ALGEBRAIC STRUCTURES IN KNOT THEORY

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Declaration

The work presented in this thesis has been carried out by me under the guidance of Dr. Mahender Singh at the Indian Institute of Science Education and Research Mohali. This work has not been submitted in part or in full for a degree, a diploma, or a fellowship to any other university or institute. Whenever contributions of others are involved, every effort is made to indicate this clearly, with due acknowledgement of collaborative research and discussions. This thesis is a bonafide record of original work done by me and all sources listed within have been detailed in the bibliography.

Manpreet Singh

In my capacity as the supervisor of the candidate's thesis work, I certify that the above statements by the candidate are true to the best of my knowledge.

Dr. Mahender Singh
(Supervisor)

Dedicated

to

My Mama and Papa

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Abstract

Knot theory is the study of embedded circles in the 3-sphere. A central problem in the subject is to develop computational invariants that can distinguish two knots. One such almost complete invariant that surfaced independently in the works of Matveev and Joyce in 1982 is what is called a link quandle, which is basically a minimal algebraic structure that encodes the three Reidemeister moves of planar diagrams of links in the 3-sphere. One of the fundamental results is that two non-split tame links have isomorphic link quandles if and only if there is a homeomorphism of the 3-sphere that maps one link onto the other, not necessarily preserving the orientations of the ambient space and that of links. Many classical topological, combinatorial and geometric knot invariants such as the knot group, the knot coloring, the Conway polynomial, the Alexander polynomial and the volume of the complement in the 3-sphere of a hyperbolic knot can be retrieved from the knot quandle. Thus, understanding of knot quandles is of fundamental importance for the classification problem for knots.

The first and major component of the thesis is a fusion of ideas from combinatorial group theory into the theory of quandles. More precisely, we introduce residual finiteness and orderability in quandles. One of our main results is that every link quandle is residually finite, a proof of which uses the idea of subgroup separability in fundamental groups of 3-manifolds. As immediate consequences of this result, it follows that the word problem is solvable for link quandles, and that every link admits a non-trivial coloring by a finite quandle. We also develop a general theory of orderability of quandles with a focus on link quandles and give some general constructions of orderable quandles. We prove that knot quandles of many fibered prime knots are right-orderable, whereas link quandles of many non-trivial torus links are not right-orderable. We prove that link quandles of certain non-trivial positive (or negative) links are not bi-orderable, which includes some alternating knots of prime determinant and alternating Montesinos links. The results show that orderability of link quandles behave quite differently than that of corresponding link groups.

Viewing classical knots as knots in the thickened 2-sphere, it is natural to explore knot theory in thickened surfaces of higher genera. This idea led to what is now known as virtual knot theory, a subject pioneered by Kauffman in 1999 with a completely different set-up. Though

many invariants from the classical knot theory extend to the virtual setting, a lot is still unknown, and the second component of the thesis focuses on this theme. We define virtually symmetric representations of virtual braid groups by automorphism groups. We prove that many known representations of these groups such as the generalized Artin representation, the Silver-Williams representation, the Boden-Dies representation and the Wada representation are equivalent to virtually symmetric representations. We use one such representation to define new virtual link groups which are extensions of link groups known due to Kauffman. Finally, we introduce marked Gauss diagrams as a generalization of Gauss diagrams and extend the definition of virtual link groups to marked Gauss diagrams.

List of notations

$\mathbb{R}^n$	Euclidean n-space
$\sqcup_n \mathbb{S}^1$	Disjoint union of n copies of $\mathbb{S}^1$
$\mathbb{S}^n$	n-sphere
$\mathbb{D}^2$	Unit disk centred at the origin of $\mathbb{R}^2$
$\lvert A \rvert$	Number of elements in set A
$A \sqcup B$	Disjoint union of sets A and B
$\mathbb{Z}$	Group of integers
$\mathbb{Z} \oplus \mathbb{Z}$	Free abelian group of rank 2
$\mathbb{Z}^n$	Free abelian group of rank n
$\mathbb{Z}_n$	Cyclic group of order n
S_n	Symmetric group on n symbols
S_X	Symmetric group on set X
$F(X)$	Free group on set X
F_n	Free group of rank n
$G * H$	Free product of groups G and H
$F_{n,m}$	Free product of F_n and $\mathbb{Z}^m$
$GL_n(R)$	General linear group over an integral domain R
$H_1 \leq H_2$	H_1 is a subgroup of H_2
$\mathrm{C}_G(x)$	Centralizer of an element x of group G
$\mathrm{Z}(G)$	Centre of group G
$H_2(G)$	Second homology of group G with integer coefficients
$\mathrm{Ker}(\phi)$	Kernel of a homomorphism ϕ
$FQ(X)$	Free quandle on set X
FQ_∞	Free quandle generated by a countably infinite set
S_x	Inner automorphism of quandle induced by element x
L_x	Left multiplication map of quandle induced by element x
$\mathrm{Inn}(X)$	Inner automorphism group of quandle or group X
$\mathrm{Aut}(X)$	Automorphism group of quandle or group X
$K(G,1)$	Eilenberg-Maclane space of group G

$\mathrm{Conj}_n(G)$	n-conjugation quandle of group G, where $n \in \mathbb{Z}$
$\mathrm{Conj}(G)$	Conjugation quandle of group G
$\mathrm{Core}(G)$	Core quandle of group G
$\mathrm{Alex}(G,\alpha)$	Generalized Alexander quandle of group G, where $\alpha \in \mathrm{Aut}(G)$
$\sqcup_{i\in I}(G,H_i,z_i)$	Quandle of disjoint union of right cosets of H_i in G, where $z_i \in G$ and $H_i \leq \mathrm{C}_G(z_i)$
R_n	Dihedral quandle of order n
$R(G,A)$	(G,A)-rack, where A is a subset of group G
$Q(G,A)$	(G,A)-quandle, where A is a subset of group G
$\prod_{i\in I} X_i$	Product of quandles X_i
$\star_{i\in I} Q_i$	Free product of quandles Q_i
$\partial(M)$	Boundary of manifold M
$\mathrm{int}(M)$	Interior of manifold M
$V(K)$	Tubular neighbourhood of knot K
$m(L)$	Mirror image of link L
$r(L)$	Link L with opposite orientation
$C(L)$	Link exterior of link L
$\pi_1(C(L))$	Link group of link L
$Q(L)$	Link quandle of link L
$IQ(L)$	Involutory link quandle of link L
$D(L)$	(Virtual) Link diagram of (virtual) link L on a plane
Σ	Compact oriented surface
I	Unit interval $[0,1]$
$\Sigma \times I$	Thickened surface Σ
B_n	Braid group on n strands
$\cup_{n\geq 1} B_n$	Infinite braid group
VB_n	Virtual braid group on n strands
$\cup_{n\geq 1} VB_n$	Infinite virtual braid group
$Cl(\beta)$	Closure of (virtual) braid diagram on a plane
π_D	Group associated to Gauss diagram D
Π_D	Group associated to marked Gauss diagram D
$G_0(L)$	Kauffman group associated to virtual link L
$G_S(L)$	Group associated to virtual link L

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Chapter 1

Introduction

Knot theory is the study of links which are embedded circles in the Euclidean 3-space or equivalently the 3-sphere. Two links are said to be equivalent if there exists an ambient isotopy mapping one link onto the other. If links are oriented, then the ambient isotopy is required to preserve the orientation of each component. A knot is simply a link with one component. All links considered in this thesis are tame, that is, have finitely many crossings in a regular projection which can be guaranteed, for example, by assuming our embeddings to be smooth.

Each link gives a regular projection on a plane after a small perturbation, if necessary. This gives what is referred as the link diagram, which is a planar 4-valent graph equipped with the information of over and under arcs at each vertex. Figure 1.1 illustrates some examples of knot diagrams.

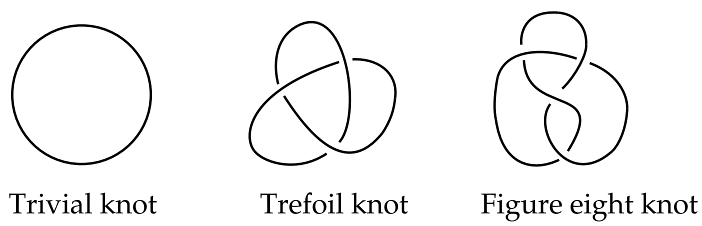

Fig. 1.1 Examples of knot diagrams.

Reidemeister [106] redefined equivalence of links in terms of link diagrams using local moves known as Reidemeister moves as shown in Figure 1.2. More precisely, he proved that two links are equivalent if and only if any link diagram of one can be transformed to any link diagram of the other by a finite sequence of Reidemeister moves and planar isotopies.

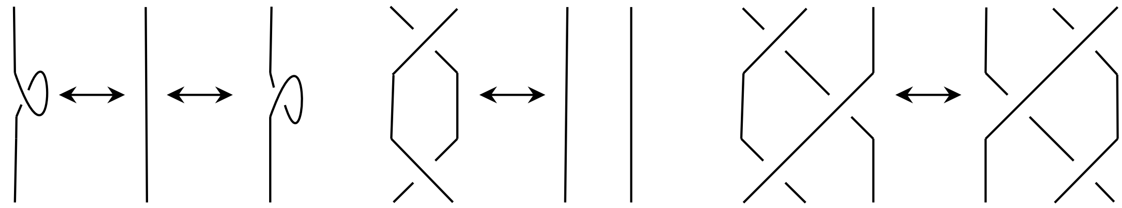

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Fig. 1.2 Reidemeister moves for link diagrams.

Artin [2, 3] introduced the notion of an n-braid as a collection of n smooth non-intersecting strands in $\mathbb{R}^2 \times [0,1]$ connecting n marked points on each of the planes $\mathbb{R}^2 \times \{0\}$ and $\mathbb{R}^2 \times \{1\}$ such that for each $t \in [0,1]$, the plane $\mathbb{R}^2 \times \{t\}$ intersects each string at exactly one point. Two braids on n strands are said to be equivalent if one can be transformed into the other by an ambient isotopy fixing the boundary. Artin proved that the set of equivalence classes of braids on n strands form a group under the operation of placing one over the other and squeezing the interval back to $[0,1]$. This group is known as the Artin braid group on n strands, and is denoted by B_n. Alexander [77, Theorem 2.3] proved that every oriented link can be represented by closure of some n-braid. Later, Markov [77, Theorem 2.8] proved that the set of equivalence classes of links in the 3-sphere is in bijection with the set of Markov equivalence classes of elements in the infinite braid group $\cup_{n\geq 1} B_n$.

One of the central problems in knot theory is to classify all knots. This problem is known as the knot recognition problem in the literature. A link invariant is a mathematical object assigned to a link that remains the same for the equivalence class of each link. One of the earliest known link invariants is the link group, which is defined as the fundamental group of the complement of a link in $\mathbb{R}^3$. It is known to be a strong invariant for prime knots. More precisely, knot groups of two oriented prime knots are isomorphic if and only if either they are equivalent or they are equivalent after changing the orientation of one of them and/or the ambient space [58, 59, 120]. This result fails for composite knots, for example, square and granny knots have isomorphic knot groups but their knot complements are not homeomorphic, and hence they cannot be equivalent. Thus, the link group is not a complete invariant for all links. Another approach to link invariants is via braid groups. Particularly, representations of braid groups have been used extensively to construct link invariants. For instance, the well-known Artin and Burau representations of the braid group can be used to calculate link groups and Alexander polynomials, respectively [25, 119].

One almost complete link invariant that surfaced independently in the works of Joyce [70, 71] and Matveev [94] in 1982 is what is called a link quandle, which is basically a minimal algebraic structure that encodes the three Reidemeister moves of planar diagrams of links in the 3-sphere. They proved that two non-split links have isomorphic link quandles if and only if there is a homeomorphism of the 3-sphere that maps one link onto the other, not necessarily preserving the orientation of the ambient space and that of links. Many classical knot invariants such as the knot group [71, 94], the knot coloring, the Alexander polynomial [71], and the volume of the complement of a hyperbolic knot in the 3-sphere can be retrieved from the link quandle [66]. Thus, understanding of knot quandles is of fundamental importance for the classification problem for knots.

It is of relevance to have a solution of the isomorphism problem for quandles in order to use them effectively as link invariants. Recently, Brooke-Taylor and Miller [28] asserted that the isomorphism problem of quandles is difficult in the sense of Borel reducibility. Thus, a more fruitful approach is to use quandles themselves to construct more convenient and computationally relevant link invariants. The past decade has seen a large number of new knot invariants arising from quandles. Most notably, as is the case with any algebraic system, a (co)homology theory for quandles and racks has been developed in [32, 48, 49], which, as applications has led to stronger invariants for links.

Apart from the classification problem, understanding the relationship between link quandles and other known invariants is of paramount interest. For instance, Inoue and Kabaya [66] introduced a new cohomology theory of quandles, and developed a method for computing the complex volume of hyperbolic links from diagrams using quandle cocycles. One of the classical problems in computational topology is the unknot detection problem, which asks whether a given knot in the 3-sphere is equivalent to the trivial knot. Eisermann [46] characterized the unknot in terms of the second cohomology group of knot quandles. Winker [121] proved that a knot is trivial if and only if the associated involutory knot quandle is trivial. Based on Winker's work, Fish and Lisitsa [52] conjectured the following:

Conjecture 1.0.1. *The involutory knot quandle of a knot is residually finite.*

The study of residual properties of algebraic objects arising in low dimensional topology is an active area of research. For groups, such properties consist of being a finite group, finitely solvable group, finitely nilpotent group and a finite p-group. A group G is said to be residually $\mathcal{P}$ if for every non-identity element in G, there exists a homomorphism to a group having property $\mathcal{P}$ such that the image of the said element is the non-identity element. Residual finiteness of groups is useful in deciding the solvability of the word problem. Given a group G with a presentation, the word problem is to decide whether a given word written in a sequence of generators and their formal inverses is the identity in G. Dehn [41] solved the unknot detection problem by proving that a knot is trivial if and only if its knot group is isomorphic to integers. He introduced the word problem to detect the unknot from a presentation of the knot group. In particular, to detect the unknot using a presentation of the knot group, one needs to check whether all the generators commute with each other. Novikov [103] proved that, in general, the word problem is not solvable for groups. This motivated search for families of groups for which the word problem is solvable. It is now known that the word problem is solvable for a finitely presented residually finite groups [107, p.55]. Further, residual finiteness has been proved for fundamental groups of compact surfaces [64] and mapping class groups of orientable surfaces [61]. Neuwirth [101] proved residual finiteness of knot groups of fibered knots and conjectured the same for each knot. Mayland [95] and Stebe [114] confirmed the conjecture for torus knots and twisted knots, respectively. Finally, as a consequence of the seminal work of Hempel [65] on residual finiteness of fundamental groups of geometric 3-manifolds and Thurston's proof [116] of the geometrization of Haken 3-manifolds, it follows that all link groups are residually finite. In view of Conjecture 1.0.1 and noting that link groups can be recovered from link quandles, the following problem seems natural.

Problem 1.0.2. *Is the link quandle of every link residually finite?*

In our works [18, 19], we investigate residual finiteness of quandles. Besides establishing some closure properties of residually finite quandles, we prove that free quandles are residually finite. We also prove that the word problem is solvable for a finitely presented residually finite quandle. We provide a solution to Problem 1.0.2 by proving that all link quandles are residually finite. We first establish this result for non-split links using a representation of the link quandle as the quandle of cosets of peripheral subgroups in the link group and subgroup separability in fundamental groups of 3-manifolds. We then investigate residual finiteness of free products of residually finite quandles and extend the proof to all links. In doing so, we also prove that a free product of finite quandles is residually finite.

Lately, linear orders on groups appearing in low-dimensional topology have received much attention. In terms of structural implications, left-orderable groups do not have torsion, and bi-orderable groups cannot have generalized torsion. As an application, integral group rings of left-orderable groups have no zero-divisors. The well-known Kaplansky conjecture asserts that the same is true for all torsion-free groups. Concerning groups arising in low dimensional topology, many of them are known to be left-orderable. For instance, the fundamental group of any connected surface except the projective plane and the Klein bottle is known to be bi-orderable [27]. Artin braid groups are left-orderable [42], and the result has been extended to the mapping class groups of all Riemann surfaces with boundary by Rourke and Wiest [111]. Since the Artin braid groups have generalized torsion, they are not bi-orderable, whereas the pure braid groups are bi-orderable [109]. It is worth looking at [37, 43, 99] for interactions between orderability of the fundamental group and topological properties of a 3-manifold.

The space of left orders on an arbitrary magma is studied in [40], where it is observed that every bi-order on a group induces a right-order on its associated conjugation quandle. Quite interestingly, contrasting to the groups, left(right)-orderability of a quandle does not ensure right(left)-orderability of the same. In [63], the space of right orders on the conjugation quandle of the countably infinite rank free group has been shown to be the Cantor set. Recently, applications of orderability in quandle rings have been explored in [17].
In our work [105], we develop a general theory of orderability of quandles with a focus on link quandles. We prove that knot quandles of many fibered prime knots are right-orderable. Considering the fact that all link groups are left-orderable [27], it is reasonable to speculate that link quandles are left(right)-orderable. In contrast, we prove that the knot quandle of the trefoil knot is neither left nor right-orderable. We also prove that link quandles of many non-trivial torus links are not right-orderable. As an application, we recover a result of Perron and Rolfsen [104] which states that the knot group of a non-trivial torus knot is not bi-orderable. We also prove that link quandles of certain non-trivial positive (or negative) links are not bi-orderable, which includes some alternating knots of prime determinant and alternating Montesinos links.

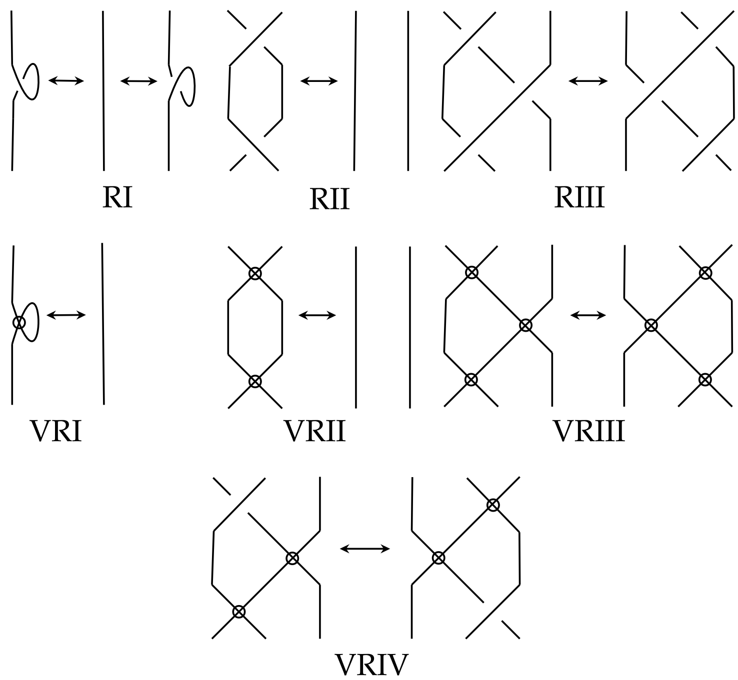


Fig. 1.3 Generalized Reidemeister moves for virtual link diagrams.

In his pioneering work [79], Kauffman gave a generalization of classical knot theory by introducing a new type of crossing in link diagrams which neither indicates which arc passes over nor which arc passes under. This new crossing is decorated with a small circle and referred as a virtual crossing. Further, the diagram is known as a virtual link diagram. These diagrams are considered up to the equivalence relation generated by finite sequences of planar isotopies and generalized Reidemeister moves which is a collection of (classical) Reidemeister moves and local virtual moves as shown in Figure 1.3. A virtual link is then defined as the equivalence class of a virtual link diagram.

It later turned out that virtual links can be thought of as links in thickened surfaces of higher genera up to a more robust definition of equivalence. In [33, 72], virtual knot theory has been interpreted as a study of links in thickened compact oriented surfaces up to ambient isotopy fixing the boundary, orientation preserving homeomorphisms of surfaces, and addition and removal of handles to surfaces. Even though virtual knot theory arose as a pictorial study of diagrams on the plane, Kuperberg [85] later on proved that virtual knot theory is indeed the study of links in 3-manifolds. To be precise, he proved that every virtual link has a unique representative as an ambient isotopy class of a link in a thickened compact oriented surface of minimal genus up to orientation preserving homeomorphisms. Considering classical knot theory as a study of links in the thickened 2-sphere, the preceding result implies that virtual knot theory is a true generalization of the same. Apart from this, Goussarov-Polyak-Viro [60] interpreted virtual knot theory in terms of equivalence classes of Gauss diagrams, where the equivalence is generated by finite sequences of abstract Reidemeister moves. These diagrams play an important role in the study of finite type invariants.

Kauffman [79] also generalized the definition of Artin braids to virtual braids via diagrams. Algebraically, the virtual braid group VB_n is generated by the Artin braid group B_n and the symmetric group S_n, satisfying some mixed relations. Analogous to the classical case, Alexander and Markov theorems for oriented virtual knots are known due to Kamada [73] and Kauffman-Lambropoulou [78, 81], independently. To be precise, it was shown that the set of equivalence classes of virtual links is in bijection with the set of appropriate Markov equivalence classes of elements of the infinite virtual braid group $\cup_{n\geq 1} VB_n$.

Though, many invariants from the classical knot theory extend to the virtual setting, a lot is still unknown, and the last component of this thesis focuses on this theme. Kauffman extended the notion of knot group and knot quandle to the virtual setting via diagrams, whose topological interpretation is given in [50, 72]. Since then various definitions of virtual link groups have been introduced in the literature [8–10, 12–15, 26, 35, 91, 113], some of which use representations of virtual braid groups into the automorphisms of appropriate groups or modules.

In our work [16], we define virtually symmetric representations of virtual braid groups. We prove that some well-known representations of VB_n are equivalent to virtually symmetric representations. One of the advantages of virtually symmetric representations is that once

the virtual link group is defined, it can be described using Gauss diagrams too. Further, we ask whether it is the case for every representation of VB_n. For a specific representation of VB_n, we associate a group to each virtual link which we refer as a virtual link group. These groups belong to a certain class of C-groups [83, 84]. Kulikov [84] proved that every C-group can be realized as the fundamental group of complement of some n-dimensional ($n \geq 2$) compact orientable manifold without boundary embedded in an $(n+2)$-sphere. In particular, the fundamental group of complement of any classical link is a C-group. Furthermore, we introduce the notion of marked virtual link diagrams as a generic immersion of marked cycles in the plane with information of virtual and classical crossings at double points. Analogous to Gauss diagrams in virtual knot theory, we define marked Gauss diagrams. We show that the study of marked Gauss diagrams (up to appropriate equivalence) is a proper generalization of combinatorial virtual knot theory. In addition to this, we extend our definition of virtual link group to marked virtual link diagrams.
In the direction of classifying knots, it has been established that the knot group of a classical knot, its peripheral subgroup along with the meridian is a complete knot invariant up to the orientation of the knot and its ambient space. Due to lack of a well-defined notion of link groups in case of virtual links as fundamental groups of spaces, the peripheral structure cannot be defined analogously. Kauffman used his definition of virtual knot groups to define a peripheral subgroup, which was further studied by Kim [82] who observed many new and unexpected results. This motivated us to formulate and study the notion of peripheral structure for marked Gauss diagrams, in particular, for virtual knots using virtual knot groups. The following subsections give a brief outline of the thesis.

1.1 Residual finiteness of quandles

A *quandle* is a non-empty set Q with a binary operation $* : Q \times Q \longrightarrow Q$ satisfying the following axioms:

(Q1) $x * x = x$ for all $x \in Q$,

(Q2) for each $x, y \in Q$, there exists a unique $z \in Q$ such that $x = z * y$,

(Q3) $(x * y) * z = (x * z) * (y * z)$ for all $x, y, z \in Q$.

The axiom (Q2) is equivalent to the existence of a dual binary operation $*^{-1}$ on Q such that $x *^{-1} y = z$ if and only if $x = z * y$ for all $x, y, z \in Q$. A non-empty set with a binary operation satisfying the axioms (Q2) and (Q3) is known as a *rack*. Thus, a quandle is an idempotent rack. For each element $x \in Q$, the map

$$S_x : Q \to Q$$

given by $S_x(y) = y * x$ is an automorphism of Q known as the *inner automorphism* induced by x. The group $\mathrm{Inn}(Q)$ generated by all such automorphisms is known as the *inner automorphism group* of Q.

Groups are a natural source of quandles. Let G be a group and $n \in \mathbb{Z}$ a fixed integer. If we define $x * y = y^{-n}xy^n$ for all $x, y \in G$, then G turns into a quandle $\mathrm{Conj}_n(G)$ called the *n-conjugation quandle*. For $n = 1$, it is denoted by $\mathrm{Conj}(G)$ and called the *conjugation quandle* of G. If G is equipped with the binary operation $x * y = yx^{-1}y$, then we get the *core quandle* $\mathrm{Core}(G)$ of G. Moreover, if $\alpha \in \mathrm{Aut}(G)$, then the binary operation $x * y = \alpha(xy^{-1})y$ gives a quandle structure on G, which is denoted by $\mathrm{Alex}(G, \alpha)$ and called the *generalized Alexander quandle* of G. In fact, all the preceding constructions are functorial.

Suppose L is an oriented link in the 3-sphere. Joyce [70, 71] and Matveev [94] associated a quandle $Q(L)$ to L called the *link quandle* of L. We fix a diagram $D(L)$ of L and label its arcs. Then the link quandle $Q(L)$ is generated by labellings of arcs of $D(L)$ with a defining relation at each crossing in $D(L)$ given as shown in Figure 1.4. The link quandle of a link L is independent of the diagram chosen, that is, the quandles obtained from any two diagrams of L are isomorphic.

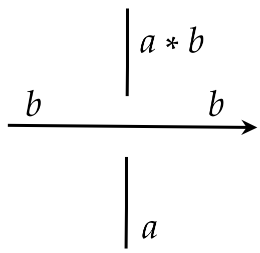


Fig. 1.4 Crossing relation.

We define a quandle Q to be *residually finite* if for all x, y in Q with $x \neq y$, there exists a finite quandle F and a quandle homomorphism $\phi : Q \to F$ such that $\phi(x) \neq \phi(y)$.

The preceding definition is motivated from the residual finiteness of groups which is defined in the same manner.

1.1.1 Basic properties of residually finite quandles

It is evident that subquandles and direct products of residually finite quandles are residually finite. Moreover, the following hold.

Proposition 1.1.1. *The following statements are equivalent for a quandle Q:*

1. *Q is residually finite.*
2. *There exists a family $\{X_i\}_{i\in I}$ of finite quandles such that the quandle Q is isomorphic to a subquandle of the product quandle $\prod_{i\in I} X_i$.*

Quandles arising from residually finite groups are residually finite.

Proposition 1.1.2. *Let G be a residually finite group. Then the following hold:*

1. $\mathrm{Conj}_n(G)$ *and* $\mathrm{Core}(G)$ *are residually finite quandles.*
2. *If $\alpha : G \to G$ is an inner automorphism, then* $\mathrm{Alex}(G, \alpha)$ *is a residually finite quandle.*

We have the following results concerning residual finiteness of automorphism groups of quandles.

Theorem 1.1.3. *The following statements hold:*

1. *If G is a finitely generated abelian group with no 2-torsion, then* $\mathrm{Aut}\left(\mathrm{Core}(G)\right)$ *is a residually finite group.*
2. *If G is a finitely generated residually finite group with trivial centre, then* $\mathrm{Aut}\left(\mathrm{Conj}(G)\right)$ *is a residually finite group.*
3. *If Q is a residually finite quandle, then* $\mathrm{Inn}(Q)$ *is a residually finite group.*

1.1.2 Residual finiteness of free products of quandles

A *free quandle* is a free object in the category of quandles and has an explicit model in terms of conjugacy classes of generators in a free group with respect to the conjugation operation. Notice that a free quandle can be viewed as a free product of one element quandles. Following is an analogue of a similar result for free groups.

Theorem 1.1.4. *Every free quandle is residually finite.*

A quandle Q is said to be *Hopfian* if every surjective quandle endomorphism of Q is injective. The following results stresses the importance of residual finiteness in quandles.

Theorem 1.1.5. *Every finitely generated residually finite quandle is Hopfian.*

Theorem 1.1.6. *Every finitely presented residually finite quandle has a solvable word problem.*

Let Q be a quandle. Then the enveloping group $\mathrm{Env}(Q)$ of Q is given by a presentation

$$\langle e_x \ \ (x \in Q) \mid e_{x*y} = e_y^{-1} e_x e_y \ \ (x, y \in Q) \rangle.$$

In the following result, we establish residual finiteness of free product of residually finite quandles.

Theorem 1.1.7. *Let $\{Q_i\}_{i \in I}$ be a family of residually finite quandles. If each $\mathrm{Env}(Q_i)$ is a residually finite group, then the free product $\star_{i \in I} Q_i$ is residually finite.*

Specializing to finite quandles, we have the following result.

Theorem 1.1.8. *If Q is a finite quandle, then $\mathrm{Env}(Q)$ is a residually finite group.*

Consequently, it follows that if $\{Q_i\}_{i \in I}$ is a family of finite quandles, then the free product $\star_{i \in I} Q_i$ is a residually finite quandle.

1.1.3 Residual finiteness of link quandles

Let $\{z_i \mid i \in I\}$ be elements of a group G, and $\{H_i \mid i \in I\}$ subgroups of G such that each H_i is a subgroup of the centralizer $\mathrm{C}_G(z_i)$ of z_i in G. Then the disjoint union $\sqcup_{i \in I}(G, H_i, z_i)$ becomes a quandle with

$$H_i x * H_j y = H_i z_i^{-1} x y^{-1} z_j y.$$

A subgroup H of a group G is said to be *finitely separable* if for any $g \in G \setminus H$, there exists a finite group F and a group homomorphism $\phi : G \to F$ such that $\phi(g) \notin \phi(H)$.

Proposition 1.1.9. *Let G be a group, $\{z_i \mid i \in I\}$ be a finite set of elements of G, and $\{H_i \mid i \in I\}$ subgroups of G such that $H_i \leq \mathrm{C}_G(z_i)$. If each H_i is finitely separable in G, then the quandle $\sqcup_{i \in I}(G, H_i, z_i)$ is residually finite.*

A connected 3-manifold M is said to be *irreducible* if every embedded 2-sphere in M bounds a 3-ball in M. A link L in the 3-sphere $\mathbb{S}^3$ is said to be *non-split* if its link exterior is irreducible. Using a result of Long and Niblo [87, Theorem 1] on finitely separable subgroups of fundamental groups of irreducible 3-manifolds and the fact that link quandles of non-split links can be written as coset quandles as in Proposition 1.1.9, we prove that the link quandle of a non-split link is residually finite. By observing that link groups are residually finite and that the link quandle of a split link is a free product of link quandles of its non-split components, using Theorem 1.1.7, we prove the following.

Theorem 1.1.10. *The link quandle of any link is residually finite.*

Since link quandles are finitely presented, we have the following result.

Corollary 1.1.11. *The link quandle of a link is Hopfian, has solvable word problem, and has residually finite inner automorphism group.*

Since enveloping groups of finite quandles, free quandles and link quandles are residually finite, we propose the following conjecture.

Conjecture 1.1.12. *The enveloping group of a finitely presented residually finite quandle is a residually finite group.*

1.2 Orderability of quandles

A quandle Q is said to be *left-orderable* if there is a (strict) linear order $<$ on Q such that $x < y$ implies $z * x < z * y$ for all x, y and z in Q. Similarly, one can define the notion of a *right-orderable* quandle. A quandle is said to be *bi-orderable* if it has a linear order with respect to which it is both left and right ordered.

1.2.1 Properties of linear orderings on quandles

Let $<$ be a linear order on a quandle Q and $\mathcal{O}$ be the set $\{=, <, >\}$. For a quadruple $(\blacklozenge_1, \blacklozenge_2, \blacklozenge_3, \blacklozenge_4) \in \mathcal{O}^4$, the order $<$ is said to be of type $(\blacklozenge_1, \blacklozenge_2, \blacklozenge_3, \blacklozenge_4)$ if the following hold for all $x, y, z \in Q$ with $x < y$:

1. $x * z \blacklozenge_1 y * z$,
2. $x *^{-1} z \blacklozenge_2 y *^{-1} z$,
3. $z * x \blacklozenge_3 z * y$,
4. $z *^{-1} x \blacklozenge_4 z *^{-1} y$.

We prove that a linear order on a quandle is of restricted type.

Theorem 1.2.1. *Let $<$ be a linear order on a quandle Q of the type $(\blacklozenge_1, \blacklozenge_2, \blacklozenge_3, \blacklozenge_4)$ for some $(\blacklozenge_1, \blacklozenge_2, \blacklozenge_3, \blacklozenge_4) \in \mathcal{O}^4$. Then we have the following:*

1. $\blacklozenge_1, \blacklozenge_2 \in \{<, >\}$.

2. $\blacklozenge_1$ *and* $\blacklozenge_2$ *are the same.*

3. *The quandle Q is trivial $\Leftrightarrow$ $\blacklozenge_3$ is the equality '$=$' $\Leftrightarrow$ $\blacklozenge_4$ is the equality '$=$'.*

4. *The quadruple $(\blacklozenge_1, \blacklozenge_2, \blacklozenge_3, \blacklozenge_4)$ is one of the following*
   $(<,<,=,=)$, $(<,<,<,>)$, $(<,<,>,<)$ *or* $(>,>,<,<)$.

Proposition 1.2.2. *Let $<$ be a linear order on a quandle Q. Then the order $<$ is a bi-ordering on Q if and only if it is of the type $(<,<,<,>)$.*

1.2.2 Constructions of orderable quandles

An *action* of a quandle Q on a quandle X is a quandle homomorphism

$$\phi : Q \to \mathrm{Conj}_{-1}(\mathrm{Aut}(X)),$$

where $\mathrm{Aut}(X)$ is the group of quandle automorphisms of X.

Theorem 1.2.3. *If a semi-latin quandle is right-orderable, then it acts faithfully on a linearly ordered set by order-preserving bijections. Conversely, if a quandle acts faithfully on a well-ordered set by order-preserving bijections, then it is right-orderable.*

The next result constructs orderable quandles from unions of orderable quandles.

Proposition 1.2.4. *Let $(Q_1, *)$ and $(Q_2, \circ)$ be right-orderable quandles, and $\sigma : Q_1 \to \mathrm{Conj}_{-1}(\mathrm{Aut}(Q_2))$ and $\tau : Q_2 \to \mathrm{Conj}_{-1}(\mathrm{Aut}(Q_1))$ be order-preserving quandle actions. Suppose that*

1. $\tau(z)(x) * y = \tau(\sigma(y)(z))(x * y)$ *for $x, y \in Q_1$ and $z \in Q_2$,*

2. $\sigma(z)(x) \circ y = \sigma(\tau(y)(z))(x \circ y)$ *for $x, y \in Q_2$ and $z \in Q_1$.*

Then $Q = Q_1 \sqcup Q_2$ with the operation

$$x \star y = \begin{cases} x * y, & x, y \in Q_1, \\ x \circ y, & x, y \in Q_2, \\ \tau(y)(x), & x \in Q_1, y \in Q_2, \\ \sigma(y)(x), & x \in Q_2, y \in Q_1, \end{cases}$$

is a right-orderable quandle.

Let Q be a quandle and A a set. Following [1, Section 2.1], a *dynamical* 2*-cocycle* is a map $\alpha : Q \times Q \to \mathrm{Map}(A \times A, A)$ such that

$$\alpha_{x,x}(s,s) = s, \tag{1.2.2.1}$$

$$\alpha_{x,y}(-,t) : A \to A \text{ is a bijection} \tag{1.2.2.2}$$

and the *cocycle condition*

$$\alpha_{x*y,z}(\alpha_{x,y}(s,t),\ u) = \alpha_{x*z,y*z}(\alpha_{x,z}(s,u),\ \alpha_{y,z}(t,u)) \tag{1.2.2.3}$$

holds for all $x,y,z \in Q$ and $s,t,u \in A$. Given a dynamical 2-cocycle α, the set $Q \times A$ can then be turned into a quandle denoted as $Q \times_\alpha A$ by defining

$$(x,s) * (y,t) = (x * y,\ \alpha_{x,y}(s,t)). \tag{1.2.2.4}$$

If A is an abelian group, then a *normalized quandle* 2*-cocycle* is a map $\alpha : Q \times Q \to A$ satisfying

$$\alpha_{x,y}\ \alpha_{x*y,z} = \alpha_{x,z}\ \alpha_{x*z,y*z}$$

and

$$\alpha_{x,x} = 1$$

for all $x,y,z \in Q$. A normalized quandle 2-cocycle $\alpha : Q \times Q \to A$ gives rise to a dynamical 2-cocycle $\alpha' : Q \times Q \to \mathrm{Map}(A \times A, A)$ defined as

$$\alpha'_{x,y}(s,t) = s\ \alpha_{x,y}.$$

Proposition 1.2.5. *The following statements hold:*

1. *Let Q be a right-orderable quandle, A an ordered set and $\alpha : Q \times Q \to \mathrm{Map}(A \times A, A)$ a dynamical 2-cocycle. If $\alpha_{x,y} : A \times A \to A$ is order-preserving for all $x,y \in Q$, then the quandle $Q \times_\alpha A$ is right-orderable.*

2. *If Q is a right-orderable quandle, A a right-orderable abelian group and $\alpha : Q \times Q \to A$ a normalized 2-cocycle, then the quandle $X \times_\alpha A$ is right-orderable.*

3. *If Q is a quandle, A a non-trivial abelian group and $\alpha : Q \times Q \to A$ a normalized 2-cocycle, then the quandle $X \times_\alpha A$ cannot be left-orderable.*

In [5], Bardakov and Nasybullov define (G,A)-racks/quandles generalizing the construction of free racks due to Fenn and Rourke [47] and free quandles due to Kamada [74].

Theorem 1.2.6. *Let G be a group and A be a subset of G.*

1. *If G is right-orderable, then the rack $R(G,A)$ is right-orderable.*
2. *If G is bi-orderable, then the quandle $Q(G,A)$ is right-orderable.*

If X is a set, then the free quandle $FQ(X)$ on the set X is $Q(F(X),X)$-quandle, where $F(X)$ is the free group on the set X. Since free groups are known to be bi-orderable [118], we recover the following result of [17, Theorem 3.5].

Corollary 1.2.7. *Free quandles are right-orderable. In particular, link quandles of trivial links are right-orderable.*

1.2.3 Orderability of link quandles

For a given quandle Q, there is a natural quandle homomorphism

$$\eta : Q \to \mathrm{Conj}(\mathrm{Env}(Q)),$$

to its enveloping group $\mathrm{Env}(Q)$ mapping $q \to e_q$ for each q in Q. The following result gives a connection between orderability of a quandle and its enveloping group.

Proposition 1.2.8. *Let Q be a quandle such that the natural map $\eta : Q \to \mathrm{Conj}(\mathrm{Env}(Q))$ is injective. If $\mathrm{Env}(Q)$ is a bi-orderable group, then Q is a right-orderable quandle.*

Using a result [104, Theorem 1.1] of Perron and Rolfsen, we obtain the following.

Corollary 1.2.9. *If all the roots of the Alexander polynomial of a fibered prime knot are real and positive, then its knot quandle is right-orderable.*

Thus, the knot quandle of the figure eight knot is right-orderable.
We present a result on bi-orderability of link quandle of a connected sum of two links which is crucial in proving subsequent results on alternating positive (negative) links.

Theorem 1.2.10. *Let L_1 be any link and L_2 a non-trivial positive $($negative$)$ link. Suppose there exists a minimal positive $($negative$)$ diagram $D(L_2)$ of L_2 such that the generators of the link quandle $Q(L_2)$ corresponding to the arcs in $D(L_2)$ are pairwise distinct. Then the link quandle of a connected sum of links L_1 and L_2 is not bi-orderable. In particular, the link quandle $Q(L_2)$ is not bi-orderable.*

Mattman and Solis [93] proved the following conjecture.

