Bounding quantum advantages in weak value metrology

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Certificate of Examination

This is to certify that the dissertation titled "Bounding quantum advantages in weak value metrology" submitted by Mr. Sourav Das (Reg. No. MS16015) for the partial fulfillment of BS-MS dual degree programme of the institute, has been examined by the thesis committee duly appointed by the institute. The committee finds the work done by the candidate satisfactory and recommends that the report be accepted.

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Date: May 11, 2021

Declaration

The work presented in this dissertation has been carried out by me under the guidance of Dr. Manabendra Nath Bera at the Indian Institute of Science Education and Research Mohali.

This work has not been submitted in part or in full for a degree, a diploma, or a fellowship to any other university or institute. Whenever contributions of others are involved, every effort is made to indicate this clearly, with due acknowledgement of collaborative research and discussions. This thesis is a bonafide record of original work done by me and all sources listed within have been detailed in the bibliography.

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In my capacity as the supervisor of the candidate's project work, I certify that the above statements by the candidate are true to the best of my knowledge.

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-Sourav Das

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Symbol	Meaning
$\mathcal{L}(\epsilon/\theta)$	Log likelihood function
$\mathcal{F}(\theta)$	Classical Fisher information
$\mathcal{I}_Q$	Quantum Fisher information
η	Efficiency of weak value amplification
ξ	Efficiency of postselection metrology

Abstract

Weak Value Amplification and Post-selection based quantum protocols have been extensively used to enhance the precision of estimating small parameters. However, the benefit of these protocols are largely constrained by the fact that higher enhancements come with a cost of very low probability of successful post-selection. Here we propose a geometric relation between the absolute value of the Weak Value and corresponding probability of successful post-selection which characterizes the condition to obtain a given amount of amplification with minimal cost and vice versa. We further implement this relation in the recently developed method of postselected metrology to find a similar relationship between the postselected quantum Fisher Information and the postselection probability. Finally we provide a preparation and postlection procedure in which we obtain the optimally enhanced postselected quantum Fisher Information using a three level non-degenerate quantum system.

Contents

Chapter 1

Overview

Information constitutes the foundation of knowledge, which sets the limits of the domain of our imagination about the physical world. Imagine a blind philosopher, cramped in a dungeon located in a place he knows nothing about, wakes up in the morning and was asked whether it will be a rainy day. He answers, the probability distribution of the day to be a sunny day or rainy day is $\{1/2, 1/2\}$. The philosopher imagines he can be at any space-time point on earth; thus, he can say nothing about the place's weather forecast. Theoretically also we can state that the distribution carries no information. The philosopher just said in a fancy way that he does not know. However, if he is provided with some information, for example, the dungeon's location, say somewhere in the Atacama desert, he would answer differently. Now, his imagination about his location in space limits within one of the driest places in the world, and he deducts his knowledge that most likely it will be a sunny day.

In the same way we collect information from nature, then we process it with our intellect and disposition. After that we draw some inference; many equivalent inferences from different kind of informations collectively constitute knowledge. Thus knowledge becomes profound, objective and explicit. On the other hand, imagination is an arbitrary entity. One can imagine the earth is flat, universe has no space-time boundary or Michael Jackson will take rebirth one day. The knowledge bounds the arbitrariness of our imagination within an accessible region.

The fundamental laws in physics also constitute some profound knowledge of the world, which also demarcate the boundary of the domain of our imagination. For example, the attempt to make a perpetual motion machine led to the discovery of laws of thermodynamics which negated its possibility[Angrist n]; "one can not overcome the speed of light", constitutes the foundation of the special theory of relativity; Bell's theorems negated the existence of hidden variables[Bell 64], etc. Each of these discoveries constrained the domain of our imagination. A byproduct of these fundamental "bounds" is that, as we recognize our imagination's boundaries, it sets more physically allowed goals for technology. Thus, the discovery of laws of thermodynamics shifted the goal of the technologists from perpetual motion machine to Carnot efficiency[Tsaousis 08].

This thesis also concerns some fundamental "bounds" in advantage of some quantum technologies based on the principle of postselection, a classical phenomenon that becomes interesting when it is associated with Quantum mechanics.

Classically, the postselection is described as conditioning a probability distribution. For example, when we toss two coins, we can postselect the outcome of the second coin to 'Heads', which obtains a conditional probability distribution of the first coin's outcomes. In quantum theory, the postselection is modeled as a final projective measurement performed on the system with some selected orthonormal projectors. The scheme is akin to selecting the system's state *a posteriori* after the operations required for the task are performed. Multiple lines of research have shown that when a quantum protocol for some technological task is associated with the postselection of the system, it can surpass the standard quantum limits for the overall gain of the protocol, set by the fundamental laws of physics[Hensen 15][Resch 04]. The discourse on this topic began with the discovery of weak values in weak measurements in 1984[Aharonov 88]. When a system is weakly measured by a meter via a weak unitary interaction followed by a postselection onto the system, the meter's deflection can go way beyond the measurement operator's eigen spectrum. The deflection happens because of the appearance of weak values. weak values are a kind of average value of some operator measured on a system that is preselected in a state before and postselected on a state after the measurement, which can lie outside the operator's eigenspectrum.The effects of weak values have been observed in many experimental situations[Hosten 08][Zhou 12][Pfeifer 11a][Dixon 09][Pfeifer 11b][Xu 13]. Although the physical meaning of the weak values is a point of debate till now[Dressel 14], the effect of this quantum phenomenon is known to be crucial in the enhancement of precision in the measurement of small parameters using quantum systems, a field of quantum technologies widely known as Quantum Metrology. The scheme for weak value based amplification of small parameters is called the Weak Value Amplification. This scheme enhances the precision of the parameter estimation enormously via amplifying the sensitivity of meter deflection on minute changes in the parameter. However this quantum advantage comes with a cost, with low probability of successful postselection. It means that the high precision trials are obtained but very rarely. This trade off can be aided by incorporating quantum resources in the system e.g entanglement[Pang 14], squeezed quadrature variables[Pang 15a], non-linear Hamiltonian[Vetrivelan 20], etc. Incorporating unsuccessful postselections into the estimation is also known to be helpful for this purpose[Dressel 13]. These techniques provides more advantage over using classical resources to mitigate this trade off.

One of the central quantities in Quantum metrology is Quantum Fisher information. This quantity also concerns a fundamental "bound" called as Quantum Cramer-Rao lower bound. This bound sets the maximum precision we can obtain using a specific estimation protocol. In weak value metrology, the postselection plays a central role in determining the Fisher information. There the maximum amount of Fisher information is preserved in the postselections with very low probabilities[Pang 14]. Which is why we observe the trade off in Weak Value Amplification. The motivation of this thesis stems from a simple question, that under what condition, this trade off works minimally and we invest minimum cost in postselection probability and in return we obtain highest gain in weak value amplification. We address this question by defining a measure of efficiency which calculates the amount of gain of amplification per unit cost of probability and then finding a fundamental "bound" for this efficiency which finally leads us to the conditions to achieve optimal weak value amplification. As I have discussed earlier, the fundamental "bounds" sets some practical goals to achieve for technology, here also we obtain such conditions of optimality in weak value amplification which have great technological significance. We also show that the probability in the optimal weak value amplification represent the ratio of classical Fisher information and the quantum Fisher information in weak value amplification which quantifies the amount of advantage the postselection provides over classical estimation.

The weak value metrology provides significant advantage in quantum estimation process, however it always works within the standard quantum limit of quantum metrology2.5, this means the Fisher information is always less than certain value fixed by the limit. However the technique of postselected metrology has the ability to surpass this bound. This scheme is little different than weak value amplification. In weak value amplification, after the postselection is performed, the state of the system is completely discarded. The information of the parameter is imprinted in the meter via the weak interaction, we do the final processing of information from the meter. However, in postselected metrology the system is not discarded after postselection, the final processing of information is done directly on the system. Here we don't associate any meter with the process. In this case the postselected Fisher information can be shown to surpass standard quantum limit of quantum metrology. This happens if and only if the initial state of the system exhibit quantum features arising from noncommutativity of observables. Surprisingly, this advantage also comes with the cost of very low probability of successful postselection. This indicates that we can implement the similar treatment as we did with the weak value amplification i.e defining the measure of efficiency and finding out its bound to quantify the conditions of achieving the optimal situation where we obtain maximum gain with minimum cost. We show that despite the two schemes are structurally different from one another, there are ontological relations between them. We don't obtain the effects of weak values in postselected metrology directly, however we see their great influence in determining the gain and cost of the protocol. This result helped us to extend the results we derived for the weak value amplification, to postselected metrology resulting in similar bounds in efficiency of this protocol. We aid this formulation with a simple preparation-measurement scheme where we obtain the optimal postselected quantum Fisher information using a three level quantum system.

Finally, we explored the optimal conditions formulated for the two schemes about what implications they have in the geometry of quantum states.

1.1 Outline

This thesis presents the results of the following three objectives which are related to each other.

- To find the general condition for optimal weak value amplification where we obtain maximum amplification with minimum trade off in postselection probability.
- The effect of weak values in postselected metrology, and the condition to achieve optimal postselected Fisher information without loss of information due to postselection.
- The geometric interpretation of optimal weak value amplification in terms of Pancharatnam phase.

The results of these projects are still unpublished.

Here we give a short summary of the chapters.

- **Chapter 1:** In this chapter we discuss briefly the theory of classical estimation theory in a quantum preparation. We introduce some of the fundamental quantities in estimation theory e.g the likelihood function, unbiased estimator, score function, Fisher information and the Cramer-Rao bound. Later we discuss the Standard quantum limit of quantum metrology and it's significance in estimation theory.
- **Chapter 2:** Here we introduce the concept of Weak measurement, Weak values and Weak Value Amplification. We discuss the trade off in probability of successful postselection to gain high Weak Value Amplification and review the advantage of using entanglement as a resource to control this trade off.
- **Chapter 3:** In this chapter we present the quantification of efficiency for a Weak Value Amplification protocol and the fundamental bound of this efficiency. We point out the condition to achieve the highest efficiency. We also reproduce the results entanglement assisted weak value amplification protocol as a special case of this optimization scheme.

- **Chapter 4:** Here we start with a brief review of the scheme of Postselected Metrology and how it differs from the technique of Weak Value Amplification. We show the influence of weak values is postselected Fisher information. Using the results of optimal weak value amplification we quantify the conditions to achieve the optimal regime in postselected metrology. We end up with a simple example where we reach the optimal quantities using a three level quantum system.

- **Chapter 5:** In this section we briefly introduce the concept of geometric phase in quantum systems and then we discuss the geometric implication of optimal weak value amplification in terms of geometric phase.

Quantum Metrology

Chapter 2

Quantum Metrology

Every system in our world have some physical parameters associated which are used to quantify the kinematics and dynamics of the system. Metrology is a field where the estimation processes of these parameters and their statistical properties are discussed. It concerns the highest precision limits we can achieve in estimation through a specified class of measurement settings. Thus it plays a central role in physics and engineering of measurement. In this section, I shall discuss the basic structure of a metrological setting, how it's precision is quantified and bounded by Fisher Information, how quantum resources provide advantage over classical resources in precision. Quantum metrology has been theoretically proven supremacy over their classical correspondents in many estimation tasks, for simplicity here we take the case where small phases in the quantum states are to be measured.

2.1 Basic protocol

We start with a given quantum system S which is prepared in a state $\hat{\rho}(\theta)$ where the information of the parameter θ is encoded. For example the system is in a thermal state and the temperature of the system is to be estimated. One way to estimate the parameter is to do a full tomography of the state $\hat{\rho}(\theta)$ and find out the map $\theta \to \hat{\rho}(\theta)$. This will give us a huge precision in our estimation, however it is an enormously expensive technique for this task. State tomography itself is a costly task. For each value of θ we have to do a full state tomography, which increases the difficulty of the task multiple folds. It is comparable to estimating the temperature of a system by estimating kinetics of every atom in the system. A more practicable method for this task is to design a measurement, whose results are directly dependent upon the parameter. For example here we design an observable $\hat{E}$ and we get the expectation value of our measurement as follows,

$$\langle \hat{E} \rangle_{\hat{\rho}(\theta)} = tr(\hat{E}\hat{\rho}(\theta)) = f(\theta). \tag{2.1}$$

Now when we invert the function $f(\theta)$ we get the information of θ. This is a simpler and more practicable method to estimate θ[Osborne]. However, here our precision reduces significantly since the observable we design will have it's own variance which will be associated with the variance of our parameter θ.

2.2 Maximum likelihood estimation (classical estimation theory)

The maximum likelihood estimation is a process of estimating parameter dependence of a probability distribution of some random variables using optimization of some constructed function. This sets the theoretical foundation of classical estimation theory. The modern version of the theory was discovered by R.A Fisher between 1912 and 1922 and extended further by many well known statisticians like Harald Cramer, C.R Rao. We will look into the theory of maximum likelihood estimation briefly in a quantum preparation which later will help us to extend the discussion to the theory of quantum estimation.

So far we have a quantum system prepared in a state $\hat{\rho}(\theta)$. Now we design a complete set of POVMs $\{\epsilon, \hat{E}(\epsilon)\}$ as our measurement setting where ϵ is the measurement outcome and $\hat{E}(\epsilon)$ is the corresponding POVM operator. When we measure the system with $\hat{E}$, we obtain a probability distribution of different measurement outcomes depending upon θ.

This distribution is called the likelihood function. This distribution carries the information of θ. To decode this information we construct an estimator function $\Phi(\epsilon)$

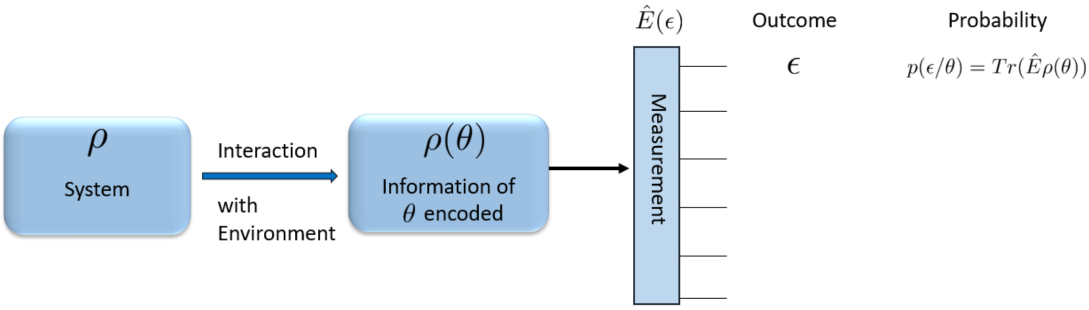

Figure 2.1: Schematic diagram of a metrological setting

such that its average with respect to the estimator function obtains the value of θ.

$$\langle \Phi \rangle_\theta = \sum_\epsilon p(\epsilon_{/\theta})\Phi(\epsilon) = \theta. \tag{2.2}$$

These kind of estimator functions are called unbiased estimators. The terms of these quantities are directly inherited from classical estimation theory since they play the same role. This $\langle \Phi \rangle_\theta$ we can consider as the inverse function of $f(\theta)$,

$$\langle \Phi \rangle_\theta = f^{-1}(\theta). \tag{2.3}$$

$$p(\epsilon_{/\theta}) = tr(\hat{\rho}(\theta)\hat{E}(\epsilon)). \tag{2.4}$$

However the condition in Eq 2.2 is a strong condition for estimators and often we find it difficult to find such estimators. A weaker notion on this purpose is to find out a locally unbiased estimator. When we estimate the parameter near a fixed value namely $\theta = \theta_0$ in this case, a locally unbiased estimator requires the condition,

$$\left. \frac{d\langle \Phi \rangle_\theta}{d\theta} \right|_{\theta_0} = 1. \tag{2.5}$$

2.3 Score function, Fisher Information and Cramer-Rao lower bound

Likelihood function is an indicator of the quality of our estimation protocol. The peak of the likelihood function is the characteristics point of our estimation since that controls the expectation value of our estimator. A good quality of the likelihood function must have low variance and high sensitivity on the value of the parameter. The sensitivity of the likelihood function is represented by the score function which is the derivative of the log-likelihood function with respect to θ.

$$S(\epsilon/_{\theta}) = \frac{\partial \mathcal{L}}{\partial \theta}. \tag{2.6}$$

Where $\mathcal{L}(\epsilon/_{\theta}) = ln(p(\epsilon/_{\theta}))$ is the log-likelihood function.

