb{D}^\circ \to \mathbb{S}^1$ is the radial projection and $k : \sqcup\, \mathbb{S}^1 \to \mathbb{R}^2\setminus\mathbb{D}^\circ$ the underlying immersion of K, then

$$\pi\circ k : \sqcup\, \mathbb{S}^1 \to \mathbb{S}^1$$

is an n-fold covering, where $\mathbb{S}^1$ is the boundary of $\mathbb{D}$ and we assume it to be oriented counterclockwise.

(3) The map π restricted to $V(K)$, the set of all crossings of K, is injective.

(4) The orientation of K is compatible with a fixed orientation of $\mathbb{S}^1$.

Definition 3.2.1. *A closed virtual twin diagram of degree n is an oriented virtual doodle diagram satisfying conditions (1)-(4) defined above.*

Figure 3.6 shows an example of a closed virtual twin diagram of degree 3. Consider a point $p \in \mathbb{S}^1$ such that $\pi^{-1}(p) \cap V(K) = \phi$. Then cutting along the ray emanating from the origin and passing through p gives a virtual twin diagram on n strands.

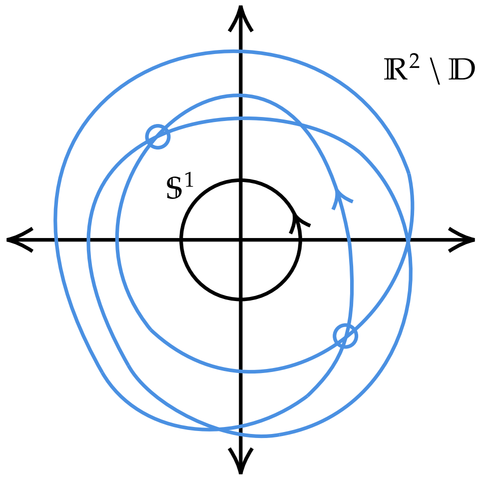

Fig. 3.6. Closed virtual twin diagram of degree 3

Definition 3.2.2. *The closure of a virtual twin diagram on the plane is defined to be the doodle obtained from the diagram by joining the end points with non-intersecting curves as shown in Figure 3.7.*

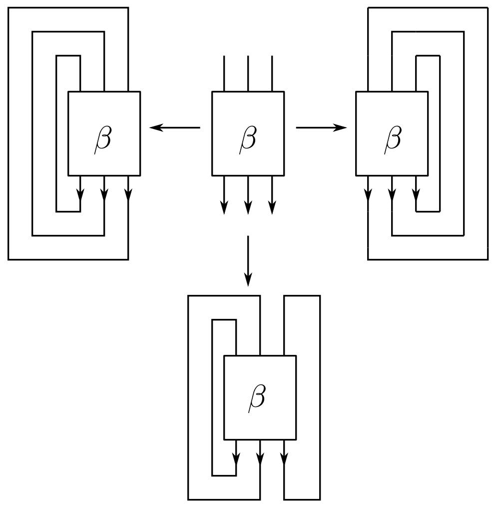

Fig. 3.7. Different closures of a virtual twin diagram

We observe that in the case of classical twins, due to forbidden move $s_i s_{i+1} s_i \neq s_{i+1} s_i s_{i+1}$, taking closure of a twin diagram on a plane is not well-defined. Note that there are many ways of taking closure of a virtual twin diagram. The following result shows that the operation of taking closure on a plane in the virtual setting is well-defined.

Lemma 3.2.3. *Any two closures of a virtual twin diagram on the plane gives equivalent virtual doodle diagrams on the plane.*

Proof. Let β be a virtual twin diagram and K and K' two different closures of β. Note that $V_R(K) = V_R(K')$. Taking $\sigma = \text{id}$ we see that whenever $(a,b) \in X(K)$, then $(a,b) \in X(K')$. Thus, K and K' have the same Gauss data. By Lemma 2.5.2, K and K' are related by a finite sequence of VR_1, VR_2, VR_3 and M moves. Consequently, K and K' are equivalent virtual doodle diagrams on the plane. It can also be observed that K' can be obtained from K by a finite sequence of Kauffman's detour move depicted in Figure 2.18. □

We now prove Alexander Theorem for virtual doodles.

Theorem 3.2.4. *Every oriented virtual doodle on the plane is equivalent to closure of a virtual twin diagram.*

Proof. Let K be a virtual doodle diagram with n real crossings. The idea is to construct a closed virtual twin diagram with the same Gauss data as that of K. The proof then follows from Lemma 2.5.2. We label each real crossing of K as in Figure 2.17. Next, we consider $\mathbb{R}^2 \setminus \mathbb{D}^\circ$ and orient the boundary $\mathbb{S}^1$ of $\mathbb{D}$, say, counterclockwise. Considering the real crossings of K with the information assigned as in Figure 2.17, we place them in $\mathbb{R}^2 \setminus \mathbb{D}$ such that $\pi(c_i) \cap \pi(c_j) = \phi$ for all $i \neq j$ and the orientation is compatible with the orientation of $\mathbb{S}^1$. Next, we join these crossings in $\mathbb{R}^2 \setminus \mathbb{D}$ according to the Gauss data such that each intersection of arcs is marked as a virtual crossing and the orientation of arcs/loops are compatible with the orientation of $\mathbb{S}^1$, as illustrated in Figure 3.8. In other words, for each $(a,b) \in X(K)$ the orientation of the arc joining a to b should be compatible with the orientation of $\mathbb{S}^1$, that is, there is a possibility that we will have to wind the arc around $\mathbb{S}^1$ to join a and b. Also, whenever it intersects with some other arc, then the intersection point should be marked as a virtual crossing. Note that this process is well defined upto detour moves shown in Figure 2.18, and virtual doodle so obtained is a closed virtual twin diagram which has the same Gauss data as that of K. Finally, cutting along $\pi^{-1}(p)$ for a point $p \in \mathbb{S}^1$ such that $\pi^{-1}(p)$ does not pass through any crossing gives the desired virtual twin diagram whose closure is K. □

Following [46], for convenience in writing, we refer the process of construction of a virtual twin in Theorem 3.2.4 as the *braiding process*, which is illustrated for the virtual Kishino doodle in Figure 3.9.

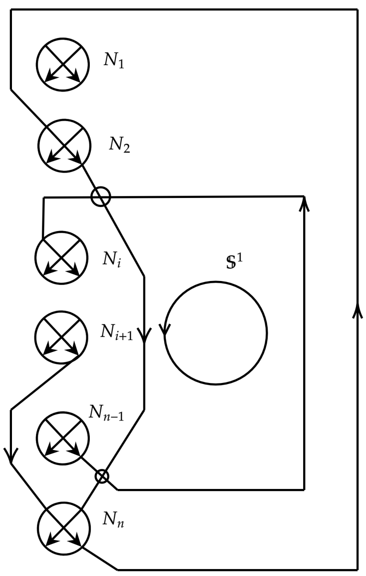

Fig. 3.8

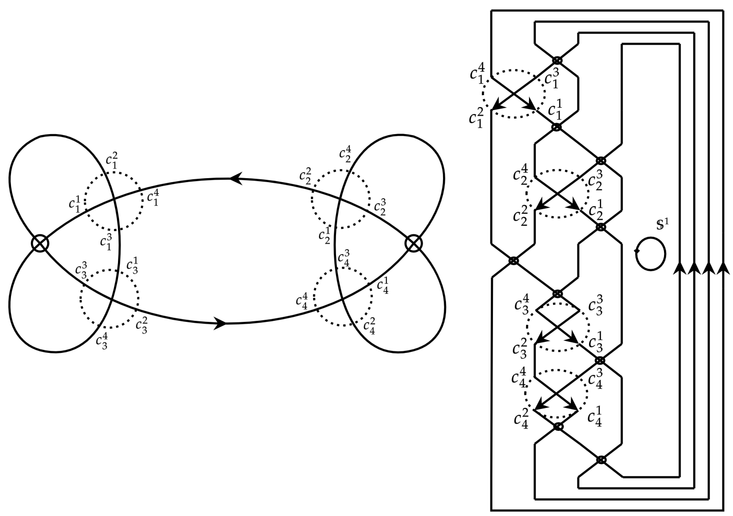

Fig. 3.9. Application of braiding process on virtual Kishino doodle

3.3 Markov Theorem for virtual doodles

For $\beta \in VT_n$, let $m \otimes \beta \in VT_{n+m}$ denote the virtual twin obtained by putting trivial m strands on the left of β as shown in Figure 3.10.

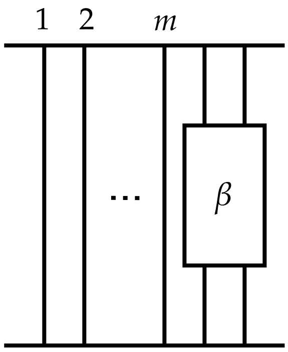

Fig. 3.10. The virtual twin $m \otimes \beta$

For $n \geq 2$ and virtual twins $\alpha, \beta, \beta_1, \beta_2 \in VT_n$, consider the following moves as illustrated in Figures 3.11 and 3.12:

$(M0)$ Defining relations 3.1 in VT_n (cf. Figure 3.2).

$(M1)$ Conjugation: $\alpha^{-1}\beta\alpha \sim \beta$.

$(M2)$ Right stabilisation of real or virtual type: $\beta s_n \sim \beta$ or $\beta \rho_n \sim \beta$.

$(M3)$ Left stabilisation of real type: $(1 \otimes \beta)s_1 \sim \beta$.

$(M4)$ Right exchange: $\beta_1 s_n \beta_2 s_n \sim \beta_1 \rho_n \beta_2 \rho_n$.

$(M5)$ Left exchange: $s_1(1 \otimes \beta_1)s_1(1 \otimes \beta_2) \sim \rho_1(1 \otimes \beta_1)\rho_1(1 \otimes \beta_2)$.

We observe that the left stabilisation of virtual type $(1 \otimes \beta)\rho_1 \sim \beta$ is a consequence of the other moves as shown in Figure 3.13.
The following results are crucial in the proof of Markov theorem for oriented virtual doodles.

Lemma 3.3.1. *Let $n \geq 2$ and $1 \leq i \leq n$. Under the assumption of moves $M0 - M5$, the following hold:*

(1) $\beta s_n s_{n-1} \dots s_{i+1} s_i s_{i+1} \dots s_{n-1} s_n \sim \beta$, *where* $\beta \in VT_n$.

(2) $s_n s_{n-1} \dots s_{i+1} s_i \beta_1 s_i s_{i+1} \dots s_n \beta_2 \sim \rho_n \rho_{n-1} \dots \rho_{i+1} \rho_i \beta_1 \rho_i \rho_{i+1} \dots \rho_n \beta_2$, *where* $\beta_1 \in VT_i$ *and* $\beta_2 \in VT_n$.

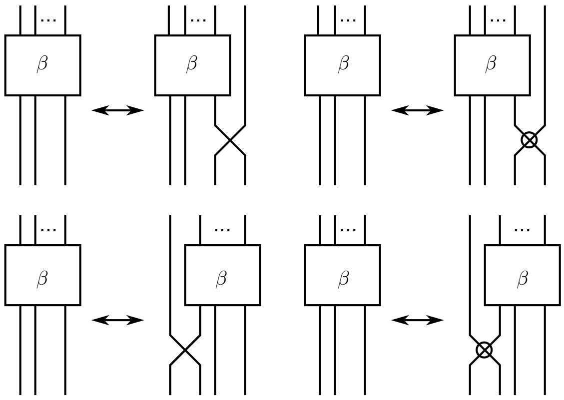

Fig. 3.11. Left and right stabilisation of real and virtual type

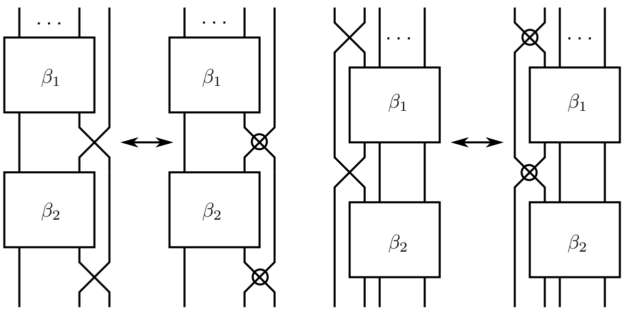

Fig. 3.12. Left and right exchange

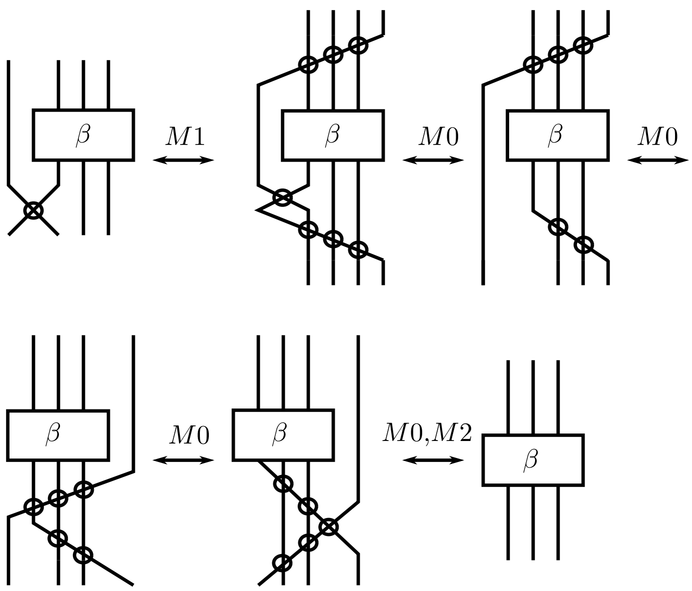

Fig. 3.13. Left stabilisation of virtual type as a consequence of $M0-M5$

(3) $\tau_n\tau_{n-1}\ldots\tau_{i+1}\tau_i\beta_1\tau_i\tau_{i+1}\ldots\tau_{n-1}\tau_n\beta_2 \sim \rho_n\rho_{n-1}\ldots\rho_{i+1}\rho_i\beta_1\rho_i\rho_{i+1}\ldots\rho_n\beta_2$*, where* $\beta_1 \in VT_i$*,* $\beta_2 \in VT_n$ *and* $\tau_j = s_j$ *or* ρ_j *for each* j*.*

(4) $\beta\tau_n\tau_{n-1}\ldots\tau_i\tau_{i-1}\tau_i\ldots\tau_{n-1}\tau_n \sim \beta$*, where* $\beta \in VT_n$ *and* $\tau_j = s_j$ *or* ρ_j *for each* j*.*

Proof. We begin by observing that the case $i = n$ holds due to move $M2$. Also, for $i = n-1$, we have

$$\begin{aligned}
\beta\underline{s_n}s_{n-1}\underline{s_n} &\overset{M4}{\sim} \beta\underline{\rho_n s_{n-1}\rho_n}\\
&\overset{M0}{\sim} \beta\rho_{n-1}s_n\underline{\rho_{n-1}}\\
&\overset{M1}{\sim} \rho_{n-1}\beta\rho_{n-1}\underline{s_n}\\
&\overset{M2}{\sim} \underline{\rho_{n-1}}\beta\underline{\rho_{n-1}}\\
&\overset{M1}{\sim} \beta.
\end{aligned}$$

Let us suppose that

$$\beta s_n s_{n-1}\ldots s_{i+2}s_{i+1}s_{i+2}\ldots s_{n-1}s_n \sim \beta \tag{3.2}$$

for $1 \le i \le n-2$ and for any $\beta \in VT_n$. Then, we have

$$\begin{aligned}
&\beta\underline{s_n}s_{n-1}\ldots s_{i+1}s_is_{i+1}\ldots s_{n-1}\underline{s_n}\\
\overset{M4}{\sim}\ &\beta\rho_ns_{n-1}\underline{s_{n-2}\ldots s_{i+1}s_is_{i+1}\ldots s_{n-2}}s_{n-1}\rho_n\\
\overset{M0}{\sim}\ &\beta\underline{\rho_ns_{n-1}\rho_n}\ldots s_{i+1}s_is_{i+1}\ldots\underline{\rho_ns_{n-1}\rho_n}\\
\overset{M0}{\sim}\ &\beta\rho_{n-1}s_n\rho_{n-1}\underline{s_{n-2}\ldots s_{i+1}s_is_{i+1}\ldots s_{n-2}}\rho_{n-1}s_n\rho_{n-1}\\
\overset{M0}{\sim}\ &\beta\rho_{n-1}s_n\underline{\rho_{n-1}s_{n-2}\rho_{n-1}}\ldots s_{i+1}s_is_{i+1}\ldots\underline{\rho_{n-1}s_{n-2}\rho_{n-1}}s_n\rho_{n-1}\\
\overset{M0}{\sim}\ &\beta\rho_{n-1}\underline{s_n\rho_{n-2}}s_{n-1}\rho_{n-2}\ldots s_{i+1}s_is_{i+1}\ldots\rho_{n-2}s_{n-1}\underline{\rho_{n-2}s_n}\rho_{n-1}\\
\overset{M0}{\sim}\ &\beta\rho_{n-1}\rho_{n-2}s_ns_{n-1}\rho_{n-2}\ldots s_{i+1}s_is_{i+1}\ldots\rho_{n-2}s_{n-1}s_n\rho_{n-2}\rho_{n-1}.
\end{aligned}$$

Repeating the above steps give

$$\begin{aligned}
&\beta s_ns_{n-1}\ldots s_{i+1}s_is_{i+1}\ldots s_{n-1}s_n\\
\sim\ &\beta\rho_{n-1}\rho_{n-2}\ldots\rho_{i+1}s_ns_{n-1}\ldots s_{i+2}\underline{\rho_{i+1}s_i\rho_{i+1}}s_{i+2}\ldots s_{n-1}s_n\rho_{i+1}\ldots\rho_{n-2}\rho_{n-1}\\
\overset{M0}{\sim}\ &\beta\rho_{n-1}\rho_{n-2}\ldots\rho_{i+1}s_ns_{n-1}\ldots s_{i+2}\underline{\rho_i}s_{i+1}\underline{\rho_i}s_{i+2}\ldots s_{n-1}s_n\rho_{i+1}\ldots\rho_{n-2}\rho_{n-1}\\
\overset{M0}{\sim}\ &\beta\rho_{n-1}\rho_{n-2}\ldots\rho_is_ns_{n-1}\ldots s_{i+2}s_{i+1}s_{i+2}\ldots s_{n-1}s_n\underline{\rho_i\ldots\rho_{n-2}\rho_{n-1}}\\
\overset{M1}{\sim}\ &\rho_i\ldots\rho_{n-2}\rho_{n-1}\beta\rho_{n-1}\rho_{n-2}\ldots\rho_i\underline{s_ns_{n-1}\ldots s_{i+2}s_{i+1}s_{i+2}\ldots s_{n-1}s_n}.
\end{aligned}$$

Since $\rho_i \dots \rho_{n-2}\rho_{n-1}\beta\rho_{n-1}\rho_{n-2}\dots\rho_i \in VT_n$, by (3.2) and move $M1$, we get

$$\beta s_n s_{n-1}\dots s_{i+1}s_i s_{i+1}\dots s_{n-1}s_n \sim \underline{\rho_i \dots \rho_{n-2}\rho_{n-1}}\beta\,\underline{\rho_{n-1}\rho_{n-2}\dots\rho_i} \overset{M1}{\sim} \beta.$$

This proves assertion (1).

For assertion (2), note that the case $i = n$ follows from moves $M1$ and $M4$. Let us suppose that for any $\beta_1 \in VT_{i+1}$ and $\beta_2 \in VT_n$, we have

$$s_n s_{n-1}\dots s_{i+2}s_{i+1}\beta_1 s_{i+1}s_{i+2}\dots s_n\beta_2 \sim \rho_n\rho_{n-1}\dots\rho_{i+2}\rho_{i+1}\beta_1\rho_{i+1}\rho_{i+2}\dots\rho_n\beta_2. \tag{3.3}$$

We claim that

$$s_n s_{n-1}\dots s_{i+1}s_i\beta_1 s_i s_{i+1}\dots s_{n-1}s_n\beta_2 \sim \rho_n\rho_{n-1}\dots\rho_{i+1}\rho_i\beta_1\rho_i\rho_{i+1}\dots\rho_{n-1}\rho_n\beta_2$$

for $\beta_1 \in VT_i$ and $\beta_2 \in VT_n$. For $1 \le i \le n-1$, we have

$$\begin{aligned}
& s_n s_{n-1}\dots s_{i+1}s_i\beta_1 s_i s_{i+1}\dots s_{n-1}s_n\underline{\beta_2}\\
\overset{M1}{\sim}\; & \beta_2\underline{s_n}s_{n-1}\dots s_{i+1}s_i\beta_1 s_i s_{i+1}\dots s_{n-1}\underline{s_n}\\
\overset{M4}{\sim}\; & \underline{\beta_2}\rho_n s_{n-1}\dots s_{i+1}s_i\beta_1 s_i s_{i+1}\dots s_{n-1}\rho_n\\
\overset{M1}{\sim}\; & \rho_n s_{n-1}\underline{s_{n-2}\dots s_{i+1}s_i\beta_1 s_i s_{i+1}\dots s_{n-2}}s_{n-1}\rho_n\beta_2\\
\overset{M0}{\sim}\; & \underline{\rho_n s_{n-1}\rho_n}\dots s_{i+1}s_i\beta_1 s_i s_{i+1}\dots\underline{\rho_n s_{n-1}\rho_n}\beta_2\\
\overset{M0}{\sim}\; & \rho_{n-1}s_n\rho_{n-1}\dots s_{i+1}s_i\beta_1 s_i s_{i+1}\dots\rho_{n-1}s_n\rho_{n-1}\beta_2\\
\overset{M0}{\sim}\; & \rho_{n-1}s_n\underline{\rho_{n-1}s_{n-2}\rho_{n-1}}\dots s_i\beta_1 s_i\dots\underline{\rho_{n-1}s_{n-2}\rho_{n-1}}s_n\rho_{n-1}\beta_2\\
\overset{M0}{\sim}\; & \rho_{n-1}\underline{s_n\rho_{n-2}}s_{n-1}\rho_{n-2}\dots s_i\beta_1 s_i\dots\rho_{n-2}s_{n-1}\underline{\rho_{n-2}s_n}\rho_{n-1}\beta_2\\
\overset{M0}{\sim}\; & \rho_{n-1}\rho_{n-2}s_n s_{n-1}\rho_{n-2}\dots s_i\beta_1 s_i\dots\rho_{n-2}s_{n-1}s_n\rho_{n-2}\rho_{n-1}\beta_2.
\end{aligned}$$

Repeating the preceding process yields

$$\begin{aligned}
& s_n s_{n-1}\dots s_{i+1}s_i\beta_1 s_i s_{i+1}\dots s_{n-1}s_n\beta_2\\
\sim\; & \rho_{n-1}\rho_{n-2}\dots\rho_i\underline{s_n s_{n-1}\dots s_{i+1}\rho_i\beta_1\rho_i s_{i+1}\dots s_{n-1}s_n}\rho_i\dots\rho_{n-2}\rho_{n-1}\beta_2.
\end{aligned}$$

Notice that $\rho_i\beta_1\rho_i \in VT_{i+1}$ and $\rho_i\dots\rho_{n-2}\rho_{n-1}\beta_2\rho_{n-1}\rho_{n-2}\dots\rho_i \in VT_n$. By (3.3) and $M1$, we get

$$\begin{aligned}
& s_ns_{n-1}\dots s_{i+1}s_i\beta_1 s_is_{i+1}\dots s_{n-1}s_n\beta_2 \\
\sim\ & \underline{\rho_{n-1}\rho_{n-2}\dots\rho_i} s_ns_{n-1}\dots s_{i+1}\rho_i\beta_1\rho_i s_{i+1}\dots s_{n-1}s_n\rho_i\dots\rho_{n-2}\rho_{n-1}\beta_2 \\
\stackrel{M1}{\sim}\ & (s_ns_{n-1}\dots s_{i+1})(\rho_i\beta_1\rho_i)(s_{i+1}\dots s_{n-1}s_n)(\rho_i\dots\rho_{n-2}\rho_{n-1}\beta_2\rho_{n-1}\rho_{n-2}\dots\rho_i) \\
\sim\ & (\rho_n\rho_{n-1}\dots\rho_{i+1})(\rho_i\beta_1\rho_i)(\rho_{i+1}\dots\rho_{n-1}\rho_n)(\rho_i\dots\rho_{n-2}\rho_{n-1}\beta_2\underline{\rho_{n-1}\rho_{n-2}\dots\rho_i}) \\
\stackrel{M1}{\sim}\ & \rho_{n-1}\rho_{n-2}\dots\underline{\rho_i}\rho_n\rho_{n-1}\dots\rho_{i+1}\rho_i\beta_1\rho_i\rho_{i+1}\dots\rho_{n-1}\rho_n\underline{\rho_i}\dots\rho_{n-2}\rho_{n-1}\beta_2 \\
\stackrel{M0}{\sim}\ & \rho_{n-1}\rho_{n-2}\dots\rho_{i+1}\rho_n\rho_{n-1}\dots\underline{\rho_i\rho_{i+1}\rho_i}\beta_1\underline{\rho_i\rho_{i+1}\rho_i}\dots\rho_{n-1}\rho_n\rho_{i+1}\dots\rho_{n-2}\rho_{n-1}\beta_2 \\
\stackrel{M0}{\sim}\ & \rho_{n-1}\rho_{n-2}\dots\rho_{i+1}\rho_n\rho_{n-1}\dots\rho_{i+1}\rho_i\underline{\rho_{i+1}\beta_1\rho_{i+1}}\rho_i\rho_{i+1}\dots\rho_{n-1}\rho_n\rho_{i+1}\dots\rho_{n-1}\beta_2 \\
\sim\ & \rho_{n-1}\dots\underline{\rho_{i+1}}\rho_n\rho_{n-1}\dots\rho_{i+1}\rho_i\beta_1\rho_i\rho_{i+1}\dots\rho_{n-1}\rho_n\underline{\rho_{i+1}}\dots\rho_{n-1}\beta_2, \\
& (\rho_{i+1}\text{'s gets canceled as } \beta_1 \in VT_i).
\end{aligned}$$

Repeating the above steps finally gives

$$s_ns_{n-1}\dots s_{i+1}s_i\beta_1 s_is_{i+1}\dots s_{n-1}s_n\beta_2 \sim \rho_n\rho_{n-1}\dots\rho_{i+1}\rho_i\beta_1\rho_i\rho_{i+1}\dots\rho_{n-1}\rho_n\beta_2,$$

which proves assertion (2).
Repeatedly applying (2) on the expression $\tau_n\tau_{n-1}\dots\tau_{i+1}\tau_i\beta_1\tau_i\tau_{i+1}\dots\tau_{n-1}\tau_n\beta_2$ yields assertion (3). For example,

$$\begin{aligned}
& \underline{s_n}\rho_{n-1}s_{n-2}\rho_{n-3}\beta_1\rho_{n-3}s_{n-2}\rho_{n-1}\underline{s_n}\beta_2 \\
\sim\ & \underline{\rho_n\rho_{n-1}}s_{n-2}\rho_{n-3}\beta_1\rho_{n-3}s_{n-2}\underline{\rho_{n-1}\rho_n}\beta_2 \\
\sim\ & \underline{s_ns_{n-1}s_{n-2}}\rho_{n-3}\beta_1\rho_{n-3}\underline{s_{n-2}s_{n-1}s_n}\beta_2 \\
\sim\ & \rho_n\rho_{n-1}\rho_{n-2}\rho_{n-3}\beta_1\rho_{n-3}\rho_{n-2}\rho_{n-1}\rho_n\beta_2.
\end{aligned}$$

For assertion (4), if we put $\beta_1 = \tau_{i-1}$ and $\beta_2 = \beta$ in assertion (3), then we get

$$\begin{aligned}
& \underline{\tau_n\tau_{n-1}\dots\tau_{i+1}\tau_i\tau_{i-1}\tau_i\tau_{i+1}\dots\tau_{n-1}\tau_n}\beta \\
\stackrel{M1}{\sim}\ & \beta\,\underline{\tau_n\tau_{n-1}\dots\tau_{i+1}\tau_i\tau_{i-1}\tau_i\tau_{i+1}\dots\tau_{n-1}\tau_n} \\
\sim\ & \beta\rho_n\rho_{n-1}\dots\rho_{i+1}\rho_i\tau_{i-1}\rho_i\rho_{i+1}\dots\rho_{n-1}\rho_n \ \ \big(\text{by taking } \beta_1 = \tau_{i-1} \text{ and } \beta_2 = \beta \text{ in (3)}\big).
\end{aligned}$$

If $\tau = \rho$, then

$$\begin{aligned}
& \beta\rho_n\rho_{n-1}\ldots\rho_i\rho_{i-1}\rho_i\ldots\rho_{n-1}\rho_n \\
\sim\ & \beta\underline{\rho_n\rho_{n-1}\ldots\rho_{i-1}\rho_i\rho_{i+2}\rho_{i+1}\rho_{i+2}\rho_i\rho_{i-1}\ldots\rho_{n-1}\rho_n} \text{ (by repeated application of } M0) \\
\sim\ & \beta\rho_{i-1}\rho_i\ldots\rho_{n-1}\rho_n\underline{\rho_{n-1}\ldots\rho_i\rho_{i-1}} \text{ (by repeated application of the preceding step)} \\
\overset{M1}{\sim}\ & \rho_{n-1}\ldots\rho_i\rho_{i-1}\beta\rho_{i-1}\rho_i\ldots\rho_{n-1}\underline{\rho_n} \\
\overset{M2}{\sim}\ & \underline{\rho_{n-1}\ldots\rho_i\rho_{i-1}}\beta\underline{\rho_{i-1}\rho_i\ldots\rho_{n-1}} \\
\overset{M1}{\sim}\ & \beta.
\end{aligned}$$

Finally if $\tau = s$, then we get

$$\begin{aligned}
& \beta\underline{\rho_n\rho_{n-1}\ldots\rho_i s_{i-1}\rho_i\ldots\rho_{n-1}\rho_n} \\
\overset{(2)}{\sim}\ & \beta\underline{s_n s_{n-1}\ldots s_i s_{i-1} s_i\ldots s_{n-1}s_n} \\
\overset{(1)}{\sim}\ & \beta,
\end{aligned}$$

which completes the proof. □

Recall that for $\beta \in VT_n$, $m \otimes \beta \in VT_{n+m}$ denotes the virtual twin obtained by putting trivial m strands on the left of β.

Lemma 3.3.2. *Let $n \geq 2$ and $1 \leq i \leq n$. Under the assumption of moves $M0 - M5$, the following hold:*

1. $(1 \otimes \beta)s_1 s_2 \ldots s_{i-1} s_i s_{i-1} \ldots s_2 s_1 \sim \beta$*, where* $\beta \in VT_n$.
2. $s_1 s_2 \ldots s_{i-1} s_i (i \otimes \beta_1) s_i s_{i-1} \ldots s_2 s_1 (1 \otimes \beta_2) \sim$ $\rho_1\rho_2\ldots\rho_{i-1}\rho_i(i\otimes\beta_1)\rho_i\rho_{i-1}\ldots\rho_2\rho_1(1\otimes\beta_2)$*, where* $\beta_1 \in VT_{n+1-i}$ *and* $\beta_2 \in VT_n$.
3. $\tau_1\tau_2\ldots\tau_{i-1}\tau_i(i\otimes\beta_1)\tau_i\tau_{i-1}\ldots\tau_2\tau_1(1\otimes\beta_2) \sim$ $\rho_1\rho_2\ldots\rho_{i-1}\rho_i(i\otimes\beta_1)\rho_i\rho_{i-1}\ldots\rho_2\rho_1(1\otimes\beta_2)$*, where* $\beta_1 \in VT_{n+1-i}$*,* $\beta_2 \in VT_n$ *and* $\tau_j = s_j$ *or* ρ_j *for each* j.
4. $(1\otimes\beta)\tau_1\tau_2\ldots\tau_{i-1}\tau_i\tau_{i-1}\ldots\tau_2\tau_1 \sim \beta$*, where* $\beta \in VT_n$ *and* $\tau_j = s_j$ *or* ρ_j *for each* j.

