Topology, Geometry and Analysis on Surfaces

Thesis to be submitted in partial fulfillment of the requirements for the degree

of

BS-MS

by

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Under the guidance of

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CERTIFICATE

This is to certify that the dissertation titled **Topology, Geometry and Analysis on Surfaces**, submitted by **Nandagopal S A**(Roll Number: *MS16019*) a postgraduate student of the **Department of Mathematical Sciences** in partial fulfilment of BS-MS dual degree program of the Indian Institute of Science Education and Research, Mohali has been examined by the thesis committee duly appointed by the Institute. The committee finds the work done by the candidate satisfactory and recommends that the report be accepted. The thesis has fulfilled all the requirements as per the institute's regulations and has reached the standard needed for submission.

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Soma Maity

Dr Soma Maity

Place: Mohali
Date: 19-05-2021

DECLARATION

The work presented in this dissertation has been carried out by me under the guidance of Dr Pranab Sardar at the Indian Institute of Science Education and Research, Mohali.

This work has not been submitted in part or in full for a degree, a diploma, or a fellowship to any other university or institute. Whenever contributions of others are involved, every effort is made to indicate this clearly, with due acknowledgement of collaborative research and discussions. This thesis is a bonafide record of original work done by me, and all sources listed within have been detailed in the bibliography.

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In my capacity as the supervisor of the candidate's project work, I certify that the above statements made by the candidate are true to the best of my knowledge.

Dr Pranab Sardar
(Supervisor)

List of Figures

NOTATIONS

$\mathbb{R}$: Real numbers

$\mathbb{C}$: Complex numbers

M^n: n-dimensional smooth manifold

$Bd(I)$: Boundary of $[0,1]$

$\mathscr{X}(M)$: the set of all differentiable vector fields on M.

T_p^*M: the dual of tangent space T_pM

$\mathscr{T}^k(V)$: the set of all k-tensors, for vector space V.

nbd denotes neighbourhood

Contents

Chapter 1

Classification of Compact Surfaces

1.1 Introduction

The aim of this chapter is to discuss the classification of compact surfaces. First we will see observe the fundamental group of certain class of closed surfaces followed by the first homology groups of the same class. Elementary operations on schemes prepare us for the classification theorem for compact surfaces. The first chapter ends by discussing the paper by Doyle and Moran which gives a proof that compact 2-manifolds can be triangulated.

1.2 Fundamental group of surfaces

Our motive in this section is to study fundamental group of compact surfaces. For this we first look at construction of polygonal regions in plane.
We are interested in polygonal regions since we obtain surfaces by carrying out a particular ”quotienting” on the polygonal region in the plane.

1.2.1 Construction of polygonal region in a plane

Observe the following figures below.
Here, $c \in \mathbb{R}^2$ $a > 0$

$$\theta_0 \leqslant \theta_1 .... \leqslant \theta_n \qquad \theta_n = \theta_0 + 2\pi, \qquad n \geqslant 3, \tag{1.1}$$

Line through p_{i-1}, p_i splits the plane into two closed half-planes.

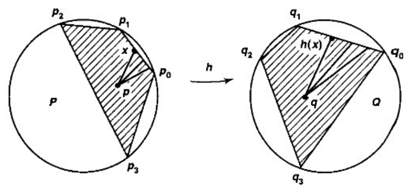

Figure 1.1: Example of a polygonal region in a plane, where x is a boundary point while "p" an interior point, h is the positive linear map.

Call H_{i-1} which contains all the points $p_k,\ k \neq i, i-1$. Then the space,

$$P = H_0 \cap H_1 \cap \ldots \cap H_{n-1} \tag{1.2}$$

Given a line segment L in $\mathbb{R}^2$, an *orientation* of L is simply an ordering of its end points.

Given two line segments say L and L', where orientations are from a to b and c to d respectively, then a *positive linear map* from L onto L' is the homeomorphism that carries, $x = (1-t)a + tb$ of L to $h(x) = (1-t)c + td$ of L'

If 2 polygonal regions are having same number of vertices then there exist a homeomorphism between them

Hint- Use positive linear map

An interesting question at this point is, how does the 'quotienting' happens in polygonal region, for this we define *labelling* and *labelling scheme*.

Definition 1.2.1. *A labelling of edges of P is a map from set of edges of P to set of labels say S.*

For labelled oriented edges we define equivalence relation on points of P as follows

- Each point of *int(P)* is equivalent to itself only.
- Given two edges of same label, any point of one is mapped to a point of other via a *positive linear map*.

Quotient of P modulo this equivalence relation (*pasting edges together*) gives us a surface.

Let us see some examples,

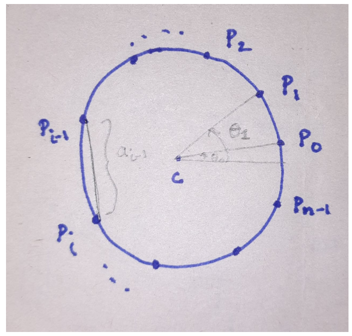

Figure 1.2: Circle with ordered points

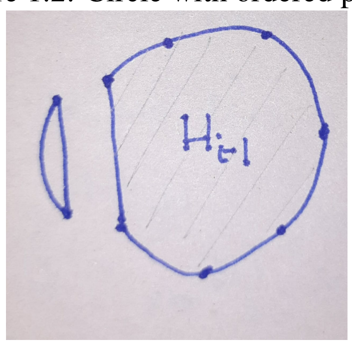

Figure 1.3: Half plane H_{i-1} between point p_i and p_{i-1}

Definition 1.2.2. *Given labelling, say $a_1, a_2, .., a_k$ and orientation,say $\varepsilon_1, \varepsilon_2, .., \varepsilon_k$, of edges of P, where $\varepsilon_i = +1$ if orientation is from p_{i-1} to p_i and $\varepsilon_i = -1$otherwise.*

$$\omega = (a_{i_1})^{\varepsilon_1}(a_{i_2})^{\varepsilon_2}...(a_{i_n})^{\varepsilon_n} \tag{1.3}$$

is labelling scheme of length n. The number of edges, orientation of edges and labelling of edges of P are specified by this symbol.

Theorem 1.2.1. *Let X be a space obtained from a finite number of polygonal regions by pasting edges together according to some labelling scheme. Then X is a* ***compact, Hausdorff space****.*

Proof. Let's consider the case when X is obtained from a single P.

Clearly $\pi : P \to X$, the quotient map, is continuous. P is compact. Hence X is compact.

To show X is Hausdorff we use the following lemma.

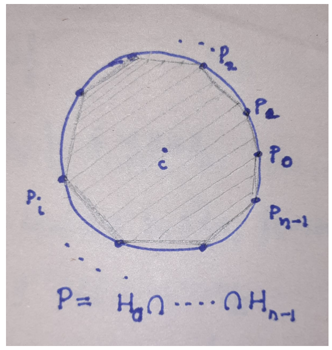

Figure 1.4: Polygonal Region 'P', $P = H_0 \cap H_1 \cap \ldots \cap H_{n-1}$

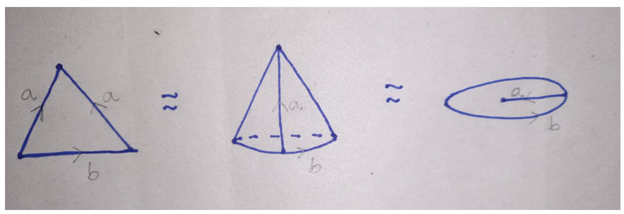

Figure 1.5: Triangular region(under the given label)$\longrightarrow$ Cone $\longrightarrow$ Disc, note $\{S\} = \{a, b\}$

Lemma 1.2.2. *If $\pi : E \to X$, is a closed quotient map, then X is normal if E is normal.*

Since the proof is standard we skip it.

So, in our case, to start with P is definitely normal. Hence we show that π is a closed map, then we are done.

For any closed set C of P, it is enough to show that $\pi^{-1}(\pi(C))$ is closed, since π is a quotient map. Observe, preimage of $\pi(C)$ contains C along with edges pasted together to edges of C under π.

Define $C_e = C \cap e$, this is a compact subset of C, if it exists, hence closed. For each i we have homeomorphism $h_i : e_i \to e$, also define $D_e = \pi^{-1}(\pi(C)) \cap e$, now D_e is union of C_e and $h_i(C_{e_i})$ for all i (finitely many), clearly C_e and $h_i(C_{e_i})$ are both closed in e, since they are union of finitely many closed sets and hence closed in P.