The precision in the estimation of θ is determined by the variance of Φ with respect to the likelihood function since it's average gives us the result of θ. The value of this variance is bounded by the Cramer-Rao lower bound.

$$\Delta\Phi_{\theta}^{2} \geq \frac{1}{\mathcal{F}(\theta)}. \tag{2.7}$$

Where $\mathcal{F}(\theta)$ is the Fisher information. This bound limits the amount of precision we can achieve to estimate θ. Fisher information is actually the variance of the score function with respect to the likelihood function.

$$\mathcal{F}(\theta) = \left\langle \left(\frac{\partial \mathcal{L}}{\partial \theta} \right)^{2} \right\rangle. \tag{2.8}$$

The Fisher Information measures the sensitivity of the likelihood function on changing the parameter and thus it quantifies the amount of information of θ residing in the likelihood function.

2.4 Quantum Fisher Information and Quantum Cramer-Rao lower bound

It is discussed earlier in subsection 2.2 that for each choice of our POVM $\{\epsilon, \hat{E}(\epsilon)\}$ we get a specific likelihood function. Specific likelihood functions give specific Fisher information which bounds the amount of precision in our choice of estimator($\langle\Phi\rangle_\theta$). Now when we maximize the Fisher information over all available choices of the POVMs, we get the Quantum Fisher Information(QFI) which is independent of the Likelihood function.

$$\mathcal{I}_Q(\theta) = \max_{\hat{E}(\epsilon)} \mathcal{F}(\theta). \tag{2.9}$$

The quantum Fisher Information sets the Quantum Cramer-Rao bound. In the following part, I shall discuss the formula to calculate the Quantum Fisher Information and how it obtains the Quantum Cramer-Rao bound.

First we have to assume the state evolves with the parameter via a unitary interaction.

$$\hat{\rho}(\theta) = exp(-i\hat{A}\theta)\hat{\rho}_0 exp(i\hat{A}\theta). \tag{2.10}$$

where $\hat{\rho_0}$ is the state for $\theta = 0$ and $\hat{A}$ is a hermitian operator onto the state space of the system. The quantum Fisher Information for this setting is as follows,

$$\mathcal{I}_Q(\hat{\rho}(\theta), \hat{A}) = tr\Big(\Omega_\rho^{-1}\Big((\partial_\theta\hat{\rho})^\dagger\Big)\partial_\theta\hat{\rho}\Big). \tag{2.11}$$

Here Ω is a linear super operator on the operator space of the Hilbert space defined as follows,

$$\Omega_X(Y) = \frac{1}{2}(XY + YX)$$

where X and Y are two operators.

In appendix 1 we prove that this is the optimized Fisher information over all the complete POVMs of the system state.

The expression for quantum Fisher information can be written in simple form when we define the following operation called symmetric logerithmic derrivative (Λ_θ),

$$\partial_\theta\hat{\rho} = \frac{1}{2}(\Lambda_\theta\hat{\rho} + \hat{\rho}\Lambda_\theta)$$

We can write the expression for Fisher information simply in terms of the Symmetric Logarithmic derivative,

$$\mathcal{I}_Q(\hat{\rho}(\theta), \hat{A}) = tr(\hat{\rho}\Lambda_\theta^2) \tag{2.12}$$

If we take pure states as the initial state of the system, the QFI can be written in the following form,

$$\mathcal{I}_Q(|\psi_i\rangle, \hat{A}) = 4[\langle\dot{\psi_\theta}|\dot{\psi_\theta}\rangle - |\langle\dot{\psi_\theta}|\psi_\theta\rangle|^2] = 4Var(\hat{A})_{|\psi_\theta\rangle} \tag{2.13}$$

Here $|\psi_i\rangle$ is the initial state of the system, $|\psi_\theta\rangle = exp(-i\theta\hat{A})\,|\psi_i\rangle$ is the evolved state.

2.5 Heisenberg scaling of Quantum Fisher Information

The result in Eq 2.13 seems to be surprising in the first glance. In order to increase the precision or to decrease the variance of our parameter we need to increase the variance of the operator $\hat{A}$ and this seems to be counter intuitive in understanding. The reason behind this dichotomy lies in the algebra of conjugate variables in Quantum mechanics. We know that generator of continuous symmetry transformations and the operator, whose eigen-space is where the transformations are performed, act as conjugate variables, or in other words their eigen-spaces can be written as the Fourier transform of each other. For example the position and momentum variables act as generator of translation on the eigen-space of one another and they are conjugate variables. In this case we encode the information of θ into the system via a unitary transformation generated by $\hat{A}$. And the estimator Φ act as the operator variable corresponding to θ on average(Eq 2.2). Thus $\hat{A}$ and Φ act here as a set of conjugate variables. That is why their corresponding uncertainties are bounded by the Heisenberg uncertainty relation. Quantum Cramer-Rao bound is nothing but a form of Heisenberg uncertainty relation. Thus when we measure θ precisely the uncertainty of A has to be large.

When we take a combination of N systems and the variable $\hat{A}$ is direct sum of the variables of each subsystem, the maximum variance that we can achieve scales with N quadratically as N^2. That is the maximum scaling of number of subsystem we can achieve on Quantum Fisher Information and it was widely believed that it is physically impossible to surpass this bound. This is called as the Heisenberg scaling for QFI. Here we give an example where this bound is achieved.

We have a quantum system S which is composed of N number of subsystems namely S_i where i goes from 1 to N. Each of S_i has observable $\hat{A}_i = \mathbf{1} \otimes .... \otimes \hat{a} \otimes ... \otimes \mathbf{1}$ where $\hat{a}$ operates in thei^{th} subsystem. $\hat{a}$ has maximum and minimum eigenvalues λ_{max} and λ_{min} The global observable of the system is written as follows,

$$\hat{A} = \hat{A}_1 + \hat{A}_2 + ... + \hat{A}_N \tag{2.14}$$

We take the initial state one of the maximally entangled states, as follows,

$$|\psi_i\rangle = \frac{|\lambda_{min}\rangle^{\otimes N} + e^{i\chi} |\lambda_{max}\rangle^{\otimes N}}{\sqrt{2}} \tag{2.15}$$

Where χ is an arbitrary phase. The state evolves via the unitary operation $exp(-i\theta\hat{A})$. At the limit $\theta \to 0$ the Quantum Fisher Information of this setting can be given as,

$$\mathcal{I}_Q(|\psi_i\rangle, \hat{A}) = N^2(\lambda_{max} - \lambda_{min})^2 \tag{2.16}$$

Thus in this setting we achieve the Heisenberg scaling in quantum Fisher information. Heisenberg scaling can be achieved by a number of other experimental settings. It can also be achieved via quantum error correcting codes which makes the system robust in a noisy environment. This limit concerns the Heisenberg inequality in quantum mechanics, it is needless to explain how profoundly this bound is associated with our foundational understanding of quantum mechanics. Any alteration in this bound will lead us to points of contradictions where we have to reconsider our imagination.

Weak Values and Weak Measurement

Chapter 3

Weak Measurement, Weak Values and Weak Value Amplification

Weak values for some Observable $\hat{A}$ with respect to a preselected state $|\psi_i\rangle$ and a postselected state $|\psi_f\rangle$ is defined as follows,

$$A_w = \frac{\langle\psi_f|\,\hat{A}\,|\psi_i\rangle}{\langle\psi_f \mid \psi_i\rangle} \tag{3.1}$$

Weak values are considered as a different form of average values of the observables which appear in the meter when a system initially in state $|\psi_i\rangle$ is subjected to a weak measurement and then postselected into a state $|\psi_f\rangle$. The meter of the weak measurement device deflects according to the weak value.

It is notable that the numerator of the weak value is one of the coherence term of the measurement operator and denominator is the inner product between the initial and final state. So basically if we start from a state $|\psi_i\rangle$ and then do a projective measurement with $\hat{A}$ the final state is proportional to $\hat{A}\,|\psi_i\rangle$ and then doing a postselection with state $|\psi_f\rangle$ basically gives us the modulus square of the coherence term in the numerator as the probability amplitude for the post-selection. It is because of the weak measurement that we get the denominator which has tremendous significance in quantum technologies. Noticeably rather than the ordinary average values of the observables, weak values can be complex and it can go way beyond the spectrum of the observable. Thus the weak values carry more information than the average values. Besides, weak values have tremendous technological advantage in many arenas e.g Quantum state tomography, Quantum thermometry, parameter estimation etc. In this section we discuss how the weak values are obtained and how it gives advantage in small parameter estimations.

3.1 Generating weak values via weak measurement

In weak measurement we take a meter(an ancilla) and couple that with the system via a Von-Neumann type interaction. Then we do a projective measurement on the system or in other words postselect the system in a particular state. The meter state undergoes a change to another state state($|\phi'\rangle$). After that we do a projective measurement onto the meter which shows a humongous deflection going beyond the spectrum of the observable depending upon the weak value. The following is a relatively general framework for weak value amplification scheme which is based on this reference[Pang 14].

Suppose we have a system with observable $\hat{A}$ and a meter with observable $\hat{F}$. System and meter is interacting with a Hamiltonian,

$$\hat{H}_{int} = \hbar g \delta(t - t_0) \hat{A} \otimes \hat{F} \tag{3.2}$$

Here it is an impulsive interaction which happens in a faster time scale than the natural dynamics of the system and meter. "g" is the interaction strength. Now suppose the experimenter prepares the system in state $|\psi_i\rangle$ and the meter in $|\phi\rangle$ and finally postselects the system in state $|\psi_f\rangle$. Then the meter evolves with the Kraus operator

$$\hat{M} = \langle\psi_f| e^{-ig\hat{A}\otimes\hat{F}} |\psi_i\rangle \tag{3.3}$$

So the final meter state will be,

$$|\phi'\rangle = \frac{\hat{M}|\phi\rangle}{||\, \hat{M}|\phi\rangle\, ||} = \frac{\hat{M}|\phi\rangle}{\sqrt{\langle\phi|\hat{M}^\dagger\hat{M}|\phi\rangle}} \tag{3.4}$$

Now if we do a measurement on the meter with a meter observable $\hat{R}$ then the expectation value can be approximated to the second order in g(the coupling constant) as following,

$$\langle\hat{R}\rangle_{|\phi'\rangle} \approx \frac{2gIm(\alpha A_w) + g^2\beta \mid A_w \mid^2}{1 + g^2\sigma^2 \mid A_w \mid^2} \tag{3.5}$$

Here $\alpha = \langle \hat{R}\hat{F} \rangle_{|\phi\rangle}$, $\beta = \langle \hat{F}\hat{R}\hat{F} \rangle_{|\phi\rangle}$, $\sigma^2 = \langle \hat{F}^2 \rangle_{|\phi\rangle}$ are the correlation parameters between the Initial meter observable $\hat{F}$ which is used to couple with the system and final meter observable $\hat{R}$ which is used to perform the measurement.

We have taken all the initial pointer variables unbiased, thus initial expectation value of all the pointer variables are zero.

Now usually the amplification experiments are performed in Linear response region so that we can easily extract out the amplification factor. That allows us to neglect the second order terms in Eq 3.5. The condition is that g is very small.

Thus we obtain the following,

$$\langle \hat{R} \rangle_{|\phi'\rangle} \approx 2gIm(\alpha A_w) = 2g[Re(A_w)Im(\alpha) + Im(A_w)Re(\alpha)] \tag{3.6}$$

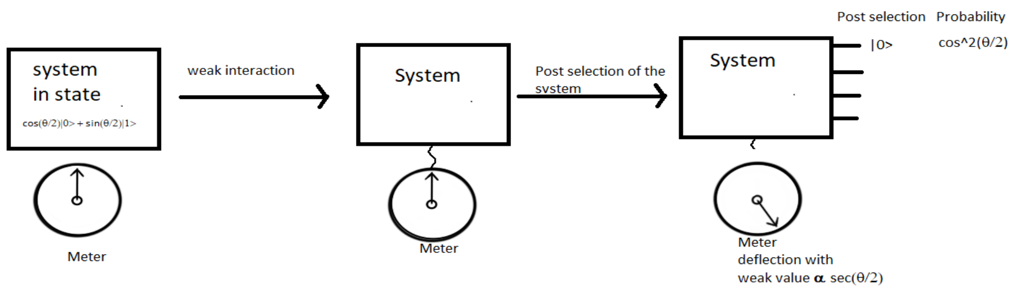

Figure 3.1: Weak Value Amplification schematic diagram

Here we clearly see that the meter result is very much dependent upon the weak value of the system observable and if the weak value is very high the meter result is also very high.

In the schematic 3.1 the weak measurement and subsequent appearance of weak value is shown here for a one qubit system. If we start from $|\psi_i\rangle = cos(\theta/2)|0\rangle + sin(\theta/2)|1\rangle$ and finally post select the state in $|0\rangle$ we get meter deflection proportional to $sec(\theta/2)$ due to the weak value and subsequent post-selection probability is $cos^2(\theta/2)$. So if θ is close to π then we have tremendous amplification in the meter reading due to $sec(\theta/2)$ this factor. But at the same time post-selection probability is also very low.

3.2 Weak Value amplification

The significance of the denominator in weak values is that it gives us tremendous amplification when the initial state and the final state are almost mutually orthogonal. This was the motivation of the very first article in this field by Aharonov, Albert and Vaidman which was titled as "How the result of a measurement of a component of the spin of a spin-1/2 particle can turn out to be 100"[Aharonov 88]. This article triggered a lot of discourse on this matter as it was very much counter intuitive. It was argued that the original PRL paper contains some shortcomings. However a lot of people defended that despite that the point raised by AAV is true. Sudarshan and his coauthors discussed the importance of this result and provided the weakness limit of the coupling parameter where we encounter this effect along with an experimental protocol with Mach-Zehnder interferometer[Duck 89].

Amidst of this dispute The Quantum metrologists have seen a great possibility in Weak values. It's that in parameter estimation protocols using Quantum technologies, the weak values can be used to avail great amount of amplification.