Proof. The proof is similar to that of Lemma 3.3.1. □

Recall that for a virtual doodle diagram K on the plane, $W(K)$ denotes the closure of the complement of union of closed disk neighbourhoods of real crossings of K. The proofs of the following two lemmas are similar to [46, Lemma 5 and Lemma 6]. We give proofs in our setting for the sake of completeness.

Lemma 3.3.3. *Let K and K' be two closed virtual twin diagrams such that K' is obtained from K by replacing $K \cap W(K)$ by $K' \cap W(K')$. Then K and K' are related by a finite sequence of $M0$ and $M2$ moves.*

Proof. We use notation from sections 2.5 and 3.2. Let π be the radial projection. Let $N_1, N_2, \ldots, N_n$ be closed 2-disks enclosing real crossings of K and hence of K' such that $\pi(N_i) \cap \pi(N_j) = \phi$ for all $i \neq j$, that is, real crossings lie at separate levels. Let $a_1, a_2, \ldots, a_s$ be arcs/loops in $K \cap W(K)$ and $a'_1, a'_2, \ldots, a'_s$ be the corresponding arcs/loops in $K' \cap W(K')$. Consider a point $p \in \mathbb{S}^1$ such that $\pi^{-1}(p)$ does not intersect either of the crossing sets $V(K)$ and $V(K')$. If there exists some arc/loop a_i and its corresponding arc/loop a'_i such that $|a_i \cap \pi^{-1}(p)| \neq |a'_i \cap \pi^{-1}(p)|$, then we bring a segment of a_i or a'_i closer to the origin by repeated use of $\rho_i^2 = 1$ and some $M2$ moves of virtual type such that $|a_i \cap \pi^{-1}(p)| = |a'_i \cap \pi^{-1}(p)|$. Thus, we can assume that $|a_i \cap \pi^{-1}(p)| = |a'_i \cap \pi^{-1}(p)|$ for all i.
Let k and k' be the underlying immersions $\sqcup \mathbb{S}^1 \to \mathbb{R}^2 \setminus \mathbb{D}^\circ$ of K and K', respectively, such that they are identical in preimage of each N_i. Let $I_1, I_2, \ldots, I_s$ be intervals/circles in $\sqcup \mathbb{S}^1$ such that $k(I_i) = a_i$ and $k'(I_i) = a'_i$. We note that $\pi \circ k|_{I_i}$ and $\pi \circ k'|_{I_i}$ are orientation preserving immersions with $\pi \circ k|_{\partial I_i} = \pi \circ k'|_{\partial I_i}$. Since $|a_i \cap \pi^{-1}(p)| = |a'_i \cap \pi^{-1}(p)|$ for any i, there exists a homotopy $k_i^t : I_i \to \mathbb{R}^2 \setminus \mathbb{D}^\circ$ relative to boundary ∂I_i such that $k_i^0 = k|_{I_i}$ and $k_i^1 = k'|_{I_i}$ and $\pi \circ k_i^t$ is an orientation preserving immersion. If we take the homotopy generically with respect to $K \cap W(K)$, $K' \cap W(K')$ and the 2-disks N_j, we see that a'_i can be transformed to a_i by a sequence of VR_2, VR_3 and M moves in $\mathbb{R}^2 \setminus \mathbb{D}^\circ$. Consequently, K and K' are related by a finite sequence of $M0$ and $M2$ moves. □

Lemma 3.3.4. *Let K and K' be closed virtual twin diagrams having the same Gauss data. Then K and K' are related by a finite sequence of $M0$ and $M2$ moves.*

Proof. Let $N_1, N_2, \ldots, N_n$ be closed 2-disks enclosing real crossings of K and $N'_1, N'_2, \ldots, N'_n$ be the corresponding closed 2-disks enclosing real crossings of K'. We consider two cases depending on the position of N_i and N'_j with respect to the map π.
Case I. Suppose that $\pi(N_1), \pi(N_2), \ldots, \pi(N_n)$ and $\pi(N'_1), \pi(N'_2), \ldots, \pi(N'_n)$ appear in the same cyclic order on boundary $\mathbb{S}^1$. Then we deform K by isotopies of the plane such that $N_i = N'_i$ for all i and diagrams of K and K' are identical in N_i for all i. Thus, K' can be obtained from K by replacing $K \cap W(K)$ by $K' \cap W(K')$, and we are done by Lemma 3.3.3.
Case II. Suppose that $\pi(N_1), \pi(N_2), \ldots, \pi(N_n)$ and $\pi(N'_1), \pi(N'_2), \ldots, \pi(N'_n)$ do not appear in the same cyclic order on $\mathbb{S}^1$. Without loss of generality, we may assume that the two sequences of sets appear in the same order except $\pi(N_1)$ and $\pi(N_2)$. Notice that the diagram K looks as shown in the leftmost part in Figure 3.14, where β_1 is a virtual twin diagram with no real crossing and β_2 a virtual twin diagram. As shown in Figure 3.14, we can make

$\pi(N_1), \pi(N_2), \ldots, \pi(N_n)$ and $\pi(N_1'), \pi(N_2'), \ldots, \pi(N_n')$ to appear in the same cyclic order on $\mathbb{S}^1$ using $M0$ and $M2$ moves. Thus, we get back to Case I and we are done. □

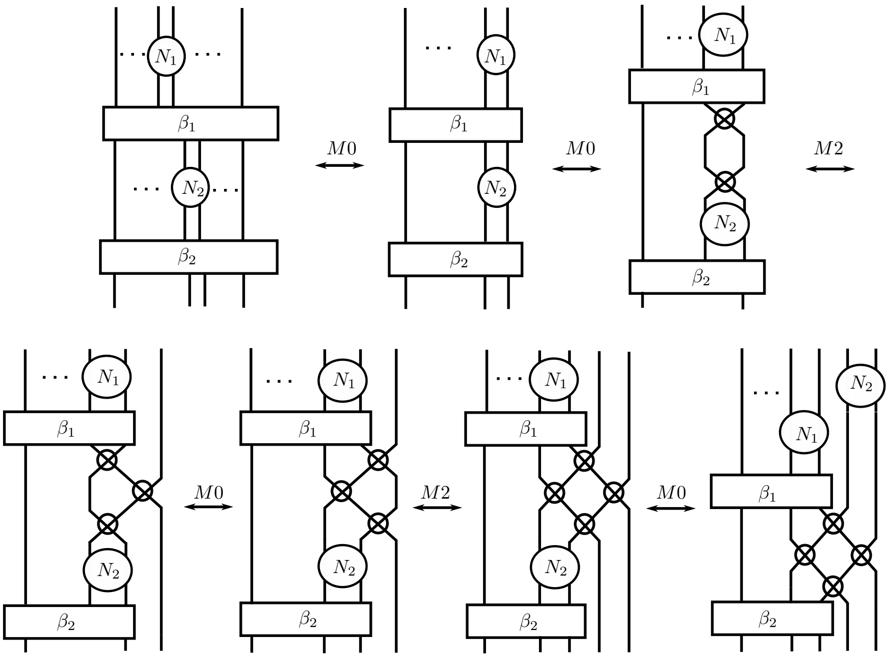

Fig. 3.14

Corollary 3.3.5. *A closed virtual twin diagram for any oriented virtual doodle is uniquely determined upto $M0$ and $M2$ moves.*

Proof. It follows from the fact that any two closed virtual twin diagrams for a virtual doodle have the same Gauss data (as in the proof of Theorem 3.2.4). The result then follows from Lemma 3.3.4. □

We now state and prove Markov Theorem for virtual doodles.

Theorem 3.3.6. *Two virtual twin diagrams on the plane (possibly on different number of strands) have equivalent closures if and only if they are related by a finite sequence of moves $M0-M5$.*

Proof. The proof of the converse implication is immediate. For the forward implication, let K and K' be two closed virtual twin diagrams which are equivalent as virtual doodles. That is, there is a finite sequence of virtual doodle diagrams, say, $K = K_0, K_1, \ldots, K_n = K'$ such that K_i is obtained from K_{i-1} by one of the moves as shown in Figure 3.15. Note that the virtual doodle diagrams obtained in the intermediate steps may not be closed virtual twin diagrams. Let $\widetilde{K_i}$ be a closed virtual twin diagram for K_i obtained by the braiding process as

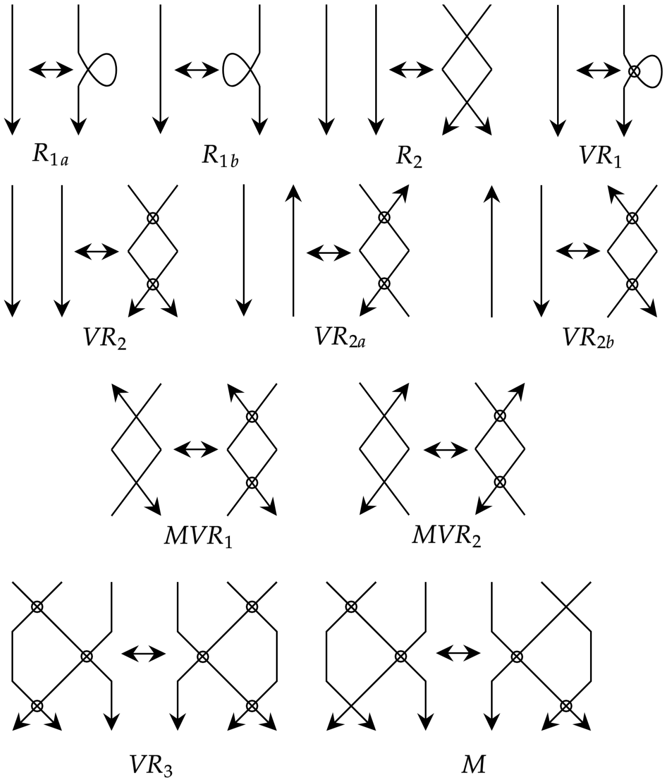

Fig. 3.15. Moves for oriented virtual doodle diagrams

in the proof of Theorem 3.2.4. Without loss of generality, we can assume that $\widetilde{K_0} = K_0$ and $\widetilde{K_n} = K_n$. By Corollary 3.3.5, we know that each $\widetilde{K_i}$ is uniquely determined up to $M0$ and $M2$ moves. Thus, it suffices to prove that $\widetilde{K_{i-1}}$ and $\widetilde{K_i}$ are related by $M0 - M5$ moves. We proceed by considering each move in Figure 3.15.

Case I. Let K_i be obtained from K_{i-1} by applying any one of the VR_1, VR_2, VR_3 or M moves. Then K_i and K_{i-1} have the same Gauss data, which means that $\widetilde{K_i}$ and $\widetilde{K_{i-1}}$ also have the same Gauss data. Then, by Lemma 3.3.4, $\widetilde{K_{i-1}}$ and $\widetilde{K_i}$ are related by $M0$ and $M2$ moves.

Case II. If K_i is obtained from K_{i-1} by an R_2 move, then $\widetilde{K_{i-1}}$ and $\widetilde{K_i}$ are related by a $M0$ move and we are done.

For the remaining moves, let $\mathbb{D}$ to be the closed 2-disk in the plane where one of the remaining moves is applied so that $K_{i-1} \cap (\mathbb{R}^2 \setminus \mathbb{D}) = K_i \cap (\mathbb{R}^2 \setminus \mathbb{D})$. We apply the braiding process to $K_{i-1} \cap (\mathbb{R}^2 \setminus \mathbb{D}) = K_i \cap (\mathbb{R}^2 \setminus \mathbb{D})$ to get diagrams $\widetilde{K}'_{i-1}$ and $\widetilde{K}'_i$ such that $\widetilde{K}'_{i-1} \cap \mathbb{D} = K_{i-1} \cap \mathbb{D}$, $\widetilde{K}'_i \cap \mathbb{D} = K_i \cap \mathbb{D}$ and $\widetilde{K}'_{i-1} \cap (\mathbb{R}^2 \setminus \mathbb{D}) = \widetilde{K}'_i \cap (\mathbb{R}^2 \setminus \mathbb{D})$.

Case III. If K_i is obtained from K_{i-1} by an R_{1a} or R_{1b} move, then after the braiding process, the diagrams $\widetilde{K}'_{i-1}$ and $\widetilde{K}'_i$ looks like as in Figure 3.16. Note that up to conjugation, virtual twins obtained from $\widetilde{K}'_{i-1}$ and $\widetilde{K}'_i$ are either of the following forms

$$\beta \text{ and } \beta\tau_n\tau_{n-1}\ldots\tau_i\tau_{i-1}\tau_i\ldots\tau_{n-1}\tau_n$$

or

$$\beta \text{ and } (1\otimes\beta)\tau_1\tau_2\ldots\tau_{i-1}\tau_i\tau_{i-1}\ldots\tau_2\tau_1,$$

where $\beta \in VT_n$, $\tau_j = s_j$ or ρ_j and $1 \leq i \leq n$. In each case, both the virtual twins are equivalent to each other by Lemma 3.3.1 or Lemma 3.3.2. Thus, $\widetilde{K}_{i-1}$ and $\widetilde{K}_i$ are related by $M0-M5$ moves.

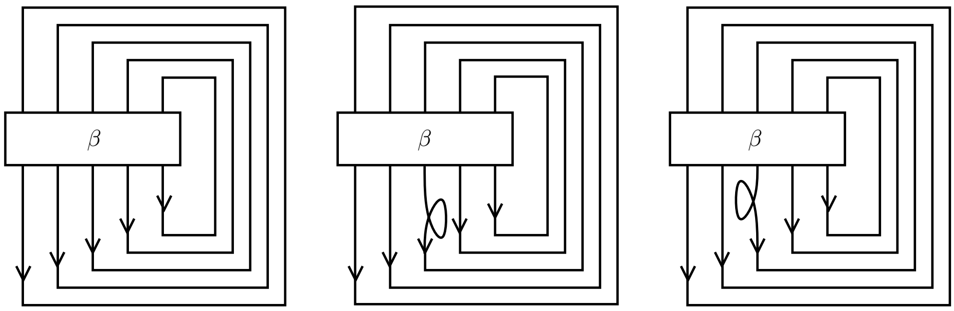

Fig. 3.16. $\widetilde{K}'_{i-1}$ and $\widetilde{K}'_i$ corresponding to R_{1a} or R_{1b} move

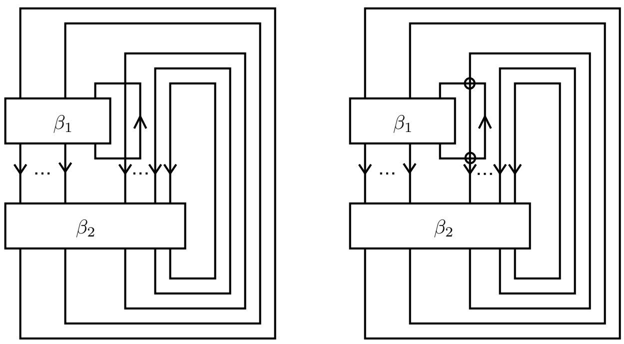

Fig. 3.17. $\widetilde{K}'_{i-1}$ and $\widetilde{K}'_i$ corresponding to MVR_1 move

Case IV. If K_i is obtained from K_{i-1} by an MVR_1 move, then after braiding process, the diagrams $\widetilde{K}'_{i-1}$ and $\widetilde{K}'_i$ looks as in Figure 3.17. The virtual twins obtained from $\widetilde{K}'_{i-1}$ and $\widetilde{K}'_i$ are of the form

$$\tau_n\tau_{n-1}\ldots\tau_{i+1}s_i\beta_1 s_i\tau_{i+1}\ldots\tau_{n-1}\tau_n\beta_2$$

and

$$\tau_n\tau_{n-1}\ldots\tau_{i+1}\rho_i\beta_1\rho_i\tau_{i+1}\ldots\tau_{n-1}\tau_n\beta_2,$$

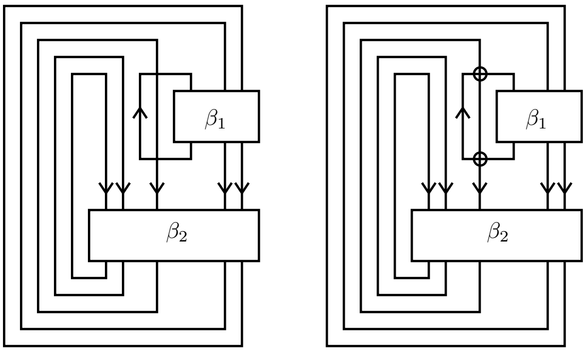

Fig. 3.18. $\widetilde{K}'_{i-1}$ and $\widetilde{K}'_i$ corresponding to MVR_2 move

respectively. By Lemma 3.3.1, both these virtual twins are equivalent, and hence $\widetilde{K}_{i-1}$ and $\widetilde{K}_i$ are related by $M0-M5$ moves.

Case V. If the move applied is MVR_2, then after the braiding process, the diagrams $\widetilde{K}'_{i-1}$ and $\widetilde{K}'_i$ looks as in Figure 3.18. The virtual twins obtained from $\widetilde{K}'_{i-1}$ and $\widetilde{K}'_i$ are of the form

$$\tau_1\tau_2\ldots\tau_{i-1}s_i(i\otimes\beta_1)s_i\tau_{i-1}\ldots\tau_2\tau_1(1\otimes\beta_2)$$

and

$$\tau_1\tau_2\ldots\tau_{i-1}\rho_i(i\otimes\beta_1)\rho_i\tau_{i-1}\ldots\tau_2\tau_1(1\otimes\beta_2),$$

respectively. By Lemma 3.3.2, both of these virtual twins are equivalent, and hence $\widetilde{K}_{i-1}$ and $\widetilde{K}_i$ are related by $M0-M5$ moves. $\square$

Chapter 4

Conjugacy classes in twin groups

This chapter focuses on conjugacy classes of involutions and centralisers in twin groups. We note that the conjugacy problem for Coxeter groups is already solved in the thesis of Moussong [65]. See also [54]. We derive a recursive formula for the number of conjugacy classes of involutions, which relates to the well-known Fibonacci sequence. We also derive a recursive formula for the number of z-classes of involutions in twin groups [70]. We conclude the chapter by addressing the algebraic doodle problem [71].

4.1 Conjugacy problem in twin groups

It is evident from the presentation of T_n that an element of T_n can have more than one expression. For example, the words $s_1, s_5s_1s_5$ and $s_1s_3s_5s_3s_5$ represent the same element in T_n. In this section, we recall some ideas from combinatorial group theory that would be needed to ease our computations throughout the thesis. The preliminaries in this section are motivated from [58, Chapter 1].

Elementary transformations

We define three elementary transformations of a word $w \in T_n$ as follows:

(i) **Deletion.** Replace the word w by deleting a subword of the form s_is_i in w.

(ii) **Insertion.** Replace the word w by inserting a word of the form s_is_i in w.

(iii) **Flip.** Replace a subword of w of the form s_is_j by s_js_i whenever $\mid i - j \mid \geq 2$.

Word equivalence and length

We say that two words w_1 and w_2 are *equivalent*, written as $w_1 \sim w_2$, if there is a finite sequence of elementary transformations turning w_1 into w_2. It is easy to check that $\sim$ is an equivalence relation on T_n. Obviously, two words are equivalent if and only if both of them represent the same element of T_n.
For a given word $w = s_{i_1}s_{i_2}\dots s_{i_k}$, let $\ell(w) = k$ be the *length* of w. For $1 \leq i \leq n-1$, we define $\eta_i(w) :=$ number of s_i's present in the expression w. Note that

$$\ell(w) = \sum_{i=1}^{n-1} \eta_i(w).$$

If $w_1 \sim w_2$, then $\eta_i(w_1) \equiv \eta_i(w_2) \pmod 2$ for each $1 \leq i \leq n-1$, and subsequently $\ell(w_1) \equiv \ell(w_2) \pmod 2$.

Reduced words

We say that a word $w \in T_n$ is *reduced* if $\ell(w) \leq \ell(w')$ for all $w' \sim w$. The existence of a reduced word in an equivalence class of a word follows from the well-ordering Principle. It is possible to have more than one reduced word representing the same element. Moreover, two reduced words represent the same element if and only if one can be obtained from the other by finitely many flip transformations, for example, s_1s_3 and s_3s_1. Obviously, any two reduced words in the same equivalence class have the same length. This allows us to define the *length* of an element $w \in T_n$ as the length of a reduced word representing w.

For each $1 \leq i \leq n-1$, we define the following subset of S:

$$s_i^* = \{s_j \mid [s_i,\, s_j] \neq 1\}.$$

More precisely, it is easy to check that $s_1^* = \{s_2\}$, $s_2^* = \{s_1, s_3\}, \dots, s_{n-2}^* = \{s_{n-3}, s_{n-1}\}$ and $s_{n-1}^* = \{s_{n-2}\}$. Then the following are easy observations:

(i) $s_i \in s_j^*$ if and only if $s_j \in s_i^*$.

(ii) $[s_i,\, s_j] = 1$ if and only if $s_j \notin s_i^*$.

Below is an easy characterisation of a reduced word in T_n.

Lemma 4.1.1. *A word w is reduced if and only if w satisfies the property that whenever two s_i's appear in w for some $1 \leq i \leq n-1$, there always exists an $s_j \in s_i^*$ in between them.*

Proof. Suppose that w is a reduced word and that there exist two s_i's in w such no $s_j \in s_i^*$ appears in between them. Then, by successive application of the flip transformation, we can bring the two s_i's together, and then delete them by the deletion transformation. Thus, the resulting word, which is equivalent to w, has length strictly less than $\ell(w)$, contradicting the fact that w is reduced.
Conversely, suppose that the word w satisfies the desired property. We note that a word obtained by flip transformations on w also satisfies the desired property. Since deletion cannot be performed on words with this property, it follows that w must be reduced. □

Cyclic permutation

A *cyclic permutation* of a word $w = s_{i_1}s_{i_2}\dots s_{i_k}$ (not necessarily reduced) is a word w' (not necessarily distinct from w) of the form $s_{i_t}s_{i_{t+1}}s_{i_{t+2}}\dots s_{i_k}s_{i_1}s_{i_2}\cdots s_{i_{t-1}}$ for some $1 \leq t \leq k$. If $t = 1$, then $w' = w$. It is easy to see that $w' = (s_{i_1}s_{i_2}\dots s_{i_{t-1}})^{-1}w(s_{i_1}s_{i_2}\dots s_{i_{t-1}})$, that is, w and w' are conjugates of each other in T_n.

Cyclically reduced words

A word w is called *cyclically reduced* if each cyclic permutation of w is reduced. It is immediate that a cyclically reduced word is reduced, but the converse is not true. For example, $s_2s_1s_2$ is reduced but not cyclically reduced.

Lemma 4.1.2. *If w is a cyclically reduced word and w' is a word obtained from w by finitely many flip transformations, then w' is also cyclically reduced.*

Proof. By induction, it suffices to prove the assertion for only one flip transformation on w. We begin by noting that any cyclic permutation of a cyclically reduced word is again a cyclically reduced word. Thus, without loss of generality, we can assume that $w = s_{i_1}s_{i_2}s_{i_3}\dots s_{i_k}$ and $w' = s_{i_2}s_{i_1}s_{i_3}\dots s_{i_k}$. We observe that except the word $s_{i_1}s_{i_3}\dots s_{i_k}s_{i_2}$, all other cyclic permutations of w' differ by a flip transformation from some cyclic permutation of w, and hence are reduced. Thus, it only remains to show that the word $s_{i_1}s_{i_3}\dots s_{i_k}s_{i_2}$ is reduced. Since w' is reduced, so are all its subwords, in particular, $s_{i_1}s_{i_3}\dots s_{i_k}$ and $s_{i_3}\dots s_{i_k}$ are reduced. If $s_{i_1}s_{i_3}\dots s_{i_k}s_{i_2}$ is not reduced, then the only reduction possible is in its subword $s_{i_3}\dots s_{i_k}s_{i_2}$, but then the word $s_{i_3}\dots s_{i_k}s_{i_2}s_{i_1}$ is not reduced, which is a contradiction. □

The following is an analogue of Lemma 4.1.1 for cyclically reduced words.

Lemma 4.1.3. *A reduced word w is cyclically reduced if and only if we cannot obtain a word of the form $s_iw's_i$ from w by applying finitely many flip transformations on w.*

Proof. Suppose that w is a cyclically reduced word and $s_iw's_i$ is obtained from w by applying finitely many flip transformations. Then, by Lemma 4.1.2, $s_iw's_i$ is also cyclically reduced. Since a cyclic permutation of a cyclically reduced word is cyclically reduced, it follows that $w's_is_i$ is cyclically reduced, which is a contradiction.
Conversely, suppose that a reduced word w is not cyclically reduced. That is, some cyclic permutation of w is not reduced. We may assume that w is of the form w_1w_2 so that its cyclic permutation w_2w_1 is not reduced. Since w is reduced, both of its subwords w_1 and w_2 are also reduced. On the other hand, the word w_2w_1 is not reduced. This is possible only if, by applying finitely many flip transformations, w_1 and w_2 can be written in the form s_iw_1' and $w_2's_i$, respectively, for some $1 \leq i \leq n-1$. Thus, by applying finitely many flip transformations on the word $w = w_1w_2$, we obtain the word $s_iw_1'w_2's_i$, which is a contradiction. $\square$

Corollary 4.1.4. *Each word in T_n is conjugate to some cyclically reduced word.*

We now investigate conjugacy problem in twin groups. In view of Corollary 4.1.4, it is enough to focus on cyclically reduced words to study conjugacy problem in T_n. The following is the main result of this section.

Theorem 4.1.5. *Suppose w_1, w_2 are two cyclically reduced words in T_n. Then w_1 is conjugate to w_2 if and only if they are cyclic permutation of each other modulo finitely many flip transformations.*

Proof. The converse is obvious. Let us assume that $w_1, w_2 \in T_n$ are two cyclically reduced conjugate words. Let $w_1 = w^{-1}w_2w$, where w is a reduced word. We need to show that w_1 and w_2 are cyclic permutation of each other modulo finitely many flip transformations. We proceed using induction on $\ell(w)$.
Suppose $\ell(w) = 1$, that is, $w = s_i$ for some $i = 1, 2, \ldots, n-1$. Then $w_1 = s_iw_2s_i$. Since w_1 is cyclically reduced, the two s_i's should get cancelled. The following are the three possibilities:

(i) There is cancellation in the subword w_2s_i.

(ii) There is cancellation in the subword s_iw_2.

(iii) Both the rightmost and the leftmost s_i cancel each other after finitely many flip transformations.

In Case (iii), $w_1 = w_2$ and we are done. In Case (i), by successive application of flip transformations on w_2, we obtain $w_2's_i$. This implies that by successive application of flip transformations on the word $s_iw_2s_i$, we get $s_iw_2's_is_i$. By deletion transformation this gives

$w_1 = s_i w_2 s_i = s_i w_2'$. Note that it is a cyclic permutation of $w_2' s_i$, which we obtained by flip transformations on w_2. Case (ii) can be treated in the same manner.
Now suppose that $\ell(w) = k > 1$, where $w = s_{i_1} s_{i_2} \dots s_{i_k}$. Then we can write

$$w_1 = (s_{i_1} s_{i_2} \dots s_{i_k})^{-1} w_2 (s_{i_1} s_{i_2} \dots s_{i_k}) = s_{i_k} s_{i_{k-1}} \dots s_{i_2} s_{i_1} w_2 s_{i_1} s_{i_2} \dots s_{i_k}.$$

Since w_1 is cyclically reduced, s_{i_k} should get cancelled. Following the steps of the case $k = 1$, we have the following possibilities:

(i′) There is cancellation of rightmost s_{i_k} in the word $w_2 s_{i_1} s_{i_2} \dots s_{i_k}$.

(ii′) There is cancellation of leftmost s_{i_k} in the word $s_{i_k} s_{i_{k-1}} \dots s_{i_2} s_{i_1} w_2$.

(iii′) Both the rightmost and the leftmost s_{i_k} cancel each other after finitely many flip transformations.

In Case (iii′) it is easy to see that $w_1 = s_{i_{k-1}} s_{i_{k-2}} \cdots s_{i_2} s_{i_1} w_2 s_{i_1} s_{i_2} \cdots s_{i_{k-1}}$ modulo finitely many flip transformations. Thus, we are done by induction hypothesis. For Case (ii′), by successive application of flip transformations on w_2, $s_{i_k} s_{i_{k-1}} \dots s_{i_2} s_{i_1}$ and $s_{i_1} s_{i_2} \dots s_{i_k}$, we obtain subwords $s_{i_k} w_2'$, $s_{i_{k-1}} s_{i_{k-2}} \dots s_{i_2} s_{i_1} s_{i_k}$ and $s_{i_k} s_{i_1} s_{i_2} \dots s_{i_{k-1}}$, respectively. Thus, after finitely many flip transformations, we get

$$w_1 = s_{i_{k-1}} s_{i_{k-2}} \dots s_{i_2} s_{i_1} s_{i_k} s_{i_k} w_2' s_{i_k} s_{i_1} s_{i_2} \dots s_{i_{k-1}}.$$

Consequently, by deletion transformation, we have

$$w_1 = s_{i_{k-1}} s_{i_{k-2}} \dots s_{i_2} s_{i_1} w_2' s_{i_k} s_{i_1} s_{i_2} \dots s_{i_{k-1}} = (s_{i_1} s_{i_2} \dots s_{i_{k-1}})^{-1} w_2' s_{i_k} (s_{i_1} s_{i_2} \dots s_{i_{k-1}}).$$

Thus, by induction hypothesis, $w_2' s_{i_k}$ (and hence $s_{i_k} w_2'$) is a cyclic permutation of w_1 modulo finitely many flip transformations. Since $s_{i_k} w_2'$ is obtained from w_2 by finitely many flip transformations, the proof of the assertion follows. Case (i′) can be dealt with along similar lines. □

Corollary 4.1.6. *A word $w \in T_n$ is cyclically reduced if and only if $\ell(w)$ is minimal in its conjugacy class.*

4.2 Conjugacy classes of involutions in twin groups

In this section, we study conjugacy classes of involutions in T_n. Since conjugate elements have the same order, in view of Corollary 4.1.4, it suffices to study cyclically reduced involutions in T_n. Specifically, we derive a formula for the number of conjugacy classes of involutions in T_n. Quite interestingly, it is closely related to the well-known Fibonacci sequence.

Proposition 4.2.1. *Let $w = s_{i_1}s_{i_2}\dots s_{i_k}$ be a cyclically reduced word in T_n. Then w is an involution if and only if $[s_{i_j}, s_{i_l}] = 1$ for all $1 \leq j, l \leq k$.*

Proof. Let us suppose that w is an involution and that it does not satisfy the desired condition. Since w is cyclically reduced, without loss of generality, we may assume that w can be written as $w = s_i w_1 s_{i+1} w_2$ such that $\eta_i(w_1) = \eta_{i+1}(w_1) = 0$ for some $1 \leq i \leq n-2$. Since w is an involution, we have

$$w^2 = s_i w_1 s_{i+1} w_2 s_i w_1 s_{i+1} w_2 = 1.$$

Thus, every letter (in particular, s_i and s_{i+1}) on the left hand side of the preceding expression should get cancelled by a finite sequence of flip and deletion transformations. But, as $w = s_i w_1 s_{i+1} w_2$ is reduced, we cannot use deletion transformation on w. Hence, cancellation of leftmost s_i in the expression of w^2 is possible only with the other s_i appearing in the expression of w^2 by repeated application of the flip transformation. This happens only if the leftmost s_{i+1} occuring between the two s_i's in the expression of w^2 cancel. But that is not possible since there is a s_i between the two s_{i+1}'s. Thus, $s_i w_1 s_{i+1} w_2 s_i w_1 s_{i+1} w_2 \neq 1$, a contradiction. The proof of the converse is immediate. □

As a consequence of Corollary 4.1.4 and Proposition 4.2.1, we obtain the following result.