Theorem 1.2.11. (Kauffman-Harary conjecture) *Let $D(K)$ be a reduced alternating diagram of the knot K having prime determinant p. Then, every non-trivial p-coloring of $D(K)$ assigns different colors to different arcs in $D(K)$.*

Using the preceding result and Theorem 1.2.10, we prove the following.

Corollary 1.2.12. *Let K be an alternating and positive (or negative) knot of prime determinant. Then the link quandle of a connected sum of K with any link is not bi-orderable. In particular, the knot quandle of K is not bi-orderable.*

In [4], Asaeda, Przytycki and Sikora proposed the following generalized Kauffman-Harary conjecture and proved the same for Montesinos links.

Conjecture 1.2.13. (The Generalized Kauffman-Harary (GKH) Conjecture) *If $D(L)$ is an alternating diagram of a prime link L without reduced crossings, then different arcs of $D(L)$ represent different elements of the first homology group of the double cover of $\mathbb{S}^3$ branched along L.*

Using their result and Theorem 1.2.10, we prove the following.

Corollary 1.2.14. *Let M be a non-trivial Montesinos link that is alternating and positive (or negative). Then the link quandle of a connected sum of M with any link is not bi-orderable. In particular, the link quandle of M is not bi-orderable.*

1.2.4 Orderability of torus link quandles and involutory quandles

We prove that link quandles of many non-trivial torus links are not right-orderable.

Theorem 1.2.15. *Let $m,n \geq 2$ be integers such that one is not a multiple of the other. Then the link quandle of the torus link $T(m,n)$ is not right-orderable.*

As an application of the preceding result, we recover the following result of Perron and Rolfsen [104, Proposition 3.2].

Corollary 1.2.16. *The knot group of a non-trivial torus knot is not bi-orderable.*

It is proved in [17, Proposition 3.7] that non-trivial involutory quandles are not right-orderable. The following result shows that there are links whose involutory link quandles are not left-orderable.

Theorem 1.2.17. *Let L be a non-trivial alternating link. If there exists a reduced alternating diagram $D(L)$ of L such that the generators of the involutory quandle $IQ(L)$ of L corresponding to the arcs in $D(L)$ are pairwise distinct, then $IQ(L)$ is not left-orderable.*

Corollary 1.2.18. *Let M be a non-trivial alternating Montesinos link. Then the involutory quandle $IQ(M)$ of M is not left-orderable.*

Corollary 1.2.19. *Let K be an alternating knot of prime determinant. Then the involutory quandle $IQ(K)$ of K is not left-orderable.*

1.3 Virtually symmetric representations and virtual link groups

The *virtual braid group* VB_n is the group generated by $\sigma_1, \sigma_2, \dots, \sigma_{n-1}, \rho_1, \dots, \rho_{n-1}$ which satisfy the following relations:

- relations of the braid group:

$$\begin{aligned} \sigma_i \sigma_{i+1} \sigma_i &= \sigma_{i+1} \sigma_i \sigma_{i+1} \text{ for } i \in \{1, 2, \dots, n-2\}, \\ \sigma_i \sigma_j &= \sigma_j \sigma_i \text{ where } |i-j| \geq 2 \text{ for } i, j \in \{1, 2, \dots, n-1\}, \end{aligned}$$

- relations of the symmetric group:

$$\begin{aligned} \rho_i^2 &= 1 \text{ for } i \in \{1, 2, \dots, n-1\}, \\ \rho_i \rho_j &= \rho_j \rho_i \text{ where } |i-j| \geq 2 \text{ for } i, j \in \{1, 2, \dots, n-1\}, \\ \rho_i \rho_{i+1} \rho_i &= \rho_{i+1} \rho_i \rho_{i+1} \text{ for } i \in \{1, 2, \dots, n-2\}, \end{aligned}$$

- mixed relations:

$$\begin{aligned} \sigma_i \rho_j &= \rho_j \sigma_i \text{ where } |i-j| \geq 2 \text{ for } i, j \in \{1, 2, \dots, n-1\}, \\ \rho_i \rho_{i+1} \sigma_i &= \sigma_{i+1} \rho_i \rho_{i+1} \text{ for } i \in \{1, 2, \dots, n-2\}. \end{aligned}$$

1.3.1 Virtually symmetric representations

Let $\varphi : VB_n \to \mathrm{Aut}(H)$ be a representation of the virtual braid group VB_n into the automorphism group of some group (or module) $H = \langle h_1, h_2, \dots, h_m \mid \mathcal{R} \rangle$. We say that the representation φ is *virtually symmetric* if the image $\varphi(\rho_i)$ of each generator ρ_i is a permutation of the generators $h_1, h_2, \dots, h_m$ of H.

Let $F_{n,n} = F_n * \mathbb{Z}^n$, where $F_n = \langle x_1, x_2, \dots, x_n \rangle$ is the free group of rank n and $\mathbb{Z}^n = \langle v_1, v_2, \dots, v_n \rangle$ is the free abelian group of rank n. In [12, Theorem 4.1], an extension φ_M of the Artin representation is defined for virtual braid groups, where $\varphi_M : VB_n \to \mathrm{Aut}(F_{n,n})$ is defined by its action on generators as follows:

$$\varphi_M(\sigma_i):\begin{cases} x_i \mapsto x_i x_{i+1} x_i^{-1}, \\ x_{i+1} \mapsto x_i, \\ x_j \mapsto x_j, \text{ for } j \neq i, i+1, \end{cases} \qquad \varphi_M(\sigma_i):\begin{cases} v_i \mapsto v_{i+1}, \\ v_{i+1} \mapsto v_i, \\ v_j \mapsto v_j, \text{ for } j \neq i, i+1, \end{cases}$$

$$\varphi_M(\rho_i):\begin{cases} x_i \mapsto x_{i+1}^{v_i^{-1}}, \\ x_{i+1} \mapsto \ x_i^{v_{i+1}}, \\ x_j \mapsto x_j, \text{ for } j \neq i, i+1, \end{cases} \qquad \varphi_M(\rho_i):\begin{cases} v_i \mapsto v_{i+1}, \\ v_{i+1} \mapsto v_i, \\ v_j \mapsto v_j, \text{ for } j \neq i, i+1. \end{cases}$$

Proposition 1.3.1. *The representation $\varphi_M : VB_n \to \mathrm{Aut}(F_{n,n})$ is equivalent to the virtually symmetric representation $\varphi_S : VB_n \to \mathrm{Aut}(F_{n,n})$, which is defined by its action on generators as follows*

$$\varphi_S(\sigma_i):\begin{cases} x_i \mapsto x_i \ x_{i+1}^{v_i} \ x_i^{-1}, \\ x_{i+1} \mapsto x_i^{v_{i+1}^{-1}}, \end{cases} \qquad \varphi_S(\sigma_i):\begin{cases} v_i \mapsto v_{i+1}, \\ v_{i+1} \mapsto v_i, \end{cases}$$

$$\varphi_S(\rho_i):\begin{cases} x_i \mapsto x_{i+1}, \\ x_{i+1} \mapsto x_i, \end{cases} \qquad \varphi_S(\rho_i):\begin{cases} v_i \mapsto v_{i+1}, \\ v_{i+1} \mapsto v_i. \end{cases}$$

Apart from this, we show that the generalized Artin representation [8, 119], the Silver-Williams representation [12, 113] and the Boden-Dies representation [26] are equivalent to virtually symmetric representations. We also define extensions of Wada representations [119], and prove that they too are equivalent to virtually symmetric representations.

Bartholomew and Fenn [20, Section 7] considered a linear, local and homogeneous representation $\varphi : VB_n \to GL_n(\mathbb{Z}[t^{\pm 1}, \lambda^{\pm 1}])$ defined on generators as

$$\sigma_i \mapsto I^{i-1} \oplus \begin{pmatrix} 1-t & \lambda^{-1} t \\ \lambda & 0 \end{pmatrix} \oplus I^{n-i-1},$$
$$\rho_i \mapsto I^{i-1} \oplus \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus I^{n-i-1}.$$

Clearly, the representation $\varphi : VB_n \to GL_n(\mathbb{Z}[t^{\pm 1}, \lambda^{\pm 1}])$ is virtually symmetric. We construct a linear, local and non-homogeneous representation of VB_n.

Proposition 1.3.2. *The map $\psi : VB_n \to GL_n(\mathbb{Z}[t^{\pm 1}, \lambda^{\pm 1}, t_1^{\pm 1}, t_2^{\pm 1}, \ldots, t_{n-1}^{\pm 1}])$ defined on generators by*

$$\psi(\sigma_i) = I^{i-1} \oplus \begin{pmatrix} 1-t & t t_i \lambda^{-1} \\ \lambda t_i^{-1} & 0 \end{pmatrix} \oplus I^{n-i-1} \ \textit{for } i = 1, 2, \ldots, n-1,$$

$$\psi(\rho_i) = I^{i-1} \oplus \begin{pmatrix} 0 & t_i \\ t_i^{-1} & 0 \end{pmatrix} \oplus I^{n-i-1} \textit{ for } i = 1,2,\ldots,n-1,$$

is a representation of VB_n and is equivalent to a virtually symmetric representation which is local and homogeneous.

1.3.2 Virtual link groups

Let $\varphi : VB_n \to \mathrm{Aut}(H)$ be a representation of the virtual braid group VB_n into the automorphism group of some group (module) $H = \langle h_1, h_2, \ldots, h_m \mid \mathcal{R}\rangle$. For a given $\beta \in VB_n$, we associate the group

$$G_\varphi(\beta) = \langle h_1, h_2, \ldots, h_m \mid \mathcal{R}, h_i = \varphi(\beta)(h_i), i = 1,2,\ldots,m\rangle.$$

For each $\beta \in VB_n$, let $G_M(\beta)$ and $G_S(\beta)$ be groups corresponding to representations $\varphi_M : VB_n \to \mathrm{Aut}(F_{n,n})$ and $\varphi_S : VB_n \to \mathrm{Aut}(F_{n,n})$, respectively.

Theorem 1.3.3. *$G_M(\beta) \cong G_S(\beta)$ for each virtual braid β. Further, if β and β' are two virtual braids whose closures give the same virtual link L, then $G_S(\beta) \cong G_S(\beta')$.*

Thus, the group $G_S(\beta)$ is a virtual link invariant. Let L be a virtual link and $D(L)$ a virtual link diagram representing L. Then we also associate a group $G_S(D(L))$ to $D(L)$ and prove the following result.

Proposition 1.3.4. *If $D(L)$ and $D'(L)$ are two diagrams representing a virtual link L, then $G_S(D(L)) \cong G_S(D'(L))$. Further, if β is a braid whose closure is equivalent to L, then $G_S(D(L)) \cong G_S(\beta)$.*

By putting relations $v_i = 1$ for all i in the presentation of $G_S(\beta)$, we recover the group defined by Kauffman [79]. Henceforth, we denote $G_S(D(L))$ by $G_S(L)$.

1.3.3 Marked Gauss diagrams

A *Gauss diagram* is a collection of a finite number of circles oriented anticlockwise with finite number of signed arrows whose heads and tails lie on circles. Gauss diagrams are considered up to the equivalence relation generated by finite sequences of moves shown in Figure 1.5, where $\varepsilon = \pm 1$.
To each Gauss diagram D, we associate a group π_D and prove the following result.

Proposition 1.3.5. *If D is a Gauss diagram representing virtual link L, then $\pi_D \cong G_S(L)$.*

For a given virtual link L, the group $G_S(L)$ is called the virtual link group of L.
We define a *marked Gauss diagram* as a collection of a finite number of circles oriented anticlockwise having finite number of signed arrows whose heads and tails lie on circles along with a finite number of signed nodes lying on circles which are not attached to arrows.

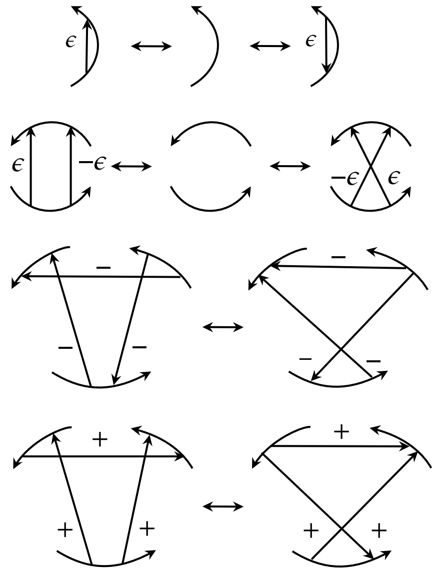


Fig. 1.5 Reidemeister moves on Gauss diagrams.

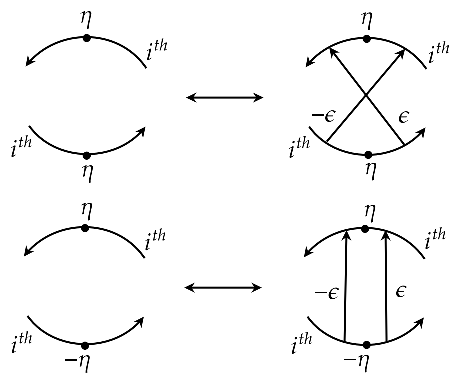


Fig. 1.6 Additional moves on marked Gauss diagrams.

We study them up to the equivalence relation generated by finite sequences of moves shown in figures 1.5 and 1.6, where ε and η can take values ± 1.
It is clear that marked Gauss diagrams are proper generalization of Gauss diagrams. To each marked Gauss diagram D, we associate a group Π_D and show that if D is a Gauss diagram,

then $\Pi_D \cong \pi_D$.

For a positive negative integer m, a group G is called a *C_m-group* if it can be defined by a set of generators $Y = X \sqcup V_m$, where $X = \{x_1, x_2, \ldots, x_n\}$, $V_m = \{v_1, v_2, \ldots, v_m\}$ and a set of relations $\mathcal{R}$ given by

$$\begin{aligned} w_{i,j}^{-1} x_i w_{i,j} = x_j, &\ \text{ for some } x_i, x_j \in X \text{ and some words } \ w_{i,j} \text{ in } Y^{\pm 1}, \\ v_i v_j = v_j v_i, &\ \text{ for all } v_i, v_j \in V_m. \end{aligned}$$

We call the presentation $\langle Y \mid \mathcal{R} \rangle$ as a C_m-presentation. A C_m-group is said to be *irreducible* if its abelianization is of rank $2m$.

Proposition 1.3.6. *The group associated to a given* 1*-circle marked Gauss diagram is an irreducible C_1-group of deficiency* 1 *or* 2 *and its second integral homology group is cyclic.*

Theorem 1.3.7. *Any irreducible C_1-presentation of deficiency* 1 *or* 2 *can be realized as the group of a marked Gauss diagram.*

Using the group Π_D for a marked Gauss diagram D, we define the notion of a meridian, a longitude, a peripheral subgroup and the peripheral structure combinatorially. In particular, we define these notions for virtual links too.

Proposition 1.3.8. *The peripheral pair and the peripheral subgroup of a marked Gauss diagram are unique up to conjugacy. Moreover, the peripheral structure is invariant under the marked Reidemeister moves.*

In classical knot theory, it is well-known that the longitude is trivial if and only if the knot is trivial. In contrast to this, we have the following result.

Theorem 1.3.9. *Let G be a group with an irreducible C_1-presentation of deficiency* 2*. Then G is the group of a marked Gauss diagram with trivial longitude. In particular, if K is a classical knot, then $G_S(K)$ is the group of some marked Gauss diagram with a trivial longitude.*

The thesis is organized as follows. In Chapter 2, we develop necessary background required for the subsequent chapters. In Chapter 3, we study residual finiteness of quandles. We establish residual finiteness of quandles arising from residually finite groups. We investigate residual finiteness of automorphism groups of some residually finite quandles. We then prove that the word problem is solvable for finitely presented residually finite quandles. We also study residual finiteness of free quandles and free product of residually finite quandles, and conclude by proving that all link quandles are residually finite. In Chapter 4, we develop a general theory of orderability of quandles with a focus on link quandles and give some general constructions of orderable quandles. We prove that knot quandles of many fibered prime knots are right-orderable, whereas link quandles of many non-trivial torus links are not right-orderable. As a consequence, we deduce that the knot quandle of the trefoil is neither left nor right orderable. Further, we prove that link quandles of certain non-trivial positive (or negative) links are not bi-orderable, which includes some alternating knots of prime determinant and alternating Montesinos links. We also explore interconnections between orderability of quandles and that of their enveloping groups. The results show that orderability of link quandles behave quite differently than that of corresponding link groups. Chapter 5 is of a different flavour. We define virtually symmetric representations of virtual braid groups, and prove that most of the representations known in the literature are equivalent to virtually symmetric representations. Using one such representation, we define a virtual link group which is an extension of the virtual link group defined by Kauffman. Moreover, we introduce the concept of marked Gauss diagrams as a generalization of Gauss diagrams. We extend our definition of virtual link group to marked Gauss diagrams, and define the peripheral structure of marked Gauss diagrams. We prove that every group with an irreducible C_1-presentation of deficiency 1 or 2 can be realized as the group of a marked Gauss diagram.

Chapter 2

Preliminaries

In this chapter, we recall some basic notions and results which will be required in the subsequent chapters. The results stated in this chapter can be found in [25, 31, 74, 77, 92].

2.1 Classical knot theory

A *knot* is the image of an embedding of a circle $\mathbb{S}^1$ into the 3-sphere $\mathbb{S}^3$. If the embedding is smooth, then the knot is said to be a *tame* knot. Throughout the thesis, knots are considered to be tame. A knot is said to be *oriented* if there is a preferred direction of motion along the string. Two knots are considered to be equivalent if one can be transformed into the other by deforming the space $\mathbb{S}^3$. More precisely, we have the following definition.

Definition 2.1.1. Let K_1 and K_2 be two knots. We say that K_1 is *equivalent* to K_2 if there exists an ambient isotopy $H : \mathbb{S}^3 \times [0,1] \to \mathbb{S}^3$ such that $H(K_1,0) = K_1$ and $H(K_1,1) = K_2$.

Clearly, an ambient isotopy of knots preserves the orientation of the ambient space. For equivalence of oriented knots, we require that the ambient isotopy must preserve orientations of knots.

A *link* is a collection of disjoint union of finitely many knots. Each knot in a link is termed as a *component*. Thus, a knot is a link with one component. Equivalence of links can be defined in the same manner as that of knots.

It is easy to note that each link can be projected on the plane $\mathbb{R}^2$ or on the 2-sphere $\mathbb{S}^2$. A projection is said to be *generic* if there are only finitely many multiple points, and that the multiple points are only transversal double points.

Definition 2.1.2. A *link diagram* is a generic projection of a link with the information of over- and under-crossing arcs at the double points.

It is easy to see that such a diagram always exists, see, for example [39, p.7].

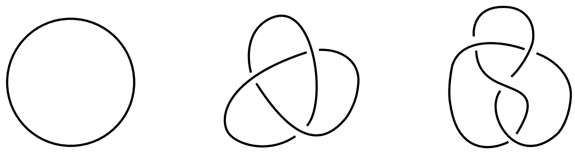


Fig. 2.1 Examples of knot diagrams.

For an oriented link L, let $r(L)$ be the *reverse* of the link L, that is, the link obtained from L by reversing the orientation of each component of L. Further, let $m(L)$ be the *mirror image* of L, that is, the link obtained from L by reflecting it across some plane. A link L is said to be *invertible* if it is isotopic to the link $r(L)$. A link is *alternating* if it has a link diagram such that the double points alternate between under- and over-crossings.
In an oriented link diagram, the crossings are distinguished as *positive* $(+)$ or *negative*$(-)$, as depicted in Figure 2.2.

Fig. 2.2 Positive and negative crossings.

In 1920s, Reidemeister [106] showed that the study of equivalence classes of links in $\mathbb{S}^3$ is equivalent to the study of link diagrams on the plane modulo three local moves known as the *Reidemeister moves* (see Figure 2.3).

Theorem 2.1.3. [106] *Two links are equivalent if and only if their link diagrams are related by a finite sequence of Reidemeister moves and planar isotopies* (*orientation preserving homeomorphisms of plane onto itself*).

The above interpretation of links in terms of their diagrams is one of the most important results in knot theory which has lead to the study of links from a combinatorial perspective. As a result, various invariants have been constructed for the classifications of knots.

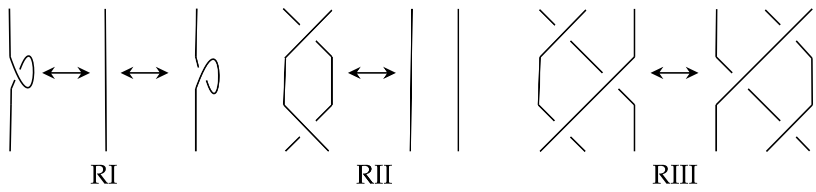


Fig. 2.3 Reidemeister moves for link diagrams.

Definition 2.1.4. Let K_1 and K_2 be two oriented knots. Then the *connected sum* of K_1 and K_2, denoted by $K_1\#K_2$, is defined as shown in Figure 2.4.

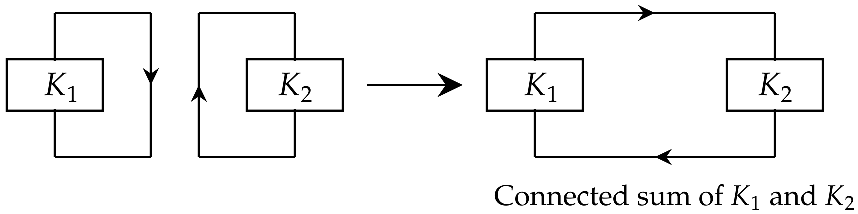


Fig. 2.4 Connected sum of oriented knots.

One can think of defining connected sum of two oriented links, but it is not well-defined up to ambient isotopy. For our purpose, we define the following.

Definition 2.1.5. A connected sum of two oriented links is defined by fixing a component in each of the links and then taking the connected sum of these components as defined for knots.

Definition 2.1.6. A knot is said to be *prime* if it cannot be written as a connected sum of two non-trivial knots. For example, torus knots are prime.

Definition 2.1.7. A crossing in a link diagram is said to be *reducible* or *nugatory* if there exists a circle in the projection plane meeting the diagram only at that crossing, transversely. For example, see Figure 2.5.

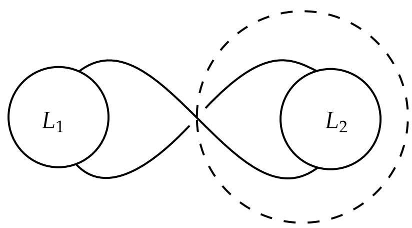


Fig. 2.5 Reducible or nugatory crossing.

Definition 2.1.8. A link diagram with no nugatory crossings is called a *reduced link diagram*.

2.2 Virtual knot theory

Virtual knot theory was introduced by Kauffman [79] and rediscovered around the same time by Goussarov, Polyak and Viro [60]. Topologically speaking, it is the study of smooth embeddings of circles in thickened compact oriented surfaces up to ambient isotopy and (de)stabilization process [33, 72]. Virtual knot theory has been proved to be a proper generalization of classical knot theory by Kauffman [79, Theorem 2] through a combinatorial approach, and by Goussarov-Polyak-Viro [60, Theorem 1.B] using algebraic methods. Later on, Kuperberg [85, Theorem 1] proved the same in a more topological sense. We give a brief description of all the three approaches.

First approach

This approach is due to Kauffman [79]. It is well-known that not every 4-valent graph can be embedded in the plane. Every link diagram can be seen as a planar 4-valent graph (including disjoint circles in the plane) with vertices carrying the information of which strand passes over and which strand passes under. A *virtual link diagram* is a generic immersion of a finite 4-valent graph into the Euclidean plane $\mathbb{R}^2$ or the two sphere $\mathbb{S}^2$, where the image of vertices are enhanced with over- and under-crossing information and the other double points are decorated with a small circle, known as *virtual crossings*. By a *generic* immersion, we mean that the multiple points are transverse double points and there are only finitely many of them.
Figure 2.6 illustrates an example of a virtual knot diagram.

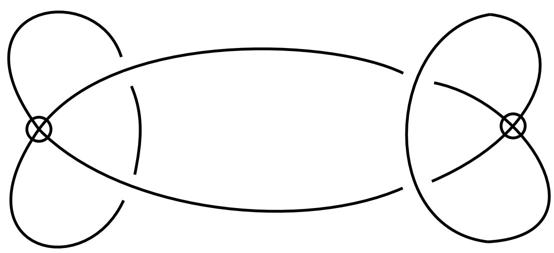


Fig. 2.6 Kishino knot

Two virtual link diagrams are said to be *equivalent* if they are related by a finite sequence of planar isotopies and generalized Reidemeister moves as shown in Figure 2.7. The moves

VRI−VRIII are collectively known as *virtual Reidemeister moves*, whereas the move VRIV is called a *semi-virtual move*.

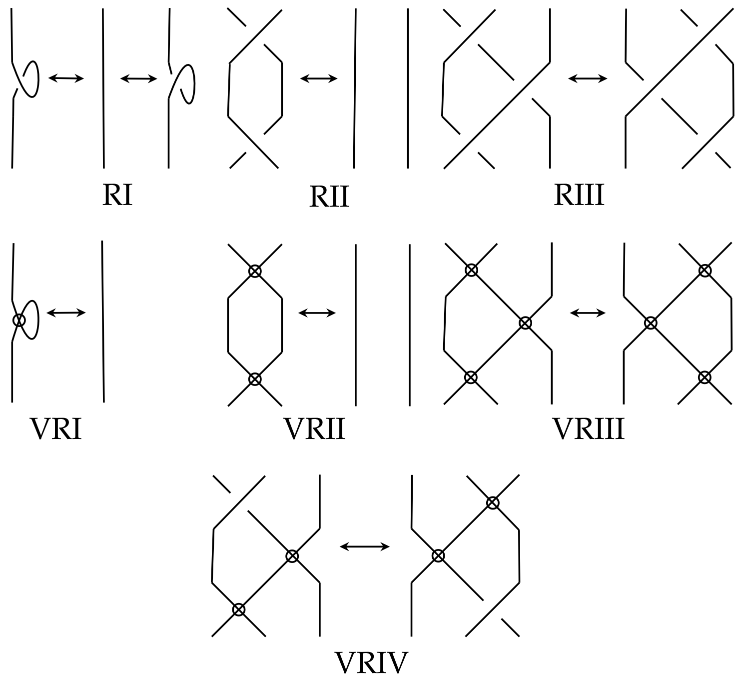


Fig. 2.7 Generalized Reidemeister moves for virtual link diagrams.

Similar to the semi-virtual move, one can think of two more moves as shown in Figure 2.8. These two moves are known as *forbidden moves*. It can be proved that all virtual knot diagrams are equivalent if one allows forbidden moves along with the generalized Reidemeister moves. See [76, 100] for more details.

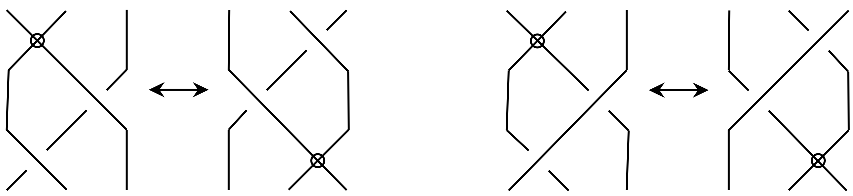


Fig. 2.8 Forbidden moves for virtual link diagrams.

Second approach

A *Gauss diagram* consists of a finite number of circles oriented anticlockwise with a finite number of signed arrows whose heads and tails lie on circles. If head and tail of an arrow lie on the same circle, then it is said to be a *chord*.

For every oriented link diagram, one can construct a Gauss diagram as follows. We first parametrize each link component with a starting point (■) and label each crossing. Then we consider disjoint anticlockwise oriented circles on a plane which are in one-to-one correspondence with link components, and each circle is parametrized with a starting point depicted by (■). We then note down the sequence of labellings along with the over and under information in each link diagram from starting points on the corresponding set of circles. We then join the preimage of each double point in the circles with arrows oriented from over-crossing to under-crossing, and assign the sign of crossing to the corresponding arrow. The process is illustrated in Figure 2.9.

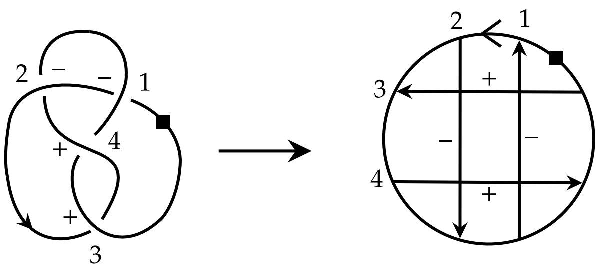


Fig. 2.9 A Gauss diagram associated to the figure eight knot.

Gauss diagrams are another way of studying links from a combinatorial point of view. It turns out that, there are Gauss diagrams which do not represent any link diagram. For example, see the Gauss diagram in Figure 2.10 and [60].

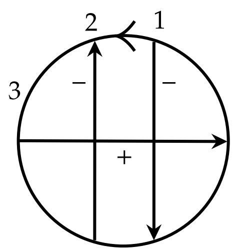


Fig. 2.10 A Gauss diagram not corresponding to any knot diagram.

One can associate a Gauss diagram to any virtual link diagram in the same way as done in the preceding paragraphs by ignoring the virtual crossings. Goussarov, Polyak and Viro [60] considered Gauss diagrams up to the equivalence relation generated by finite sequences of abstract Reidemeister moves for Gauss diagrams as shown in Figure 2.11.

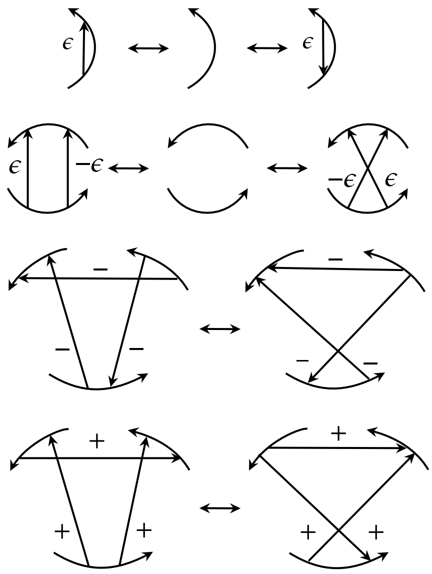


Fig. 2.11 Reidemeister moves on Gauss diagrams.

It is easy to notice that the virtual Reidemeister moves and semi-virtual move have no affect on Gauss diagrams. In particular, we have the following.

Theorem 2.2.1. [60] *A Gauss diagram defines a virtual link diagram up to virtual Reidemeister moves and semi-virtual move.*

As a consequence, a virtual link can be defined as the equivalence class of a Gauss diagram.

Third approach

Let Σ be a compact oriented surface. A *link* in $\Sigma \times I$ is the image of a smooth embedding $l : \sqcup_n \mathbb{S}^1 \to \mathrm{int}(\Sigma \times I)$ of disjoint union of circles $\sqcup_n \mathbb{S}^1$ in the interior of thickened surface $\Sigma \times I$, where I is the unit interval $[0,1]$. Two links l_1 and l_2 in $\Sigma \times I$ are *equivalent* if there exists an ambient isotopy $H : (\Sigma \times I) \times I \to \Sigma \times I$ such that $H(l_1, 0) = l_1, H(l_1, 1) = l_2$ and $H(x,t) = x$ for all $x \in \partial(\Sigma \times I)$ and $t \in I$.
Let l be a knot in $\Sigma \times I$ and $D(l)$ a generic projection of l on $\Sigma \times \{0\}$. Suppose that D_1 and D_2 are two disjoint discs in $(\Sigma \times \{0\}) \setminus D(l)$. A *stabilization* of l in $\Sigma \times I$ is attaching an oriented handle (oriented cylinder) in place of D_1 and D_2 to form a new surface $\Sigma^{'}$. The inverse of this operation is known as *destabilization*. As a result, we obtain a new link $l^{'}$ in the thickened surface $\Sigma^{'} \times I$. Two links $l_1 \subset \Sigma_1 \times I$ and $l_2 \subset \Sigma_2 \times I$ are said to be *stably equivalent* if they are related by a finite sequence of ambient isotopies, orientation preserving homeomorphisms of surfaces and (de)stabilizations. There is a one-to-one correspondence

between stable equivalence classes of links in thickened surfaces and virtual links. For a proof and more details, see [33, 72, 79, 92]. The following result of Kuperberg [85, Theorem 1] builds a bridge between the combinatorial interpretation of virtual knot theory and its topological counterpart.

Theorem 2.2.2. [85, Theorem 1] *Every virtual link has a unique representative as an irreducible link in a thickened compact orientable surface up to orientation preserving homeomorphism.*

By an *irreducible* link, we mean that no destabilization is possible. Considering classical links as links in the thickened 2-sphere $\mathbb{S}^2 \times I$, the above result implies that any two classical links which are equivalent under generalized Reidemeister moves are equivalent under Reidemeister moves. Therefore, classical knot theory is a part of virtual knot theory.

2.3 Classical braids

A *geometric braid* on n strands is a subset β of $\mathbb{R}^2 \times I$ consisting of n disjoint closed intervals such that following conditions are satisfied:

(1) $\beta \cap (\mathbb{R}^2 \times \{0\}) = \{(1,0,0),(2,0,0),\dots,(n,0,0)\}$,

(2) $\beta \cap (\mathbb{R}^2 \times \{1\}) = \{(1,0,1),(2,0,1),\dots,(n,0,1)\}$,

(3) each strand of β intersects $\mathbb{R}^2 \times \{t\}$ on a point for all $t \in [0,1]$.

Two geometric braids β_1 and β_2 are said to be *isotopic* if there exists an ambient isotopy

$$H : (\mathbb{R}^2 \times I) \times I \to \mathbb{R}^2 \times I$$

such that $H(\beta_1, 0) = \beta_1$, $H(\beta_1, 1) = \beta_2$ and $H(\beta_1, t)$ is a geometric braid at each time t. Also, $H(x,t) = x$ for all $x \in \partial(\mathbb{R}^2 \times I)$ and for all $t \in [0,1]$.
Clearly, isotopy induces an equivalence relation on the set of geometric braids on n strands. These equivalence classes are called *braids*.
As in case of links, geometric braids can be studied via diagrams on the plane.
A *braid diagram* on n strands is a subset of $\mathbb{R} \times I$ consisting of n monotone smooth arcs starting from $(1,1),(2,1),\dots,(n,1)$ and ending at $(1,0),(2,0),\dots,(n,0)$ in an arbitrary order. The arcs are allowed to cross each other as classical crossings. Figure 2.12 illustrates a braid diagram with 3 strands.

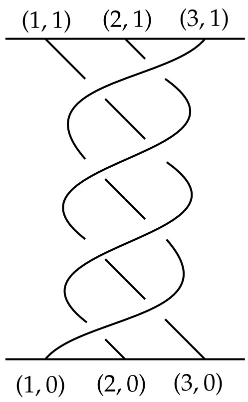


Fig. 2.12 A braid diagram on 3 strands.

Two braid diagrams D_1 and D_2 on n strands are said to be *planar isotopic* if there exists an ambient isotopy

$$H : (\mathbb{R} \times I) \times I \to \mathbb{R} \times I$$

such that $H(D_1, 0) = D_1$, $H(D_1, 1) = D_2$ and $H(D_1, t)$ is a braid diagram at each t. Two braid diagrams are said to be *equivalent* if they are related by a finite sequence of planar isotopies and local moves shown in Figure 2.13.

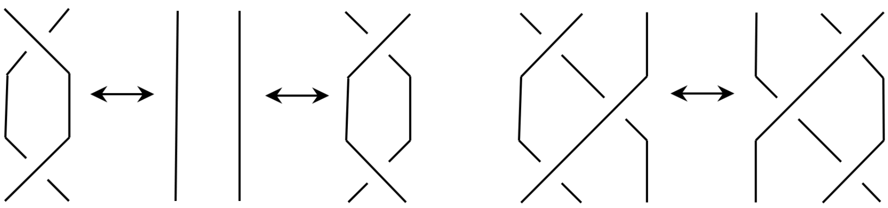


Fig. 2.13 Local moves on braid diagrams.

Evidently, two geometric braids are equivalent if and only if their braid diagrams are equivalent. We also refer the equivalence classes of braid diagrams as *braids*. Let β_1 and β_2 be two braids represented by diagrams D_1 and D_2, respectively. Then the product $\beta_1\beta_2$ can be defined by placing D_1 over D_2 and shrinking the interval to $[0, 1]$ as shown in Figure 2.14. We note that it is a well-defined operation. It is easy to notice that after small perturbations each braid diagram on n strands can be decomposed into elementary braid diagrams σ_i and σ_i^{-1} as shown in Figure 2.15, where $1 \leq i \leq n-1$. Thus, the set of equivalence classes of braid diagrams on n strands forms a group under this operation, and is isomorphic to the braid group B_n defined below.

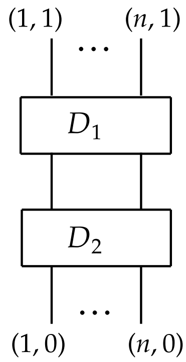


Fig. 2.14 The braid diagram D_1D_2.

Definition 2.3.1. The braid group B_n is the group with a presentation having $n-1$ generators $\sigma_1,\dots,\sigma_{n-1}$ and following set of relations:

(B1) $\sigma_i\sigma_j = \sigma_j\sigma_i$ for $|i-j|\geq 2$ and $i,j\in\{1,\dots,n-1\}$,

(B2) $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$ for $i\in\{1,\dots,n-2\}$.

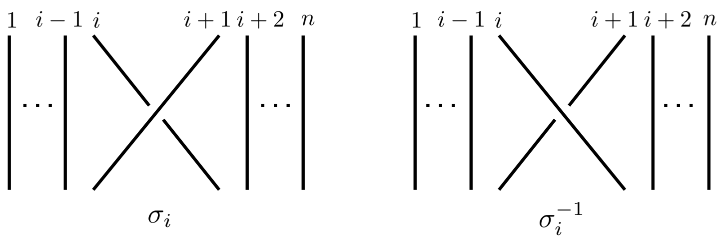


Fig. 2.15 Generators of the braid group B_n.

One can obtain a link diagram from a braid diagram. By *closure* of a braid diagram D, we mean a diagram obtained by connecting the boundary points of D having the same second coordinate with smooth non-intersecting arcs. Obviously, closure of a braid is a well-defined operation as closures of any two equivalent braid diagrams give equivalent link diagrams. From now onwards, we will denote the closure of a braid β by $Cl(\beta)$. Figure 2.16 illustrates closure of a braid diagram given in Figure 2.12.
The following folklore theorem relates braids with oriented links in the Euclidean 3-space.

Theorem 2.3.2. (Alexander Theorem [77, Theorem 2.3]) *For any oriented link L, there exists a braid β whose closure $Cl(\beta)$ is equivalent to L.*

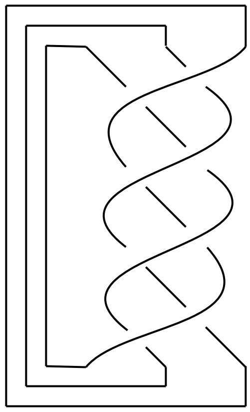


Fig. 2.16 Closure of a braid diagram.

It should be noted that closures of braids on different number of strands can give equivalent links. The following theorem characterizes braids whose closures are equivalent.

Theorem 2.3.3. (Markov Theorem [77, Theorem 2.8]) *Closures of two braid diagrams are equivalent as links if and only if they are related by a finite sequence of the following moves:*

(1) braid equivalence,

(2) conjugation in the braid group,

(3) right stabilization and destabilization (*see Figure* 2.17).