In practice plenty of classical resources are used in order to amplify the measurement outcomes of Quantum systems to our observation limits. For example to observe effects in atomic scale huge amount of magnetic field is used to split the quantum states using Zeeman effect before it is interacted with electromagnetic waves. In weak measurement we obtain the amplifications directly from the system within the framework of Quantum mechanics without any external classical resource.

3.3 The cost of WVA

So far it is clear that as small as $\langle\psi_f \mid \psi_i\rangle$ is, so much amplification we obtain in the weak value. But it comes with a cost. It is that $\langle\psi_f \mid \psi_i\rangle$ this factor also denotes the post selection probability.

$$P_s = \mid \langle\psi_f \mid \psi_i\rangle \mid^2 \tag{3.7}$$

This means higher the weak value is lower is the post selection probability of the system. To interpret this fact in practical language we can put it like: if we wish to have a high weak value deflection in the meter, the meter is less likely to give any reading.

A lot of effort has been made so far to enhance this probability for a fixed weak value. One of such methods use Quantum resource for this enhancement[Pang 14]. We are going to look at this method in detail in the following sections.

3.4 Review of the entanglement assisted enhancement of WVA

Here the system is considered to be comprised of n no of subsystems. This opens up access to a huge Quantum resource encoded into the correlations between the subsystems. Here I mention that WVA already uses one of the Quantum resources present in unfragmented $\hat{A}$ that is the Coherence. It is the grace of Quantum mechanics that allows superposition of Quantum states which let us control the orthogonality between the two states to achieve desired WVA. Allowing the system to have fragments or subsystems let us access entanglement between the subsystems.

Here the system observable is described by following non interacting variable,

$$\hat{A} = \sum_{k} \mathbf{I}^{\otimes k-1} \otimes \hat{a} \otimes \mathbf{I}^{\otimes n-k} \tag{3.8}$$

Where all the subsystem observables of $\hat{A}$ have same set of eigenvalues and eigenvectors but with different support and that is why direct sum sign is used. Now if $\hat{a}$ is the subsystem observable having set of eigenvalues and eigenvectors $\{\lambda, |\lambda\rangle\}$ then we have the following bound on eigenvalues of $\hat{A}$.

$$\Lambda_{max(min)} = n(\lambda_{max(min)}) \tag{3.9}$$

$$|\Lambda_{max(min)}\rangle = |\lambda_{max(min)}\rangle^{\otimes n} \tag{3.10}$$

Now with the help of the subsystems we have a bigger set of aspirants of pre-selection and post-selection states. We have to maximize the post-selection probability as in Eq 3.7

over these aspirants for a given weak value. So it comes down to an optimization problem at last. Here we have a very interesting constraint which makes this problem easier. The constraint is that, $|\psi_f\rangle$ is always orthogonal to $|\nu\rangle$

$$|\nu\rangle = (\hat{A} - A_w)\,|\psi_i\rangle \tag{3.11}$$

This is quite obvious and comes directly from the definition of the weak value. Although it is worth great importance since now we can consider $|\psi_f\rangle$ to be in orthogonal subspace of $|\nu\rangle$ to have maximum P_s.

The optimization is done first over $|\psi_f\rangle$ and then $|\psi_i\rangle$.

The part of $|\psi_i\rangle$ orthogonal to $|\nu\rangle$ can be written as follows,

$$|\psi_i\rangle_{\nu\perp} = |\psi_i\rangle - \frac{(\hat{A} - A_w)\,|\psi_i\rangle\,(\langle\psi_i|\,(\hat{A} - A_w^*)\,|\psi_i\rangle)}{[\langle\psi_i|\,(\hat{A} - A_w^*)(\hat{A} - A_w)\,|\psi_i\rangle]^{\frac{1}{2}}\mid\langle\psi_i|\,(\hat{A} - A_w^*)\,|\psi_i\rangle\mid} \tag{3.12}$$

As discussed earlier we can consider $|\psi_f\rangle$ to be parallel to $|\psi_i\rangle_{\nu\perp} = |\psi_i\rangle$. After some simplification we can derive,

$$|\psi_f\rangle \propto |\psi_i\rangle - \frac{(\hat{A} - A_w)\,|\psi_i\rangle\,(\langle\hat{A}\rangle - A_w^*)}{[\langle\hat{A}^2\rangle - 2\langle\hat{A}\rangle Re(A_w) + \mid A_w\mid^2]^{\frac{1}{2}}\mid\langle\hat{A}\rangle - A_w^*\mid} \tag{3.13}$$

Here $\langle.\rangle = \langle\psi_i|\,.\,|\psi_i\rangle$.
After some algebra we get the following relation,

$$P_s \leq \frac{Var(\hat{A})}{\langle\hat{A}^2\rangle - 2\langle\hat{A}\rangle Re(A_w) + \mid A_w^2\mid^2} \tag{3.14}$$

It is interesting that maximum value for P_s is dependent upon the variance of $\hat{A}$. Change in variance of a joint observable is dependent upon the correlations between the subsystems. For continuous variables the diffusive velocity or rate of change of variance contributes in covariance matrix of two particle momentum[Valdés-Hernández 19]. Here[Giovannetti 11] is shown that $Var(\hat{A})$ is maximum when $|\psi_i\rangle$ is one of the maximally entangled states. That's where the entanglement come into this formalism for the first time. It has been shown that for the state which maximizes this variance can be written as follows,

$$|\psi_i\rangle = \frac{1}{\sqrt{2}}(|\lambda_{max}\rangle^{\otimes n} + e^{i\theta}\,|\lambda_{min}\rangle^{\otimes n}) \tag{3.15}$$

The maximum value of $Var(\hat{A})$ that we obtain from this state is,

$$\underset{|\psi_i\rangle}{max}\; Var(\hat{A}) = \frac{n^2}{4}(\lambda_{max} - \lambda_{min})^2 \tag{3.16}$$

Based on this $|\psi_i\rangle$ we get the maximal $|\psi_f\rangle$ using Eq. 3.13

$$|\psi_f\rangle \propto -(n\lambda_{min} - A_w^*)\,|\lambda_{max}\rangle^{\otimes n} + e^{i\theta}(n\lambda_{max} - A_w^*)\,|\lambda_{min}\rangle^{\otimes n} \tag{3.17}$$

So we get the preselection and postselection states maximizing the P_s. Both are entangled state. It is difficult to prepare and do measurement on maximally entangled states but given we can do that, we can perform this enhancement.

3.5 The quantum Fisher information of weak value amplification protocol

In the weak value amplification scheme as soon as the postselection is performed in the system, the information of the parameter encoded in the system is completely erased which is why the Fisher information in the system equals zero. But the information of the parameter θ remains in the meter due to the weak interaction which is associated with the amplification factor due to the weak value. The meter variable $\hat{R}$ acts in this process as an estimator 3.1. Thus we have to calculate the Fisher information in the meter state. The quantum Fisher information of the final meter state can be approximately written as[Pang 15b],

$$\mathcal{I}_Q^{wk} \approx 4p_s|A_w|^2 Var(\hat{F})_{|\phi\rangle}. \tag{3.18}$$

It is interesting to see that the meter variable which is used to generate the weak interaction must have a large variance in order to have significant amount of Fisher information. This Fisher information never exceeds the standard quantum limit of quantum metrology. However a recently developped postselected metrology scheme can exceed this limit. We shall discuss this in the next to next chapter. In the following chapter we discuss the efficiency and optimal weak value amplification regime and how it compart with other optimization schemes of weak values and corresponding probabilities.

Optimal weak value amplification

Chapter 4

Optimal weak value amplification

Large and anomalous weak values provide significant amplification in parameter estimations with the cost of very low probability of successful post-selection. Which is why it becomes very hard to practically obtain the advantage of this quantum phenomenon. The question which appears directly from this point is that in this cost-gain dichotomy where do we get the maximum advantage with minimal cost. To find the answer to that question, one first needs to define the efficiency of a general weak value amplification protocol. Now efficiency in general is defined as the rate of gain per unit cost. To increase the efficiency one needs to maximize the gain and minimize the cost. In this scenario the gain is quantified as the absolute value of the weak value $|A_w|$. However p_s doesn't quantify the cost properly, since the cost here is the low probability. Here we quantify the cost of WVA as p_s^{-1} which minimizes when the probability is maximized. Thus we define the efficiency of weak value amplification as follows,

Definition 1. *Efficiency of a Weak Value Amplification protocol η with Weak Value A_w and corresponding postselection probability p_s is given by,*

$$\eta(A_w, p_s) = p_s |A_w|^2. \tag{4.1}$$

This efficiency quantifies the comparative speed of Weak Value getting large and the probability getting small. If we break the A_w and p_s in terms of the initial state, postselection state and the operator $\hat{A}$ we get the following result,

$$\eta(A_w, p_s) = |\langle \psi_i | \hat{A} | \psi_f \rangle|^2. \tag{4.2}$$

We can presume that this term can not go outside the spectrum of A thus it is bounded from below and above. In the following theorem we shall see the upper bound and lower bound of this efficiency and under which condition the efficiency reaches the maximum.

Theorem 1 (Bound in efficiency of Weak Value Amplification)**.** *Efficiency of a Weak Value Amplification protocol is bounded in accordance with the following inequality:*

$$0 \leq \eta(A_w, p_s) \leq \langle\psi_i|\hat{A}^2|\psi_i\rangle, \tag{4.3}$$

Thus the maximum efficiency that we can achieve in weak value amplification is $\langle\psi_i|\hat{A}^2|\psi_i\rangle$. This is quantified completely by the operator $\hat{A}$ and the initial state $|psi_i\rangle$. Thus the condition to achieve this optimal situation imposes a constraint in the choice of our final postselection state. We present this condition as a corollary of this theorem. Before that we give a simple proof of this theorem.

Proof.
- First inequality is very evident. It only states that efficiency is a positive real number.$p_s \geq 0$ and $|A_w| \geq 0 \implies \eta(A_w, p_s) \geq 0$.
- The second inequality is a crucial result of this thesis. To prove this first we write the efficiency in the following form,

$$\eta(A_w, p_s) = \langle\Psi_f|\hat{A}|\Psi_i\rangle\langle\Psi_i|\hat{A}|\Psi_f\rangle. \tag{4.4}$$

Let $\{|\psi_k\rangle\}$ forms a complete ortho-normal basis where $|\psi_i\rangle \in \{|\psi_k\rangle\}$. We can write,

$$\begin{aligned} \eta(A_w, p_s) &\leq \sum_k \langle\psi_i|\hat{A}|\psi_k\rangle\langle\psi_k|\hat{A}|\psi_i\rangle \\ &\leq \langle\psi_i|\hat{A}\Big(\sum_k |\psi_k\rangle\langle\psi_k|\Big)\hat{A}|\psi_i\rangle \\ &\leq \langle\psi_i|\hat{A}^2|\psi_i\rangle. \end{aligned} \tag{4.5}$$

□

We have proved that the efficiency of the weak value amplification is bounded. Now we shall derive some important corollaries from this theorem.

Corollary 1. *The efficiency of a Weak Value amplification protocol has a universal bound which is independent of the initial and final state of the protocol in terms of the operator norm of the observable* $\hat{A}$.

$$\eta(A_w, p_s) \leq ||\hat{A}^2||_{op}. \tag{4.6}$$

Operator norm of an operator is defined as $||\hat{T}||_{op} := \sup_{|\phi\rangle \in \mathcal{H}}\{\langle\phi|\hat{T}|\phi\rangle : \langle\phi|\phi\rangle = 1\}$. This upper bound is independent of the initial state and the final state of the system, thus it sets a stronger bound for the efficiency.

Corollary 2. *The upper bound of the efficiency of Weak Value Amplification is achieved when the final state* $|\psi_f\rangle$ *is parallel to* $\hat{A}|\psi_i\rangle$.

This corollary states the condition to achieve the optimal weak value amplification. The condition imposes a constraint in the choice of the final state. Under this condition we can write the expression for the optimal weak value and the optimal probability of successful post selection.

When $|\psi_f\rangle$ is parallel to $\hat{A}|\psi_i\rangle$ we can write the final state as follows,

$$|\psi_f\rangle = \frac{\hat{A}|\psi_i\rangle}{\sqrt{\langle\psi_i|\hat{A}^2|\psi_i\rangle}}. \tag{4.7}$$

Putting this $|\psi_f\rangle$ in the expression of the weak value and the post selection probability, we get the following expressions,

$$A_w = \frac{\langle\psi_i|\hat{A}^2|\psi_i\rangle}{\langle\psi_i|\hat{A}|\psi_i\rangle}. \tag{4.8}$$

$$p_s = \frac{\langle\psi_i|\hat{A}|\psi_i\rangle^2}{\langle\psi_i|\hat{A}^2|\psi_i\rangle}. \tag{4.9}$$

Evidently in optimal limit the weak value, the postselection probability and the efficiency, all are independent of the final state. The anomalous weak value amplification condition is determined by the average value of $\hat{A}$ in the initial state. When this average is very small, we get anomalously large weak values and simultaneously the postselection probability becomes very small. The weak value is always real in this regime. In the next section we take an example where we explore how these conditions perform in physical systems.

4.1 Optimal weak value amplification with an entangled initial state

This example starts with the system constructed in the way same as the section 2.5, where we have n combined systems with the global variable,

$$\hat{A} = \sum_k \mathbf{I}^{\otimes k-1} \otimes \hat{a} \otimes \mathbf{I}^{\otimes n-k} \tag{4.10}$$

Where $\hat{a}$ has eigen spectrum$\{\lambda_i, |\lambda_i\rangle\}$. We prepare the system in the following pre-selected state,

$$|\psi_i\rangle = \frac{|\lambda_x\rangle^{\otimes n} + |\lambda_y\rangle^{\otimes n}}{\sqrt{2}} \tag{4.11}$$

Where λ_x and λ_y are two different eigenvalues of $\hat{a}$. Note that each choice of λ_x and λ_y specifies the choice of our initial state.

Now in optimal weak value amplification regime, we get the following expressions for the optimal weak value and the optimal probability of post selection using Eq 4.8 and 4.9,

$$p_s = \frac{(\lambda_x + \lambda_y)^2}{2(\lambda_x^2 + \lambda_y^2)}; \tag{4.12}$$

$$A_w = \frac{n(\lambda_x^2 + \lambda_y^2)}{\lambda_x + \lambda_y}. \tag{4.13}$$

Plotting these two functions gives us insight about the choice of the eigenvalues of $\hat{a}$ and the choice of the initial state. Now we examine two cases,

- Optimizing the probability for fixed weak value.
- Optimizing the Weak value for fixed probability.