Corollary 4.2.2. *Let w be an element of T_n. Then w is an involution if and only if w is conjugate to a cyclically reduced word of the form $s_{i_1}s_{i_2}\dots s_{i_k}$ such that $i_{t+1} - i_t \geq 2$. Furthermore, any two distinct cyclically reduced words of this form are not conjugates.*

Note that a cyclically reduced word w is an involution if and only if it can be written in the form $s_{i_1}s_{i_2}\dots s_{i_k}$ such that $i_{t+1} - i_t \geq 2$. Set

$$\mathcal{A}_n = \left\{ s_{i_1}s_{i_2}\dots s_{i_k} \mid 1 \leq i_t \leq n-1,\ i_{t+1} - i_t \geq 2 \right\} \tag{4.1}$$

and recall

$$s_i^* = \{ s_j \mid [s_i,\ s_j] \neq 1 \}.$$

The following result, whose proof is immediate from the presentation of T_n, gives ranks of the centralisers of cyclically reduced involutions.

Lemma 4.2.3. *Let $w = s_{i_1}s_{i_2}\dots s_{i_k}$ be an involution in T_n, where $i_{t+1} - i_t \geq 2$ for all $1 \leq t \leq k-1$. Then $C_{T_n}(w) = \left\langle S \setminus \bigcup_{t=1}^{k} s_{i_t}^* \right\rangle$, and consequently rank$(C_{T_n}(w)) = (n-1) - |\bigcup_{t=1}^{k} s_{i_t}^*|$.*

We now state and prove the main result of this section.

Theorem 4.2.4. *Let ρ_n denote the number of conjugacy classes of involutions in T_n. Then*

$$\rho_n = 1 + \rho_{n-1} + \rho_{n-2}$$

for all $n \geq 4$, where $\rho_2 = 1$ and $\rho_3 = 2$.

Proof. Consider the set $\mathcal{A}_n$ as defined in (4.1). Then, by Corollary 4.2.2, we have $\rho_n = |\mathcal{A}_n|$. Note that $\mathcal{A}_2 = \{s_1\}$ and $\mathcal{A}_3 = \{s_1, s_2\}$, which implies that $\rho_2 = 1$ and $\rho_3 = 2$. We now proceed to compute ρ_n for $n \geq 4$. We define three mutually disjoint subsets of $\mathcal{A}_n$ as follows:

(i) $\mathcal{B}_n = \{s_{n-1}\}$.

(ii) $\mathcal{C}_n = \{s_{i_1} s_{i_2} \dots s_{i_k} \mid k > 1, i_k = n-1\}$.

(iii) $\mathcal{D}_n = \{s_{i_1} s_{i_2} \cdots s_{i_k} \mid i_k < n-1\}$.

It is easy to see that $\mathcal{A}_n = \mathcal{B}_n \sqcup \mathcal{C}_n \sqcup \mathcal{D}_n$, and hence

$$|\mathcal{A}_n| = |\mathcal{B}_n| + |\mathcal{C}_n| + |\mathcal{D}_n| = 1 + |\mathcal{C}_n| + |\mathcal{D}_n|.$$

Now, the map sending $s_{i_1} s_{i_2} \dots s_{i_k}$ to $s_{i_1} s_{i_2} \dots s_{i_{k-1}}$ gives a bijection between the sets $\mathcal{C}_n$ and $\mathcal{A}_{n-2}$, and hence $|\mathcal{C}_n| = |\mathcal{A}_{n-2}|$. Also, note that $\mathcal{D}_n = \mathcal{A}_{n-1}$. Thus, we have

$$|\mathcal{A}_n| = 1 + |\mathcal{A}_{n-1}| + |\mathcal{A}_{n-2}|,$$

which implies that

$$\rho_n = 1 + \rho_{n-1} + \rho_{n-2}.$$

□

Corollary 4.2.5. *For each $n \geq 2$, $\rho_n + 1 = \boldsymbol{F}_{n+1}$, where $(\boldsymbol{F}_n)_{n \geq 1}$ is the well-known Fibonacci sequence with $\boldsymbol{F}_1 = \boldsymbol{F}_2 = 1$. In particular,*

$$\rho_n = \sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} \binom{n-k}{k}.$$

Proof. Observe that $\rho_n + 1 = (\rho_{n-1} + 1) + (\rho_{n-2} + 1)$. The first assertion is clear from Theorem 4.2.4. The formula for ρ_n can be derived from the well-known value of $(n+1)$-term of the Fibonacci sequence [79, Chapter 3, Section 3.1.2]. □

4.3 z-classes of involutions in twin groups

Two elements x, y of a group G are said to be *z-equivalent* if their centralisers $C_G(x)$ and $C_G(y)$ are conjugates in G. A z-equivalence class is called a *z-class*. These classes also appear naturally in geometry and topology. We refer the reader to [55] for a quick review of the same.
It is clear that conjugate elements are z-equivalent and the converse is not true. For example, $C_{T_4}(s_1s_3) = C_{T_4}(s_3)$, but s_1s_3 and s_3 are not conjugate. Thus, to investigate z-classes in T_n, it is sufficient to study centralisers of cyclically reduced words (by Corollary 4.1.4). Note that every element of T_n is either torsion-free or of order 2. We first show that only T_2 and T_3 have finitely many z-classes, and then compute number of z-classes of involutions in T_n for $n \geq 2$.

Proposition 4.3.1. *T_n has finitely many z-classes if and only if $n = 2$ or 3.*

Proof. Since $T_2 \cong \mathbb{Z}_2$, there are two conjugacy classes and only one z-class. In T_3, there are infinitely many conjugacy classes, namely, $s_1^{T_3}, s_2^{T_3}, (s_1s_2)^{T_3}, ((s_1s_2)^2)^{T_3}, ((s_1s_2)^3)^{T_3}$, and so on. We note that

$$
\begin{aligned}
C_{T_3}(s_1) &= \langle s_1 \rangle, \\
C_{T_3}(s_2) &= \langle s_2 \rangle, \\
C_{T_3}(s_1s_2) &= \langle s_1s_2 \rangle = C_{T_3}((s_1s_2)^m),\ m \geq 2.
\end{aligned}
$$

By Theorem 4.1.5, it follows that $C_{T_3}(s_1)$, $C_{T_3}(s_2)$ and $C_{T_3}(s_1s_2)$ are pairwise not conjugate. Therefore, there are three z-classes in T_3.
Now, we proceed to prove that T_n has infinitely many z-classes for $n \geq 4$. It suffices to construct an infinite sequence of cyclically reduced words in T_n such that their centralisers are not pairwise conjugate in T_n. We define $X_1 = s_1s_2$, $X_2 = s_1s_2s_3$, $X_3 = s_1s_2s_3s_2$, $X_{2i} = X_{2i-1}s_3$, $X_{2i+1} = X_{2i}s_2$ for $i \geq 2$. It is easy to check that $C_{T_n}(X_1) = \langle X_1 \rangle \times H$ and $C_{T_n}(X_j) = \langle X_j \rangle \times K$ for $j \geq 2$, where $H = \langle s_4, s_5, \ldots, s_{n-1} \rangle$ and $K = \langle s_5, s_6, \ldots, s_{n-1} \rangle$. It can be easily deduced that if $C_{T_n}(X_i)$ is conjugate to $C_{T_n}(X_j)$ for some $i \neq j$, then X_i is conjugate to X_j. But this is not possible due to Theorem 4.1.5. $\square$

Now, we proceed to compute the number of z-classes of involutions in T_n. As mentioned earlier, it is sufficient to consider centralisers of cyclically reduced involutions in T_n. Thus, for the rest of this section, by an involution, we mean a cyclically reduced involution, that is, an element of $\mathcal{A}_n = \left\{ s_{i_1}s_{i_2}\ldots s_{i_k} \mid 1 \leq i_t \leq n-1,\ i_{t+1} - i_t \geq 2 \right\}$. We begin with the following observation.

Lemma 4.3.2. *Let w_1 and w_2 be two involutions in T_n. Then either $C_{T_n}(w_1) = C_{T_n}(w_2)$ or $C_{T_n}(w_1)$ and $C_{T_n}(w_2)$ are not conjugates of each other.*

Proof. Let us suppose $C_{T_n}(w_1) \neq C_{T_n}(w_2)$. Then, without loss of generality, we can assume that there exists some $s_j \in C_{T_n}(w_1) \setminus C_{T_n}(w_2)$. Thus, we can write $C_{T_n}(w_2) = \langle s_{i_1}, s_{i_2}, \ldots, s_{i_k} \rangle$ such that $j \notin \{i_1, i_2, \ldots, i_k\}$. Consequently, for each $g \in T_n$, $C_{T_n}(w_2)^g = \langle s_{i_1}^g, s_{i_2}^g, \ldots, s_{i_k}^g \rangle$, where $s_{i_j}^g = g^{-1} s_{i_j} g$. Thus, each word in $C_{T_n}(w_2)^g$ contains s_j even number of times, and hence $s_j \notin C_{T_n}(w_2)^g$ for any $g \in T_n$. Therefore, $C_{T_n}(w_1)$ and $C_{T_n}(w_2)$ are not conjugates of each other. □

By virtue of the preceding lemma, the number of z-classes of involutions in T_n is equal to the number of distinct centralisers of cyclically reduced involutions in T_n.
Let λ_n denote the number of distinct centralisers of involutions in T_n, $n \geq 2$. A direct computation yields $\lambda_2 = 1$, $\lambda_3 = 2$, $\lambda_4 = 2$, $\lambda_5 = 5$ and $\lambda_6 = 8$. The following main result of this section gives a recursive formula for λ_n, $n \geq 7$.

Theorem 4.3.3. *Let λ_n be as defined above. Then, for $n \geq 7$,*

$$\lambda_n = \Big(\sum_{i=3}^{n-2} \lambda_i\Big) - \lambda_{n-4} + n - 2.$$

We establish the preceding theorem through a sequence of lemmas.

Lemma 4.3.4. *The number of distinct centralisers of involutions ending with s_i in T_n is equal to number of distinct centralisers of involutions ending with s_i in T_m, for all $n, m \geq i+1$.*

Proof. It is sufficient to prove the assertion for $n = i+1$ and $m \geq n$. Let ws_i be an involution ending with s_i. Then

$$C_{T_m}(ws_i) = \begin{cases} C_{T_n}(ws_i), \text{ if } m = n+1 = i+2, \\ \langle C_{T_n}(ws_i), s_{n+1}, s_{n+2}, \ldots, s_{m-1} \rangle, \text{ if } m \geq n+2 = i+3. \end{cases}$$

Hence $C_{T_m}(w_1 s_i) = C_{T_m}(w_2 s_i)$ if and only if $C_{T_n}(w_1 s_i) = C_{T_n}(w_2 s_i)$. This completes the proof. □

The preceding lemma allows us to define α_i as the number of distinct centralisers of involutions ending with s_i in T_n for all $n \geq i+1$.

Lemma 4.3.5. *In T_n, the centralizer of an involution ending with s_i is not equal to the centraliser of an involution ending with s_j, for $i < j$, unless $i = n-3$ and $j = n-1$. Moreover, the centraliser of an involution ending with s_{n-3} is equal to the centraliser of some involution ending with s_{n-1}.*

Proof. Let w_1s_i and w_2s_j be two involutions ending with s_i and s_j, respectively, such that $i < j$. If $j \leq n-2$, then $s_{j+1} \in C_{T_n}(w_1s_i)$, but $s_{j+1} \notin C_{T_n}(w_2s_j)$. If $j = n-1$, then unless $i = n-3$, $s_{n-2} \in C_{T_n}(w_1s_i)$, but $s_{n-2} \notin C_{T_n}(w_2s_j)$. This proves the first assertion of the lemma. For the second assertion, if ws_{n-3} be an involution ending with s_{n-3}, then $C_{T_n}(ws_{n-3}) = C_{T_n}(ws_{n-3}s_{n-1})$. □

Lemma 4.3.6. *For $n \geq 4$,*

$$\lambda_n = (\sum_{i=1}^{n-1} \alpha_i) - \alpha_{n-3}.$$

Proof. From the preceding lemma, we see that

$$\begin{aligned} \lambda_n &= \sum_{i=1}^{n-1} \big(\textit{number of distinct centralisers of involutions ending with } s_i\big) \\ &\quad -\textit{number of distinct centralisers of involutions ending with } s_{n-3} \\ &= (\sum_{i=1}^{n-1} \alpha_i) - \alpha_{n-3}, \end{aligned}$$

which is desired. □

Lemma 4.3.7. *In T_n, the centralizer of an involution ending with s_is_{n-1} is not equal to the centraliser of any involution ending with s_js_{n-1}, for $1 \leq i < j \leq n-3$, unless $i = n-5$ and $j = n-3$. Moreover, the centraliser of an involution ending with $s_{n-5}s_{n-1}$ is equal to the centraliser of some involution ending with $s_{n-3}s_{n-1}$.*

Proof. Let $w_1s_is_{n-1}$ and $w_2s_js_{n-1}$ be two involutions ending with s_is_{n-1} and s_js_{n-1}, respectively, with $1 \leq i < j \leq n-3$. If $j \leq n-4$, then $s_{j+1} \in C_{T_n}(w_1s_is_{n-1})$, but $s_{j+1} \notin C_{T_n}(w_2s_js_{n-1})$. If $j = n-3$, then unless $i = n-5$, $s_{n-4} \in C_{T_n}(w_1s_is_{n-1})$, but $s_{n-4} \notin C_{T_n}(w_2s_js_{n-1})$. This proves the first part of the lemma. For the second assertion, if $ws_{n-5}s_{n-1}$ is an involution ending with $s_{n-5}s_{n-1}$, then $C_{T_n}(ws_{n-5}s_{n-1}) = C_{T_n}(ws_{n-5}s_{n-3}s_{n-1})$. □

Lemma 4.3.8. *For all $i \leq n-3$, the number of distinct centralisers of involutions ending with s_is_{n-1} is equal to the number of distinct centralisers of involutions ending with s_i in T_n.*

Proof. Note that $C_{T_n}(ws_{n-3}) = C_{T_n}(ws_{n-3}s_{n-1})$. But for $i \leq n-4$, we have $s_{n-2} \notin C_{T_n}(ws_is_{n-1})$ and $C_{T_n}(ws_i) = \langle C_{T_n}(ws_is_{n-1}), s_{n-2}\rangle$. Thus, $C_{T_n}(w_1s_is_{n-1}) = C_{T_n}(w_2s_is_{n-1})$ if and only if $C_{T_n}(w_1s_i) = C_{T_n}(w_2s_i)$. □

Lemma 4.3.9. *For $n \geq 5$,*

$$\alpha_{n-1} = 1 + (\sum_{i=1}^{n-3} \alpha_i) - \alpha_{n-5}.$$

Proof. The set of centralisers of involutions ending with s_{n-1} in T_n can be divided into two disjoint subsets, namely, $\{C_{T_n}(s_{n-1})\}$ and the set of centralisers of involutions ending with s_{n-1} and of length strictly greater than 1. The proof now follows from lemmas 4.3.7 and 4.3.8. □

Proof of Theorem 4.3.3. Replacing n by $n+2$ in the preceding result and using Lemma 4.3.6, we get $\alpha_{n+1} = 1+\lambda_n$ for $n \geq 3$. A repeated use of this identity in Lemma 4.3.6 and some simplifications yields

$$\lambda_n = \Big(\sum_{i=3}^{n-2} \lambda_i\Big) - \lambda_{n-4} + n - 2$$

for $n \geq 7$, which is the desired formula. □

4.4 Algebraic doodle problem

In case of classical links and braids, the algebraic link problem asks whether given two braids are equivalent under the classical Markov moves. The algebraic doodle problem can be formulated along the similar lines, that is, to determine whether two twins are equivalent under the Markov moves $\mathrm{M}_1 - \mathrm{M}_4$.

$\mathrm{M}_1 \ : \beta \otimes 1 \to 1 \otimes \ \beta,$

$\mathrm{M}_2 \ : \beta \to \alpha^{-1}\beta\alpha,$

$\mathrm{M}_3 \ : \beta \to (\beta \otimes 1)s_n s_{n-1} \dots s_{i+1} s_i s_{i+1} \dots s_{n-1} s_n,$

$\mathrm{M}_4 \ : \beta \to (1 \otimes \beta)s_1 s_2 \dots s_{i-1} s_i s_{i-1} \dots s_2 s_1,$

where $s_i \in T_{n+1}$. We use the previous setup to address the algebraic doodle problem. We first note that the M_1-move is equivalent to saying that whenever the reduced expression of $\alpha = s_{i_1} s_{i_2} \dots s_{i_k} \in T_{n+1}$ does not contain s_n, we can replace α by $1 \otimes \alpha = s_{i_1+1} s_{i_2+1} \dots s_{i_k+1}$. Next, checking the equivalence of twins under the M_2-move is same as the conjugacy problem which is solvable in T_n (see [54, 65] and Section 4.1). Thus, we consider the moves M_3 and M_4 and prove the following result.

Theorem 4.4.1. *Given a twin $\alpha \in T_{n+1}$, there is an algorithm to determine whether*

(i) α *can be written as* $(\beta \otimes 1)s_n s_{n-1} \dots s_{i+1} s_i s_{i+1} \dots s_{n-1} s_n$,

(ii) α *can be written as* $(1 \otimes \beta)s_1 s_2 \dots s_{i-1} s_i s_{i-1} \dots s_2 s_1$,

for some $\beta \in T_n$ and $1 \leq i \leq n$.

Proof. Case (i). We need to determine whether an element $\alpha \in T_{n+1}$ can be written as $(\beta \otimes 1)s_n s_{n-1} \dots s_{i+1} s_i s_{i+1} \dots s_{n-1} s_n$, for some $\beta \in T_n$ and $1 \leq i \leq n$. Upon applying Lemma 4.1.1, we get a reduced word equivalent to α and have the following possibilities:

(a) If there is only one s_n present in the reduced expression of α, then we can write α as $\alpha' s_n \alpha''$, where $\alpha', \alpha'' \in T_n$. Such an α can be written in the desired form if and only if there is no s_{n-1} present in α''.

(b) Suppose that there are two s_n's present in the reduced expression of α. If the expression does not have a subword of the form $s_n s_{n-1} \dots s_{i+1} s_i s_{i+1} \dots s_{n-1} s_n$ for any $1 \leq i \leq n-1$, then we cannot write α in the desired form. On the other hand, if the reduced expression of α can be written as $\alpha' s_n s_{n-1} \dots s_{i+1} s_i s_{i+1} \dots s_{n-1} s_n \alpha''$, then α has the desired form if and only if α'' is a word in s_j for $1 \leq j \leq i-2$.

(c) If the number of s_n's present in the expression is greater than equal to 3, then we cannot write α in the desired form. For, if we get a subword of the form $s_n s_{n-1} \dots s_{i+1} s_i s_{i+1} \dots s_{n-1} s_n$ for some i and we move this subword to the rightmost position in the reduced expression of α by flip transformations, there will be an s_n present in the expression of β which is not possible since $\beta \in T_n$.

Case (ii). It is easy to note that if $\beta \in T_n$ is written as $s_{i_1} s_{i_2} \dots s_{i_k}$, then $1 \otimes \beta$ which is a twin obtained by putting a trivial strand on the left of β, can be written as $s_{i_1+1} s_{i_2+1} \dots s_{i_k+1} \in T_{n+1}$. We determine whether $\alpha \in T_{n+1}$ can be written as $\alpha = (1 \otimes \beta) s_1 s_2 \dots s_{i-1} s_i s_{i-1} \dots s_2 s_1$, for some $\beta \in T_n$ and $1 \leq i \leq n$. On applying Lemma 4.1.1, we get a reduced word equivalent to α and have the following possibilities:

(a) If there is only one s_1 present in the reduced expression of α, then we can write α as $\alpha' s_1 \alpha''$, where $\alpha', \alpha'' \in T_{n+1}$. Such an α can be written in the desired form if and only if there is no s_2 present in the expression of α''.

(b) Suppose that there are two s_1's present in the reduced expression of α. If the expression does not have a subword of the form $s_1 s_2 \dots s_{i-1} s_i s_{i-1} \dots s_2 s_1$ for any $2 \leq i \leq n$, then we cannot write α in the desired form. On the other hand, if we can write reduced expression of α as $\alpha' s_1 s_2 \dots s_{i-1} s_i s_{i-1} \dots s_2 s_1 \alpha''$, then α has the desired form if and only if α'' is a word in s_j for $i+2 \leq j \leq n$.

(c) If number of s_1's present in the expression is greater than equal to 3, then we cannot write α in the desired form. For, if we get a subword of the form $s_1 s_2 \dots s_{i-1} s_i s_{i-1} \dots s_2 s_1$

for some i, there will be an s_1 present in the expression of $1 \otimes \beta$ which is not possible, since it is an expression in $s_2, s_3, \dots, s_n$.

$\square$

We now define split doodles and split twins analogous to split links and braids. A doodle on a 2-sphere is said to be *split* if it contains two disjoint open disks each containing at least one component of the doodle. We define a twin to be *split* if its closure is a split doodle on a 2-sphere. The following figure is an example of a split doodle which is the closure of a twin $(s_1 s_2)^3 (s_4 s_5)^4$.

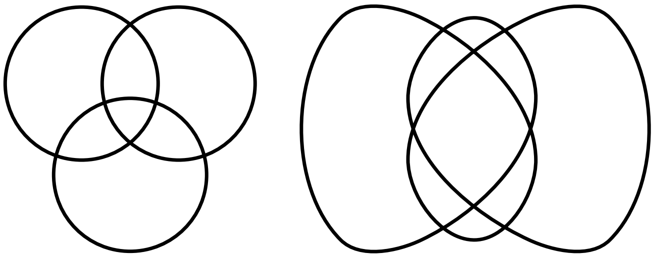

Fig. 4.1. A split doodle

For each $1 \leq i \leq n-1$, let T_n^i be the subgroup of T_n generated by $\{s_1, s_2, \dots, s_{i-1}, s_{i+1}, \dots, s_{n-1}\}$. We conclude this chapter by stating the following proposition, whose proof is immediate, which gives sufficient conditions for a twin to be split.

Proposition 4.4.2. *If $\alpha \in T_n$ satisfy one of the following conditions:*

(i) *α is conjugate to a word in T_n^i for some $1 \leq i \leq n-1$,*

(ii) *$\alpha = (\beta \otimes 1) s_{n-1} s_{n-2} \dots s_{j+1} s_j s_{j+1} \dots s_{n-2} s_{n-1}$, where β is conjugate of a word in T_{n-1}^i, $1 \leq i \leq n-2$ and $1 \leq j \leq n-1$,*

(iii) *$\alpha = (1 \otimes \beta) s_1 s_2 \dots s_{j-1} s_j s_{j-1} \dots s_2 s_1$, where β is conjugate of a word in T_{n-1}^i, $1 \leq i \leq n-2$ and $1 \leq j \leq n-1$,*

then α is a split twin.

Chapter 5

Automorphism groups of twin groups

Using the set up built in the preceding chapter, we determine automorphism groups of twin groups in full generality. Note that, the automorphism group of $T_3 \cong \mathbb{Z}_2 * \mathbb{Z}_2$ is well-known, and structure of $\mathrm{Aut}(T_n)$ is determined in [42] for $n \geq 4$. However, our approach is elementary and yields an alternate proof for all $n \geq 3$ which can be found in [70]. The following is the main result of this chapter.

Theorem 5.0.1. *Let T_n be the twin group with $n \geq 3$. Then the following hold:*

(1) $\mathrm{Aut}(T_3) \cong T_3 \rtimes \mathbb{Z}_2$.

(2) $\mathrm{Aut}(T_4) \cong T_4 \rtimes S_3$.

(3) $\mathrm{Aut}(T_n) \cong T_n \rtimes D_8$ *for $n \geq 5$, where D_8 is the dihedral group of order 8.*

Our plan is to prove Theorem 5.2.8 in two parts. In the Section 5.1, we show that any automorphism that preserves conjugacy classes of generators is an inner automorphism. It is well-known that the center $\mathrm{Z}(T_n) = 1$, and hence $\mathrm{Inn}(T_n) \cong T_n$. We then determine all the non-inner automorphisms of T_n in Section 5.2, and show that $\mathrm{Out}(T_3) \cong \mathbb{Z}_2$, $\mathrm{Out}(T_4) \cong S_3$ and $\mathrm{Out}(T_n) \cong D_8$ for $n \geq 5$.

5.1 Characterisation of inner automorphisms of twin groups

In this section, we characterise inner automorphisms of twin groups.

Proposition 5.1.1. *Let ϕ be an automorphism of T_n for $n \geq 3$. Then ϕ is inner if and only if $\phi(s_i) \in s_i^{T_n}$ for all $1 \leq i \leq n-1$.*

Proof. The forward implication is obvious. For the converse, suppose that $\phi(s_i) \in s_i^{T_n}$ for all $1 \leq i \leq n-1$. We complete the proof in the following steps:

Step 1: *There exists some $u \in T_n$ such that $\widehat{u}\phi(s_{2i-1}) = s_{2i-1}$ for all* $1 \leq i \leq \lfloor n/2 \rfloor$, where $\lfloor . \rfloor$ is the floor function.

We begin by setting $\phi_1 := \phi$. Without loss of generality, we may assume that $\phi_1(s_1) = s_1$. Let us suppose that $\phi_1(s_3) = w_3^{-1} s_3 w_3$, where w_3 is a reduced word. We claim that w_3 does not contain s_2. Let us, on the contrary, suppose that w_3 contains s_2. Then s_1 does not commute with $w_3^{-1} s_3 w_3$, but s_1 commutes with s_3. This is a contradiction to the fact that automorphisms preserve commuting relations. Thus, our claim is true. Next, we define $\phi_3 := \widehat{w_3}^{-1}\phi_1$. Note that $\phi_3(s_1) = s_1$ and $\phi_3(s_3) = s_3$.
Let us now suppose that $\phi_3(s_5) = w_5^{-1} s_5 w_5$, where w_5 is a reduced word. Suppose that the word w_5 contains s_2 or s_4 or both. Then s_3 and s_5 commute but their images do not commute under the automorphism ϕ_3, leading to a contradiction. Hence, w_5 contains neither s_2 nor s_4. Now, we define $\phi_5 = \widehat{w_5}^{-1}\phi_3$. Note that $\phi_5(s_1) = s_1$, $\phi_5(s_3) = s_3$ and $\phi_5(s_5) = s_5$.
Again, suppose $\phi_5(s_7) = w_7^{-1} s_7 w_7$, where w_7 is a reduced word. Repeating the argument, we can show that w_7 does not contain s_2, s_4 and s_6. Define $\phi_7 := \widehat{w_7}^{-1}\phi_5$. Note that $\phi_7(s_1) = s_1$, $\phi_7(s_3) = s_3$, $\phi_7(s_5) = s_5$ and $\phi_7(s_7) = s_7$. Continuing this process, we finally get $\phi_{2k-1}(s_{2i-1}) = s_{2i-1}$, for all $1 \leq i \leq \lfloor n/2 \rfloor$ and $k = \lfloor n/2 \rfloor$. This completes Step 1.

Step 2: *There exists some $v \in T_n$ such that $\widehat{v}\phi(s_{2i}) = s_{2i}$ for all* $1 \leq i \leq \lfloor n-1/2 \rfloor$.

Proof of this step goes along the same lines as that of Step 1.

Step 3: *Without loss of generality, we can assume that there exists a reduced word $w \in T_n$ such that $\phi(s_{2i-1}) = s_{2i-1}$ for all* $1 \leq i \leq \lfloor n/2 \rfloor$ *and* $\phi(s_{2i}) = w^{-1} s_{2i} w$ *for all* $1 \leq i \leq \lfloor n-1/2 \rfloor$.

This follows immediately from steps 1 and 2.

Step 4: If w is the reduced word as in Step 3, then ϕ is an inner automorphism induced by some subword of w.

For a word $w = s_{i_1} s_{i_2} \ldots s_{i_k}$, if i_1 is even, then $w^{-1} s_{2i} w = w'^{-1} s_{2i} w'$ for all $1 \leq i \leq \lfloor n-1/2 \rfloor$, where $w' = s_{i_2} s_{i_3} \ldots s_{i_k}$. On the other hand, if i_k is odd, then $\widehat{s_{i_k}}^{-1}\phi(s_{2i-1}) = s_{2i-1}$ for all $1 \leq i \leq \lfloor n/2 \rfloor$ and $\widehat{s_{i_k}}^{-1}\phi(s_{2i}) = w''^{-1} s_{2i} w''$ for all $1 \leq i \leq \lfloor n-1/2 \rfloor$, where $w'' = s_{i_1} s_{i_2} \ldots s_{i_k - 1}$.

It follows from the preceding observation that if s_{i_t}'s, $1 \leq t \leq k$ are all even indexed, then ϕ is the identity automorphism. Similarly, if s_{i_t}'s, $1 \leq t \leq k$ are all odd indexed, then ϕ is the inner automorphism induced by w. Further, if by applying finitely many flip transformations, we can write $w = w_1 w_2$, where w_1 is subword with even indexed generators and w_2 a subword with odd indexed generators, then ϕ is the inner automorphism induced by w_2.

Now suppose that i_1 is odd, i_k is even and that we cannot bring an even indexed generator to the leftmost position and an odd indexed generator to rightmost position in the expression of w by finitely many flip transformations on w. . We would derive a contradiction by proving that ϕ is not surjective in this case.

We note that $s_{i_k} \neq \phi(s_j)$ for all $j \neq i_k$. Suppose $s_{i_k} = \phi(s_{i_k}) = s_{i_k} s_{i_{k-1}} \dots s_{i_1} s_{i_k} s_{i_1} s_{i_2} \dots s_{i_k}$. This implies $s_{i_{k-1}} \dots s_{i_1} s_{i_k} s_{i_1} s_{i_2} \dots s_{i_k} = 1$. Thus, every generator (in particular s_{i_k}) appearing in the expression $s_{i_{k-1}} \dots s_{i_1} s_{i_k} s_{i_1} s_{i_2} \dots s_{i_k}$ should get cancelled by some elementary transformation. But as $w = s_{i_1} s_{i_2} \dots s_{i_k}$ is a reduced word, deletion of s_{i_k} is only possible if we can bring the s_{i_k} to the leftmost position in the expression of w by some flip transformations. But this is not possible, and hence $s_{i_k} \neq \phi(s_{i_k})$.

Now suppose that $s_{i_k} = \phi(x)$ for some reduced word $x = s_{j_1} s_{j_2} \dots s_{j_t}$ of length greater than 1, i.e., $t > 1$. There are four possibilities on the choice of indices of s_{j_1} and s_{j_t} to be even or odd. Here, we consider one case, i.e. j_1 is odd and j_t is even. Rest of the cases follow similarly. Now, we can write $x = x_1 x_2 \dots x_{2l}$, where $2 \leq 2l \leq t$, the odd indexed subwords (i.e. x_1, $x_3, \dots, x_{2l-1}$) contain generators of odd index ($s_1, s_3, \dots$, etc.) and even indexed subwords (i.e. x_2, $x_4, \dots, x_{2l}$) contain generators of even index ($s_2, s_4, \dots$, etc.). We have

$$s_{i_k} = \phi(x_1 x_2 \dots x_{2l}) = x_1 (w^{-1} x_2 w) x_3 (w^{-1} x_4 w) \dots x_{2l-1} (w^{-1} x_{2l} w).$$

Note that no deletion is possible in the expression $x_1 (w^{-1} x_2 w) x_3 (w^{-1} x_4 w) \dots x_{2l-1} (w^{-1} x_{2l} w)$, because of the assumption that i_1 is odd, i_k is even and that we cannot bring an even indexed generator to the leftmost position and an odd indexed generator to rightmost position in the expression of w by finitely many flip transformations on w. Thus,

$$\ell(s_{i_k}) = 1 < 2 \leq \ell\big(x_1 (w^{-1} x_2 w) x_3 (w^{-1} x_4 w) \dots x_{2l-1} (w^{-1} x_{2l} w)\big),$$

and hence $s_{i_k} \neq \phi(x_1 x_2 \dots x_{2l})$ showing that ϕ is not surjective. This completes the proof of the proposition. $\square$

5.2 Outer automorphisms of twin groups

In this section, we describe the group of automorphisms of T_n in full generality. As a consequence, we get an explicit description of outer automorphisms of T_n.