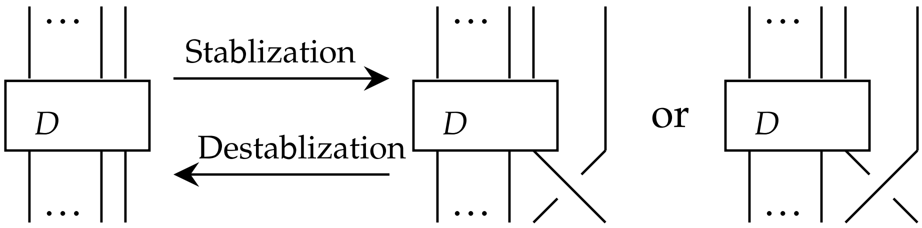


Fig. 2.17

2.4 Virtual braids

The notion of braid diagrams was generalized to virtual braid diagrams by Kauffman [79]. A *virtual braid diagram* on n strands is a subset of $\mathbb{R} \times I$ consisting of n monotone smooth arcs each homeomorphic to the unit interval starting from $(1,1),(2,1),\ldots,(n,1)$ and ending at $(1,0),(2,0),\ldots,(n,0)$ in an arbitrary order. The arcs are allowed to cross each other as classical crossings or virtual crossings.

Two virtual braid diagrams D_1 and D_2 on n strands are said to be *planar isotopic* if there exists an ambient isotopy $H : (\mathbb{R}\times I)\times I \to \mathbb{R}\times I$ such that $H(D_1,0) = D_1$, $H(D_1,1) = D_2$ and $H(D_1,t)$ is a virtual braid diagram at each t.

Two virtual braid diagrams are said to be *equivalent* if they are related by a finite sequence of planar isotopies and local moves shown in Figure 2.18. The equivalence class of a virtual braid diagram is known as a *virtual braid*.

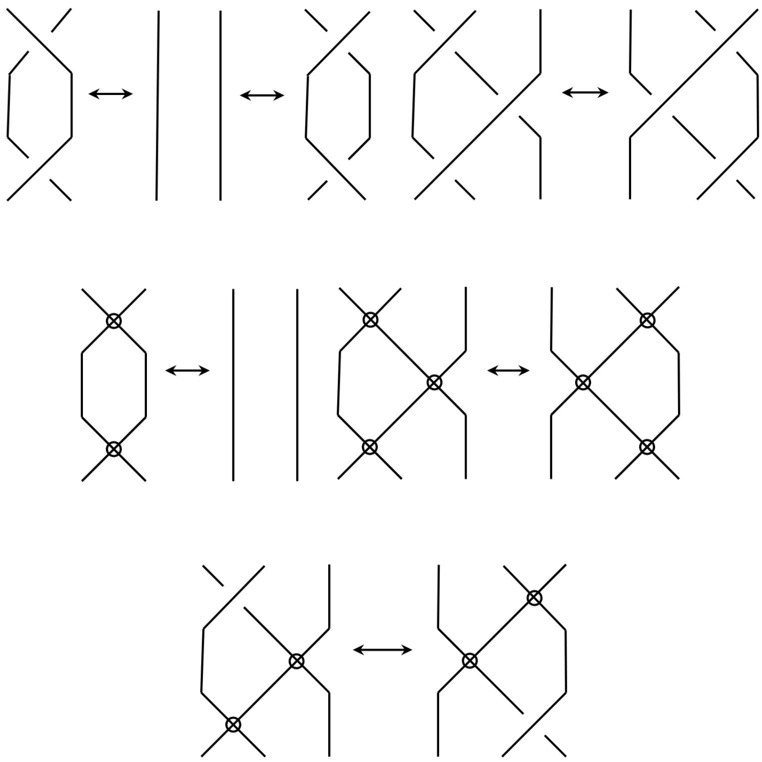


Fig. 2.18 Local moves on virtual braid diagrams.

As in case of classical braids, the set of virtual braids on n strands forms a group under the operation of juxtaposition of one virtual braid over the other and squeezing them vertically to the unit interval. This group is known as the *virtual braid group on n strands*, and is denoted by VB_n. Any virtual braid diagram on n strands can be decomposed into elementary virtual braid diagrams on n strands shown in Figure 2.19.

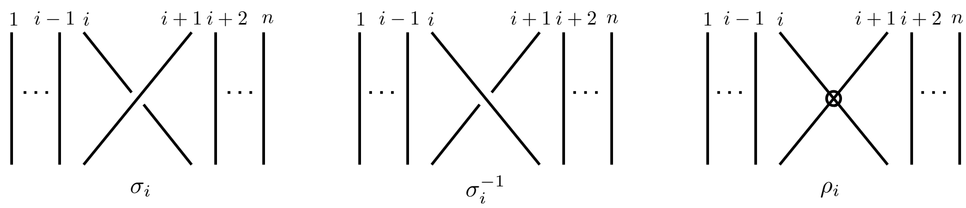


Fig. 2.19 Generators of the virtual braid group VB_n.

Thus, the virtual braid group can be defined algebraically as follows.

Definition 2.4.1. The n-strand virtual braid group VB_n is the group with a presentation having generators $\sigma_1, \dots, \sigma_{n-1}$, $\rho_1, \dots, \rho_{n-1}$ and following set of relations:

- relations of the braid group on n strands:

(B1) $\sigma_i \sigma_j = \sigma_j \sigma_i$ for $|i-j| \geq 2$ and $i, j \in \{1, \dots, n-1\}$,

(B2) $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$ for $i \in \{1, \dots, n-2\}$,

- relations of the symmetric group:

(S1) $\rho_i^2 = 1$ for $i \in \{1, 2, \dots, n-1\}$,

(S2) $\rho_i \rho_j = \rho_j \rho_i$ where $|i-j| \geq 2$ for $i, j \in \{1, 2, \dots, n-1\}$,

(S3) $\rho_i \rho_{i+1} \rho_i = \rho_{i+1} \rho_i \rho_{i+1}$ for $i \in \{1, 2, \dots, n-2\}$,

- mixed relations:

(M1) $\sigma_i \rho_j = \rho_j \sigma_i$ where $|i-j| \geq 2$ for $i, j \in \{1, 2, \dots, n-1\}$,

(M2) $\rho_i \rho_{i+1} \sigma_i = \sigma_{i+1} \rho_i \rho_{i+1}$ for $i \in \{1, 2, \dots, n-2\}$.

The *closure* of a virtual braid diagram can be defined analogously as in the case of classical braids. It is not difficult to check that closures of equivalent virtual braid diagrams are equivalent as virtual link diagrams. Kauffman and Lambropoulou established Alexander [81] and Markov theorems [78] for oriented virtual links. An independent approach to Alexander and Markov theorems was given by Kamada in [73].

Theorem 2.4.2. (Alexander Theorem) *Let L be an oriented virtual link. Then there exists a virtual braid diagram whose closure is equivalent to L.*

Theorem 2.4.3. (Markov Theorem) *Closures of two virtual braid diagrams are equivalent as virtual links if and only if they are related by a finite sequence of following moves:*

(1) virtual braid equivalence,

(2) conjugation in the virtual braid group,

(3) right stabilization and destabilization of real and virtual type (*see Figure* 2.20)*,*

(4) right/left virtual exchange move (*see Figure* 2.21)*.*

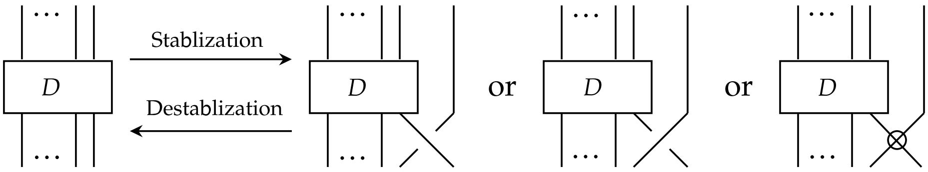


Fig. 2.20

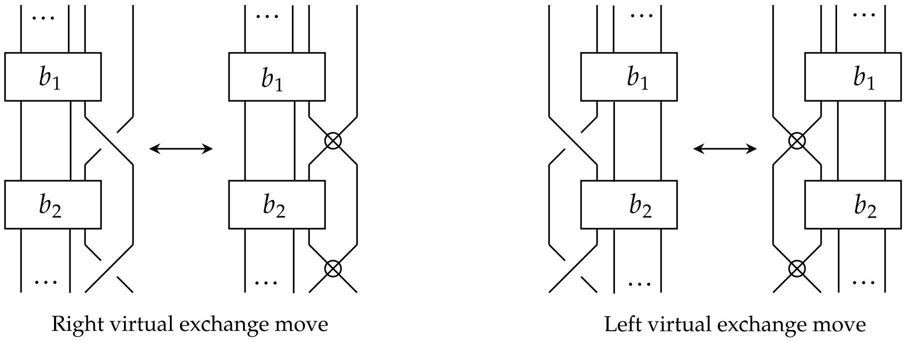


Fig. 2.21

2.5 Quandle theory

Next, we introduce main objects of our study.

Definition 2.5.1. A *quandle* is a non-empty set X endowed with a binary operation $*: X \times X \to X$ satisfying the following axioms:

(Q1) *idempotency*: $x * x = x$ for all $x \in X$,

(Q2) *existence of left inverse*: for each $x, y \in X$, there exists a unique $z \in X$ such that $x = z * y$,

(Q3) *right self-distributivity*: $(x * y) * z = (x * z) * (y * z)$ for all $x, y, z \in X$.

The axiom (Q2) is equivalent to the existence of a dual binary operation $*^{-1}$ on X such that

$$x *^{-1} y = z \text{ if and only if } x = z * y$$

for all $x, y, z \in X$. A subset S of Q is called a *subquandle* of $(Q, *)$ if S is a quandle under the same binary operation $*$. A non-empty set with a binary operation satisfying axioms (Q2) and (Q3) is known as *rack*. Thus, a quandle is an idempotent rack.

Example 2.5.2. Following are some examples of quandles:

(1) Let X be a non-empty set. Then the binary operation $x * y := x$ defines a quandle structure on X, known as a *trivial* quandle.

(2) If G is a group and $n \in \mathbb{Z}$, then the binary operation $x * y := y^{-n} x y^{n}$ turns G into the quandle $\mathrm{Conj}_n(G)$ called the *n-conjugation quandle* of G. For $n = 1$, the quandle is simply denoted by $\mathrm{Conj}(G)$.

(3) A group G with the binary operation $x * y := y x^{-1} y$ turns G into the quandle $\mathrm{Core}(G)$ called the *core quandle* of G. In particular, if G is a cyclic group of order n, then it is called the *dihedral quandle* and is denoted by R_n. Usually, one writes $\mathrm{R}_n = \{0, 1, \ldots, n-1\}$ with $i * j = 2j - i \mod n$.

(4) If G is a group and $\phi \in \mathrm{Aut}(G)$, then G with the binary operation $x * y := \phi\left(xy^{-1}\right) y$ forms a quandle $\mathrm{Alex}(G, \phi)$ referred as the *generalized Alexander quandle* of G with respect to ϕ. In particular, if G is an abelian group, then $\mathrm{Alex}(G, \phi)$ is known as the *Alexander quandle* of G with respect to ϕ.

(5) Let $\{z_i \mid i \in I\}$ be elements of a group G, and $\{H_i \mid i \in I\}$ subgroups of G such that $H_i \leq \mathrm{C}_G(z_i)$ for all $i \in I$. Then the disjoint union $\sqcup_{i \in I}(G, H_i, z_i)$ becomes a quandle with
$$H_i x * H_j y := H_i z_i^{-1} x y^{-1} z_j y.$$

Remark 2.5.3. The constructions in examples (2), (3) and (4) can be seen as functors from the category of groups to that of quandles. In fact, all these functors have appropriate left adjoint functors from the category of quandles to that of groups [7, 70].

Let X and Y be two quandles. A map $\phi : X \to Y$ is termed as *quandle homomorphism* if $\phi(x * y) = \phi(x) * \phi(y)$ for all $x, y \in X$. By axiom (Q2), we have $\phi(x *^{-1} y) = \phi(x) *^{-1} \phi(y)$ for all $x, y \in X$. We denote the group of all automorphisms of X by $\mathrm{Aut}(X)$. For a given element $x \in X$, the inner automorphism induced by x is a map $S_x : X \to X$ such that $S_x(y) = y * x$. By

axiom (Q3), S_x is an automorphism of X fixing x. The subgroup of $\mathrm{Aut}(X)$ generated by the set $\{S_x \mid x \in X\}$ is known as the *inner automorphism* group of X, and is denoted by $\mathrm{Inn}(X)$. Henceforth, the word *orbit* would correspond to an orbit in Q under the action of $\mathrm{Inn}(Q)$.
An *action* of a quandle Q on a quandle X is a quandle homomorphism

$$\phi : Q \to \mathrm{Conj}_{-1}\left(\mathrm{Aut}(X)\right),$$

where $\mathrm{Aut}(X)$ is the group of quandle automorphisms of X, and the operation in $\mathrm{Conj}_{-1}\left(\mathrm{Aut}(X)\right)$ is nothing but $x * y = yxy^{-1}$. Viewing any set X as a trivial quandle, we have $\mathrm{Aut}(X) = \mathrm{S}_X$, the symmetric group on X, and we obtain the definition of an action of a quandle Q on a set X.

Example 2.5.4. Some basic examples of quandle actions are:

(1) If Q is a quandle, then the map $\phi : Q \to \mathrm{Conj}_{-1}\left(\mathrm{Aut}(Q)\right)$ given by $q \mapsto S_q$ is a quandle homomorphism. Thus, every quandle acts on itself by inner automorphisms.

(2) Let G be a group acting on a set X. That is, there is a group homomorphism $\phi : G \to \mathrm{S}_X$. Viewing both G and S_X as conjugation quandles and observing that a group homomorphism is also a quandle homomorphism between corresponding conjugation quandles, it follows that the quandle $\mathrm{Conj}_{-1}(G)$ acts on the set X.

Definition 2.5.5. A quandle X is said to be

(1) *connected* if the inner automorphism group $\mathrm{Inn}(X)$ acts transitively on X. For example, the dihedral quandle R_{2n+1} is connected, whereas R_{2n} is not.

(2) *involutory* if for each $x \in X$, the inner automorphism S_x is an involution, that is, $(y * x) * x = y$ for all $x, y \in X$. For example, for any group G, the core quandle $\mathrm{Core}(G)$ is involutory, whereas $\mathrm{Conj}(F_n)$ is not involutory for a free group F_n of rank $n \geq 2$.

(3) *commutative* if $x * y = y * x$ for all $x, y \in X$. For example, R_3 is commutative but R_4 is not.

(4) *latin* if for all $x \in X$, the left multiplication map $L_x : X \to X$ defined as $L_x(y) := x * y$ is a bijection. The dihedral quandle R_3 is latin, whereas R_4 is not.

(5) *semi-latin* if for each $x \in X$, the left multiplication map $L_x : X \to X$ is injective. Each latin quandle is semi-latin, but the converse is not true, in general. For example, the quandle $\mathrm{Core}(\mathbb{Z})$ is semi-latin but not latin.

(6) *simple* if for any quandle Y, every quandle homomorphism $X \to Y$ is either injective or constant. Let G be a simple group. Then $\mathrm{Core}(G)$ is a simple quandle. On the other hand, the dihedral quandle R_{2n} is neither latin nor simple.

(7) *quasi-commutative* if for given $x, y \in X$, at least one of the following holds:

(i) $x * y = y * x$,

(ii) $x * y = y *^{-1} x$,

(iii) $x *^{-1} y = y * x$,

(iv) $x *^{-1} y = y *^{-1} x$.

Every commutative quandle is quasi-commutative. Consider group $(\mathbb{R}, +)$ and its automorphism $\phi : \mathbb{R} \to \mathbb{R}$ defined as $\phi(x) = 2x$. Then, the Alexander quandle $\mathrm{Alex}(\mathbb{R}, \phi_2)$ is quasi-commutative but not commutative. On the other hand, $\mathrm{Core}(\mathbb{Z})$ is not quasi-commutative.

If X is any quandle, then the *left association identity*

$$x *^{d} (y *^{e} z) = \Big(\big(x *^{-e} z \big) *^{d} y \Big) *^{e} z$$

holds for all $x, y, z \in X$ and $d, e \in \{-1, 1\}$. Henceforth, we write a left-associated product

$$((\cdots((a_0 *^{e_1} a_1) *^{e_2} a_2) *^{e_3} \cdots) *^{e_{n-1}} a_{n-1}) *^{e_n} a_n$$

simply as

$$a_0 *^{e_1} a_1 *^{e_2} \cdots *^{e_n} a_n.$$

A repeated use of left association identity gives the following result.

Lemma 2.5.6. [121, Lemma 4.4.8] *The product*

$$(a_0 *^{d_1} a_1 *^{d_2} \cdots *^{d_m} a_m) *^{e_0} (b_0 *^{e_1} b_1 *^{e_2} \cdots *^{e_n} b_n)$$

of two left-associated forms $a_0 *^{d_1} a_1 *^{d_2} \cdots *^{d_m} a_m$ and $b_0 *^{e_1} b_1 *^{e_2} \cdots *^{e_n} b_n$ in a quandle can again be written in a left-associated form as

$$a_0 *^{d_1} a_1 *^{d_2} \cdots *^{d_m} a_m *^{-e_n} b_n *^{-e_{n-1}} b_{n-1} *^{-e_{n-2}} \cdots *^{-e_1} b_1 *^{e_0} b_0 *^{e_1} b_1 *^{e_2} \cdots *^{e_n} b_n.$$

Thus, any product of elements of a quandle Q can be expressed in the canonical left-associated form $a_0 *^{e_1} a_1 *^{e_2} \cdots *^{e_n} a_n$, where $a_0 \neq a_1$, and for $i = 1, 2, \ldots, n-1$, $e_i = e_{i+1}$ whenever $a_i = a_{i+1}$.

Definition 2.5.7. A *free quandle* on a non-empty set X is a quandle $FQ(X)$ together with a map $\phi : X \to FQ(X)$ such that for any other map $\rho : X \to Q$, where Q is a quandle, there exists a unique quandle homomorphism $\bar{\rho} : FQ(X) \to Q$ such that the following diagram commutes

$$\begin{tikzcd} X \arrow[r, "\phi"] \arrow[dr, "\rho"'] & FQ(X) \arrow[d, "\bar{\rho}"] \\ & Q. \end{tikzcd}$$

A *free rack* can be defined analogously. It follows from the definition that the free (rack) quandle on a set X is unique up to isomorphism, and that every (rack) quandle is a quotient of a free (rack) quandle.

Construction of free racks and free quandles

We consider the construction of free racks given by Fenn and Rourke [47, p.351]. For a given set X, consider the free group $F(X)$ on X and define a binary operation $*$ on the set $X \times F(X)$ as follows

$$(x,v) * (y,w) := (x, vw^{-1}yw) \text{ for all } (x,v),(y,w) \in X \times F(X).$$

The algebraic system $FR(X) := (X \times F(X), *)$ is the free rack on the given set X. For given $(x,v),(y,w) \in FR(X)$, the inverse operation is given by

$$(x,v) *^{-1} (y,w) = (x, vw^{-1}y^{-1}w).$$

Kamada [75, Section 8.6] constructed the free quandle $FQ(X)$ on the set X as a quotient of the free rack $FR(X)$ modulo the equivalence relation generated by $(x,w) \sim (x,xw)$ for all $x \in X$ and $w \in F(X)$. For convenience, we denote elements of the free quandle $FQ(X)$ by (x,w).

Construction of (G,A)-racks/quandles

In a recent work of Bardakov and Nasybullov [5, Section 4], the construction of free racks and free quandles has been extended which are referred as (G,A)-racks/quandles. Interestingly, many well-known quandles can be viewed as (G,A)-quandles.
Consider a group G and its subset A. The set $A \times G$ can be seen as a rack under the following operation

$$(a,u) * (b,v) = (a, uv^{-1}bv) \text{ for } a,b \in A \text{ and } u,v \in G.$$

The rack defined above is known as (G,A)-*rack*, and is denoted by $R(G,A)$. Let $Q(G,A)$ be the quotient of the set $A \times G$ by the equivalence relation

$$(a, vu) \sim (a, u)$$

if and only if

$$v \in \mathrm{C}_G(a) = \{x \in G \mid xa = ax\}.$$

If $[(a,u)]$ is the equivalence class of (a,u) in $Q(G,A)$, then the set $Q(G,A)$ becomes a quandle under the operation

$$[(a,u)] * [(b,v)] = [(a, uv^{-1}bv)] \text{ for } a,b \in A \text{ and } u,v \in G.$$

This quandle is known as a (G,A)-*quandle*. It is easy to check that if A is the set of representatives of conjugacy classes of a group G, then $Q(G,A) \cong \mathrm{Conj}(G)$.
Hereafter, by (a,u) we mean $[(a,u)]$, which will be used in the subsequent chapters.

Presentation of a quandle

One can define quandles via presentations. The notion of a presentation of a quandle used in this thesis can be found in [5, Section 6]. Let X be a set of symbols. Consider the set $WQ(X)$ of words in $X \sqcup \{*, *^{-1}, (,)\}$, which is defined inductively as follows:

(1) $X \subset WQ(X)$,

(2) if $a,b \in WQ(X)$, then $(a)*(b), (a)*^{-1}(b) \in WQ(X)$.

Let S be a subset of $WQ(X) \times WQ(X)$ and

$$\mathcal{R} = \{s_1 = s_2 \mid (s_1, s_2) \in S\}$$

the set of formal equalities. For every $x,y,z \in WQ(X)$, define an equivalence relation $\sim$ on $WQ(X)$ given by the following rules:

(1) $(x) \sim (x)*(x)$,

(2) $(x) \sim ((x)*(y))*^{-1}(y) \sim ((x)*^{-1}(y))*(y)$,

(3) $((x)*(y))*(z) \sim ((x)*(z))*((y)*(z))$,

(4) if $w_1 = w_2 \in \mathcal{R}$ for some $w_1, w_2 \in WQ(X)$, then $w_1 \sim w_2$,

(5) for given $w \in WQ(X)$, if $u_1 \sim u_2$, then $(w)*(u_1) \sim (w)*(u_2)$, $(w)*^{-1}(u_1) \sim (w)*^{-1}(u_2)$, $(u_1)*(w) \sim (u_2)*(w)$ and $(u_1)*^{-1}(w) \sim (u_2)*^{-1}(w)$.

The quotient set $WQ(X)/\sim$ forms a quandle under the operation $*$ defined as $(x,y) \mapsto x*y$. The quandle $WQ(X)/\sim$ is said to be presented by a generating set X and relation set $\mathcal{R}$. By a quandle corresponding to the presentation $\langle X \mid \mathcal{R} \rangle$, we mean the quandle constructed as above. Clearly, every quandle Q has a presentation, where $X = Q$ and $\mathcal{R}$ the set of equalities $x*y=z$ in Q. Moreover, the free quandle on the set X can be presented as $\langle X \mid \mathcal{R} \rangle$, where $\mathcal{R}$ is the empty set.
A quandle Q is said to be *finitely generated* if it has a presentation $\langle X \mid \mathcal{R} \rangle$ such that X is a finite set. Moreover, if there exists a presentation $\langle X \mid \mathcal{R} \rangle$ of Q such that both X and $\mathcal{R}$ are finite sets, then Q is said to be *finitely presented* quandle.

Enveloping groups of quandles

The *enveloping group* $\mathrm{Env}(Q)$ of a quandle Q is defined as the group with the set of generators

$$\{e_x \mid x \in Q\}$$

and defining relations

$$e_{x*y} = e_y^{-1} e_x e_y$$

for all $x,y \in Q$. Taking the enveloping group of a quandle is a functor from the category of quandles to that of groups and is left adjoint of the functor of taking conjugation quandle of a group from the category of groups to that of quandles. There is a natural map

$$\eta : Q \to \mathrm{Env}(Q)$$

defined as

$$\eta(x) := e_x,$$

which is not injective, in general. The following result gives a presentation of the enveloping group $\mathrm{Env}(Q)$ for a given presentation $\langle X \mid \mathcal{R} \rangle$ of a quandle Q.

Theorem 2.5.8. [121, Theorem 5.1.7] *If Q is a quandle with a presentation $\langle X \mid \mathcal{R} \rangle$, then its enveloping group has a presentation $\langle e_x,\ x \in X \mid \overline{\mathcal{R}} \rangle$, where $\overline{\mathcal{R}}$ consists of relations in $\mathcal{R}$ with the expression $x*y$ replaced by $e_y^{-1} e_x e_y$ and the expression $x*^{-1} y$ replaced by $e_y e_x e_y^{-1}$.*

It is easy to note that any quandle homomorphism $f : Q \to P$ induces a group homomorphism $f_\# : \mathrm{Env}(Q) \to \mathrm{Env}(P)$ defined as

$$f_\#(e_x) := e_{f(x)},$$

where $x \in Q$.

Proposition 2.5.9. [75, Proposition 8.8.4] *For a quandle Q, the following holds:*

(1) The natural map $\eta : Q \to \mathrm{Conj}(\mathrm{Env}(Q))$ is a quandle homomorphism.

(2) For any group G and quandle homomorphism $f : Q \to \mathrm{Conj}(G)$, there exists a unique group homomorphism $\bar{f} : \mathrm{Env}(Q) \to G$ such that the following diagram commutes

$$\begin{tikzcd} Q \arrow[r, "\eta"] \arrow[d, "f"'] & \mathrm{Env}(Q) \arrow[d, "\bar{f}"] \\ \mathrm{Conj}(G) \arrow[r, "\mathrm{Id}"'] & G. \end{tikzcd}$$

Remark 2.5.10. A trivial quandle homomorphism $f : Q \to \{a\}$ induces a group homomorphism $f_\# : \mathrm{Env}(Q) \to \mathrm{Env}(\{a\}) \cong \mathbb{Z}$, where $f_\#(e_x) = 1$ for all $x \in Q$. Thus, under the natural map $\eta : Q \to \mathrm{Env}(Q)$ none of the elements of Q map to the identity of the enveloping group $\mathrm{Env}(Q)$.

For a given quandle Q, there is a group homomorphism

$$\psi_Q : \mathrm{Env}(Q) \to \mathrm{Inn}(Q)$$

defined as $\psi_Q(e_x) = S_x$, where $x \in Q$, $e_x \in \mathrm{Env}(Q)$ and $S_x \in \mathrm{Inn}(Q)$. It is easy to check that the kernel $\mathrm{Ker}(\psi_Q)$ of ψ_Q is contained in the centre of the enveloping group $\mathrm{Env}(Q)$, and hence gives rise to the central extension

$$1 \to \mathrm{Ker}(\psi_Q) \to \mathrm{Env}(Q) \to \mathrm{Inn}(Q) \to 1$$

of groups. Also, the homomorphism ψ_Q induces a right action of $\mathrm{Env}(Q)$ on Q by setting $x.e_y := x * y$.

2.6 Invariants of classical and virtual links

An *invariant* of (virtual) links is a function from the set of (virtual) links to a set S such that the function maps equivalent (virtual) links to the same element in S. The major thrust of the subject has been on finding computable invariants. See [32, 54, 69, 92, 102] for details.
A *tubular neighbourhood* $V(K)$ of a knot K is a smooth embedding $s : \mathbb{S}^1 \times \mathbb{D}^2 \to \mathbb{S}^3$ such that $s(\mathbb{S}^1 \times \{0\}) = K$. For a given knot K, the *knot exterior* is the closure of $\mathbb{S}^3 \setminus V(K)$ in $\mathbb{S}^3$, and is denoted by $C(K)$.
The knot exterior is an orientable compact 3-manifold with torus $\mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1$ as its boundary. Similarly, the *link exterior* $C(L)$ for a t-component link $L = K_1 \sqcup K_2 \sqcup \cdots \sqcup K_t$ is the closure of $\mathbb{S}^3 \setminus (\sqcup_{i=1}^t (V(K_i))$ in $\mathbb{S}^3$, where $V(K_i)$ is the tubular neighbourhood of i^{th} component knot K_i in L. By definition, if two links are ambient isotopic, then their link exteriors are homeomorphic. This implies that the link exterior is an invariant of links. Using link exteriors, we define the following invariants of links, namely link groups and link quandles.

2.6.1 Link groups

Let us consider a link L in the 3-sphere and a point x_0 in $C(L)$. Let $\pi_1(C(L), x_0)$ be the fundamental group of the 3-manifold $C(L)$. It is well-known that the fundamental group of a path connected topological space is independent of the base point up to isomorphism. Thus, we sometimes omit the base point and just write $\pi_1(C(L))$.

Definition 2.6.1. The group $\pi_1(C(L))$ is called the *link group* of the link L.

Obviously, if two links are ambient isotopic, then their link groups are isomorphic. Thus, link group is an invariant of classical links. Notice that $\pi_1(C(L)) \cong \pi_1(\mathbb{S}^3 \setminus L)$ for any link L. It is also easy to observe that if $m(K)$ is the mirror image of a knot K, then $\pi_1(C(K)) \cong \pi_1(C(m(K))$.
For a given oriented link, there is a well-known presentation for link groups known as the *Wirtinger presentation*. We consider a diagram $D(L)$ of a link L and label its arcs as $x_1, x_2, \ldots, x_n$. These labels are generators for $\pi_1(C(L))$. The defining relations for the presentation are obtained from each crossing by the rule shown in Figure 2.22.

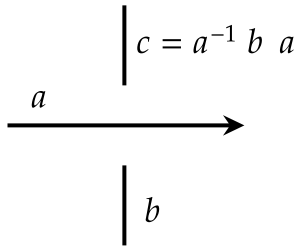


Fig. 2.22 Relations for Wirtinger presentation of link groups.

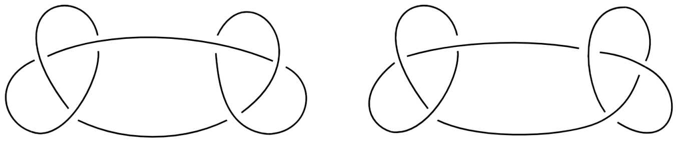


Fig. 2.23 Square knot and Granny knot.

Note that there exists non-equivalent knots which are not mirror images with isomorphic knot groups. For example, using the Wirtinger presentation, one can check that the square knot and the granny knot (see Figure 2.23) have isomorphic knot groups. But, Dehn proved that the knot group is a strong invariant in the sense that it can detect the unknot.

Theorem 2.6.2. (Dehn [41]) *An n-component link L is trivial if and only if $\pi_1(C(L))$ is isomorphic to the free group of rank n.*

Let K be an oriented knot and x_1 a point on the boundary $\partial(V(K))$ of $V(K)$. Consider a path $s: I \to C(K)$ such that $s(0) = x_1$ and $s(1) = x_0$. Since, there is a natural inclusion map

$$\partial(V(K)) \hookrightarrow C(K),$$

it induces a group homomorphism

$$\hat{s}: \pi_1(\partial(V(K), x_1)) \to \pi_1(C(K), x_0)$$

defined as

$$[\alpha] \mapsto [s^{-1}\alpha s].$$

This map is a monomorphism unless K is the trivial knot. The image of $\hat{s}$ is called a *peripheral subgroup* of the knot group $\pi_1(C(K))$. For non-trivial knots, peripheral subgroup is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$. A *meridian* of K is a simple closed curve lying on $\partial(V(K))$ which bounds a disc in $V(K)$, and is oriented as shown in Figure 2.24.

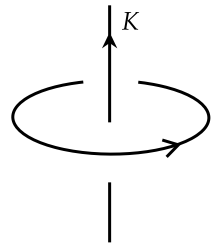


Fig. 2.24 Orientation of a meridian.

2.6.2 Link quandles

Let L be an oriented link in $\mathbb{S}^3$ with components $K_1, K_2, \ldots, K_t$. Let $V(L)$ be the disjoint union of tubular neighbourhoods of K_i for each $1 \leq i \leq t$. Fix a base point x_0 in $C(L)$. Let $S(L)$ be the set of all paths $a : I \to C(L)$ such $a(0) \in \partial(C(L))$ and $a(1) = x_0$.
Given $a, b \in S(L)$, we say that a is *homotopic* to b if there exists a continuous function $H : I \times I \to C(L)$ such that

(1) $H(I, 0) = a$,

(2) $H(I, 1) = b$,

(3) $H(I, t) \in S(L)$ for each $t \in I$.

The notion of homotopy defines an equivalence relation $\sim$ on $S(L)$. We denote the equivalence class of $a \in S(L)$ by $[a]$. Let $Q(L)$ be the set $S(L)$ modulo the relation $\sim$. We define a binary operation $*$ on $Q(L)$ as

$$[a] * [b] := [ab^{-1} m_{b(0)} b],$$

where $[a], [b] \in Q(L)$ and $m_{b(0)}$ is a meridian at point $b(0)$. One can check that $(Q(L), *)$ is a quandle. It follows that if L and L' are equivalent then $Q(L) \cong Q(L')$.

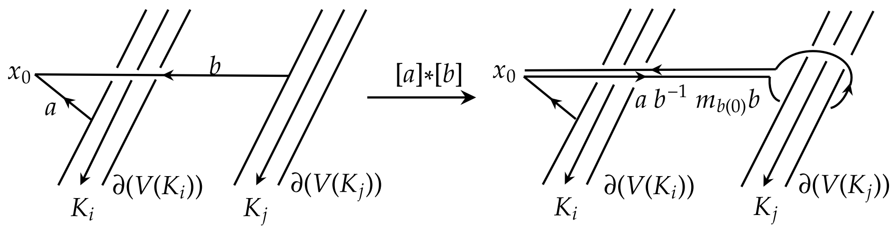


Fig. 2.25 Illustration of link quandle operation.

Definition 2.6.3. The quandle $(Q(L),*)$ is called the link quandle of the link L.

We now state fundamental results of Matveev [94] and Joyce [70, 71].

Theorem 2.6.4. *For an oriented link L, the enveloping group* $\mathrm{Env}(Q(L))$ *is isomorphic to* $\pi_1(C(L))$.

Theorem 2.6.5. *If K_1 and K_2 are two oriented knots such that $Q(K_1)$ is isomorphic to $Q(K_2)$, then either K_1 is equivalent to K_2 or K_1 is equivalent to $r(m(K_2))$.*

A connected 3-manifold M is said to be *irreducible* if every embedded 2-sphere in M bounds a 3-ball in M. A link L in $\mathbb{S}^3$ is said to be *non-split* if the link exterior $C(L)$ is an irreducible 3-manifold. Fenn and Rourke [47, Theorem 5.2 and Corollary 5.3] extended the result of Matveev and Joyce to non-split links.

Theorem 2.6.6. *If L_1 and L_2 are two oriented non-split links such that $Q(L_1)$ is isomorphic to $Q(L_2)$, then either L_1 is equivalent to L_2 or L_1 is equivalent to $r(m(L_2))$.*

Observe that the preceding result does not hold for split links. For instance, let K be the oriented trefoil knot. Then link quandles of links $L_1 = K \sqcup K$ and $L_2 = K \sqcup m(K)$ are isomorphic, but neither L_1 is equivalent to L_2 nor L_1 is equivalent to $r(m(L_2))$.
Similar to the case of link groups, we can define a presentation of a link quandle. We consider a diagram $D(L)$ of an oriented link L and label the arcs as $x_1, x_2, \ldots, x_n$. These labels correspond to the generators for $Q(L)$. The defining relations for the presentation are given by the information on each crossing by the rule depicted in Figure 2.26.

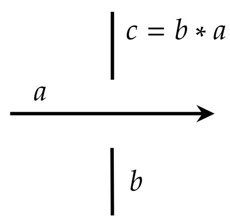


Fig. 2.26 Quandle relation at a crossing.

2.6.3 Invariants of virtual links

We conclude this chapter by recalling two invariants for virtual links. Kauffman [79] extended the notion of a knot group and a knot quandle to the setting of virtual links via diagrams. Let $D(L)$ be a virtual link diagram representing an oriented virtual link L. By a *long arc* in diagram $D(L)$, we mean an arc from one under-crossing to the next under-crossing. We label each long arc in $D(L)$ as $x_1, x_2, \ldots, x_n$.

Definition 2.6.7. The *Kauffman group* for the link diagram $D(L)$ is defined to be the group generated by elements $x_1, x_2, \ldots, x_n$ with defining relations from each classical crossing by rule given in Figure 2.27.

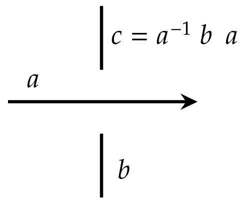


Fig. 2.27 Relation at a classical crossing.

It is straightforward to check that the Kauffman group is invariant under generalized Reidemeister moves. For a virtual link L, we denote the associated Kauffman group by $G_0(L)$. It is easy to observe that if L is a classical link, then $G_0(L) \cong \pi_1(C(L))$.
We conclude this chapter with the following definition.

Definition 2.6.8. The *Kauffman quandle* for the link diagram $D(L)$ is the quandle generated by $x_1, x_2, \ldots, x_n$ and relations from each classical crossing by rule given in Figure 2.28.

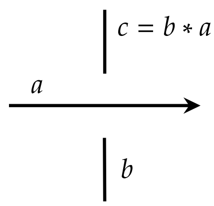


Fig. 2.28 Relation at a classical crossing.

One can check that the Kauffman quandle is an invariant for virtual links. We denote the Kauffman quandle for the virtual link L by $Q_0(L)$. Note that if L is a classical link, then $Q_0(L) \cong Q(L)$.

Chapter 3

Residual finiteness of quandles

In this chapter, we study residual finiteness of quandles. The chapter is organized as follows. In Section 3.1, we record some basic observations of residually finite quandles. In Section 3.2, we study residual finiteness of core, conjugation and Alexander quandles arising from residually finite groups. In Section 3.3, we prove that free quandles and free product of residually finite quandles under certain conditions are residually finite. We also establish that the word problem for finitely presented residually finite quandles is solvable. In Section 3.4, we prove our main result that all link quandles are residually finite, and hence the word problem is solvable for link quandles. The results are from our works [18, 19].

3.1 Basic properties of residually finite quandles

Recall that a group G is said to be *residually finite* if for each $g,h \in G$ with $g \neq h$, there exists a finite group F and a homomorphism $\phi : G \to F$ such that $\phi(g) \neq \phi(h)$. For example, finite groups, free groups and link groups [116] are known to be residually finite.
Residual finiteness of quandles can be defined in the same manner.

Definition 3.1.1. A quandle X is said to be *residually finite* if for all $x,y \in X$ with $x \neq y$, there exists a finite quandle F and quandle homomorphism $\phi : X \to F$ such that $\phi(x) \neq \phi(y)$.

In [90], Mal′cev gave the definition of a residually finite algebra, and proved that for some algebras, residual finiteness implies that the word problem is solvable. The preceding definition is a particular case of Mal′cev's definition.
Obviously, every finite quandle is residually finite, and every subquandle of a residually finite quandle is residually finite. We begin with some elementary observations.

Proposition 3.1.2. *Every trivial quandle is residually finite.*

Proof. Let X be a trivial quandle. If X has only one element, then there is nothing to prove. Suppose that X has at least two elements. Let $x, y \in X$ with $x \neq y$. Consider the trivial subquandle $\{x, y\}$ of X and define $\phi : X \to \{x, y\}$ by $\phi(x) = x$ and $\phi(z) = y$ for all $z \neq x$. Then it is easy to see that ϕ is a quandle homomorphism with $\phi(x) \neq \phi(y)$, and hence X is residually finite. □

Next, we investigate some closure properties of residually finite quandles. Let $\{X_i\}_{i \in I}$ be an indexed family of quandles and $X = \prod_{i \in I} X_i$ their Cartesian product. Then X is itself a quandle, called the *product quandle*, with binary operation given by

$$(x_i) * (y_i) = (x_i * y_i)$$

for $(x_i), (y_i) \in X$. Further, for each $j \in I$, the projection map

$$\pi_j : X \to X_j$$

given by $\pi_j((x_i)) = x_j$ is a quandle homomorphism.