Finally D_e over all e, union C equals $\pi^{-1}(\pi(C))$, hence this is closed in P $\square$

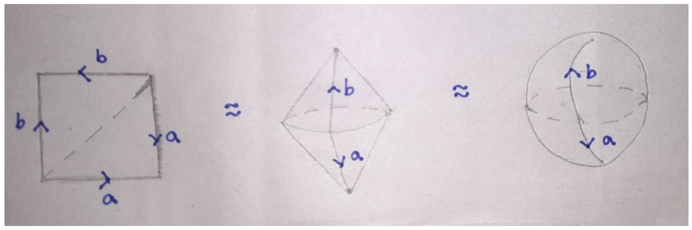

Figure 1.6: Rectangular region(under the given label)⟶Two cones intersecting at the circle⟶ Sphere

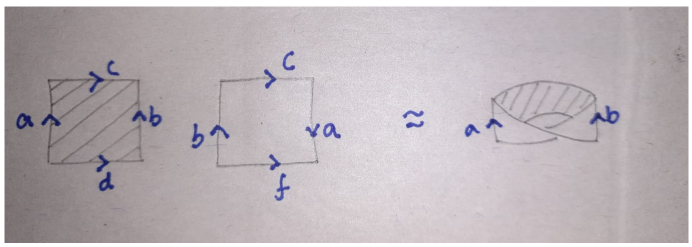

Figure 1.7: Two disjoint polygonal regions results in a connected space on quotienting

Theorem 1.2.3. *Let P be a polygonal region*

$$\omega = (a_{i_1})^{\varepsilon_1}(a_{i_2})^{\varepsilon_2}...(a_{i_n})^{\varepsilon_n} \tag{1.4}$$

be a labelling scheme for edges of P. Let X be the resulting quotient space under $\pi : P \to X$*.If* π *maps all the vertices of P to a single point* x_0 *of X, and if* $a_1, ..., a_k$ *are the distinct labels that appear in the labelling scheme, then* $\pi_1(X, x_0)$ *is isomorphic to the quotient of the free group on k generators* $\alpha_1, ..., \alpha_k$ *by the smallest*

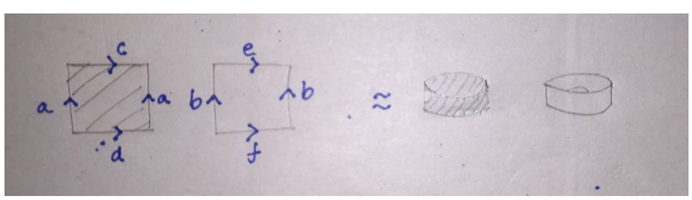

Figure 1.8: Two disjoint polygonal regions results in a disconnected space on quotienting

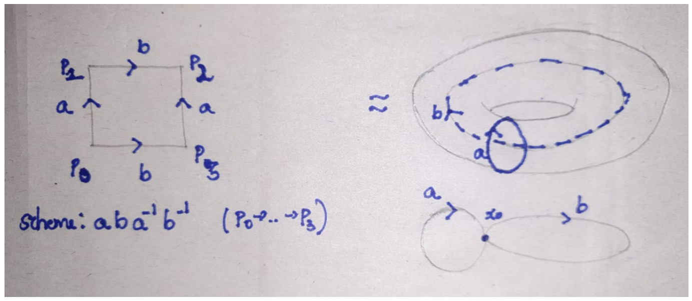

Figure 1.9: A scheme for torus

normal subgroup containing the element

$$(\alpha_{i_1})^{\varepsilon_1}(\alpha_{i_2})^{\varepsilon_2}\ldots(\alpha_{i_n})^{\varepsilon_n} \tag{1.5}$$

Proof. A rough sketch, using Van Kampen theorem is given below

Immediate observations,

- π maps *Bd(P)* to a closed set of X, call it A.
- π maps all vertices of P to a single point x_0 in X(maybe, example of torus, is easy to see)
- $\pi(Bd(P))$ is a wedge of k circles.
- So $\pi_1(\pi(Bd(P)), x_0)$, is free group on k generators
- If $g_i = \pi \cdot f_i$,the loops $g_1, \ldots, g_k$ represent a set of free generators for $\pi_1(A, x_0)$

Consider this figure, conditions for Van Kampen theorem are satisfied, so Van Kampen $\Longrightarrow$

$$i_* : *\pi_1(A_{\alpha_i}, x_\alpha) \to \pi_1(X, x_0)$$

is a surjection, and we can observe that loops in $\pi_1(A_{\alpha_1} \cap A_{\alpha_2}, x_\alpha)$ are exactly of the form $(g_{i_1})^{\varepsilon_1}(g_{i_2})^{\varepsilon_2}\ldots(g_{i_n})^{\varepsilon_n}$.

Hence $\pi_1(X, x_0)$ is isomorphic to the quotient of $g_1 * g_2 * \ldots * g_k$ by the normal closure of $(g_{i_1})^{\varepsilon_1}(g_{i_2})^{\varepsilon_2}\ldots(g_{i_n})^{\varepsilon_n}$. □

Proof using another theorem.

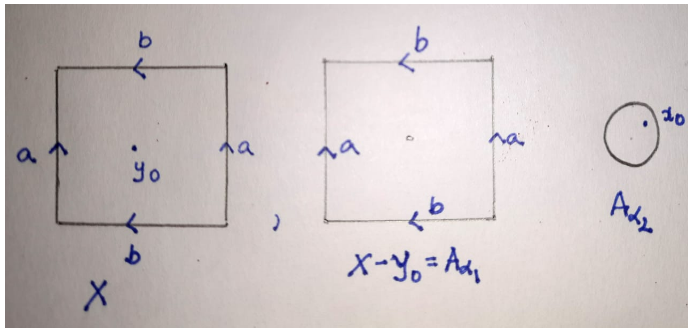

Figure 1.10:

Theorem 1.2.4. *Let X be a Hausdorff space and A be a closed path connected subspace of X. Suppose that there exist continuous map $h : B^2 \to X$ that maps $Int(B)^2$ bijectively onto X-A and $Bd(B)^2 = S^1$ into A. Let $p \in S^1$ and h(p)=a, let k : $(S^1, p) \to (A, a)$ be the map obtained by restricting h. Then the homomorphism*

$$i_* : \pi_1(A, a) \to \pi_1(X, a)$$

induced by inclusion is surjective and its kernel is the least normal subgroup of $\pi_1(A, a)$, containing the image of $k_* : \pi_1(S^1, p) \to \pi_1(X, a)$

We will see an application of this theorem

Theorem 1.2.5. *The fundamental group of the torus has a presentation consisting of two generators α, β and a single relator $\alpha\beta\alpha^{-1}\beta^{-1}$.*

Proof. Let X be the torus under consideration, so we know that $X = S^1 \times S^1$.

If p is a covering map from $\mathbb{R} \to S^1$. Now restriction of the covering map ,

$$p \times p : \mathbb{R} \times \mathbb{R} \to S^1 \times S^1 \; to \; I \times I \qquad \text{is a continuous map.}$$

Also let $A = h(Bd(I^2))$ if we consider the point (0,0) in I^2 and let $a = h(0,0)$, then this matches conditions mentioned in above theorem.
Let $a_0(t) = (t, 0)$, $b_0(t) = (0, t)$, let $\alpha = h \cdot a_0(t)$ and $\beta = h \cdot b_0(t)$, clearly α and β are loops in A, A is wedge of 2 circles, so α, β are generators for $\pi_1(A, a)$.

Also, let $a_1(t) = (1, t)$, $b_1(t) = (1, t)$, as mentioned in the figure. Now,

$$f = a_0 * b_0 * \overline{a_1} * \overline{b_1}$$

is a loop generating $Bd(I^2)$, and image of this loop is clearly $\alpha_0 * \beta_0 * \overline{\alpha_1} * \overline{\beta_1}$, so applying by above theorem

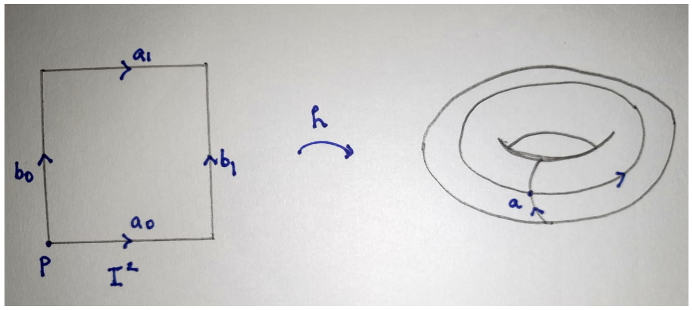

Figure 1.11:

$$\pi_1(X,a) = \langle \alpha, \beta | \alpha_0 \beta_0 \alpha_1^{-1} \beta_1^{-1} = 1 \rangle$$

□

Definition 1.2.3. *Consider the space obtained from a 4n-sided polygonal region P by means of the labelling scheme*

$$(a_1 b_1 a^{-1} b^{-1})(a_2 b_2 a^{-1} b^{-1})..(a_n b_n a^{-1} b^{-1})$$

This space is called the ***n-fold connected sum of tori****, or simply the* ***n-fold torus****, and denoted T#T#..#T*