[1] The significance of these two cases are, often we are given with a required amount of amplification to detect the changes in the small parameter within the error range of the measuring apparatus and we want to find the best situation to avail that amount of

[1]This graph is plotted in GeoGebra

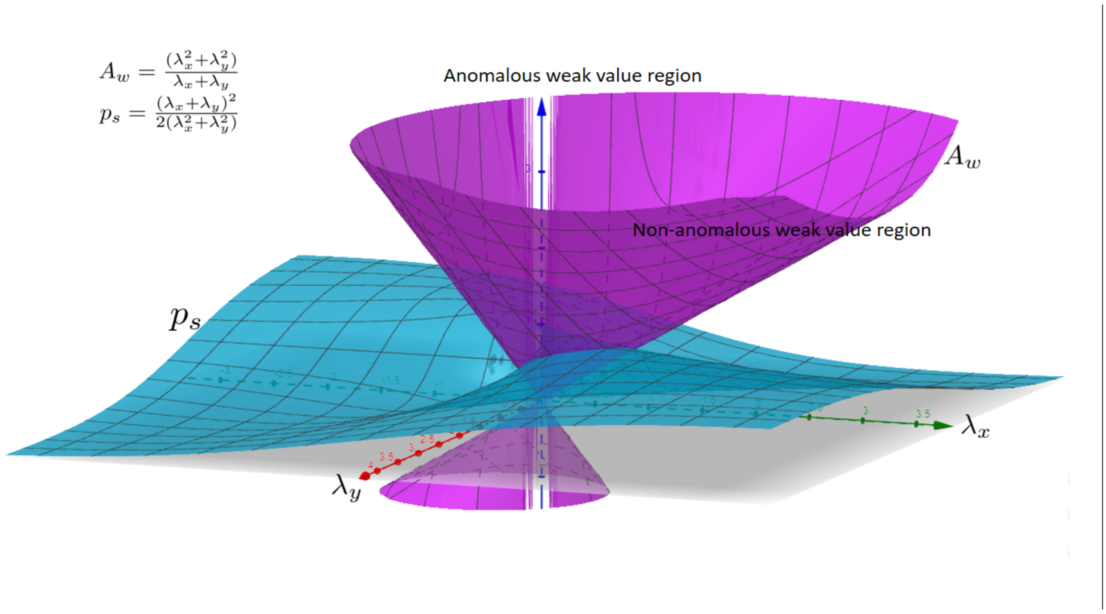

Figure 4.1: Weak value and postselection probability for different initial states characterized by λ_x and λ_y

amplification with minimal cost. We can also encounter the reverse situation where the cost we can bear is given and we are asked to find the best situation where we get the best amplification within the cost range.

In our optimization we get a constraint in the choice of the two eigen values λ_x and λ_y in both the situations. This fixes the initial state of the protocol.

4.2 Optimizing probability with fixed weak value

For fixed weak value we get the following constraint in the choice of λ_x and λ_y from Eq 4.13.

$$\lambda_y = \frac{A_w \pm \sqrt{A_w^2 - 4n^2\lambda_x^2 + 4nA_w\lambda_x}}{2n}. \tag{4.14}$$

For large weak values we can approximate this relation as follows,

$$\lambda_y \approx \frac{A_w - n\lambda_x}{2n}, -\lambda_x. \tag{4.15}$$

Point to mention here is that since in the optimized regime the weak value is always real, we don't get any complex valued choices for λ_y which has to be real by definition.

Now, the first choice for λ_y corresponds to non-anomalous weak value amplification where the eigenvalues of the operator A is of the order of the weak value. In figure 4.1 the outer region specifies the weak values where λ_x and λ_y are large, this region corresponds to the non-anomalous weak value region where the amplification arise due to higher eigenvalues of the observable. The second choice of the constraint corresponds to the anomalous weak value amplification where the average of A in the initial state becomes close to zero and thus the weak value becomes anomalously large. In the figure 4.1 we have a singularity in the region corresponding to $\lambda_x = -\lambda_y$ where the probability is also very low. This region corresponds to the anomalous weak value region where the amplification arise due to the denominator in weak value. This choice is of our interest since it gives us quantum advantages in weak value amplification. Thus focusing on the second choice of the constraint we can derive the following expression for the postselection probability,

$$p_s = \frac{n^2 \lambda_x^2}{A_w^2}. \tag{4.16}$$

This probability scales with n quadratically. The bet enhancement of the postselection probability using classical resources yields linear scaling with n. Thus using quantum resources it gives significant enhancement on this purpose.

4.3 Optimizing the Weak Value with fixed probability of successful postslection

In this case we involve Eq 4.12 to get the appropriate constraint on the choice of λ_x and λ_y. Doing this we get the following constraint,

$$\lambda_y = \frac{\lambda_x \pm \sqrt{\lambda_x^2 - (2p_s - 1)^2 \lambda_x^2}}{2p_s - 1}. \tag{4.17}$$

Now we take low p_s approximation. Interesting point here is that, this approximation leads us to the case of anomalous weak value amplification. We get the same condition as Eq 4.15.

$$\lambda_y \approx -\lambda_x. \tag{4.18}$$

This yields the optimum weak value for a given amount of postselection probability,

$$|A_w| = \frac{n\lambda_x}{\sqrt{p_s}}. \tag{4.19}$$

This scales with n linearly. Both of these cases are examined by Shengsi Pang et al obtaining the similar scaling with n. Each of the cases were taken as two different optimization tasks. However in this work we treat it as two special cases of a single optimization task which gives the same result for anomalously large weak value limit where the postselection probability becomes small and weak value becomes very large.

4.4 Discussion

In the optimal regime which we explored in this project obtains only real weak values. A simple question arises from this point do we loose information because of this restriction? The whole manifold of possible weak values is larger since it contains imaginary values as well, thus it has a higher content of information. A simple answer to this question is, as far as the Weak value amplification is concerned, the information of the coupling parameter is not encoded in the weak value. Rather it is encoded in the system state and the meter state after the interaction (Eq 3.2) is performed, and when we postselect the system, although the information of the parameter in the system is lost, the information in the meter is not. The weak value brings an amplification factor which enhances the sensitivity of the meter under small changes in the parameter value. Which is why, whether the weak value is real or complex is not a concern of weak value amplification, what matters is the magnitude of the weak value. Thus performing weak value amplification in the optimal regime don't loose information of the parameter whatsoever.

Another point to debate is the advantage of our optimization over the approach of entanglement assisted weak value amplification. We defend this advantage with the following arguments. Firstly, the approach of entanglement assisted weak value amplification treats the two optimization scheme specified in section 4.1 differently, we have shown the two optimization as a special case of a same optimization. Secondly, the entanglement assisted weak value amplification approach concerns the very large weak value and very small postselection probability limit, which basically talks about the anomalous weak value amplification case. However, in our case we can make our choice between these two according to our requirement. Thirdly, we do our optimization in a more general scenario by defining a measure of efficiency of Weak value amplification p rotocol. This provides us with a wide variety of situations where we can achieve this optimality.

Bounding quantum advantages in postselected metrology

Chapter 5

Bounding quantum advantages in postselected metrology

The scheme for postselected metrology is a little different from the scheme of weak value metrology. In this case, we prepare the system in a preselected state, then evolve it via a unitary interaction and up to this point the standard protocol of an estimation task is followed. After that we postselect the evolved system in a set of projectors and then the final post processing is done on to the collapsed state. Here, by "final post-processing" I mean operating the unbiased estimator to extract the value of the parameter. The difference of this scheme from the Weak Value metrology is that, there we weakly interact the system with a meter and after postselecting the system we discard the system and we do the final post-processing in the meter state where we obtain the amplification due to the weak value. In this case we don't discard the postselected system, instead we do the post-processing on to that postselected state.

In this chapter we first review the idea of "Quantum advantages in postselected metrology"[Arvidsson-Shukur 20] and then we shall discuss how the Fisher information in this scheme is related with the weak values of the evolution generator $\hat{A}$. We shall also prove that even if the Fisher Information in this case can go way beyond the Heisenberg limit using states with nonclassical features[Arvidsson-Shukur 20], this phenomenon always comes with a cost of low postselection probability as in WVA. We shall extend our discussion further in order to find the optimal scenario for this purpose as we did in the previous chapter and present an example where we can optimally enhance the Fisher information via nonclassical states using a three level quantum system.

5.1 Review of Postselected quantum metrology

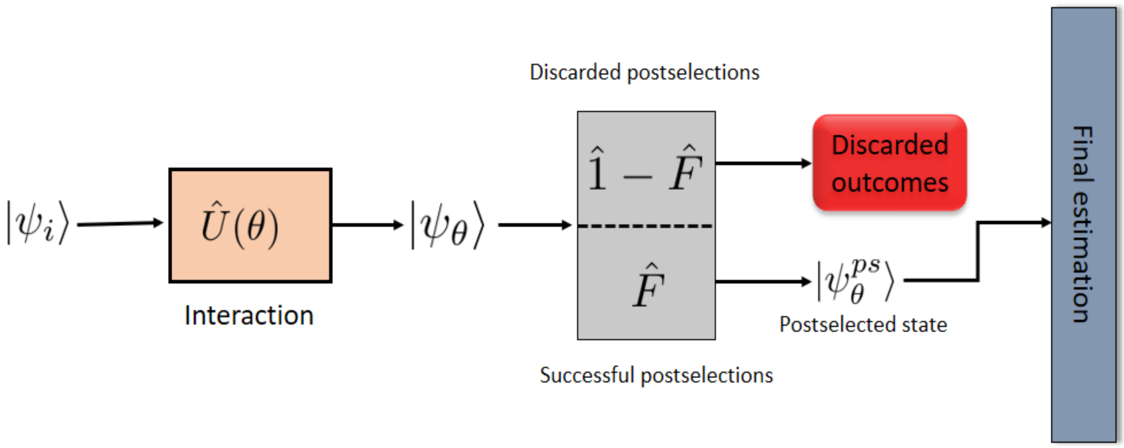

Figure 5.1: Schematic of Postselected metrology

In this section we first give the detailed setting of postselected metrology. We first prepare the system in a pure state $|\psi_i\rangle$. Then we evolve the state via a unitary interaction $exp(-i\theta\hat{A})$ generated by the operator $\hat{A}$. The evolved state becomes $|\psi_\theta\rangle = exp(-i\theta\hat{A})\,|\psi_i\rangle$. Now we postselect the system with the following projector,

$$\hat{F} = \sum_{|f\rangle\in\mathcal{F}_{ps}} |f\rangle\langle f|\,. \tag{5.1}$$

Here $\mathcal{F}_{ps}$ is a set of orthonormal basis vectors specifying the states allowed by the postselection. This gives us the unnormalized postselected state as follows,

$$|\psi_\theta^{ps}\rangle = \hat{F}\,|\psi_\theta\rangle\,. \tag{5.2}$$

The probability of successful postselection in this case is given by,

$$p_\theta^{ps} = Tr(\hat{F}\,|\psi_\theta\rangle\langle\psi_\theta|). \tag{5.3}$$

When we normalize this state we get the following postselected state,

$$|\Psi_\theta^{ps}\rangle = \frac{|\psi_\theta^{ps}\rangle}{\sqrt{p_\theta^{ps}}}. \tag{5.4}$$

On this postselected state we perform the final estimation. Now, one might think on this point that after the system is postselected it is collapsed into some state; how a collapsed state can carry information about the parameter. This is a valid point which gives good insight into this scheme. It so happens that the collapsed state does not carry any information when the system is post-selected in a rank 1 projector. However in other cases the state does not discard all the information of θ and gives a non zero quantum Fisher information. Which is why we have taken a projector for postselection with arbitrary rank apart from the scheme of weak value measurement scheme. This state gives us the following Quantum Fisher Information which can be calculated directly using Eq 2.11.

$$\mathcal{I}_Q(\theta/\,|\Psi_\theta^{ps}\rangle) = 4\Big[\langle\dot{\psi_\theta^{ps}}|\dot{\psi_\theta^{ps}}\rangle\frac{1}{p_\theta^{ps}} - |\langle\dot{\psi_\theta^{ps}}|\psi_\theta^{ps}\rangle|^2\frac{1}{(p_\theta^{ps})^2}\Big]. \tag{5.5}$$

The formula can be generalized to get the expression of postselected quantum Fisher Information for a mixed state $\hat{\rho}_0$ as the initial state,

$$\mathcal{I}_Q(\theta/\rho_\theta^{ps}) = 4\Big[\frac{Tr(\hat{F}\hat{A}\hat{U}(\theta)\hat{\rho}_0\hat{U}(\theta)^\dagger\hat{A})}{p_\theta^{ps}} - \frac{|Tr(\hat{F}\hat{U}(\theta)\hat{\rho}_0\hat{U}(\theta)^\dagger\hat{A})|^2}{(p_\theta^{ps})^2}\Big]. \tag{5.6}$$

5.1.1 Phase-space quasiprobability distribution

The advantage of the postselected metrology stems from the idea of complex quasiprobability distribution as the phase space representation of quantum states. In classical mechanics, the state of a point particle is described by it's position $\vec{x}$ and momentum $\vec{p}$ at some point of time, thus it is represented by a point in phase space which is spanned by $\vec{x}$ and $\vec{p}$. The state of an ensemble of classical point particles are described by a phase space density function which is a joint probability distribution of $\vec{x}$ and $\vec{p}$. In quantum mechanics the position and momentum becomes non-commuting operators and it was shown by Von-Neumann that we can't express the quantum states unambiguously with joint probability distributions of a set of observables. However we can represent the states by a quasiprobability distribution of a set of noncommuting observables. The quasiprobability distributions are a form of probability distribution functions which does not follow the first two Kolmogorov axioms; in other words they can assume negative and complex values, but they add up to 1 which means the total probability of the distribution is one. Quasi-probability distribution functions have been widely used in quantum optics. However the true statistical meaning of these distributions are still an apple of discord. The quasi-probability distributions are considered as signature for nonclassicality. Whenever we have such states with quasiprobability distributions, we consider the state posses non-classical feature which means that the complete statistical description of the state is not possible within the domain of classical mechanics. These non-trivialities actually comes from the noncommutativity of proposals in quantum mechanics, where definiteness in one proposal can hamper the definiteness of another proposal which is practically impossible to imagine in our classical understanding.