Proposition 3.1.3. *Let $\{X_i\}_{i \in I}$ be an indexed family of residually finite quandles. Then the product quandle $X = \prod_{i \in I} X_i$ is residually finite.*

Proof. Let $x = (x_i), y = (y_i) \in X$ such that $x \neq y$. Then there exists an $i_0 \in I$ such that $x_{i_0} \neq y_{i_0}$. Since X_{i_0} is residually finite, there exists a finite quandle F and a homomorphism $\phi : X_{i_0} \to F$ such that $\phi(x_{i_0}) \neq \phi(y_{i_0})$. The homomorphism $\phi' := \phi \circ \pi_{i_0}$ satisfy $\phi'(x) \neq \phi'(y)$, and hence X is a residually finite quandle. □

Proposition 3.1.4. *The following statements are equivalent for a quandle X:*

1. *X is residually finite,*
2. *there exists a family $\{W_i\}_{i \in I}$ of finite quandles such that the quandle X is isomorphic to a subquandle of the product quandle $\prod_{i \in I} W_i$.*

Proof. The implication (2) $\implies$ (1) follows from Proposition 3.1.3 and the fact that a subquandle of a residually finite quandle is residually finite. Conversely, suppose that X is residually finite. For each pair $(x, y) \in X \times X$ such that $x \neq y$, there exists a finite quandle $W_{(x,y)}$ and a homomorphism $\phi_{(x,y)} : X \to W_{(x,y)}$ such that $\phi_{(x,y)}(x) \neq \phi_{(x,y)}(y)$. Now consider the quandle

$$W = \prod_{(x,y) \in X \times X,\ x \neq y} W_{(x,y)},$$

and define a homomorphism $\psi : X \to W$ by

$$\psi = \prod_{(x,y) \in X \times X,\ x \neq y} \phi_{(x,y)},$$

which is clearly injective. Hence X is residually finite being isomorphic to a subquandle of W. $\square$

3.2 Residual finiteness of quandles arising from groups

In this section, we investigate residual finiteness of conjugation, core and Alexander quandles of residually finite groups. We also discuss residual finiteness of certain automorphism groups of residually finite quandles.

Proposition 3.2.1. *If G is a residually finite group, then $\mathrm{Conj}_n(G)$ and $\mathrm{Core}(G)$ are both residually finite quandles.*

Proof. The proof follows from the fact that Conj_n and Core are functors from the category of groups to that of quandles. $\square$

For generalized Alexander quandles, we observe the following.

Proposition 3.2.2. *Let G be a residually finite group. If $\alpha : G \to G$ is an inner automorphism, then $\mathrm{Alex}(G, \alpha)$ is a residually finite quandle.*

Proof. Let α be the inner automorphism induced by $g_0 \in G$. If $g_1, g_2 \in G$ such that $g_1 \neq g_2$, then there exists a finite group F and a group homomorphism $\psi : G \to F$ such that $\psi(g_1) \neq \psi(g_2)$. Let β be the inner automorphism of F induced by $\psi(g_0)$. It follows that ψ viewed as a map $\psi : \mathrm{Alex}(G, \alpha) \to \mathrm{Alex}(F, \beta)$ is a quandle homomorphism with $\psi(g_1) \neq \psi(g_2)$, and hence $\mathrm{Alex}(G, \alpha)$ is residually finite. $\square$

It is well-known that the automorphism group of a finitely generated residually finite group is residually finite [88, p.414]. For the inner automorphism group of residually finite quandles, we have the following result.

Theorem 3.2.3. *If X is a residually finite quandle, then $\mathrm{Inn}(X)$ is a residually finite group.*

Proof. Let $S_{a_1}^{e_1} S_{a_2}^{e_2} \cdots S_{a_m}^{e_m} \neq 1$ be an element of $\mathrm{Inn}(X)$, where $a_j \in X$ and $e_j \in \{1, -1\}$ for $1 \leq j \leq m$. Then there exists an element $x \in X$ such that

$$S_{a_1}^{e_1} S_{a_2}^{e_2} \cdots S_{a_m}^{e_m}(x) \neq x.$$

Equivalently,

$$(((x *^{e_m} a_m) *^{e_{m-1}} a_{m-1}) \cdots) *^{e_1} a_1 \neq x.$$

Since X is residually finite, there exists a finite quandle F and an onto quandle homomorphism $\phi : X \to F$ such that

$$\phi((((x *^{e_m} a_m) *^{e_{m-1}} a_{m-1}) \cdots) *^{e_1} a_1) \neq \phi(x). \tag{3.2.0.1}$$

Define a map

$$\widetilde{\phi} : \{S_x^{\pm 1} \mid x \in X\} \to \mathrm{Inn}(F)$$

by setting

$$\widetilde{\phi}(S_x^{\pm 1}) = S_{\phi(x)}^{\pm 1}.$$

It is easy to see that if $S_{x_1}^{e'_1} S_{x_2}^{e'_2} \dots S_{x_n}^{e'_n} = 1$ in $\mathrm{Inn}(X)$, then $S_{\phi(x_1)}^{e'_1} S_{\phi(x_2)}^{e'_2} \dots S_{\phi(x_n)}^{e'_n} = 1$ in $\mathrm{Inn}(F)$, where $e'_i \in \{1, -1\}$ for $1 \leq i \leq n$, and hence $\widetilde{\phi}$ extends to a group homomorphism $\widetilde{\phi}$: $\mathrm{Inn}(X) \to \mathrm{Inn}(F)$. If $\widetilde{\phi}(S_{a_1}^{e_1} S_{a_2}^{e_2} \cdots S_{a_m}^{e_m}) = 1$, then evaluating both the sides at $\phi(x)$ contradicts (3.2.0.1). Hence, $\mathrm{Inn}(X)$ is a residually finite group. □

Next, we present some observations on automorphism groups of core and conjugation quandles of residually finite groups.

Proposition 3.2.4. *Let G be a finitely generated abelian group with no 2-torsion. Then* $\mathrm{Aut}(\mathrm{Core}(G))$ *is a residually finite group.*

Proof. Since G is a finitely generated abelian group, it is residually finite, and hence $\mathrm{Aut}(G)$ is also residually finite. Moreover, a semi-direct product of a finitely generated residually finite group by a residually finite group is residually finite. By [11, Theorem 4.2], $\mathrm{Aut}(\mathrm{Core}(G)) \cong G \rtimes \mathrm{Aut}(G)$, and hence $\mathrm{Aut}(\mathrm{Core}(G))$ is residually finite. □

Proposition 3.2.5. *If G is a finitely generated residually finite group with trivial centre, then* $\mathrm{Aut}(\mathrm{Conj}(G))$ *is residually finite.*

Proof. Since G has trivial centre, by [6, Corollary 4.2], $\mathrm{Aut}(\mathrm{Conj}(G)) = \mathrm{Aut}(G)$, which is residually finite as G is so. □

3.3 Residual finiteness of free products of quandles

Recall the construction of free quandles described in Section 2.5. There is another model [102, Example 2.16] for the free quandle on a set S, which is defined as the subquandle of

$\mathrm{Conj}\,(F(S))$ consisting of all conjugates of elements of S. For the benefit of readers, we present an explicit isomorphism between the two models.

Proposition 3.3.1. *The map* $\Phi : FQ(S) \to \mathrm{Conj}\,(F(S))$ *given by* $\Phi(a,w) = w^{-1}aw$ *is an embedding of quandles.*

Proof. Let $(a_1,w_1),(a_2,w_2) \in FQ(S)$. Then $\Phi(a_1,w_1) = w_1^{-1}a_1w_1$, $\Phi(a_2,w_2) = w_2^{-1}a_2w_2$ and $(a_1,w_1) * (a_2,w_2) = (a_1, w_1w_2^{-1}a_2w_2)$. Further,

$$\begin{aligned}
\Phi((a_1,w_1) * (a_2,w_2)) &= \Phi(a_1, w_1w_2^{-1}a_2w_2)\\
&= (w_1w_2^{-1}a_2w_2)^{-1}a_1(w_1w_2^{-1}a_2w_2)\\
&= w_2^{-1}a_2^{-1}w_2w_1^{-1}a_1w_1w_2^{-1}a_2w_2\\
&= (w_2^{-1}a_2w_2)^{-1}(w_1^{-1}a_1w_1)(w_2^{-1}a_2w_2)\\
&= \Phi(a_1,w_1) * \Phi(a_2,w_2),
\end{aligned}$$

and hence Φ is a quandle homomorphism. Let $(a_1,w_1),(a_2,w_2) \in FQ(S)$ such that $(a_1,w_1) \neq (a_2,w_2)$.

Case 1: Suppose that $a_1 \neq a_2$. If $\Phi(a_1,w_1) = \Phi(a_2,w_2)$, then $w_1^{-1}a_1w_1 = w_2^{-1}a_2w_2$, which contradicts the fact that $F(S)$ is a free group. Hence $\Phi(a_1,w_1) \neq \Phi(a_2,w_2)$.

Case 2: Suppose that $a_1 = a_2 = a$. If $\Phi(a,w_1) = \Phi(a,w_2)$, then $w_1^{-1}aw_1 = w_2^{-1}aw_2$, which further implies that $w_1w_2^{-1}$ commutes with a in $F(S)$. Since $F(S)$ is a free group, only powers of a can commute with a, and hence $w_1w_2^{-1} = a^i$ for some integer i. Thus $w_1 = a^iw_2$, which implies that $(a,w_1) = (a,a^iw_2) = (a,w_2)$ in $FQ(S)$, a contradiction. Hence $\Phi(a,w_1) \neq \Phi(a,w_2)$, and Φ is an embedding of quandles. $\square$

Theorem 3.3.2. *Every free quandle is residually finite.*

Proof. Let $FQ(S)$ be the free quandle on the set S. It is well-known that the free group $F(S)$ is residually finite [36, Theorem 2.3.1]. By Proposition 3.2.1, the quandle $\mathrm{Conj}\,(F(S))$ is residually finite. Since $FQ(S)$ is a subquandle of $\mathrm{Conj}\,(F(S))$, it follows that $FQ(S)$ is residually finite. $\square$

The following is a well-known result for free groups [86, p.42].

Theorem 3.3.3. *If* $F(S)$ *is a free group on a set* S *and* $g \neq 1$ *an element of* $F(S)$*, then there is a homomorphism* $\rho : F(S) \to \mathrm{S}_n$ *for some* n *such that* $\rho(g) \neq 1$*, where* S_n *is the symmetric group on* n *elements.*

We prove an analogue of the preceding result for free quandles.

Theorem 3.3.4. *Let $FQ(S)$ be a free quandle on a set S and $x, y \in FQ(S)$ such that $x \neq y$. Then there is a quandle homomorphism $\phi : FQ(S) \to \mathrm{Conj}(\mathrm{S}_n)$ for some n such that $\phi(x) \neq \phi(y)$.*

Proof. Recall that the map $\Phi : FQ(S) \to \mathrm{Conj}\,(F(S))$ in Theorem 3.3.2 is an injective quandle homomorphism. Let $(a_1, w_1) \neq (a_2, w_2) \in FQ(S)$. Then $g_1 \neq g_2 \in F(S)$, where $g_1 = \Phi(a_1, w_1)$ and $g_2 = \Phi(a_2, w_2)$. Thus, $g_2^{-1} g_1$ is a non-trivial element of $F(S)$. By Theorem 3.3.3, there exists a symmetric group S_n for some n and a group homomorphism $\rho : F(S) \to \mathrm{S}_n$ such that $\rho(g_1) \neq \rho(g_2)$. Then ρ can also be viewed as a quandle homomorphism $\mathrm{Conj}\,(F(S)) \to \mathrm{Conj}(\mathrm{S}_n)$. Taking $\phi := \mathrm{Conj}(\rho) \circ \Phi : FQ(S) \to \mathrm{Conj}(\mathrm{S}_n)$, we see that $\phi(a_1, w_1) \neq \phi(a_2, w_2)$. □

Definition 3.3.5. A quandle X is called *Hopfian* if every surjective quandle endomorphism of X is injective.

It is well-known that finitely generated residually finite groups are Hopfian [89]. We prove a similar result for quandles.

Theorem 3.3.6. *Every finitely generated residually finite quandle is Hopfian.*

Proof. Let X be a finitely generated residually finite quandle and $\phi : X \to X$ a surjective quandle homomorphism. Suppose that ϕ is not injective. Let $x_1, x_2 \in X$ such that $x_1 \neq x_2$ and $\phi(x_1) = \phi(x_2)$. Since X is residually finite, there exist a finite quandle F and a quandle homomorphism $\tau : X \to F$ such that $\tau(x_1) \neq \tau(x_2)$.
We claim that the maps $\tau \circ \phi^n : X \to F$ are distinct quandle homomorphisms for all $n \geq 0$. Let $0 \leq m < n$ be integers. Since
$$\phi^m : X \to X$$
is surjective, there exist $y_1, y_2 \in X$ such that $\phi^m(y_1) = x_1$ and $\phi^m(y_2) = x_2$. Thus, we have
$$\tau \circ \phi^m(y_1) \neq \tau \circ \phi^m(y_2),$$
whereas
$$\tau \circ \phi^n(y_1) = \tau \circ \phi^n(y_2),$$
which proves our claim. Thus, there are infinitely many quandle homomorphisms from X to F, which is a contradiction, since X is finitely generated and F is finite. Hence, ϕ is an automorphism, and X is Hopfian. □

By theorems 3.3.2 and 3.3.6, we obtain

Corollary 3.3.7. *Every finitely generated free quandle is Hopfian.*

Remark 3.3.8. The preceding result is not true for infinitely generated free quandles (free racks). Indeed, if FQ_∞ is a free quandle that is freely generated by an infinite set $\{x_1, x_2, \ldots\}$, then we can define a homomorphism $\varphi : FQ_\infty \to FQ_\infty$ by setting

$$\varphi(x_1) = x_1 \text{ and } \varphi(x_i) = x_{i-1}$$

for $i \geq 2$. It is easy to see that φ is an epimorphism which is not an automorphism since $\varphi(x_1) = \varphi(x_2)$.

Proposition 3.3.9. *Let $FQ(S)$ and $FQ(T)$ be free quandles on sets S and T, respectively. If $FQ(S) \cong FQ(T)$, then $|S| = |T|$.*

Proof. By Theorem 2.5.8, if Q is a quandle with a presentation $Q = \langle X \mid \mathcal{R} \rangle$, then its enveloping group has a presentation $\mathrm{Env}(Q) \cong \langle e_x \quad (x \in X) \mid \overline{\mathcal{R}} \rangle$, where $\overline{\mathcal{R}}$ consists of relations in R with each expression $x * y$ replaced by $e_y^{-1} e_x e_y$. Consequently, since $FQ(S)$ and $FQ(T)$ are free quandles, it follows that $\mathrm{Env}(FQ(S)) \cong F(S)$ and $\mathrm{Env}\,(FQ(T)) \cong F(T)$ are free groups on the sets S and T, respectively. Since $FQ(S) \cong FQ(T)$, we must have $\mathrm{Env}(FQ(S)) \cong \mathrm{Env}(FQ(T))$, and hence $|S| = |T|$. $\square$

In view of Proposition 3.3.9, we can define the *rank of a free quandle* as the cardinality of its any free generating set.
Analogous to groups, we define the *word problem* for quandles as the problem of determining whether two given elements of a quandle are the same. The word problem is solvable for finitely presented residually finite groups [107, p.55]. Below is an analogous result for quandles.

Theorem 3.3.10. *Every finitely presented residually finite quandle has a solvable word problem.*

Proof. Let $Q = \langle X \mid \mathcal{R} \rangle$ be a finitely presented residually finite quandle, and w_1, w_2 two words in the generators X. We describe two procedures which tell us whether or not $w_1 = w_2$ in Q. The first procedure lists all the words that we obtain by using the relations of Q on the word w_1. If the word w_2 turns up at some stage, then $w_1 = w_2$, and we are done.
The second procedure lists all the finite quandles. Since Q is finitely generated, for each finite quandle F, the set $\mathrm{Hom}(Q, F)$ of all quandle homomorphisms is finite. Now for each homomorphism $\phi \in \mathrm{Hom}(Q, F)$, we look for $\phi(w_1)$ and $\phi(w_2)$ in F, and check whether or not $\phi(w_1) = \phi(w_2)$. Since Q is residually finite, the above procedure must stop at some point. That is, there exists a finite quandle F and $\phi \in \mathrm{Hom}(Q, F)$ such that $\phi(w_1) \neq \phi(w_2)$ in F, and hence $w_1 \neq w_2$ in Q. $\square$

Remark 3.3.11. In a recent work [23], Belk and McGrail showed that the word problem for quandles is unsolvable in general by giving an example of a finitely presented quandle with unsolvable word problem. In view of Theorem 3.3.10, such a quandle cannot be residually finite.

We define the free product of quandles as follows. Let

$$A = \langle X \mid \mathcal{R} \rangle \text{ and } B = \langle Y \mid \mathcal{S} \rangle$$

be two quandles with non-intersecting sets of generators. Then the free product $A \star B$ is a quandle that is defined by the presentation

$$A \star B = \langle X \sqcup Y \mid \mathcal{R} \sqcup \mathcal{S} \rangle.$$

For example, if FQ_n is the free n-generated quandle, then

$$FQ_n = \underbrace{T_1 \star T_1 \star \cdots \star T_1}_{n \text{ copies}},$$

the free product of n copies of trivial one element quandles. We refer the reader to the recent work [5, Section 7] for more on free products of quandles. Free product of racks can be defined analogously.

Lemma 3.3.12. *If Q_1, Q_2 are quandles, then* $\operatorname{Env}(Q_1 \star Q_2) \cong \operatorname{Env}(Q_1) * \operatorname{Env}(Q_2)$.

Proof. If Q_1 and Q_2 have presentations $Q_1 = \langle X_1 \mid \mathcal{R}_1 \rangle$ and $Q_2 = \langle X_2 \mid \mathcal{R}_2 \rangle$, then $Q_1 \star Q_2 = \langle X_1 \sqcup X_2 \mid \mathcal{R}_1 \sqcup \mathcal{R}_2 \rangle$. Now, by Theorem 2.5.8, we have

$$\begin{aligned} \operatorname{Env}(Q_1 \star Q_2) &\cong \langle e_x \ (x \in X_1 \sqcup X_2) \mid \overline{\mathcal{R}_1} \sqcup \overline{\mathcal{R}_2} \rangle \\ &\cong \langle e_x \ (x \in X_1) \mid \overline{\mathcal{R}_1} \rangle * \langle e_x \ (x \in X_2) \mid \overline{\mathcal{R}_2} \rangle \\ &\cong \operatorname{Env}(Q_1) * \operatorname{Env}(Q_2). \end{aligned}$$

□

The following result is well-known in combinatorial group theory, first proved by Gruenberg [62, Theorem 4.1]. See also [21, 38].

Theorem 3.3.13. *A free product of residually finite groups is residually finite.*

We prove an analogue of the preceding theorem for quandles provided their enveloping groups are residually finite.

We recall that every element of a quandle X can be written in a left associated form $x_0 *^{e_1} x_1 *^{e_2} x_2 *^{e_3} \cdots *^{e_n} x_n$. Moreover, the expression $x_0 *^{e_1} x_1 *^{e_2} x_2 *^{e_3} \cdots *^{e_n} x_n$ is called a *reduced form* when $x_0 \neq x_1$ and if $x_i = x_{i+1}$, then $e_i = e_{i+1}$. Notice that the reduced form is not unique. For example, if $Q = \{t\} \star \mathrm{R}_3$ is the free product of one element trivial quandle and the dihedral quandle $\mathrm{R}_3 = \{a_0, a_1, a_2\}$, then

$$(t * a_1) * a_2 = (t * a_2) * (a_1 * a_2) = (t * a_2) * a_0.$$

Theorem 3.3.14. *Let $Q_1, Q_2, \ldots, Q_n$ be residually finite quandles. If each associated group $\mathrm{Env}(Q_i)$ is residually finite, then $Q_1 \star Q_2 \star \cdots \star Q_n$ is a residually finite quandle.*

Proof. It is enough to consider the case $n = 2$. Set $Q = Q_1 \star Q_2$. Let x and x' be two distinct elements of Q, where

$$\begin{aligned} x &= a_0 *^{e_1} a_1 *^{e_2} a_2 *^{e_3} \ldots *^{e_n} a_n, \\ x' &= b_0 *^{e'_1} b_1 *^{e'_2} b_2 *^{e'_3} \ldots *^{e'_m} b_m \end{aligned}$$

are their reduced expressions, and a_i, b_j lie in $Q_1 \sqcup Q_2$.

Case 1: $x, x' \in Q_1$ or $x, x' \in Q_2$. Suppose that $x, x' \in Q_1$. Since Q_1 is a residually finite quandle, there exist a finite quandle F and a quandle homomorphism $\phi : Q_1 \to F$ such that $\phi(x) \neq \phi(x')$. Define a map $\tilde{\phi} : Q \to F$ by setting

$$\tilde{\phi}(q) = \begin{cases} \phi(q) & \text{if } q \in Q_1, \\ a & \text{if } q \in Q_2, \text{ where } a \text{ is some fixed element of } F. \end{cases}$$

Since $\tilde{\phi}$ preserve all the relations in Q, it extends to a quandle homomorphism with $\tilde{\phi}(x) \neq \tilde{\phi}(x')$ in F.

Case 2: $x \in Q_1$ and $x' \in Q_2$. Consider a map $\phi : Q \to FQ(X)$, where $X = \{a, b\}$ and $FQ(X)$ is the free quandle on X, defined as

$$\phi(q) = \begin{cases} a & \text{if } q \in Q_1, \\ b & \text{if } q \in Q_2. \end{cases}$$

Since ϕ preserve all the relations in Q, it extends to a quandle homomorphism with $\phi(x) \neq \phi(x')$ in $FQ(X)$.

Case 3: $x \in Q \setminus (Q_1 \sqcup Q_2)$ and $x' \in Q_1$. We can assume that either $a_0 \in Q_1$, $a_1 \in Q_2$ and $a_2 \ldots, a_n \in Q_1 \sqcup Q_2$ or $a_0 \in Q_2$, $a_1 \in Q_1$ and $a_2, \ldots, a_n \in Q_1 \sqcup Q_2$ i.e.,

$$x = \begin{cases} q_1 *^{e_1} q_2 *^{e_2} a_2 *^{e_3} \dots *^{e_n} a_n & \text{where } \; q_1 \in Q_1, \;\; q_2 \in Q_2, \\ \text{or} & \\ q_2 *^{e_1} q_1 *^{e_2} a_2 *^{e_3} \dots *^{e_n} a_n & \text{where } \; q_1 \in Q_1, \;\; q_2 \in Q_2. \end{cases}$$

It follows from Lemma 3.3.12, Theorem 3.3.13 and 3.2.1 that $\mathrm{Conj}\,(\mathrm{Env}(Q))$ is a residually finite quandle. Let

$$\eta : Q \to \mathrm{Conj}\,(\mathrm{Env}(Q))$$

be the natural quandle homomorphism. Then, we have

$$\eta(x_0 *^{e_1} x_1 *^{e_2} x_2 *^{e_3} \dots *^{e_n} x_n) = (e_{x_1}^{e_1} e_{x_2}^{e_2} \dots e_{x_n}^{e_n})^{-1} e_{x_0} (e_{x_1}^{e_1} e_{x_2}^{e_2} \dots e_{x_n}^{e_n}).$$

We claim that $\eta(x) \neq \eta(x')$.

Subcase 3.1: If $x = q_1 *^{e_1} q_2 *^{e_2} a_2 *^{e_3} \dots *^{e_n} a_n$, where $q_1 \in Q_1$, $q_2 \in Q_2$, and $a_2, \dots, a_n \in Q_1 \sqcup Q_2$, then

$$\begin{aligned} \eta(x) &= (e_{q_2}^{e_1} e_{a_2}^{e_2} \dots e_{a_n}^{e_n})^{-1} e_{q_1} (e_{q_2}^{e_1} e_{a_2}^{e_2} \dots e_{a_n}^{e_n}) \\ &= e_{a_n}^{-e_n} \dots e_{a_2}^{-e_2} e_{q_2}^{-e_1} e_{q_1} e_{q_2}^{e_1} e_{a_2}^{e_2} \dots e_{a_n}^{e_n}. \end{aligned}$$

Suppose that $\eta(x) = \eta(x')$. Then by the Remark 2.5.10 and the fact that elements of $\mathrm{Env}(Q_1)$ have no relations with elements of $\mathrm{Env}(Q_2)$ in the group $\mathrm{Env}(Q)$, it follows that either $e_{q_2}^{e_1} e_{a_2}^{e_2} \dots e_{a_n}^{e_n} = 1$ in $\mathrm{Env}(Q)$ or $e_{q_2}^{e_1} e_{a_2}^{e_2} \dots e_{a_n}^{e_n} = e_{q_{i_1}}^{\varepsilon_1} e_{q_{i_2}}^{\varepsilon_2} \dots e_{q_{i_k}}^{\varepsilon_k}$, where $q_{i_1}, q_{i_2}, \dots, q_{i_k}$ belongs to Q_1 and $\varepsilon_j = \pm 1$ for $1 \leq j \leq k$. Since $\mathrm{Env}(Q)$ has a right action on the quandle Q, this implies that in either situation $q_1.(e_{q_2}^{e_1} e_{a_2}^{e_2} \dots e_{a_n}^{e_n})$ belongs to Q_1. Thus, $x = q_1 *^{e_1} q_2 *^{e_2} a_2 *^{e_3} \dots *^{e_n} a_n \in Q_1$, which is a contradiction. Hence we must have $\eta(x) \neq \eta(x')$.

Subcase 3.2: If $x = q_2 *^{e_1} q_1 *^{e_2} a_2 *^{e_3} \dots *^{e_n} a_n$, where $q_1 \in Q_1$, $q_2 \in Q_2$ and $a_2, \dots, a_n \in Q_1 \sqcup Q_2$, then

$$\begin{aligned} \eta(x) &= (e_{q_1}^{e_1} e_{a_2}^{e_2} \dots e_{a_n}^{e_n})^{-1} e_{q_2} (e_{q_1}^{e_1} e_{a_2}^{e_2} \dots e_{a_n}^{e_n}) \\ &= e_{a_n}^{-e_n} \dots e_{a_2}^{-e_2} e_{q_1}^{-e_1} e_{q_2} e_{q_1}^{e_1} e_{a_2}^{e_2} \dots e_{a_n}^{e_n}. \end{aligned}$$

Clearly $\eta(x) \neq \eta(x')$ since they belong to different conjugacy classes in $\mathrm{Env}(Q)$.

Case 4: $x \in Q \setminus (Q_1 \sqcup Q_2)$ and $x' \in Q_2$. This is similar to Case 3.

Case 5: $x, x' \in Q \setminus (Q_1 \sqcup Q_2)$. This case can be reduced to one of the Cases (1–4) by repeated use of the second quandle axiom. More precisely, we can replace the element x by y and x'

by y', where

$$y = a_0 *^{e_1} a_1 *^{e_2} \cdots *^{e_n} a_n *^{-e'_m} b_m *^{-e'_{m-1}} b_{m-1} *^{-e'_{m-2}} \cdots *^{-e'_1} b_1,$$
$$y' = b_0.$$

Since finite quandles, free quandles (Theorem 3.3.2) and $\mathrm{Conj}\,(\mathrm{Env}(Q))$ are residually finite, we conclude that $Q = Q_1 \star Q_2$ is a residually finite quandle. $\square$

We know extend the preceding result to arbitrary family of quandles.

Theorem 3.3.15. *Let $\{Q_i\}_{i \in I}$ be a family of residually finite quandles. If each* $\mathrm{Env}(Q_i)$ *is a residually finite group, then the free product $\star_{i \in I} Q_i$ is a residually finite quandle.*

Proof. Let $Q = \star_{i \in I} Q_i$ be the free product of residually finite quandles Q_i. Let $x, x' \in Q$ be two distinct elements such that

$$x = a_0 *^{e_1} a_1 *^{e_2} a_2 *^{e_3} \cdots *^{e_n} a_n,$$
$$x' = b_0 *^{e'_1} b_1 *^{e'_2} b_2 *^{e'_3} \cdots *^{e'_m} b_m.$$

Consider the set $S = \{a_i, b_j \mid 1 \leq i \leq n, \; 1 \leq j \leq m\}$. Then S is a finite set contained in $Q_{i_1} \sqcup Q_{i_2} \sqcup \cdots \sqcup Q_{i_k}$ for some $i_1, i_2, \ldots, i_k \in I$. Define a map

$$\phi : Q \to Q_{i_1} \star Q_{i_2} \star \cdots \star Q_{i_k}$$

by setting

$$\phi(q) = \begin{cases} q & \text{if } \; q \in Q_{i_1} \sqcup Q_{i_2} \sqcup \cdots \sqcup Q_{i_k}, \\ a & \text{if } q \in \sqcup_{i \in I} Q_i \setminus (Q_{i_1} \sqcup Q_{i_2} \sqcup \cdots \sqcup Q_{i_k}), \end{cases}$$

where a is some fixed element in $\sqcup_{i \in I} Q_i \setminus (Q_{i_1} \sqcup Q_{i_2} \sqcup \cdots \sqcup Q_{i_k})$. Since ϕ preserves all the relations in Q, it extends to a quandle homomorphism with $\phi(x) \neq \phi(x')$. Hence by Theorem 3.3.14, Q is a residually finite quandle. $\square$

We note that the preceding theorem gives an alternate proof of Theorem 3.3.2. Finally we discuss residual finiteness of the enveloping groups of quandles. The following results are well-known in combinatorial group theory.

Theorem 3.3.16. *If G is a finitely generated group with infinitely generated centre* $\mathrm{Z}(G)$*, then the quotient $G/\mathrm{Z}(G)$ is not finitely presented.*

Theorem 3.3.17. [36, Proposition 2.2.12] *Let G be a group. If N is a normal subgroup of finite index in G and is a residually finite group, then G is residually finite group.*

Proposition 3.3.18. *If X is a finite quandle, then its enveloping group* $\mathrm{Env}(X)$ *is a residually finite group.*

Proof. Consider the natural group homomorphism $\psi_X : \mathrm{Env}(X) \to \mathrm{Inn}(X)$. Since X is a finite quandle, the inner automorphism group $\mathrm{Inn}(X)$ of X is finite, and hence $\mathrm{Env}(X)/\mathrm{Ker}(\psi_X)$ is finite. Moreover, $\mathrm{Ker}(\psi_X)$ is contained in the centre $\mathrm{Z}(\mathrm{Env}(X))$ of $\mathrm{Env}(X)$, and hence $\mathrm{Env}(X)/\mathrm{Z}(\mathrm{Env}(X))$ is finite. By Theorem 3.3.16, $\mathrm{Z}(\mathrm{Env}(X))$ is a finitely generated abelian group, and hence residually finite. The result now follows from Theorem 3.3.17. □

As a consequence of the above result and Theorem 3.3.15, we note the following result.

Corollary 3.3.19. *The free product of finite quandles is residually finite.*

3.4 Residual finiteness of link quandles

In this section, we prove that the link quandle of a link is residually finite. We recall the following definition from [90].

Definition 3.4.1. A subgroup H of a group G is said to be *finitely separable* if for any $g \in G \setminus H$, there exists a finite group F and a group homomorphism $\phi : G \to F$ such that $\phi(g) \notin \phi(H)$.

For example, if G is a residually finite group and H a finite subgroup of G, then H is finitely separable in G.

Definition 3.4.2. A subquandle Y of a quandle X is said to be *finitely separable* in X if for each $x \in X \setminus Y$, there exists a finite quandle F and a quandle homomorphism $\phi : X \to F$ such that $\phi(x) \notin \phi(Y)$.

The following result might be of independent interest.

Proposition 3.4.3. *Let X be a residually finite quandle and $\alpha \in \mathrm{Aut}(X)$. If* $\mathrm{Fix}(\alpha) := \{x \in X \mid \alpha(x) = x\}$ *is non-empty, then it is a finitely separable subquandle of X.*

Proof. Clearly $\mathrm{Fix}(\alpha)$ is a subquandle of X. Let $x_0 \in X \setminus \mathrm{Fix}(\alpha)$, that is, $\alpha(x_0) \neq x_0$. Since X is residually finite, there exists a finite quandle F and a quandle homomorphism $\phi : X \to F$ such that $\phi(\alpha(x_0)) \neq \phi(x_0)$. Define a map $\eta : X \to F \times F$ by $\eta(x) = (\phi(x), \phi(\alpha(x)))$. Clearly η is a quandle homomorphism with $\eta(x_0) \notin \eta(\mathrm{Fix}(\alpha))$, and hence $\mathrm{Fix}(\alpha)$ is finitely separable in X. □

The following result concerning finitely separable subgroups of fundamental groups of irreducible 3-manifolds is due to Long and Niblo [87].

Theorem 3.4.4. [87, Theorem 1] *Suppose that M is an orientable, irreducible compact 3-manifold and X an incompressible connected subsurface of a component of $\partial(M)$. If $p \in X$ is a base point, then $\pi_1(X,p)$ is a finitely separable subgroup of $\pi_1(M,p)$.*

We begin with the following result which will be used in the sequel.

Proposition 3.4.5. *Let G be a group, $\{z_i \mid i \in I\}$ be a finite set of elements of G, and $\{H_i \mid i \in I\}$ subgroups of G such that $H_i \leq \mathrm{C}_G(z_i)$. If each H_i is finitely separable in G, then the quandle $\sqcup_{i\in I}(G,H_i,z_i)$ is residually finite.*

Proof. Let $H_k a \neq H_j b$ be two elements of $\sqcup_{i\in I}(G,H_i,z_i)$.
Case 1: $k \neq j$. Let $F = \{a', b'\}$ be a two element trivial quandle. Define

$$\phi : \sqcup_{i\in I}(G,H_i,z_i) \to F$$

by

$$\phi(H_i x) = \begin{cases} a' & \text{if } i = k, \\ b' & \text{if } i \neq k. \end{cases}$$

Then ϕ is a quandle homomorphism with $\phi(H_k a) \neq \phi(H_j b)$.
Case 2: $k = j$. Since $H_k a \neq H_k b$, $a \neq hb$ for any $h \in H_k$. Further, since H_k is finitely separable in G, there exists a finite group F and a group homomorphism $\phi : G \to F$ such that $\phi(a) \neq \phi(hb)$ for each $h \in H_k$. Let $\overline{H_i} := \phi(H_i)$ and $\bar{z}_i := \phi(z_i)$ for each $i \in I$. Then $\overline{H_i} \leq \mathrm{C}_F(\bar{z}_i)$, and $\sqcup_{i\in I}(F,\overline{H_i},\bar{z}_i)$ is a finite quandle. Further, the group homomorphism $\phi : G \to F$ induces a map

$$\bar{\phi} : \sqcup_{i\in I}(G,H_i,z_i) \to \sqcup_{i\in I}(F,\overline{H_i},\bar{z}_i)$$

given by

$$\bar{\phi}(H_i x) = \overline{H_i}\phi(x),$$

which is a quandle homomorphism. Also, $\bar{\phi}(H_k a) \neq \bar{\phi}(H_k b)$, otherwise $\phi(a) = \phi(hb)$ for some $h \in H_k$, which is a contradiction. Hence, $\sqcup_{i\in I}(G,H_i,z_i)$ is residually finite. □

Let L be an oriented link in $\mathbb{S}^3$ with components $K_1, K_2, \ldots, K_t$. Recall the construction of the link quandle $Q(L)$ as described in Subsection 2.6.2. For each $1 \leq i \leq t$, define $Q(K_i)$ to be the set of homotopy classes of paths in $C(L)$ starting on the boundary $\partial V(K_i)$ of a tubular neighbourhood $V(K_i)$ of K_i and ending at x_0. Then each $Q(K_i)$ is a subquandle of $Q(L)$, and in fact, $Q(L) = \sqcup_{i=1}^{t} Q(K_i)$.

We note that there is a natural action of the link group $\pi_1(C(L),x_0)$ on $Q(L)$ defined as

$$[a].[\alpha] = [a\alpha],$$

where $[\alpha] \in \pi_1(C(L),x_0)$ and $[a] \in Q(L)$. One can easily check that the action keeps each $Q(K_i)$ invariant.
For each $1 \leq i \leq t$, let $x_i \in \partial(V(K_i))$ be a fixed base point, and s_i a path from x_i to x_0 in $C(L)$. Then each

$$\hat{s}_i : \pi_1(\partial V(K_i),x_i) \to \pi_1(C(L),x_0)$$

defined as $\hat{s}_i([\alpha]) = [s_i^{-1}\alpha s_i]$ is a group homomorphism. If H_i denotes the image of $\hat{s}_i$, then we have the following result whose proof is analogous to the one worked out in [94, Lemma 2] for knots.

Lemma 3.4.6. *The action of $G(L)$ on $Q(K_i)$ is transitive and stabilizer of $[s_i]$ is H_i.*

For each $1 \leq i \leq t$, let m_i be the image of a meridian at point x_i in H_i. Then $\sqcup_{i=1}^t(\pi_1(C(L)),H_i,m_i)$ becomes a quandle under the operation defined as

$$H_ig * H_jg' = H_igg'^{-1}m_jg'.$$

Theorem 3.4.7. *Link quandles are residually finite.*

Proof. Note that for trivial knot, the knot quandle is residually finite. Let L be a non-trivial non-split link. Then for each i, the map $(\pi_1(C(L)),H_i,m_i) \to Q(K_i)$ given by

$$H_ig \mapsto [s_ig]$$

is bijective (by Lemma 3.4.6), and is also a quandle homomorphism. Since $Q(L) = \sqcup_{i=1}^t Q(K_i)$, we obtain an isomorphism of quandles

$$\sqcup_{i=1}^t(\pi_1(C(L)),H_i,m_i) \to Q(L).$$

As L is non-split, it follows from Theorem 3.4.4 that each H_i is finitely separable in $\pi_1(C(L))$. Thus, by Proposition 3.4.5, the link quandle $Q(L)$ is residually finite.
Now suppose that L is a split link. Then the link quandle of L is a free product of link quandles of its non-split components. Using the fact that all link groups are residually finite, the result now follows from the preceding case and theorems 2.6.4 and 3.3.14. □

Corollary 3.4.8. *The link quandle of a link is Hopfian, has solvable word problem, and has residually finite inner automorphism group.*

We now give an alternate proof of residual finiteness of quandles of split links whose each component is a prime knot. We note the following result due to Ryder [112, Corollary 3.6].

Theorem 3.4.9. *The fundamental quandle of a knot in $\mathbb{S}^3$ embeds into its enveloping group if and only if the knot is prime.*

It is interesting to know which other quandles embeds into their enveloping groups. In this direction, we refer to a recent result of Bardakov and Nasybullov [5, Lemma 7.1].

Lemma 3.4.10. *Let Q and P be quandles. If the natural maps $Q \to \mathrm{Env}(Q)$ and $P \to \mathrm{Env}(P)$ are injective, then the natural map $Q \star P \to \mathrm{Env}(Q) * \mathrm{Env}(P)$ is injective.*

As a consequence of above results, we have

Theorem 3.4.11. *If L is a link consisting of untangled components each of which is a prime knot, then $Q(L)$ is a residually finite quandle.*

Proof. Observe that the link quandle $Q(L)$ of the link L is a free product of knot quandles of its constituent prime knots. Further, recall that the enveloping group of a knot quandle is the knot group (Theorem 2.6.4), which is residually finite. The result now follows from theorems 3.3.13, 3.4.9 and Lemma 3.4.10. □

Since enveloping groups of finite quandles, free quandles and link quandles are residually finite, the following seems to be the case in general.

Conjecture 3.4.12. *The enveloping group of a finitely presented residually finite quandle is a residually finite group.*

If the above conjecture is true, then by Theorem 3.3.15, a free product of finitely presented residually finite quandles is residually finite, which is an analogue of Theorem 3.3.13 for quandles.