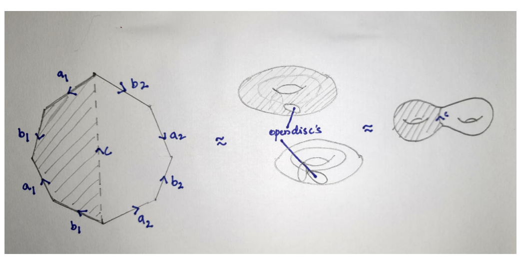

Figure 1.12: 2 Torus

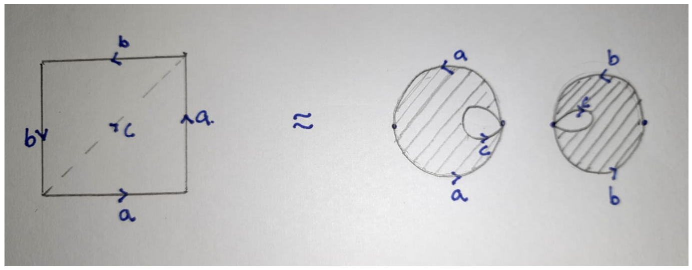

Figure 1.13: P^2 can be obtained by pasting the two discs

Theorem 1.2.6. *Let X denote the n-fold torus. Then $\pi_1(X, x_0)$ is isomorphic to the quotient of the free group on 2n generators $\alpha_1, \beta_1, ..., \alpha_n, \beta_n$ by the least normal subgroup containing the element*

$$[\alpha_1, \beta_1]....[\alpha_n, \beta_n]$$

where $[\alpha_1, \beta_1] = \alpha_1\beta_1\alpha^{-1}\beta^{-1}$

Proof. Given $n-fold\ torus$ we know that labelling scheme take every vertex to a single point(positive linear map helps). Now we can apply Theorem 1.2. □

Definition 1.2.4. *Let $m > 1$. Consider the space obtained from a 2m-sided polygonal region P in the plane by means of the labelling scheme*

$$(a_1a_1)(a_2a_2)...(a_ma_m)$$

This space is called the ***m-fold connected sum of projective planes****, or simply the* ***m-fold projective plane****, and denoted $P^2\#...\#P^2$.*

The 2-fold projective plane $P^2\#P^2$ is pictured below.

Theorem 1.2.7. *Let X denote the m-fold projective plane. Then $\pi_1(X, x_0)$ is isomorphic to the quotient of the free group on m generators $\alpha_1, \alpha_2, .., \alpha_m$ by the least normal subgroup containing the element*

$$(\alpha_1)^2(\alpha_2)^2..(\alpha_m)^2$$

Proof. As before, observe all vertices are mapped to a single point, then apply Theorem 1.2. □

Qn 1.3. *Find a presentation for the fundamental group of $P^2\#T$.*

Sol: $\pi_1(P^2\#T, x_0) = \langle \alpha, \beta, \gamma | (\alpha\alpha)(\beta\gamma\beta^{-1}\gamma^{-1}) = 1 \rangle$

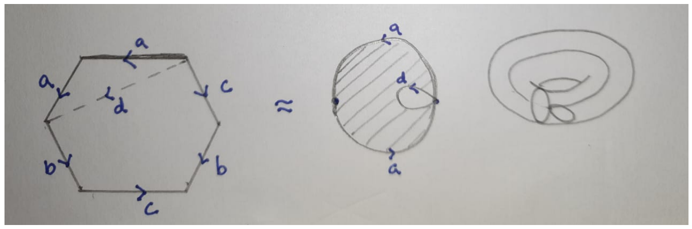

Figure 1.14: Six sided polygon with scheme $a^2bcb^{-1}c^{-1}$ cut along d

1.4 Homology of surfaces

If X is a path-connected space, and if α is a path in X from x_0 to x_1, then there is an isomorphism $\hat{\alpha}\colon \pi_1(X,x_0) \to \pi_1(X,x_1)$, but the isomorphism depends on the choice of the path α.

However, the isomorphism of the "abelianized fundamental group" based at x_0 with the one based at x_1, induced by α, is in fact independent of the choice of the path α.

To verify this fact consider two paths α, β from x_0, x_1, and we show that, for a loop γ based at x_1, $[\alpha\gamma\alpha^{-1}] = \frac{[\beta\gamma\beta^{-1}]}{[\pi_1,\pi_1]}$.

Definition 1.4.1. *If X is a path-connected space, let*

$$H_1(X) = \pi_1(X,x_0)/[\pi_1(X,x_0),\pi_1(X,x_0)]$$

We call $H_1(X)$ the first homology group of X.

Theorem 1.4.1. *Let F be a group and N be a normal subgroup of F. Let $q : F \to F/N$ be the quaotient map. The quotient homomorphism*

$$p : F \to F/[F,F]$$

induces an isomorphism

$$\phi : q(F)/[q(F),q(F)] \to p(F)/p(N)$$

Proof. p, q, r, s, are quotient homomorphisms

$$q : F \to q(F) = F/N, q(f) = f \circ N$$
$$p : F \to p(F) = F/[F,F], p(f) = f \circ [F,F]$$
$$r : p(F) \to p(F)/p(N), r(p(f)) = p(f) \circ p(N)$$

similarly s

$$
\begin{tikzcd}
 & q(F) \arrow[r, "s"] \arrow[dr, "u" near start] & q(F)/[q(F),q(F)] \arrow[d, shift right, "\phi"'] \\
F \arrow[ur, "q"] \arrow[dr, "p"'] & & \\
 & p(F) \arrow[r, "r"'] \arrow[ur, "v" near start] & p(F)/p(N) \arrow[u, shift right, "\psi"']
\end{tikzcd}
$$

Figure 1.15:

$$
\begin{tikzcd}
G \arrow[d, "\pi"'] \arrow[dr, "\phi"] & \\
G/H \arrow[r, "\tilde{\phi}"'] & K
\end{tikzcd}
$$

Figure 1.16: Universal property

Because r○p maps N to 1, it induces a homomorphism u. As p(F)/p(N) is abelian, the homomorphism u induces a homomorphism ϕ of q(F)/[q(F), q(F)]. On the other hand, because $s \circ q$ maps F into an abelian group, it induces a homomorphism $v : p(F) \rightarrow q(F)/[q(F), q(F)]$. Since $s \circ q$ carries N to 1, so does $v \circ p$, hence v induces a homomorphism ψ of p(F)/p(N).

The homomorphism ϕ can be described as follows: Given an element y of the group $q(F)/[q(F), q(F)]$, choose an element x of F such that $s(q(x)) = y$; then

$$\phi(y) = r(p(x)).$$

The homomorphism ϕ can be described similarly. It follows that ϕ and ψ are inverse to each other. □

Put it simply, if one takes quotient of F by N and then abelianizes the quotient, one obtains the same result as if one first abelianizes F and then divides by the image of N in this abelianization.

Corollary 1.4.1.1. *Let F be a free group with free generators $\alpha_1, \alpha_2 ..., \alpha_n$; let N be the least normal subgroup of F containing the element x of F; let G = F/N. Let $p : F \rightarrow F/[F, F]$ be quotient. Then G/[G, G] is isomorphic to the quotient of F/[F,F], which is free abelian with basis $p(\alpha_1), ..., p(\alpha_n)$, by the subgroup generated by p(x).*

Theorem 1.4.2. *If X is the n-fold connected sum of tori, then $H_1(X)$ is a free abelian group of rank 2n.*

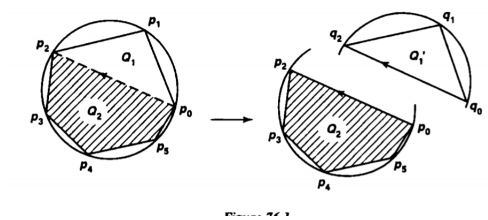

Figure 1.17: Cutting

Proof. Recall $\pi_1(X,x_0)$ is quotient of free group on 2n generators,

say $\alpha_1,\beta_1,\alpha_2,\beta_2...,\alpha_n,\beta_n$, by the subgroup generated by the element $[\alpha_1,\beta_1][\alpha_2,\beta_2]...[\alpha_n,\beta_n]$.