The quasi-probability distribution which we shall use in this case was discovered by J.G Kirkwood and is called as the Kirkwood distribution. It is a complex quasi-probability distribution. It contains the same amount of information as the Wigner quasi-probability distribution, a very popular, well behaved, quasiprobability distribution which contains the full information of the corresponding quantum state. The Kirkwood distribution is obtained by a similarity transformation of the phase space variables from the Wigner quasi-probability distribution. To define the Kirkwood distribution of a state we first define two set of orthonormal basis states $\{|a_m\rangle\}$ and $\{|b_n\rangle\}$.Now the Kirkwood distribution of the state $\hat{\rho}$ at the point $\{a_m, b_n\}$ can be expressed as follows,

$$K_\rho(a_m, b_n) = Tr(\hat{\rho} \, |a_m\rangle \langle a_m| \, |b_n\rangle \langle b_n|). \tag{5.7}$$

After decomposition we get,

$$K_\rho(a_m, b_n) = \langle a_m| \, \hat{\rho} \, |b_n\rangle \, \langle b_n|.a_m\rangle \tag{5.8}$$

This can also be expressed as follows,

$$K_\rho(a_m, b_n) = \sum_k \langle a_m| \, \hat{\rho} \, |a_k\rangle \, \langle a_k|b_n\rangle\langle b_n|a_m\rangle. \tag{5.9}$$

The distribution inside the summation sign is termed as the doubly extended Kirkwood distribution[Arvidsson-Shukur 20].

$$K^d_\rho(a_m, a_k, b_n) = \langle a_m| \, \hat{\rho} \, |a_k\rangle \, \langle a_k|b_n\rangle\langle b_n|a_m\rangle. \tag{5.10}$$

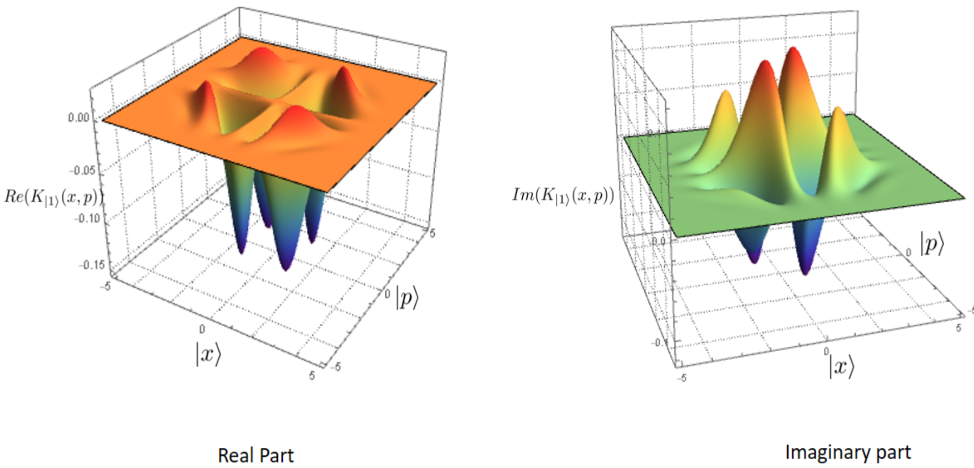

Figure 5.2: Real and Imaginary part of Kirkwood distribution in $|x\rangle$ and $|p\rangle$ basis for n=1 Fock state which is a nonclassical state in position-momentum basis

The state can be expressed in terms of the doubly extended Kirkwood distribution as follows,

$$\hat{\rho} = \sum_{m,k,n} K^{d}_{\rho}(a_m, a_k, b_n) \frac{|a_m\rangle \langle b_n|}{\langle b_n|a_m\rangle}. \tag{5.11}$$

Now we go back to our discussion on the postselected Fisher information. In our scheme we have two set of orthonormal basis, one is the eigen basis of $\hat{A}$ $\{|a\rangle\}$ and the other one is the orthonormal basis set $\{|f\rangle\}$ where we do the postselection. Which means $\mathcal{F}_{ps} \subset \{|f\rangle\}$. We express the state $\hat{\rho}$ in this doubly extended Kirkwood distribution as follows,

$$\hat{\rho} = \sum_{a,a',f} K^{d}_{\rho}(a, a', f) \frac{|a\rangle \langle f|}{\langle f|a\rangle}. \tag{5.12}$$

Here $\hat{\rho}$ is the updated system state after evolution. At the points where $\hat{A}$ and $\hat{F}$ do not commute, the doubly kirkwood distribution gives complex values, otherwise it gives values within 0 and 1.

Now when we put this form of $\hat{\rho}$ in Eq 5.6, we get the form of the Fisher information in terms of the Kirkwood distribution,

$$\mathcal{I}_Q(\theta/\rho_\theta^{ps}) = 4\Big[\sum_{\substack{a,a',\\ f\in\mathcal{F}^{ps}}} K_\rho^d(a,a',f)\frac{aa'}{p_\theta^{ps}} - \Big|\sum_{\substack{a,a',\\ f\in\mathcal{F}^{ps}}} K_\rho^d(a,a',f)\frac{a}{p_\theta^{ps}}\Big|^2\Big]. \tag{5.13}$$

When we have a classically commuting theory which means $\hat{A}$ and $\hat{F}$ commutes at every point in the phase space, the postselected Fisher information does not exceed the standard quantum limit, set by the Heisenberg limit as discussed in section 2.5. However, when we access the quantum states where $\hat{A}$ and $\hat{F}$ don't commute, we have complex nonclassical values of the doubly extended Kirkwood distribution, the postselected Fisher information may exceed the Heisenberg limit. The statement can be proved via examples [Arvidsson-Shukur 20]. Here the negativity of the nonclassical kirkwood distribution is used as a resource to enhance the precision of the estimation. Only the selected initial states which does not commute with the projectors for postselection and the operator $\hat{A}$ at every point in phase space, shows this effect. Another point to mention is that the selected postselections where this enhancement is obtained have very low probability for successful postselection, this behaviour coincides with the behaviour of the Weak value amplification. This happens because the bulk of the information is preserved in some of the postselected states with very low postselection probability. Which means that the postselection probability has to become very small in order to enhance the precision of the estimation significantly. The part of the information in the discarded states of post-selection is lost. This situation can be overcome by introducing some degeneracy into the generator of evolution $\hat{A}$. However this brings some non-trivialities in the estimation process since preparing a degenerate operator is a difficult task because in a noisy environment the degeneracy of the operator breaks and the estimation process becomes clumsy. Our treatment of this protocol of postselected metrology yields an example where we can perform the information preserving parameter estimation using a three level quantum system with no degeneracy. In the next section we shall look into these matters in detail.

5.2 Significance of weak values in postselected metrology

Although the schemes of Weak value amplification and Postselected metrology are very much different from each other there are ontological relations between them. We will show that the postselected quantum Fisher information is related with different weak values of the generator observable $\hat{A}$. Which indirectly shows that their existence is intertwined. The weak values and the postselected quantum metrology, despite being structurally different from one another, they have one thing in common, both of them breaks our standard understanding of quantum mechanics and by extension, the whole physical world itself. One shows that the expectation values of the observables can go way beyond the spectrum of the observable, the other one shows the precision can go beyond the standard Heisenberg limit set by the Heisenberg uncertainty relation. Thus understanding the meaning of both of these techniques means a great deal in formulating the philosophical and operational foundation of quantum physics.

We start with a pure state as the initial state, $\hat{\rho}_0 = |\psi_i\rangle\langle\psi_i|$. Let us first assume that the postselection is done in a rank two projector $\hat{F} = |f_1\rangle\langle f_1| + |f_2\rangle\langle f_2|$. The updated system state after the evolution is $\hat{\rho}_\theta = \hat{U}(\theta)\hat{\rho}_0\hat{U}(\theta)^\dagger$. $\hat{U}(\theta) = exp(-i\theta\hat{A})$. After the postselection is performed on $\hat{\rho}_\theta$ the probability for successful postselection can be calculated using Eq 5.3,

$$p_\theta^{ps} = |\langle\psi_\theta|f_1\rangle|^2 + |\langle\psi_\theta|f_2\rangle|^2. \tag{5.14}$$

This is basically the total of the two probabilities corresponding the two ends of the postselection. Let us denote the probability of successful postselection corresponding to ith measurement projector with $p_{\theta_k}^{ps} := |\langle\psi_\theta|f_k\rangle|^2$. Now we put these quantities in Eq 5.6 and after some algebra we derive the following relation. See the appendix 9.2 for a detailed derivation.

$$\mathcal{I}_Q(\theta/\rho_\theta^{ps}) = 4\left(\frac{p_{\theta_1}^{ps}p_{\theta_2}^{ps}}{(p_\theta^{ps})^2}\right)\left|A_{\theta_1}^w - A_{\theta_2}^w\right|^2. \tag{5.15}$$

$$A_{\theta_k}^w = \frac{\langle\psi_\theta|A|f_k\rangle}{\langle\psi_\theta|f_k\rangle}$$

$A_{\theta_k}^w$ is the weak value of the observable $\hat{A}$ between the preselected state $|\psi_\theta\rangle$ and the postselected state $|f_k\rangle$. In Eq 5.15, the probability factors inside the big bracket, the numerator and the denominator are in quadratic order of the probabilities for successful postselection. Since they are of the same order, they does not actually contribute much to determine the magnitude of Fisher information in this setting. Thus the determining factors for the Fisher information in this case are the weak values. Thus, in order to obtain significant amount of Fisher information out of this protocol we need to look into the cases where we get anomalous weak values.

It is interesting to notice that despite being so different in structure, without introducing any meter and weak interaction in the preparation, we get significant influence of weak values in postselected metrology. The theoretical reason why we get no enhancement from postselection metrology when we postselect the system in a rank one projector, can be drawn from this result. Postselecting in a rank one projector means we put the condition $|f_1\rangle = |f_2\rangle$. This implies that $A^w_{\theta_1} = A^w_{\theta_2}$. When we put this condition in Eq 5.15, we end up getting zero quantum Fisher information. This coincides with our expectations from physical understanding as mentioned earlier in section 5.1. When we postselect the system in a rank one projector, the state of the system collapses and the information of θ in the system is lost.

When we do the postselection in an arbitrary rank projector, the expression for the postselection becomes,

$$\mathcal{I}_Q(\theta/\rho_\theta^{ps}) = \frac{4}{(p_\theta^{ps})^2} \sum_{m<n} p^{ps}_{\theta_m} p^{ps}_{\theta_n} \left| A^w_{\theta_m} - A^w_{\theta_n} \right|^2 \tag{5.16}$$

The derivation is given in appendix 9.3.

5.3 Optimal quantum enhancement in postselected metrology

As we optimized the weak value amplification protocol, we can also find the optimization conditions for postselected metrology. We have already established the relation between the postselected metrology and the weak values. This helps us to link the results we have derived in the chapter of optimal weak value amplification with this scheme. Before that we have to quantify the efficiency of postselected metrology. So far it is clear that the cost incurred against the enhancement of precision in postselected metrology is same as the weak value amplification, the low postselection probability. Which is why we define the efficiency in this case, in the same manner.

Definition 2. *Efficiency of a postselected metrology protocol ξ with postselected Fisher information $\mathcal{I}_Q$ and total probability of successful postselection p_θ^{ps} is defined as follows,*

$$\xi(\mathcal{I}_Q; p_\theta^{ps}) = p_\theta^{ps}\mathcal{I}_Q \tag{5.17}$$

At this point a question arises that why in this definition the postselection probability is taken in linear order. In reality, we quantify the efficiency in order to find a bound for the advantages these protocols provide. Efficiency is a quantity which quantifies the amount of gain and advantage per unit cost. Furthermore, it is always bounded for any protocol, be it in thermodynamics or in advantages of quantum resources. All these bounds provide profound understanding of our physical world, for example the bound in efficiency of heat engines establish the second law of thermodynamics. Here we also get a fundamental bound on the efficiency of a postselected metrology protocol which is presented in the following theorem.

Theorem 2 (Bound in efficiency of postselected metrology)**.** *The efficiency of a protocol for postselected metrology is bounded according to the following inequality.*

$$0 \leq \xi(\mathcal{I}_Q; p_\theta^{ps}) \leq 4||\hat{A}^2||_{op}. \tag{5.18}$$

Proof. We start with the result is Eq 5.15

$$\mathcal{I}_Q(p_\theta^{ps})^2 = 4p_{\theta_1}^{ps}p_{\theta_2}^{ps}\left|A_{\theta_1}^w - A_{\theta_2}^w\right|^2. \tag{5.19}$$

Now after decomposing the terms inside modulus square we get the following result,

$$\left|A_{\theta_1}^w - A_{\theta_2}^w\right|^2 \leq \left(\left|A_{\theta_1}^w\right|^2 + \left|A_{\theta_2}^w\right|^2\right). \tag{5.20}$$

1. $\Big[$In this case the $=$ is achieved under the condition: $Re(A_{\theta_1}^w(A_{\theta_2}^w)^*) = 0\Big]$.

Now we apply the corollary 1 of the theorem of bounded efficiency in weak value amplification (theorem 1) for our two weak values and we get the following result,

$$p_{\theta_1}^{ps}\left|A_{\theta_1}^w\right|^2 \leq ||\hat{A}^2||_{op} \quad , \quad p_{\theta_2}^{ps}\left|A_{\theta_2}^w\right|^2 \leq ||\hat{A}^2||_{op}. \tag{5.21}$$

Putting these two results in Eq 5.19 we obtain the following result,

$$\begin{aligned}\mathcal{I}_Q(p_\theta^{ps})^2 &\leq 4\big(p_{\theta_2}^{ps}||\hat{A}^2||_{op} + p_{\theta_1}^{ps}||\hat{A}^2||_{op}\big)\\ &= 4||\hat{A}^2||_{op}(p_{\theta_2}^{ps} + p_{\theta_1}^{ps}).\end{aligned} \tag{5.22}$$

2. $\Big[$The equality condition holds when $\langle f_1|\hat{A}^2|f_1\rangle = \langle f_2|\hat{A}^2|f_2\rangle = ||\hat{A}^2||_{op}$ and one of the postselection states are parallel to $\hat{A}\,|\psi_\theta\rangle\Big]$

Now $p_\theta^{ps} = p_{\theta_2}^{ps} + p_{\theta_1}^{ps}$. So we get,

$$\mathcal{I}_Q\; p_\theta^{ps} \leq 4||\hat{A}^2||_{op} \quad \Longrightarrow \quad \xi(\mathcal{I}_Q; p_\theta^{ps}) \leq 4||\hat{A}||_{op}^2 \tag{5.23}$$

□

We can see, that the optimal bound of this efficiency only depends upon the observable $\hat{A}$. It is interesting to notice that this is the optimal standard quantum limit for the quantum Fisher information (Eq 2.13) which is achieved when the average of the observable becomes zero and the state of the system at that instance obtains the operator norm of the operator $\hat{A}^2$ when it is averaged over the state. At that point the variance of $\hat{A}$ equals the operator norm of $\hat{A}^2$. What we can infer from this observation is that although the standard quantum limit does not work for QFI in postselected metrology, it works at the level of efficiency of the protocol. Thus we see that this measure of efficiency plays a central role in postselected metrology.

Now we point out the conditions to achieve this optimal bound,

1. We have to take one of the weak values zero, say $A_{\theta_2}^w$. This implies that $|f_2\rangle$ becomes orthogonal to $\hat{A}\,|\psi_\theta\rangle$. This condition ensures the first equality condition in the proof of previous theorem is satisfied.

2. $|f_1\rangle$ obtains the operator norm of $\hat{A}^2$ and $|\psi_\theta\rangle$ becomes parallel to $\hat{A}\,|f_1\rangle$ asymptotically. This choice ensures the condition in second equality condition of the proof.

3. The probability of postselection of the other weak value, here $p_{\theta_1}^{ps}$ tends to zero asymptotically. This choice obtains the maximal efficiency.

These conditions are the direct consequence of the proof of Theorem 2. In the next part we give an example where we satisfy all these conditions and show how we achieve optimal quantum enhancement in this postselected metrology.

5.3.1 Information preserving postselected metrology using a three level quantum system

The state space of our system is spanned by the basis set $\{|\lambda\rangle, |\tilde{\lambda}\rangle, |-\lambda\rangle\}$. They are the eigen-kets of $\hat{A}$ with respective eigenvalues $\{\lambda, \tilde{\lambda}, -\lambda\}$. We do the parameter estimation of θ in a close proximity region around $\theta = \theta_0$ where $\theta - \theta_0 = \delta_\theta$ and $|\delta\theta| << 1$.

Now we prepare our system in the following preselected state,

$$|\psi_i\rangle = \frac{1}{\sqrt{2}}\hat{U}^\dagger(\theta_0)[(cos(\phi) + sin(\phi))|\lambda\rangle - (cos(\phi) - sin(\phi))|-\lambda\rangle], \tag{5.24}$$

where ϕ is a real parameter and $\hat{U}(\theta) = exp(-i\theta\hat{A})$ is the evolution operator.