Chapter 4

Orderability of quandles

In this chapter, we develop a general theory of orderability of quandles with a focus on link quandles and give some general constructions of orderable quandles.
The chapter is organized as follows. In Section 4.1, we derive some basic properties of orderings on quandles, and prove that any linear order on a quandle must be of restricted type. In Section 4.2, we show that certain disjoint unions, direct products and extensions of orderable quandles are orderable. Section 4.3 focusses on orderability of some general quandles. As a consequence, we prove that free quandles, in particular, quandles of trivial links are right-orderable. In Section 4.4, we then prove that if all the roots of the Alexander polynomial of a fibered prime knot are real and positive, then its knot quandle is right-orderable. In another main result of this section, we prove that link quandles of certain non-trivial positive (or negative) links are not bi-orderable. In Section 4.5, we prove that if $m,n \geq 2$ are integers such that one is not a multiple of the other, then the link quandle of the torus link $T(m,n)$ is not right-orderable. As a consequence, we recover a result of Perron-Rolfsen that the knot group of a non-trivial torus knot is not bi-orderable. Finally, in Section 4.6, we discuss left-orderability of involutory quandles of alternating links. The results are from our work [105].
We begin with the following definition.

Definition 4.0.1. A group G is said to be *left-orderable* if there is a (strict) linear order $<$ on G such that $g < h$ implies $fg < fh$ for all $f,g,h \in G$. Similarly, G is said to be *right-orderable* if there is a linear order $<'$ on G such that $g <' h$ implies $gf <' hf$ for all $f,g,h \in G$. A group is *bi-orderable* if it has a linear order with respect to which it is both left and right ordered.

It is easy to see that if a group G is left-orderable under a linear order $<$, then G can be seen as right-orderable group under the linear order $<'$ defined by $g <' h$ if $g^{-1} < h^{-1}$ for all $g, h \in G$.

Braid groups [42] and link groups [27] are known to be left-orderable. Moreover, free groups and fundamental groups of surfaces except the projective plain and the Klein bottle are known to be bi-orderable [27].

One can define orderability of quandles in a similar manner.

Definition 4.0.2. A quandle Q is said to be *left-orderable* if there is a (strict) linear order $<$ on Q such that $x < y$ implies $z * x < z * y$ for all $x, y, z \in Q$. Similarly, a quandle Q is *right-orderable* if there is a linear order $<'$ on Q such that $x <' y$ implies $x * z <' y * z$ for all $x, y, z \in Q$. A quandle is *bi-orderable* if it has a linear order with respect to which it is both left and right ordered.

For example, a trivial quandle can be right-orderable but not left-orderable. If $Q = \{x_1, x_2, \ldots\}$ is a trivial quandle with more than one element, then it is clear that the linear order $x_1 < x_2 < \cdots < x_i < \cdots$ is preserved under multiplication on the right, but is not preserved under multiplication on the left. Notice the contrast to groups where left-orderability implies right-orderability and vice-versa.

Proposition 4.0.3. *Any non-trivial left or right orderable quandle is infinite.*

Proof. Let Q be a non-trivial quandle that is right-orderable. Then there exist elements $x \neq y$ in Q such that $S_y(x) \neq x$. It follows from [17, Proposition 3.7] that the $\langle S_y \rangle$-orbit of x is infinite, and thus Q must be infinite. On the other hand, if Q is left-orderable, then by [17, Proposition 3.7], the set $\{L_y^n(x) \mid n = 1, 2, \ldots\}$ is infinite for any $x \neq y$ in Q, and hence Q must be infinite. $\square$

It also follows from [17, Proposition 3.7] that a non-trivial involutory quandle is not right-orderable. A large number of left or right-orderable quandles can be constructed from bi-orderable groups. See [40, Proposition 7] and [17, Proposition 3.4].

Proposition 4.0.4. *The following hold for any bi-orderable group G:*

1. $\mathrm{Conj}_n(G)$ *is a right-orderable quandle.*
2. $\mathrm{Core}(G)$ *is a left-orderable quandle.*
3. *If* $\phi \in \mathrm{Aut}(G)$ *is an order reversing automorphism, then* $\mathrm{Alex}(G, \phi)$ *is a left-orderable quandle.*

The other sided orderability of these quandles fails in general [17, Corollaries 3.8 and 3.9].

Proposition 4.0.5. *The following hold for any non-trivial group G:*

1. *The quandle* $\mathrm{Conj}_n(G)$ *is not left-orderable.*
2. *The quandle* $\mathrm{Core}(G)$ *is not right-orderable.*
3. *If* $\phi \in \mathrm{Aut}(G)$ *an involution, then the quandle* $\mathrm{Alex}(G,\phi)$ *is not right-orderable.*

An immediate consequence of Proposition 4.0.4 is the following.

Corollary 4.0.6. *The following hold for any quandle Q:*

1. *If Q is a subquandle of* $\mathrm{Conj}_n(G)$ *for some bi-orderable group G, then Q is right-orderable.*
2. *If Q is a subquandle of* $\mathrm{Core}(G)$ *for some bi-orderable group G, then Q is left-orderable.*

4.1 Properties of linear orderings on quandles

In this section, we analyze some basic properties of linear orderings on quandles. Observe that a quandle essentially has two binary operations $*$ and $*^{-1}$. Thus, it is necessary to understand the behaviour of a linear order with respect to both of these binary operations.

Definition 4.1.1. Let $<$ be a linear order on a quandle Q and $\mathcal{O}$ be the set $\{=,<,>\}$. For a quadruple $(\blacklozenge_1,\blacklozenge_2,\blacklozenge_3,\blacklozenge_4) \in \mathcal{O}^4$, the order $<$ is said to be of *type* $(\blacklozenge_1,\blacklozenge_2,\blacklozenge_3,\blacklozenge_4)$ if the following hold for $x,y,z \in Q$ with $x<y$:

(1) $x*z \blacklozenge_1 y*z$,

(2) $x*^{-1}z \blacklozenge_2 y*^{-1}z$,

(3) $z*x \blacklozenge_3 z*y$,

(4) $z*^{-1}x \blacklozenge_4 z*^{-1}y$.

We say that the order $<$ is of type $(\_,\blacklozenge_2,\_,\_)$ if the second condition is true, it is of the type $(\blacklozenge_1,\_,\blacklozenge_3,\_)$ if the first and third conditions are true, it is of the type $(\blacklozenge_1,\blacklozenge_2,\_,\blacklozenge_4)$ if the first, second and fourth conditions are true, etc.

The quandle axiom (Q2) implies that if $<$ is a linear order on a quandle Q, then

$$x*z \neq y*z \qquad \text{and} \qquad x*^{-1}z \neq y*^{-1}z \tag{4.1.0.1}$$

for all $x,y,z \in Q$ with $x<y$.

Lemma 4.1.2. *Let $<$ be a linear order on a quandle Q and let $\blacklozenge \in \{<,>\}$. Then the order $<$ is of the type $(\blacklozenge,\_,\_,\_)$ if and only if it is of the type $(\_,\blacklozenge,\_,\_)$.*

Proof. Let $\blacklozenge \in \{<,>\}$. Define $\blacklozenge^{-1}$ to be $>$ if $\blacklozenge$ is $<$ and define it as $<$ if $\blacklozenge$ is $>$. Furthermore, define $\blacklozenge^{1}$ as $\blacklozenge$. By (4.1.0.1), we note that $x*z \blacklozenge^{d} y*z$ and $x*^{-1}z \blacklozenge^{e} y*^{-1}z$ for some $d,e \in \{-1,1\}$ whenever $x,y,z \in Q$ and $x<y$.
$\Rightarrow$: Suppose on the contrary that $x*^{-1}z \blacklozenge^{-1} y*^{-1}z$ for some $x,y,z \in Q$ with $x<y$. This implies that $\left(x*^{-1}z\right)*z \blacklozenge^{-1} \left(y*^{-1}z\right)*z$ if $\blacklozenge$ is $<$ and $\left(x*^{-1}z\right)*z \blacklozenge \left(y*^{-1}z\right)*z$ if $\blacklozenge$ is $>$, since the order $<$ is of the type $(\blacklozenge,\_,\_,\_)$. In other words, $\left(x*^{-1}z\right)*z > \left(y*^{-1}z\right)*z$, that is, $x>y$, which is a contradiction.
$\Leftarrow$: Suppose on the contrary that $x*z \blacklozenge^{-1} y*z$ for some $x,y,z \in Q$ with $x<y$. This implies that $(x*z)*^{-1}z \blacklozenge^{-1} (y*z)*^{-1}z$ if $\blacklozenge$ is $<$ and $(x*z)*^{-1}z \blacklozenge (y*z)*^{-1}z$ if $\blacklozenge$ is $>$, since the order $<$ is of the type $(\_,\blacklozenge,\_,\_)$. This gives $(x*z)*^{-1}z > (y*z)*^{-1}z$, that is, $x>y$, which is a contradiction. $\square$

Lemma 4.1.3. *Let $<$ be a linear order on a quandle Q. Then the following statements are equivalent:*

(1) The quandle Q is trivial.

(2) The order $<$ is of the type $(\_,\_,=,\_)$.

(3) The order $<$ is of the type $(\_,\_,\_,=)$.

Proof. It is trivial that (1) $\Rightarrow$ (2) and (1) $\Rightarrow$ (3).
(2) $\Rightarrow$ (1): Let x and y be any elements in Q. If $x=y$, then by idempotency, $x*y=x$. If $x<y$ or $y<x$, then by (2), $x*y=x*x=x$. This proves that $x*y=x$ for all $x,y \in Q$.
(3) $\Rightarrow$ (1): Let x and y be any elements in Q. If $x=y$, then by idempotency, $x*y=x$. If $x<y$ or $y<x$, then by (3), $x*^{-1}y=x*^{-1}x=x$. This implies that $(x*^{-1}y)*y=x*y$, and thus by cancellation, we get $x*y=x$. This proves that $x*y=x$ for all $x,y \in Q$. $\square$

Theorem 4.1.4. *Let $<$ be a linear order on a quandle Q of the type $(\blacklozenge_1,\blacklozenge_2,\blacklozenge_3,\blacklozenge_4)$ for some $(\blacklozenge_1,\blacklozenge_2,\blacklozenge_3,\blacklozenge_4) \in \mathcal{O}^4$. Then we have the following:*

(1) $\blacklozenge_1,\blacklozenge_2 \in \{<,>\}$.

(2) $\blacklozenge_1$ *and* $\blacklozenge_2$ *are the same.*

(3) The quandle Q *is trivial* $\Leftrightarrow$ $\blacklozenge_3$ *is the equality '$=$'* $\Leftrightarrow$ $\blacklozenge_4$ *is the equality '$=$'.*

(4) The quadruple $(\blacklozenge_1, \blacklozenge_2, \blacklozenge_3, \blacklozenge_4)$ *is one of the following:*
$(<,<,=,=)$, $(<,<,<,>)$, $(<,<,>,<)$ *or* $(>,>,<,<)$.

Proof. Statement (1) follows from (4.1.0.1), statement (2) follows from Lemma 4.1.2, and statement (3) follows from Lemma 4.1.3.
If $\blacklozenge_3$ or $\blacklozenge_4$ is the equality '$=$', then by (3), the quandle Q is trivial. In this case, $(\blacklozenge_1, \blacklozenge_2, \blacklozenge_3, \blacklozenge_4)$ must be $(<,<,=,=)$. If $\blacklozenge_3, \blacklozenge_4 \in \{<,>\}$, then by (1) and (2), $(\blacklozenge_1, \blacklozenge_2, \blacklozenge_3, \blacklozenge_4)$ must be one of the following quadruples:

(a) $(<,<,<,<)$, (b) $(<,<,<,>)$, (c) $(<,<,>,<)$, (d) $(<,<,>,>)$,
(e) $(>,>,<,<)$, (f) $(>,>,<,>)$, (g) $(>,>,>,<)$, (h) $(>,>,>,>)$.

To prove the assertion (4), we have to rule out the cases (a), (d), (f), (g) and (h). Let's begin by ruling out the case (g) first. Assume contrary that $(\blacklozenge_1, \blacklozenge_2, \blacklozenge_3, \blacklozenge_4) = (>,>,>,<)$. Let $x, y, z \in Q$ and let $x < y$. Then we have the following:

$$\begin{aligned} & z*x > z*y && \text{(since } \blacklozenge_3 \text{ is } >) && (4.1.0.2)\\ \Rightarrow\ & (z*x)*^{-1}x < (z*y)*^{-1}x && \text{(since } \blacklozenge_2 \text{ is } >) && (4.1.0.3)\\ \Rightarrow\ & z < z*y*^{-1}x && \text{(by right cancellation)} && (4.1.0.4) \end{aligned}$$

Furthermore, we have

$$\begin{aligned} & (z*y)*^{-1}x < (z*y)*^{-1}y && \text{(since } \blacklozenge_4 \text{ is } <) && (4.1.0.5)\\ \Rightarrow\ & z*y*^{-1}x < z && \text{(by right cancellation)} && (4.1.0.6) \end{aligned}$$

This is a contradiction to (4.1.0.4). The cases (a), (d) and (f) can be ruled out similarly.
Finally, we rule out the case (h). Assume contrary that $(\blacklozenge_1, \blacklozenge_2, \blacklozenge_3, \blacklozenge_4) = (>,>,>,>)$. Let $x, y, z \in Q$ and let $x < y$. Then we have the following:

$$\begin{aligned} & z*x > z*y && \text{(since } \blacklozenge_3 \text{ is } >) && (4.1.0.7)\\ \Rightarrow\ & x*^{-1}(z*x) < x*^{-1}(z*y) && \text{(since } \blacklozenge_4 \text{ is } >) && (4.1.0.8)\\ \Rightarrow\ & x*^{-1}z*x < x*^{-1}y*^{-1}z*y && \text{(by Lemma 2.5.6)} && (4.1.0.9) \end{aligned}$$

Furthermore, we have

$$
\begin{aligned}
& \left(x *^{-1} z\right) * x > \left(x *^{-1} z\right) * y && \text{(since } \blacklozenge_3 \text{ is } >) && (4.1.0.10)\\
\Rightarrow \quad & x *^{-1} z * x > x *^{-1} z * y && && (4.1.0.11)
\end{aligned}
$$

Combining (4.1.0.9) with (4.1.0.11), we get

$$
\begin{aligned}
& x *^{-1} z * y < x *^{-1} y *^{-1} z * y && && (4.1.0.12)\\
\Rightarrow \quad & \left(x *^{-1} z * y\right) *^{-1} y > \left(x *^{-1} y *^{-1} z * y\right) *^{-1} y && \text{(since } \blacklozenge_2 \text{ is } >) && (4.1.0.13)\\
\Rightarrow \quad & x *^{-1} z > x *^{-1} y *^{-1} z && \text{(by right cancellation)} && (4.1.0.14)\\
\Rightarrow \quad & \left(x *^{-1} z\right) * z < \left(x *^{-1} y *^{-1} z\right) * z && \text{(since } \blacklozenge_1 \text{ is } >) && (4.1.0.15)\\
\Rightarrow \quad & x < x *^{-1} y && \text{(by right cancellation)} && (4.1.0.16)
\end{aligned}
$$

We also have the following:

$$
\begin{aligned}
& x *^{-1} x > x *^{-1} y && \text{(since } \blacklozenge_4 \text{ is } >) && (4.1.0.17)\\
\Rightarrow \quad & x > x *^{-1} y && && (4.1.0.18)
\end{aligned}
$$

This is a contradiction to (4.1.0.16). □

Corollary 4.1.5. *Let $<$ be a linear order on a quandle Q. Then the quandle Q is trivial if and only if the order $<$ is of the type $(<,<,=,=)$.*

We remark that all the four possibilities for the quadruple $(\blacklozenge_1, \blacklozenge_2, \blacklozenge_3, \blacklozenge_4)$ in Theorem 4.1.4 (4) can be realized as we shall see in the following example.

Example 4.1.6. Consider the group $(\mathbb{R},+)$. For a non-zero $u \in \mathbb{R}$, let ϕ_u be the automorphism of $\mathbb{R}$ given by $\phi_u(x) = ux$. Then for the Alexander quandle $\operatorname{Alex}(\mathbb{R},\phi_u)$, the quandle operation $*$ and the dual operation $*^{-1}$ are given by $x * y = ux + (1-u)y$ and $x *^{-1} y = u^{-1}x + \left(1-u^{-1}\right) y$ for $x, y \in \operatorname{Alex}(\mathbb{R},\phi_u)$. With the usual linear order $<$ on $\mathbb{R}$, one can check the following:

- If $0 < u < 1$, then $<$ is a bi-ordering for $\operatorname{Alex}(\mathbb{R},\phi_u)$.
- If $u < 1$, then $<$ is a left-ordering for $\operatorname{Alex}(\mathbb{R},\phi_u)$.
- If $0 < u$, then $<$ is a right-ordering for $\operatorname{Alex}(\mathbb{R},\phi_u)$.

Further, the following properties of the ordering on $\operatorname{Alex}(\mathbb{R},\phi_u)$ can be checked easily.

- If $u=1$, then the order $<$ is of the type $(<,<,=,=)$.
- If $0<u<1$, then the order $<$ is of the type $(<,<,<,>)$.
- If $1<u$, then the order $<$ is of the type $(<,<,>,<)$.
- If $u<0$, then the order $<$ is of the type $(>,>,<,<)$.

Remark 4.1.7. Question 3.6 in [17] asks whether there exists an infinite non-commutative bi-orderable quandle. One can see that for $u \in (0,1) \setminus \{1/2\}$, the quandle $\mathrm{Alex}(\mathbb{R}, \phi_u)$ with the usual order $<$ on $\mathbb{R}$ is an infinite non-commutative bi-orderable quandle, thereby answering the question in an affirmative.

Proposition 4.1.8. *Let $<$ be a linear order on a quandle Q. Then the order $<$ is a bi-ordering on Q if and only if it is of the type $(<,<,<,>)$.*

Proof. It is trivial that if the ordering $<$ is of the type $(<,<,<,>)$, then it is a bi-ordering on Q. Conversely, suppose that $<$ is a bi-ordering on Q. Then we can say that the ordering is of the type $(<,\_,<,\_)$. Hence, by Lemma 4.1.2, the ordering $<$ is of the type $(<,<,<,\_)$. Now, suppose on the contrary that $<$ is not of the type $(<,<,<,>)$. Then $z*^{-1}x < z*^{-1}y$ for some $x,y,z \in Q$ with $x<y$. We have

$$\begin{aligned} & \left(z*^{-1}y\right)*x < \left(z*^{-1}y\right)*y && \text{(since $<$ is a left ordering on Q)}, && (4.1.0.19)\\ \Rightarrow\quad & z*^{-1}y*x < z && \text{(by right cancellation)}. && (4.1.0.20)\end{aligned}$$

Also, we have

$$\begin{aligned} & \left(z*^{-1}x\right)*x < \left(z*^{-1}y\right)*x && \text{(since $<$ is a right ordering on Q)}, && (4.1.0.21)\\ \Rightarrow\quad & z < z*^{-1}y*x && \text{(by right cancellation)}. && (4.1.0.22)\end{aligned}$$

But, this contradicts (4.1.0.20). $\square$

Proposition 4.1.9. *Let $<$ be a bi-ordering on a quandle Q. If $x,y \in Q$ are distinct elements, then*

1. $x*^{-1}y \blacklozenge x \blacklozenge x*y \blacklozenge y \blacklozenge y*^{-1}x$ *and*
2. $x*^{-1}y \blacklozenge x \blacklozenge y*x \blacklozenge y \blacklozenge y*^{-1}x$

for some $\blacklozenge \in \{<,>\}$.

Proof. Since $x \neq y$, we have $x \bullet y$ for some $\bullet \in \{<,>\}$. By Proposition 4.1.8 and axiom (Q1), we have

$$\text{(a) } x = x*^{-1}x \bullet^{-1} x*^{-1}y, \qquad \text{(b) } x = x*x \bullet x*y, \qquad \text{(c) } x*y \bullet y*y = y,$$
$$\text{(d) } y*^{-1}x \bullet^{-1} y*^{-1}y = y, \qquad \text{(e) } x = x*x \bullet y*x, \qquad \text{(f) } y*x \bullet y*y = y.$$

Combining these inequalities gives the desired result. $\square$

Corollary 4.1.10. *A quasi-commutative bi-orderable quandle is commutative.*

Proof. Let Q be a quasi-commutative quandle that is not commutative and $<$ be a bi-ordering on Q. Then there exist distinct elements x and y in Q such that at least one of the following hold: $x*y = y*^{-1}x$, $x*^{-1}y = y*x$ or $x*^{-1}y = y*^{-1}x$. By Proposition 4.1.9, $x*y \bullet y*^{-1}x$, $x*^{-1}y \bullet y*x$ and $x*^{-1}y \bullet y*^{-1}x$ for some $\bullet \in \{<,>\}$. This is a contradiction. $\square$

4.2 Constructions of orderable quandles

We begin with the following observation.

Theorem 4.2.1. *If a semi-latin quandle is right-orderable, then it acts faithfully on a linearly ordered set by order-preserving bijections. Conversely, if a quandle acts faithfully on a well-ordered set by order-preserving bijections, then it is right-orderable.*

Proof. Let Q be a semi-latin quandle that is right-ordered with respect to a linear order $<$. Taking $X = Q$ and defining $\phi : Q \to \mathrm{Conj}_{-1}(\mathrm{S}_X)$ by $\phi(q) = S_q$, we see that ϕ is an action of Q on the ordered set X. Further, if $q \in Q$ and $x, y \in X$ such that $x < y$, then right-orderability of Q implies that

$$\phi(q)(x) = S_q(x) = x*q < y*q = S_q(y) = \phi(q)(y).$$

Further, if $p, q \in Q$ such that $\phi(p) = \phi(q)$, then Q being semi-latin implies that $p = q$. Hence, Q acts faithfully on X by order-preserving bijections.

Conversely, suppose that $\phi : Q \to \mathrm{Conj}_{-1}(\mathrm{S}_X)$ is a faithful action of Q on a well-ordered set X by order-preserving bijections. Let $<$ be the well-order on X. We use the order $<$ to define an order on the quandle Q as follows. For $p, q \in Q$ with $p \neq q$, consider the set $A_{p,q} = \{x \in X \mid \phi(p)(x) \neq \phi(q)(x)\}$. Faithfulness of the action implies that $\phi(p) \neq \phi(q)$, and hence $A_{p,q}$ is a non-empty subset of X. Since $<$ is a well-ordering on X, the set $A_{p,q}$ must admit the smallest element, say x_0, with respect to $<$. We define $p \prec q$ if $\phi(p)(x_0) < \phi(q)(x_0)$ and $q \prec p$ if $\phi(q)(x_0) < \phi(p)(x_0)$.

It is enough to check transitivity of $\prec$ on Q. Let $p \prec q$ and $q \prec r$. Let $A_{p,q} = \{x \in X \mid \phi(p)(x) \neq \phi(q)(x)\}$, $A_{q,r} = \{x \in X \mid \phi(q)(x) \neq \phi(r)(x)\}$ and $A_{p,r} = \{x \in X \mid \phi(p)(x) \neq \phi(r)(x)\}$. Since $p \prec q$ and $q \prec r$, it follows that $A_{p,r}$ is non-empty. Let x_0, y_0 and z_0 be the smallest elements of the sets $A_{p,q}$, $A_{q,r}$ and $A_{p,r}$, respectively. Then we have the following cases:

- If $x_0 < y_0$, then $\phi(p)(x_0) < \phi(q)(x_0) = \phi(r)(x_0)$, which implies that $z_0 \leq x_0$. If $z_0 < x_0$, then $\phi(p)(z_0) = \phi(q)(z_0) \neq \phi(r)(z_0)$, which contradicts the fact that y_0 is the smallest element of $A_{q,r}$. Hence, $x_0 = z_0$ and $p \prec r$.
- If $x_0 = y_0$, then $\phi(p)(x_0) < \phi(q)(x_0) < \phi(r)(x_0)$, which gives $z_0 \leq x_0$. If $z_0 < x_0$, then $\phi(p)(z_0) = \phi(q)(z_0) \neq \phi(r)(z_0)$, which is a contradiction to the fact that y_0 is the smallest element of $A_{q,r}$. Hence, $z_0 = x_0$ and $p \prec r$.
- If $x_0 > y_0$, then $\phi(p)(y_0) = \phi(q)(y_0) < \phi(r)(y_0)$, which further gives $z_0 \leq y_0$. If $z_0 < y_0$, then $\phi(p)(z_0) = \phi(q)(z_0) \neq \phi(r)(z_0)$, which is again a contradiction to the fact that y_0 is the smallest element of $A_{q,r}$. Hence, $z_0 = y_0$ and $p \prec r$.

Now, suppose that $p,q,r \in Q$ such that $p \prec q$. Let $A_{p,q} = \{x \in X \mid \phi(p)(x) \neq \phi(q)(x)\}$ and $A_{p*r,q*r} = \{x \in X \mid \phi(p*r)(x) \neq \phi(q*r)(x)\}$. Since both $A_{p,q}$ and $A_{p*r,q*r}$ are non-empty, they admit smallest elements, say x_0 and y_0, respectively. Notice that the bijection $\phi(r)$ maps $A_{p,q}$ onto $A_{p*r,q*r}$. Since $\phi(r)$ is order-preserving with respect to $<$, we have $\phi(r)(x_0) = y_0$. Since $p \prec q$, we have $\phi(p)(x_0) < \phi(q)(x_0)$, which implies that $\phi(p)\phi(r)^{-1}(y_0) < \phi(q)\phi(r)^{-1}(y_0)$. Since ϕ is a quandle homomorphism and $\phi(r)$ is order-preserving, this gives

$$
\begin{aligned}
\phi(p*r)(y_0) &= \phi(p)*\phi(r)(y_0) \\
&= \phi(r)\phi(p)\phi(r)^{-1}(y_0) \\
&< \phi(r)\phi(q)\phi(r)^{-1}(y_0) \\
&= \phi(q)*\phi(r)(y_0) \\
&= \phi(q*r)(y_0),
\end{aligned}
$$

and hence $p*r \prec q*r$. Thus, Q is a right orderable quandle. □

Next, we give three constructions of orderable quandles.

Proposition 4.2.2. *Let $(Q_1,*)$ and $(Q_2,\circ)$ be right-orderable quandles, and $\sigma : Q_1 \to \mathrm{Conj}_{-1}(\mathrm{Aut}(Q_2))$ and $\tau : Q_2 \to \mathrm{Conj}_{-1}(\mathrm{Aut}(Q_1))$ be order-preserving quandle actions. Suppose that*

1. $\tau(z)(x) * y = \tau(\sigma(y)(z))(x * y)$ *for* $x, y \in Q_1$ *and* $z \in Q_2$,

2. $\sigma(z)(x) \circ y = \sigma(\tau(y)(z))(x \circ y)$ *for* $x, y \in Q_2$ *and* $z \in Q_1$.

Then $Q = Q_1 \sqcup Q_2$ *with the operation*

$$x \star y = \begin{cases} x * y, & x, y \in Q_1, \\ x \circ y, & x, y \in Q_2, \\ \tau(y)(x), & x \in Q_1, y \in Q_2, \\ \sigma(y)(x), & x \in Q_2, y \in Q_1, \end{cases}$$

is a right-orderable quandle.

Proof. That Q is a quandle follows from [6, Proposition 11]. Let $<_1$ and $<_2$ be the right-orders on Q_1 and Q_2, respectively. Define an order $<$ on Q by setting $x < y$ iff $x, y \in Q_1$ and $x <_1 y$ or $x, y \in Q_2$ and $x <_2 y$ or $x \in Q_1$ and $y \in Q_2$. A direct check shows that $<$ is indeed a linear order on Q. We claim that $<$ turns Q into a right orderable quandle. Let $x, y, z \in Q$ such that $x < y$. We have the following cases:

- $x, y, z \in Q_1$ or $x, y, z \in Q_2$: In this case, since Q_1 and Q_2 are right-orderable, we get $x \star z < y \star z$.

- $x, y \in Q_1$ and $z \in Q_2$: In this case, since $\tau(z)$ is order preserving, we have $x \star z = \tau(z)(x) <_1 \tau(z)(y) = y \star z$, and hence $x \star z < y \star z$.

- $x, y \in Q_2$ and $z \in Q_1$: In this case, $\sigma(z)$ being order preserving implies that $x \star z = \sigma(z)(x) <_2 \sigma(z)(y) = y \star z$, and hence $x \star z < y \star z$.

- $x, z \in Q_1$ and $y \in Q_2$: In this case, $x \star z = x * z \in Q_1$ and $y \star z = \sigma(z)(y) \in Q_2$, and hence $x \star z < y \star z$.

- $x \in Q_1$ and $y, z \in Q_2$: In this case, $x \star z = \tau(z)(x) \in Q_1$ and $y \star z = y \circ z \in Q_2$, and hence $x \star z < y \star z$.

Thus, Q is a right-orderable quandle. $\square$

If $\sigma : Q_1 \to \mathrm{Id}_{Q_2}$ and $\tau : Q_2 \to \mathrm{Id}_{Q_1}$ are the trivial actions, then conditions (1) and (2) of Proposition 4.2.2 always hold. Thus, the disjoint union of two right-orderable quandles is right-orderable. It is clear that the disjoint union of two left-orderable quandles is not left-orderable.

Let $\{Q_i, *_i\}_{i\in\Lambda}$ be a family of quandles and $Q = \prod_{i\in\Lambda} Q_i$ their Cartesian product. Then Q is a quandle with $(x_i)*(y_i) = (x *_i y_i)$ and called the *product quandle*. The following observation is rather immediate, but we give a proof for the sake of completeness.

Proposition 4.2.3. *The product of right-orderable quandles is a right-orderable quandle. Similarly, the product of left-orderable (bi-orderable) quandles is a left-orderable (bi-orderable) quandle.*

Proof. Let $\{Q_i, *_i\}_{i\in\Lambda}$ be a family of right-orderable quandles. Let $<_i$ be the right-order on Q_i for $i \in \Lambda$ and Q their quandle. By axiom of choice, we can take a well-ordering $<$ on the indexing set Λ. Let $(x_i),(y_i) \in Q$ such that $(x_i) \neq (y_i)$. Then there exists the least index $\ell \in \Lambda$ such that $x_\ell \neq y_\ell$. We define $(x_i) \prec (y_i)$ if $x_\ell <_\ell y_\ell$ and $(y_i) \prec (x_i)$ if $y_\ell <_\ell x_\ell$. It is easy to check that $\prec$ is a linear order on Q.
Let $(x_i),(y_i),(z_i) \in Q$ such that $(x_i) \prec (y_i)$. Then $x_\ell <_\ell y_\ell$, where ℓ is the least index such that $x_\ell \neq y_\ell$. The second quandle axiom in Q implies that $(x_i *_i z_i) = (x_i)*(z_i) \neq (y_i)*(z_i) = (y_i *_i z_i)$. It turns out that ℓ is also the least index for which $x_\ell *_\ell z_\ell \neq y_\ell *_\ell z_\ell$. Since $x_\ell <_\ell y_\ell$ and Q_ℓ is right orderable, it follows that $x_\ell *_\ell z_\ell <_\ell y_\ell *_\ell z_\ell$. By definition of $\prec$, we have $(x_i)*(z_i) \prec (y_i)*(z_i)$. Thus, Q is a right-orderable quandle. The second assertion follows analogously. $\square$

Let Q be a quandle and A a set. Following [1, Section 2.1], a *dynamical 2-cocycle* is a map $\alpha : Q\times Q \to \mathrm{Map}(A\times A, A)$ such that

$$\alpha_{x,x}(s,s) = s, \tag{4.2.0.1}$$

$$\alpha_{x,y}(-,t) : A \to A \text{ is a bijection} \tag{4.2.0.2}$$

and the *cocycle condition*

$$\alpha_{x*y,z}(\alpha_{x,y}(s,t),\ u) = \alpha_{x*z,y*z}(\alpha_{x,z}(s,u),\ \alpha_{y,z}(t,u)) \tag{4.2.0.3}$$

holds for all $x,y,z \in Q$ and $s,t,u \in A$. Given a dynamical 2-cocycle α, the set $Q\times A$ can then be turned into a quandle denoted as $Q\times_\alpha A$ by defining

$$(x,s)*(y,t) = (x*y,\ \alpha_{x,y}(s,t)). \tag{4.2.0.4}$$

The equations (4.2.0.1), (4.2.0.2) and (4.2.0.3) give the quandle axioms (Q1), (Q2) and (Q3), respectively. The quandle $Q\times_\alpha A$ is called the *extension* of Q by A through α.

If A is an abelian group, then a *normalized quandle 2-cocycle* is a map $\alpha : Q \times Q \to A$ satisfying

$$\alpha_{x,y}\ \alpha_{x*y,z} = \alpha_{x,z}\ \alpha_{x*z,y*z}$$

and

$$\alpha_{x,x} = 1$$

for all $x,y,z \in Q$. A normalized quandle 2-cocycle $\alpha : Q \times Q \to A$ gives rise to a dynamical 2-cocycle $\alpha' : Q \times Q \to \mathrm{Map}(A \times A, A)$ by setting

$$\alpha'_{x,y}(s,t) = s\ \alpha_{x,y}.$$

In this case, the quandle $X \times_\alpha A$ is called the *abelian extension* of Q by A through α. Such extensions appeared first in [34].

Proposition 4.2.4. *The following statements hold:*

(1) Let Q be a right-orderable quandle, A an ordered set and $\alpha : Q \times Q \to \mathrm{Map}(A \times A, A)$ a dynamical 2-cocycle. If $\alpha_{x,y} : A \times A \to A$ is order-preserving for all $x,y \in Q$, then the quandle $Q \times_\alpha A$ is right-orderable.

(2) If Q is a right-orderable quandle, A a right-orderable abelian group and $\alpha : Q \times Q \to A$ a normalized 2-cocycle, then the quandle $X \times_\alpha A$ is right-orderable.

(3) If Q is a quandle, A a non-trivial abelian group and $\alpha : Q \times Q \to A$ a normalized 2-cocycle, then the quandle $X \times_\alpha A$ cannot be left-orderable.

Proof. Let $<$ be a right-order on Q and $<'$ an order on A. Consider the set $Q \times A$ with the induced lexicographic order $\prec$ and $A \times A$ equipped with the lexicographic order $\prec'$. Let $(x,s),(y,t),(z,u) \in Q \times A$. By (4.2.0.4), we have $(x,s)*(z,u) = (x*z, \alpha_{x,z}(s,u))$ and $(y,t)*(z,u) = (y*z, \alpha_{y,z}(t,u))$. If $(x,s) \prec (y,t)$, then we have two cases:

- If $x < y$, then right-orderability of Q implies that $x*z < y*z$, and hence $(x,s)*(z,u) \prec (y,t)*(z,u)$.
- If $x = y$ and $s <' t$, then $(s,u) \prec' (t,u)$ and $\alpha_{x,z}$ being order-preserving implies that $\alpha_{x,z}(s,u) <' \alpha_{x,z}(t,u) = \alpha_{y,z}(t,u)$.

Hence, $(x,s)*(z,u) \prec (y,t)*(z,u)$, and the quandle $X \times_\alpha A$ is right-orderable proving (1). Define $\alpha'_{x,y}(s,t) = s\ \alpha_{x,y}$ for all $x,y \in Q$ and $s,t \in A$. Then α' is a dynamical 2-cocycle. The right-orderability of the abelian group A implies that

$$\alpha'_{x,z}(s,u) = s\ \alpha_{x,z} <' t\ \alpha_{x,z} = \alpha'_{x,z}(t,u)$$

for all $x,z \in Q$ and $s,t,u \in A$ with $s <' t$. The proof of assertion (2) now follows along the lines of that of assertion (1).
For assertion (3), notice that any left-orderable quandle must be semi-latin. But, for any $(x,s),(x,t),(z,u) \in Q \times A$ with $s \neq t$, we have

$$(z,u)*(x,s) = (z*x, u\ \alpha_{z,x}) = (z,u)*(x,t).$$

Hence, the abelian extension $X \times_{\alpha} A$ cannot be left-orderable. □

We conclude this section with some observations on order-preserving automorphisms of orderable quandles. Let $\mathrm{Aut}^{\circ}(Q)$ denote the group of order-preserving automorphisms of a quandle Q equipped with an order. Similarly, let $\mathrm{Aut}^{\circ}(G)$ denote the group of order-preserving automorphisms of a group G equipped with an order.

Proposition 4.2.5. *If Q is a right-orderable quandle, then* $\mathrm{Inn}(Q)$ *is a subgroup of* $\mathrm{Aut}^{\circ}(Q)$.

Proof. Let $<$ be a right-order on Q, $x,y \in Q$ with $x<y$ and $S_{z_1}^{d_1} S_{z_2}^{d_2} \cdots S_{z_k}^{d_k} \in \mathrm{Inn}(Q)$, where each $d_i \in \{1,-1\}$ and $z_i \in Q$. Then, right-orderability of Q and Lemma 4.1.2 implies that

$$S_{z_1}^{d_1} S_{z_2}^{d_2} \cdots S_{z_k}^{d_k}(x) = x *^{d_k} z_k *^{d_{k-1}} \cdots *^{d_1} z_1 < y *^{d_k} z_k *^{d_{k-1}} \cdots *^{d_1} z_1 = S_{z_1}^{d_1} S_{z_2}^{d_2} \cdots S_{z_k}^{d_k}(y),$$

and hence $\mathrm{Inn}(Q) \leq \mathrm{Aut}^{\circ}(Q)$. □

Note that Proposition 4.2.5 fails if Q is a left-orderable quandle. In Example 4.1.6, if we take $u<0$, then the quandle $\mathrm{Alex}(\mathbb{R},\phi_u)$ is left-orderable. However, if $x,y,z \in \mathrm{Alex}(\mathbb{R},\phi_u)$ with $x<y$, then $S_z(y) < S_z(x)$.

Proposition 4.2.6. *The following hold for any bi-orderable group G:*

1. *There is an embedding of groups* $\mathrm{Z}(G) \rtimes \mathrm{Aut}^{\circ}(G) \hookrightarrow \mathrm{Aut}^{\circ}(\mathrm{Conj}(G))$.
2. *If G has trivial centre, then* $\mathrm{Aut}^{\circ}(\mathrm{Conj}(G)) = \mathrm{Aut}^{\circ}(G)$.

Proof. Let $<$ be a bi-ordering on the group G. Then, by Proposition 4.0.4 (1), $\mathrm{Conj}(G)$ is a right-orderable quandle with respect to $<$. By [11, Proposition 4.7], $\mathrm{Z}(G) \rtimes \mathrm{Aut}(G) \hookrightarrow \mathrm{Aut}(\mathrm{Conj}(G))$, where each central element $z \in \mathrm{Z}(G)$ act on $\mathrm{Conj}(G)$ by left translation t_z by z. Left orderability of G implies that $t_z \in \mathrm{Aut}^{\circ}(\mathrm{Conj}(G))$ for all $z \in \mathrm{Z}(G)$. Since $\mathrm{Aut}^{\circ}(G) \leq \mathrm{Aut}^{\circ}(\mathrm{Conj}(G))$, we obtain $\mathrm{Z}(G) \rtimes \mathrm{Aut}^{\circ}(G) \hookrightarrow \mathrm{Aut}^{\circ}(\mathrm{Conj}(G))$.
It follows from [6, Corollary 1] that if G has trivial centre, then $\mathrm{Aut}(\mathrm{Conj}(G)) = \mathrm{Aut}(G)$, and hence $\mathrm{Aut}^{\circ}(\mathrm{Conj}(G)) = \mathrm{Aut}^{\circ}(G)$. □

Proposition 4.2.7. *Let G be a bi-orderable group and $\varphi \in \mathrm{Aut}(G)$ an order-reversing automorphism. Then the following hold:*

1. *There is an embedding* $\mathrm{Z}(G) \rtimes \mathrm{C}_{\mathrm{Aut}^\circ(G)}(\varphi) \hookrightarrow \mathrm{Aut}^\circ\left(\mathrm{Alex}(G,\varphi)\right)$, *where* $\mathrm{C}_{\mathrm{Aut}^\circ(G)}(\varphi) = \{f \in \mathrm{Aut}^\circ(G) \mid f\phi = \phi f\}$.

2. *If G is a torsion free abelian group, then* $\mathrm{Aut}^\circ\left(\mathrm{Core}(G)\right) \cong G \rtimes \mathrm{Aut}^\circ(G)$.