Apply corollary above, so $H_1(X)$ is isomorphic to the quotient of the free abelian group F' on the set $\alpha_1,\beta_1,\alpha_2,\beta_2...,\alpha_n,\beta_n$ by the subgroup generated by the element $[\alpha_1,\beta_1]...[\alpha_n,\beta_n]$, where $[\alpha,\beta]=\alpha\beta\alpha^{-1}\beta^{-1}$ as usual. Because the group F' is abelian, this element equals the identity element. □

Theorem 1.4.3. *If X is the m-fold connected sum of projective planes, then the torsion subgroup $T(X)$ of $H_1(X)$ has order 2, and $H_1(X)/T(X)$ is a free abelian group of rank m-1.*

Proof. Corollary $\implies$ $H_1(X)$ is quotient of free abelian group F' on the set $\alpha_1,\alpha_2,...\alpha_m$ by the subgroup generated by the element $(\alpha_1)^2(\alpha_2)^2...(\alpha_m)^2$

A small trick, switch to additive notation , and let us change bases in the group F'. If we let$\beta=\alpha_1+...+\alpha_m$,then the elements $\alpha_1,..,\alpha_{m-1},\beta$ form a basis for F', any element of F' can be written uniquely in terms of these elements.

The group $H_1(X)$ is isomorphic to the quotient of the free abelian group on $\alpha_1,..,\alpha_{m-1},\beta$ by the subgroup generated by 2β. Said differently, $H_1(X)$ is isomorphic to the quotient of the m-fold cartesian product $\mathbb{Z}\times\mathbb{Z}\times...\times\mathbb{Z}$ by the subgroup $0\times...\times 0\times 2\mathbb{Z}$. The theorem follows. □

1.5 Elementary operations on schemes

The regions Q_1' and Q_2 in figure 1.17 are said to have been obtained by ***cutting*** P apart along the line from p_o to p_k. The region P is homeomorphic to the quotient space of Q_1 and Q_2 obtained by pasting the edge of Q_1' going from q_o to q_k to the edge of Q_2 going from p_o to p_k, by the positive linear map.

Sort of reverse of the above operation is pasting. Given two polygonal regionals we paste them to form a single polygonal region.

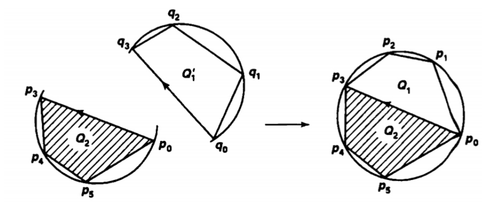

Figure 1.18: Pasting

This task is accomplished as follows: The points of Q_2 lie on a circle and are arranged in counterclockwise fashion. Let us choose points $p_1, ..., p_{k-1}$ on this same circle in such a way that $p_0, p_1, ..., P_{k-1}$, p_k are arranged in counterclockwise order, and let Q_1 be the polygonal region with these as successive vertices. There is a homeomorphism of Q_1' onto Q_1 that carries q_i to p_i, for each i and maps the edge $q_0 q_k$ of Q_1' linearly onto the edge $p_0 p_K$ of Q_2. Therefore, the quotient space in question is homeomorphic to the region P that is the union of Q_1' and Q_2. We say that P is obtained by pasting Q_1' and Q_2 together along the indicated edges.

Theorem 1.5.1. *Suppose X is the space obtained by pasting the edges of m polygonal regions together according to the labelling scheme*

$$(*)\ \ y_0 y_1, w_2, ..., w_m.$$

Let c be a label not appearing in this scheme. If both y_0 and y_1 have length at least two, then X can also be obtained by pasting the edges of m + 1 polygonal regions together according to the scheme

$$(**)\ \ y_0 c^{-1}, c y_1, w_2, ..., w_m.$$

Conversely, if X is the space obtained from m + 1 polygonal regions by means of the scheme (**), it can also be obtained from m polygonal regions by means of the scheme (*), providing that c does not appear in scheme (*).

Elementary operations on scheme

- *Cut*. One can replace the scheme $w_1 = y_0 y_1$ by the scheme $y_0 c^{-1}$ and $c y_1$, provided c does not appear elsewhere in the total scheme and y_0 and y_1 have length at least two.

- *Paste*. One can replace the scheme $y_0 c^{-1}$ and $c y_1$ by the scheme $y_0 y_1$, provided c does not appear elsewhere in the total scheme.

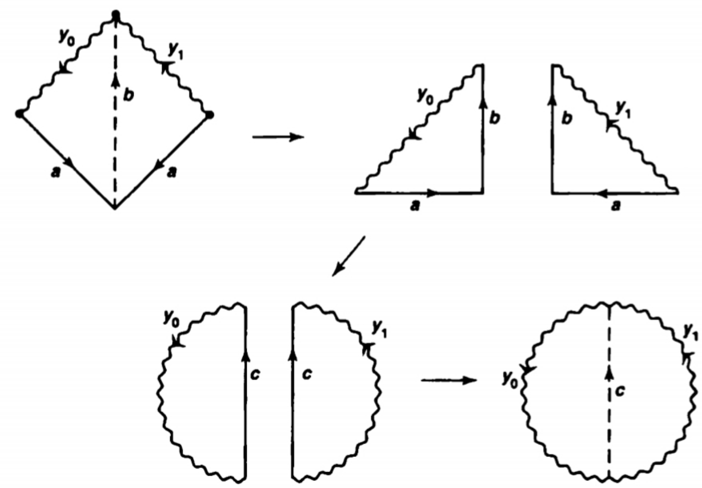

Figure 1.19: Cancel

- *Relabel*. One can replace all occurrences of any given label by some other label that does not appear elsewhere in the scheme. Similarly, one can change the sign of the exponent of all occurrences of a given label a; this amounts to reversing the orientations of all the edges labelled ”a”. Neither of these alterations affects the pasting map.

- *Permute*. One can replace any one of the schemes w, by a cyclic permutation of w. Specifically, if $w_i = y_0 y_1$, we can replace w_i,- by $y_1 y_0$- This amount to renumbering the vertices of the polygonal region P_i, so as to begin with a different vertex; it does not affect the resulting quotient space.

- *Flip*. One can replace the scheme by its inverse

- *Cancel*. One can replace the scheme $w_i = y_0 a a^{-1} y_1$ by the scheme $y_1 y_0$, provided a does not appear elsewhere in the total scheme and both y_0 and y_1 have length at least two.

- *Uncancel*. Reverse of cancel

Definition 1.5.1. *Two labelling schemes for collections of polygonal regions to be equivalent if one can be obtained from the other by a sequence of elementary scheme operations. Since each elementary operation has as its inverse another such operation, this is an equivalence relation.*

Example Klein bottle K is the space obtained from the labelling scheme $aba^{-1}b$. We give a geometric argument showing that K is homeomorphic to the 2-fold projective plane $P^2 \# P^2$.

$$\begin{aligned}
Sol: aba^{-1}b &\longrightarrow abc^{-1} \text{ and} ca^{-1}b && \text{by cutting}\\
&\longrightarrow c^{-1}ab \text{ and } b^{-1}ac^{-1}b && \text{by permuting first and flipping second}\\
&\longrightarrow c^{-1}aac^{-1} && \text{by pasting}\\
&\longrightarrow aacc && \text{by permuting and relabelling}
\end{aligned}$$

1.6 Classification of compact surfaces

We will see some definitions prior to classification theorem.

Definition 1.6.1. *If $w_1, w_2 ..., w_m$ be the labelling schemes of polygonal regions $P_1, P_2, .., P_m$, if each label appears twice in this scheme we call it a **proper** labelling scheme*

Remark. *If one applies any elementary operation to a proper scheme, one obtains another proper scheme.*

Definition 1.6.2. *If w is a proper labelling scheme and if every label is appearing once with exponent +1 and once with -1, we say w is of **torus** type, otherwise it is said to be of **projective** type.*

We will see some useful equivalent schemes, of some general proper labelling schemes. Following series of lemmas helps us in proving classification theorem.

Lemma 1.6.1. *Let ω be a proper scheme of the form*

$$\omega = [y_0]a[y_1]a[y_2],$$

where some of the y_i may be empty. Then one has the equivalence

$$\omega \sim aa[y_0 {y_1}^{-1} y_2].$$

Proof. Step1 Consider the case where y_0 is empty.

We show that

$$a[y_1]a[y_2] \sim \; aa[y_1^{-1}y_2]$$

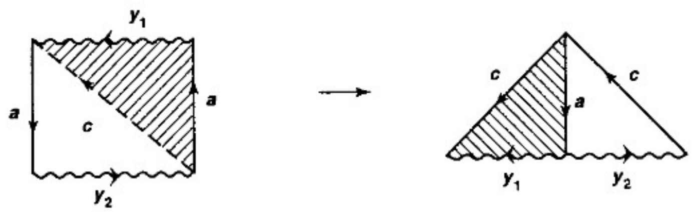

Figure 1.20:

If y_1 is empty then it is trivial otherwise we follow Figure1.19.

$$\begin{aligned} a[y_1]a[y_2] &\sim ay_1c^{-1}\ cay_2 && \text{by cutting} \\ &\sim y_1c^{-1}a\ a^{-1}c^{-1}y_2^{-1} && \text{by permuting 1st and flipping 2nd} \\ &\sim y_1c^{-1}c^{-1}y_1 && \text{by pasting along a} \\ &\sim aay_2^{-1}y_1 && \text{by permuting and relabelling} \end{aligned}$$

Step2 Consider the general case, $\omega = [y_0]a[y_1]a[y_2]$, if both $y_1$and y_2 are empty then permuting gives result. Otherwise we apply cutting and pasting sequence given in Figure1.20 to show that

$$\omega \sim b[y_2]b[y_1y_0^{-1}].$$

Then,

$$\begin{aligned} \omega \sim bb[y_2^{-1}y_1y_0^{-1}] && \text{by step 1} \\ [y_0y_1^{-1}y_2]b^{-1}b^{-1} && \text{by flipping} \\ aa[y_0y_1^{-1}y_2] && \text{by permuting and relabelling} \end{aligned}$$

□

Corollary 1.6.1.1. *Given a projective type scheme ω, it is equivalent to a scheme of same length having the form*

$$(a_1a_1)(a_2a_2)...(a_ka_k)\omega_1,$$

where ω_1 is either torus type or empty.