For postselection we choose th projectors generated from the following two orthonormal states,

$$|f_1\rangle = \frac{|\lambda\rangle + |-\lambda\rangle}{\sqrt{2}}, \tag{5.25}$$

$$|f_2\rangle = \frac{cos(\psi)[|\lambda\rangle - |-\lambda\rangle]}{\sqrt{2}} + sin(\psi)|\tilde{\lambda}\rangle. \tag{5.26}$$

Here ψ is a real parameter which controls the alignment of the postselection.

Thus in this setting the evolved state will be,

$$|\psi_\theta\rangle = \hat{U}(\theta)|\psi_i\rangle = \frac{1}{\sqrt{2}}[(cos(\phi) + sin(\phi))e^{-i\delta_\theta\lambda}|\lambda\rangle - (cos(\phi) - sin(\phi))e^{i\delta_\theta\lambda}|-\lambda\rangle]. \tag{5.27}$$

Calculating the postselected Fisher information using Eq 5.6 we get,

$$\mathcal{I}_Q(\theta/|\psi_\theta^{ps}\rangle) = \frac{4\lambda^2cos^2(\psi)\{cos^2(\phi+\lambda\delta_\theta) - sin^2(\phi-\lambda\delta_\theta)\}^2}{sin^2(\phi-\lambda\delta_\theta) + cos^2(\psi)cos^2(\phi+\lambda\delta_\theta)}. \tag{5.28}$$

The total postselection probability comes out for this estimation as follows (using Eq 5.14),

$$p_\theta^{ps} = sin^2(\phi-\lambda\delta_\theta) + cos^2(\psi)cos^2(\phi+\lambda\delta_\theta). \tag{5.29}$$

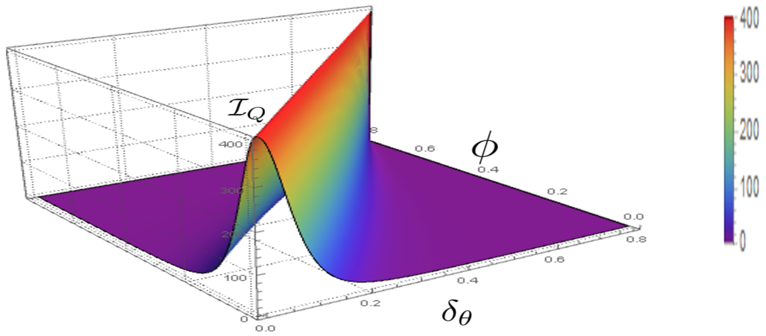

Figure 5.3: The parameter dependence of postselected quantum Fisher information ($\psi = \frac{\pi}{2} - 0.1$) and $\lambda = 1$

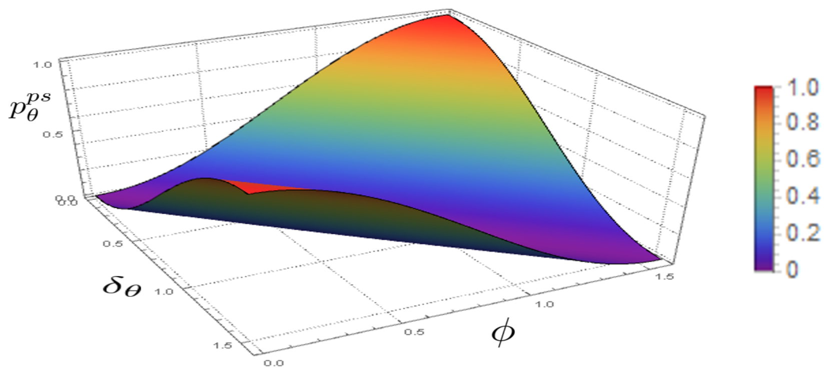

Figure 5.4: The parameter dependence of total postselection probability ($\psi = \frac{\pi}{2} - 0.1$) and $\lambda = 1$

The weak values and their corresponding probabilities are,

$$A^w_{\theta_1} = \frac{\lambda cos(\phi + \lambda\delta_\theta)}{sin(\phi - \lambda\delta_\theta)}, \quad p^{ps}_{\theta_1} = sin^2(\phi - \lambda\delta_\theta) \tag{5.30}$$

and

$$A^w_{\theta_2} = \frac{\lambda sin(\phi - \lambda\delta_\theta)}{cos(\phi + \lambda\delta_\theta)}, \quad p^{ps}_{\theta_1} = cos^2(\psi)cos^2(\phi - \lambda\delta_\theta) \tag{5.31}$$

When we take the limit $\theta \to 0$ and $\phi \to 0$ the conditions for optimal quantum advantage conditions (as described in 5.3) are satisfied. In this limit the quantities take the following form.

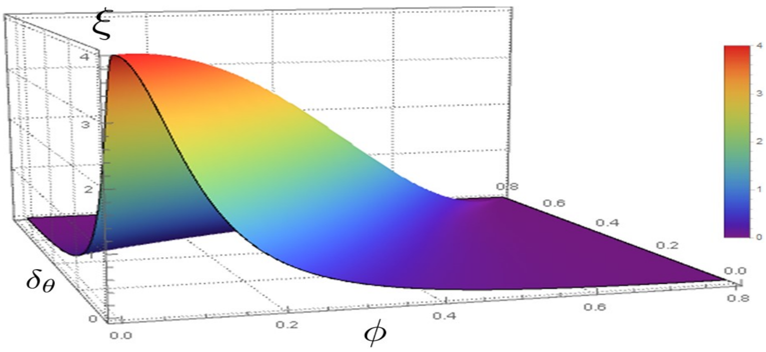

Figure 5.5: The parameter dependence of efficiency ($\psi = \frac{\pi}{2} - 0.1$) and $\lambda = 1$

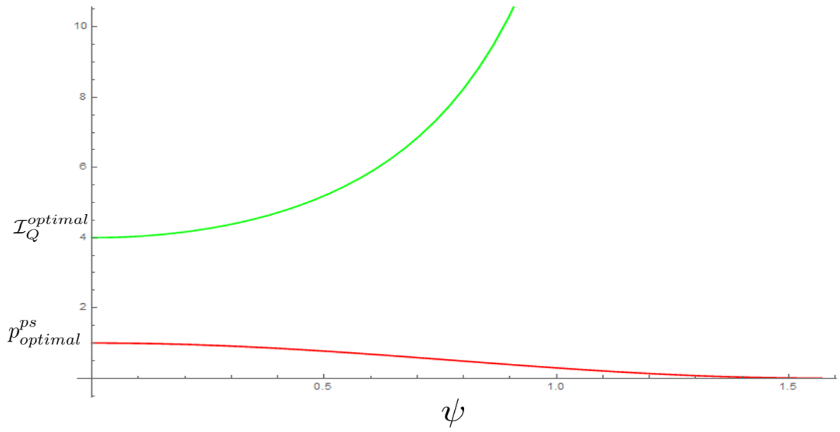

Figure 5.6: Optimal quantum advantage limit, as probability decreases the QFI increases

The Fisher information obtains,

$$\lim_{\delta_\theta \to 0} \lim_{\phi \to 0} \mathcal{I}_Q(\theta / |\psi_\theta^{ps}\rangle) = 4\lambda^2 sec^2(\psi), \tag{5.32}$$

the postselection probability becomes

$$\lim_{\delta_\theta \to 0} \lim_{\phi \to 0} \; p_\theta^{ps} = cos^2(\psi) \tag{5.33}$$

and the efficiency of quantum advantages of this protocol becomes,

$$\lim_{\delta_\theta \to 0} \lim_{\phi \to 0} \xi(\mathcal{I}_Q, p_\theta^{ps}) = 4\lambda^2. \tag{5.34}$$

Thus we see that in the aforementioned limit, we achieve the information preserving protocol quantum enhancement of postselected metrology.

Now if we wish to amply amplify the Fisher information, we just have to set the parameter ψ accordingly so that the Fisher information inflates and the probability decreases. In figure 5.3 and 5.4 we have taken $\psi = \pi/2 - 0.1$. In that limit we get anomalous amount of Fisher information according to result in Eq 5.34. We can see in the scale in figure 5.3 which plots the quantum Fisher information v/s ϕ and δ_θ shows that the maximum Fisher information reaches 400 which is well beyond the Heisenberg limit.

In the following diagram we show how quantum Fisher information, postselection probability and the efficiency of the protocol are dependent upon the parameters $\{\delta_\theta, \phi, \psi\}$. It opens a choice for us to choose the amount of precision we require (by fixing the value of ψ) within the regime of no loss of information.

When we take the limit $\delta_\theta \to 0$, it seems that we work in very close proximity region of the actual value of the parameter. It means that we have to have a lot of information about the parameter θ beforehand. However, for our discussion, this approximation is legitimate since the purpose of quantum metrology is to measure very minute fluctuations of the parameter. That is why the approximated region is the domain of the parameter values we are interested in.

5.4 Discussion

It is evident from the recent lines of research that associating postselection with the standard quantum metrology provides significant advantage in parameter estimation. It is somewhat related with the nonclassical features of quantum states since only the states with complex Kirkwood distribution exhibit this effect. Although we don't fully understand this particular phase space distribution and it's correspondence with our physical understanding. Thus to answer why do we avail this quantum advantage is not an easy task.

To understand the reason behind postselected metrology one needs to explore the significance of postselection in this method. The meaning of postselection in the context of quantum mechanics is understood only through it's effects. We know that the effect of postselection in a classical distribution is akin to classical conditioning of the distribution. The quantum advantages in postselected metrology can be viewed as the effect of postselection on a quasiprobability distribution. That might be starting point of another line of research to find the meaning of postselection in quantum world.

Our argument on this matter have mainly three focal points,

1. There exist relation between two distinguished realities of postselection, the weak values and the postselected metrology. We reason the inflation of postselected Fisher information with the appearance of weak values. Weak values provide significant precision in parameter estimation which comes here directly in the expression of Fisher information. Thus the cost-gain relation is also same in these two significantly different scheme.

2. The effect of weak values appear in this scheme without any weak measurement. This reveals a very interesting fact that the existence of weak values is independent of it's measurement via weak interaction indicating that the ontology of weak value is a major point of concern in order to truly understand the physical meaning behind these phenomena.

3. The Kirkwood distribution plays a key role to determine the consequentiality of postselected metrology. The elements of the Kirkwood distribution has geometric meaning in terms of Pancharatnam phase in quantum systems, which is also closely related with the weak values. Thus a geometric perspective on to this problem would be rather helpful to visualize and understand these effects better. We found that the optimization schemes which we have taken so far especially in weak value amplification exhibit interesting effects in the geometry of state space. This part is explored in the following section.

Geometric interpretation of optimal weak value amplification

Chapter 6

Geometric interpretation of optimal weak value amplification

Suppose in a quantum system the state of the system is evolved unitarily via a cyclic Hamiltonian (Hamiltonian with a cyclic parameter dependence) and comes back to the same point in Bloch sphere, does the state keeps information of the path? The answer to this question is "Yes". The state assumes a global phase which keeps the information of the path including it's geometrical properties and topological invariants. The concept of geometric phase in quantum mechanics originates from Aharanov-Bohm effect but it's actual inception dates back to 1950s when Pancharatnam introduced the idea in the context of classical optics[Pancharatnam 56]. Later on this concept was generalised by Berry, Aharanov and Anandan. Berry's construction of geometric phase was very much dependent upon the adiabatic approximation of quantum evolutions which are cyclic in nature[Victor 84]. Later on it was found that the occurrence of geometric phase was independent of quantum adiabaticity, it is only dependent upon the intrinsic geometry of the state space and cyclicity of the evolution[Aharonov 87].Geometric phase has become a popular topic since last couple of decades because of it's applications in solid state physics, thermodynamics, quantum phase transitions[ZHU 08][Ma 09][Carollo 20][Ma 15]. The topological nature of geometric phase is also crucial and have proved its usefulness in understanding the topological phase transitions in recently reported results [Henriet 18] [Wang 21][Zhu 14].

In this chapter we briefly review the idea of geometric phase and the geometric implications of optimal weak value amplification in terms of pancharatnam's geometric phase.

6.1 Brief review of geometric phase in a one qubit Bloch sphere

A one qubit Bloch sphere is the simplest geometry of state space, that is why we shall discuss the concept of geometric phase using this system which will help us to elaborate profound concepts in an efficient manner. This review is based on these references[Simon 92] [Sjöqvist 15][Sjöqvist 06].

In general the Bloch sphere or Poincare sphere is the geometric representation of a complex projective Hilbert space $\mathbf{CP}^{n-1}$. In simple term this is the set of pure states in a n dimensional quantum system. The one qubit Bloch sphere is represented by a hollow sphere of radius one embedded in a three dimensional euclidean space. The boundary of the region represents the pure states of the system which can be parametrized by two real parameters (θ, ϕ) representing the azimuthal and polar angle of a coordinate on the surface of the sphere. The state corresponding to that point is,

$$|\theta, \phi\rangle = cos\frac{\theta}{2}|0\rangle + e^{i\phi}sin\frac{\theta}{2}|1\rangle \tag{6.1}$$

The (x,y,z) coordinates of an arbitrary state $\hat{\rho}$ in Bloch sphere can be obtained by averaging the Pauli spin matrices: $x = Tr(\hat{\sigma_x}\hat{\rho})$, $y = Tr(\hat{\sigma_y}\hat{\rho})$,$z = Tr(\hat{\sigma_z}\hat{\rho})$.

This Bloch geometry has a natural metric associated defining the topology of this manifold. This is the Fubini-Study metric where the distance between two states $|\psi\rangle$ and $|\phi\rangle$ is given as,

$$(|\psi\rangle, |\phi\rangle) := cos^{-1}\sqrt{\frac{\langle\phi|\psi\rangle\langle\psi|\phi\rangle}{\langle\phi|\phi\rangle\langle\psi|\psi\rangle}} \tag{6.2}$$

This implies that the orthonormal states are located farthest apart from each other. Thus the two basis states, $|0\rangle$ and $|1\rangle$ remain in the two poles of the 1 qubit Bloch sphere.