Proof. Let $<$ be a bi-ordering on the group G and $\phi \in \mathrm{Aut}(G)$ an order-reversing automorphism. Then, by Proposition 4.0.4 (3), $\mathrm{Alex}(G,\phi)$ is a left-orderable quandle with respect to the order $<$. Further, by [11, Proposition 4.1], $\mathrm{Z}(G) \rtimes \mathrm{C}_{\mathrm{Aut}(G)}(\varphi) \hookrightarrow \mathrm{Aut}\left(\mathrm{Alex}(G,\varphi)\right)$, where an element $z \in \mathrm{Z}(G)$ act on the quandle $\mathrm{Alex}(G,\phi)$ by left translation t_z. Since G is left-orderable, the translation $t_z \in \mathrm{Aut}^\circ\left(\mathrm{Alex}(G,\varphi)\right)$ for each $z \in \mathrm{Z}(G)$. Further, if $f \in \mathrm{C}_{\mathrm{Aut}^\circ(G)}(\varphi)$, then $f \in \mathrm{Aut}^\circ\left(\mathrm{Alex}(G,\varphi)\right)$, and hence $\mathrm{Z}(G) \rtimes \mathrm{C}_{\mathrm{Aut}^\circ(G)}(\varphi) \hookrightarrow \mathrm{Aut}^\circ\left(\mathrm{Alex}(G,\varphi)\right)$. For the second assertion, note that every torsion free abelian group G is bi-orderable. Taking $\varphi(g) = g^{-1}$ for all $g \in G$, we have $\mathrm{Alex}(G,\varphi) = \mathrm{Core}(G)$ and $\mathrm{C}_{\mathrm{Aut}^\circ(G)}(-\mathrm{Id}_G) = \mathrm{Aut}^\circ(G)$. By [11, Theorem 4.2], $\mathrm{Aut}(\mathrm{Core}(G)) \cong G \rtimes \mathrm{Aut}(G)$, and hence $\mathrm{Aut}^\circ(\mathrm{Core}(G)) \cong G \rtimes \mathrm{Aut}^\circ(G)$. □

4.3 Orderability of some general quandles

In this section, we discuss orderability of some general quandles. Let G be a group and A a subset of G. Let $R(G,A)$ be a (G,A)-rack and $Q(G,A)$ be the corresponding (Q,A)-quandle as defined in Section 2.5. There is a natural rack homomorphism $\varepsilon : R(G,A) \to \mathrm{Conj}(G)$ defined as $\varepsilon(a,u) = u^{-1}au$. Moreover, this map induces a quandle homomorphism

$$\overline{\varepsilon} : Q(G,A) \to \mathrm{Conj}(G)$$

defined as $\overline{\varepsilon}(a,u) = u^{-1}au$.

Theorem 4.3.1. *Let G be a group and A be a subset of G.*

(1) If G is right-orderable, then the rack $R(G,A)$ is right-orderable.

(2) If G is bi-orderable, then the quandle $Q(G,A)$ is right-orderable.

Proof. (1) Let $<$ be a right ordering on G. We define a linear order $<'$ on $R(G,A)$ as follows. Let (a,u) and (b,v) be two distinct elements of $R(G,A)$.

- If $a \neq b$, define $(a,u) <' (b,v)$ if $a < b$ and $(b,v) <' (a,u)$ if $b < a$.

- If $a = b$, define $(a,u) <' (a,v)$ if $u < v$ and $(a,v) <' (a,u)$ if $v < u$.

Let $(a,u),(b,v),(c,w) \in R(G,A)$ such that $(a,u) <' (b,v)$. If $a \neq b$, then $a < b$, and hence $(a,u) * (c,w) <' (b,v) * (c,w)$. If $a = b$, then $u < v$. Since G is right-ordered with respect to $<$, it follows that $uw^{-1}cw < vw^{-1}cw$, and hence $(a,u) * (c,w) <' (a,v) * (c,w)$. This shows that $R(G,A)$ is a right-orderable rack.

(2) Let $<$ be a bi-ordering on G. Define a linear order $<'$ on $Q(G,A)$ as follows. Let (a,u) and (b,v) be two distinct elements of $Q(G,A)$.

- If $a \neq b$, then define $(a,u) <' (b,v)$ if $a < b$ and $(b,v) <' (a,u)$ if $b < a$.
- If $a = b$, then we define the order using the image of (a,u) and (a,v) under the map $\overline{\varepsilon} : Q(G,A) \to G$. Notice that, if $(a,u) \neq (a,v)$ in $Q(G,A)$, then $\overline{\varepsilon}(a,u) \neq \overline{\varepsilon}(a,v)$. For, if $u^{-1}au = v^{-1}av$, then $vu^{-1}a = avu^{-1}$; this implies that $vu^{-1} \in \mathrm{C}_G(a)$ and hence $(a,u) = (a,v)$. Now, define $(a,u) <' (a,v)$ if $u^{-1}au < v^{-1}av$ and $(a,v) <' (a,u)$ if $v^{-1}av < u^{-1}au$.

We claim that $Q(G,A)$ is right-ordered with respect to $<'$. Let $(a,u),(b,v),(c,w) \in R(G,A)$ such that $(a,u) <' (b,v)$. If $a \neq b$, then $a < b$, and hence $(a,u) * (c,w) <' (b,v) * (c,w)$. If $a = b$, then $u^{-1}au < v^{-1}av$. Since G is bi-ordered with respect to $<$, we have $w^{-1}c^{-1}wu^{-1}auw^{-1}cw < w^{-1}c^{-1}wv^{-1}avw^{-1}cw$, and hence $(a,u) * (c,w) <' (a,v) * (c,w)$. This shows that $Q(G,A)$ is right-orderable. □

If A is the set of representatives of conjugacy classes of a group G, then $Q(G,A) \cong \mathrm{Conj}(G)$. Thus, we recover Proposition 4.0.4 (1). Further, since free groups are bi-orderable [118], we retrieve the following result of [17, Theorem 3.5].

Corollary 4.3.2. *Free quandles are right-orderable. In particular, link quandles of trivial links are right orderable.*

Next, we give a sufficient condition for the failure of left-orderability in quandles.

Proposition 4.3.3. *Let Q be a quandle generated by a set X such that the map $\eta : Q \to \mathrm{Conj}(\mathrm{Env}(Q))$ is injective. If there exist two distinct commuting elements in $\mathrm{Env}(Q)$ that are not inverses of each other and that are conjugates of elements from $\eta(X)^{\pm 1}$, then the quandle Q is not left-orderable.*

Proof. Recall from Theorem 2.5.8 that the set $\eta(X) = \{e_x \mid x \in X\}$ is a generating set for the enveloping group $\mathrm{Env}(Q)$. Let $\eta(X)^{-1}$ denote the set of inverses of elements in $\eta(X)$, and let $\tilde{a}, \tilde{b} \in \mathrm{Env}(Q)$ with $\tilde{a}^{\pm 1} \neq \tilde{b}$ be two commuting elements that are conjugates of elements from $\eta(X)^{\pm 1}$. Then we can write

$$\begin{aligned}\tilde{a} &= e_{x_1}^{-d_1} e_{x_2}^{-d_2} \cdots e_{x_{m-1}}^{-d_{m-1}} e_{x_m}^{d_m} e_{x_{m-1}}^{d_{m-1}} \cdots e_{x_2}^{d_2} e_{x_1}^{d_1} \text{ and}\\ \tilde{b} &= e_{y_1}^{-e_1} e_{y_2}^{-e_2} \cdots e_{y_{n-1}}^{-e_{n-1}} e_{y_n}^{e_n} e_{y_{n-1}}^{e_{n-1}} \cdots e_{y_2}^{e_2} e_{y_1}^{e_1},\end{aligned}$$

where $e_{x_i}, e_{y_i} \in \eta(X)$ and $d_i, e_i \in \{-1, 1\}$ for all i. For each i, there exist $x_i, y_i \in X$ such that $e_{x_i} = \eta(x_i)$ and $e_{y_i} = \eta(y_i)$. We get

$$\begin{aligned}\tilde{a}^{d_m} &= \eta(x_1)^{-d_1}\eta(x_2)^{-d_2} \cdots \eta(x_{m-1})^{-d_{m-1}}\eta(x_m)\eta(x_{m-1})^{d_{m-1}} \cdots \eta(x_2)^{d_2}\eta(x_1)^{d_1}\\ &= \eta(x_m) *^{d_{m-1}} \eta(x_{m-1}) *^{d_{m-2}} \cdots *^{d_1} \eta(x_1), \text{ by quandle operation in } \mathrm{Conj}(\mathrm{Env}(Q))\\ &= \eta\left(x_m *^{d_{m-1}} x_{m-1} *^{d_{m-2}} \cdots *^{d_1} x_1\right), \text{ since } \eta \text{ is a quandle homomorphism}\\ &= \eta(a)\end{aligned}$$

and similarly

$$\tilde{b}^{e_n} = \eta(b),$$

where $a = x_m *^{d_{m-1}} x_{m-1} *^{d_{m-2}} \cdots *^{d_1} x_1$ and $b = y_n *^{e_{n-1}} y_{n-1} *^{e_{n-2}} \cdots *^{e_1} y_1$.
Suppose on the contrary that the quandle Q is left-ordered with respect to a linear order $<$. Since $\tilde{a}^{\pm 1} \neq \tilde{b}$, we get $\tilde{a}^{d_m} \neq \tilde{b}^{e_n}$, and thus $\eta(a) \neq \eta(b)$. This implies that $a \neq b$. In other words, we have $a \blacklozenge b$ for some $\blacklozenge \in \{<, >\}$, and hence $a = a * a \blacklozenge a * b$. Since $\tilde{b}^{-1}\tilde{a}\tilde{b} = \tilde{a}$, we have $\tilde{b}^{-1}\tilde{a}^{d_m}\tilde{b} = \tilde{a}^{d_m}$, and thus

$$\eta(a *^{e_n} b) = \eta(a) *^{e_n} \eta(b) = \eta(b)^{-e_n}\eta(a)\eta(b)^{e_n} = \tilde{b}^{-1}\tilde{a}^{d_m}\tilde{b} = \tilde{a}^{d_m} = \eta(a).$$

The map η being a monomorphism gives $a *^{e_n} b = a$, and hence $a * b = a$. This is a contradiction, since we have $a \blacklozenge a * b$. □

If Q is a trivial quandle with more than one element, then its enveloping group $\mathrm{Env}(Q)$ is the free abelian group of rank $|Q|$. Thus, if $x, y \in Q$ are two distinct elements, then $e_x, e_y \in \mathrm{Env}(Q)$ are two distinct commuting elements that are not inverses of each other. Thus, Q is not left-orderable, which can also be checked directly.

Corollary 4.3.4. *Let K be a prime knot such that $Q(K)$ is generated by a set X. If there exist two distinct commuting elements in $\pi_1(C(K))$ that are not inverses of each other and that are conjugates of elements from $\eta(X)^{\pm 1}$, then $Q(K)$ is not left-orderable.*

Proof. If K is a prime knot, then by [112, Corollary 3.6], the map $\eta : Q(K) \to \mathrm{Conj}(\pi_1(C(K)))$ is a monomorphism of quandles. The result now follows from Proposition 4.3.3. □

4.4 Orderability of some link quandles

Problem 3.16 in [17] asks to determine whether link quandles are orderable. We investigate orderability of link quandles in the remaining two sections and provide solution to this problem in some cases. The next result relates orderability of the enveloping group of a quandle to that of the quandle itself.

Proposition 4.4.1. *Let Q be a quandle such that the natural map $\eta : Q \to \mathrm{Conj}(\mathrm{Env}(Q))$ is injective. If $\mathrm{Env}(Q)$ is a bi-orderable group, then Q is a right-orderable quandle.*

Proof. Since $\mathrm{Env}(Q)$ is a bi-orderable group, by Proposition 4.0.4 (1), $\mathrm{Conj}(\mathrm{Env}(Q))$ is a right-orderable quandle. Since η is injective, it follows that Q is right-orderable. □

Corollary 4.4.2. *If Q is a commutative, latin or simple quandle such that $\mathrm{Env}(Q)$ is a bi-orderable group, then Q is right-orderable.*

Proof. It is not difficult to see that the map η is injective for a commutative, latin or simple quandle. □

Corollary 4.4.3. *If the knot group of a prime knot is bi-orderable, then its knot quandle is right-orderable.*

Proof. Let K be a prime knot such that its knot group $\pi_1(C(K))$ is bi-orderable. Since K is prime, by [112, Corollary 3.6], the map $\eta : Q(K) \to \mathrm{Conj}(\pi_1(C(K)))$ is injective. Thus, by Proposition 4.4.1, the knot quandle $Q(K)$ is right-orderable. □

A knot K is a *fibered knot* if there is a fibration from its complement $\mathbb{S}^3 \setminus K$ to the circle $\mathbb{S}^1$ with fiber a surface. For example, all torus knots are fibered knots [108, p.336].

Corollary 4.4.4. *If all the roots of the Alexander polynomial of a fibered prime knot are real and positive, then its knot quandle is right-orderable.*

Proof. Let K be a fibered prime knot all the roots of whose Alexander polynomial are real and positive. Then, by [104, Theorem 1.1], $\pi_1(C(K))$ is a bi-orderable group. The result now follows from Corollary 4.4.3. □

As a special case, it follows that the knot quandle of the figure eight knot is right-orderable.

Definition 4.4.5. A link L is said to be *positive* if there exists a diagram $D(L)$ of L such that all its crossings are positive.

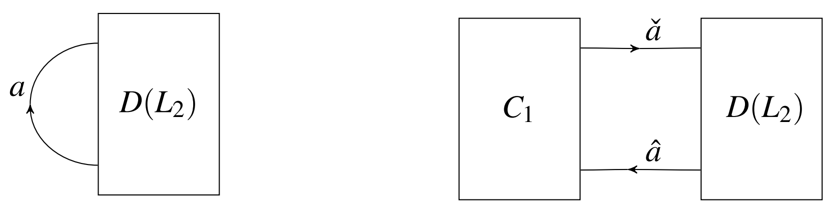
A diagram $D(L)$ of a link L is said to be

- *minimal* if it is having the minimal number of crossings among all diagrams of L.
- *positive* if all its crossings are positive.
- *positive minimal* if it is both positive as well as minimal.
- *minimal positive* if it is positive and having the minimal number of crossings among all positive diagrams of L.

The terms *negative link*, *negative diagram*, *negative minimal diagram* and *minimal negative diagram* are defined analogously.
If a positive minimal diagram exists for a positive link L, then it is always a minimal positive diagram of L. There are examples of positive links for which positive minimal diagrams do not exist. For example, the number of crossings in a minimal positive diagram of the knot 11_{550} is 12 while its crossing number is 11. In other words, a positive minimal diagram does not exist for this knot. See [98, 115] for more details.

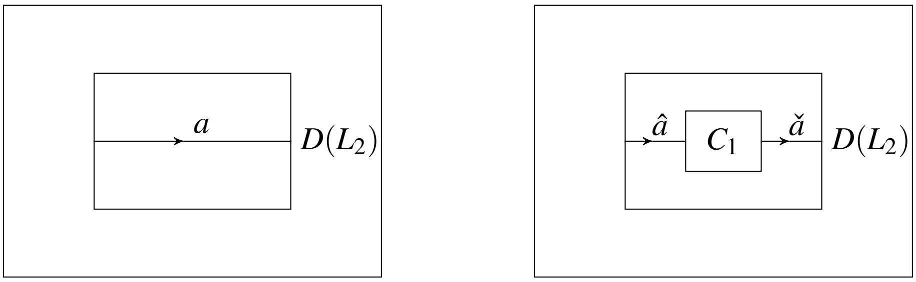
Theorem 4.4.6. *Let L_1 be any oriented link and L_2 a non-trivial positive (negative) oriented link. Suppose there exists a minimal positive (negative) diagram $D(L_2)$ of L_2 such that the generators of the link quandle $Q(L_2)$ corresponding to the arcs in $D(L_2)$ are pairwise distinct. Then the link quandle of a connected sum of links L_1 and L_2 is not bi-orderable. In particular, the link quandle $Q(L_2)$ is not bi-orderable.*

Proof. Let $L = L_1 \# L_2$ be the link obtained by taking the connected sum of a component K_1 of L_1 with a component K_2 of L_2. Suppose $D(L_1)$ be a diagram of L_1 such that the component K_1 of L_1 has an exterior arc in $D(L_1)$, and $D(L_2)$ be a diagram of L_2 as described in the hypothesis of the theorem. Let $D(L)$ be a diagram of L obtained using diagrams $D(L_1)$ and $D(L_2)$ without introducing any extra crossing and possibly turning over the diagram $D(L_1)$ if required. The diagram $D(L)$ looks as shown in Figure 4.1 or in Figure 4.2 depending on whether the component K_2 of L_2 has an exterior arc in $D(L_2)$ or not. In both the figures, the diagram C_1 is either $D(L_1)$ or it is obtained by turning over $D(L_1)$.



(a) A diagram $D(L_2)$ of L_2 (b) A diagram $D(L)$ of L

Fig. 4.1 If the component K_2 of L_2 has an exterior arc in $D(L_2)$.



(a) A diagram $D(L_2)$ of L_2. (b) A diagram $D(L)$ of L.

Fig. 4.2 If the component K_2 of L_2 has no exterior arc in $D(L_2)$.

Let $x_0, x_1, \ldots, x_n$ be the generators of the link quandle $Q(L_2)$ corresponding to the arcs in $D(L_2)$. We may assume that x_0 corresponds to the arc a in $D(L_2)$ that splits into the connecting arcs $\check{a}$ and $\hat{a}$ in $D(L)$. Looking at Figure 4.1 and Figure 4.2, the arc $\check{a}$ is an incoming arc to $D(L_2)$ and the arc $\hat{a}$ is an outgoing arc from $D(L_2)$. Let $\check{x}_0$ and $\hat{x}_0$ be the elements in the link quandle $Q(L)$ that correspond to the arcs $\check{a}$ and $\hat{a}$ respectively. By the hypothesis of the theorem, the generators $x_0, x_1, \ldots, x_n$ are pairwise distinct in $Q(L_2)$, and thus the elements $\check{x}_0, \hat{x}_0, x_1, x_2, \ldots, x_n$ are pairwise distinct in $Q(L)$ except possibly for the pair $\check{x}_0$ and $\hat{x}_0$.

Suppose on the contrary that the quandle $Q(L)$ is bi-ordered with respect to a linear order $<$. Then we have the smallest and the largest elements in any finite subset of $Q(L)$. Let us consider the following cases:

Case 1: L_2 is a positive link: Let $\hat{y}_1$ and $\hat{y}_2$ be the smallest and largest elements, respectively, in the set $\{\hat{x}_0, x_1, x_2, \ldots, x_n\}$. Since L_2 is a non-trivial link, we have $n \geq 1$, and hence $\hat{y}_1 < \hat{y}_2$. For $i = 1, 2$, consider the crossing $\hat{c}_i$ where the arc corresponding to $\hat{y}_i$ is an outgoing arc (see Figure 4.3). Note that $\hat{c}_i$ must be a crossing in $D(L_2)$. Let $\hat{u}_i \in \{\check{x}_0, x_1, x_2, \ldots, x_n\}$ and $\hat{v}_i \in \{\hat{x}_0, x_1, x_2, \ldots, x_n\} \cup \{\check{x}_0\}$ be the elements corresponding to the incoming arc and the over arc at $\hat{c}_i$, respectively (see Figure 4.3). We claim that $\hat{u}_i \neq \hat{v}_i$. Suppose on the contrary that $\hat{u}_i = \hat{v}_i$. Since $\check{x}_0, \hat{x}_0, x_1, x_2, \ldots, x_n$ are pairwise distinct except possibly for the pair $\check{x}_0$ and $\hat{x}_0$, we must have either (a) $\hat{u}_i = \hat{v}_i = x_j$ for some j, or (b) $\hat{u}_i = \check{x}_0$ and $\hat{v}_i \in \{\check{x}_0\} \cup \{\hat{x}_0\}$. If $\hat{u}_i = \hat{v}_i = x_j$, then the arc corresponding to x_j is the incoming as well as over arc at $\hat{c}_i$. This contradicts to the fact that $D(L_2)$ is a minimal positive diagram of L_2. If $\hat{u}_i = \check{x}_0$ and $\hat{v}_i \in \{\check{x}_0\} \cup \{\hat{x}_0\}$, then the arc $\check{a}$ is the incoming arc at $\hat{c}_i$, and one of the arc among $\check{a}$ and $\hat{a}$ is the over arc at $\hat{c}_i$. In other words, in the diagram $D(L_2)$, the arc a is the incoming as well as over arc at $\hat{c}_i$. This is again a contradiction, and hence $\hat{u}_i \neq \hat{v}_i$. Note that $\hat{y}_i = \hat{u}_i * \hat{v}_i$. By Proposition 4.1.9, we have $\hat{u}_i \blacklozenge_i \hat{y}_i \blacklozenge_i \hat{v}_i$ for some $\blacklozenge_i \in \{<, >\}$. This implies that $\hat{z}_1 < \hat{y}_1$ for some $\hat{z}_1 \in \{\hat{u}_1, \hat{v}_1\}$ and $\hat{y}_2 < \hat{z}_2$ for some $\hat{z}_2 \in \{\hat{u}_2, \hat{v}_2\}$. In other words $\hat{z}_1 < \hat{y}_1 < \hat{y}_2 < \hat{z}_2$ for some $\hat{z}_1, \hat{z}_2 \in \{\hat{x}_0, x_1, x_2, \ldots, x_n\} \cup \{\check{x}_0\}$. But, then at least one of the elements $\hat{z}_1$ or $\hat{z}_2$ must belong to $\{\hat{x}_0, x_1, x_2, \ldots, x_n\}$. This contradicts the choice of at least one of $\hat{y}_1$ or $\hat{y}_2$.

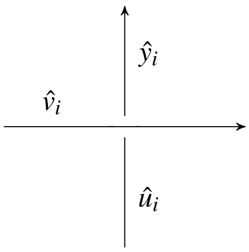


Fig. 4.3 At the crossing $\hat{c}_i$.

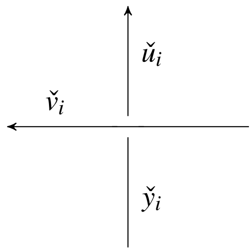


Fig. 4.4 At the crossing $\check{c}_i$.

Case 2: L_2 is a negative link: Let $\check{y}_1$ and $\check{y}_2$ be the smallest and largest elements, respectively, in $\{\check{x}_0, x_1, x_2, \ldots, x_n\}$. For $i = 1, 2$, consider the crossing $\check{c}_i$ where the arc corresponding to $\check{y}_i$ is an incoming arc (see Figure 4.4). Note that $\check{c}_i$ must be a crossing in $D(L_2)$. Let $\check{u}_i \in \{\hat{x}_0, x_1, x_2, \ldots, x_n\}$ and $\check{v}_i \in \{\check{x}_0, x_1, x_2, \ldots, x_n\} \cup \{\hat{x}_0\}$ be the elements corresponding to the outgoing arc and the over arc at $\check{c}_i$, respectively (see Figure 4.4). By the similar argument as in the first case, we have $\check{y}_i \neq \check{v}_i$. Note that $\check{u}_i = \check{y}_i *^{-1} \check{v}_i$. By Proposition 4.1.9, we have $\check{u}_i \blacklozenge_i \check{y}_i \blacklozenge_i \check{v}_i$ for some $\blacklozenge_i \in \{<, >\}$. Now, arguing as in the first case leads to a contradiction. $\square$

Corollary 4.4.7. *Let K be an oriented alternating and positive (or negative) knot of prime determinant. Then the link quandle of a connected sum of K with any oriented link is not bi-orderable. In particular, the knot quandle of K is not bi-orderable.*

Proof. Consider a minimal diagram $D(K)$ of K. By [97, Corollary 2], the diagram $D(K)$ is positive, and hence it is a minimal positive diagram of K. Let $x_0, x_1, \ldots, x_n$ be the generators of the knot quandle $Q(K)$ corresponding to the arcs in $D(K)$. Then, by [93, Proposition 4.4], there exists a Fox coloring that assigns different colors to different arcs of the diagram $D(K)$. Thus, the elements $x_0, x_1, \ldots, x_n$ in $Q(K)$ are also distinct. The result now follows from Theorem 4.4.6. $\square$

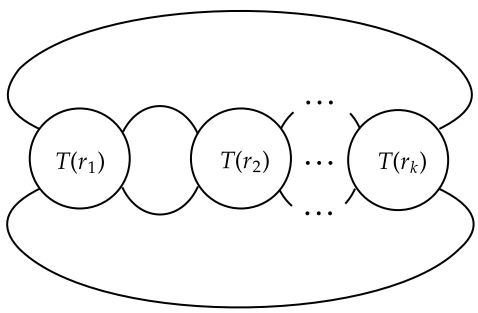


Fig. 4.5 Montesinos link $M(r_1, r_2, \ldots, r_k)$.

For rational numbers $r_1, r_2, \dots, r_k$, the *Montesinos link* $M(r_1, r_2, \dots, r_k)$ is the link shown in Figure 4.5, where $T(r_i)$ is the rational tangle [4, 80] associated with r_i for $i = 1, 2, \dots, k$. If $n_1, n_2, \dots, n_k$ are integers, then the Montesinos link $M(1/n_1, 1/n_2, \dots, 1/n_k)$ is called the *pretzel link* of type $(n_1, n_2, \dots, n_k)$. Note that any 2-bridge link (i.e. a rational link) is a Montesinos link.

Corollary 4.4.8. *Let M be a non-trivial oriented Montesinos link that is alternating and positive* (*or negative*)*. Then the link quandle of a connected sum of M with any oriented link is not bi-orderable. In particular, the link quandle of M is not bi-orderable.*

Proof. Consider an alternating diagram $D(M)$ of M without a nugatory crossing (i.e. $D(M)$ is a minimal diagram of M). By [97, Corollary 2], the diagram $D(M)$ is positive, and hence it is a minimal positive diagram of M. Let $x_0, x_1, \dots, x_n$ be the generators of the link quandle $Q(M)$ corresponding to the arcs in $D(M)$. Suppose $H_1(X_M, \mathbb{Z})$ be the first homology group of the double branched cover X_M of $\mathbb{S}^3$ branched along M. Then, by [4, Theorem 4.2], different arcs of $D(M)$ represent different elements of $H_1(X_M, \mathbb{Z})$. This is equivalent to the statement that for any pair of arcs of the diagram $D(M)$, there is a coloring by elements of $\mathrm{Core}(H_1(X_M, \mathbb{Z}))$ distinguishing them. Hence, the elements $x_0, x_1, \dots, x_n$ in $Q(M)$ are all distinct. Taking M in place of L_2 and $D(M)$ in place of $D(L_2)$, the result now follows from Theorem 4.4.6. □

As examples, knot quandles of knots 3_1, 5_1 and 5_2 (and of their mirror images) are not bi-orderable, since each of them is a positive (or a negative) alternating rational knot.

4.5 Orderability of link quandles of torus links

Two links L_1 and L_2 are called *weakly equivalent* if L_1 is ambient isotopic to either L_2 or the reverse of the mirror image of L_2. It is known that link quandles of weakly equivalent links are isomorphic (see [47, Theorem 5.2 and Corollary 5.3]). For non-zero integers m and n, a *torus link* $T(m,n)$ is the link obtained from the closure of the braid $(\sigma_1^{\varepsilon}\sigma_2^{\varepsilon}\dots\sigma_{m-1}^{\varepsilon})^n$, where $\varepsilon = \frac{m}{|m|}$. Since the torus link $T(m,n)$ is invertible, it is weakly equivalent to its reverse, mirror image and the reverse of its mirror image, and hence link quandles of all of them are isomorphic to that of $T(m,n)$. Recall that a torus link $T(m,n)$ is a knot (a one component link) if and only if $\gcd(m,n) = 1$.

Proposition 4.5.1. *The link quandle of a torus link $T(m,n)$ has a presentation with generators $a_1, a_2, \dots, a_m$ and relations*

$$a_i = a_{n+i} * a_n * a_{n-1} * \cdots * a_1 \quad \textit{for } \; i = 1, 2, \dots, m,$$

where $a_{mj+k} = a_k$ for $j \in \mathbb{Z}$ and $k \in \{1,2,\dots,m\}$.

Proof. Since a torus link $T(m,n)$, with $m,n \geq 1$, is the closure of the braid $\tau(m,n) = (\sigma_1\sigma_2\cdots\sigma_{m-1})^n$, with reference to Figure 4.6, it is enough to prove that

$$c_i = a_{n+i} * a_n * a_{n-1} * \cdots * a_1 \quad \text{for } i = 1,2,\dots,m. \tag{4.5.0.1}$$

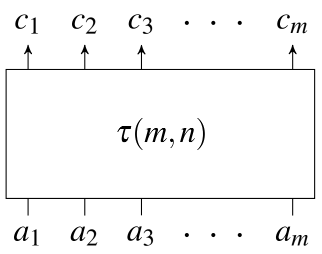


Fig. 4.6 Toric braid $\tau(m,n)$.

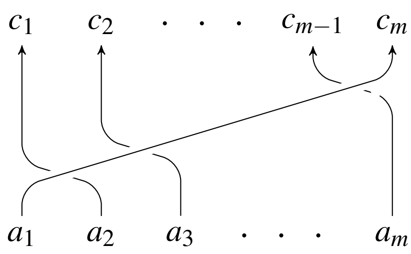


Fig. 4.7 Toric braid $\tau(m,1)$.

We prove (4.5.0.1) by induction on n. By looking at Figure 4.7, one can see that $c_i = a_{i+1} * a_1$ for $i = 1,2,\dots,m$. Thus, the equations given by (4.5.0.1) hold for $n = 1$. Assume that the equations given by (4.5.0.1) hold for a positive integer $n-1$.

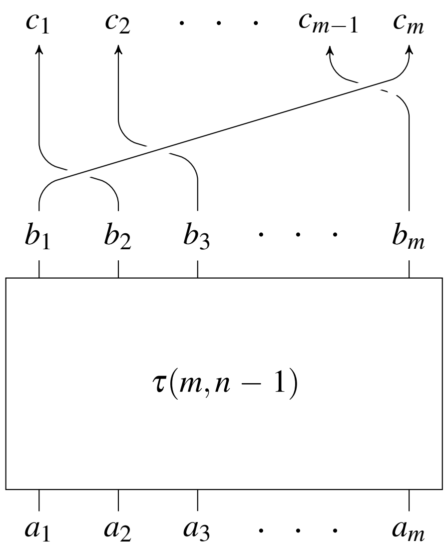


Fig. 4.8 Toric braid $\tau(m,n)$ seen as $\tau(m,n-1)\tau(m,1)$.

Since $\tau(m,n) = \tau(m,n-1)\tau(m,1)$ (see Figure 4.8), we have

$$c_i = b_{i+1} * b_1 \quad \text{for } i = 1,2,\dots,m, \tag{4.5.0.2}$$

where $b_{m+1} = b_1$. By induction hypothesis,

$$b_{i+1} = a_{n+i} * a_{n-1} * a_{n-2} * \cdots * a_1 \quad \text{for } i = 1, 2, \ldots, m. \tag{4.5.0.3}$$

Using (4.5.0.3) in (4.5.0.2), we get

$$c_i = (a_{n+i} * a_{n-1} * a_{n-2} * \cdots * a_1) * (a_n * a_{n-1} * a_{n-2} * \cdots * a_1) \tag{4.5.0.4}$$
$$= a_{n+i} * a_{n-1} * a_{n-2} * \cdots * a_1 *^{-1} a_1 *^{-1} a_2 *^{-1} \cdots *^{-1} a_{n-1} * a_n * a_{n-1} * \cdots * a_1 \tag{4.5.0.5}$$
$$= a_{n+i} * a_n * a_{n-1} * \cdots * a_1 \quad \text{for } i = 1, 2, \ldots, m, \tag{4.5.0.6}$$

where the second equality follows from Lemma 2.5.6 and the third follows by cancellation. This proves that the equations given by (4.5.0.1) hold for all n. $\square$

If $<$ is a right ordering on a quandle Q and $x, y, z_1, z_2, \ldots, z_n \in Q$ with $x \blacklozenge y$ for $\blacklozenge \in \{<, >\}$, then

$$x * z_1 * z_2 * \cdots * z_n \blacklozenge y * z_1 * z_2 * \cdots * z_n \text{ and } x *^{-1} z_1 *^{-1} z_2 *^{-1} \cdots *^{-1} z_n \blacklozenge y *^{-1} z_1 *^{-1} \cdots *^{-1} z_n. \tag{4.5.0.7}$$

Theorem 4.5.2. *Let $m, n \geq 2$ be integers such that one is not a multiple of the other. Then the link quandle of the torus link $T(m,n)$ is not right-orderable.*

Proof. Recall that the torus links $T(m,n)$ and $T(n,m)$ are ambient isotopic. Thus, we can assume that $m < n$ by switching m and n if required. Let $d = \gcd(m,n)$. Then we have $d < m$. By Proposition 4.5.1, the link quandle $Q(T(m,n))$ is generated by $a_1, a_2, \ldots, a_m$ and has the following relations:

$$a_i = a_{n+i} * a_n * a_{n-1} * \cdots * a_1 \quad \text{for } i = 1, 2, \ldots, m, \tag{4.5.0.8}$$

where $a_{mj+k} = a_k$ for $j \in \mathbb{Z}$ and $k \in \{1, 2, \ldots, m\}$. Using (4.5.0.8), one can obtain the following:

$$a_i = a_{n+i} * a_n * a_{n-1} * \cdots * a_1 \quad \text{for all } i \in \mathbb{Z}, \tag{4.5.0.9}$$

where $a_{mj+k} = a_k$ for $j \in \mathbb{Z}$ and $k \in \{1, 2, \ldots, m\}$. We can rewrite (4.5.0.9) as

$$a_{i-n} = a_i * a_n * a_{n-1} * \cdots * a_1 \quad \text{for all } i \in \mathbb{Z}. \tag{4.5.0.10}$$

Also, (4.5.0.9) can be written as

$$a_{n+i} = a_i *^{-1} a_1 *^{-1} a_2 *^{-1} \cdots *^{-1} a_n \quad \text{for all } i \in \mathbb{Z}. \tag{4.5.0.11}$$

Suppose on the contrary that the quandle $Q(T(m,n))$ is right-ordered with respect to a linear order $<$. By the proof of Proposition 4.5.1 (see Figures 4.6, 4.7 and 4.8), the generators $a_1, a_2, \ldots, a_m$ of $Q(T(m,n))$ correspond to some of the arcs in the standard diagram of the closed toric braid representing $T(m,n)$. Note that $\eta(a_1), \eta(a_2), \ldots, \eta(a_m)$ are the meridional elements that generate the link group $\pi_1(C(T(m,n)))$, where $\eta : Q(T(m,n)) \to \pi_1(C(T(m,n)))$ is the natural map. The elements $\eta(a_1), \eta(a_2), \ldots, \eta(a_m)$ must be pairwise distinct in $\pi_1(C(T(m,n)))$ according to [110, Corollary 1.5], and hence so are the elements $a_1, a_2, \ldots, a_m$ in $Q(T(m,n))$. In particular, we have $a_1 \neq a_{d+1}$, and hence $a_1 \blacklozenge a_{d+1}$ for some $\blacklozenge \in \{<, >\}$. A repeated application of (4.5.0.7) together with (4.5.0.10) and (4.5.0.11) yields

$$a_{nk+1} \blacklozenge a_{nk+d+1} \quad \text{for all } k \in \mathbb{Z}. \tag{4.5.0.12}$$

Let l be an integer. Since $\gcd(m,n) = d$, we have $dl = mj + nk$ for some integers j and k. This implies that $nk + 1 \equiv dl + 1 \pmod{m}$ and $nk + d + 1 \equiv dl + d + 1 \pmod{m}$. By (4.5.0.12), we have $a_{dl+1} \blacklozenge a_{dl+d+1}$. Thus, $a_{dl+1} \blacklozenge a_{dl+d+1}$ for any integer l. Using this repeatedly, we get $a_1 \blacklozenge a_{d+1} \blacklozenge a_{2d+1} \blacklozenge \cdots \blacklozenge a_{cd+1} \blacklozenge a_1$, where $c = \frac{m}{d} - 1$. This implies that $a_1 < a_1$ or $a_1 > a_1$, a contradiction. □

As a consequence of the preceding theorem, we retrieve the following result of Perron and Rolfsen [104, Proposition 3.2].

Corollary 4.5.3. *The knot group of a non-trivial torus knot is not bi-orderable.*

Proof. Let K be a non-trivial torus knot. Then, by Theorem 4.5.2, the knot quandle of K is not right-orderable, and hence by Corollary 4.4.3, the knot group of K is not bi-orderable. □

We conclude with the following result.

Corollary 4.5.4. *The knot quandle of the trefoil knot is neither left nor right orderable.*

Proof. Note that the trefoil knot is the torus knot $T(2,3)$. By Theorem 4.5.2, the knot quandle of the trefoil knot is not right-orderable. We claim that the knot quandle of the trefoil knot is not left-orderable as well. Using Proposition 4.5.1, the knot quandle $Q(T(2,3))$ is generated by a_1 and a_2 with the relations $a_1 = a_2 * a_1 * a_2 * a_1$ and $a_2 = a_1 * a_1 * a_2 * a_1$. These relations can be rewritten as follows:

$$a_1 = a_2 * a_1 * a_2 \quad \text{and} \tag{4.5.0.13}$$
$$a_2 = a_1 * a_2 * a_1. \tag{4.5.0.14}$$

Assume contrary that the quandle $Q(T(2,3))$ is left-ordered with respect to a linear order $<$. Since the quandle $Q(T(2,3))$ is non-trivial, we must have $a_1 \neq a_2$. Hence $a_1 \blacklozenge a_2$ for some $\blacklozenge \in \{<,>\}$. Consider

$$
\begin{array}{llll}
 & a_1 \blacklozenge a_2 & & (4.5.0.15)\\
\Rightarrow & a_1 * a_1 \blacklozenge a_1 * a_2 & \text{(since $<$ is left ordering)} & (4.5.0.16)\\
\Rightarrow & a_1 \blacklozenge a_1 * a_2 & \text{(by idempotency)} & (4.5.0.17)\\
\Rightarrow & a_2 * a_1 \blacklozenge a_2 * (a_1 * a_2) & \text{(since $<$ is left ordering)} & (4.5.0.18)\\
\Rightarrow & a_2 * a_1 \blacklozenge a_2 * a_1 * a_2 & \text{(by Lemma 2.5.6)} & (4.5.0.19)\\
\Rightarrow & a_2 * a_1 \blacklozenge a_1 & \text{(by (4.5.0.13))} & (4.5.0.20)\\
\Rightarrow & a_1 * (a_2 * a_1) \blacklozenge a_1 * a_1 & \text{(since $<$ is left ordering)} & (4.5.0.21)\\
\Rightarrow & a_1 * a_2 * a_1 \blacklozenge a_1 & \text{(by Lemma 2.5.6)} & (4.5.0.22)\\
\Rightarrow & a_2 \blacklozenge a_1 & \text{(by (4.5.0.14)).} & (4.5.0.23)
\end{array}
$$

This is a contradiction, since we cannot have $a_1 \blacklozenge a_2$ and $a_2 \blacklozenge a_1$ together. □

4.6 Orderability of involutory quandles of alternating links

We know that a non-trivial involutory quandle is not right-orderable whereas there are many involutory quandles that are left-orderable. For example, the quandle $\mathrm{Core}(G)$ is left-orderable for any bi-orderable group G. We conclude with some observations on left-orderability of involutory quandles of alternating links.

Definition 4.6.1. For a given link L, the *involutory link quandle* $IQ(L)$ is a quandle obtained from $Q(L)$ by adding relations $(x*y)*y = x$ for all $x, y \in Q(L)$.