Proposition 5.2.1. *The map $\psi : T_n \to T_n$ given by $\psi(s_i) = s_{n-i}$, $1 \leq i \leq n-1$, extends to an order 2 non-inner automorphism of T_n for all $n \geq 3$.*

Proof. The proof follows from the definition of ψ. □

Proposition 5.2.2. *The following hold in T_4:*

(i) The map $\tau : T_4 \to T_4$ given by $\tau(s_1) = s_1s_3$, $\tau(s_2) = s_2$ and $\tau(s_3) = s_1$, extends to an order 3 non-inner automorphism of T_4.

(ii) The subgroup generated by τ and ψ is isomorphic to S_3.

Proof. That τ is a non-inner automorphism of order 3 is obvious. Since τ satisfies the relation

$$\psi\tau\psi = \tau^2,$$

we have $\langle \psi, \tau \rangle \cong S_3$. □

Proposition 5.2.3. *The following hold in T_n for $n \geq 5$:*

(i) The map $\kappa : T_n \to T_n$ given by $\kappa(s_3) = s_{n-3}s_{n-1}$ and $\kappa(s_i) = s_{n-i}$ for $i \neq 3$ extends to an order 4 non-inner automorphism of T_n.

(ii) The subgroup generated by κ and ψ is isomorphic to D_8.

Proof. It is easy to check that κ extends to a non-inner automorphism of order 4. Since κ satisfies the relation

$$\psi\kappa\psi = \kappa^{-1},$$

it follows that $\langle \psi, \kappa \rangle \cong D_8$.

□

Lemma 5.2.4. *Let ϕ be an automorphism of T_4. Then $\phi(s_1), \phi(s_3) \in s_1^{T_4}, s_3^{T_4}$ or $(s_1s_3)^{T_4}$ and $\phi(s_2) \in s_2^{T_4}$.*

Proof. It follows from Corollary 4.2.3 that s_1, s_3 and s_1s_3 are the only involutions (upto conjugation) with centralisers of rank 2 and s_2 is the only involution (upto conjugation) with centraliser of rank 1. The result follows since their images under any automorphism should again be involutions with centralisers of the same rank. □

Lemma 5.2.5. *Let ϕ be an automorphism of T_n for $n \geq 3$ and $n \neq 4$. Then either $\phi(s_1) \in s_1^{T_n}$ and $\phi(s_{n-1}) \in s_{n-1}^{T_n}$ or $\psi\phi(s_1) \in s_1^{T_n}$ and $\psi\phi(s_{n-1}) \in s_{n-1}^{T_n}$.*

Proof. It follows from Corollary 4.2.3 that s_1 and s_{n-1} are the only involutions with centralisers (upto conjugation) of rank $n-2$ in T_n. The result follows since their images under any automorphism should again be involutions with centralisers of the same rank $n-2$. □

Lemma 5.2.6. *Let $n \geq 5$ and $\phi \in \mathrm{Aut}(T_n)$. Then for all $2 \leq i \leq n-2$, $\phi(s_i) \in s_j^{T_n}$ for some $2 \leq j \leq n-2$ or $\phi(s_i) \in (s_1s_3)^{T_n}$ or $(s_{n-3}s_{n-1})^{T_n}$.*

Proof. Fix an i such that $2 \leq i \leq n-2$. We observe that s_i is an involution and its centraliser has rank $n-3$. From Corollary 4.2.3, it is clear that only $s_2, s_3, \ldots, s_{n-2}$ and s_1s_{n-1}, s_1s_3 and $s_{n-3}s_{n-1}$ are cyclically reduced involutions whose centralisers have rank $n-3$. Further, from Lemma 5.2.5, it follows that $\phi(s_i) \notin (s_1s_{n-1})^{T_n}$ for $2 \leq i \leq n-2$. □

Lemma 5.2.7. *Let $n \geq 5$ and $\phi \in \mathrm{Aut}(T_n)$ be an automorphism such that $\phi(s_1) \in s_1^{T_n}$. Then the following hold:*

(i) $\phi(s_i) \in s_i^{T_n}$ *for all* $2 \leq i \leq n-2$ *and* $3 \neq i \neq n-3$.

(ii) $\phi(s_3) \in s_3^{T_n}$ *or* $(s_1s_3)^{T_n}$.

(iii) $\phi(s_{n-3}) \in s_{n-3}^{T_n}$ *or* $(s_{n-3}s_{n-1})^{T_n}$.

Proof. Suppose $\phi(s_2) \in g^{T_n}$ for some $g \in \{s_2, s_3, \ldots, s_{n-2}, s_1s_3, s_{n-3}s_{n-1}\}$. Choosing an appropriate inner automorphism say $\hat{w}$ and a reduced word w', we get $\hat{w}(\phi(s_2)) = g$ and $\hat{w}(\phi(w'^{-1}s_1w')) = s_1$. We note that $w'^{-1}s_1w'$ and s_2 do not commute. Since automorphisms preserve commuting relations, s_1 and g also should not commute, and hence $g = s_2$. The proof can now be completed by repeating the argument. □

We now state and prove the main result of this chapter.

Theorem 5.2.8. *Let T_n be the twin group with $n \geq 3$. Then the following hold:*

(1) $\mathrm{Aut}(T_3) \cong T_3 \rtimes \mathbb{Z}_2$.

(2) $\mathrm{Aut}(T_4) \cong T_4 \rtimes S_3$.

(3) $\mathrm{Aut}(T_n) \cong T_n \rtimes D_8$ *for $n \geq 5$, where D_8 is the dihedral group of order 8.*

Recall from propositions 5.2.2 and 5.2.3 that $\langle \psi, \tau \rangle \cong S_3$ and $\langle \psi, \kappa \rangle \cong D_8$. We observe that $\mathrm{Inn}(T_3) \cap \langle \psi \rangle$, $\mathrm{Inn}(T_4) \cap \langle \psi, \tau \rangle$ and $\mathrm{Inn}(T_n) \cap \langle \psi, \kappa \rangle$, $n \geq 5$, are all trivial. Thus, $\mathrm{Inn}(T_3) \rtimes \langle \psi \rangle \leq \mathrm{Aut}(T_3)$, $\mathrm{Inn}(T_4) \rtimes \langle \psi, \tau \rangle \leq \mathrm{Aut}(T_4)$ and $\mathrm{Inn}(T_n) \rtimes \langle \psi, \kappa \rangle \leq \mathrm{Aut}(T_n)$ for $n \geq 5$. It now remains to prove the reverse inclusions. Let ϕ be an automorphism of T_n. It follows from Proposition 5.1.1 and lemmas 5.2.4, 5.2.5, 5.2.6, 5.2.7 that

(a) $\psi^t \phi \in \mathrm{Inn}(T_3)$ for some $0 \leq t \leq 1$,

(b) $\psi^{m_1} \tau^{m_2} \phi \in \mathrm{Inn}(T_4)$ for some $0 \leq m_1 \leq 1$ and $0 \leq m_2 \leq 2$,

(c) $\psi^{n_1} \kappa^{n_2} \phi \in \mathrm{Inn}(T_n)$ for some $0 \leq n_1 \leq 1$ and $0 \leq n_2 \leq 3$, where $n \geq 5$.

This completes the proof of the theorem. $\square$

Corollary 5.2.9. *The following hold in T_n:*

(i) $\mathrm{Out}(T_3) \cong \mathbb{Z}_2 \cong \langle \psi \rangle$.

(ii) $\mathrm{Out}(T_4) \cong S_3 \cong \langle \psi, \tau \rangle$.

(iii) $\mathrm{Out}(T_n) \cong D_8 \cong \langle \psi, \kappa \rangle$ *for* $n \geq 5$.

A consequence of our preceding analysis is the following result.

Proposition 5.2.10. *PT_n is characteristic in T_n if and only if $n = 2, 3$.*

Proof. PT_2 being trivial is obviously characteristic in T_2. We observe that PT_n is invariant under ψ. This follows since the set $\{((s_i s_{i+1})^3)^g \mid 1 \leq i \leq n-2,\ g \in T_n\}$ generates PT_n ([4, Theorem 4]) and $\psi(((s_i s_{i+1})^3)^g) = ((s_{n-i} s_{n-i-1})^3)^{\psi(g)} \in PT_n$. This together with Theorem 5.2.8(1) implies that PT_3 is characteristic in T_3.
For the reverse implication, first consider the element $(s_1 s_2)^3 \in PT_4$. Then $\tau\big((s_1 s_2)^3\big) = (s_1 s_3 s_2)^3 \notin PT_4$ (since $\pi((s_1 s_3 s_2)^3) \neq 1$), and hence PT_4 is not invariant under τ. Similarly, $\kappa((s_2 s_3)^3) = (s_{n-2} s_{n-3} s_{n-1})^3 \notin PT_n$ for $n \geq 5$, and we are done. $\square$

An IA *automorphism* of a group is an automorphism that acts as identity on the abelianisation of the group. Note that inner automorphisms are IA automorphisms. It is easy to check that non-inner automorphisms of T_n for $n \geq 3$ are not IA automorphisms. Therefore, we have the following result.

Proposition 5.2.11. *Every* IA *automorphism of T_n is inner for $n \geq 3$.*

An automorphism of a group is said to be *normal* if it maps every normal subgroup onto itself. The following is an analogue of a similar result of Neshchadim for braid groups [76].

Proposition 5.2.12. *Every normal automorphism of T_n is inner for $n \geq 3$.*

Proof. Note that every inner automorphism is a normal automorphism. Thus, in view of Theorem 5.2.8, it suffices to prove that no automorphism in the sets $\{\psi\}$, $\{\psi, \tau, \tau^2, \tau\psi, \tau^2\psi\}$ and $\{\psi, \kappa, \kappa^2, \kappa^3, \kappa\psi, \kappa^2\psi, \kappa^3\psi\}$ is normal for $n = 3$, $n = 4$ and $n \geq 5$, respectively.

We first prove that ψ is not a normal automorphism of T_n for all $n \geq 3$. Take N to be the normal closure of s_1 in T_n. Note that for each element $g \in N$ and each generator s_i, $i \geq 2$, number of s_i's present in the expression of g is even. This implies that $s_1 \in N$ whereas $\psi(s_1) = s_{n-1} \notin N$, and hence ψ is not normal.
It follows from the proof of Proposition 5.2.10 that PT_4 is not invariant under τ, and so under its inverse τ^2. Hence, both τ and τ^2 are not normal. Similarly, by Proposition 5.2.10, it follows that κ and its inverse κ^3 are not normal. Further, κ^2 is not normal, since $(s_2s_3)^3 \in PT_n$ whereas $\kappa^2((s_2s_3)^3) = (s_2s_4s_3s_1)^3$ for $n = 5$, $\kappa^2((s_2s_3)^3) = (s_2s_3s_5s_1)^3$ for $n = 6$ and $\kappa^2((s_2s_3)^3) = (s_2s_3s_1)^3$ for $n \geq 7$. In each of these cases, $\kappa^2((s_2s_3)^3) \notin PT_n$.
For the remaining cases, we recall that PT_n is invariant under ψ. Consequently, if PT_4 is invariant under $\tau^i\psi$, then it is so under τ^i, a contradiction. Similarly, if PT_n, $n \geq 5$, is invariant under $\kappa^j\psi$, then it is so under κ^j, again a contradiction. Thus, the only normal automorphisms of T_n are the inner automorphisms. $\square$

5.3 R_∞-property for twin groups

For any group G and an automorphism ϕ of G, we say that two elements $x, y \in G$ are (ϕ-twisted conjugate) ϕ-conjugate if there exists an element $g \in G$ such that

$$x = gy\phi(g)^{-1}.$$

It can be easily verified that the relation of ϕ-conjugation is an equivalence relation which divides the group into ϕ-conjugacy classes. In particular, if ϕ is the identity automorphism, then we get the usual conjugacy classes. The number of ϕ-conjugacy classes $R(\phi) \in \mathbb{N} \cup \{\infty\}$ is called the *Reidemeister number* of the automorphism ϕ. We say that a group G has R_∞-property if $R(\phi) = \infty$ for each $\phi \in \mathrm{Aut}(G)$. Obviously, finite groups (in particular T_2) do not satisfy R_∞-property.
It is known that braid groups B_n [24] and pure braid groups P_n [21] have the R_∞-property for all $n \geq 3$. We refer the reader to [19, 20, 22, 34, 35, 43, 66, 67, 74, 75] for some recent works on the topic. Since we established a complete description of $\mathrm{Aut}(T_n)$ in the previous section, we now investigate twisted conjugacy classes of T_n for $n \geq 3$. More precisely, we prove that twin groups T_n have R_∞-property for each $n \geq 3$. The results of this section are from the work [71].
Firstly, recall a basic result on twisted conjugacy classes [25, Corollary 3.2].

Lemma 5.3.1. *Let ϕ be an automorphism and $\hat{g}$ an inner automorphism of a group G. Then $R(\hat{g}\phi) = R(\phi)$.*

The following result relates twisted conjugacy with usual conjugacy.

Lemma 5.3.2. *Let G be a group and ϕ an order k automorphism of G. If $x, y \in G$ are ϕ-conjugates, then the elements $x\phi(x)\phi^2(x)\cdots\phi^{k-1}(x)$ and $y\phi(y)\phi^2(y)\cdots\phi^{k-1}(y)$ are conjugates (in the usual sense). The converse is not true in general.*

Proof. Since $x, y \in G$ are ϕ-conjugates, there exists $z \in G$ such that $x = zy\phi(z^{-1})$. Applying ϕ^i, $1 \leq i \leq k-1$, to this equality gives

$$\begin{aligned}
\phi(x) &= \phi(z)\phi(y)\phi^2(z^{-1}),\\
\phi^2(x) &= \phi^2(z)\phi^2(y)\phi^3(z^{-1}),\\
&\vdots\\
\phi^{k-1}(x) &= \phi^{k-1}(z)\phi^{k-1}(y)\phi^k(z^{-1}) = \phi^{k-1}(z)\phi^{k-1}(y)z^{-1}.
\end{aligned}$$

Multiplying all the preceding equalities gives

$$x\phi(x)\phi^2(x)\cdots\phi^{k-1}(x) = z\big(y\phi(y)\phi^2(y)\cdots\phi^{k-1}(y)\big)z^{-1},$$

which is the first assertion.
For the second assertion, consider the extra-special p-group

$$\mathcal{G} = \big\langle a, b, c \mid a^p = b^p = c^p = 1, ab = bac, ac = ca, bc = cb \big\rangle$$

of order p^3 and exponent p, where p is an odd prime. Note that $\mathcal{G}' = \mathrm{Z}(\mathcal{G}) = \langle c \rangle$ is of order p. It is easy to check that the map $\phi : \mathcal{G} \to \mathcal{G}$ given by $\phi(a) = ac$ and $\phi(b) = bc$ extends to an order p automorphism of $\mathcal{G}$. Then, we have $a\phi(a)\phi^2(a)\cdots\phi^{p-1}(a) = 1 = b\phi(b)\phi^2(b)\cdots\phi^{p-1}(b)$. Suppose that there exists $g \in \mathcal{G}$ such that $a = gb\phi(g^{-1})$. This gives $a = gbg^{-1}c^l$ for some $l \in \mathbb{Z}$. Thus, $a = bb^{-1}gbg^{-1}c^l = b[b, g^{-1}]c^l \in b\mathcal{G}'$, which is not possible. Hence, a cannot be ϕ-conjugate to b. □

Recall from Theorem 5.2.8 that

(i) $\mathrm{Aut}(T_3) = \mathrm{Inn}(T_3) \rtimes \langle \psi \rangle \cong \mathrm{Inn}(T_3) \rtimes \mathbb{Z}_2$,

(ii) $\mathrm{Aut}(T_4) = \mathrm{Inn}(T_4) \rtimes \langle \psi, \tau \rangle \cong \mathrm{Inn}(T_4) \rtimes S_3$,

(iii) $\mathrm{Aut}(T_n) = \mathrm{Inn}(T_n) \rtimes \langle \psi, \kappa \rangle \cong \mathrm{Inn}(T_n) \rtimes D_8$ for $n \geq 5$,

where

D_8 is the dihedral group of order 8,

$\psi : T_n \to T_n$ is given by $\psi(s_i) = s_{n-i}$ for each $1 \leq i \leq n-1$,

$\tau : T_4 \to T_4$ is given by $\tau(s_1) = s_1 s_3$, $\tau(s_2) = s_2$ and $\tau(s_3) = s_1$,

$\kappa : T_n \to T_n$ is given by $\kappa(s_3) = s_{n-3}s_{n-1}$ and $\kappa(s_i) = s_{n-i}$ for $i \neq 3$.

We now proceed to prove the main theorem of this section.

Theorem 5.3.3. *T_n satisfy R_∞-property for all $n \geq 3$.*

Proof. It follows from Proposition 4.3.1 that T_n has infinitely many conjugacy classes for all $n \geq 3$. Hence, due to Lemma 5.3.1, it is enough to show that T_n has infinitely many ϕ-conjugacy classes for automorphisms ϕ in the groups $\langle \psi \rangle$, $\langle \psi, \tau \rangle$ and $\langle \psi, \kappa \rangle$, which are all finite. Our plan is to use Lemma 5.3.2. We divide the proof into four cases, namely, $n \geq 6$, $n = 5$, $n = 4$ and $n = 3$.
Case $n \geq 6$. Consider the sequence of elements $x_i = (s_1 s_2)^i$, $i \geq 1$. We claim that for any automorphism $\phi \in \langle \psi, \kappa \rangle$, x_i is not ϕ-conjugate to x_j whenever $i \neq j$. Let us, on the contrary, suppose that x_i is ϕ-conjugate to x_j for some $i \neq j$. Then, by Lemma 5.3.2, $x_i \phi(x_i) \phi^2(x_i) \cdots \phi^{k-1}(x_i)$ and $x_j \phi(x_j) \phi^2(x_j) \cdots \phi^{k-1}(x_j)$ are conjugates, where k is the order of the automorphism ϕ.
Note that the subgroup $H = \langle s_1, s_2, s_{n-2}, s_{n-1} \rangle$ of T_n is invariant under all automorphisms in $\langle \psi, \kappa \rangle$. In fact, $\psi(x) = \kappa(x)$ for all $x \in H$. Thus, it is sufficient to show that $x_i \psi(x_i)$ and $x_j \psi(x_j)$ are not conjugates in T_n. Observe that $x_i \psi(x_i) = (s_1 s_2)^i (s_{n-1} s_{n-2})^i$ and $x_j \psi(x_j) = (s_1 s_2)^j (s_{n-1} s_{n-2})^j$. It is easy to see that the words $(s_1 s_2)^i (s_{n-1} s_{n-2})^i$ and $(s_1 s_2)^j (s_{n-1} s_{n-2})^j$ are cyclically reduced, and hence by Theorem 4.1.5 they are not conjugates of each other for $i \neq j$. Thus, we have infinitely many ϕ-conjugacy classes in T_n, $n \geq 6$, for any automorphism ϕ of T_n.
Case $n = 5$. For this case we consider the sequence of elements $x_i = (s_1 s_2)^{2i}$, $i \geq 1$. We claim that for any automorphism $\phi \in \langle \psi, \kappa \rangle$, x_i is not ϕ-conjugate to x_j for $i \neq j$. Note that there are two automorphisms of order 4, namely κ and κ^3, and five automorphisms of order 2, namely κ^2, $\psi\kappa$, $\psi\kappa^2$, $\psi\kappa^3$ and ψ.
Direct computations give

$$\begin{aligned} x_i \kappa(x_i) \kappa^2(x_i) \kappa^3(x_i) &= (s_1 s_2)^{2i} (s_4 s_3)^{2i} (s_1 s_2 s_4)^{2i} (s_4 s_1 s_3)^{2i} \\ &= (s_1 s_2)^{2i} (s_4 s_3)^{2i} (s_1 s_2)^{2i} (s_4 s_3)^{2i}. \end{aligned}$$

Again, by Theorem 4.1.5, we note that the elements $(s_1 s_2)^{2i} (s_4 s_3)^{2i} (s_1 s_2)^{2i} (s_4 s_3)^{2i}$ and $(s_1 s_2)^{2j} (s_4 s_3)^{2j} (s_1 s_2)^{2j} (s_4 s_3)^{2j}$ are not conjugate for $i \neq j$. Similarly, notice that $x_i \psi\kappa^2(x_i) = (s_1 s_2)^{2i} (s_4 s_1 s_3)^{2i} = (s_1 s_2)^{2i} (s_4 s_3)^{2i}$. As before, $x_i \psi\kappa^2(x_i)$ is not conjugate to $x_j \psi\kappa^2(x_j)$ for

$i \neq j$. The remaining automorphisms can be considered in the same manner, and hence there are infinitely many ϕ-conjugacy classes in T_5 for any automorphism ϕ.
Case $n = 4$. We again consider the sequence of elements $x_i = (s_1 s_2)^i$, $i \geq 1$, and prove that for any automorphism $\phi \in \langle \psi, \tau \rangle$, x_i is not ϕ-conjugate to x_j whenever $i \neq j$. Note that there are two automorphisms of order 3, namely τ and τ^2, and three automorphisms of order 2, namely $\psi\tau$, $\psi\tau^2$ and ψ. Direct computations yield $x_i \tau(x_i) \tau^2(x_i) = (s_1 s_2)^i (s_1 s_3 s_2)^i (s_3 s_2)^i$. Again by Theorem 4.1.5, $x_i \tau(x_i) \tau^2(x_i)$ is not conjugate to $x_j \tau(x_j) \tau^2(x_j)$ whenever $i \neq j$. The remaining automorphisms can be dealt with similarly, and the assertion follows.
Case $n = 3$. Unlike the earlier cases, here we consider the sequence of elements $x_i = (s_1 s_2)^i s_1$, $i \geq 1$. In this case we need to consider only one automorphism ψ which is of order 2. We have $x_i \psi(x_i) = (s_1 s_2)^i s_1 (s_2 s_1)^i s_2 = (s_1 s_2)^{2i+1}$. By Theorem 4.1.5, $x_i \psi(x_i)$ is not conjugate to $x_j \psi(x_j)$ whenever $i \neq j$, and hence there are infinitely many ψ-conjugacy classes in T_3. This completes the proof of the theorem. $\square$

Remark 5.3.4. *It is interesting to see whether the pure twin group PT_n has R_∞-property. It is known that $PT_3 \cong \mathbb{Z}$, $PT_4 \cong F_7$ and $PT_5 \cong F_{31}$ [4, 36]. It follows from [20, Theorem 2.1] and [67, Lemma 2.1] that non-abelian free groups of finite rank have R_∞-property. Thus, PT_4 and PT_5 have R_∞-property. A precise description of PT_6 has been obtained recently in [63, Theorem 2] where it is proved that* $PT_6 \cong F_{71} * (*_{20}(\mathbb{Z} \oplus \mathbb{Z}))$. *On the other hand, a complete presentation of PT_n is still unknown for $n \geq 7$. Thus, determining whether PT_n has R_∞-property for $n \geq 6$ remains an open problem.*

5.4 Representations of twin groups by automorphisms

Since PT_n is normal in T_n and center of T_n is trivial for $n \geq 3$, there is a natural homomorphism

$$\phi_n : T_n \cong \mathrm{Inn}(T_n) \to \mathrm{Aut}(PT_n),$$

obtained by restricting the inner automorphisms on PT_n. It is proved in [4] that $\mathrm{Ker}(\phi_3) \neq 1$ and ϕ_4 is injective. We show that this is the case for all $n \geq 4$.

Proposition 5.4.1. *The homomorphism $\phi_n : T_n \to \mathrm{Aut}(PT_n)$ is injective if and only if $n \geq 4$.*

Proof. Note that $\mathrm{Ker}(\phi_n) = C_{T_n}(PT_n)$. It is easy to check that

$$C_{T_n}((s_i s_{i+1})^3) = \langle s_1, s_2, \ldots, s_{i-2}, s_i s_{i+1}, s_{i+3}, s_{i+4}, \ldots, s_{n-1} \rangle,$$

and

$$C_{T_n}(PT_n) \le \bigcap_{i=1}^{n-2} C_{T_n}((s_i s_{i+1})^3) = 1.$$

□

Therefore, it follows that for $n = 4,5,6$ we have faithful representations

$$T_4 \hookrightarrow \mathrm{Aut}(F_7),$$

$$T_5 \hookrightarrow \mathrm{Aut}(F_{31}),$$

and

$$T_6 \hookrightarrow \mathrm{Aut}\big(F_{71} * (*_{20}(\mathbb{Z} \oplus \mathbb{Z}))\big).$$

We do not know whether there exists a faithful representation of T_n into $\mathrm{Aut}(F_n)$ analogous to the classical Artin representation of the braid group. However, we have the following representation.

Theorem 5.4.2. *The map $\mu_n : T_n \to \mathrm{Aut}(F_n)$ defined by the action of generators of T_n by*

$$\mu_n(s_i) : \begin{cases} x_i \mapsto x_i x_{i+1}, \\ x_{i+1} \mapsto x_{i+1}^{-1}, \\ x_j \mapsto x_j, \quad j \neq i, i+1, \end{cases}$$

is a representation of T_n. Moreover, μ_n is faithful if and only if $n = 2,3$.

Proof. We begin by proving that μ_n is a representation. Clearly, s_i^2 act as identity automorphism of F_n. Moreover, the action of $\mu_n(s_i)\mu_n(s_j)$ and $\mu_n(s_j)\mu_n(s_i)$ on generators $x_1, \ldots, x_n$ is

$$\mu_n(s_i)\mu_n(s_j) : \begin{cases} x_i \mapsto x_i x_{i+1}, \\ x_{i+1} \mapsto x_{i+1}^{-1}, \\ x_j \mapsto x_j x_{j+1}, \\ x_{j+1} \mapsto x_{j+1}^{-1}, \end{cases} \qquad \mu_n(s_j)\mu_n(s_i) : \begin{cases} x_i \mapsto x_i x_{i+1}, \\ x_{i+1} \mapsto x_{i+1}^{-1}, \\ x_j \mapsto x_j x_{j+1}, \\ x_{j+1} \mapsto x_{j+1}^{-1}, \end{cases}$$

for all $|i-j| \geq 2$.

Faithfulness for $n = 2$ is obvious. For the case $n = 3$, we know that any arbitrary element of T_3 is of the form $(s_1 s_2)^m$ or $(s_1 s_2)^m s_1$ or $s_2 (s_1 s_2)^m$ for some integer m. If $\mathrm{Ker}(\mu_3) \neq 1$, then there exists a non-trivial element $s \in T_3$ such that $\mu_3(s)(x_i) = x_i$ for $i = 1,2,3$. We show that no such element exists. We first consider elements of the form $(s_1 s_2)^m$. It is easy to see that the action of all odd powers of $s_1 s_2$ gives a non-identity automorphism of F_3, since it sends x_3 to x_3^{-1}. On the other hand, for even powers of $s_1 s_2$, we have

$$\mu_3((s_1 s_2)^{2k}) : \begin{cases} x_1 \mapsto x_1 x_3^k, \\ x_2 \mapsto x_3^{-k} x_2 x_3^{-k}, \\ x_3 \mapsto x_3, \end{cases}$$

for all integer k. Next we consider $(s_1 s_2)^m s_1$. Again, if m is odd, the action is non-trivial and if $m = 2k$, then we have

$$\mu_3((s_1 s_2)^{2k} s_1) : \begin{cases} x_1 \mapsto x_1 x_2 x_3^{-k}, \\ x_2 \mapsto x_3^k x_2^{-1} x_3^k, \\ x_3 \mapsto x_3. \end{cases}$$

Similarly, for $s_2(s_1 s_2)^m$, we have a non-trivial action if m is even. If $m = 2k - 1$, then we have

$$\mu_3(s_2(s_1 s_2)^{2k-1}) : \begin{cases} x_1 \mapsto x_1 x_2 x_3^k, \\ x_2 \mapsto x_3^{-k} x_2^{-1} x_3^{-k}, \\ x_3 \mapsto x_3. \end{cases}$$

Thus, $\mu_3 : T_3 \to \mathrm{Aut}(F_3)$ is faithful.
Finally, we show that μ_n is not faithful for $n \geq 4$. Consider the element

$$x = (s_2 s_3)^{-2} s_1 (s_2 s_3)^2 s_1 (s_2 s_3)^2 s_1 (s_2 s_3)^{-2} s_1 \in T_n, \ n \geq 4.$$

Since $\pi(x) \neq 1$, it follows that $x \neq 1$. It is easy to check that

$$\mu_n((s_2 s_3)^2) : \begin{cases} x_1 \mapsto x_1, \\ x_2 \mapsto x_2 x_4, \\ x_3 \mapsto x_4^{-1} x_3 x_4^{-1}, \\ x_j \mapsto x_j \ , \ j \geq 4, \end{cases}$$

$$\mu_n((s_2s_3)^{-2}):\begin{cases} x_1 \mapsto x_1, \\ x_2 \mapsto x_2x_4^{-1}, \\ x_3 \mapsto x_4x_3x_4, \\ x_j \mapsto x_j\ ,\ j \geq 4, \end{cases}$$

$$\mu_n((s_2s_3)^2s_1(s_2s_3)^{-2}):\begin{cases} x_1 \mapsto x_1x_2x_4, \\ x_2 \mapsto x_4^{-1}x_2^{-1}x_4^{-1}, \\ x_j \mapsto x_j\ ,\ j \geq 3, \end{cases} \tag{5.1}$$

and

$$\mu_n((s_2s_3)^{-2}s_1(s_2s_3)^2):\begin{cases} x_1 \mapsto x_1x_2x_4^{-1}, \\ x_2 \mapsto x_4x_2^{-1}x_4, \\ x_j \mapsto x_j\ ,\ j \geq 3. \end{cases} \tag{5.2}$$

Using 5.1, 5.2 and action of s_1, we conclude that $x \in \mathrm{Ker}(\mu_n)$. □

We note that μ_n extends easily to a representation of the virtual twin group VT_n.

Proposition 5.4.3. *The map $\mu_n : VT_n \to \mathrm{Aut}(F_n)$ defined by the action of generators of VT_n by*

$$\mu_n(s_i):\begin{cases} x_i \mapsto x_ix_{i+1}, \\ x_{i+1} \mapsto x_{i+1}^{-1}, \\ x_j \mapsto x_j, \quad j \neq i, i+1, \end{cases}$$

$$\mu_n(\rho_i):\begin{cases} x_i \mapsto x_{i+1}, \\ x_{i+1} \mapsto x_i \\ x_j \mapsto x_j, \quad j \neq i, i+1, \end{cases}$$

is a representation of VT_n.

As a consequence of Proposition 5.4.3, it follows that the forbidden moves in Figure 3.4 cannot be obtained from the moves in Figure 3.2.

Proposition 5.4.4. *The following holds in VT_n:*

1. $s_is_{i+1}s_i \neq s_{i+1}s_is_{i+1}$.
2. $\rho_is_{i+1}s_i \neq s_{i+1}s_i\rho_{i+1}$.

Proof. An easy check gives

$$\mu_n(s_i s_{i+1} s_i)(x_i) \neq \mu_n(s_{i+1} s_i s_{i+1})(x_i)$$

and

$$\mu_n(\rho_i s_{i+1} s_i)(x_i) \neq \mu_n(s_{i+1} s_i \rho_{i+1})(x_i)$$

for each i. □

5.5 (Co-)Hopfian property for twin groups

A group is said to be *co-Hopfian* (respectively *Hopfian*) if every injective (respectively surjective) endomorphism is an automorphism. For example, the infinite cyclic group is not co-Hopfian whereas all finite groups (in particular T_2) are Hopfian as well as co-Hopfian. The investigation of residual properties of groups is an active area of research. In particular, considering the fact that Coxeter groups are linear [13], and hence residually finite [84], implies that T_n is residually finite. These groups, being finitely generated residually finite, are Hopfian [58, Chapter 4, Theorem 4.10]. The property of co-Hopfianity cannot be inherited from this fact.

These properties are closely related to R_∞-property, see, for example [67, Lemma 2.3]. The classical Artin braid groups B_n are known to be Hopfian being residually finite [47, Chapter I, Corollary 1.22] and are not co-Hopfian for $n \geq 2$ [11]. In fact, the map which sends each standard generator of B_n to itself times a fixed power of the central element extends to an injective endomorphism which is not surjective. For twin groups T_n with $n \geq 3$, we have the following result which is proved in [71].