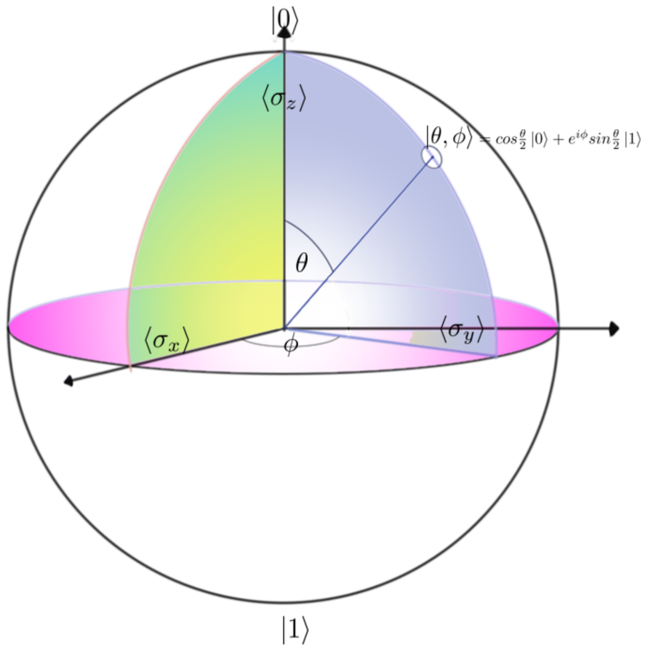

Figure 6.1: Bloch diagram of one qubit optimal weak value amplification with $\hat{A} = \hat{\sigma}_z$

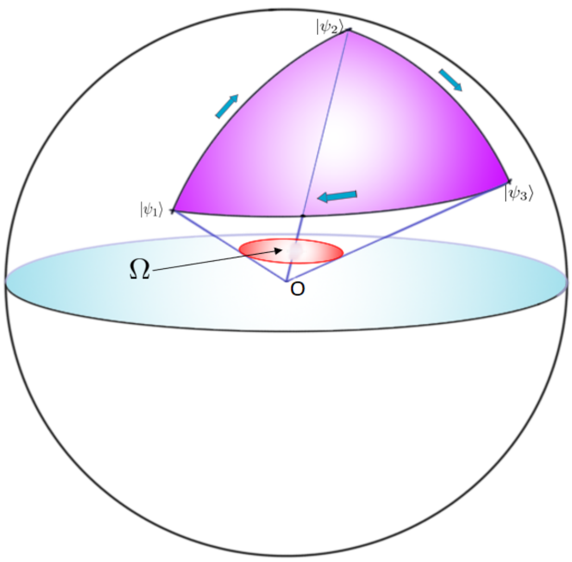

Figure 6.2: Geometric phase in 1 qubit Bloch sphere.

Suppose we have three states on the surface of a Bloch sphere: $|\psi_1\rangle$, $|\psi_2\rangle$, $|\psi_3\rangle$. The three vertex Bergman invariant of these three states is given as,

$$\Delta_3(|\psi_1\rangle, |\psi_2\rangle, |\psi_3\rangle) = \langle\psi_1|\psi_2\rangle\langle\psi_2|\psi_3\rangle\langle\psi_3|\psi_1\rangle \tag{6.3}$$

This Bergman invariant is directly connected with the geometric phase which is given as the argument of the Bergman invariant.

$$\Phi_g(|\psi_1\rangle, |\psi_2\rangle, |\psi_3\rangle) = arg(\Delta_3(|\psi_1\rangle, |\psi_2\rangle, |\psi_3\rangle)) \tag{6.4}$$

What is geometric in geometric phase?

The answer is that this phase is dependent upon the intrinsic geometry of the state space. When we make a cyclic state transition from $|\psi_1\rangle$ to $|\psi_2\rangle$, then $|\psi_2\rangle$ to $|\psi_3\rangle$ and then back to $|\psi_1\rangle$, this quantity Φ_g remains invariant over all cyclic Hamiltonians which generate this transition and it is invariant over all cyclic permutations of this transition. The state assumes a phase which is just addition of a dynamical phase which is dependent upon the path of transition (the cyclic Hamiltonian which specify this path) and this geometric phase. When we do the state transitions via the geodesic lines i.e the great circles in Bloch sphere, the dynamic phase becomes zero and we are left with only the geometric phase. Every cyclic pure state transitions traverse the boundary of a closed region in Bloch sphere. This closed region subtends a solid angle at the centre of Bloch sphere. When the transition is done through the geodesic paths, this solid angle is proportional to the geometric phase.

$$\Phi_g(|\psi_1\rangle, |\psi_2\rangle, |\psi_3\rangle) = -\frac{\Omega}{2} \tag{6.5}$$

Here Ω is the solid angle subtended by the enclosed region at the centre of the Bloch sphere. See the image 6.2.

The relation between the weak values and geometric phase derives through the Bergman invariant. Let us take the operator $\hat{A} := |a\rangle\langle a|$ then from Eq 3.1 and Eq 3.7 we can see that when we multiply the weak value with the postselection probability we get the following relation,

$$p_s A_w = \Delta_3(|\psi_i\rangle, |a\rangle, |\psi_f\rangle) \tag{6.6}$$

The Kirkwood distribution is also related with Bergman invariant and the geometric phase. In Eq 5.8 if we take the state ρ as a pure state $|\psi\rangle$ we can write the Kirkwood distribution in form of Bergman invariant,

$$K_{|\psi\rangle}(a_m, b_n) = \Delta_3(|a_m\rangle, |\psi\rangle, |b_n\rangle) \tag{6.7}$$

6.2 Geometric significance of optimal Weak Value Amplification

Here we show that the efficiency of weak value amplification as we have defined earlier has connection with the geometric phase. To establish that we first write the operator $\hat{A}$ in spectral decomposition in Eq 4.1 and 4.2 to get the following relation,

$$\begin{aligned} p_s|A_w|^2 = |\sum_k a_k |\langle\psi_f|a_k\rangle\langle a_k|\psi_i\rangle| \\ exp(i\Phi_g(|\psi_i\rangle, |a_k\rangle, |\psi_f\rangle))|^2 \end{aligned} \tag{6.8}$$

When we make a cyclic transition from the preselected state $|\psi_i\rangle$ to the same state via a path connecting $|a_k\rangle$ and $|\psi_f\rangle$ through geodesic lines which are specified by the Fubini-Study metric in Bloch sphere, The final state acquires a phase factor $exp(i\Phi_g(|\psi_i\rangle, |a_k\rangle, |\psi_f\rangle))$ over $|\psi_i\rangle$ where this phase Φ_g is proportional to the Solid angle subtended by the area enclosed by the transition path at the centre of the Bloch Sphere. Thus this phase is associated with the geometry of the state space of the system. Eq 6.8 shows how the efficiency of the Weak Value Amplification protocol depends on the geometric phases between the preselected state, postselected state and the eigen states of the system variable. We can argue in this point that the factor which actually controls the efficiency of weak value amplification is the dependence of the geometric phase Φ_g on the eigenvalues a_k. Additionally we can state that the right side of the Eq 6.8 maximizes when the geometric phase factor($e^{i\Phi_g}$) is dependent on the eigenvalues a_k via a sign function. This condition ensures that all the terms inside the summation sign is always positive. The following theorem shows that when we make the choice for our postselection state as given in corollary 1 the condition is satisfied.

Theorem 3. *In a Weak Value Amplification protocol if the postselection state $|\psi_f\rangle$ is* *proportional to* $\hat{A}\left|\psi_i\right\rangle$*, then*

$$exp(i\Phi_g(\left|\psi_i\right\rangle, \left|a_k\right\rangle, \left|\psi_f\right\rangle)) = sgn(a_k)exp(i\phi) \tag{6.9}$$

where sgn(.) is the sign function defined as, $sgn(x) = \frac{|x|}{x}$ *and* ϕ *is a constant phase which doesn't depend on* a_k*.*

Proof. $\left|\psi_f\right\rangle$ is parallel to $\hat{A}\left|\psi_i\right\rangle$. This implies we can write the post selected state as follows,

$$\left|\psi_f\right\rangle = \frac{\hat{A}\left|\psi_i\right\rangle}{[\left\langle\psi_i\right|\hat{A}^2\left|\psi_i\right\rangle]^{1/2}} \tag{6.10}$$

Using this relation we can derive the following relation,

$$\langle\psi_f|a_k\rangle\langle a_k|\psi_i\rangle = a_k\frac{|\langle\psi_i|a_k\rangle|^2}{[\left\langle\psi_i\right|\hat{A}^2\left|\psi_i\right\rangle]^{1/2}} \tag{6.11}$$

The geometric phase writes,

$$\begin{aligned} \Phi_g(\left|\psi_i\right\rangle, \left|a_k\right\rangle, \left|\psi_f\right\rangle) &= arg(\langle\psi_i|a_k\rangle\langle a_k|\psi_f\rangle\langle\psi_f|\psi_i\rangle) \\ &= arg(\langle\psi_f|a_k\rangle\langle a_k|\psi_i\rangle) + arg(\langle\psi_f|\psi_i\rangle) \\ &= arg(a_k\frac{|\langle\psi_i|a_k\rangle|^2}{[\left\langle\psi_i\right|\hat{A}^2\left|\psi_i\right\rangle]^{1/2}}) + arg(\langle\psi_f|\psi_i\rangle) \end{aligned} \tag{6.12}$$

Now instead of a_k all the terms inside the first argument function is positive. So we can write the phase as follows,

$$\Phi_g(\left|\psi_i\right\rangle, \left|a_k\right\rangle, \left|\psi_f\right\rangle) = arg(sgn(a_k)) + arg(\langle\psi_f|\psi_i\rangle) \tag{6.13}$$

This implies that,

$$exp(i\Phi_g(\left|\psi_i\right\rangle, \left|a_k\right\rangle, \left|\psi_f\right\rangle)) = sgn(a_k)exp(i\phi) \tag{6.14}$$

where $\phi = arg(\langle\psi_f|\psi_i\rangle)$ which is independent of a_k.

□

We explore this result with a simple example with one qubit system. Suppose the operator $\hat{A}$ is $\hat{\sigma_z}$ operator. The initial preselected state $\left|\psi_i\right\rangle$ is given by a point in Bloch sphere $\left|\theta, \phi\right\rangle$. Thus our final postselection state for optimal weak value amplification is proportional to $\hat{A}\left|\psi_i\right\rangle$. So in this case the final postselected state is, $\left|\psi_f\right\rangle = \left|\theta, \phi + \pi\right\rangle$.

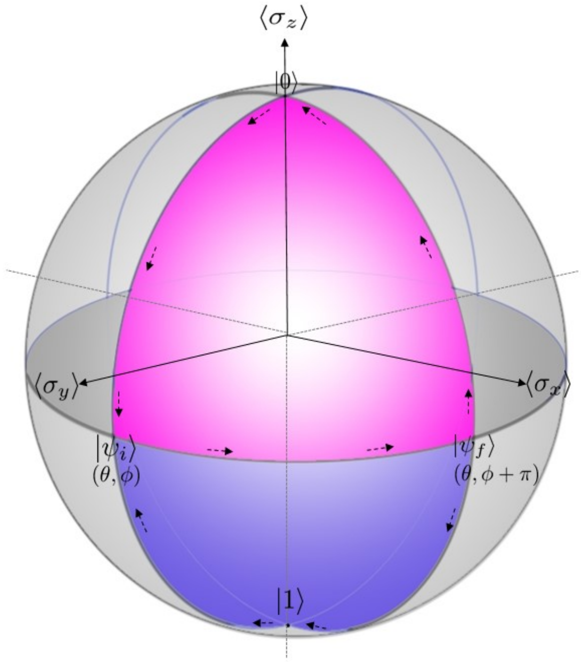

Figure 6.3: Bloch diagram of one qubit optimal weak value amplification with $\hat{A} = \sigma_z$

The two closed region covered by the two geometric phases in this case is given in the figure 6.3. The two geometric phases are, $\Phi_g(|\theta,\phi\rangle, |0\rangle, |\theta,\phi+\pi\rangle)$ which equals -1 and $\Phi_g(|\theta,\phi\rangle, |1\rangle, |\theta,\phi+\pi\rangle)$ which equals 1. The first geometric phase encloses the blue region and the second one encloses the pink region. It is evident that the blue region covers a smaller area than the pink region. The areas are subtended accordingly such that the phase factor induced by the blue region is 1 and the pink region induces the phase factor -1.

6.3 Discussion

In this chapter we showed that the optimization of weak value scheme can also be shown from a geometrical perspective where we are led to the same conditions for optimization. From the geometric phase perspective we showed that under what condition we achieve the highest efficiency of weak value amplification. The optimization condition imposes a constraint in the choice of postselection. In this result we get a geometrical insight into this constraint via theorem 3.

Conclusion and Future prospects

Chapter 7

Conclusion

The fundamental bounds in nature are crucial in understanding the laws of nature itself. This project yields some fundamental bounds which is crucial in understanding the power of weak values and postselection in quantum metrology. In the first part we showed how the efficiency of weak value amplification is bounded and how we achieve this bound. In the second part we showed how the postselected metrology scheme is dependent upon the internal weak values. With our observations from this part we arrive at a conclusion that the effects of weak values exist even when we don't perform any weak measurement. This result takes us one step further towards a realization that weak values are universal[Dressel 12]. Using this observation we formulated the example where we achieve arbitrary Fisher information without loss of information with a simple three level system. This allows us to device efficient experimental schemes to realize the postselected quantum metrology. The effects of weak values in geometric phase has also became a point of research in recent years. Our treatment of weak value amplification is also consequential in this context. We showed that the optimization of weak value amplification can be drawn from a geometric perspective which obtains interesting results in terms of geometric phase. The results we derived in this project comes from three different lines of research, the weak value, the postselected metrology and geometric phase and they confluence in one main conclusion, the technological advantage in quantum metrology we obtain based on weak values and postselection is not unbounded when we compare it with it's cost. It is crucial to specify the conditions to achieve the best situation where we obtain maximum yield with a minimum amount of cost.

Chapter 8

Future prospects

This project opens up many possible lines of research in both theory and experiment. We list them as follows,

1. The postselection plays a crucial role in weak values and postselected metrology. Recently it has been reported that one can obtain anomalous weak values even without postselection[Abbott 19]. Thus it is worthwhile to explore the role of postselection in these two schemes from the perspective of quantum foundations.

2. The weak values and geometric phase has become a crucial ingredient to understand the topological phase transitions in solid state materials. The geometric phases are known to set order parameters for these phase transitions[Ma 13]. Thus it would prove consequential to explore the physical consequences of theorem 3 in various contexts where geometric phases are realized physically.

3. The nonclassical features of Kirkwood distribution is essential in order to obtain quantum advantages in postselected metrology. The nonclassicality in Kirkwood distribution is fundamentally concerned with it's complex values. The resource theory of complex numbers in quantum mechanics has become a point of research in recent times[Wu 21a][Wu 21b]. In future we plan to explore how quantum imaginary values of Kirkwood distribution is used as a resource in postselected metrology.

4. In multiparameter estimation protocols how the postselected metrology works and

how quantum systems provide advantages for this purpose is another point we plan to explore in future.

5. The postselected metrology scheme we propose in section 5.3.1 can be materialized using spin systems in NMRQC. This method is simpler than the proposed scheme in here [Arvidsson-Shukur 20]. This will possibly give advantage to device experiments for realizing this phenomenon.

Bibliography

[Abbott 19] Alastair A. Abbott, Ralph Silva, Julian Wechs, Nicolas Brunner & Cyril Branciard. *Anomalous Weak Values Without Post-Selection.* Quantum, vol. 3, page 194, Oct 2019.

[Aharonov 87] Y. Aharonov & J. Anandan. *Phase change during a cyclic quantum evolution.* Phys. Rev. Lett., vol. 58, pages 1593–1596, Apr 1987.

[Aharonov 88] Yakir Aharonov, David Z. Albert & Lev Vaidman. *How the result of a measurement of a component of the spin of a spin-1/2 particle can turn out to be 100.* Phys. Rev. Lett., vol. 60, pages 1351–1354, Apr 1988.

[Angrist n] Stanley Angrist. *Perpetual Motion Machines.* Scientific American, vol. 218, no. 1, pages 114–122, 1968(Jan).