Theorem 4.6.2. *Let L be a non-trivial alternating link. If there exists a reduced alternating diagram $D(L)$ of L such that the generators of the involutory quandle $IQ(L)$ of L corresponding to the arcs in $D(L)$ are pairwise distinct, then $IQ(L)$ is not left-orderable.*

Proof. First suppose that L is a non-trivial non-split alternating link. Let $D(L)$ be a reduced alternating diagram of L such that the generators of the involutory quandle $IQ(L)$ of L

corresponding to the arcs in $D(L)$ are pairwise distinct. Suppose on the contrary that the involutory quandle $IQ(L)$ is left-ordered with respect to a linear order $<$. Let y be the smallest element among the generators of $IQ(L)$ corresponding to the arcs in $D(L)$. Since L is non-trivial and non-split, and $D(L)$ is alternating, there is a crossing, say c, in D such that the arc corresponding to y is the over arc at c (see Figure 4.9). Let x and z be the elements of $IQ(L)$ corresponding to the other two arcs meeting at c (see Figure 4.9).

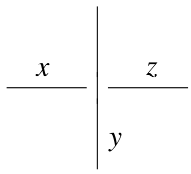


Fig. 4.9 At the crossing c.

By the hypothesis, we have $x \neq y$ and $y \neq z$. Now, y being the smallest element implies that $y < x$ and $y < z$. Since $<$ is a left-ordering on $IQ(L)$ this implies that $x * y < x * x = x$ and $z * y < z * z = z$. Since the quandle is involutory, we have $x * y = z$ and $z * y = x$, and hence $z < x$ and $x < z$, a contradiction.
Now suppose that L is an arbitrary non-trivial alternating link. Then there exists a non-trivial non-split component, say L', of L. Let $D(L')$ be the diagram of L' obtained from $D(L)$ by throwing away the components of L that do not belong to L'. Note that the involutory quandle $IQ(L')$ of L' is a subquandle of $IQ(L)$. By the preceding paragraph, it follows that $IQ(L')$ is not left-orderable, and hence so is $IQ(L)$. $\square$

Corollary 4.6.3. *Let K be an alternating knot of prime determinant. Then the involutory quandle $IQ(K)$ of K is not left-orderable.*

Proof. Consider a reduced alternating diagram $D(K)$ of K. Arguing as in the proof of Theorem 4.4.7, the elements of $IQ(M)$ corresponding to any two arcs in $D(K)$ are distinct. The proof now follows from Theorem 4.6.2. $\square$

Corollary 4.6.4. *Let M be a non-trivial alternating Montesinos link. Then the involutory quandle $IQ(M)$ of M is not left-orderable.*

Proof. Consider a reduced alternating diagram $D(M)$ of M. By the same argument as in the proof of Theorem 4.4.8, the elements of $IQ(M)$ corresponding to the arcs in $D(M)$ are pairwise distinct. The result now follows from Theorem 4.6.2. $\square$

Let $m \geq 3$ and $n \geq 2$ be relatively prime integers. The *Turk's head knot* $THK(m,n)$ is the closure of the braid $\left(\sigma_1\sigma_2^{-1}\sigma_3\sigma_4^{-1}\cdots\sigma_{m-1}^{\delta}\right)^n$, where $\delta = +1$ if m is even and $\delta = -1$ if m is odd. Note that Turk's head knots are alternating knots.

Corollary 4.6.5. *Let $m \geq 3$ and $n \geq 2$ be relatively prime integers. If $n = 2$ or $m = 3$ or m is even, then the involutary quandle of the Turk's head knot $THK(m,n)$ is not left-orderable.*

Proof. Let D be a reduced alternating diagram of $THK(m,n)$. By [44, Theorems 3 and 4], there exists a Fox coloring that assigns different colors to different arcs of the diagram D. Hence the elements of the involutary quandle of $THK(m,n)$ corresponding to the arcs in D are pairwise distinct. The result now follows from Theorem 4.6.2. □

Chapter 5

Virtually symmetric representations and virtual link groups

This final chapter, which is of a different flavour, focuses on virtual knots and marked Gauss diagrams. In Section 5.1, we define virtually symmetric representations of virtual braid groups and prove that most of the representations known in the literature are equivalent to virtually symmetric representations. We also construct linear representations of virtual braid groups, which are equivalent to virtually symmetric representations. In Section 5.2, using one such representation, we define a virtual link group and give various approaches for the same. In Section 5.3, we introduce the concept of marked Gauss diagrams as a generalization of Gauss diagrams. We then extend the definition of virtual link groups introduced in this chapter to marked Gauss diagrams. Further, in Section 5.4, we give an interpretation of marked Gauss diagrams as planar diagrams. We then define C_m-groups in Section 5.5 and prove that every group with an irreducible C_1-presentation of deficiency 1 or 2 can be realized as the group of a marked Gauss diagram. In Section 5.6, we define the peripheral structure of 1-circle marked Gauss diagrams and study some of its properties. Lastly, in Section 5.7, we study peripherally specified homomorphic images of groups associated with marked Gauss diagrams. The results are from our work [16].

5.1 Virtually symmetric representations

In this section, we define virtually symmetric representations of virtual braid groups, and prove that most of the known representations are equivalent to virtually symmetric representations.

Definition 5.1.1. A representation $\varphi : VB_n \to \mathrm{Aut}(H)$ of the virtual braid group VB_n into the automorphism group of some group (or module) $H = \langle h_1, h_2, \dots, h_m \mid \mathcal{R} \rangle$ is called *virtually symmetric* if for any generator ρ_i, $i = 1, 2, \dots, n-1$, its image $\varphi(\rho_i)$ is a permutation of generators $h_1, h_2, \dots, h_m$.

Let $\varphi_i : VB_n \to \mathrm{Aut}(H)$ $(i = 1, 2)$ be two representations. We say that φ_1 and φ_2 are *equivalent* if there exists an automorphism $\phi : H \to H$ such that $\varphi_1(\beta) = \phi^{-1} \circ \varphi_2(\beta) \circ \phi$ for all $\beta \in VB_n$. Let $F_{n,n} = F_n * \mathbb{Z}^n$, where $F_n = \langle x_1, x_2, \dots, x_n \rangle$ is the free group of rank n and $\mathbb{Z}^n = \langle v_1, v_2, \dots, v_n \rangle$ is the free abelian group of rank n. In [12, Theorem 4.1], an extension φ_M of Artin representation is defined for virtual braid groups, where $\varphi_M : VB_n \to \mathrm{Aut}(F_{n,n})$ is defined by its action on generators as follows

$$\varphi_M(\sigma_i) : \begin{cases} x_i \mapsto x_i x_{i+1} x_i^{-1}, \\ x_{i+1} \mapsto x_i, \\ x_j \mapsto x_j, \text{ for } j \neq i, i+1, \end{cases} \qquad \varphi_M(\sigma_i) : \begin{cases} v_i \mapsto v_{i+1}, \\ v_{i+1} \mapsto v_i, \\ v_j \mapsto v_j, \text{ for } j \neq i, i+1, \end{cases}$$

$$\varphi_M(\rho_i) : \begin{cases} x_i \mapsto x_{i+1}^{v_i^{-1}}, \\ x_{i+1} \mapsto x_i^{v_{i+1}}, \\ x_j \mapsto x_j, \text{ for } j \neq i, i+1, \end{cases} \qquad \varphi_M(\rho_i) : \begin{cases} v_i \mapsto v_{i+1}, \\ v_{i+1} \mapsto v_i, \\ v_j \mapsto v_j, \text{ for } j \neq i, i+1. \end{cases}$$

In particular, if we put $v_1 = \cdots = v_n = 1$, we get a representation $\varphi_0 : VB_n \to \mathrm{Aut}(F_n)$ defined by Vershinin [117]. Hereafter, we only write non-trivial actions on generators assuming that all other generators are fixed.

Proposition 5.1.2. *The representation $\varphi_M : VB_n \to \mathrm{Aut}(F_{n,n})$ is equivalent to a virtually symmetric representation.*

Proof. We define an automorphism $\phi : F_{n,n} \to F_{n,n}$ by setting

$$\begin{aligned} \phi(x_i) &= x_i^{(v_i v_{i+1} \dots v_n)}, \; i = 1, \dots, n, \\ \phi(v_i) &= v_i, \; i = 1, \dots, n. \end{aligned}$$

Thus, we have a new representation $\varphi_S : VB_n \to \mathrm{Aut}(F_{n,n})$ of the virtual braid group VB_n to the automorphism group of $F_{n,n}$ by setting

$$\varphi_S(\beta) = \phi^{-1} \circ \varphi_M(\beta) \circ \phi, \text{ for } \beta \in VB_n.$$

In particular,

$$\varphi_S(\sigma_i) : \begin{cases} x_i \mapsto x_i \ x_{i+1}^{v_i} \ x_i^{-1}, \\ x_{i+1} \mapsto x_i^{v_{i+1}^{-1}}, \end{cases} \qquad \varphi_S(\sigma_i) : \begin{cases} v_i \mapsto v_{i+1}, \\ v_{i+1} \mapsto v_i, \end{cases}$$

$$\varphi_S(\rho_i) : \begin{cases} x_i \mapsto x_{i+1}, \\ x_{i+1} \mapsto x_i, \end{cases} \qquad \varphi_S(\rho_i) : \begin{cases} v_i \mapsto v_{i+1}, \\ v_{i+1} \mapsto v_i. \end{cases}$$

Notice that

$$\varphi_S(\sigma_i^{-1}) : \begin{cases} x_i \mapsto x_{i+1}^{v_i}, \\ x_{i+1} \mapsto x_{i+1}^{-v_i v_{i+1}^{-1}} \ x_i^{v_{i+1}^{-1}} \ x_{i+1}^{v_i v_{i+1}^{-1}}, \end{cases} \qquad \varphi_S(\sigma_i^{-1}) : \begin{cases} v_i \mapsto v_{i+1}, \\ v_{i+1} \mapsto v_i. \end{cases}$$

It follows that φ_S is a virtually symmetric representation. □

5.1.1 Generalized Artin representation

Let $F_{n+1} = \langle x_1, x_2, \ldots, x_n, v \rangle$ be the free group of rank $n+1$. In [8, 91], a representation $\varphi_A : VB_n \to \mathrm{Aut}(F_{n+1})$ is defined by its action on generators as follows

$$\varphi_A(\sigma_i) : \begin{cases} x_i \mapsto x_i x_{i+1} x_i^{-1}, \\ x_{i+1} \mapsto x_i, \end{cases} \qquad \varphi_A(\rho_i) : \begin{cases} x_i \mapsto x_{i+1}^{v^{-1}}, \\ x_{i+1} \mapsto x_i^{v}. \end{cases}$$

We define an automorphism $\phi : F_{n+1} \to F_{n+1}$ by setting

$$\begin{aligned} \phi(x_i) &= x_i^{v^{n-i}}, \ i = 1, \ldots, n, \\ \phi(v) &= v. \end{aligned}$$

Thus, we have a new representation $\tilde{\varphi}_A : VB_n \to \mathrm{Aut}(F_{n+1})$ by setting

$$\tilde{\varphi}_A(\beta) = \phi^{-1} \circ \varphi_A(\beta) \circ \phi, \ \text{ where } \ \beta \in VB_n.$$

In particular,

$$\tilde{\varphi}_A(\sigma_i) : \begin{cases} x_i \mapsto x_i \ x_{i+1}^{v} \ x_i^{-1}, \\ x_{i+1} \mapsto x_i^{v^{-1}}, \end{cases} \qquad \tilde{\varphi}_A(\rho_i) : \begin{cases} x_i \mapsto x_{i+1}, \\ x_{i+1} \mapsto x_i. \end{cases}$$

Therefore, φ_A is equivalent to a virtually symmetric representation.

5.1.2 The Silver-Williams representation

Let $F_{n,n+1} = F_n * \mathbb{Z}^{n+1}$, where $F_n = \langle x_1, x_2, \dots, x_n \rangle$ is the free group of rank n and $\mathbb{Z}^{n+1} = \langle u_1, u_2, \dots, u_n, v \rangle$ is the free abelian group of rank $n+1$. Using the definition of the generalized Alexander group for virtual links [113], a representation $\varphi_{SW} : VB_n \longrightarrow \mathrm{Aut}(F_{n,n+1})$ is constructed in [12] which is defined on generators as follows

$$\varphi_{SW}(\sigma_i) : \begin{cases} x_i \mapsto x_i x_{i+1}^{u_i} x_i^{-vu_{i+1}}, \\ x_{i+1} \mapsto x_i^{v}, \end{cases} \qquad \varphi_{SW}(\sigma_i) : \begin{cases} u_i \mapsto u_{i+1}, \\ u_{i+1} \mapsto u_i, \end{cases}$$

$$\varphi_{SW}(\rho_i) : \begin{cases} x_i \mapsto x_{i+1}, \\ x_{i+1} \mapsto x_i, \end{cases} \qquad \varphi_{SW}(\rho_i) : \begin{cases} u_i \mapsto u_{i+1}, \\ u_{i+1} \mapsto u_i. \end{cases}$$

This representation is a virtually symmetric representation.

5.1.3 Boden-Dies representation

Let $F_{n,2} = F_n * \mathbb{Z}^2$, where $F_n = \langle x_1, x_2, \dots, x_n \rangle$ is the free group of rank n and $\mathbb{Z}^2 = \langle u, v \rangle$ is the free abelian group of rank 2. In [26], a representation $\varphi_{BD} : VB_n \to \mathrm{Aut}(F_{n,2})$ is defined by its action on generators as follows

$$\varphi_{BD}(\sigma_i) : \begin{cases} x_i \mapsto x_i x_{i+1} x_i^{-u}, \\ x_{i+1} \mapsto x_i^{u}, \end{cases} \qquad \varphi_{BD}(\rho_i) : \begin{cases} x_i \mapsto x_{i+1}^{v^{-1}}, \\ x_{i+1} \mapsto x_i^{v}. \end{cases}$$

We define an automorphism $\phi : F_{n,2} \to F_{n,2}$ by setting

$$\begin{aligned} \phi(x_i) &= x_i^{v^{n-i}}, \text{ for } i = 1, \dots, n, \\ \phi(u) &= u, \\ \phi(v) &= v. \end{aligned}$$

Thus, we have a new representation $\tilde{\varphi}_{BD} : VB_n \to \mathrm{Aut}(F_{n,2})$ by defining

$$\tilde{\varphi}_{BD}(\beta) = \phi^{-1} \circ \varphi_{BD}(\beta) \circ \phi, \ \text{ where } \ \beta \in VB_n.$$

We see that

$$\tilde{\varphi}_{BD}(\sigma_i) : \begin{cases} x_i \mapsto x_i \ x_{i+1}^{v} \ x_i^{-u}, \\ x_{i+1} \mapsto x_i^{uv^{-1}}, \end{cases} \qquad \tilde{\varphi}_{BD}(\rho_i) : \begin{cases} x_i \mapsto x_{i+1}, \\ x_{i+1} \mapsto x_i. \end{cases}$$

Therefore, φ_{BD} is equivalent to a virtually symmetric representation.

5.1.4 Extended Wada representation

Let $F_{n,n} = F_n * \mathbb{Z}^n$, where $F_n = \langle x_1, x_2, \ldots, x_n \rangle$ is the free group of rank n and $\mathbb{Z}^n = \langle v_1, v_2, \ldots, v_n \rangle$ is the free abelian group of rank n. Wada [119] defined representations $w_{1,r}, r \in \mathbb{Z}, w_2$, w_3 of the braid group B_n into $\mathrm{Aut}(F_n)$. We define extensions of Wada representations $w : VB_n \to \mathrm{Aut}(F_{n,n})$, where $w = w_{1,r}, w_2$ or w_3 to the virtual braid group VB_n by its action on generators as follows

$$w_{1,r}(\sigma_i) : \begin{cases} x_i \mapsto x_i^r x_{i+1} x_i^{-r}, \\ x_{i+1} \mapsto x_i, \end{cases} \qquad w_{1,r}(\sigma_i) : \begin{cases} v_i \mapsto v_{i+1}, \\ v_{i+1} \mapsto v_i, \end{cases}$$

$$w_{1,r}(\rho_i) : \begin{cases} x_i \mapsto x_{i+1}^{v_i^{-1}}, \\ x_{i+1} \mapsto x_i^{v_{i+1}}, \end{cases} \qquad w_{1,r}(\rho_i) : \begin{cases} v_i \mapsto v_{i+1}, \\ v_{i+1} \mapsto v_i, \end{cases}$$

$$w_2(\sigma_i) : \begin{cases} x_i \mapsto x_i x_{i+1}^{-1} x_i, \\ x_{i+1} \mapsto x_i, \end{cases} \qquad w_2(\sigma_i) : \begin{cases} v_i \mapsto v_{i+1}, \\ v_{i+1} \mapsto v_i, \end{cases}$$

$$w_2(\rho_i) : \begin{cases} x_i \mapsto x_{i+1}^{v_i^{-1}}, \\ x_{i+1} \mapsto x_i^{v_{i+1}}, \end{cases} \qquad w_2(\rho_i) : \begin{cases} v_i \mapsto v_{i+1}, \\ v_{i+1} \mapsto v_i, \end{cases}$$

$$w_3(\sigma_i) : \begin{cases} x_i \mapsto x_i^2 x_{i+1}, \\ x_{i+1} \mapsto x_{i+1}^{-1} x_i^{-1} x_{i+1}, \end{cases} \qquad w_3(\sigma_i) : \begin{cases} v_i \mapsto v_{i+1}, \\ v_{i+1} \mapsto v_i, \end{cases}$$

$$w_3(\rho_i) : \begin{cases} x_i \mapsto x_{i+1}^{v_i^{-1}}, \\ x_{i+1} \mapsto x_i^{v_{i+1}}, \end{cases} \qquad w_3(\rho_i) : \begin{cases} v_i \mapsto v_{i+1}, \\ v_{i+1} \mapsto v_i. \end{cases}$$

The case $v_1 = v_2 = \cdots = v_n$ is studied in [96]. We define an automorphism of $F_{n,n}$ by setting

$$\phi(x_i) = x_i^{(v_i v_{i+1} \ldots v_n)},$$
$$\phi(v_i) = v_i, \ i = 1, \ldots, n.$$

Thus, we have new representations $\tilde{w} : VB_n \to \mathrm{Aut}(F_{n+1})$ by defining

$$\tilde{w}(\beta) = \phi^{-1} \circ w(\beta) \circ \phi, \ \text{ where } \ \beta \in VB_n, \quad w = w_{1,r}, w_2 \ \textit{or} \ w_3.$$

One can check that

$$\tilde{w}_{1,r}(\sigma_i):\begin{cases} x_i \mapsto x_i^r x_{i+1}^{v_i} x_i^{-r}, \\ x_{i+1} \mapsto x_i^{v_{i+1}^{-1}}, \end{cases} \qquad \tilde{w}_{1,r}(\sigma_i):\begin{cases} v_i \mapsto v_{i+1}, \\ v_{i+1} \mapsto v_i, \end{cases}$$

$$\tilde{w}_{1,r}(\rho_i):\begin{cases} x_i \mapsto x_{i+1}, \\ x_{i+1} \mapsto x_i, \end{cases} \qquad \tilde{w}_{1,r}(\rho_i):\begin{cases} v_i \mapsto v_{i+1}, \\ v_{i+1} \mapsto v_i, \end{cases}$$

$$\tilde{w}_2(\sigma_i):\begin{cases} x_i \mapsto x_i x_{i+1}^{-v_i} x_i, \\ x_{i+1} \mapsto x_i^{v_{i+1}^{-1}}, \end{cases} \qquad \tilde{w}_2(\sigma_i):\begin{cases} v_i \mapsto v_{i+1}, \\ v_{i+1} \mapsto v_i, \end{cases}$$

$$\tilde{w}_2(\rho_i):\begin{cases} x_i \mapsto x_{i+1}, \\ x_{i+1} \mapsto x_i, \end{cases} \qquad \tilde{w}_2(\rho_i):\begin{cases} v_i \mapsto v_{i+1}, \\ v_{i+1} \mapsto v_i, \end{cases}$$

$$\tilde{w}_3(\sigma_i):\begin{cases} x_i \mapsto x_i^2 x_{i+1}^{v_i}, \\ x_{i+1} \mapsto x_{i+1}^{-v_i v_{i+1}^{-1}} x_i^{-v_{i+1}^{-1}} x_{i+1}^{v_i v_{i+1}^{-1}}, \end{cases} \qquad \tilde{w}_3(\sigma_i):\begin{cases} v_i \mapsto v_{i+1}, \\ v_{i+1} \mapsto v_i, \end{cases}$$

$$\tilde{w}_3(\rho_i):\begin{cases} x_i \mapsto x_{i+1}, \\ x_{i+1} \mapsto x_i, \end{cases} \qquad \tilde{w}_3(\rho_i):\begin{cases} v_i \mapsto v_{i+1}, \\ v_{i+1} \mapsto v_i. \end{cases}$$

Therefore, extended Wada representations are equivalent to virtually symmetric representations.

5.1.5 Linear representations of braid groups

In this section, we construct a linear, local and non-homogeneous representation of B_n and prove that it is equivalent to the well-known Burau representation. A linear representation $\varphi : B_n \to GL_n(R)$ is called *local* if

$$\varphi(\sigma_i) = I^{i-1} \oplus M_i \oplus I^{n-i-1} \text{ for } i \in \{1,2,\dots,n-1\},$$

where I^k is the $k\times k$ identity matrix and M_i is a 2×2 matrix, $i = 1,2,\dots,n-1$, with entries from an integral domain R. If $M_1 = M_2 = \cdots = M_{n-1}$, then $\varphi : B_n \to GL_n(R)$ is called a *homogeneous* representation. A linear, local and homogeneous representation of S_n is defined similarly, where S_n is the symmetric group of degree n. A linear representation $\varphi : VB_n \to GL_n(R)$ is called local (homogeneous) if its restrictions to B_n and S_n are local (homogeneous).

Proposition 5.1.3. *The map* $\varphi : B_n \to GL_n(\mathbb{Z}[t^{\pm 1}, t_1^{\pm 1}, t_2^{\pm 1}, \dots, t_{n-1}^{\pm 1}])$ *defined on generators by*

$$\varphi(\sigma_i) = I^{i-1} \oplus \begin{pmatrix} 1-t & tt_i \\ t_i^{-1} & 0 \end{pmatrix} \oplus I^{n-i-1} \text{ for } i = 1, 2, \dots, n-1,$$

is a representation of B_n*. In particular, if* $t_i = 1$ *for every* $i = 1, 2, \dots, n-1$*, then it is the Burau representation. Moreover,* φ *is equivalent to the Burau representation.*

Proof. The fact that φ is a representation can be easily deduced. Let us now consider ϕ to be the automorphism of the free module V with the basis $\{e_1, e_2, \dots, e_n\}$ over the ring $R = \mathbb{Z}[t^{\pm 1}, t_1^{\pm 1}, t_2^{\pm 1}, \dots, t_{n-1}^{\pm 1}]$ which is defined on the basis as follows.

$$\phi : \begin{cases} e_1 \to e_1, \\ e_2 \to t_1 e_2, \\ \vdots \\ e_n \to t_1 t_2 \dots t_{n-1} e_n. \end{cases}$$

Next, we consider the representation $\tilde{\varphi}$ defined as $\tilde{\varphi}(\beta) := \phi \varphi(\beta) \phi^{-1}$, where $\beta \in B_n$. Then

$$\tilde{\varphi}(\sigma_i) = \phi \varphi(\sigma_i) \phi^{-1} = \begin{cases} e_i \to (1-t)e_i + e_{i+1}, \\ e_{i+1} \to te_i, \\ e_j \to e_j, \text{ for } j \neq i, i+1, \end{cases}$$

Hence, $\tilde{\varphi}$ is the Burau representation.

□

It is well-known that the Burau representation is not faithful for $n > 4$. Bigelow [24] proved the existence of non-trivial elements $b_1 \in B_5$ and $b_2 \in B_6$ in the kernel of the Burau representation. These elements are

$$b_1 = [c_1^{-1} \sigma_4 c_1, c_2^{-1} \sigma_4 \sigma_3 \sigma_2 \sigma_1^2 \sigma_2 \sigma_3 \sigma_4 c_2], \quad b_2 = [d_1^{-1} \sigma_3 d_1, d_2^{-1} \sigma_3 d_2],$$

where

$$c_1 = \sigma_3^{-1} \sigma_2 \sigma_1^2 \sigma_2 \sigma_4^3 \sigma_3 \sigma_2, \quad c_2 = \sigma_4^{-1} \sigma_3 \sigma_2 \sigma_1^{-2} \sigma_2 \sigma_1^2 \sigma_2^2 \sigma_1 \sigma_4^5,$$
$$d_1 = \sigma_4 \sigma_5^{-1} \sigma_2^{-1} \sigma_1, \quad d_2 = \sigma_4^{-1} \sigma_5^2 \sigma_2 \sigma_1^{-2}.$$

This means that the Burau representation is a linear, local and homogenous representation which is not faithful. Furthermore, from the result [51, Theorem 5.3], it follows that there does not exist any linear, local, homogeneous and faithful representation of B_n for $n > 4$. This naturally leads us to ask the following question.

Question 5.1.4. *Does there exist a linear, local and faithful representation of B_n for $n > 4$?*

5.1.6 Linear representations of virtual braid groups

Bartholomew and Fenn [14, Section 7] considered a linear, local and homogeneous representation $\varphi : VB_n \to GL_n(\mathbb{Z}[t^{\pm 1}, \lambda^{\pm 1}])$ defined on generators as

$$\sigma_i \mapsto I^{i-1} \oplus \begin{pmatrix} 1-t & \lambda^{-1}t \\ \lambda & 0 \end{pmatrix} \oplus I^{n-i-1},$$
$$\rho_i \mapsto I^{i-1} \oplus \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus I^{n-i-1}.$$

Clearly, the representation $\varphi : VB_n \to GL_n(\mathbb{Z}[t^{\pm 1}, \lambda^{\pm 1}])$ is virtually symmetric. We now construct a linear, local and non-homogeneous representation of VB_n whose proof is immediate. It suffices to check that the map satisfies defining relations of VB_n.

Proposition 5.1.5. *The map $\psi : VB_n \to GL_n(\mathbb{Z}[t^{\pm 1}, \lambda^{\pm 1}, t_1^{\pm 1}, t_2^{\pm 1}, \ldots, t_{n-1}^{\pm 1}])$ defined on generators by*

$$\psi(\sigma_i) = I^{i-1} \oplus \begin{pmatrix} 1-t & tt_i\lambda^{-1} \\ \lambda t_i^{-1} & 0 \end{pmatrix} \oplus I^{n-i-1} \text{ for } i = 1, 2, \ldots, n-1,$$

$$\psi(\rho_i) = I^{i-1} \oplus \begin{pmatrix} 0 & t_i \\ t_i^{-1} & 0 \end{pmatrix} \oplus I^{n-i-1} \text{ for } i = 1, 2, \ldots, n-1,$$

is a representation of VB_n.

Let us consider the ring $R = \mathbb{Z}[t^{\pm 1}, \lambda^{\pm 1}, t_1^{\pm 1}, t_2^{\pm 1}, \ldots, t_{n-1}^{\pm 1}]$ and a free R-module V with basis $\{e_1, e_2, \ldots, e_n\}$. Then we can rewrite the above representation $\psi : VB_n \to GL(V)$ as

$$\psi(\sigma_i) = \begin{cases} e_i \to (1-t)e_i + \lambda t_i^{-1} e_{i+1}, \\ e_{i+1} \to tt_i\lambda^{-1} e_i, \\ e_j \to e_j, \text{ for } j \neq i, i+1, \end{cases}$$
$$\psi(\rho_i) = \begin{cases} e_i \to t_i^{-1} e_{i+1}, \\ e_{i+1} \to t_i e_i, \\ e_j \to e_j, \text{ for } j \neq i, i+1. \end{cases}$$

The following proposition generalizes the result from [14, Theorem 7.1, part 3].

Proposition 5.1.6. *The representation ψ is equivalent to a virtually symmetric representation which is local and homogeneous.*

Proof. Let θ be the automorphism of V, which is defined on the basis by

$$\theta = \begin{cases} e_1 \to e_1, \\ e_2 \to t_1 e_2, \\ \vdots \\ e_n \to t_1 t_2 \dots t_{n-1} e_n. \end{cases}$$

Consider the representation $\tilde{\psi} = \theta \psi \theta^{-1}$. By definition, this representation is equivalent to ψ and is a virtually symmetric representation. Indeed,

$$\tilde{\psi}(\sigma_i) = \begin{cases} e_i \to (1-t)e_i + \lambda e_{i+1}, \\ e_{i+1} \to t\lambda^{-1} e_i, \\ e_j \to e_j, \text{ for } j \neq i, i+1, \end{cases}$$

$$\tilde{\psi}(\rho_i) = \begin{cases} e_i \to e_{i+1}, \\ e_{i+1} \to e_i, \\ e_j \to e_j, \text{ for } j \neq i, i+1. \end{cases}$$

□

5.2 Virtual link groups

In this section, we use our previously defined virtually symmetric representation $\varphi_S : VB_n \to \mathrm{Aut}(F_{n,n})$, to associate a group to each virtual link by various approaches.

5.2.1 Braid approach

It is known [9, 12] that for a given representation of VB_n into the automorphism group of some group or module, one can assign a group to any braid $\beta \in VB_n$. Let $\varphi : VB_n \to \mathrm{Aut}(H)$ be a representation of VB_n into the automorphism group of some group $H = \langle h_1, h_2, \dots, h_m \mid \mathcal{R} \rangle$, where $\mathcal{R}$ is the set of defining relations. For a given $\beta \in VB_n$, we associate the group

$$G_\varphi(\beta) = \langle h_1, h_2, \dots, h_m \mid \mathcal{R}, h_i = \varphi(\beta)(h_i), i = 1, 2, \dots, m \rangle.$$

For each $\beta \in VB_n$, let $G_M(\beta)$ and $G_S(\beta)$ be groups corresponding to representations $\varphi_M : VB_n \to \mathrm{Aut}(F_{n,n})$ and $\varphi_S : VB_n \to \mathrm{Aut}(F_{n,n})$, respectively.

The following result follows from [12, Theorem 4.1 and Theorem 6.1].

Theorem 5.2.1. *If β and β' are two virtual braids such that their closures define the same link L, then $G_M(\beta) \cong G_M(\beta')$.*

The above theorem implies that the group $G_M(\beta)$ is an invariant of the link L.

Proposition 5.2.2. *Let $\beta \in VB_n$. Then the group $G_M(\beta)$ is isomorphic to the group $G_S(\beta)$. In particular, $G_S(\beta)$ is a link invariant.*

Proof. For $\beta \in VB_n$, we have the following presentation

$$G_S(\beta) = \langle x_1,\dots,x_n,v_1,\dots,v_n \mid [v_i,v_j]=1, \varphi_S(\beta)(x_i)=x_i, \varphi_S(\beta)(v_j)=v_j\rangle.$$

Consider the map $\phi : F_{n,n} \to F_{n,n}$ defined in Proposition 5.1.2. So we have

$$\begin{aligned}
G_S(\beta) &\cong \langle x_1,\dots,x_n,v_1,\dots,v_n \mid [v_i,v_j]=1, \phi(\varphi_S(\beta)(x_i)) = \phi(x_i), \phi(\varphi_S(\beta)(v_j)) = v_j\rangle \\
&\cong \langle x_1,\dots,x_n,v_1,\dots,v_n \mid [v_i,v_j]=1, \varphi_M(\beta)(\phi(x_i)) = \phi(x_i), \varphi_M(\beta)(\phi(v_j)) = v_j\rangle \\
&\cong \langle x_1,\dots,x_n,v_1,\dots,v_n \mid [v_i,v_j]=1, (\varphi_M(\beta)(x_i))^{v_i\dots v_n} = x_i^{v_i\dots v_n}, \varphi_M(\beta)(v_j) = v_j\rangle \\
&\cong \langle x_1,\dots,x_n,v_1,\dots,v_n \mid [v_i,v_j]=1, \varphi_M(\beta)(x_i) = x_i, \varphi_M(\beta)(v_j) = v_j\rangle \\
&\cong G_M(\beta).
\end{aligned}$$

□

5.2.2 Diagram approach

Let $D(L)$ be a virtual link diagram of a virtual m-component link L. We begin by enumerating all the components of $D(L)$ with numbers from 1 to m and label the i^{th} component with v_i. Then we divide the diagram into arcs from one classical crossing to the next classical crossing and label them as $x_1, x_2, \dots, x_n$.

A *virtual link group* $G_S(D(L))$ of the link L corresponding to the diagram $D(L)$ is the group generated by elements $x_1,x_2,\dots,x_n,v_1,v_2,\dots,v_m$ with defining relations corresponding to each classical crossing as shown in Figure 5.1, along with the commutativity of all v_i's $(1 \le i \le m)$ with each other. Note that there are no relations at virtual crossings.

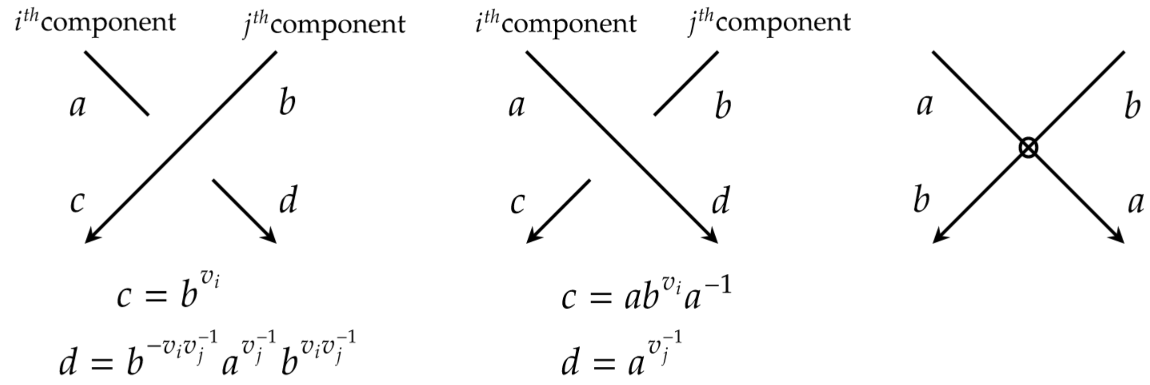


Fig. 5.1 Crossing relations.

Note that if $D'(L)$ is another diagram representing L, then $D(L)$ and $D'(L)$ are equivalent under a finite sequence of *generalized Reidemeister moves* as shown in Figure 2.7 up to planar isotopies.
The following statement is similar to the case of classical links and is not difficult to prove.

Proposition 5.2.3. *Let $D(L)$ and $D'(L)$ be two diagrams representing a virtual link L. Then groups $G_S(D(L))$ and $G_S(D'(L))$ are isomorphic. Hence, $G_S(D(L))$ is an invariant of L.*

The following result whose proof is immediate relates the groups constructed using braid and diagram approaches.

Proposition 5.2.4. *Let L be a virtual link and $D(L)$ its diagram and β be a braid whose closure is equivalent to L. Then $G_S(D(L)) \cong G_S(\beta)$.*

Hereafter, we denote $G_S(D(L))$ by $G_S(L)$.

Remark 5.2.5. We note that if the closure of $\beta \in VB_n$ is a virtual knot, then $G_S(\beta) \cong G_{\tilde{\varphi}_A}(\beta)$, where $\tilde{\varphi}_A$ is the generalized Artin representation considered in Subsection 5.1.1.

Remark 5.2.6. By putting each relation $v_i = 1$ in the presentation of $G_S(L)$, we recover the group $G_0(L)$ defined by Kauffman [79], which corresponds to the representation φ_0 mentioned in Section 5.1.

5.2.3 Gauss diagram approach

We recall the definition of a Gauss diagram from Section 2.2. An advantage of using the virtually symmetric representation $\varphi_S : VB_n \to \mathrm{Aut}(F_{n,n})$ over the representation $\varphi_M : VB_n \to \mathrm{Aut}(F_{n,n})$ is that we can describe the virtual link group $G_S(L)$ in terms of the Gauss diagram. Let D be the Gauss diagram with m circles corresponding to the virtual link L and label these circles with symbols $v_1, v_2, \ldots, v_m$. If we cut the circles at head and tail of each arrow, then circles of D are divided into arcs which are assigned symbols $x_1, x_2, \ldots, x_n$. Next, we define a group

$$\pi_D = \langle x_1, \ldots, x_n, v_1, \ldots, v_m \mid \mathcal{R}, v_i v_j = v_j v_i \text{ where } 1 \leq i, j \leq m \rangle,$$

where $\mathcal{R}$ is the set of relations defined for each signed arrow as depicted in Figure 5.2.

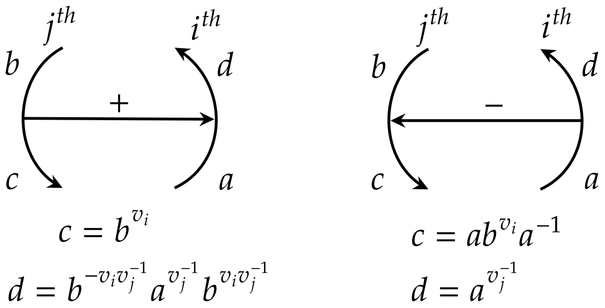


Fig. 5.2 Relations for the group π_D.

Since relations due to positive and negative arrows in Figure 5.2 are same as relations induced at positive and negative crossings in Figure 5.1, respectively, we have the following result.

Proposition 5.2.7. *If D is a Gauss diagram representing virtual link L, then $\pi_D \cong G_S(L)$.*

5.3 Marked Gauss diagrams

In this section, we define and study Gauss diagrams with additional structure, and extend the notion of virtual link group to these diagrams using similar approach as used in Section 5.2.3.

Definition 5.3.1. A *marked Gauss diagram* is a collection of a finite number of circles oriented anticlockwise having finite number of signed arrows whose heads and tails lie on circles along with a finite number of signed nodes lying on circles which are not attached to arrows. If head and tail of an arrow lie on the same circle, then it is said to be a *chord*.

By 1-*circle marked Gauss diagram*, we mean a marked Gauss diagram consisting of one circle. The following figures illustrates some examples of marked Gauss diagrams.

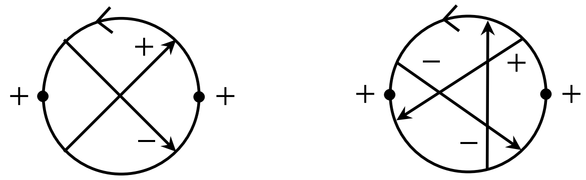


Fig. 5.3 Examples of 1-circle marked Gauss diagrams.

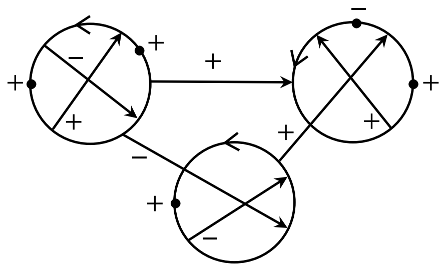


Fig. 5.4 A marked Gauss diagram having three components.

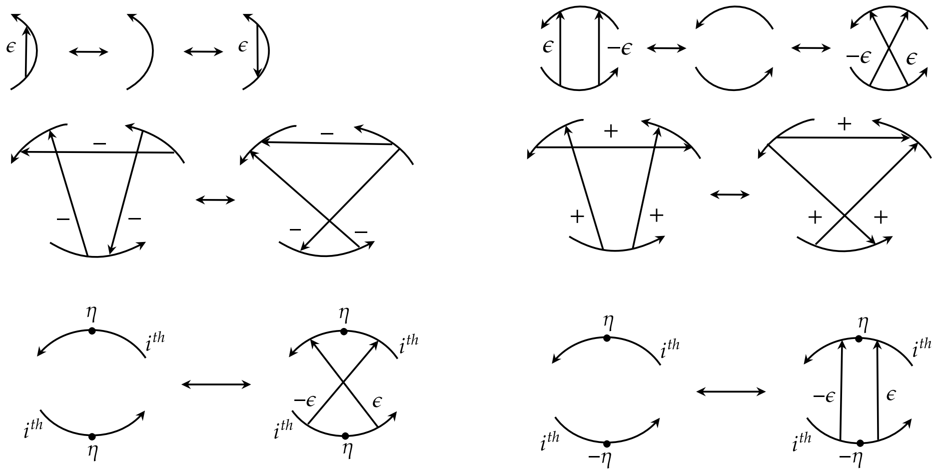


Fig. 5.5 Marked Reidemeister moves.