Theorem 5.5.1. *T_n is not co-Hopfian for $n \geq 3$.*

Proof. We construct a map $\psi_n : T_n \to T_n$, $n \geq 3$, by setting

$$\psi_n(s_i) = \begin{cases} s_i & \text{for } i \neq 2, \\ s_2 s_1 s_2 & \text{for } i = 2. \end{cases}$$

It is easy to check that ψ_n is a group homomorphism. It is evident from the definition of ψ_n that for any element $w \in T_n$, the expression $\psi_n(w)$ contains an even number of s_2. Hence, $s_2 \notin \psi_n(T_n)$ and the map ψ_n is not surjective. We now proceed to prove that ψ_n is injective by induction on n.

We note that T_3 can be generated by elements s_1 and $s_1 s_2$. Since the cyclic subgroup $\langle s_1 s_2 \rangle$ is normal in T_3, any element of T_3 is of the form either $(s_1 s_2)^m$ or $(s_1 s_2)^m s_1$ for some integer

m. The images of these elements under ψ_3 are:

$$\psi_3((s_1s_2)^m) = (s_1s_2s_1s_2)^m = (s_1s_2)^{2m},$$
$$\psi_3((s_1s_2)^m s_1) = (s_1s_2s_1s_2)^m s_1 = (s_1s_2)^{2m} s_1,$$

It is clear that none of the non-trivial elements of T_3 belong to $\mathrm{Ker}(\psi_3)$, and hence ψ_3 is injective.

Suppose that $1 \neq w \in \mathrm{Ker}(\psi_4)$. Without loss of generality we may assume that w is a reduced word. Suppose that $w = w_1 s_3 w_2 s_3 \cdots w_k s_3 w_{k+1}$, where w_i's are words in T_3. Then, we have

$$\psi_4(w) = \psi_4(w_1) s_3 \psi_4(w_2) s_3 \cdots \psi_4(w_k) s_3 \psi_4(w_{k+1}) = 1.$$

Notice that all the w_i's are non-trivial words in T_3, since w is a reduced word. Also, the map ψ_4 restricted to T_3 is ψ_3 which is injective. Thus, $\psi_4(w_i) = \psi_3(w_i) \neq 1$ for all $1 \leq i \leq k+1$. For $\psi_4(w) = 1$ to be true, all the s_3's must get cancelled. But there will always be at least one s_2 in between any two s_3's, which is a contradiction. Therefore, the map ψ_4 is injective.

Let us now assume that ψ_{n-1} is injective for $n \geq 5$. Consider a non-trivial reduced word w in $\mathrm{Ker}(\psi_n)$. Let $w = w_1 s_{n-1} w_2 s_{n-1} \cdots w_k s_{n-1} w_{k+1}$, where $w_i \in T_{n-1}$ for all $1 \leq i \leq k+1$. This implies that

$$\psi_n(w) = \psi_{n-1}(w_1) s_{n-1} \psi_{n-1}(w_2) s_{n-1} \cdots \psi_{n-1}(w_k) s_{n-1} \psi_{n-1}(w_{k+1}) = 1.$$

For the above equality to be true, all the s_{n-1}'s must get cancelled. In particular, the two s_{n-1}'s in the subword $s_{n-1}\psi_{n-1}(w_j)s_{n-1}$ must get cancelled. This implies that $\psi_{n-1}(w_j)$ does not have s_{n-2}, which means that w_j does not have s_{n-2}, which contradicts the fact that w is reduced. Hence, ψ_n is injective. □

Remark 5.5.2. *Note that the infinite cyclic group and a free product of any two non-trivial groups is not co-Hopfian. Thus, PT_n is not co-Hopfian for $3 \leq n \leq 6$. Whether PT_n is co-Hopfian for $n \geq 7$ remains unknown.*

Remark 5.5.3. *It is well-known that the braid group B_n is not co-Hopfian for $n \geq 2$ [11]. In fact, the map $\phi_n : B_n \to B_n$, $n \geq 2$, defined on the standard generators by*

$$\phi_n(\sigma_i) = \sigma_i z,$$

where $\langle z \rangle = \mathrm{Z}(B_n)$, is an injective homomorphism which is not surjective. Let P_n be the pure braid group on n strands. Since $\phi_n(P_n) \subset P_n$ and $z \in \mathrm{Z}(B_n) = \mathrm{Z}(P_n)$ does not have a *preimage under ϕ_n, it follows that the restriction of ϕ_n on P_n is injective but not surjective, and hence P_n is not co-Hopfian for $n \geq 2$.*

Chapter 6

Structure of pure virtual twin groups

The kernel of the natural surjection of VT_n onto S_n which trails the end points of the strands of virtual twins, is known as pure virtual twin group PVT_n. In this chapter, we give a precise presentation of PVT_n, which in turn proves that it is an irreducible right-angled Artin group [73]. Furthermore, we describe these groups as iterated semidirect product of infinitely generated free groups, which is crucial in proving the triviality of centers of VT_n and PVT_n.

6.1 Presentation of pure virtual twin groups

In this section, we give a presentation of PVT_n. We show that the rank of PVT_n is $n(n-1)/2$, which, interestingly, coincides with the rank of the pure braid group P_n.

We shall use the standard presentation of VT_n and the Reidemeister-Schreier method described in Section 2.6. We also set some notations for this section. We begin by recalling the defining relations in the presentation of VT_n.

$$\begin{aligned}
s_i^2 &= 1 \quad \text{for } i = 1,2,\ldots,n-1, && (6.1)\\
s_is_j &= s_js_i \quad \text{for } |i-j| \geq 2, && (6.2)\\
\rho_i^2 &= 1 \quad \text{for } i = 1,2,\ldots,n-1, && (6.3)\\
\rho_i\rho_j &= \rho_j\rho_i \quad \text{for } |i-j| \geq 2, && (6.4)\\
\rho_i\rho_{i+1}\rho_i &= \rho_{i+1}\rho_i\rho_{i+1} \quad \text{for } i = 1,2,\ldots,n-2, && (6.5)\\
\rho_is_j &= s_j\rho_i \quad \text{for } |i-j| \geq 2, && (6.6)\\
\rho_i\rho_{i+1}s_i &= s_{i+1}\rho_i\rho_{i+1} \quad \text{for } i = 1,2,\ldots,n-2. && (6.7)
\end{aligned}$$

Let us take the set

$$\mathrm{M}_n = \left\{ m_{1,i_1} m_{2,i_2} \ldots m_{n-1,i_{n-1}} \mid m_{k,i_k} = \rho_k \rho_{k-1} \ldots \rho_{i_k+1} \text{ for each } 1 \le k \le n-1 \text{ and } 0 \le i_k < k \right\}$$

as the Schreier system of coset representatives of PVT_n in VT_n. We set $m_{kk} = 1$ for $1 \le k \le n-1$. For an element $w \in VT_n$, recall that $\overline{w}$ denote the unique coset representative of the coset of w in the Schreier set M_n. By Reidemeister-Schreier method, the group PVT_n is generated by the set

$$\left\{ \gamma(\mu, a) = (\mu a)(\overline{\mu a})^{-1} \mid \mu \in \mathrm{M}_n \text{ and } a \in \{s_1, \ldots, s_{n-1}, \rho_1, \ldots, \rho_{n-1}\} \right\}.$$

with defining relations

$$\left\{ \tau(\mu r \mu^{-1}) \mid \mu \in \mathrm{M}_n \text{ and } r \text{ is a defining relation in } VT_n \right\},$$

where τ is the rewriting process. We set

$$\lambda_{i,i+1} = s_i \rho_i$$

for each $1 \le i \le n-1$ and

$$\lambda_{i,j} = \rho_{j-1} \rho_{j-2} \cdots \rho_{i+1} \lambda_{i,i+1} \rho_{i+1} \cdots \rho_{j-2} \rho_{j-1}$$

for each $1 \le i < j \le n$ and $j \ne i+1$.

Theorem 6.1.1. *The pure virtual twin group PVT_n on $n \ge 2$ strands is generated by*

$$\mathcal{S} = \left\{ \lambda_{i,j} \mid 1 \le i < j \le n \right\}.$$

Proof. The case $n = 2$ is immediate, and hence we assume $n \ge 3$. Note that PVT_n is generated by the elements $\gamma(\mu, a)$, where $\mu \in \mathrm{M}_n$ and $a \in \{s_1, \ldots, s_{n-1}, \rho_1, \ldots, \rho_{n-1}\}$. We observe that

$$\begin{aligned} \overline{\alpha} &= \alpha, \\ \overline{\alpha_1 s_{i_1} \ldots \alpha_k s_{i_k}} &= \alpha_1 \rho_{i_1} \ldots \alpha_k \rho_{i_k} \end{aligned}$$

in VT_n for words α, α_j in the generators $\{\rho_1, \ldots, \rho_{n-1}\}$. Therefore, we have

$$\gamma(\mu, \rho_i) = (\mu \rho_i)(\mu \rho_i)^{-1} = 1 \tag{6.8}$$

and

$$\gamma(\mu, s_i) = (\mu s_i)(\mu \rho_i)^{-1} = \mu s_i \rho_i \mu^{-1} = \mu \lambda_{i,i+1} \mu^{-1}$$

for each $\mu \in \mathrm{M}_n$ and $i = 1, 2, \ldots, n-1$. Let $\mathcal{S} \sqcup \mathcal{S}^{-1} = \{\lambda_{i,j}^{\pm 1} \mid \lambda_{i,j} \in \mathcal{S}\}$. We claim that each $\gamma(\mu, s_i)$ lie in $\mathcal{S} \sqcup \mathcal{S}^{-1}$. For this, we analyse the conjugation action of $S_n = \langle \rho_1, \ldots, \rho_{n-1} \rangle$ on the set $\mathcal{S}$.

First consider $\lambda_{i,i+1}$ for $i = 1, 2, \ldots, n-1$.

(i) If $1 \leq k \leq i-2$ or $i+2 \leq k \leq n-1$, then

$$\rho_k \lambda_{i,i+1} \rho_k = \lambda_{i,i+1}.$$

(ii) If $k = i-1$, then

$$\begin{aligned}
\rho_k \lambda_{i,i+1} \rho_k &= \rho_{i-1} \lambda_{i,i+1} \rho_{i-1} \\
&= \rho_{i-1} s_i \rho_i \rho_{i-1} \\
&= \rho_{i-1} s_i \rho_{i-1} \rho_{i-1} \rho_i \rho_{i-1} \\
&= \rho_i s_{i-1} \rho_i \rho_{i-1} \rho_i \rho_{i-1} \\
&= \rho_i s_{i-1} \rho_{i-1} \rho_i \rho_{i-1} \rho_{i-1} \\
&= \rho_i (s_{i-1} \rho_{i-1}) \rho_i \\
&= \lambda_{i-1,i+1}.
\end{aligned}$$

(iii) If $k = i$, then

$$\rho_k \lambda_{i,i+1} \rho_k = \lambda_{i,i+1}^{-1}.$$

(iii) If $k = i+1$, then

$$\rho_k \lambda_{i,i+1} \rho_k = \lambda_{i,i+2}.$$

Next, we consider $\lambda_{i,j}$ for each $1 \leq i < j \leq n$ and $j \neq i+1$.

(i) If $1 \leq k \leq i-2$ or $j+1 \leq k \leq n-1$, then

$$\rho_k \lambda_{i,j} \rho_k = \lambda_{i,j}.$$

(ii) For $k=i-1$, we have $\rho_{i-1}\lambda_{i,j}\rho_{i-1}=\lambda_{i-1,j}$ since

$$\begin{aligned}
\rho_{i-1}\lambda_{i,j}\rho_{i-1} &= \rho_{i-1}\rho_{j-1}\rho_{j-2}\dots\rho_{i+1}\lambda_{i,i+1}\rho_{i+1}\dots\rho_{j-2}\rho_{j-1}\rho_{i-1}\\
&= \rho_{j-1}\rho_{j-2}\dots\rho_{i+1}\rho_{i-1}\lambda_{i,i+1}\rho_{i-1}\rho_{i+1}\dots\rho_{j-2}\rho_{j-1}\\
&= \rho_{j-1}\rho_{j-2}\dots\rho_{i+1}\rho_{i-1}s_i\rho_i\rho_{i-1}\rho_{i+1}\dots\rho_{j-2}\rho_{j-1}\\
&= \rho_{j-1}\rho_{j-2}\dots\rho_{i+1}\rho_i\rho_i\rho_{i-1}s_i\rho_i\rho_{i-1}\rho_{i+1}\dots\rho_{j-2}\rho_{j-1}\\
&= \rho_{j-1}\rho_{j-2}\dots\rho_{i+1}\rho_i s_{i-1}\rho_i\rho_{i-1}\rho_i\rho_{i-1}\rho_{i+1}\dots\rho_{j-2}\rho_{j-1}\\
&= \rho_{j-1}\rho_{j-2}\dots\rho_{i+1}\rho_i s_{i-1}\rho_i\rho_i\rho_{i-1}\rho_i\rho_{i+1}\dots\rho_{j-2}\rho_{j-1}\\
&= \rho_{j-1}\rho_{j-2}\dots\rho_{i+1}\rho_i(s_{i-1}\rho_{i-1})\rho_i\rho_{i+1}\dots\rho_{j-2}\rho_{j-1}\\
&= \lambda_{i-1,j}.
\end{aligned}$$

(iii) For $k=i$, we have $\rho_i\lambda_{i,j}\rho_i=\lambda_{i+1,j}$, since

$$\begin{aligned}
\rho_i\lambda_{i,j}\rho_i &= \rho_i\rho_{j-1}\rho_{j-2}\dots\rho_{i+1}\lambda_{i,i+1}\rho_{i+1}\dots\rho_{j-2}\rho_{j-1}\rho_i\\
&= \rho_{j-1}\rho_{j-2}\dots\rho_i\rho_{i+1}\lambda_{i,i+1}\rho_{i+1}\rho_i\dots\rho_{j-2}\rho_{j-1}\\
&= \rho_{j-1}\rho_{j-2}\dots\rho_i\rho_{i+1}s_i\rho_i\rho_{i+1}\rho_i\dots\rho_{j-2}\rho_{j-1}\\
&= \rho_{j-1}\rho_{j-2}\dots\rho_{i+2}s_{i+1}\rho_i\rho_{i+1}\rho_i\rho_{i+1}\rho_i\dots\rho_{j-2}\rho_{j-1}\\
&= \rho_{j-1}\rho_{j-2}\dots\rho_{i+2}s_{i+1}\rho_i\rho_i\rho_{i+1}\rho_i\rho_i\dots\rho_{j-2}\rho_{j-1}\\
&= \rho_{j-1}\rho_{j-2}\dots\rho_{i+2}(s_{i+1}\rho_{i+1})\rho_{i+2}\dots\rho_{j-2}\rho_{j-1}\\
&= \lambda_{i+1,j}.
\end{aligned}$$

(iv) If $i+1\leq k\leq j-2$, then

$$\begin{aligned}
\rho_k\lambda_{i,j}\rho_k &= \rho_k\rho_{j-1}\dots\rho_{k+1}\rho_k\dots\rho_{i+1}\lambda_{i,i+1}\rho_{i+1}\dots\rho_k\rho_{k+1}\dots\rho_{j-1}\rho_k\\
&= \rho_{j-1}\dots\rho_k\rho_{k+1}\rho_k\dots\rho_{i+1}\lambda_{i,i+1}\rho_{i+1}\dots\rho_k\rho_{k+1}\rho_k\dots\rho_{j-1}\\
&= \rho_{j-1}\dots\rho_{k+1}\rho_k\rho_{k+1}\rho_{k-1}\dots\rho_{i+1}\lambda_{i,i+1}\rho_{i+1}\dots\rho_{k-1}\rho_{k+1}\rho_k\rho_{k+1}\dots\rho_{j-1}\\
&= \rho_{j-1}\dots\rho_{k+1}\rho_k\rho_{k-1}\dots\rho_{i+1}\lambda_{i,i+1}\rho_{i+1}\dots\rho_{k-1}\rho_k\rho_{k+1}\dots\rho_{j-1}\\
&= \lambda_{i,j}.
\end{aligned}$$

(v) If $k=j-1$, then

$$\rho_k\lambda_{i,j}\rho_k=\lambda_{i,j-1}.$$

(vi) If $k=j$, then

$$\rho_k\lambda_{i,j}\rho_k=\lambda_{i,j+1}.$$

Hence, each generator $\gamma(\mu, s_i)$ lie in the set $\mathcal{S} \sqcup \mathcal{S}^{-1}$. Conversely, if $\lambda_{i,j} \in \mathcal{S}$ is an arbitrary element, then we see that conjugation by $(\rho_{i-1}\rho_{i-2}\dots\rho_2\rho_1)(\rho_{j-1}\rho_{j-2}\dots\rho_3\rho_2)$ maps $\lambda_{1,2}$ to $\lambda_{i,j}$, whereas conjugation by $(\rho_{i-1}\rho_{i-2}\dots\rho_2\rho_1)(\rho_{j-1}\rho_{j-2}\dots\rho_3\rho_2\rho_1)$ maps $\lambda_{1,2}$ to $\lambda_{i,j}^{-1}$. That is, each $\lambda_{i,j} = \mu\lambda_{1,2}\mu^{-1} = \gamma(\mu, s_1)$ for some $\mu \in \mathrm{M}_n$, and hence $\mathcal{S}$ generates the group PVT_n. □

Remark 6.1.2. *We can summarise the action of S_n on the set $\mathcal{S} \sqcup \mathcal{S}^{-1}$ as*

$$\rho_i : \begin{cases} \lambda_{i,i+1} \longleftrightarrow \lambda_{i,i+1}^{-1}, \\ \lambda_{i,j} \longleftrightarrow \lambda_{i+1,j} & \textit{for all } \ i+2 \leq j \leq n, \\ \lambda_{j,i} \longleftrightarrow \lambda_{j,i+1} & \textit{for all } \ 1 \leq j < i, \\ \lambda_{k,l} \longleftrightarrow \lambda_{k,l} & \textit{otherwise, i.e., } \ k \geq i+2, \textit{or } \ k < i \textit{ and } i \neq l \neq i+1. \end{cases}$$

The action can be further simplified as

$$\rho_i : \begin{cases} \lambda_{i,i+1} \longleftrightarrow \lambda_{i,i+1}^{-1}, \\ \lambda_{k,l} \longleftrightarrow \lambda_{k',l'} & \textit{for all } \ (k,l) \neq (i,i+1), \end{cases}$$

where the transposition (k',l') equals $(i,i+1)(k,l)(i,i+1)$ and $k' < l'$. As seen in the proof of Theorem 6.1.1, the action of S_n on the set $\mathcal{S} \sqcup \mathcal{S}^{-1}$ is transitive.

We now prove the main result of this section.

Theorem 6.1.3. *The pure virtual twin group PVT_n on $n \geq 2$ strands has the presentation*

$$\big\langle \lambda_{i,j},\ 1 \leq i < j \leq n \mid \lambda_{i,j}\lambda_{k,l} = \lambda_{k,l}\lambda_{i,j} \textit{ for distinct integers } i,j,k,l \big\rangle.$$

Proof. Theorem 6.1.1 already gives a generating set $\mathcal{S}$ for PVT_n. Geometrically, a generator $\lambda_{i,j}$ looks as in Figure 6.1.
The defining relations are given by

$$\tau(\mu r \mu^{-1}),$$

where τ is the rewriting process, $\mu \in \mathrm{M}_n$ and r is a defining relation in VT_n. Let us take

$$\mu = \rho_{i_1}\rho_{i_2}\dots\rho_{i_k} \in \mathrm{M}_n\,.$$

Recall that by (6.8), $\gamma(\mu, \rho_i) = 1$ for all i. Thus, no non-trivial relations for PVT_n can be obtained from the relations (6.3)–(6.5) of VT_n. We consider the remaining relations one by one.

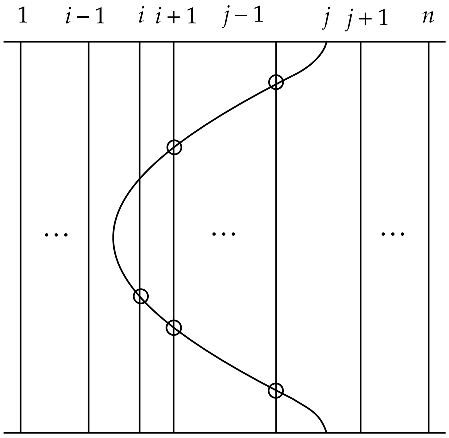

Fig. 6.1. The generator $\lambda_{i,j}$

(i) First consider the relations $s_i^2 = 1$, $1 \le i \le n-1$, of VT_n. Then we have

$$\begin{aligned}
\tau(\mu s_i^2 \mu^{-1}) &= \gamma(\rho_{i_1} \dots \rho_{i_k} s_i s_i \rho_{i_k} \dots \rho_{i_1}) \\
&= \gamma(1, \rho_{i_1})\gamma(\overline{\rho_{i_1}}, \rho_{i_2}) \dots \gamma(\overline{\mu}, s_i)\gamma(\overline{\mu s_i}, s_i) \dots \gamma(\overline{\mu s_i s_i \rho_{i_k} \dots \rho_{i_2}}, \rho_{i_1}) \\
&= \gamma(\overline{\mu}, s_i)\gamma(\overline{\mu s_i}, s_i) \\
&= \gamma(\mu, s_i)\gamma(\mu \rho_i, s_i) \\
&= (\mu s_i \rho_i \mu^{-1})(\mu \rho_i s_i \mu^{-1}) \\
&= (\mu s_i \rho_i \mu^{-1})(\mu s_i \rho_i \mu^{-1})^{-1},
\end{aligned}$$

which does not yield any non-trivial relation in PVT_n.

(ii) Next we consider the relations $(s_i \rho_j)^2 = 1$ for $|i-j| > 1$. Then we have

$$\begin{aligned}
\tau(\mu s_i \rho_j s_i \rho_j \mu^{-1}) &= \gamma(\overline{\mu}, s_i)\gamma(\overline{\mu s_i}, \rho_j)\gamma(\overline{\mu s_i \rho_j}, s_i)\gamma(\overline{\mu s_i \rho_j s_i}, \rho_j) \\
&= \gamma(\overline{\mu}, s_i)\gamma(\overline{\mu s_i \rho_j}, s_i) \\
&= \gamma(\mu, s_i)\gamma(\mu \rho_i \rho_j, s_i) \\
&= (\mu s_i \rho_i \mu^{-1})(\mu \rho_i \rho_j s_i \rho_i \rho_j \rho_i \mu^{-1}) \\
&= (\mu s_i \rho_i \mu^{-1})(\mu \rho_i s_i \mu^{-1}) \\
&= (\mu s_i \rho_i \mu^{-1})(\mu s_i \rho_i \mu^{-1})^{-1},
\end{aligned}$$

which gives a trivial relation in PVT_n.

(iii) Now we consider the relations $\rho_i s_{i+1} \rho_i \rho_{i+1} s_i \rho_{i+1} = 1$, where $1 \leq i \leq n-2$. Computing

$$\begin{aligned}
\tau(\mu\rho_i s_{i+1}\rho_i\rho_{i+1}s_i\rho_{i+1}\mu^{-1}) &= \gamma(\overline{\mu\rho_i}, s_{i+1})\gamma(\overline{\mu\rho_i s_{i+1}\rho_i\rho_{i+1}}, s_i) \\
&= \gamma(\mu\rho_i, s_{i+1})\gamma(\mu\rho_i\rho_{i+1}\rho_i\rho_{i+1}, s_i) \\
&= (\mu\rho_i s_{i+1}\rho_{i+1}\rho_i\mu^{-1})(\mu\rho_{i+1}\rho_i s_i\rho_i\rho_i\rho_{i+1}\mu^{-1}) \\
&= (\mu\rho_i s_{i+1}\rho_{i+1}\rho_i\mu^{-1})(\mu\rho_{i+1}\rho_i s_i\rho_{i+1}\mu^{-1}) \\
&= (\mu\rho_i s_{i+1}\rho_{i+1}\rho_i\mu^{-1})(\mu\rho_{i+1}\rho_i\rho_{i+1}\rho_{i+1}s_i\rho_{i+1}\mu^{-1}) \\
&= (\mu\rho_i s_{i+1}\rho_{i+1}\rho_i\mu^{-1})(\mu\rho_{i+1}\rho_i\rho_{i+1}\rho_i s_{i+1}\rho_i\mu^{-1}) \\
&= (\mu\rho_i s_{i+1}\rho_{i+1}\rho_i\mu^{-1})(\mu\rho_{i+1}\rho_{i+1}\rho_i\rho_{i+1}s_{i+1}\rho_i\mu^{-1}) \\
&= (\mu\rho_i s_{i+1}\rho_{i+1}\rho_i\mu^{-1})(\mu\rho_i\rho_{i+1}s_{i+1}\rho_i\mu^{-1}) \\
&= (\mu\rho_i s_{i+1}\rho_{i+1}\rho_i\mu^{-1})(\mu\rho_i s_{i+1}\rho_{i+1}\rho_i\mu^{-1})^{-1}
\end{aligned}$$

gives only trivial relations in PVT_n.

(iv) Finally we consider the relations $(s_i s_j)^2 = 1$ for $|i-j| > 1$. If $\mu = 1$, then we get

$$\begin{aligned}
\tau(s_i s_j s_i s_j) &= \gamma(1, s_i)\gamma(\overline{s_i}, s_j)\gamma(\overline{s_i s_j}, s_i)\gamma(\overline{s_i s_j s_i}, s_j) \\
&= \gamma(1, s_i)\gamma(\rho_i, s_j)\gamma(\rho_i\rho_j, s_i)\gamma(\rho_i\rho_j\rho_i, s_j) \\
&= (s_i\rho_i)(s_j\rho_j)(\rho_i s_i)(\rho_j s_j) \\
&= \lambda_{i,i+1}\lambda_{j,j+1}\lambda_{i,i+1}^{-1}\lambda_{j,j+1}^{-1}.
\end{aligned}$$

For $\mu \neq 1$, we have

$$\begin{aligned}
\tau(\mu s_i s_j s_i s_j \mu^{-1}) &= \gamma(\overline{\mu}, s_i)\gamma(\overline{\mu s_i}, s_j)\gamma(\overline{\mu s_i s_j}, s_i)\gamma(\overline{\mu s_i s_j s_i}, s_j) \\
&= \gamma(\mu, s_i)\gamma(\mu\rho_i, s_j)\gamma(\mu\rho_i\rho_j, s_i)\gamma(\mu\rho_i\rho_j\rho_i, s_j) \\
&= (\mu s_i\rho_i\mu^{-1})(\mu s_j\rho_j\mu^{-1})(\mu\rho_i s_i\mu^{-1})(\mu\rho_j s_j\mu^{-1}) \\
&= (\mu s_i\rho_i\mu^{-1})(\mu s_j\rho_j\mu^{-1})(\mu s_i\rho_i\mu^{-1})^{-1}(\mu s_j\rho_j\mu^{-1})^{-1} \\
&= (\mu\lambda_{i,i+1}\mu^{-1})(\mu\lambda_{j,j+1}\mu^{-1})(\mu\lambda_{i,i+1}\mu^{-1})^{-1}(\mu\lambda_{j,j+1}\mu^{-1})^{-1}.
\end{aligned} \tag{6.9}$$

For $n \geq 4$, we set

$$\mathcal{T} = \left\{ (\lambda_{i,j}^{\varepsilon}, \lambda_{k,l}^{\varepsilon'}) \mid i,j,k,l \text{ are distinct integers with } 1 \leq i < j \leq n,\ 1 \leq k < l \leq n \text{ and } \varepsilon, \varepsilon' \in \{1,-1\} \right\}.$$

If $\rho \in S_n$ and $(\lambda_{i,j}, \lambda_{k,l}) \in \mathcal{T}$, then Remark 6.1.2 implies that

$$\rho \lambda_{i,j} \rho^{-1} = \lambda^{\varepsilon}_{i',j'}$$

and

$$\rho \lambda_{k,l} \rho^{-1} = \lambda^{\varepsilon'}_{k',l'}$$

for some $\varepsilon, \varepsilon' \in \{1, -1\}$ and distinct integers i', j', k', l' with $1 \leq i' < j' \leq n$ and $1 \leq k' < l' \leq n$. Thus, there is an induced diagonal action of S_n on $\mathcal{T}$ given as

$$\rho \cdot (\lambda^{\varepsilon}_{i,j},\ \lambda^{\varepsilon'}_{k,l}) := (\rho \lambda^{\varepsilon}_{i,j} \rho^{-1},\ \rho \lambda^{\varepsilon'}_{k,l} \rho^{-1}).$$

We claim that this action is transitive. Since $|\mathcal{T}| = n(n-1)(n-2)(n-3)$, it is enough to show that the stabiliser of some element of $\mathcal{T}$ has $(n-4)!$ elements. In fact, the stabiliser of the element $(\lambda_{1,2},\ \lambda_{n-1,n})$ equals $\langle \rho_3, \rho_4, \ldots, \rho_{n-1} \rangle \cap \langle \rho_1, \rho_2, \ldots, \rho_{n-3} \rangle = \langle \rho_3, \rho_4, \ldots, \rho_{n-3} \rangle \cong S_{n-4}$, which is of order $(n-4)!$. Thus, the action is transitive and the defining relations of PVT_n obtained from (6.9) are precisely of the form

$$\lambda_{i,j} \lambda_{k,l} = \lambda_{k,l} \lambda_{i,j}$$

for distinct integers i, j, k, l with $1 \leq i < j \leq n$ and $1 \leq k < l \leq n$. This completes the proof of the theorem. □

Recall that, a group is called a *right-angled Artin group* if it has a presentation in which the only relations are the commuting relations among the generators. Further, a right-angled Artin group is called *irreducible* if it cannot be written as a direct product of two non-trivial subgroups.

Corollary 6.1.4. *The pure virtual twin group PVT_n is an irreducible right-angled Artin group for each $n \geq 2$.*

Proof. That PVT_n is a right-angled Artin group follows from Theorem 6.1.3. Since $PVT_2 \cong \mathbb{Z}$ and $PVT_3 \cong F_3$, they are irreducible. Let $n \geq 4$ and suppose that $PVT_n = A \times B$ for non-trivial subgroups A and B of PVT_n. Assume that $\lambda_{1,2} \in A$. Since $[\lambda_{1,2}, \lambda_{1,j}] \neq 1$ for all $3 \leq j \leq n$, it follows that $\lambda_{1,j} \in A$ for all $2 \leq j \leq n$. Consider an arbitrary generator $\lambda_{i,j}$ of PVT_n with $1 < i < j \leq n$. Since $[\lambda_{1,i}, \lambda_{i,j}] \neq 1$, it follows that $\lambda_{i,j} \in A$, and hence $B = 1$, a contradiction. □

Corollary 6.1.5. *The virtual twin group VT_n is residually finite and Hopfian for each $n \geq 2$.*

Proof. It is well-known that a right-angled Artin group is linear [40, Corollary 3.6] and that a finitely generated linear group is residually finite [61]. Thus, PVT_n is linear, and hence residually finite. Since any extension of a residually finite group by a finite group is residually finite, it follows that VT_n is residually finite. The second assertion follows from the fact that every finitely generated residually finite group is Hopfian [61]. $\square$

The conjugation action of S_n on PVT_n gives a group homomorphism $\phi : S_n \to \mathrm{Aut}(PVT_n)$. We conclude the section with the following observation.