[Arvidsson-Shukur 20] David R. M. Arvidsson-Shukur, Nicole Yunger Halpern, Hugo V. Lepage, Aleksander A. Lasek, Crispin H. W. Barnes & Seth Lloyd. *Quantum advantage in postselected metrology.* Nature Communications, vol. 11, no. 1, Jul 2020.

[Bell 64] John S. Bell. *On Einstein Podolsky Rosen paradox.* Physics Physique Fizika, vol. 1, no. 3, page 195, 1964.

[Carollo 20] Angelo Carollo, Davide Valenti & Bernardo Spagnolo. *Geometry of quantum phase transitions.* Physics Reports, vol. 838, pages 1–72, 2020. Geometry of quantum phase transitions.

[Dixon 09] P. Ben Dixon, David J. Starling, Andrew N. Jordan & John C. Howell. *Ultrasensitive Beam Deflection Measurement via Inter-*

ferometric Weak Value Amplification. Phys. Rev. Lett., vol. 102, page 173601, Apr 2009.

[Dressel 12] Justin Dressel & Andrew N. Jordan. *Weak Values are Universal in Von Neumann Measurements.* Physical Review Letters, vol. 109, no. 23, Dec 2012.

[Dressel 13] Justin Dressel, Kevin Lyons, Andrew N. Jordan, Trent M. Graham & Paul G. Kwiat. *Strengthening weak-value amplification with recycled photons.* Phys. Rev. A, vol. 88, page 023821, Aug 2013.

[Dressel 14] Justin Dressel, Mehul Malik, Filippo M. Miatto, Andrew N. Jordan & Robert W. Boyd. *Colloquium: Understanding quantum weak values: Basics and applications.* Rev. Mod. Phys., vol. 86, pages 307–316, Mar 2014.

[Duck 89] I. M. Duck, P. M. Stevenson & E. C. G. Sudarshan. *The sense in which a "weak measurement" of a spin-$\frac{1}{2}$ particle's spin component yields a value 100.* Phys. Rev. D, vol. 40, pages 2112–2117, Sep 1989.

[Giovannetti 11] Vittorio Giovannetti, Seth Lloyd & Lorenzo Maccone. *Advances in quantum metrology.* Nature Photonics, vol. 5, no. 4, page 222–229, Mar 2011.

[Henriet 18] Loïc Henriet. *Geometrical properties of the ground state manifold in the spin boson model.* Phys. Rev. B, vol. 97, page 195138, May 2018.

[Hensen 15] Bernien H. Dréau A. et al Hensen B. *Loophole-free Bell inequality violation using electron spins separated by 1.3 kilometres.* Nature, vol. 526, pages 682–686, 2015.

[Hosten 08] Onur Hosten & Paul Kwiat. *Observation of the Spin Hall Effect of Light via Weak Measurements.* Science, vol. 319, no. 5864, pages 787–790, 2008.

[Ma 09] Yu-Quan Ma & Shu Chen. *Geometric phase and quantum phase transition in an inhomogeneous periodic XY spin-$\frac{1}{2}$ model.* Phys. Rev. A, vol. 79, page 022116, Feb 2009.

[Ma 13] Yu-Quan Ma, Zhao-Xian Yu, Deng-Shan Wang & Xiang-Gui Li. *Quantized Berry phase in twisted Bloch momentum space as a topological order parameter for spin chains.* Physics Letters A, vol. 377, no. 18, pages 1250–1254, 2013.

[Ma 15] Yu-Quan Ma. *Ground-state information geometry and quantum criticality in an inhomogeneous spin model.* Chinese Physics B, vol. 24, no. 9, page 090301, sep 2015.

[Osborne] Tobias Osborne. *Quantum parameter estimation, Fisher information, and the Cramér-Rao bound.*

[Pancharatnam 56] S. Pancharatnam. *Generalized theory of interference, and its applications.* Proceedings of Indian Academy of Sciences- Section A, vol. 44, no. 10, pages 247–262, 1956.

[Pang 14] Shengshi Pang, Justin Dressel & Todd A. Brun. *Entanglement-Assisted Weak Value Amplification.* Physical Review Letters, vol. 113, no. 3, Jul 2014.

[Pang 15a] Shengshi Pang & Todd A. Brun. *Improving the Precision of Weak Measurements by Postselection Measurement.* Physical Review Letters, vol. 115, no. 12, Sep 2015.

[Pang 15b] Shengshi Pang & Todd A. Brun. *Improving the Precision of Weak Measurements by Postselection Measurement.* Physical Review Letters, vol. 115, no. 12, Sep 2015.

[Pfeifer 11a] Marcel Pfeifer & Peer Fischer. *Weak value amplified optical activity measurements.* Opt. Express, vol. 19, no. 17, pages 16508–16517, Aug 2011.

[Pfeifer 11b] Marcel Pfeifer & Peer Fischer. *Weak value amplified optical activity measurements*. Opt. Express, vol. 19, no. 17, pages 16508–16517, Aug 2011.

[Resch 04] K.J. Resch, J.S. Lundeen & A.M. Steinberg. *Experimental realization of the quantum box problem*. Physics Letters A, vol. 324, no. 2-3, page 125–131, Apr 2004.

[Simon 92] R Simon & N Mukunda. *Hamilton's turn and geometric phase for two-level systems*. Journal of Physics A: Mathematical and General, vol. 25, no. 22, pages 6135–6144, nov 1992.

[Sjöqvist 06] Erik Sjöqvist. *Geometric phase in weak measurements*. Physics Letters A, vol. 359, no. 3, page 187–189, Nov 2006.

[Sjöqvist 15] Erik Sjöqvist. *Geometric phases in quantum information*. International Journal of Quantum Chemistry, vol. 115, no. 19, page 1311–1326, May 2015.

[Tsaousis 08] D. Tsaousis. *Perpetual Motion Machine*. Journal of Engineering Science and Technology Review, vol. 1, no. 1, pages 53–57, 2008.

[Valdés-Hernández 19] A. Valdés-Hernández, L. de la Peña & A.M. Cetto. *Strong entanglement criterion involving momentum weak values*. Physics Letters A, vol. 383, no. 9, page 838–842, Feb 2019.

[Vetrivelan 20] Muthumanimaran Vetrivelan & Sai Vinjanampathy. *Near-Deterministic Weak-Value Metrology via Collective non-Linearity*, 2020.

[Victor 84] Berry Michael Victor. *Quantal phase factors accompanying adiabatic changes*. Proc. R. Soc. Lond., vol. A, pages 45–57, Mar 1984.

[Wang 21] Yunzhao Wang, Kyrylo Snizhko, Alessandro Romito, Yuval Gefen & Kater Murch. *Observing a Topological Transition in Weak-Measurement-Induced Geometric Phases*, 2021.

[Wu 21a] Kang-Da Wu, Tulja Varun Kondra, Swapan Rana, Carlo Maria Scandolo, Guo-Yong Xiang, Chuan-Feng Li, Guang-Can Guo & Alexander Streltsov. *Operational Resource Theory of Imaginarity.* Phys. Rev. Lett., vol. 126, page 090401, Mar 2021.

[Wu 21b] Kang-Da Wu, Tulja Varun Kondra, Swapan Rana, Carlo Maria Scandolo, Guo-Yong Xiang, Chuan-Feng Li, Guang-Can Guo & Alexander Streltsov. *Resource theory of imaginarity: Quantification and state conversion.* Phys. Rev. A, vol. 103, page 032401, Mar 2021.

[Xu 13] Xiao-Ye Xu, Yaron Kedem, Kai Sun, Lev Vaidman, Chuan-Feng Li & Guang-Can Guo. *Phase Estimation with Weak Measurement Using a White Light Source.* Phys. Rev. Lett., vol. 111, page 033604, Jul 2013.

[Zhou 12] Xinxing Zhou, Xiaohui Ling, Hailu Luo & Shuangchun Wen. *Identifying graphene layers via spin Hall effect of light.* Applied Physics Letters, vol. 101, no. 25, page 251602, 2012.

[ZHU 08] SHI-LIANG ZHU. *GEOMETRIC PHASES AND QUANTUM PHASE TRANSITIONS.* International Journal of Modern Physics B, vol. 22, no. 06, page 561–581, Mar 2008.

[Zhu 14] Ling-Li Zhu, Qian Huang, Hui-Lin Dai, Cui Kong, Yu lin Feng & Chuan-Jia Shan. *Detecting topological phase transition in 1D superconducting systems with next nearest neighbor hopping.* Laser Physics Letters, vol. 12, no. 1, page 015202, dec 2014.

Chapter 9

Appendix

9.1 Appendix 1

Derivation of the Quantum Fisher Information and Quantum Cramer-Rao bound

This proof is taken from [Osborne]. First we construct an inner product on the operator space of the system which is dependent on the system.

$$(\hat{A}, \hat{B})_\rho = tr(\hat{A}^\dagger \Omega_\rho(\hat{B})) \tag{9.1}$$

This is a valid sesquilinear inner product since it satisfies all the properties of inner product. This inner product has the following properties,

- $(\hat{A}, \hat{A})_\rho = Var(\hat{A})_\rho$
- $(\hat{A}, \mathbf{I})_\rho = \langle \hat{A} \rangle_\rho$

Now let Φ be our unbiased estimator. Thus we have,

$$\langle \Phi \rangle_\rho = \theta \tag{9.2}$$

From the above Eq we derive the following result,

$$\begin{aligned} 1 &= \partial_\theta tr(\Phi\hat{\rho}) \\ &= tr(\Phi\partial_\theta\hat{\rho}) \\ &= tr(\Phi\Omega_\rho(\Omega_\rho^{-1}(\partial_\theta\hat{\rho}))) \end{aligned} \tag{9.3}$$

The symmetric logarithmic derivative of θ is expressed as,

$$\Lambda_\theta := \Omega_\rho^{-1}(\partial_\theta\hat{\rho})$$

Thus we get,

$$tr(\Phi\Omega_\rho(\Lambda_\theta)) = (\Phi, \Lambda_\theta)_\rho = 1 \tag{9.4}$$

Now we apply Cauchy-Schwartz inequality on this inner product to get the following,

$$(\Phi, \Phi)_\rho(\Lambda_\theta, \Lambda_\theta)_\rho \geq |(\Phi, \Lambda_\theta)_\rho|^2 = 1 \tag{9.5}$$

From the previous results it is easy to deduct that,

$$Var(\theta) = Var(\Phi)_\rho = (\Phi, \Phi)_\rho \tag{9.6}$$

and,

$$(\Lambda_\theta, \Lambda_\theta)_\rho = \mathcal{I}_Q(\rho(\theta), \hat{A}) \tag{9.7}$$

is the Quantum Fisher information. Thus we deduct the Quantum Cramer-Rao bound which is completely independent of the choice of the likelihood function thus sets a stronger limit in the precision of the parameter estimation than the Classical Cramer-Rao bound.,

$$Var(\theta) \geq \frac{1}{\mathcal{I}_Q(\rho(\theta), \hat{A})} \tag{9.8}$$

9.2 Appendix 2

Derivation of the relation between postselected Fisher information and weak values

We start with a pure initial state $\hat{\rho}_0 = |\psi_i\rangle\langle\psi_i|$. The evolved state becomes, $|\psi_\theta\rangle = \hat{U}|\psi_i\rangle$ where $\hat{U} := exp(-i\theta\hat{A})$. First we do postselection with a rank 2 projector as in 5.14. Using Eq 5.6 we get the following result,

$$
\begin{aligned}
\mathcal{I}(p_\theta^{ps})^2 &= 4p_\theta^{ps} Tr(\hat{A}\hat{F}\hat{A}\hat{U}\rho_0\hat{U}^\dagger) - 4|Tr(\hat{A}\hat{F}\hat{U}\rho_0\hat{U}^\dagger)|^2 \\
&= 4\Big[\big(|\langle\psi_\theta|f_1\rangle|^2 + |\langle\psi_\theta|f_2\rangle|^2\big)\big(|\langle\psi_\theta|\hat{A}|f_1\rangle|^2 + |\langle\psi_\theta|\hat{A}|f_2\rangle|^2\big) \\
&\quad -\big|\langle\psi_\theta|\hat{A}|f_1\rangle\langle f_1|\psi_\theta\rangle + \langle\psi_\theta|\hat{A}|f_2\rangle\langle f_2|\psi_\theta\rangle\big|^2\Big] \\
&= 4\Big[p_{\theta_1}^{ps}|\langle\psi_\theta|\hat{A}|f_2\rangle|^2 + p_{\theta_2}^{ps}|\langle\psi_\theta|\hat{A}|f_1\rangle|^2 - \langle\psi_\theta|\hat{A}|f_1\rangle\langle f_1|\psi_\theta\rangle\langle\psi_\theta|f_2\rangle\langle f_2|\hat{A}|\psi_\theta\rangle \\
&\quad -\langle\psi_\theta|\hat{A}|f_2\rangle\langle f_2|\psi_\theta\rangle\langle\psi_\theta|f_1\rangle\langle f_1|\hat{A}|\psi_\theta\rangle\Big] \\
&= 4[p_{\theta_1}^{ps}p_{\theta_2}^{ps}|A_{\theta_1}^w|^2 + p_{\theta_1}^{ps}p_{\theta_2}^{ps}|A_{\theta_2}^w|^2 - p_{\theta_1}^{ps}p_{\theta_2}^{ps}(A_{\theta_1}^w)^* A_{\theta_2}^w - p_{\theta_1}^{ps}p_{\theta_2}^{ps}(A_{\theta_2}^w)^* A_{\theta_1}^w] \\
&= 4p_{\theta_1}^{ps}p_{\theta_2}^{ps}|A_{\theta_1}^w - A_{\theta_2}^w|^2
\end{aligned}
\tag{9.9}
$$

9.3 Appendix 3

Postselected Fisher information for for n projectors For two projectors we derived the formulae,

$$
\mathcal{I}_Q(\theta|\Psi_\theta^{\text{ps}})(p_\theta^{\text{ps}})^2 = 4p_{\theta 1}^{\text{ps}}p_{\theta 2}^{\text{ps}}|A_{\theta_1}^w - A_{\theta_2}^w|^2 \tag{9.10}
$$

For three projectors we get the following result.

$$
\mathcal{I}_Q(p_\theta^{\text{ps}})^2 = 4\Big[p_{\theta 1}^{\text{ps}}p_{\theta 2}^{\text{ps}}|A_{\theta_1}^w - A_{\theta_2}^w|^2 + p_{\theta 2}^{\text{ps}}p_{\theta 3}^{\text{ps}}|A_{\theta_2}^w - A_{\theta_3}^w|^2 + \ p_{\theta 3}^{\text{ps}}p_{\theta 1}^{\text{ps}}|A_{\theta_3}^w - A_{\theta_1}^w|^2\Big] \tag{9.11}
$$

From these two results we can easily extrapolate this formulae for n number of projectors as follows,

$$
\mathcal{I}_Q(\theta/\rho_\theta^{ps}) = \frac{4}{(p_\theta^{ps})^2}\sum_{m<n} p_{\theta_m}^{ps}p_{\theta_n}^{ps}\big|A_{\theta_m}^w - A_{\theta_n}^w\big|^2 \tag{9.12}
$$