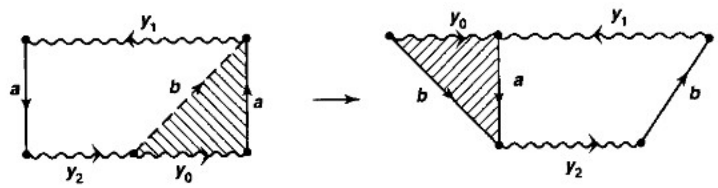

Figure 1.21:

Proof. Let ω be the projective type scheme therefore

$$\omega = [y_0]a[y_1]a[y_2]$$

$$\omega \sim aa\omega_1$$

by applying previous lemma. If ω_1 is of torus type or empty then we are done otherwise ω_1 is of projective type therefore

$$\omega_1 = [z_0]b[z_1]b[z_2],$$

applying previous lemma again $\omega_1 \sim bb\omega'$ if ω' is empty or torus type then we are done otherwise we proceed as before. $\square$

All the results above tells us that if ω is a proper labelling scheme then,ω is of the following type

- Torus type –(1)
- $\omega \sim (a_1a_1)(b_1b_1)...(n_1n_1)\omega_1$, where ω_1 is of torus type–(2)
- $\omega = (a_1a_1)(a_2a_2)..(a_na_n)$–(3)

If it is of type (3) then scheme represents n fold connected sum of projective planes, otherwise the following lemma helps classify the schemes.

Lemma 1.6.2. *Given a proper scheme ω , $\omega = \omega_0\omega_1$ where ω_1 of torus type (after cancellation), then ω is equivalent to $\omega_0\omega_2$, where*

$$\omega_2 = aba^{-1}b^{-1}\omega_3$$

is of this form and ω_3 is of torus type or empty.

Proof. *Step1-* $\omega = \omega_0\omega_1 = \omega_0[y_0]a[y_1]b[y_2]a^{-1}[y_3]b^{-1}[y_4]$

By assumption ω_1 is of torus type(empty case is trivial), so let

$$\omega_1 = [y_0]a[y_1]b[y_2]a^{-1}[y_3]b^{-1}[y_4],$$

by this we avoid cancellation of labels if we use 1.6.1. Also if switch a, a^{-1} and b, b^{-1} then the scheme is similar to the above one and also general in nature.

First cutting and pasting Apply cutting pasting techniques given in fig 1.21 directly so

$$\begin{aligned}
\omega_1 &= [y_0]a[y_1by_2]a^{-1}[y_3b^{-1}y_4] \\
\omega &\sim \omega_0c[y_1]b[y_2]c^{-1}[y_0y_3]b^{-1}[y_4] \\
&\sim \omega_0a[y_1]b[y_2]a^{-1}[y_0y_3]b^{-1}[y_4] \qquad \text{by relabelling.}
\end{aligned}$$

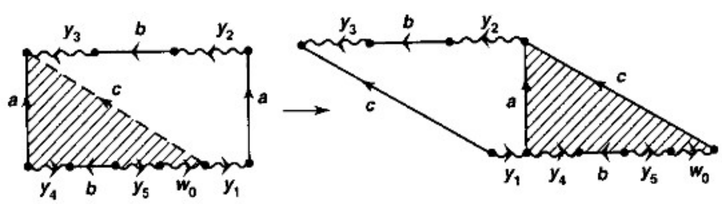

Figure 1.22:

Step 2- second cutting and pasting
Suppose

$$\begin{aligned}
\omega' &= \omega_0a[y_1]b[y_2]a^{-1}[y_0y_3]b^{-1}[y_4] \\
Claim1: \omega' &\sim \omega_0a[y_0y_3y_2]ba^{-1}b^{-1}[y_1y_4] \\
Proof1: If y_4\ y_0y_3 &\text{and} \omega_1 \text{are empty then} \\
\omega' &\sim a[y_1]b[y_2]a^{-1}b^{-1} \\
&= b[y_2]a^{-1}b^{-1}a[y_1] \qquad \text{by permuting} \\
&= a[y_2]ba^{-1}b^{-1}[y_1] \qquad \text{by relabelling.}
\end{aligned}$$

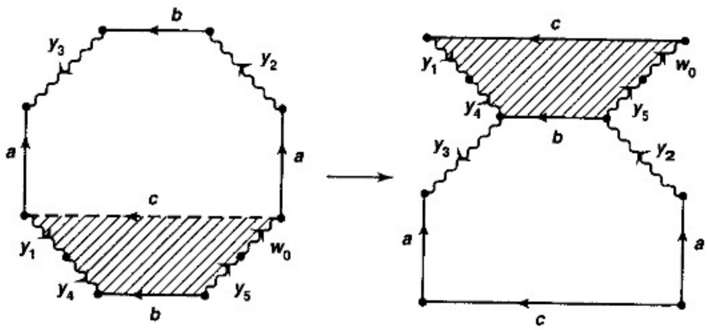

Figure 1.23:

Otherwise we apply the operations indicated in figure1.22

$$\begin{aligned} \omega'' &= \omega_0 a[y_1]b[y_2]a^{-1}[y_0y_3]b^{-1}[y_4] \\ &\sim \omega_0 c[y_0y_3y_2]a^{-1}c^{-1}a[y_1y_4] \\ &\sim \omega_0 a[y_0y_3y_2]ba^{-1}b^{-1}[y_1y_4] \qquad \text{by relabelling.} \end{aligned}$$

Step 3- third cutting and pasting

$$\omega'' = \omega_0 a[y_0y_3y_2]ba^{-1}b^{-1}[y_1y_4].$$

We show that ω'' is equivalent to the scheme

$$\omega''' = \omega_0 aba^{-1}b^{-1}[y_0y_3y_2y_1y_4]$$

as before if the schemes ω_0, y_1y_4 are empty then permuting and relabelling $\Longrightarrow$

$$\omega' = aba^{-1}b^{-1}[y_0y_3y_2]$$

Otherwise we apply the operations indicated in figure1.23

$$\begin{aligned} \omega'' &= \omega_0 a[y_0y_3y_2]ba^{-1}b^{-1}[y_1y_4] \\ &\sim ca^{-1}c^{-1}a[y_0y_3y_2y_1y_4] \\ &\sim ab^{-1}a^{-1}b[y_0y_3y_2y_1y_4] \end{aligned}$$

by relabelling as desired.

$\square$

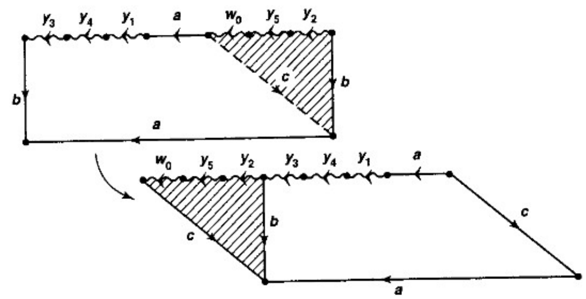

Figure 1.24:

Next lemma shows that connected sum of projective planes and torii is equivalent to connected sum of projective planes alone.