Proposition 6.1.6. *The homomorphism $\phi : S_n \to \mathrm{Aut}(PVT_n)$ is injective for each $n \geq 2$.*

Proof. The assertion is immediate for the case $n = 2$. First suppose that $n \geq 3$ and $n \neq 4$. Recall that the only normal subgroups of S_n are 1, A_n and S_n. Note that the automorphisms $\phi(\rho_1)$, $\phi(\rho_2)$ and $\phi(\rho_1\rho_2)$ are all distinct. Thus, order of $\mathrm{Im}(\phi)$ is strictly greater than 2, and hence $\mathrm{Ker}(\phi)$ must be trivial.
Now suppose that $n = 4$. In this case, the normal subgroups of S_4 are 1, S_4, A_4 and $K_4 = \{1, (1,\ 2)(3,\ 4) = \rho_1\rho_3, (1,\ 3)(2,\ 4), (1,\ 4)(2,\ 3)\}$ (the Klein four group). As in the previous case, $\mathrm{Im}(\phi)$ has more than two elements, and hence $\mathrm{Ker}(\phi)$ is either trivial or K_4. Since $\phi(\rho_1\rho_3)(\lambda_{1,2}) = \lambda_{1,2}^{-1} \neq 1$, we have $\rho_1\rho_3 \notin \mathrm{Ker}(\phi)$. Hence, $\mathrm{Ker}(\phi)$ must be trivial in this case as well. $\square$

6.2 Decomposition of pure virtual twin groups

Let $i_n : PVT_{n-1} \to PVT_n$ be the natural inclusion obtained by adding a strand to the rightmost side of the diagram of an element of PVT_{n-1}. In the reverse direction, we have a well-defined homomorphism $f_n : PVT_n \to PVT_{n-1}$ obtained by removing the rightmost strand from the diagram of an element of PVT_n. Algebraically, f_n is defined on generators of PVT_n by

$$f_n(\lambda_{i,j}) = \begin{cases} \lambda_{i,j} & \text{if } j \neq n, \\ 1 & \text{if } j = n. \end{cases}$$

Furthermore, we have $f_n \circ i_n = \mathrm{id}_{PVT_{n-1}}$, and hence f_n is surjective. For each $n \geq 2$, let U_n denotes $\mathrm{Ker}(f_n)$. Then we have the split short exact sequence

$$\begin{tikzcd} 1 \arrow[r] & U_n \arrow[r] & PVT_n \arrow[r, "f_n"] & PVT_{n-1} \arrow[r] \arrow[l, bend right, "i_n"'] & 1, \end{tikzcd}$$

that is,

$$PVT_n \cong U_n \rtimes PVT_{n-1}.$$

Theorem 6.2.1. *For $n \geq 2$, we have*

$$PVT_n \cong U_n \rtimes (U_{n-1} \rtimes (\cdots \rtimes (U_4 \rtimes (U_3 \rtimes U_2))\cdots)),$$

where $U_2 = PVT_2 \cong \mathbb{Z}$ and $U_i = \mathrm{Ker}(f_i : PVT_i \to PVT_{i-1})$ are infinitely generated free groups for $i \geq 3$.

Proof. It is clear that $U_2 = PVT_2 \cong \mathbb{Z}$. We use the Reidemeister-Schreier method for $n \geq 3$, for which we take the Schreier system to be PVT_{n-1}. Note that for $\mu \in PVT_{n-1}$ and $\lambda_{i,j} \in \mathcal{S}$, we have

$$\gamma(\mu, \lambda_{i,j}) = \begin{cases} 1 & \text{if } j \neq n, \\ \mu\lambda_{i,j}\mu^{-1} & \text{if } j = n. \end{cases}$$

This implies that U_n is generated by the set

$$X = \{\mu\lambda_{i,n}\mu^{-1} \mid \mu \in PVT_{n-1} \text{ and } i = 1, 2, \ldots, n-1\}.$$

Since $PVT_3 \cong F_3$, it follows that U_3 is an infinitely generated free group with generators

$$\{\mu\lambda_{1,3}\mu^{-1},\ \mu\lambda_{2,3}\mu^{-1} \mid \mu \in PVT_2\}.$$

For $n \geq 4$, in PVT_n we have relations of the form

$$\lambda_{i,j}\lambda_{k,l}\lambda_{i,j}^{-1}\lambda_{k,l}^{-1} = 1,$$

where i, j, k, l are distinct integers with $i < j$ and $k < l$. First, consider the case when none of i, j, k, l is equal to n. Since $\gamma(\mu, \lambda_{i,j}) = 1$ for $\mu \in PVT_{n-1}$ and $\lambda_{i,j} \in \mathcal{S}$ with $j \neq n$, we have

$$\tau(\mu\lambda_{i,j}\lambda_{k,l}\lambda_{i,j}^{-1}\lambda_{k,l}^{-1}\mu^{-1}) = 1,$$

and hence there is no non-trivial relation in this case. Next, we consider the case when exactly one of i, j, k, l is equal to n. Without loss of generality, we can assume that $j = n$. Applying the rewriting process to the relations $\lambda_{i,n}\lambda_{k,l}\lambda_{i,n}^{-1}\lambda_{k,l}^{-1} = 1$ of PVT_n gives

$$\begin{aligned}
\tau(\mu\lambda_{i,n}\lambda_{k,l}\lambda_{i,n}^{-1}\lambda_{k,l}^{-1}\mu^{-1}) &= \gamma(\overline{\mu}, \lambda_{i,n})\gamma(\overline{\mu\lambda_{i,n}}, \lambda_{k,l})\gamma(\overline{\mu\lambda_{i,n}\lambda_{k,l}}, \lambda_{i,n}^{-1})\gamma(\overline{\mu\lambda_{i,n}\lambda_{k,l}\lambda_{i,n}^{-1}}, \lambda_{k,l}^{-1}) \\
&= \gamma(\overline{\mu}, \lambda_{i,n})\gamma(\overline{\mu\lambda_{i,n}\lambda_{k,l}}, \lambda_{i,n}^{-1}) \\
&= \gamma(\mu, \lambda_{i,n})\gamma(\mu\lambda_{k,l}, \lambda_{i,n}^{-1}) \\
&= \lambda_{i,n}^{\mu^{-1}}(\lambda_{i,n}^{\lambda_{k,l}^{-1}\mu^{-1}})^{-1}.
\end{aligned}$$

This gives the relations

$$\mu\lambda_{i,n}\mu^{-1} = \mu\lambda_{k,l}\lambda_{i,n}\lambda_{k,l}^{-1}\mu^{-1}$$

in U_n, which simply identifies two generators of U_n. Finally, to prove that there are still infinitely many distinct generators in the set X, we consider the sequence of elements

$$\alpha_\varepsilon = \lambda_{n-2,n-1}^{-\varepsilon}\lambda_{n-1,n}\lambda_{n-2,n-1}^{\varepsilon}$$

in X. Notice that $\alpha_\varepsilon \neq \alpha_{\varepsilon'}$ for $\varepsilon \neq \varepsilon'$. Hence, U_n is an infinite rank free group for each $n \geq 3$. $\square$

Corollary 6.2.2. $\mathrm{Z}(PVT_n) = 1$ *for* $n \geq 3$ *and* $\mathrm{Z}(VT_n) = 1$ *for* $n \geq 2$.

Proof. We use the elementary fact that if $G = N \rtimes H$ is an internal semidirect product with $\mathrm{Z}(H) = 1$, then $\mathrm{Z}(G) \leq \mathrm{Z}(N)$. Recall that $PVT_2 \cong \mathbb{Z}$ and $PVT_3 \cong F_3$. Since $PVT_4 = U_4 \rtimes PVT_3$, we have

$$\mathrm{Z}(PVT_4) \leq \mathrm{Z}(U_4).$$

By Theorem 6.2.1, $\mathrm{Z}(U_4) = 1$, and hence $\mathrm{Z}(PVT_4) = 1$. An easy induction gives $\mathrm{Z}(PVT_n) = 1$ for all $n \geq 4$.
Since $VT_2 \cong \mathbb{Z}_2 * \mathbb{Z}_2$, we have $\mathrm{Z}(VT_2) = 1$. For $n \geq 3$, since $\mathrm{Z}(S_n) = 1$ and $VT_n = PVT_n \rtimes S_n$, we have $\mathrm{Z}(VT_n) \leq \mathrm{Z}(PVT_n) = 1$. $\square$

A right-angled Artin group is called *spherical* if its corresponding Coxeter group is finite. PVT_n is clearly non-spherical for $n \geq 3$. It is a well-known conjecture that every irreducible non-spherical Artin group has trivial center [31, Conjecture B]. In view of Corollary 6.1.4, PVT_n is irreducible and non-spherical. Hence, by Corollary 6.2.2, PVT_n satisfies the conjecture for $n \geq 3$.

Chapter 7

Automorphism groups of pure virtual twin groups

In this chapter, we compute automorphism groups of pure virtual twin groups. The results are from [73].

7.1 Graphs of pure virtual twin groups

Given a graph Γ with the vertex set V, the right-angled Artin group associated to Γ is the group

$$A_\Gamma = \big\langle V \mid [v,\ w] = 1 \text{ if } v, w \in V \text{ are joined by an edge in } \Gamma \big\rangle.$$

Conversely, each right-angled Artin group gives a graph whose vertex set is the set of generators of the group and there is an edge between the two vertices if and only if the two generators commute. It is easy to see that the right-angled Artin group corresponding to the complete graph on n vertices is the free abelian group $\mathbb{Z}^n$ and the group corresponding to the edgeless graph on n vertices is the free group F_n.

The generating set $\mathcal{S} = \{\lambda_{i,j} \mid 1 \leq i < j \leq n\}$ is the vertex set of the associated graph of PVT_n for $n \geq 2$. Note that $PVT_2 \cong \mathbb{Z}$, $PVT_3 \cong F_3$ and $PVT_4 \cong (\mathbb{Z} \times \mathbb{Z}) * (\mathbb{Z} \times \mathbb{Z}) * (\mathbb{Z} \times \mathbb{Z})$. While the graphs of PVT_2 and PVT_3 are edgeless graphs on 1 and 3 vertices, respectively; the graphs of PVT_4 and PVT_5 are shown in figures 7.1 and 7.2, respectively. It is interesting to note that the graph of PVT_n is the Kneser graph $K(n,2)$, which is same as the commuting graph of the conjugacy class of transpositions in the symmetric group S_n.

The *link* $lk(v)$ of a vertex $v \in V$ is defined as the set of all vertices that are connected to v by an edge. The *star* $st(v)$ of v is defined as $lk(v) \cup \{v\}$. If $v \neq w$ are vertices, then we say that w dominates v, written $v \leq w$, if $lk(v) \subseteq st(w)$.

For each $\lambda_{i,j} \in \mathcal{S}$, let

$$N_{i,j} = \mathcal{S} \setminus st(\lambda_{i,j}) = \{\lambda_{k,l} \mid [\lambda_{i,j},\ \lambda_{k,l}] \neq 1\}$$

be the set of vertices that are not connected to $\lambda_{i,j}$ by an edge.

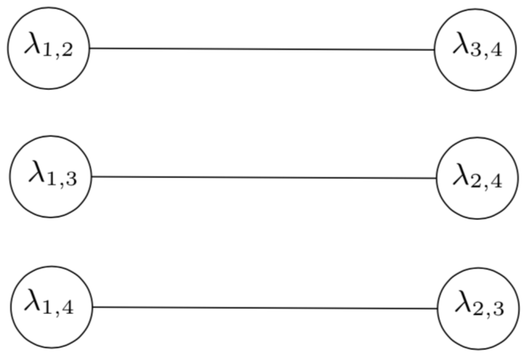

Fig. 7.1. The graph of PVT_4

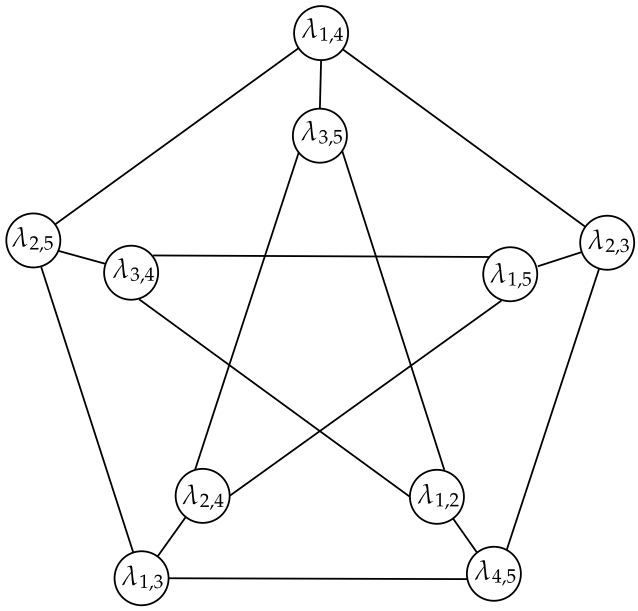

Fig. 7.2. The graph of PVT_5

Proposition 7.1.1. *The following hold for each $\lambda_{i,j} \in \mathcal{S}$:*

1. $|N_{i,j}| = 2n-4$.
2. $|st(\lambda_{i,j})| = \dfrac{(n-2)(n-4)+n}{2}$. *In particular, the graph of PVT_n is regular.*

Proof. Note that the set $N_{i,j}$ can be written as a union of four disjoint subsets as follows

$$\begin{aligned} N_{i,j} =& \{\lambda_{i,k} \mid i+1 \leq k \leq n,\ k \neq j\} \cup \{\lambda_{j,k} \mid j+1 \leq k \leq n\} \\ & \cup \{\lambda_{k,i} \mid 1 \leq k \leq i-1\} \cup \{\lambda_{k,j} \mid 1 \leq k \leq j-1,\ k \neq i\}. \end{aligned}$$

Observe that

$$|\{\lambda_{i,k} \mid i+1 \leq k \leq n,\ k \neq j\}| + |\{\lambda_{k,i} \mid 1 \leq k \leq i-1\}| = n-2,$$

$$|\{\lambda_{j,k} \mid j+1 \leq k \leq n\}| + |\{\lambda_{k,j} \mid 1 \leq k \leq j-1,\ k \neq i\}| = n-2,$$

and hence $|N_{i,j}| = 2n-4$. The second assertion is immediate. □

It is well-known (see, for example, [57, 80]) that the automorphism group $\mathrm{Aut}(A_\Gamma)$ of a right-angled Artin group A_Γ is generated by the following four types of automorphisms.

1. *Graph automorphism*: Automorphism of A_Γ induced by an automorphism of the graph Γ.
2. *Inversion* ι_a: Sends a generator a to its inverse and leaves all other generators fixed.
3. *Transvection* τ_{ab}: Sends a generator a to ab and leaves all other generators fixed, where b is another generator with $a \leq b$.
4. *Partial conjugation* $p_{b,C}$: If b is a generator and C is a union of connected components of $\Gamma \setminus \Gamma(st(b))$, then $p_{b,C}$ sends each generator a in C to a^b and leaves the other generators fixed. It follows that if $\Gamma \setminus \Gamma(st(b))$ is connected or $C = \Gamma \setminus \Gamma(st(b))$, then the partial conjugation $p_{b,C}$ is simply the inner automorphism induced by b.

We set the following notations for the subgroups of the automorphism group $\mathrm{Aut}(PVT_n)$ of PVT_n for $n \geq 2$.

- $\mathrm{Aut}_{gr}(PVT_n)$: The subgroup generated by all graph automorphisms.
- $\mathrm{Aut}_{inv}(PVT_n)$: The subgroup generated by all inversions.
- $\mathrm{Aut}_{tr}(PVT_4)$: The subgroup generated by all transvections.
- $\mathrm{Aut}_{pc}(PVT_n)$: The subgroup generated by all partial conjugations.
- $\mathrm{Inn}(PVT_n)$: The subgroup of all inner automorphisms.

7.2 Automorphism group of $PVT_n, n \neq 4$

In this section, we compute the automorphism group of PVT_n for $n \neq 4$. The case $n = 4$ is exotic and will be dealt separately in the subsequent section. The following result is the main theorem of this section.

Theorem 7.2.1. *Let $n \geq 5$. Then there exist split exact sequences*

$$1 \to \mathrm{Aut}_{inv}(PVT_n) \to \langle \mathrm{Aut}_{gr}(PVT_n), \mathrm{Aut}_{inv}(PVT_n)\rangle \to \mathrm{Aut}_{gr}(PVT_n) \to 1 \tag{7.1}$$

and

$$1 \to \mathrm{Inn}(PVT_n) \to \mathrm{Aut}(PVT_n) \to \langle \mathrm{Aut}_{gr}(PVT_n), \mathrm{Aut}_{inv}(PVT_n)\rangle \to 1. \tag{7.2}$$

In particular,

$$\langle \mathrm{Aut}_{gr}(PVT_n), \mathrm{Aut}_{inv}(PVT_n)\rangle \cong \mathbb{Z}_2^{n(n-1)/2} \rtimes S_n$$

and

$$\mathrm{Aut}(PVT_n) \cong PVT_n \rtimes (\mathbb{Z}_2^{n(n-1)/2} \rtimes S_n).$$

We begin by computing the structure of $\mathrm{Aut}_{inv}(PVT_n)$, which follows immediately as consequence of the definition.

Lemma 7.2.2. $\mathrm{Aut}_{inv}(PVT_n) \cong \mathbb{Z}_2^{n(n-1)/2}$ *for all $n \geq 2$.*

Next we compute the group $\mathrm{Aut}_{gr}(PVT_n)$. Since the graph of PVT_n is the Kneser graph $K(n,2)$, its group of graph automorphisms is well-known [32, Corollary 7.8.2]. However, we give a direct computation of $\mathrm{Aut}_{gr}(PVT_n)$ in our set-up. For $n = 2$, there is only one vertex, and hence $\mathrm{Aut}_{gr}(PVT_2) = 1$. Assume that $n \geq 3$. For each $1 \leq k \leq n-1$, define

$$\theta_k := \iota_{\lambda_{k,k+1}} \circ \phi(\rho_k),$$

where $\phi : S_n \to \mathrm{Aut}(PVT_n)$ is the map from Proposition 6.1.6. The action of θ_k on the set $\mathcal{S}$ of generators is described explicitly as follows:

$$\theta_k : \begin{cases} \lambda_{k,k+1} \longrightarrow \lambda_{k,k+1}, & \\ \lambda_{k,j} \longrightarrow \lambda_{k+1,j} & \text{for all } \ k+2 \leq j \leq n, \\ \lambda_{k+1,j} \longrightarrow \lambda_{k,j} & \text{for all } \ k+2 \leq j \leq n, \\ \lambda_{i,k} \longrightarrow \lambda_{i,k+1} & \text{for all } \ 1 \leq i < k, \\ \lambda_{i,k+1} \longrightarrow \lambda_{i,k} & \text{for all } \ 1 \leq i < k, \\ \lambda_{i,j} \longrightarrow \lambda_{i,j} & \text{else, i.e., } \ i \geq k+2, \text{ or } \ i < k \text{ and } k \neq j \neq k+1. \end{cases}$$

Since each automorphism θ_k keeps the set $\mathcal{S}$ invariant, we have

$$\langle \theta_1, \theta_2, \ldots, \theta_{n-1} \rangle \leq \mathrm{Aut}_{gr}(PVT_n). \tag{7.3}$$

Note that this action of $\langle \theta_1, \theta_2, \ldots, \theta_{n-1} \rangle$ on $\mathcal{S}$ is transitive. In fact, given any generator $\lambda_{i,j} \in \mathcal{S}$, the automorphism $(\theta_{i-1}\theta_{i-2}\cdots\theta_2\theta_1)(\theta_{j-1}\theta_{j-2}\cdots\theta_3\theta_2)$ maps $\lambda_{1,2}$ onto $\lambda_{i,j}$.

Lemma 7.2.3. $\langle \theta_1, \theta_2, \ldots, \theta_{n-1} \rangle \cong S_n$ *for all* $n \geq 3$.

Proof. Note that θ_i is an involution for each $1 \leq i \leq n-1$. It follows from the construction that $[\theta_i,\ \theta_j] = 1$ for all $|i-j| \geq 2$. We now claim that $(\theta_i\theta_{i+1})^3 = 1$ for all $1 \leq i \leq n-2$. We verify this for $i = 1$ and other cases will follow similarly. Consider

$$\theta_1 : \begin{cases} \lambda_{1,2} \longrightarrow \lambda_{1,2}, & \\ \lambda_{1,j} \longrightarrow \lambda_{2,j} & \text{for all } 3 \leq j \leq n, \\ \lambda_{2,j} \longrightarrow \lambda_{1,j} & \text{for all } 3 \leq j \leq n, \\ \lambda_{i,j} \longrightarrow \lambda_{i,j} & \text{otherwise, i.e., } i \geq 3, \end{cases} \qquad \theta_2 : \begin{cases} \lambda_{1,2} \longrightarrow \lambda_{1,3}, & \\ \lambda_{1,3} \longrightarrow \lambda_{1,2}, & \\ \lambda_{2,3} \longrightarrow \lambda_{2,3}, & \\ \lambda_{2,j} \longrightarrow \lambda_{3,j} & \text{for all } 4 \leq j \leq n, \\ \lambda_{3,j} \longrightarrow \lambda_{2,j} & \text{for all } 4 \leq j \leq n, \\ \lambda_{i,j} \longrightarrow \lambda_{i,j} & i \geq 4, \text{ or } i = 1 \text{ and} \\ & 4 \leq j \leq n. \end{cases}$$

Note that $\theta_1\theta_2$ fixes the generators $\lambda_{i,j}$ for all $i \geq 4$. Thus, we need to check only for $\lambda_{i,j}$ with $1 \leq i \leq 3$. We see that

$$\theta_1\theta_2 : \begin{cases} \lambda_{1,2} \longrightarrow \lambda_{2,3}, & \\ \lambda_{1,3} \longrightarrow \lambda_{1,2}, & \\ \lambda_{1,j} \longrightarrow \lambda_{2,j} & \text{for all } 4 \leq j \leq n, \\ \lambda_{2,3} \longrightarrow \lambda_{1,3}, & \\ \lambda_{2,j} \longrightarrow \lambda_{3,j} & \text{for all } 4 \leq j \leq n, \\ \lambda_{3,j} \longrightarrow \lambda_{1,j} & \text{for all } 4 \leq j \leq n, \\ \lambda_{i,j} \longrightarrow \lambda_{i,j} & \text{otherwise, i.e., } i \geq 4, \end{cases}$$

and it can be easily checked that $(\theta_1\theta_2)^3 = 1$. Thus, sending ρ_k to θ_k gives a surjective homomorphism from S_n onto $\langle \theta_1, \theta_2, \ldots, \theta_{n-1} \rangle$. We claim that this homomorphism is injective as well. Note that the only normal subgroups of S_n are 1, A_n and S_n, $n \geq 3$ and $n \neq 4$. We know $\rho_1\rho_2 \in A_n$. From preceding computation we have $\theta_1\theta_2 \neq 1$. Thus $\rho_1\rho_2$ does not belong to the kernel and consequently A_n and S_n cannot be the kernel. So we are done for the case $n \geq 3$ and $n \neq 4$. For $n = 4$, we have an extra normal subgroup of S_4, namely the Klein four-subgroup, $K_4 = \{1, \rho_1\rho_3 = (1,2)(3,4), (1,3)(2,4), (1,4)(2,3)\}$. Like previous case, we can check that $\theta_1\theta_3 \neq 1$. Thus $\rho_1\rho_3$ does not belong to the kernel and consequently K_4 cannot be the kernel. Thus, the kernel of the homomorphism must be trivial and $\langle \theta_1, \theta_2, \dots, \theta_{n-1} \rangle \cong S_n$. □

We refer the set $\{\lambda_{k,l} \mid k < l \leq n\}$ as the k^{th} column of the graph of PVT_n.

Lemma 7.2.4. *Let $\phi \in \mathrm{Aut}_{gr}(PVT_n)$ such that $\phi(\lambda_{1,j}) = \lambda_{1,j}$ for all $2 \leq j \leq n$. Then ϕ is the identity automorphism.*

Proof. We begin by noting that if $\phi(\lambda_{i,j}) = \lambda_{i,j}$ for some i, j, then being a graph automorphism, ϕ keeps the set $N_{i,j}$ invariant. For $1 \leq i \leq n-1$, we have

$$(i+1)^{\text{th}} \text{ column } \subseteq N_{i,i+1} \subseteq \bigcup_{k=1}^{i+1} k^{\text{th}} \text{ column}.$$

We now proceed to the main part of the proof. Since $\phi(\lambda_{1,2}) = \lambda_{1,2}$, ϕ keeps $N_{1,2}$, i.e., first two columns invariant. As the first column is already fixed pointwise, ϕ keeps the second column invariant. Now suppose that $\phi(\lambda_{2,3}) = \lambda_{2,j}$ for some $4 \leq j \leq n$. Now we have two elements $\lambda_{1,3}$ and $\lambda_{2,3}$ who do not commutes, but their images ($\lambda_{1,3}$ and $\lambda_{2,j}$, $j \geq 4$ respectively) under ϕ commutes. This contradicts the hypothesis that ϕ is an automorphism. Thus $\phi(\lambda_{2,3}) = \lambda_{2,3}$, and similarly we can show that $\phi(\lambda_{2,j}) = \lambda_{2,j}$, for all $4 \leq j \leq n$. Now consider the element $\lambda_{2,3}$. As ϕ fixes $\lambda_{2,3}$, it should keep $N_{2,3}$ invariant. But ϕ already fixes first two columns pointwise. Thus ϕ keeps the third column invariant. By repeated use of above arguments we get the desired result. □

Theorem 7.2.5. *If $n \geq 3$ and $n \neq 4$, then $\mathrm{Aut}_{gr}(PVT_n) = \langle \theta_1, \theta_2, \dots, \theta_{n-1} \rangle \cong S_n$.*

Proof. Since the graph of PVT_n has $n(n-1)/2$ vertices, it follows that

$$S_n \cong \langle \theta_1, \theta_2, \dots, \theta_{n-1} \rangle \leq \mathrm{Aut}_{gr}(PVT_n) \leq S_{n(n-1)/2}.$$

Thus, we have $\mathrm{Aut}_{gr}(PVT_3) \cong S_3$. For $n \geq 5$, it is sufficient to prove $\mathrm{Aut}_{gr}(PVT_n) \leq \langle \theta_1, \theta_2, \dots, \theta_{n-1} \rangle$. Consider an arbitrary automorphism $\phi \in \mathrm{Aut}_{gr}(PVT_n)$. Our plan is to compose ϕ with finitely many automorphism of $\langle \theta_1, \theta_2, \dots, \theta_{n-1} \rangle$ and obtain the identity automorphism.

As the automorphism group $\langle \theta_1, \theta_2, \dots, \theta_{n-1} \rangle$ acts transitively on the generating set $\mathcal{S}$, there exists $\phi_1 \in \langle \theta_1, \theta_2, \dots, \theta_{n-1} \rangle$ such that $\phi_1\phi(\lambda_{1,2}) = \lambda_{1,2}$. Thus, the set $N_{1,2} = \{\lambda_{i,j} \mid 1 \leq i \leq 2,\ 3 \leq j \leq n\}$ should be invariant under $\phi_1\phi$. Now set

$$\phi_2 : \begin{cases} 1 & \text{if } \phi_1\phi(\lambda_{1,3}) = \lambda_{1,3}, \\ \theta_1 & \text{if } \phi_1\phi(\lambda_{1,3}) = \lambda_{2,3}, \\ \theta_3\theta_4\cdots\theta_{j-1} & \text{if } \phi_1\phi(\lambda_{1,3}) = \lambda_{1,j}, j \geq 4, \\ \theta_3\theta_4\cdots\theta_{j-1}\theta_1 & \text{if } \phi_1\phi(\lambda_{1,3}) = \lambda_{2,j}, j \geq 4. \end{cases}$$

Note that $\phi_2\phi_1\phi(\lambda_{1,2}) = \lambda_{1,2}$ and $\phi_2\phi_1\phi(\lambda_{1,3}) = \lambda_{1,3}$. Now $\phi_2\phi_1\phi$ keeps the sets $N_{1,2}$ and $lk(1,3)$ invariant and thus keeps there intersection also invariant. Note that

$$N_{1,2} \cap lk(\lambda_{1,3}) = \{\lambda_{2,j} \mid 4 \leq j \leq n\}.$$

We claim that there exists $\phi_3 \in \langle \theta_4, \theta_5, \dots, \theta_{n-1} \rangle$ such that $\phi_3\phi_2\phi_1\phi(\lambda_{2,j}) = \lambda_{2,j}$ for all $4 \leq j \leq n$. Suppose that $\phi_2\phi_1\phi(\lambda_{2,4}) = \lambda_{2,j}$ for some $4 \leq j \leq n$. Take $\psi_4 = \theta_4\theta_5\dots\theta_{j-1} \in \langle \theta_4, \theta_5, \dots, \theta_{n-1} \rangle$. Note that $\psi_4\phi_2\phi_1\phi(\lambda_{2,4}) = \psi_4(\lambda_{2,j}) = \lambda_{2,4}$ and $\psi_4\phi_2\phi_1\phi$ keeps the set $\{\lambda_{2,j} \mid 5 \leq j \leq n\}$ invariant. Now suppose that $\phi_2\phi_1\phi(\lambda_{2,5}) = \lambda_{2,j}$ for some $5 \leq j \leq n$. Take $\psi_5 = \theta_5\theta_6\dots\theta_{j-1} \in \langle \theta_5, \theta_6, \dots, \theta_{n-1} \rangle \subseteq \langle \theta_4, \theta_5, \dots, \theta_{n-1} \rangle$. Repeating the argument we can take $\phi_3 = \psi_n\psi_{n-1}\dots\psi_4 \in \langle \theta_4, \theta_5, \dots, \theta_{n-1} \rangle$.
Note that, by the choice of ϕ_3, we also have $\phi_3\phi_2\phi_1\phi(\lambda_{1,2}) = \lambda_{1,2}$ and $\phi_3\phi_2\phi_1\phi(\lambda_{1,3}) = \lambda_{1,3}$. We claim that $\phi_3\phi_2\phi_1\phi(\lambda_{2,3}) = \lambda_{2,3}$ and $\phi_3\phi_2\phi_1\phi(\lambda_{1,j}) = \lambda_{1,j}$ for all $4 \leq j \leq n$. We use the commuting relations among elements of $N_{1,2}$ and the hypothesis that $n \geq 5$. There are two cases here. First, if $\phi_3\phi_2\phi_1\phi(\lambda_{2,3}) = \lambda_{1,j}$ for some $j \geq 4$, then $\phi_3\phi_2\phi_1\phi(\lambda_{1,k}) = \lambda_{2,3}$ and $\phi_3\phi_2\phi_1\phi(\lambda_{1,j}) = \lambda_{1,l}$ for some $k, l \geq 4$. Notice that $\lambda_{2,3}$ and $\lambda_{1,j}$ commute but not their images. Thus, this case does not arise. Secondly, suppose that $\phi_3\phi_2\phi_1\phi(\lambda_{2,3}) = \lambda_{2,3}$, but $\phi_3\phi_2\phi_1\phi(\lambda_{1,j}) = \lambda_{1,k}$ for some $j, k \geq 4$ with $j \neq k$. Then, the elements $\lambda_{1,j}$ and $\lambda_{2,k}$ commute, but not their images. Since the first two columns are fixed pointwise, we are done by Lemma 7.2.4. □

Theorem 7.2.6. $\mathrm{Aut}(PVT_n) = \big\langle \mathrm{Aut}_{gr}(PVT_n), \mathrm{Aut}_{inv}(PVT_n), \mathrm{Inn}(PVT_n) \big\rangle$ *for all* $n \geq 5$.

Proof. Our first claim is that PVT_n does not admit any automorphism of transvection type. Equivalently, if $\lambda_{i,j} \neq \lambda_{k,l}$, then neither $\lambda_{i,j} \leq \lambda_{k,l}$ nor $\lambda_{k,l} \leq \lambda_{i,j}$. Suppose that $\lambda_{i,j} \neq \lambda_{k,l}$. That means either $\{i,j\} \cap \{k,l\}$ is empty or a singleton set. Let us first suppose that the intersection is empty. Since $n \geq 5$, we can choose $1 \leq q \leq n$ such that $q \notin \{i,j,k,l\}$. Set $x = \lambda_{i,q}$ (if $i < q$) or $x = \lambda_{q,i}$ (if $i > q$) and $y = \lambda_{k,q}$ (if $k < q$) or $y = \lambda_{q,k}$ (if $k > q$). Observe that $x \in lk(\lambda_{k,l}) \setminus st(\lambda_{i,j})$ and $y \in lk(\lambda_{i,j}) \setminus st(\lambda_{k,l})$. Thus, neither $\lambda_{k,l} \leq \lambda_{i,j}$ nor $\lambda_{i,j} \leq \lambda_{k,l}$.
Now we suppose that the intersection is a singleton set. Without loss of generality, we may assume that $i = k$. Since $n \geq 5$, choose $m \notin \{i,j,k\}$ and set $x = \lambda_{m,l}$ (if $m < l$) or $x = \lambda_{l,m}$ (if $m > l$) and $y = \lambda_{m,j}$ (if $m < j$) or $y = \lambda_{j,m}$ (if $m > j$). Observe that $x \in lk(\lambda_{i,j}) \setminus st(\lambda_{k,l})$ and

$y \in lk(\lambda_{k,l}) \setminus st(\lambda_{i,j})$. Thus, PVT_n does not admit any automorphism of transvection type for $n \geq 5$.