We consider marked Gauss diagrams up to the equivalence relation generated by finite sequences of moves shown in Figure 5.5, and we call these moves as *marked Reidemeister moves*. It is clear from Figure 5.5 that marked Gauss diagrams are proper generalizations of Gauss diagrams, that is, there is a canonical injective map from the set of equivalence classes of Gauss diagrams to the set of equivalence classes of marked Gauss diagrams.

Next, to each marked Gauss diagram we associate a group as follows. Let D be a marked Gauss diagram with m circles where the i^{th} circle is labelled as v_i. Then we cut circles at head and tail of each arrow and at each node point which divides circles of D into arcs to which we assign symbols $x_1, x_2, \ldots, x_n$. Define a group

$$\Pi_D := \langle x_1, \ldots, x_n, v_1, \ldots, v_m \mid \mathcal{R}, v_i v_j = v_j v_i \text{ for } 1 \leq i, \leq j \leq m \rangle,$$

where $\mathcal{R}$ consists of relations for each arrow and each node in D as shown in Figure 5.6. It is easy to check that the group Π_D is invariant under the marked Reidemeister moves. We note that if D is a Gauss diagram, then $\Pi_D \cong \pi_D$ which shows that the notion of a virtual link group can be extended to marked Gauss diagrams.

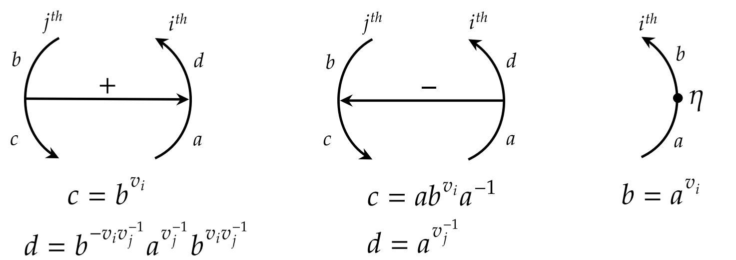


Fig. 5.6 Relations for the group Π_D.

Proposition 5.3.2. *The number of nodes, and the sum and product of signs of nodes in a given marked Gauss diagram are invariant under marked Reidemeister moves.*

Proof. It follows from the fact that the number of nodes, sum and product of signs of nodes do not change under any move given in the Figure 5.5. □

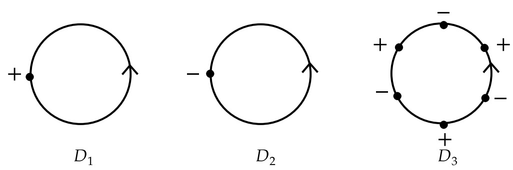


Fig. 5.7 Non-equivalent marked Gauss diagrams.

Example 5.3.3. Consider diagrams in Figure 5.7. Note that D_1 has presentation $\langle a, v \mid a = a^{v^{-1}} \rangle$ and D_2 has presentation $\langle b, v \mid b = b^v \rangle$. Clearly $\Pi_{D_1} \cong \Pi_{D_2} \cong \mathbb{Z}^2$ but diagrams D_1 and D_2 are not equivalent since the sum of sign of nodes are not equal. Furthermore, for D_3 we have equal number of positive and negative nodes and no chords, and hence $\Pi_{D_3} \cong \mathbb{Z} * \mathbb{Z}$.

For a Gauss diagram T without chords, that is, a Gauss diagram corresponding to the trivial knot, we have the following result.

Proposition 5.3.4. *There exists infinitely many marked Gauss diagrams with associated group isomorphic to* $\Pi_T = \mathbb{Z} * \mathbb{Z}$.

Proof. Let D be a 1-circle marked Gauss diagram with equal number of positive and negative nodes and having no chords. Then clearly, $\Pi_D \cong \Pi_T = \mathbb{Z} * \mathbb{Z}$. □

5.4 Marked virtual link diagrams

In this section, we give an interpretation of marked Gauss diagrams in terms of planar diagrams.
Let $G = (V, E)$ be a *directed graph*, where V denotes the set of vertices and E denotes the set of directed edges. A *diwalk* is an alternating sequence of vertices and edges $v_0, e_1, v_1, \ldots, v_{n-1}, e_n, v_n$ with edge e_i directed from v_{i-1} to v_i, for every $v_i \in V$ and $e_i \in E$. A *directed cycle* is a diwalk in which all vertices except the first and last are different. From now onwards, by a *cycle*, we mean a directed cycle.
Beineke and Harary [22] defined a *marked graph* as a directed graph in which each vertex is assigned either a positive or negative sign. We define *marked cycles* as a marked graph consisting only of cycles and no cycle shares a common vertex with another cycle. For example, see Figure 5.8.

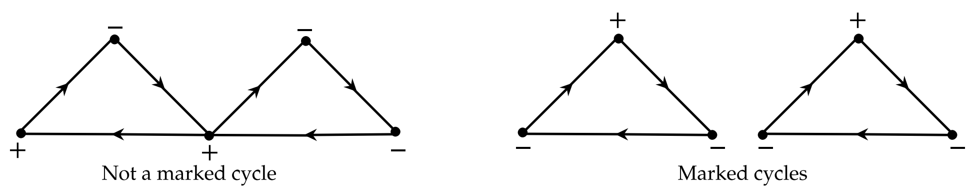


Fig. 5.8 Illustration of marked cycles.

Further, Fleming and Mellor [53] defined a *virtual spatial graph diagram* as a generic immersion of a directed graph in $\mathbb{R}^2$, where the double points are either classical crossings or virtual crossings. Analogously, we can define the following.

Definition 5.4.1. A *marked virtual link diagram* is a generic immersion of marked cycles in $\mathbb{R}^2$ with information of virtual and classical crossings at double points. If it is a one component diagram, then it is said to be a *marked virtual knot diagram*.

Note that for any given marked Gauss diagram, we can draw a marked virtual link diagram, and the converse also holds. Please refer to Figure 5.9 for an illustration.

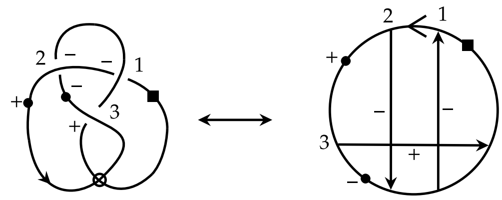


Fig. 5.9 Marked virtual knot diagram and its marked Gauss diagram.

We say that two marked virtual link diagrams are *equivalent* if they are related by a finite sequence of moves shown in Figure 5.10.
It is clear that there is a one-to-one correspondence between the set of equivalence classes of marked Gauss diagrams and the set of equivalence classes of marked virtual link diagrams.
We note that moves shown in Figure 5.11 are forbidden and cannot be obtained from moves shown in Figure 5.10.
Let D be a marked Gauss diagram and $D(L)$ its corresponding marked virtual link diagram. Then the group Π_D can be defined from $D(L)$ using a similar approach as described in Section 5.2.2 with relations defined at crossings and nodes as shown in Figure 5.12.

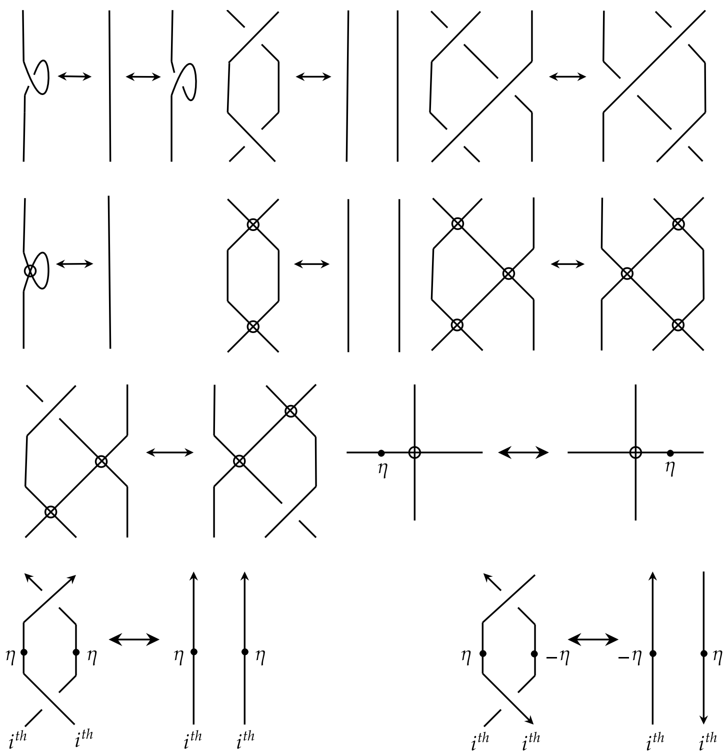


Fig. 5.10 Marked Reidemeister moves for marked virtual links.

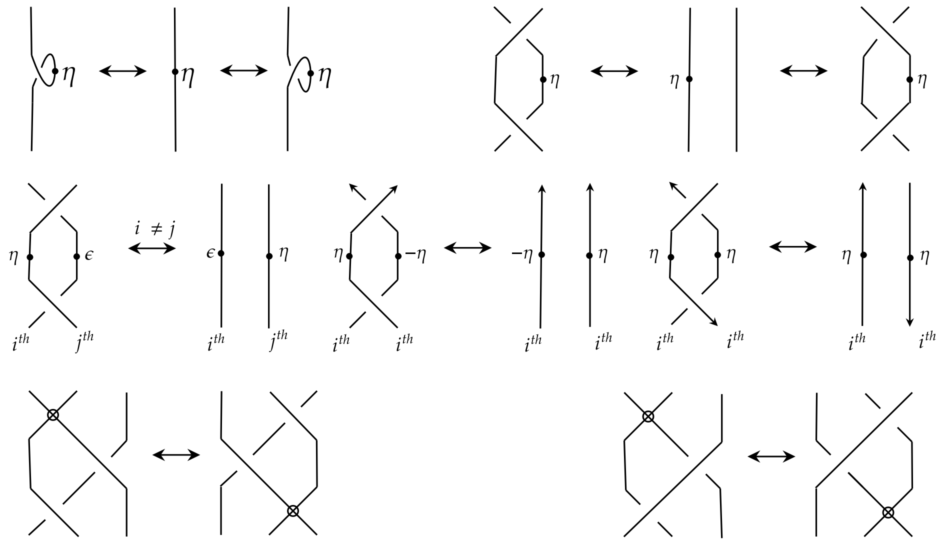


Fig. 5.11 Forbidden Reidemeister moves for marked virtual links.

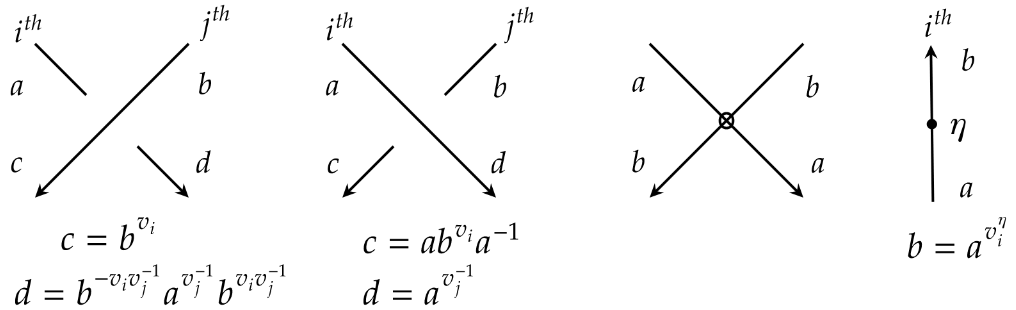


Fig. 5.12 Defining relations obtained from crossings and nodes of a marked virtual link diagram.

5.5 Realization of irreducible C_1-groups

In this section, we prove that every irreducible C_1-presentation of deficiency 1 or 2 is the group of some 1-circle marked Gauss diagram. We first recall the definition of C-groups given by Kulikov [83, 84]. A group G is called a *C-group* if it admits a presentation $\langle X \mid \mathcal{R} \rangle$, where $X = \{x_1, x_2, \ldots, x_n\}$ and relations $\mathcal{R}$ are of the type

$$w_{i,j}^{-1} x_i w_{i,j} = x_j$$

for some $x_i, x_j \in X$ and some words $w_{i,j}$ in $X^{\pm 1}$. Such a presentation is known as a *C-presentation*. Gilbert and Howie [55] called these groups as LOG groups. It is established in

[84] that every C-group can be realized as the fundamental group of complement of some n-dimensional $(n \geq 2)$ compact orientable manifold without boundary embedded in $\mathbb{S}^{n+2}$. In particular, any classical link group is a C-group.

We now define C_m-groups which are particular type of C-groups. For a non-negative integer m, a group G is called a *C_m-group* if it can be defined by a set of generators $Y = X \sqcup V_m$, where $X = \{x_1, x_2, \dots, x_n\}$, $V_m = \{v_1, v_2, \dots, v_m\}$ and a set of relations $\mathcal{R}$ given by

$$\begin{aligned} w_{i,j}^{-1} x_i w_{i,j} = x_j, &\quad \text{for some } x_i, x_j \in X \text{ and some words } \ w_{i,j} \text{ in } Y^{\pm 1}, \\ v_i v_j = v_j v_i, &\quad \text{for all } v_i, v_j \in V_m. \end{aligned}$$

We call the presentation $\langle Y \mid \mathcal{R} \rangle$ as a *C_m-presentation.*

Notice that all C_m-groups are C_{m-1}-groups for $m \geq 1$. In particular, all C_m-groups are C-groups. It is easy to see that the abelianization of a C_m-group is a free abelian group. If we put $x_i = 1$ for all $i = 1, 2, \dots, n$, then we get the free abelian group of rank m whereas, if we put $v_j = 1$ for all $j = 1, 2, \dots, m$, then we get a C-group.

A C-group is called *irreducible* if all generators in its C-presentation are conjugates of each other. Analogously, we say that a finitely generated C_1-group is *irreducible* if all its generators $x_i \in X$ in a C_1-presentation $\langle Y \mid \mathcal{R} \rangle$, where $Y = X \sqcup V_1$, are conjugate to each other. Equivalently, its abelianization is the free abelian group of rank 2. Next we extend this definition to C_m-groups for $m \geq 1$ as follows.

Associate a graph (not directed) Γ_m to a C_m-presentation $\langle Y \mid \mathcal{R} \rangle$ with vertices $x_1, x_2, \dots, x_n$ and edges e_i^j between vertices x_i and x_j if there is a relation $w_{i,j}^{-1} x_i w_{i,j} = x_j$ in $\mathcal{R}$. We say that a C_m-group with $m \geq 1$ is *irreducible* if the associated graph Γ_m has m-connected components, equivalently, its abelianization is of rank $2m$. Note that in this case $n \geq m$.

Remark 5.5.1. Note that an irreducible C_m-group $(m > 0)$ is not an irreducible C_{m-i}-group, where $1 \leq i \leq m$.

The *deficiency* of a group presentation is the number of generators minus the number of relations. The *deficiency* of a finitely presented group G is defined as the maximum deficiency of finite group presentations for G. It is easy to see that groups associated to marked Gauss diagrams are C_m-groups. The following result gives an example of an irreducible C_1-group.

Proposition 5.5.2. *The group associated to a given* 1*-circle marked Gauss diagram is an irreducible C_1-group of deficiency* 1 *or* 2 *and its second integral homology group is cyclic.*

Proof. Let D be a 1-circle marked Gauss diagram and Π_D the group associated to D having deficiency d. Then the group Π_D has a presentation $\mathcal{P}$ with $n + d$ generators x_i and n relations r_j. Let X be the 2-skeleton of the $K(\Pi_D, 1)$ space. To be precise, X is obtained by gluing n

many 2-disks to the one-point union of $n+d$ circles along the relations r_j. By construction, $\pi_1(X) \cong \Pi_D$. Then the cellular chain complex of X is

$$\ldots \to 0 \to \mathbb{Z}^n \overset{\partial_2}{\to} \mathbb{Z}^{n+d} \overset{\partial_1}{\to} \mathbb{Z},$$

where ∂_1 is the zero map. Since Π_D is an irreducible C_1-group, $H_1(X) \cong \mathbb{Z} \oplus \mathbb{Z}$. As $\mathrm{rank}(\mathrm{coker}\,\partial_2) + \mathrm{rank}(\partial_2(\mathbb{Z}^n)) = n+d$ and $\mathrm{rank}(\partial_2(\mathbb{Z}^n)) \leq n$, we have $d \leq 2$. Thus Π_D has deficiency either 1 or 2. Also, it follows that $H_2(X)$ is 0 for $d=2$ and $\mathbb{Z}$ for $d=1$. Hence, $H_2(\Pi_D) \cong H_2(X)$ is cyclic. $\square$

Corollary 5.5.3. *Every virtual knot group has deficiency* 1 *or* 2*, and its second integral homology group is cyclic.*

It is well-known that if K is a classical knot, then $\pi_1(C(K))$ has deficiency 1.

Example 5.5.4. It follows from Remark 5.2.5, [9, Proposition 4] and Theorem 5.2.2 that $G_S(K) \cong \pi_1(C(K)) * \langle v \rangle$, and hence it is of deficiency 2 and $H_2(G_S(K)) = 0$.

Example 5.5.5. Let G be a group having an irreducible C-presentation of deficiency 0 such that $H_2(G) \neq 0$. Let $G^* = G * \langle v \rangle$ be the free product of G and $\langle v \rangle$. Then $H_2(G^*) \neq 0$ since $H_2(G^*) \cong H_2(G) \oplus H_2(\mathbb{Z}) \cong H_2(G)$ (see [29, Thorem 7.2]). Assuming Theorem 5.5.11, we have that G^* is the group of some marked Gauss diagram, so we get that G^* has deficiency 1. This illustrates the existence of marked Gauss diagrams whose associated groups have deficiency 1. Gordon [57] gave a family of irreducible C-presentations of deficiency 0 whose second homology groups are $\mathbb{Z}$. Moreover, one can find an irreducible C-presentation of deficiency 0 with second homology group of order 2 in [30].

Definition 5.5.6. A *cyclic irreducible C_1-presentation* is an irreducible C_1-presentation of the form

$$\langle x_1, x_2, \ldots, x_n, v \mid r_1, r_2, \ldots, r_n \rangle,$$

where the relation r_j is of the form $x_{j+1}^{-1} x_j^{w_j}$ for $j \in \mathbb{Z}_n$, and w_j a word in alphabets $x_i^{\pm 1}$ and $v^{\pm 1}$.

Definition 5.5.7. A *realizable irreducible C_1-presentation* is a cyclic irreducible C_1-presentation where w_j belongs to $\{v^{-\varepsilon} x_i^{\varepsilon}, v^{\varepsilon} \mid i = 1, 2, \ldots, n \text{ and } \varepsilon = \pm 1\}$ and satisfy the following conditions:

(1) If $w_k = v^{-1} x_p$ for some p, then $w_p = v$ and the word w_j is not equal to $v^{-\varepsilon} x_p^{\varepsilon}$ and $v^{-\varepsilon} x_k^{\varepsilon}$ for any $j \neq k$.

(2) If $w_k = vx_p^{-1}$ for some p, then $w_p = v^{-1}$ and the word w_j is not equal to $v^{-\varepsilon}x_p^{\varepsilon}$ and $v^{-\varepsilon}x_k^{\varepsilon}$ for any $j \neq k$.

The proof of the following result is similar to [82, Lemma 2].

Proposition 5.5.8. *Any irreducible C_1-presentation of deficiency* 1 *or* 2 *can be transformed to a cyclic irreducible C_1-presentation.*

Proof. Let $\mathcal{P} = \langle x_1, \ldots, x_n, v \mid r_1, \ldots, r_m \rangle$ be an irreducible C_1-presentation of deficiency 1 or 2 that is, either $m = n$ or $n-1$. If $m = n-1$, then we add a relation $r_{m+1} = r_m$ and therefore, we assume $m = n$. Next, we consider the graph Γ associated to the presentation $\mathcal{P}$. We observe that if Γ has two edges e_j^i and e_k^j meeting at the vertex x_j, then there are relations of the form $x_i^{-1}x_j^{w_{j,i}}$ and $x_j^{-1}x_k^{w_{k,j}}$ in $\mathcal{P}$. Note that removing relation $x_i^{-1}x_j^{w_{j,i}}$ and adding relation $x_i^{-1}x_k^{w_{k,j}w_{j,i}}$ corresponds to an operation on Γ of removing the edge e_j^i and adding e_k^i. Clearly, this operation does not change the underlying group. Since Γ is a connected graph and number of edges is equal to number of its vertices, Γ has exactly one cycle C. Now, if the length $l(C)$ of cycle C is n, then $\mathcal{P}$ is cyclic irreducible C_1-presentation and if not, then because Γ is connected, there is an edge e_j^i such that x_i is in C and x_j is not in C. Using the above operation, we get a graph Γ' with cycle C' containing all vertices of C and x_j with $l(C') = l(C)+1$. Thus, after finitely many steps we obtain a graph with cycle of length n, which gives the desired result. $\square$

Proposition 5.5.9. *Every cyclic irreducible C_1-presentation can be transformed to a realizable irreducible C_1-presentation.*

Proof. A given cyclic irreducible C_1-presentation $\mathcal{P}$ can be turned into a realizable irreducible C_1-presentation in the following steps:

- *Step* 1*:* Make each w_j one of the letter in $\{x_1^{\pm 1}, \ldots, x_n^{\pm 1}, v^{\pm 1}\}$. For example, let $w_j = x_4x_3v^{-1}$ and $r_j = x_{j+1}^{-1}x_j^{w_j}$. Then introduce two more generators say x'_j and x''_j, remove the relation r_j and add three more relations ${x'_j}^{-1}x_j^{x_4}$, ${x''_j}^{-1}{x'_j}^{x_3}$ and $x_{j+1}^{-1}{x''_j}^{v^{-1}}$. Thus we get a new cyclic irreducible C_1-presentation presenting the same group. Moreover, we can assume that in the new cyclic irreducible C_1-presentation there are no relations of the type $r_j = x_{j+1}^{-1}x_j^{x_j^{\varepsilon}}$.

- *Step* 2*:* If $r_j = x_{j+1}^{-1}x_j^{x_k^{\varepsilon}}$ $(x_k \neq x_j)$, then remove relations r_j and $r_k = x_{k+1}^{-1}x_k^{w_k}$, add three generators $x_{k,1}, x_{k,2}, X_j$, five relations $x_{k,1} = x_k^{v^{\varepsilon}}, x_{k,2} = x_{k,1}^{v^{-\varepsilon}}, X_j = x_j^{v^{\varepsilon}}, x_{j+1} = X_j^{v^{-\varepsilon}x_k^{\varepsilon}}, x_{k+1} = x_{k,2}^{w_k}$. Moreover, replace x_k by $x_{k,2}$ in words w_i in presentation $\mathcal{P}$ for $i \neq j$.

□

As a consequence, we have the following result.

Corollary 5.5.10. *Any irreducible C_1-presentation of deficiency* 1 *or* 2 *can be transformed to a realizable irreducible C_1-presentation.*

The following result relates an irreducible C_1-presentation with the group of a marked Gauss diagram.

Theorem 5.5.11. *Any irreducible C_1-presentation of deficiency* 1 *or* 2 *can be realized as the group of a marked Gauss diagram.*

Proof. Let $\mathcal{P}$ be an irreducible C_1-presentation of deficiency 1 or 2. By Corollary 5.5.10, we can assume that $\mathcal{P}$ is a realizable irreducible C_1-presentation having $n+1$ generators $\{x_i, v \mid i = 1,2,\ldots,n\}$ and n relations $r_1,\ldots,r_n$. Next, we consider a circle with anticlockwise orientation and mark n points on it, thereby dividing the circle into n arcs. We then label the obtained arcs as $x_1,\ldots,x_n$ successively in the anticlockwise direction. Based on the type of relation, we perform the following steps.

- If $r_j = x_{j+1}^{-1}x_j^{v^{-\varepsilon}x_k^{\varepsilon}}$, then we attach the tail of a chord at the point where the arcs x_k and x_{k+1} meet, and we attach the head of the same chord at the point where the arcs x_j and x_{j+1} meet. Also, we assign sign ε to the chord.
- If $r_j = x_{j+1}^{-1}x_j^{v^{\varepsilon}}$ and there is no relation of the type $x_{k+1} = x_k^{v^{-\varepsilon}x_j^{\varepsilon}}$, then put a node with ε sign on the point where arcs x_j and x_{j+1} meets.

Further, it is easy to check that the group of the obtained marked Gauss diagram has the presentation $\mathcal{P}$. □

As a result of the previous theorem, it is clear that the group G^* in Example 5.5.5 corresponds to some marked Gauss diagram D.

Corollary 5.5.12. *If G has an irreducible C-presentation of deficiency* 1*, then the group $G^* = G * \langle v\rangle$ is an irreducible C_1-group of deficiency* 2*.*

5.6 Peripheral subgroup and peripheral structure

It is well established that the knot group, along with the peripheral subgroup and the meridian of a classical knot, is a complete invariant of classical knots up to the orientations of the knot and the ambient space. However, this is not the case for virtual knots (see, for example, [9, 60, 79, 82]). In particular, if we consider the virtual Kishino knot K, then $(G_{\tilde{\varphi}_A}(K);(m,l))$ (refer to Remark 5.2.5) defined in [9] does not distinguish it from the trivial knot. In this section, we extend the notion of a meridian, longitude, peripheral subgroup and peripheral structure to marked Gauss diagrams.

Let us begin by fixing a base point on the k^{th} circle of a given marked Gauss diagram such that it does not lie on the endpoints of arrows and nodes. Then we fix a *meridian* m_k to be the generator corresponding to the arc over which the base point lies. We now describe a procedure to write a *longitude* l_k corresponding to the meridian m_k. We start moving along the circle from the base point in the anticlockwise direction, and write v_t^ε when passing the tail of an arrow whose sign is ε and head lies on t^{th} circle, and when passing the head of an arrow we use the following rule.

- If the sign of an arrow is $+1$, and its tail divides the arcs x_i and x_{i+1} on the n^{th} circle, then we write $v_n^{-1}x_i^{v_k v_n^{-1}}$.
- If the sign of an arrow is -1, and its tail divides the arcs x_i and x_{i+1} on the n^{th} circle, then we write $v_n x_i^{-1}$.

If we pass a node, then we write v_k^ε, where ε is the sign of the node. On arriving at the base point, we write $m_k^{-\alpha}$, where α is the sum of signs of arrows whose head lies on the k^{th} circle. Hereafter, throughout the chapter by a marked Gauss diagram we mean a 1-circle marked Gauss diagram.

Let D be a given marked Gauss diagram, with m its meridian and l the corresponding longitude. A *peripheral pair* of a marked Gauss diagram D is the pair (m,l) and the *peripheral subgroup* corresponding to the meridian m of D is the subgroup of Π_D generated by m and l. Two pairs (m,l) and (m',l') are said to be *conjugates* if there is an element g in the group Π_D such that $m' = m^g$ and $l' = l^g$. We then define the *peripheral structure* as the conjugacy class of a peripheral pair of D.

We now prove that the peripheral structure of a marked Gauss diagram D is unique and invariant under the marked Reidemeister moves. Let us consider a presentation

$$\Pi_D = \langle x_1, \dots, x_n, v \mid r_1 = x_2^{-1}x_1^{w_1}, \dots, r_n = x_1^{-1}x_n^{w_n} \rangle,$$

of Π_D which is written as per the procedure described in Section 5.3. If x_1 is a meridian of D, then $l = w_1 \dots w_n x_1^{-\alpha}$ is the corresponding longitude, where α is the sum of signs of chords in the diagram D.

Proposition 5.6.1. *The peripheral pair and the peripheral subgroup of a marked Gauss diagram are unique up to conjugacy. Moreover, the peripheral structure is invariant under the marked Reidemeister moves.*

Proof. Let D be a marked Gauss diagram. We first choose two meridians m_1, m_2 corresponding to two different arcs of D. By construction of the group Π_D, there exists g_1 and g_2 in Π_D such that $m_1 = m_2^{g_2}$, $m_2 = m_1^{g_1}$, $l_1 = g_1g_2m_1^{-\alpha}$ and $l_2 = g_2g_1m_2^{-\alpha}$, where l_1 and l_2 are longitudes corresponding to meridians m_1 and m_2, respectively. It is not difficult to see that $l_2 = l_1^{g_1}$ and so $(m_1,l_1)^{g_1} = (m_2,l_2)$. This implies that the peripheral pair and the peripheral subgroup of D are unique up to conjugacy. Hence peripheral structure of D is independent of choice of meridian. At last, it is easy to check the invariance of the peripheral structure under marked Reidemeister moves. □

Proposition 5.6.2. *A peripheral subgroup of* Π_D *is abelian.*

Proof. Using previous proposition, it suffices to prove that the subgroup generated by x_1 and the corresponding longitude $l = w_1 \dots w_n x_1^{-\alpha}$ is abelian. By considering relations in the presentation of Π_D, we have $x_1 = x_1^{w_1 \dots w_n}$. This implies that the meridian x_1 commutes with the longitude l. □

Let G be a group with an irreducible C_1-presentation $\mathcal{P} = \langle x_1,\dots,x_n,v \mid r_1,\dots,r_m\rangle$. Let G_v the group $\langle x_1,\dots,x_n,v \mid r_1,\dots,r_m,v\rangle$, $g \in G$ and g_v the image of g in G_v. It is easy to observe that for any marked Gauss diagram D, the image l_v of longitude l belongs to the commutator subgroup of Π_{D_v}.

Theorem 5.6.3. *Let G be a group with an irreducible C_1-presentation given by*

$$\langle x_1,\dots,x_n,v \mid r_1,\dots,r_{n-1}\rangle$$

and l an element of G. If the image of l in G_v belongs to the commutator subgroup of G_v, and l commutes with some conjugate of x_1 say x_0, then G is the group of a marked Gauss diagram with peripheral pair (x_0,l).

Proof. Since x_0 is conjugate to x_1 in G, there exist some w in G such that $r_0 := x_1^{-1}x_0^{w} = 1$. Thus, $\mathcal{P} = \langle x_0,x_1,\dots,x_n,v \mid r_0,r_1,\dots,r_{n-1}\rangle$ is a presentation of the group G. We may assume that each relation in $\mathcal{P}$ is of the form $r_i = x_{i+1}^{-1}x_i^{w_i}$, $i = 0,1,\dots,n-1$. On adding a redundant relation $r_n = x_0^{-1}x_n^{(w_0w_1\dots w_{n-1})^{-1}l}$ to the presentation $\mathcal{P}$, we get a cyclic irreducible C_1-presentation of G. By Proposition 5.5.10, we can assume that $\mathcal{P}$ is a realizable irreducible C_1-presentation. Thus, it is the group of a marked Gauss diagram with a peripheral pair (x_0,l). □

As a consequence, we have the following results.

Corollary 5.6.4. *Let G be a group with an irreducible C-presentation $\langle x_1,\dots,x_n \mid r_1,\dots,r_{n-1}\rangle$ and l an element of G. If l belongs to the commutator subgroup of G and commutes with some conjugate of x_1 in G say x_0, then $G*\langle v\rangle$ is the group of a marked Gauss diagram with peripheral pair (x_0', l'), where x_0' and l' are natural images of x_0 and l in $G*\langle v\rangle$, respectively.*

Corollary 5.6.5. *Let G be a group with an irreducible C_1-presentation of deficiency 2. Then G is the group of a marked Gauss diagram with trivial longitude. In particular, if K is a classical knot, then $G_S(K)$ is the group of some marked Gauss diagram with a trivial longitude.*

Remark 5.6.6. In [82, Corollary 9], it is proved that there exists a non-trivial virtual knot K with trivial longitude in the group $G_0(K)$. We do not know an example of a non-trivial virtual knot K' having a trivial longitude in the group $G_S(K')$. But using the above corollary, one can construct a non-trivial marked Gauss diagram with a trivial longitude.

5.7 Peripherally specified homomorphs

The *weight* of a group G is defined as the minimum number of elements required to normally generate G. It was asked in [101] whether a finitely generated group of weight one is a homomorph (homomorphic image) of a knot group. This was answered positively in [56, 67] where it was proved that for any element μ in group G which is finitely generated by conjugates of μ, there exists a knot K in $\mathbb{S}^3$ and an onto homomorphism $\rho : \pi_1(C(K)) \to G$ such that $\rho(m) = \mu$, where m is a meridian of K.
Necessary and sufficient conditions for a pair of elements in the symmetric group S_n to be realized as the image of a meridian-longitude pair for some knot K in $\mathbb{S}^3$ can be found in [45] which was later extended to knot group representations into general groups [68]. Later on, Kim [82] extended these results to homomorphs of virtual knot groups $G_0(K)$.
In this section, we investigate the following analogous problem.

Problem 5.7.1. *Let G be a group, and μ and ν be elements in G. Does there exists a marked Gauss diagram D and an onto homomorphism $\rho : \Pi_D \to G$ such that $\rho(m) = \mu$ and $\rho(v) = \nu$, where m is a meridian of D and v corresponds to the component of D?*

It is easy to see that G must be finitely generated by ν and conjugates of μ. We fix μ and ν in G with these properties. We say that $\lambda \in G$ is realizable if we can find a marked Gauss diagram D and a representation $\rho : \Pi_D \to G$ with above properties such that $\rho(l) = \lambda$, where

l is the longitude of D corresponding to meridian m. We denote the set of realizable elements by Λ_G and prove the following result.

Theorem 5.7.2. *The set Λ_G is non-empty.*

Proof. Let $G = \langle \mu_1, \ldots, \mu_n, \nu \rangle$, where $\mu_1 = \mu$ and $\mu_i = \mu_1^{w_i}$, $2 \le i \le n$ and w_i are words in $\mu_1^{\pm 1}, \ldots, \mu_n^{\pm 1}, \nu^{\pm 1}$. Using techniques described in the proof of theorems 5.5.8 and 5.5.9, we can assume the following in G.

- $\mu_1 = \mu_n^{w_n}$ and $\mu_{i+1} = \mu_i^{w_i}$ for $1 \le i \le n-1$.
- Each w_i is either ν^{ε} or $\nu^{-\varepsilon}\mu_j^{\varepsilon}$, where $\varepsilon = \pm 1$. If $w_k = \nu^{-1}\mu_p$, then $w_p = \nu$ and if $w_k = \nu\mu_p^{-1}$, then $w_p = \nu^{-1}$. Moreover, $w_j \ne \nu^{-\varepsilon}\mu_p^{\varepsilon}, \nu^{-\varepsilon}\mu_k^{\varepsilon}$, where $k \ne j$.

By Theorem 5.5.11, we construct a marked Gauss diagram D corresponding to the following realizable irreducible C_1-presentation

$$\langle x_1, \ldots, x_n, v \mid x_2^{-1}x_1^{u_1}, \ldots, x_1^{-1}x_n^{u_n} \rangle,$$

where u_j is obtained by replacing μ_i and ν in w_j with x_i and v, respectively for every $1 \le i, j \le n$. Clearly, we have a well-defined onto homomorphism $\rho : \Pi_D \to G$ mapping x_1 to μ_1 and v to ν. $\square$

Let us now consider two marked Gauss diagrams D_1 and D_2. Let p_1 and p_2 be two points on D_1 and D_2, respectively, which are not meeting any chord or node. The *connected sum D* of D_1 and D_2 at p_1 and p_2 is a marked Gauss diagram obtained by removing a small interval around p_1 and p_2 not intersecting a chord or a node, and then joining end points of remaining diagrams while respecting orientations. Let $\Pi_{D_1} = \langle x_1, \ldots, x_n, v_1 \mid x_2^{-1}x_1^{\alpha_1}, \ldots, x_1^{-1}x_n^{\alpha_n} \rangle$ and $\Pi_{D_2} = \langle y_1, \ldots, y_m, v_2 \mid y_2^{-1}y_1^{\beta_1}, \ldots, y_1^{-1}y_m^{\beta_m} \rangle$ be group presentations of D_1 and D_2, respectively. If p_1 and p_2 are on arcs x_1 and y_1, respectively, then a group presentation of Π_D is

$$\langle x_1, \ldots, x_n, y_1, \ldots, y_m, v \mid x_2^{-1}x_1^{\alpha_1'}, \ldots, x_n^{-1}x_{n-1}^{\alpha_{n-1}'}, y_1^{-1}x_n^{\alpha_n'}, y_2^{-1}y_1^{\beta_1'}, \ldots, y_m^{-1}y_{m-1}^{\beta_{m-1}'}, x_1^{-1}y_m^{\beta_m'} \rangle.$$

Note that here α_i' and β_j' are obtained from α_i $(1 \le i \le n)$ and β_j $(1 \le j \le m)$ by replacing v_1 and v_2 by v.

Example 5.7.3. Figure 5.13 shows the marked Gauss diagram D_3 as a connected sum of two non-trivial marked Gauss diagrams D_1 and D_2 at p_1 and p_2. It is easy to check that $\Pi_{D_3} \cong \mathbb{Z} * \mathbb{Z}$. Let us now consider T to be a marked Gauss diagram without chords and nodes, then $\Pi_T \cong \mathbb{Z} * \mathbb{Z}$. However, it can be easily seen that D_3 is not equivalent to T.

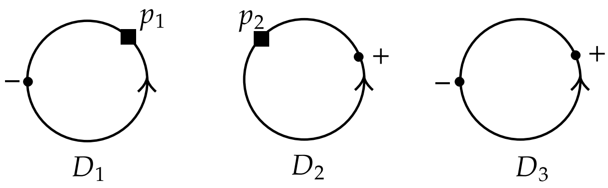


Fig. 5.13 D_3 is the connected sum of D_1 and D_2 at points p_1 and p_2.

Our final result shows that the non-empty set Λ_G is, in fact, a subgroup of G.

Theorem 5.7.4. *The set Λ_G is a subgroup of G.*

Proof. Let λ_1, λ_2 be two elements of Λ_G. Since λ_1, λ_2 are realizable, there exist marked Gauss diagrams D_1, D_2 and onto homomorphisms $\rho_1 : \Pi_{D_1} \to G$, $\rho_2 : \Pi_{D_2} \to G$ such that $\rho_1(x_1) = \rho_2(y_1) = \mu$, $\rho_1(v_1) = \rho_2(v_2) = \nu$ and $\rho_1(l_1) = \lambda_1$, $\rho_2(l_2) = \lambda_2$, where (x_1, l_1) and (y_1, l_2) are meridian-longitude pairs of D_1 and D_2, respectively. Let D be a connected sum of D_1 and D_2 made over points lying on arcs x_1 and y_1. Define a map $\rho : \Pi_D \to G$ such that $\rho(x_i) = \rho_1(x_i)$, $\rho(y_j) = \rho_2(y_j)$ and $\rho(v) = \nu$ for every $1 \leq i \leq n$ and $1 \leq j \leq m$. It is easy to see that the map ρ is a well-defined onto homomorphism and that $\rho(l) = \lambda_1\lambda_2$, where l is the longitude corresponding to the meridian x_1 for D. Let $\overline{D}_1$ be the marked Gauss diagram obtained from D_1 by reversing the orientation of circle and signs of chords and nodes. If $\bar{x}_i$ denote the arc in $\overline{D}_1$ which was labelled x_i in D_1, then the map $\bar{\rho}_1 : \Pi_{\overline{D}_1} \to G$ defined by $\bar{\rho}_1(\bar{x}_i) = \rho_1(x_i)$ is well-defined and $\bar{\rho}_1(l) = \lambda_1^{-1}$, where l is the longitude of $\overline{D}_1$ corresponding to the meridian $\bar{x}_1$ for $\overline{D}_1$. □

We conclude the thesis by listing some questions.

Question 5.7.5. *Is every representation of VB_n equivalent to a virtually symmetric representation?*

Question 5.7.6. *Under what conditions an irreducible C_m-group, $m \geq 2$, can be realized as the group of a marked Gauss diagram?*

Question 5.7.7. *Does there exists a topological interpretation of equivalence classes of marked Gauss diagrams?*

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