Lemma 1.6.3. *If ω is a proper labelling scheme of the form*

$$\omega = \omega_0 cc(aba^{-1}b^{-1})\omega_1,$$

then ω is equivalent to the scheme

$$\omega' = \omega_0(aabbcc)\omega_1$$

Proof. Recall lemma 1.6.1 for proper schemes we have

$$[y_0]a[y_1]a[y_2 \sim aa[y_0 {y_1}^{-1} y_2]] ----(*)$$

Now,

$$\begin{aligned}
\omega = {} & \omega_0 cc(ab)a^{-1}b^{-1}\omega_1 & \\
\sim {} & cc[ab][ba]^{-1}[\omega_1\omega_0] & \text{by permuting} \\
\sim {} & [ab]c[ba]c[\omega_1\omega_0] & \text{by (*) backwards} \\
\sim {} & [a]b[c]b[ac\omega_1\omega_0] & \\
\sim {} & bb[ac^{-1}ac\omega_1\omega_0] & \text{by (*)} \\
\sim {} & [bb]a[c^{-1}]a[c\omega_1\omega_0] & \text{by (*) backwards} \\
\sim {} & aa[bbcc\omega_1\omega_0] & \\
\sim {} & [\omega_0][aabbcc][\omega_1] & \text{by permuting}
\end{aligned}$$

□

Theorem 1.6.4 (The classification theorem)**.** *If X is the quotient space obtained from a polygonal region by pasting edges togather in pairs then X is homeomorphic to either S^2, n-fold connected sum of torus, or the m-fold connected sum of projective plane.*

Proof. Let ω be the scheme of polygonal region for our space X. So ω is proper scheme of length atleast four, we show it is equivalent to one of the following

1. $aa^{-1}bb^{-1}$
2. $abab$
3. $(a_1a_1)(a_2a_2)..(a_ma_m)$ with $m \geqslant 2$,
4. $(a_1b_1a_1^{-1}b_1^{-1})(a_2b_2a_2^{-1}b_2^{-1})..(a_nb_na_n^{-1}b_n^{-1})$ with $n \geqslant 1$

Step 1. Let ω be a proper scheme of torus type. We claim ω is equivalent to scheme of type (1) or (4).

If ω is of length 4, then ω is in one of the forms

$$aa^{-1}bb^{-1} \text{ or } aba^{-1}b^{-1}.$$

We proceed by induction on length of ω. Assume ω has length greater than 4. By induction any torus type scheme of length 'm' will be of the type

$$(a_1b_1a_1^{-1}b_1^{-1})(a_2b_2a_2^{-1}b_2^{-1})..(a_mb_ma_m^{-1}b_m^{-1}).$$

If ω is of length greater than 'm', then ω does not contain pair of adjacent terms having same label, $\therefore$ we apply Lemma1.6.2 to conclude that ω is equivalent to a scheme having same length as ω, of the form,

$$aba^{-1}b^{-1}\omega_3,$$

ω_3 is of torus type and is non empty since ω is of length $\geqslant 4$. We can apply the Lemma1.6.2 again to conclude the result.

Step 2. Next case is ω is a scheme of projective type.

If ω has length 4 then by Corollary1.6.1.1

$$\omega \sim \text{one of the form } aabb \text{ or } aab^{-1}b.$$

First is of type 3. Second can be written as $aay_1^{-1}y_2$, with $y_1 = y_2 = b$, then Lemma1.6.1 $\implies$ scheme is $\sim$ $ay_1ay_2 = abab$ which is of type 2.

If length of scheme is greater than 4 we proceed in the similar direction as in *Step 1*, by induction. Corollary1.6.1.1 tells us that ω of length greater than 4 is equivalent to a scheme of the form

$$\omega' = (a_1a_1)(a_2a_2)..(a_ka_k)\omega_1,$$

where $k \geqslant 1$ and ω_1 is of torus type or empty. If ω_1 is empty then we are done, otherwise if ω_1 has 2 adjacent terms of same label then induction hypothesis applies, if not Lemma1.6.2 $\implies$ ω' is equivalent to a scheme of the form

$$\omega'' = (a_1a_1)(a_2a_2)..(a_ka_k)aba^{-1}b^{-1}\omega_2,$$

where ω_2 is of torus type or empty. Applying lemma1.6.3 ω'' is equivalent to

$$(a_1a_1)(a_2a_2)..(a_ka_k)aabb\omega_2.$$

Continuing similarly we obtain a scheme of type 3. □

1.7 Constructing compact surfaces

Since we have shown that Classification theorem(for compact surfaces 1.6.4) is applicable for every surfaces that are obtained by pasting the edges together in pairs (with proper labelling schemes) from a polygonal region, we now are required to show that every compact connected surface can be obtained by pasting edges together in pairs of a polygonal region. We show something weaker than this, that is we show that surface under consideration is having a *triangulation*.

Definition 1.7.1. *Consider a compact, Hausdorff space say X. A* ***curved triangle*** *in X is a subspace A of X and a homeomorphism $h : T \to A$, where T is a closed triangular region in the plane. If e is an edge of T, then* ***h(e)*** *is an edge of A and similarly for vertex.*

Definition 1.7.2. *For a compact Hausdorff space X, a* ***triangulation*** *of X is a collection of curved triangles $A_1,A_2A_3...A_n$ in X whose union is X and for $i \neq j$, $A_i \cap A_j$ is either empty, a vertex, or an edge of both. Also,for each i since*

$$h_i : T_i \to A_i$$

is the homeomorphism associated with A_i, if $A_i \cap A_j$ is an edge e of both, then the map $h_j^{-1}h_i$ defines a linear homomorphism of the edge $h_i^{-1}(e)$ of T_i with the edge $h_j^{-1}(e)$ of T_j. If X has a triangulation then X is said to be ***triangulable***

Triangulation of a circle, sphere e.t.c are easy to observe.

Example. *The figure 1.25 given below, is a triangulation of Torus*

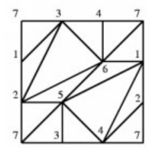

Figure 1.25: Triangulation of torus

Example. *The figure, 1.26 is not a triangulation of torus.*

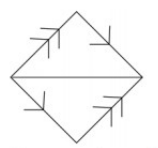

Figure 1.26: Not a triangulation of torus

Theorem 1.7.1. *For a compact triangulable surface X, X is homeomorphic to the quotient space obtained from a collection of disjoint triangular regions in the plane by pasting their edges together in pairs.*

Proof. Let $A_1, A_2, A_3 ... A_n$ be a triangulation of X with h_i as corresponding homeomorphisms. Consider $h : T_1 \cup T_2 \cup ... \cup T_n \to X$, clearly h is continuous since each h_i's are continuous and pasting lemma implies h is continuous. Furthermore, for any closed set C in X, $f^{-1}(C)$ is closed, similarly for any closed set $f^{-1}(C) \subset$ some T_i , continuity of $h_i \implies$ C is closed in X, as h is also surjective, h is a quotient map.Moreover, recall that the map $h_j^{-1} h_i$ defines a linear homeomorphism whenever $A_i \cap A_j$ is an edge e, so h pastes edges of T_i and T_j together.

If $A_i \cap A_j = \phi \ \forall \ i \neq j$ then we have nothing to prove since there are no edge pasting. So we consider the two case when $A_i \cap A_j =$ an edge e or a vertex v.

First we show that if $A_i \cap A_j = v$ some vertex then there exists a sequence of triangles having v as vertex beginning with A_i and ending with A_j, such that intersection of each triangle in the sequence with it's succesor equals an edge common to both. See figure below.

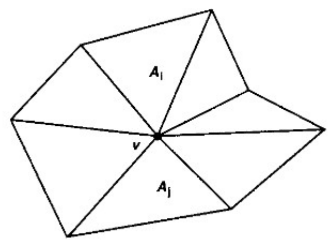

Figure 1.27: Triangular regions of a surface

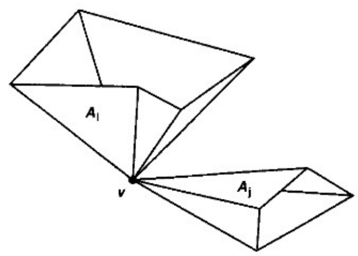

Figure 1.28:

Otherwise, if the situation is as in this figure, figure , we show since X is a surface such a situation cannot happen. Figure actually provides an intuitive idea.

First we define two triangles A_i and A_j to be equivalent if there is a sequence of triangles having a vertex in common beginning with A_i and ending with A_j. Consider two equivalent class call B and C,now $B \cap C = v$ and *for every sufficiently small neighbourhood W of v in X, the space W-v is disconnected*

But, if X is a surface, then v has neighbourhood homeomorphic to open 2-ball $\Longrightarrow$ v has sufficiently small neighbourhood that is connected. This proves the first part.

Next objective to show is that for each edge e of a triangle A_i, there is exactly one another triangle A_j such that $A_i \cap A_j = e$. First we will show the existence, and then show there is exactly one for each A_i.

The existence part is a consequence of the following claim.

Claim: *If X is a triangular region in the plane and if x is a point on the interior to one of the edges of X, then x does not have a neighbourhood homeomorphic to an open 2-ball.*

Proof: Note that x has arbitrarily small neighbourhoods W for which W- x is simply connected, clearly from figure W-x is contractible to a point.

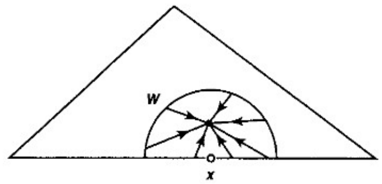

Figure 1.29:

Assume the contrary, so if U is a neighbourhood of x that is homeomorphic to an open ball in $\mathbb{R}^2$, with homeomorphism carrying x to 0. We show that x does not have arbitrarily small neighbourhoods W such that W-x is simply connected.