Our second claim is that $\mathrm{Aut}_{pc}(PVT_n) = \mathrm{Inn}(PVT_n)$. Equivalently, the subgraph $\Gamma \setminus \Gamma(st(\lambda_{i,j}))$ is connected for each $\lambda_{i,j}$. Note that the action of $\mathrm{Aut}_{gr}(PVT_n) = \langle \theta_1, \theta_2, \ldots, \theta_{n-1} \rangle$ on $\mathcal{S}$ is transitive. In fact, given any generator $\lambda_{i,j} \in \mathcal{S}$, the automorphism

$$(\theta_{i-1}\theta_{i-2}\cdots\theta_2\theta_1)(\theta_{j-1}\theta_{j-2}\cdots\theta_3\theta_2)$$

maps $\lambda_{1,2}$ onto $\lambda_{i,j}$. Thus, it suffices to prove the claim for $\lambda_{1,2}$. Note that the vertex set of $\Gamma \setminus \Gamma(st(\lambda_{1,2}))$ is $\{\lambda_{1,i}, \lambda_{2,j} \mid 3 \leq i, j \leq n\}$. Let v_1, v_2 be two vertices of $\Gamma \setminus \Gamma(st(\lambda_{1,2}))$. We find a path joining these two vertices as per the following cases:

1. $v_1 = \lambda_{1,i}, v_2 = \lambda_{1,j}, i \neq j$: Choose an integer k such that $3 \leq k \leq n$ and $i \neq k \neq j$. This is possible since $n \geq 5$. We see that there is an edge joining $\lambda_{1,i}$ and $\lambda_{2,k}$ and an edge joining $\lambda_{2,k}$ and $\lambda_{1,j}$.

2. $v_1 = \lambda_{2,i}, v_2 = \lambda_{2,j}, i \neq j$: This is analogous to the previous case.

3. $v_1 = \lambda_{1,i}, v_2 = \lambda_{2,j}, i \neq j$: Clearly there is an edge joining $\lambda_{1,i}$ and $\lambda_{2,j}$.

4. $v_1 = \lambda_{1,i}, v_2 = \lambda_{2,i}$: Choose two integers j,k such that $3 \leq j,k \leq n$ and $j \neq i \neq k \neq j$. We can see that there are edges from $\lambda_{1,i}$ to $\lambda_{2,j}$, from $\lambda_{2,j}$ to $\lambda_{1,k}$, and from $\lambda_{1,k}$ to $\lambda_{2,i}$.

Hence, the subgraph $\Gamma \setminus \Gamma(st(\lambda_{1,2}))$ is connected. Thus, $\mathrm{Aut}_{pc}(PVT_n) = \mathrm{Inn}(PVT_n)$ for $n \geq 5$. Finally, by [57, 80], we have $\mathrm{Aut}(PVT_n) = \big\langle \mathrm{Aut}_{gr}(PVT_n), \mathrm{Aut}_{inv}(PVT_n), \mathrm{Inn}(PVT_n) \big\rangle$. $\square$

We now state and prove the main theorem of this section.

Theorem 7.2.7. *Let $n \geq 5$. Then there exist split exact sequences*

$$1 \to \mathrm{Aut}_{inv}(PVT_n) \to \langle \mathrm{Aut}_{gr}(PVT_n), \mathrm{Aut}_{inv}(PVT_n) \rangle \to \mathrm{Aut}_{gr}(PVT_n) \to 1 \tag{7.4}$$

and

$$1 \to \mathrm{Inn}(PVT_n) \to \mathrm{Aut}(PVT_n) \to \langle \mathrm{Aut}_{gr}(PVT_n), \mathrm{Aut}_{inv}(PVT_n) \rangle \to 1. \tag{7.5}$$

In particular,

$$\langle \mathrm{Aut}_{gr}(PVT_n), \mathrm{Aut}_{inv}(PVT_n) \rangle \cong \mathbb{Z}_2^{n(n-1)/2} \rtimes S_n$$

and

$$\mathrm{Aut}(PVT_n) \cong PVT_n \rtimes (\mathbb{Z}_2^{n(n-1)/2} \rtimes S_n).$$

Proof. It follows from the construction of graph automorphisms that $\mathrm{Aut}_{gr}(PVT_n)$ normalises $\mathrm{Aut}_{inv}(PVT_n)$. Further, since $\mathrm{Aut}_{gr}(PVT_n) \cap \mathrm{Aut}_{inv}(PVT_n) = 1$, the short exact sequence (7.4) splits.
Recall from Theorem 7.2.6 that $\mathrm{Aut}(PVT_n) = \langle \mathrm{Aut}_{gr}(PVT_n), \mathrm{Aut}_{inv}(PVT_n), \mathrm{Inn}(PVT_n)\rangle$. Note that any automorphism $\phi \in \langle \mathrm{Aut}_{gr}(PVT_n), \mathrm{Aut}_{inv}(PVT_n)\rangle$ keeps the set $\mathcal{S} \cup \mathcal{S}^{-1}$ invariant. Since two distinct elements of $\mathcal{S} \cup \mathcal{S}^{-1}$ are not conjugates of each other in PVT_n, it follows that

$$\mathrm{Inn}(PVT_n) \cap \langle \mathrm{Aut}_{gr}(PVT_n), \mathrm{Aut}_{inv}(PVT_n)\rangle = 1.$$

This gives the split sequence (7.5). The remaining two assertions are immediate. □

Recall that $PVT_2 \cong \mathbb{Z}$ and $PVT_3 \cong F_3$. While $\mathrm{Aut}(PVT_2) \cong \mathbb{Z}_2$, the structure of $\mathrm{Aut}(F_3)$ is well-known, see, for example, [1, Corollary 1]. The case $n = 4$ is exotic and will be dealt with separately in Subsection 7.3.
Recall that a group G is said to have R_∞-property if it has infinitely many ϕ-twisted conjugacy classes for each automorphism ϕ of G, where two elements $x, y \in G$ are said to lie in the same ϕ-twisted conjugacy class if there exists $g \in G$ such that $x = gy\phi(g)^{-1}$.

Theorem 7.2.8. *PVT_n has R_∞-property if and only if $n \geq 3$.*

Proof. Clearly, PVT_2 does not have R_∞-property. Since the graphs of PVT_3 and PVT_4 are non-complete graphs on at most 7 vertices, it follows from [22, Theorem 7.1.1] that these groups have R_∞-property. For $n \geq 5$, Theorem 7.2.6 gives the structure of $\mathrm{Aut}(PVT_n)$ as $\mathrm{Aut}(PVT_n) = \big\langle \mathrm{Aut}_{gr}(PVT_n), \mathrm{Aut}_{inv}(PVT_n), \mathrm{Inn}(PVT_n)\big\rangle$. Since the graph of PVT_n is not complete, the result now follows from [22, Theorem 3.3.3]. □

7.3 Automorphism group of PVT_4

Recall that $PVT_4 \cong (\mathbb{Z}\times\mathbb{Z}) * (\mathbb{Z}\times\mathbb{Z}) * (\mathbb{Z}\times\mathbb{Z})$. For $1 \leq i \leq 3$, let H_i denote the i-th free abelian factor in the free product decomposition of PVT_4. For simplicity of notation, we set

$$H_i = \langle x_i,\, y_i \mid [x_i,\, y_i] = 1\rangle.$$

Recall that the automorphism group of a right-angled Artin group is generated by graph automorphisms, inversions, transvections and partial conjugations. By looking at the graph of PVT_4 (see Figure 7.1) one can easily see that $\mid \mathrm{Aut}_{gr}(PVT_4) \mid = 48$. In fact, $\mathrm{Aut}_{gr}(PVT_4)$ is generated by the following five graph automorphisms:

$$\sigma_1 : \begin{cases} x_1 \longleftrightarrow y_1, & \\ x_i \longleftrightarrow x_i & \text{if } i = 2,3, \\ y_j \longleftrightarrow y_j & \text{if } j = 2,3, \end{cases} \quad \sigma_2 : \begin{cases} x_2 \longleftrightarrow y_2, & \\ x_i \longleftrightarrow x_i & \text{if } i = 1,3, \\ y_j \longleftrightarrow y_j & \text{if } j = 1,3, \end{cases}$$

$$\sigma_3 : \begin{cases} x_3 \longleftrightarrow y_3, & \\ x_i \longleftrightarrow x_i & \text{if } i = 1,2, \\ y_j \longleftrightarrow y_j & \text{if } j = 1,2, \end{cases}$$

$$\psi_1 : \begin{cases} x_1 \longleftrightarrow x_2, \\ y_1 \longleftrightarrow y_2, \\ x_3 \longleftrightarrow x_3, \\ y_3 \longleftrightarrow y_3, \end{cases} \quad \text{and} \quad \psi_2 : \begin{cases} x_1 \longleftrightarrow x_1, \\ y_1 \longleftrightarrow y_1, \\ x_2 \longleftrightarrow x_3, \\ y_2 \longleftrightarrow y_3. \end{cases}$$

Let ι_{x_i} and ι_{y_i} denote the inversion automorphisms that invert the generators x_i and y_i, respectively, and fix all other generators. Let $\tau_{x_1y_1}$, $\tau_{y_1x_1}$, $\tau_{x_2y_2}$, $\tau_{y_2x_2}$, $\tau_{x_3y_3}$ and $\tau_{y_3x_3}$ be the transvection automorphisms that generate $\mathrm{Aut}_{tr}(PVT_4)$.

Let C_i denote the connected component of the graph Γ of PVT_4 corresponding to the subgroup H_i or equivalently to the vertex set $\{x_i, y_i\}$. Then, for the generator x_1 of PVT_4, there are three choices for a union C of connected components of $\Gamma \setminus \Gamma(st(x_1))$. Thus, there are 18 partial conjugations that generate $\mathrm{Aut}_{pc}(PVT_4)$. The partial conjugations corresponding to the generator x_1 are as follows:

$$p_{x_1,C_2} : \begin{cases} x_2 \to x_1^{-1}x_2x_1, & \\ y_2 \to x_1^{-1}y_2x_1, & \\ x_j \to x_j & \text{if } j = 1,3, \\ y_k \to y_k & \text{if } k = 1,3, \end{cases} \quad p_{x_1,C_3} : \begin{cases} x_3 \to x_1^{-1}x_3x_1, & \\ y_3 \to x_1^{-1}y_3x_1, & \\ x_j \to x_j & \text{if } j = 1,2, \\ y_k \to y_k & \text{if } k = 1,2, \end{cases}$$

$$p_{x_1,C_2\cup C_3} : \begin{cases} x_i \to x_1^{-1}x_ix_1 & \text{if } i = 2,3, \\ y_i \to x_1^{-1}y_ix_1 & \text{if } i = 2,3, \\ x_1 \to x_1, & \\ y_1 \to y_1. & \end{cases}$$

Notice that $p_{x_1,C_2}\, p_{x_1,C_3} = p_{x_1,C_2\cup C_3} = p_{x_1,C_3}\, p_{x_1,C_2}$. Moreover, $p_{x_1,C_2\cup C_3}$ is the inner automorphism induced by x_1. By symmetry, the remaining 15 generating partial conjugations can be defined analogously, and we have

$$\mathrm{Aut}_{pc}(PVT_4) = \big\langle \mathrm{Inn}(PVT_4), \langle p_{x_1,C_2}, p_{y_1,C_2}, p_{x_2,C_3}, p_{y_2,C_3}, p_{x_3,C_1}, p_{y_3,C_1} \rangle \big\rangle.$$

Since $\mathrm{Aut}(H_i) \cong \mathrm{Aut}(\mathbb{Z} \times \mathbb{Z}) \cong \mathrm{GL}_2(\mathbb{Z})$, it follows that $\mathrm{Aut}(PVT_4)$ contains a subgroup isomorphic to $\mathrm{GL}_2(\mathbb{Z}) \times \mathrm{GL}_2(\mathbb{Z}) \times \mathrm{GL}_2(\mathbb{Z})$.

Lemma 7.3.1.

$$\langle \mathrm{Aut}_{inv}(PVT_4), \mathrm{Aut}_{tr}(PVT_4) \rangle \cong \mathrm{GL}_2(\mathbb{Z}) \times \mathrm{GL}_2(\mathbb{Z}) \times \mathrm{GL}_2(\mathbb{Z}).$$

Proof. Recall that

$$\langle \mathrm{Aut}_{inv}(PVT_4), \mathrm{Aut}_{tr}(PVT_4) \rangle = \langle \iota_{x_1}, \iota_{x_2}, \iota_{x_3}, \iota_{y_1}, \iota_{y_2}, \iota_{y_3}, \tau_{x_1y_1}, \tau_{y_1x_1}, \tau_{x_2y_2}, \tau_{y_2x_2}, \tau_{x_3y_3}, \tau_{y_3x_3} \rangle.$$

Let us set

$$\begin{aligned} K_1 &= \langle \tau_{x_1y_1}, \tau_{y_1x_1}, \iota_{x_1}, \iota_{y_1} \rangle, \\ K_2 &= \langle \tau_{x_2y_2}, \tau_{y_2x_2}, \iota_{x_2}, \iota_{y_2} \rangle, \\ K_3 &= \langle \tau_{x_3y_3}, \tau_{y_3x_3}, \iota_{x_3}, \iota_{y_3}, \rangle. \end{aligned}$$

Notice that K_1, K_2, K_3 act trivially on $H_2 * H_3$, $H_1 * H_3$ and $H_1 * H_2$, respectively. Further, $[K_1,\ K_2] = [K_2,\ K_3] = [K_3,\ K_1] = 1$ and $K_1 \cong K_2 \cong K_3$. Thus, it suffices to prove that $K_1 \cong \mathrm{GL}_2(\mathbb{Z})$. But, this follows by recalling that

$$\mathrm{GL}_2(\mathbb{Z}) = \left\langle \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \right\rangle$$

and identifying generators of K_1 with matrices in $\mathrm{GL}_2(\mathbb{Z})$ as

$$\tau_{x_1y_1} \to \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \tau_{y_1x_1} \to \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \iota_{x_1} \to \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \text{ and } \iota_{y_1} \to \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

□

Lemma 7.3.2.

$$\mathrm{Aut}_{gr}(PVT_4) \cap \langle \mathrm{Aut}_{inv}(PVT_4), \mathrm{Aut}_{tr}(PVT_4) \rangle = \langle \sigma_1, \sigma_2, \sigma_3 \rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$$

and

$$\begin{aligned}\langle \mathrm{Aut}_{gr}(PVT_4), \mathrm{Aut}_{inv}(PVT_4), \mathrm{Aut}_{tr}(PVT_4)\rangle &= \langle \mathrm{Aut}_{inv}(PVT_4), \mathrm{Aut}_{tr}(PVT_4)\rangle \rtimes \langle \psi_1, \psi_2\rangle \\ &\cong \big(\mathrm{GL}_2(\mathbb{Z}) \times \mathrm{GL}_2(\mathbb{Z}) \times \mathrm{GL}_2(\mathbb{Z})\big) \rtimes S_3.\end{aligned}$$

Proof. Notice that ψ_1, ψ_2 permutes H_i's non-trivially, and hence $\langle \mathrm{Aut}_{inv}(PVT_4), \mathrm{Aut}_{tr}(PVT_4)\rangle \cap \langle \psi_1, \psi_2\rangle = 1$. On the other hand $\sigma_1, \sigma_2, \sigma_3 \in \langle \mathrm{Aut}_{inv}(PVT_4), \mathrm{Aut}_{tr}(PVT_4)\rangle$, and hence the first assertion follows.
For the second assertion notice that $\langle \psi_1, \psi_2\rangle \cong S_3$ and $\psi_1^2 = 1 = \psi_2^2$. It suffices to show that $\langle \psi_1, \psi_2\rangle$ normalises $\langle \mathrm{Aut}_{inv}(PVT_4), \mathrm{Aut}_{tr}(PVT_4)\rangle$. A direct check shows that

$$\begin{aligned}&\psi_1 \tau_{x_1y_1} \psi_1 = \tau_{x_2y_2}, \quad \psi_1 \tau_{y_1x_1} \psi_1 = \tau_{y_2x_2}, \quad \psi_1 \tau_{x_3y_3} \psi_1 = \tau_{x_3y_3}, \quad \psi_1 \tau_{y_3x_3} \psi_1 = \tau_{y_3x_3},\\ &\psi_1 \iota_{x_1} \psi_1 = \iota_{x_2}, \quad \psi_1 \iota_{y_1} \psi_1 = \iota_{y_2}, \quad \psi_1 \iota_{x_3} \psi_1 = \iota_{x_3}, \quad \psi_1 \iota_{y_3} \psi_1 = \iota_{y_3},\end{aligned}$$

and hence ψ_1 normalises $\langle \mathrm{Aut}_{inv}(PVT_4), \mathrm{Aut}_{tr}(PVT_4)\rangle$. By symmetry, the same assertion holds for ψ_2, and we get the desired result. $\square$

Lemma 7.3.3. $\mathrm{Aut}_{pc}(PVT_4)$ *is normal in* $\mathrm{Aut}(PVT_4)$.

Proof. Note that $\mathrm{Inn}(PVT_4) \leq \mathrm{Aut}_{pc}(PVT_4)$. Set $M = \langle p_{x_1,C_2}, p_{y_1,C_2}, p_{x_2,C_3}, p_{y_2,C_3}, p_{x_3,C_1}, p_{y_3,C_1}\rangle$.
It suffices to show that $\phi^{-1}M\phi \leq \mathrm{Aut}_{pc}(PVT_4)$ for all $\phi \in \langle \mathrm{Aut}_{gr}(PVT_4), \mathrm{Aut}_{inv}(PVT_4), \mathrm{Aut}_{tr}(PVT_4)\rangle$.
If $\phi = \psi_1$, then

$$\begin{aligned}&\psi_1 p_{x_1,C_2} \psi_1 = p_{x_2,C_1}, \quad \psi_1 p_{y_1,C_2} \psi_1 = p_{y_2,C_1}, \quad \psi_1 p_{x_2,C_3} \psi_1 = p_{x_1,C_3},\\ &\psi_1 p_{y_2,C_3} \psi_1 = p_{y_1,C_3}, \quad \psi_1 p_{x_3,C_1} \psi_1 = p_{x_3,C_2}, \quad \psi_1 p_{y_3,C_1} \psi_1 = p_{y_3,C_2}.\end{aligned}$$

Thus, ψ_1 normalises $\mathrm{Aut}_{pc}(PVT_4)$. By symmetry, ψ_2 and hence $\langle \psi_1, \psi_2\rangle$ normalises $\mathrm{Aut}_{pc}(PVT_4)$.
By Lemma 7.3.2, it remains to show that $\langle \mathrm{Aut}_{inv}(PVT_4), \mathrm{Aut}_{tr}(PVT_4)\rangle$ normalises $\mathrm{Aut}_{pc}(PVT_4)$.
If $\phi = \iota_{x_1}$, then

$$\begin{aligned}&\iota_{x_1} p_{x_1,C_2} \iota_{x_1} = p_{x_1,C_2}^{-1}, \quad \iota_{x_1} p_{y_1,C_2} \iota_{x_1} = p_{y_1,C_2}, \quad \iota_{x_1} p_{x_2,C_3} \iota_{x_1} = p_{x_2,C_3},\\ &\iota_{x_1} p_{y_2,C_3} \iota_{x_1} = p_{y_2,C_3}, \quad \iota_{x_1} p_{x_3,C_1} \iota_{x_1} = p_{x_3,C_1}, \quad \iota_{x_1} p_{y_3,C_1} \iota_{x_1} = p_{y_3,C_1}.\end{aligned}$$

Thus, ι_{x_1} normalises $\mathrm{Aut}_{pc}(PVT_4)$. By symmetry, all the other inversions also normalise $\mathrm{Aut}_{pc}(PVT_4)$, and consequently $\mathrm{Aut}_{inv}(PVT_4)$ normalises $\mathrm{Aut}_{pc}(PVT_4)$.

If $\phi = \tau_{x_1y_1}$, then

$$\begin{aligned}&\tau_{x_1y_1}^{-1} p_{x_1,C_2} \tau_{x_1y_1} = p_{y_1,C_2}^{-1} p_{x_1,C_2}, \quad \tau_{x_1y_1}^{-1} p_{y_1,C_2} \tau_{x_1y_1} = p_{y_1,C_2}, \quad \tau_{x_1y_1}^{-1} p_{x_2,C_3} \tau_{x_1y_1} = p_{x_2,C_3},\\ &\tau_{x_1y_1}^{-1} p_{y_2,C_3} \tau_{x_1y_1} = p_{y_2,C_3}, \quad \tau_{x_1y_1}^{-1} p_{x_3,C_1} \tau_{x_1y_1} = p_{x_3,C_1}, \quad \tau_{x_1y_1}^{-1} p_{y_3,C_1} \tau_{x_1y_1} = p_{y_3,C_1}.\end{aligned}$$

Thus, $\tau_{x_1y_1}$ normalises $\mathrm{Aut}_{pc}(PVT_4)$. Similarly, one can show that all other transvections also normalise $\mathrm{Aut}_{pc}(PVT_4)$, which completes the proof of the lemma. □

Finally, we determine the structure of $\mathrm{Aut}_{pc}(PVT_4)$. A presentation of the group of partial conjugations of a right-angled Artin group has been constructed in [83, Theorem 3.1].

Lemma 7.3.4. *The group* $\mathrm{Aut}_{pc}(PVT_4)$ *has a presentation with generating set*

$$\{p_{x_i,C_j}, p_{y_i,C_j} \mid i \neq j \text{ and } i,j = 1,2,3\}$$

and following defining relations:

1. $[p_{x_i,C_j},\, p_{x_i,C_k}] = [p_{y_i,C_j},\, p_{y_i,C_k}] = [p_{x_i,C_j},\, p_{y_i,C_k}] = [p_{x_i,C_j},\, p_{y_i,C_j}] = 1$ *for* $i = 1,2,3$ *with* $i \neq j \neq k \neq i$.
2. $[p_{x_i,C_j}p_{x_i,C_k},\, p_{x_j,C_k}] = [p_{y_i,C_j}p_{y_i,C_k},\, p_{y_j,C_k}] = [p_{x_i,C_j}p_{x_i,C_k},\, p_{y_j,C_k}] = [p_{y_i,C_j}p_{y_i,C_k},\, p_{x_j,C_k}] = 1$ *for* $i,j,k = 1,2,3$ *with* $i \neq j \neq k \neq i$.

In particular,

$$\mathrm{Aut}_{pc}(PVT_4) \cong (\mathbb{Z}^2 * \mathbb{Z}^2 * \mathbb{Z}^2) \rtimes (\mathbb{Z}^2 * \mathbb{Z}^2 * \mathbb{Z}^2).$$

Proof. Relations in (1) and (2) follow by direct computations together with [83, Theorem 3.1]. Note that

$$\mathrm{Inn}(PVT_4) = \langle p_{x_i,C_j}p_{x_i,C_k}, p_{y_i,C_j}p_{y_i,C_k} \mid i \neq j \neq k \neq i,\ j < k \text{ and } i,j,k = 1,2,3 \rangle.$$

Setting

$$\mathrm{Aut}_{pc\setminus inn}(PVT_4) = \langle p_{x_1,C_2}, p_{y_1,C_2}, p_{x_2,C_3}, p_{y_2,C_3}, p_{x_3,C_1}, p_{y_3,C_1} \rangle,$$

we see that

- $\mathrm{Aut}_{pc\setminus inn}(PVT_4) \cong \mathbb{Z}^2 * \mathbb{Z}^2 * \mathbb{Z}^2$.
- $\mathrm{Aut}_{pc}(PVT_4) = \mathrm{Inn}(PVT_4)\,\mathrm{Aut}_{pc\setminus inn}(PVT_4)$.

Consider the surjective homomorphism $g : \mathrm{Aut}_{pc}(PVT_4) \to \mathrm{Aut}_{pc\setminus inn}(PVT_4)$ defined on generators as

$$g : \begin{cases} p_{x_1,C_2} \mapsto p_{x_1,C_2}, \\ p_{x_1,C_3} \mapsto p_{x_1,C_2}^{-1}, \\ p_{y_1,C_2} \mapsto p_{y_1,C_2}, \\ p_{y_1,C_3} \mapsto p_{y_1,C_2}^{-1}, \end{cases} \quad g : \begin{cases} p_{x_2,C_1} \mapsto p_{x_2,C_3}^{-1}, \\ p_{x_2,C_3} \mapsto p_{x_2,C_3}, \\ p_{y_2,C_1} \mapsto p_{y_2,C_3}^{-1}, \\ p_{y_2,C_3} \mapsto p_{y_2,C_3}, \end{cases} \quad g : \begin{cases} p_{x_3,C_1} \mapsto p_{x_3,C_1}, \\ p_{x_3,C_2} \mapsto p_{x_3,C_1}^{-1}, \\ p_{y_3,C_1} \mapsto p_{y_3,C_1}, \\ p_{y_3,C_2} \mapsto p_{y_3,C_1}^{-1}. \end{cases}$$

Note that $\mathrm{Inn}(PVT_4) \subseteq \mathrm{Ker}(g)$. Let $w \in \mathrm{Ker}(g)$ and write $w = xy$ for some $x \in \mathrm{Inn}(PVT_4)$ and $y \in \mathrm{Aut}_{pc\setminus inn}(PVT_4)$. Then we have

$$1 = g(w) = g(x)g(y) = y,$$

and hence $\mathrm{Ker}(g) = \mathrm{Inn}(PVT_4)$. This implies that

$$\mathrm{Aut}_{pc}(PVT_4) = \mathrm{Inn}(PVT_4) \rtimes \mathrm{Aut}_{pc\setminus inn}(PVT_4), \tag{7.6}$$

and hence

$$\mathrm{Aut}_{pc}(PVT_4) \cong (\mathbb{Z}^2 * \mathbb{Z}^2 * \mathbb{Z}^2) \rtimes (\mathbb{Z}^2 * \mathbb{Z}^2 * \mathbb{Z}^2).$$

This completes the proof. □

Combining the preceding lemmas yield the following theorem.

Theorem 7.3.5. *There exists a split exact sequence*

$$1 \to \mathrm{Aut}_{pc}(PVT_4) \to \mathrm{Aut}(PVT_4) \to \left\langle \mathrm{Aut}_{gr}(PVT_4), \mathrm{Aut}_{inv}(PVT_4), \mathrm{Aut}_{tr}(PVT_4) \right\rangle \to 1.$$

In particular,

$$\begin{aligned}\mathrm{Aut}(PVT_4) &= \mathrm{Aut}_{pc}(PVT_4) \rtimes \left\langle \mathrm{Aut}_{gr}(PVT_4), \mathrm{Aut}_{inv}(PVT_4), \mathrm{Aut}_{tr}(PVT_4) \right\rangle \\ &\cong ((\mathbb{Z}^2 * \mathbb{Z}^2 * \mathbb{Z}^2) \rtimes (\mathbb{Z}^2 * \mathbb{Z}^2 * \mathbb{Z}^2)) \rtimes \big((\mathrm{GL}_2(\mathbb{Z}) \times \mathrm{GL}_2(\mathbb{Z}) \times \mathrm{GL}_2(\mathbb{Z})) \rtimes S_3\big).\end{aligned}$$

Proof. Note that each automorphism in $\mathrm{Aut}_{pc}(PVT_4)$ preserves conjugacy classes of generators. But, the only automorphism in $\langle \mathrm{Aut}_{gr}, \mathrm{Aut}_{inv}, \mathrm{Aut}_{tr} \rangle$ which preserves conjugacy classes of generators is the identity automorphism. Hence

$$\mathrm{Aut}_{pc}(PVT_4) \cap \left\langle \mathrm{Aut}_{gr}(PVT_4), \mathrm{Aut}_{inv}(PVT_4), \mathrm{Aut}_{tr}(PVT_4) \right\rangle = 1,$$

and the assertion follows. □

Recall that an automorphism of a group is called an IA *automorphism* if it acts as identity on the abelianisation of the group. Note that inner automorphisms are IA automorphisms.

Corollary 7.3.6. *Each* IA *automorphism of PVT_n is inner if and only if $n = 2$ or $n \geq 5$.*

Proof. Note that the IA automorphism group of PVT_2 is obviously trivial. Magnus [59] gave generators of the group of IA automorphisms of $F_3 \cong PVT_3$ and showed that it contains non-inner automorphisms. Clearly, $\mathrm{Aut}_{pc}(PVT_4)$ is a subgroup of the group of IA automorphisms of PVT_4. It follows from (7.6) that $\mathrm{Aut}_{pc}(PVT_4)$ contains non-inner automorphisms as well. For $n \geq 5$, a direct check using the description of $\mathrm{Aut}(PVT_n)$ in Theorem 7.2.7 shows that the only IA automorphisms of PVT_n are the inner automorphisms. $\square$

Chapter 8

Commutator subgroups of virtual twin groups

This final chapter deals with commutator subgroups of VT_n and PVT_n [73]. We begin by giving a reduced presentation of VT_n for $n \geq 3$, which we will use in finding a presentation of the commutator subgroup $\gamma_2(VT_n)$ of VT_n. We will also show that the lower central series of VT_n stabilises at the second term.

8.1 A reduced presentation of virtual twin groups

For the sake of convenience, we recall the defining relations in the standard presentation of VT_n.

$$
\begin{aligned}
s_i^2 &= 1 \quad \text{for } i = 1, 2, \ldots, n-1, && (8.1)\\
s_i s_j &= s_j s_i \quad \text{for } |i-j| \geq 2, && (8.2)\\
\rho_i^2 &= 1 \quad \text{for } i = 1, 2, \ldots, n-1, && (8.3)\\
\rho_i \rho_j &= \rho_j \rho_i \quad \text{for } |i-j| \geq 2, && (8.4)\\
\rho_i \rho_{i+1} \rho_i &= \rho_{i+1} \rho_i \rho_{i+1} \quad \text{for } i = 1, 2, \ldots, n-2, && (8.5)\\
\rho_i s_j &= s_j \rho_i \quad \text{for } |i-j| \geq 2, && (8.6)\\
\rho_i \rho_{i+1} s_i &= s_{i+1} \rho_i \rho_{i+1} \quad \text{for } i = 1, 2, \ldots, n-2. && (8.7)
\end{aligned}
$$

Theorem 8.1.1. *The virtual twin group has the following reduced presentation:*

1. $VT_3 = \langle s_1, \rho_1, \rho_2 \mid s_1^2 = \rho_1^2 = \rho_2^2 = (\rho_1 \rho_2)^3 = 1 \rangle$.

2. *For $n \geq 4$, VT_n is generated by $\{s_1, \rho_1, \rho_2, \ldots, \rho_{n-1}\}$ with following defining relations*

$$s_1^2 = 1, \tag{8.8}$$
$$\rho_i^2 = 1 \quad \text{for } i = 1, 2, \ldots, n-1, \tag{8.9}$$
$$\rho_i \rho_j = \rho_j \rho_i \quad \text{for } |i-j| \geq 2, \tag{8.10}$$
$$\rho_i \rho_{i+1} \rho_i = \rho_{i+1} \rho_i \rho_{i+1} \quad \text{for } i = 1, 2, \ldots, n-2, \tag{8.11}$$
$$\rho_i s_1 = s_1 \rho_i \quad \text{for } i \geq 3, \tag{8.12}$$
$$(s_1 \rho_2 \rho_1 \rho_3 \rho_2)^4 = 1. \tag{8.13}$$

Proof. The case $n = 3$ is immediate. We first use the relation (8.7) to eliminate a few generators and then we show that the rest of the relations in first presentation can be derived from relations (8.8)–(8.13). We prove the desired result in the following three steps.