Let B be the open unit ball in $\mathbb{R}^2$ centered at the origin, and suppose V is any neighbourhood of $\mathbf{0}$ that is contained in B. Choose ε so that the open ball B_ε of radius ε centered at $\mathbf{0}$ lies in V, and consider the inclusion mappings

$$\begin{tikzcd} B_\epsilon - \mathbf{0} \arrow[rr, "i"] \arrow[dr, "j"'] & & B - \mathbf{0} \\ & V - \mathbf{0} \arrow[ur, "k"'] & \end{tikzcd}$$

Figure 1.30:

The inclusion i is homotopic to the homeomorphism $h(x) = x/\varepsilon$, so it induces an isomorphism of fundamental groups. Therefore, k_* is surjective, it follows that *V*-***0*** cannot be simply connected. If it were, it will contradict the inclusion map from B_ε to V and the fact that B_ε is not simply connected.

As we have shown the existence of more than one triangle intersecting an edge, we now show that there is exactly one other triangle, say A_j intersecting edge of another triangle A_i for $i \neq j$. This indeed is the consequence of following result,

Claim: *Let X be the union of k triangles in* $\mathbb{R}^3$*, each pair of which intersect in the common edge e. Let x be an interior point of e. If* $k \geqslant 3$*, then x does not have a neighbourhood in X homeomorphic to an open 2-ball.*

Proof: The idea is to show that there is no neighbourhood W of x in X such that W-x has abelian fundamental group. It follows that no neighbourhood of x is homeomorphic to an open 2-ball.

For this consider A as union of all edges of the triangles of X that are different from e, then we claim that fundamental group of A is not abelian. Consider B as union of 3 arcs that make A, now r is retract of **A** onto B obtained by mapping

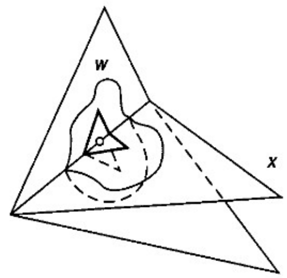

Figure 1.31:

all arc of A not in B homeomorphically to one arc of B, keeping end points fixed. Then r_* is an epimorphism, also since fundamental group of B is not abelian fundamental group of A also ceases to be abelian.

Why is fundamental group of B not abelian?

B is union of 3 arcs, so B is of homotopy type of θ space, we can either use the result that fundamental group of θ space and figure 8 are isomorphic or apply Van-Kampen theorem to show fundametal group is free group on 2 generators and hence not abelian.

In our case A is a deformation retract of X-x and it should follow that fundamental group of$X$-x is not abelian.

For convenience, assume x as the origin of $\mathbb{R}^3$. Let W be a arbitrary neighbourhood of $\nvdash$, then the shrinking map $f(x) = \varepsilon x$ for some $\varepsilon < 1$ carries X to W and the space $X_\varepsilon = f(X)$ is a copy of X lying inside W. Consider the below inclusions as before since i is homotopic to homeomorphism $h(x) = x/\varepsilon$, it induces

$$\begin{tikzcd} X_\epsilon - \mathbf{0} \arrow[rr, "i"] \arrow[dr, "j"'] & & X - \mathbf{0} \\ & W - \mathbf{0} \arrow[ur, "k"'] & \end{tikzcd}$$

Figure 1.32:

isomorphism of fundamental groups.Now k_* is surjective, so fundamental group of W-$\boldsymbol{0}$ cannot be abelian.

Thus we have shown that given an edge e of a triangle A_i there is exactly one other triangle A_j having e as a common edge. Hence the compact triangulable surface X is homeomorphic to the space obtained from pasting edges together by the linear homeomorphism $h_j^{-1}h_i$ in the collection of disjoint triangular regions in the plane. □

Theorem 1.7.2. *For a compact connected triangulable surface X, X is homeomorphic to a space obtained from a polygonal region in the plane by pasting edges together in pairs.*

Proof. From the earlier theorem we have that X is homeomorphic to a collection of disjoint triangular regions say $T_1, T_2, .., T_n$ in the plane. Moreover, since the edges are pasted in pairs by the quotient map the space in plane is having proper labelling scheme.

Application of elementary operations will give us the result. Pasting (flipping also if needed) operation of two triangular regions along the edges bearing same label will give a single four-sided polygonal region, preserving the same orientations and label. The process can be iterated as long as there are two edges with same label. If we are having single polygonal region at the end then the theorem is proved.

Otherwise one has several polygonal regions, no two which has edges bearing same label, but this give rise to a disconnected quotient space, but space X was connected, therefore such a situation cannot occur. □

1.8 Proof that compact 2-manifolds can be triangulated

Theorem 1.8.1 (Jordan-Schoenflies theorem)**.** *A simple closed J curve in E^2 separates E^2 into two regions. There exists a self-homeomorphism of E^2 under which J is mapped onto a circle.*

Definition 1.8.1. *A cellular set K is one that can be written as intersection of 2-cells, E_i.*

$$K = \cap_{i=1}^{\infty} E_i, \;\; E_i \subset int(E_{i-1}).$$

Remark. *If K is a cellular subset of a 2-manifold M, then M/K is homeomorphic to M*

Lemma 1.8.2. *Let M be a closed 2-manifold and let C be a connected subset of M which is a union of n-simple closed curves*

$$C = \cup_{i=1}^{n} C_i.$$

Then a compact totally disconnected set A lies in the interior of a closed 2-cell in M.

Remark. *A* `totally disconneced set` *is characterised by the property that each of its connected subset has a single point.*

So, if γ_1 ,γ_2 are 2 intersecting simple closed curves in a closed 2-manifold, consider an open set U containing both curves, U is locally euclidean. Now, every point on the curves are having 1 dimensional euclidean neighbourhood except at the points where the curves 'cut' one another. Call the set of points were curves `'cut'` as A.By cut it is meant that for some t, $\gamma_1(t) = \gamma_2(t)$, but $\exists$ atleast one $\delta > 0, \gamma_1(t) \neq \gamma_2(t) \forall\, t \in (t-\delta, t+\delta) - \{t\}$,$t \in [0,1]$.

Proposition 1.8.1. *Set of cut points does not have 1 dimensional euclidean neighbourhood*

Proof. Let $a \in A$, any nbd U_a of a will contain points from $\gamma_1 \cup \gamma_2$, choose a nbd $U_{a'}$ which does not contain any $a' \in A$ such that $a' \neq a$. Let t_1, t_2 be the respective time when $\gamma_1(t_1) = a = \gamma_2(t_2)$,
let $U_{a'} = \gamma_1(t_1-\delta_1, t_1+\delta_1) \cup \gamma_2(t_2-\delta_2, t_2+\delta_2)$, if $U_{a'}$ was 1 dimensional then $U_{a'}$ must be homeomorphic to (0,1)(say f is the homeomorphism), hence $U_{a'} - a'$ and (0,1)- $f(a')$ must have same number of connected components. $U_{a'} - a'$ is having 4 connected components while (0,1)- $f(a')$ is having 2 connected components. Therefore any $a \in A$ is not having 1 dim euclidean nbd. □

If 2 curves are `'touching'` each other, i.e $\gamma_1(t) = \gamma_2(t) \mid \exists$ atleast one $\delta > 0, \gamma_1(t) = \gamma_2(t) \forall\, t \in (t-\delta, t+\delta)$, clearly all such points were the curves are touching, say $\{T\}$ posses 1 dim euclidean nbd, $\therefore$ any subset of $\{T\}$ is connected.

Proposition 1.8.2. *If γ_1 ,γ_2 are 2 intersecting simple closed curves in a closed 2-manifold, then the set containing the 'cut' points, say A, is totally disconnected.*

Proof. We have already shown that no point of A is having 1 dim euclidean nbd. Suppose for some $A_1 \subset A$ with atleast two points, is connected. Then $\because A_1 \subset \gamma_1 \cup \gamma_2$, A_1 will be path connected $\implies$ points of A_1 posses 1 dim euclidean nbd, meaning they are touching hence implying they are not in A_1. Therefore any connected subset of A will be singleton, implying A is totally disconnected. □

Observe, that the totally disconnected set A can be finite or infinite.

Theorem 1.8.3. *Any closed 2-manifold M can be triangulated[1]*

Proof. Proof is attempted in 3 steps.

Recall: A closed manifold is a compact manifold without boundary.

- Step 1: Choose an irreducible cover for the manifold M.
- Step 2: Proceed to $M_1 = M/D$; it does not contain any self intersecting curves from C.*All the intersecting simple closed curves in M are now a one point union of simple closed curves in* M_1.
- Step 3: Change to $M_2 = M_1/T$ so as to obtain finitely many one point union of simple closed curves(*r-leafed rose*) in $intM_2$, then a triangulable manifold N with boundary is obtained, now extend the triangulation to whole manifold.