Claim 1. We claim that

$$s_{i+1} = (\rho_i \rho_{i-1} \ldots \rho_2 \rho_1)(\rho_{i+1} \rho_i \ldots \rho_3 \rho_2) s_1 (\rho_2 \rho_3 \ldots \rho_i \rho_{i+1})(\rho_1 \rho_2 \ldots \rho_{i-1} \rho_i)$$

for $i \geq 2$.

We note that the case $i = 2$ follows from relation in (8.7). Let us suppose that the claim holds for i and we prove it for $i+1$. For $i+1$, we have

$$s_{i+1} = \rho_i \rho_{i+1} s_i \rho_{i+1} \rho_i.$$

Substituting the value of s_i from our assumption gives

$$s_{i+1} = \rho_i \rho_{i+1} (\rho_{i-1} \rho_{i-2} \ldots \rho_2 \rho_1)(\rho_i \rho_{i-1} \ldots \rho_3 \rho_2) s_1 (\rho_2 \rho_3 \ldots \rho_{i-1} \rho_i)(\rho_1 \rho_2 \ldots \rho_{i-2} \rho_{i-1}) \rho_{i+1} \rho_i.$$

By using relation in (8.10), we get

$$s_{i+1} = (\rho_i \rho_{i-1} \ldots \rho_2 \rho_1)(\rho_{i+1} \rho_i \ldots \rho_3 \rho_2) s_1 (\rho_2 \rho_3 \ldots \rho_i \rho_{i+1})(\rho_1 \rho_2 \ldots \rho_{i-1} \rho_i).$$

Using Claim 1, we can express s_i, $i \geq 2$, in terms of s_1 and ρ_j's. This means we can eliminate the generators s_i, $i \geq 2$.

Claim 2. The relation

$$s_i \rho_j = \rho_j s_i, \ |i-j| \geq 2$$

is a consequence of Claim 1 and relations in equations 8.10, 8.11 and 8.12.

Claim 1 gives

$$s_i\rho_j = (\rho_{i-1}\rho_{i-2}\dots\rho_2\rho_1)(\rho_i\rho_{i-1}\dots\rho_3\rho_2)s_1(\rho_2\rho_3\dots\rho_{i-1}\rho_i)(\rho_1\rho_2\dots\rho_{i-2}\rho_{i-1})\rho_j.$$

If $j \geq i+2$, then we are done by relations 8.10 and 8.12. Next, we consider the case $j \leq i-2$.

$$\begin{aligned}
s_i\rho_j &= (\rho_{i-1}\rho_{i-2}\dots\rho_2\rho_1)(\rho_i\rho_{i-1}\dots\rho_3\rho_2)s_1(\rho_2\rho_3\dots\rho_{i-1}\rho_i)(\rho_1\rho_2\dots\rho_{i-2}\rho_{i-1})\rho_j\\
&\overset{(8.10)}{=} (\rho_{i-1}\rho_{i-2}\dots\rho_1)(\rho_i\rho_{i-1}\dots\rho_3\rho_2)s_1(\rho_2\dots\rho_{i-1}\rho_i)(\rho_1\rho_2\dots\rho_j\rho_{j+1}\rho_j\dots\rho_{i-1})\\
&\overset{(8.11)}{=} (\rho_{i-1}\rho_{i-2}\dots\rho_1)(\rho_i\rho_{i-1}\dots\rho_3\rho_2)s_1(\rho_2\dots\rho_{i-1}\rho_i)(\rho_1\dots\rho_{j+1}\rho_j\rho_{j+1}\dots\rho_{i-1})\\
&\overset{(8.10)}{=} (\rho_{i-1}\rho_{i-2}\dots\rho_1)(\rho_i\rho_{i-1}\dots\rho_3\rho_2)s_1(\rho_2\dots\rho_{j+1}\rho_{j+2}\rho_{j+1}\dots\rho_i)(\rho_1\dots\rho_{i-1})\\
&\overset{(8.11)}{=} (\rho_{i-1}\rho_{i-2}\dots\rho_1)(\rho_i\rho_{i-1}\dots\rho_3\rho_2)s_1(\rho_2\dots\rho_{j+2}\rho_{j+1}\rho_{j+2}\dots\rho_i)(\rho_1\dots\rho_{i-1})\\
&\overset{(8.12)}{=} (\rho_{i-1}\rho_{i-2}\dots\rho_1)(\rho_i\dots\rho_{j+2}\rho_{j+1}\rho_{j+2}\dots\rho_2)s_1(\rho_2\dots\rho_i)(\rho_1\dots\rho_{i-1})\\
&\overset{(8.11)}{=} (\rho_{i-1}\rho_{i-2}\dots\rho_1)(\rho_i\dots\rho_{j+1}\rho_{j+2}\rho_{j+1}\dots\rho_2)s_1(\rho_2\dots\rho_i)(\rho_1\dots\rho_{i-1})\\
&\overset{(8.10)}{=} (\rho_{i-1}\dots\rho_{j+1}\rho_j\rho_{j+1}\dots\rho_1)(\rho_i\dots\rho_2)s_1(\rho_2\dots\rho_i)(\rho_1\dots\rho_{i-1})\\
&\overset{(8.11)}{=} (\rho_{i-1}\dots\rho_j\rho_{j+1}\rho_j\dots\rho_1)(\rho_i\dots\rho_2)s_1(\rho_2\dots\rho_i)(\rho_1\dots\rho_{i-1})\\
&\overset{(8.10)}{=} \rho_j(\rho_{i-1}\dots\rho_1)(\rho_i\dots\rho_2)s_1(\rho_2\dots\rho_i)(\rho_1\dots\rho_{i-1})\\
&= \rho_j s_i.
\end{aligned}$$

Claim 3. The relation

$$s_is_j = s_js_i,\ |i-j| \geq 2$$

is a consequence of Claim 1 and relations in equations (8.8) – (8.13). The proof of this claim is along the similar lines as [49, Lemma 3]. □

8.2 Presentation of commutator subgroups of virtual twin groups

The lower central series of a group G is defined as

$$G = \gamma_1(G) \geq \gamma_2(G) \geq \dots \geq \gamma_i(G) \geq \gamma_{i+1}(G) \geq \dots,$$

where

$$\gamma_{\alpha+1}(G) = \langle [g_\alpha, g] \mid g_\alpha \in \gamma_\alpha(G), g \in G\rangle.$$

In particular, $\gamma_2(G)$ is the commutator subgroup of G. The group G is said to be *residually nilpotent* if

$$\gamma_\omega(G) = \bigcap_{i=1}^{\infty} \gamma_i(G) = 1.$$

We now give a presentation of $\gamma_2(VT_n)$.

Theorem 8.2.1. *The commutator subgroup of the virtual twin group has the following presentation:*

1. $\gamma_2(VT_2) = \langle (\rho_1 s_1)^2 \rangle \cong \mathbb{Z}$.
2. $\gamma_2(VT_3) = \langle \rho_2\rho_1, s_1\rho_2\rho_1 s_1, (\rho_1 s_1)^2 \mid (\rho_2\rho_1)^3 = 1 = (s_1\rho_2\rho_1 s_1)^3 \rangle \cong \mathbb{Z}_3 * \mathbb{Z}_3 * \mathbb{Z}$.
3. *For $n \geq 4$, $\gamma_2(VT_n)$ is generated by*

$$\{x_i, y, z \mid i = 2, 3, \dots, n-1\}.$$

The set $\{x_2, \dots, x_{n-1}\}$ generate a subgroup isomorphic to the alternating group A_n and has relations

$$\begin{array}{rcll} x_2^3 & = & 1, & \\ x_j^2 & = & 1 & \textit{for } 3 \leq j \leq n-1, \\ (x_i x_{i+1}^{-1})^3 & = & 1 & \textit{for } 2 \leq i \leq n-2, \\ (x_i x_j^{-1})^2 & = & 1 & \textit{for } 2 \leq i \leq n-2 \textit{ and } j \geq i+2. \end{array}$$

The other defining relations in $\gamma_2(VT_n)$ are the following mixed relations:

$$\begin{array}{rcll} y^3 & = & 1, & \\ (x_j z)^2 & = & 1 & \textit{for } 3 \leq j \leq n-1, \\ (yz^{-1}x_3^{-1})^3 & = & 1, & \\ (yz^{-1}x_j^{-1})^2 & = & 1 & \textit{for } 4 \leq j \leq n-1, \\ (yz^{-1}x_3^{-1}y^{-1}x_2x_3x_2^{-1})^2 & = & 1, & \\ (zy^{-1}x_3zyz^{-1}x_2^{-1}x_3^{-1}x_2)^2 & = & 1. & \end{array}$$

Proof. We use Theorem 8.1.1 and Reidemeister-Schreier method to give a presentation of $\gamma_2(VT_n)$. Since the abelianisation of VT_n is isomorphic to the elementary abelian 2-group of order 4, we can take a Schreier system to be

$$\mathrm{M} = \{1, s_1, \rho_1, s_1\rho_1\}.$$

In view of Theorem 8.1.1, we take the generating set for VT_n to be $S = \{s_1, \rho_1, \rho_2, \ldots, \rho_{n-1}\}$.

Generators of $\gamma_2(VT_n)$

We compute the generators $\gamma(\mu, a)$, $\mu \in \mathrm{M}$, $a \in S$ explicitly.

$$\begin{aligned}
\gamma(1, s_1) &= s_1(\overline{s_1})^{-1} = s_1 s_1 = 1,\\
\gamma(1, \rho_1) &= \rho_1(\overline{\rho_1})^{-1} = \rho_1\rho_1 = 1,\\
\gamma(1, \rho_i) &= \rho_i(\overline{\rho_i})^{-1} = \rho_i\rho_1,\ i \geq 2,
\end{aligned}$$

$$\begin{aligned}
\gamma(s_1, s_1) &= s_1 s_1(\overline{s_1 s_1})^{-1} = 1,\\
\gamma(s_1, \rho_1) &= s_1\rho_1(\overline{s_1\rho_1})^{-1} = s_1\rho_1\rho_1 s_1 = 1,\\
\gamma(s_1, \rho_i) &= s_1\rho_i(\overline{s_1\rho_i})^{-1} = s_1\rho_i\rho_1 s_1,\ i \geq 2,
\end{aligned}$$

$$\begin{aligned}
\gamma(\rho_1, s_1) &= \rho_1 s_1(\overline{\rho_1 s_1})^{-1} = (\rho_1 s_1)^2,\\
\gamma(\rho_1, \rho_1) &= \rho_1\rho_1(\overline{\rho_1\rho_1})^{-1} = 1,\\
\gamma(\rho_1, \rho_i) &= \rho_1\rho_i(\overline{\rho_1\rho_i})^{-1} = \rho_1\rho_i,\ i \geq 2,
\end{aligned}$$

$$\begin{aligned}
\gamma(s_1\rho_1, s_1) &= s_1\rho_1 s_1(\overline{s_1\rho_1 s_1})^{-1} = (s_1\rho_1)^2,\\
\gamma(s_1\rho_1, \rho_1) &= s_1\rho_1\rho_1(\overline{s_1\rho_1\rho_1})^{-1} = 1,\\
\gamma(s_1\rho_1, \rho_i) &= s_1\rho_1\rho_i(\overline{s_1\rho_1\rho_i})^{-1} = s_1\rho_1\rho_i s_1 = (s_1\rho_i\rho_1 s_1)^{-1},\ i \geq 2.
\end{aligned}$$

For $i = 2, 3, \ldots, n-1$, define

$$\begin{aligned}
x_i &:= \rho_i\rho_1,\\
y_i &:= s_1\rho_i\rho_1 s_1,\\
z &:= (\rho_1 s_1)^2.
\end{aligned}$$

Then the preceding computations show that $\gamma_2(VT_n)$ is generated by the set

$$\{x_i, y_i, z \mid i = 2, 3, \ldots, n-1\}.$$

Relations in $\gamma_2(VT_n)$

We begin by finding relations in $\gamma_2(VT_n)$ corresponding to the relation $s_1^2 = 1$ in VT_n.

$$\begin{array}{rcl}
\tau(1 s_1 s_1 1) & = & \gamma(1, s_1)\gamma(\overline{s_1}, s_1) = \gamma(1, s_1)\gamma(s_1, s_1) = 1,\\
\tau(s_1 s_1 s_1 s_1) & = & (\gamma(1, s_1)\gamma(s_1, s_1))^2 = 1,\\
\tau(\rho_1 s_1 s_1 \rho_1) & = & \gamma(1, \rho_1)\gamma(\rho_1, s_1)\gamma(s_1\rho_1, s_1)\gamma(\rho_1, \rho_1) = 1,\\
\tau(s_1\rho_1(s_1 s_1)\rho_1 s_1) & = & \gamma(1, s_1)\gamma(s_1, \rho_1)\gamma(s_1\rho_1, s_1)\gamma(\rho_1, s_1)\gamma(s_1\rho_1, \rho_1)\gamma(s_1, s_1) = 1.
\end{array}$$

Next, we find relations in $\gamma_2(VT_n)$ corresponding to relations $\rho_i^2 = 1$, $i = 1, 2, \ldots, n-1$.

$$\begin{array}{rcl}
\tau(1\rho_i\rho_i 1) & = & \gamma(1, \rho_i)\gamma(\overline{\rho_i}, \rho_i) = \gamma(1, \rho_i)\gamma(\rho_1, \rho_i) = 1,\\
\tau(s_1\rho_i\rho_i s_1) & = & \gamma(1, s_1)\gamma(s_1, \rho_i)\gamma(s_1\rho_1, \rho_i)\gamma(s_1, s_1) = 1,\\
\tau(\rho_1\rho_i\rho_i\rho_1) & = & \gamma(1, \rho_1)\gamma(\rho_1, \rho_i)\gamma(1, \rho_i)\gamma(\rho_1, \rho_1) = 1,\\
\tau(s_1\rho_1(\rho_1\rho_1)\rho_1 s_1) & = & \gamma(1, s_1)\gamma(s_1, \rho_1)\gamma(s_1\rho_1, \rho_1)\gamma(s_1, \rho_1)\gamma(s_1\rho_1, \rho_1)\gamma(s_1, s_1) = 1,\\
\tau(s_1\rho_1(\rho_i\rho_i)\rho_1 s_1) & = & \gamma(1, s_1)\gamma(s_1, \rho_1)\gamma(s_1\rho_1, \rho_i)\gamma(s_1, \rho_i)\gamma(s_1\rho_1, \rho_1)\gamma(s_1, s_1) = y_i^{-1}y_i = 1,\ i \geq 2.
\end{array}$$

Relations in $\gamma_2(VT_n)$ corresponding to relations $(\rho_i\rho_{i+1})^3 = 1$, $i = 1, 2, \ldots, n-2$, are given by

$$\begin{array}{rcl}
\tau(1(\rho_i\rho_{i+1})^3 1) & = & \big(\gamma(1, \rho_i)\gamma(\rho_1, \rho_{i+1})\big)^3\\
& = & \begin{cases} x_2^{-3} & \text{for } i = 1,\\ (x_i x_{i+1}^{-1})^3 & \text{for } 2 \leq i \leq n-2, \end{cases}
\end{array}$$

$$\begin{array}{rcl}
\tau(s_1(\rho_i\rho_{i+1})^3 s_1) & = & \gamma(1, s_1)\big(\gamma(s_1, \rho_i)\gamma(s_1\rho_1, \rho_{i+1})\big)^3\gamma(s_1, s_1)\\
& = & \big(\gamma(s_1, \rho_i)\gamma(\rho_1 s_1, \rho_{i+1})\big)^3\\
& = & \begin{cases} y_2^{-3} & \text{for } i = 1,\\ (y_i y_{i+1}^{-1})^3 & \text{for } 2 \leq i \leq n-2, \end{cases}
\end{array}$$

$$\begin{aligned}
\tau(\rho_1(\rho_i\rho_{i+1})^3\rho_1) &= \big(\gamma(\rho_1,\rho_i)\gamma(1,\rho_{i+1})\big)^3\\
&= \begin{cases} x_2^3 & \text{for } i=1,\\ (x_i^{-1}x_{i+1})^3 & \text{for } 2\le i\le n-2.\end{cases}
\end{aligned}$$

$$\begin{aligned}
\tau(s_1\rho_1(\rho_i\rho_{i+1})^3\rho_1 s_1) &= \gamma(1,s_1)\gamma(s_1,\rho_1)\big(\gamma(s_1\rho_1,\rho_i)\gamma(s_1,\rho_{i+1})\big)^3\gamma(s_1\rho_1,\rho_1)\gamma(s_1,s_1)\\
&= \big(\gamma(s_1\rho_1,\rho_i)\gamma(s_1,\rho_{i+1})\big)^3\\
&= \begin{cases} y_2^3 & \text{for } i=1,\\ (y_i^{-1}y_{i+1})^3 & \text{for } 2\le i\le n-2.\end{cases}
\end{aligned}$$

The preceding computations give the following non-trivial relations

$$\begin{aligned}
x_2^3 &= 1,\\
y_2^3 &= 1,\\
(x_i x_{i+1}^{-1})^3 &= 1 \quad \text{for } 2\le i\le n-2,\\
(y_i y_{i+1}^{-1})^3 &= 1 \quad \text{for } 2\le i\le n-2.
\end{aligned}$$

Next, we find relations in $\gamma_2(VT_n)$ corresponding to relations $(\rho_i\rho_j)^2=1$, $|i-j|\ge 2$.

$$\begin{aligned}
\tau(1(\rho_i\rho_j)^2 1) &= (\gamma(1,\rho_i)\gamma(\rho_1,\rho_j))^2\\
&= \begin{cases} x_j^{-2} & \text{for } j\ge 3,\\ (x_i x_j^{-1})^2 & \text{for } i\ge 2 \text{ and } i+2\le j\le n-1,\end{cases}
\end{aligned}$$

$$\begin{aligned}
\tau(s_1(\rho_i\rho_j)^2 s_1) &= \gamma(1,s_1)(\gamma(s_1,\rho_i)\gamma(s_1\rho_1,\rho_j))^2\gamma(s_1,s_1)\\
&= \begin{cases} y_j^{-2} & \text{for } 3\le j\le n-1,\\ (y_i y_j^{-1})^2 & \text{for } 2\le i\le n-2 \text{ and } j\ge i+2,\end{cases}
\end{aligned}$$

$$\begin{aligned}
\tau(\rho_1(\rho_i\rho_j)^2\rho_1) &= \gamma(1,\rho_1)(\gamma(\rho_1,\rho_i)\gamma(1,\rho_j))^2\gamma(\rho_1,\rho_1)\\
&= \begin{cases} x_j^{2} & \text{for } 3\le j\le n-1,\\ (x_i^{-1}x_j)^2 & \text{for } i\ge 2 \text{ and } i+2\le j\le n-1,\end{cases}
\end{aligned}$$

$$\begin{aligned}
\tau(s_1\rho_1(\rho_i\rho_j)^2\rho_1 s_1) &= \gamma(1,s_1)\gamma(s_1,\rho_1)(\gamma(s_1\rho_1,\rho_i)\gamma(s_1,\rho_j))^2\gamma(s_1\rho_1,\rho_1)\gamma(s_1,s_1)\\
&= (\gamma(s_1\rho_1,\rho_i)\gamma(s_1,\rho_j))^2\\
&= \begin{cases} y_j^2 & \text{for } j\geq 3,\\ (y_i^{-1}y_j)^2 & \text{for } i\geq 2 \text{ and } i+2\leq j\leq n-1.\end{cases}
\end{aligned}$$

Thus, the non-trivial relations are

$$\begin{aligned}
x_j^2 &= 1 \quad \text{for } 3\leq j\leq n-1,\\
y_j^2 &= 1 \quad \text{for } 3\leq j\leq n-1,\\
(x_i x_j^{-1})^2 &= 1 \quad \text{for } 2\leq i\leq n-2 \text{ and } j\geq i+2,\\
(y_i y_j^{-1})^2 &= 1 \quad \text{for } 2\leq i\leq n-2 \text{ and } j\geq i+2.
\end{aligned}$$

Next, consider relations $(\rho_i s_1)^2 = 1$, $3\leq i\leq n-1$.

$$\begin{aligned}
\tau(1(\rho_i s_1)^2 1) &= \gamma(1,\rho_i)\gamma(\rho_1,s_1)\gamma(s_1\rho_1,\rho_i)\gamma(s_1,s_1)\\
&= x_i z y_i^{-1} \text{ for } i\geq 3,\\
\tau(s_1(\rho_i s_1)^2 s_1) &= \gamma(1,s_1)\gamma(s_1,\rho_i)\gamma(s_1\rho_1,s_1)\gamma(\rho_1,\rho_i)\gamma(1,s_1)\gamma(s_1,s_1)\\
&= y_i z^{-1} x_i^{-1} \text{ for } i\geq 3,\\
\tau(\rho_1(\rho_i s_1)^2\rho_1) &= \gamma(1,\rho_1)\gamma(\rho_1,\rho_i)\gamma(1,s_1)\gamma(s_1,\rho_i)\gamma(s_1\rho_1,s_1)\gamma(\rho_1,\rho_1)\\
&= x_i^{-1} y_i z^{-1} \text{ for } i\geq 3,\\
\tau(s_1\rho_1(\rho_i s_1)^2\rho_1 s_1) &= \gamma(1,s_1)\gamma(s_1,\rho_1)\gamma(s_1\rho_1,\rho_i)\gamma(s_1,s_1)\gamma(1,\rho_i)\gamma(\rho_1,s_1)\gamma(s_1\rho_1,\rho_1)\gamma(s_1,s_1)\\
&= y_i^{-1} x_i z \text{ for } i\geq 3.
\end{aligned}$$

This gives the non-trivial relations

$$y_i = x_i z \ \text{ for } 3\leq i\leq n-1.$$

Finally, we consider the relation $(s_1\rho_2\rho_1\rho_3\rho_2)^4 = 1$.

$$\begin{aligned}
\tau(1(s_1\rho_2\rho_3\rho_1\rho_2 s_1\rho_2\rho_1\rho_3\rho_2)^2 1) &= (\gamma(1,s_1)\gamma(s_1,\rho_2)\gamma(s_1\rho_1,\rho_3)\gamma(s_1,\rho_1)\gamma(s_1\rho_1,\rho_2)\\
&\qquad \gamma(s_1,s_1)\gamma(1,\rho_2)\gamma(\rho_1,\rho_1)\gamma(1,\rho_3)\gamma(\rho_1,\rho_2))^2\\
&= (y_2 y_3^{-1} y_2^{-1} x_2 x_3 x_2^{-1})^2,
\end{aligned}$$

$$\begin{aligned}\tau(s_1(s_1\rho_2\rho_3\rho_1\rho_2 s_1\rho_2\rho_1\rho_3\rho_2)^2 s_1) &= \gamma(1,s_1)(\gamma(s_1,s_1)\gamma(1,\rho_2)\gamma(\rho_1,\rho_3)\gamma(1,\rho_1)\gamma(\rho_1,\rho_2)\\ &\quad \gamma(1,s_1)\gamma(s_1,\rho_2)\gamma(s_1\rho_1,\rho_1)\gamma(s_1,\rho_3)\gamma(s_1\rho_1,\rho_2))^2\gamma(s_1,s_1)\\ &= (x_2x_3^{-1}x_2^{-1}y_2y_3y_2^{-1})^2,\end{aligned}$$

$$\begin{aligned}\tau(\rho_1(s_1\rho_2\rho_3\rho_1\rho_2 s_1\rho_2\rho_1\rho_3\rho_2)^2 \rho_1) &= \gamma(1,\rho_1)(\gamma(\rho_1,s_1)\gamma(s_1\rho_1,\rho_2)\gamma(s_1,\rho_3)\gamma(s_1\rho_1,\rho_1)\gamma(s_1,\rho_2)\\ &\quad \gamma(s_1\rho_1,s_1)\gamma(\rho_1,\rho_2)\gamma(1,\rho_1)\gamma(\rho_1,\rho_3)\gamma(1,\rho_2))^2\gamma(\rho_1,\rho_1)\\ &= (zy_2^{-1}y_3y_2z^{-1}x_2^{-1}x_3^{-1}x_2)^2,\end{aligned}$$

$$\begin{aligned}\tau(s_1\rho_1(s_1\rho_2\rho_3\rho_1\rho_2 s_1\rho_2\rho_1\rho_3\rho_2)^2 \rho_1 s_1) &= \gamma(1,s_1)\gamma(s_1,\rho_1)(\gamma(s_1\rho_1,s_1)\gamma(\rho_1,\rho_2)\\ &\quad \gamma(1,\rho_3)\gamma(\rho_1,\rho_1)\gamma(1,\rho_2)\\ &\quad \gamma(s_1\rho_1,\rho_1)\gamma(s_1,s_1)\gamma(\rho_1,s_1)\gamma(s_1\rho_1,\rho_2)\\ &\quad \gamma(s_1,\rho_1)\gamma(s_1\rho_1,\rho_3)\gamma(s_1,\rho_2))^2\\ &= (z^{-1}x_2^{-1}x_3x_2zy_2^{-1}y_3^{-1}y_2)^2.\end{aligned}$$

We get two non-trivial relations

$$\begin{aligned}(y_2y_3^{-1}y_2^{-1}x_2x_3x_2^{-1})^2 &= 1,\\ (zy_2^{-1}y_3y_2z^{-1}x_2^{-1}x_3^{-1}x_2)^2 &= 1.\end{aligned}$$

Using the relation $y_i = x_iz$, we eliminate y_i for $3 \leq i \leq n-1$ and we put $y_2 = y$, so we get the following relations.

$$\begin{aligned}x_2^3 &= 1,\\ x_j^2 &= 1 \quad \text{for } 3 \leq j \leq n-1,\\ y^3 &= 1,\\ (x_ix_{i+1}^{-1})^3 &= 1 \quad \text{for } 2 \leq i \leq n-2,\\ (x_ix_j^{-1})^2 &= 1 \quad \text{for } 2 \leq i \leq n-2 \text{ and } j \geq i+2,\\ (x_jz)^2 &= 1 \quad \text{for } 3 \leq j \leq n-1,\\ (yz^{-1}x_3^{-1})^3 &= 1,\\ (yz^{-1}x_j^{-1})^2 &= 1 \quad \text{for } 4 \leq j \leq n-1,\\ (yz^{-1}x_3^{-1}y^{-1}x_2x_3x_2^{-1})^2 &= 1,\\ (zy^{-1}x_3zyz^{-1}x_2^{-1}x_3^{-1}x_2)^2 &= 1.\end{aligned}$$

We see that

$$\gamma_2(VT_2) = \langle (\rho_1 s_1)^2 \rangle \cong \mathbb{Z}$$

and

$$\gamma_2(VT_3) = \left\langle \rho_2\rho_1, s_1\rho_2\rho_1 s_1, (\rho_1 s_1)^2 \mid (\rho_2\rho_1)^3 = 1 = (s_1\rho_2\rho_1 s_1)^3 \right\rangle \cong \mathbb{Z}_3 * \mathbb{Z}_3 * \mathbb{Z}.$$

For $n \geq 4$, $\gamma_2(VT_n)$ is generated by $\left\{x_i, y, z \mid i = 2, 3, \ldots, n-1\right\}$. The set $\{x_2, x_3, \ldots, x_{n-1}\}$ generate a subgroup isomorphic to A_n and has relations

$$\begin{aligned}
x_2^3 &= 1, \\
x_j^2 &= 1 \quad \text{for } 3 \leq j \leq n-1, \\
(x_i x_{i+1}^{-1})^3 &= 1 \quad \text{for } 2 \leq i \leq n-2, \\
(x_i x_j^{-1})^2 &= 1 \quad \text{for } 2 \leq i \leq n-2 \text{ and } j \geq i+2.
\end{aligned}$$

In addition, $\gamma_2(VT_n)$ has the following mixed relations

$$\begin{aligned}
y^3 &= 1, \\
(x_j z)^2 &= 1 \quad \text{for } 3 \leq j \leq n-1, \\
(yz^{-1}x_3^{-1})^3 &= 1, \\
(yz^{-1}x_j^{-1})^2 &= 1 \quad \text{for } 4 \leq j \leq n-1, \\
(yz^{-1}x_3^{-1}y^{-1}x_2x_3x_2^{-1})^2 &= 1, \\
(zy^{-1}x_3zyz^{-1}x_2^{-1}x_3^{-1}x_2)^2 &= 1.
\end{aligned}$$

This completes the proof. □

It is known that the second and the third term of the lower central series of the braid group B_n and virtual braid group VB_n coincide for $n \geq 3$ and $n \geq 4$, respectively. It turns out that the same holds for VT_n.

Proposition 8.2.2. $\gamma_2(VT_n) = \gamma_3(VT_n)$ *for* $n \geq 3$.

Proof. We need to show that $\gamma_2(VT_n) \subseteq \gamma_3(VT_n)$ or equivalently that $VT_n/\gamma_3(VT_n)$ is abelian. The group $VT_n/\gamma_3(VT_n)$ is generated by

$$\overline{s}_i = s_i\gamma_3(VT_n)$$

and

$$\overline{\rho}_i = \rho_i\gamma_3(VT_n)$$

for all $i = 1, 2, \ldots, n-1$. It is easy to check that

$$\rho_{i+1} = \rho_i[\rho_i, [\rho_i, \rho_{i+1}]]$$

and

$$s_{i+1} = s_i[[\rho_{i+1}, [\rho_{i+1}, \rho_i]], s_i]^{-1}.$$

This gives

$$\overline{\rho}_{i+1} = \rho_{i+1}\gamma_3(VT_n) = \rho_i[\rho_i, [\rho_i, \rho_{i+1}]]\gamma_3(VT_n) = \rho_i\gamma_3(VT_n) = \overline{\rho}_i$$

and

$$\overline{s}_{i+1} = s_{i+1}\gamma_3(VT_n) = s_i[[\rho_{i+1}, [\rho_{i+1}, \rho_i]], s_i]^{-1}\gamma_3(VT_n) = s_i\gamma_3(VT_n) = \overline{s}_i.$$

Thus, we have

$$\overline{\rho}_i = \overline{\rho}_{i+1} \text{ and } \overline{s}_i = \overline{s}_{i+1}$$

for all $i = 1, 2, \ldots n-2$. Since

$$\overline{\rho}_1\overline{s}_1 = \overline{\rho}_3\overline{s}_1 = \overline{s}_1\overline{\rho}_3 = \overline{s}_1\overline{\rho}_1,$$

the assertion follows. □

Corollary 8.2.3. *VT_n is residually nilpotent if and only if $n = 2$.*

We conclude with a result on freeness of commutator subgroup of PVT_n. A graph is called *chordal* if each of its cycles with more than three vertices has a chord (an edge joining two vertices that are not adjacent in the cycle). A *clique* (or a complete subgraph) of a graph is a subset C of vertices such that every two vertices in C are connected by an edge. It is well-known that a graph is chordal if and only if its vertices can be ordered in such a way that the lesser neighbours of each vertex form a clique.
We conclude the thesis with the following result.

Theorem 8.2.4. *The commutator subgroup of PVT_n is free if and only if $n \leq 4$.*

Proof. The assertion is immediate for $n = 2, 3$. The graph of PVT_4 (see Figure 7.1) is vacuously chordal. By [77, Corollary 4.4], the commutator subgroup of a right-angled Artin group is free if and only if its associated graph is chordal, and hence PVT_4 is free.
For $n \geq 5$, fix an ordering on the vertex set of the graph, for example, it could be the lexicographic ordering in our case. Let $\lambda_{i,j}$ be a maximal vertex and $p, q, r \in \{1, 2, \ldots, n\} \setminus \{i, j\}$ be three distinct integers. Then both $\lambda_{p,q}$ and $\lambda_{p,r}$ are lesser neighbours of $\lambda_{i,j}$, but

there cannot be an edge between $\lambda_{p,q}$ and $\lambda_{p,r}$. Thus, lesser neighbours of the vertex $\lambda_{i,j}$ do not form a clique, and hence the graph is not chordal. $\square$

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