Step 1

Cover M irreducibly (no proper subset of these cover M)by a finite collection of closed disks say $B_1, B_2, .., B_n$.Put $C_i = Bd(B_i)$, let A be the set of cut points in C, apply lemma, so $A \subset D$. [Use homotopy of curves, $f(t,s) = (1-t)\gamma_1(s) + t\gamma_2(s)$, to say one can shift from infinite set of cut points A_∞ to a finite/countable set of cut points A_n][If two s.c curves γ_1 and γ_2 intersect at two points, then we can find 3 s.c curves with non-intersecting interiors]Observe $M - C$ is disjoint union of open 2-cells, consequence of Jordan-Schoenflies theorem, and $C - D$ is collection of countably many disjoint arcs.

Step 2

Move to $M_1 = M/D$, recall 'Remark', let $R = im\overline{(C-D)}$ under quotient map, therefore all arcs in $C - D$ becomes one point union of simple closed curves in M_1

Claim- R is one point union of a countable collection of simple closed curves

Proof- R $\subset M_1$, and $M_1 = M/D$. For each disjoint arc, say a_i in $C - D$, end-points of each arc is in D, so $\overline{a_i}$ is a compact set, let p_i andq_i be the end points of a_i, then in M/D every point in D goes to a point say p. Therefore $\forall$ i p_i, q_i is mapped to p, this shows that each a_i is now closed loop around p in M_1.

Any 2 cell nbd V of p will contain all but a finite number of the simple closed curves which comprise R. If not, then for any arbitrary open cover $V_\alpha$of R if one chooses a finite subcover $\{V_1, ..., V_n\}$, each of these V_i will contain only finitely many simple closed curves and hence R will be a union of finite union of finite curves. A contradiction to the fact that R is a countable union of s.c curves.

Step 3

Pick a cellular set T containing V, move to M_1/T if necessary, in M_1/T $im(R)$ is an r-leafed rose. Moreover, the complement of R in M_1/T is composed of finitely many components that are open 2-cells. Pick a small 2-cell, E, enclosing p, such that each simple closed curve of R meets boundary of E at two points. $E \cup R$ is now disk with finitely many closed arcs say, $A_1, A_2, .., A_r$. Each A_i may be enclosed in interior of a closed disk meeting E in a pair of arcs on its boundary. Choosing pair wise disjoint disks one obtains a triangulable manifiold N. Next is to extend the triangulation of N to the whole manifold.(paper by M A Armstrong)

Extension is possible using the following corollary from the paper by *M A Armstrong*

Corollary 1.8.3.1. *Any triangulation of a compact PL-manifold can be extended to the whole manifold.*

□

Chapter 2

A brief revision of basic Riemannian Geometry

2.1 Introduction

A revision on $Riemannian\ metric$, $Riemannian\ isometry$, $Levi-Civita\ connection$,the notion of $parallelism$ of vectors along curves, $geodesics$ and $Hopf-Rinow$ theorem are done very briefly in this chapter.

2.2 Riemannian metric and isometry

Definition 2.2.1. *A tensor $g \in \mathscr{T}^2(T_p^*M)$ is said to be*

1. **symmetric** *if $g(v,w) = g(w,v)$ for all $v, w \in T_pM$*
2. **nondegenerate** *if $g(v,w) = 0$ for all $w \in T_pM \implies v = 0$*
3. **positive definite** *if $g(v,v) = 0$ for all $v \in T_pM \setminus \{0\}$*

A covariant 2-tensor field g is said to be symmetric, positive definite or non-degenerate if g_p is symmetric, positive definite or non-degenerate $\forall p \in M$. If

$x\colon U \to \mathbb{R}^n$ is a local chart, we have

$$g = \sum_{i,j=1}^{n} g_{ij} dx^i \otimes dx^j$$

$$\text{in U, where } g_{ij} = g(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j})$$

It is easy to see that g is symmetric, positive definite or non-degenerate iff the matrix (g_{ij}) has these properties.

Definition 2.2.2. *A* **Riemannian metric** *on a smooth manifold M is a symmetric, positive definite smooth covariant 2-tensor field g. A smooth manifold M equipped with a Riemannian metric g is called a* **Riemannian manifold***, and is denoted by (M,g)*

Hence a Riemannian metric is a smooth assignment of an inner product to each tangent space.

$$g_p(v,w) = \langle v,w \rangle_p.$$

Proposition 2.2.1. *Let (N,g) be a Riemannian manifold and $f\colon M \to N$ be an immersion. Then f^*g is a Riemannian metric(called the* **induced metric***).*

Proof. Let $p \in M$ and let $v,w \in T_pM$, define

$$(f^*g)_p(v,w) = g_p(df_p(v), df_p(w)).$$

Since g is symmetric we have (f^*g) also satisfying this property. For positive definiteness, $(f^*g)p$ is clearly $\geqslant 0$.
If $(f^*g)p = 0 \implies g_p(df_p(v), df_p(w)) = 0 \implies df_p(v) = 0 \implies v = 0$ (as f is an immersion $\implies df_p(v)$ is injective.) □

Definition 2.2.3. *Let (M,g) and (N,h) be Riemannian manifolds. A diffeomorphism $f\colon M \to N$ is said to be an* **isometry** *if $f^*h = g$. Similarly, a local diffeomorphism $f'\colon U \subset M \to V \subset N$ is said to be a* **local isometry** *if $f^*h = g$.*

2.3 Affine connection

Given vector fields in Euclidean space, we can define the *directional derivative* $\nabla_X Y$ of Y along X. Connection helps us to extend this concept to an arbitrary smooth manifold.

Definition 2.3.1. *An* **affine connection** *on a smooth manifold M is a map $\nabla\colon \mathscr{X}(M) \times \mathscr{X}(M) \to \mathscr{X}(M)$ such that,*

1. $\nabla_{fX+gY} Z = f\nabla_X Z + g\nabla_Y Z$
2. $\nabla_X(Y+Z) = \nabla_X Y + \nabla_X Z$
3. $\nabla_X(fY) = (Xf)Y + f\nabla_X Y$

for all $X, Y, Z \in \mathscr{X}(M)$ and $f, g \in C^\infty(M,\mathbb{R})$

Proposition 2.3.1. *Let ∇ be a connection on a differentiable manifold M, let $X, Y \in \mathscr{X}(M)$ with $p \in M$. Then $(\nabla_X Y)_p \in T_pM$ depends only on X_p and on the values of Y along a curve tangent to X at p. Moreover, if $x\colon W \to \mathbb{R}^n$ are local coordinates on some open set $W \subset M$ and*

$$X = \sum_{i=1}^{n} x^i \frac{\partial}{\partial x^i}, Y = \sum_{i=1}^{n} y^i \frac{\partial}{\partial x^i}$$

on this set we have,

$$\nabla_X Y = \sum_{i=1}^{n}(X \cdot y^i + \sum_{j,k=1}^{n} \Gamma^i_{jk} x^j y^k)\frac{\partial}{\partial x^i}$$

where the differentiable functions $\Gamma^i_{jk}\colon W \to \mathbb{R}$, called the **Christoffel symbols** *are defined by*

$$\nabla_{\frac{\partial}{\partial x^j}} \frac{\partial}{\partial x^k} = \sum_{i=1}^{n} \Gamma^i_{jk} \frac{\partial}{\partial x^i} \tag{2.1}$$

Proof. Observe that affine connection is local, i.e, if $X, Y \in \mathscr{X}(M)$ coincide with $\tilde{X}, \tilde{Y} \in \mathscr{X}(M)$ in some open set $W \subset M$ then $\nabla_X Y = \nabla_{\tilde{X}} \tilde{Y}$ on W.

Let W be a coordinate neighbourhood for the local coordinates $x\colon W \to \mathbb{R}^n$, and define the Christoffel symbols associated with these local coordinates through

2.1, so we have :

$$\begin{aligned}
\nabla_X Y &= \nabla_{(\sum_{i=1}^n x^i \frac{\partial}{\partial x^i})}(\sum_{j=1}^n y^j \frac{\partial}{\partial x^j}) \\
\text{by properties of } \nabla \text{in2.3.1, } \nabla_X Y &= \sum_{i=1}^n x^i (\nabla_{\frac{\partial}{\partial x^i}}(\sum_{j=1}^n y^j \frac{\partial}{\partial x^j})) \\
&= \sum_{i=1}^n x^i (\sum_{j=1}^n (\frac{\partial}{\partial x^i}(y^j)\frac{\partial}{\partial x^j} + y_j \nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j})) \\
\text{from 2.1, we know} \nabla_{\frac{\partial}{\partial x^j}} \frac{\partial}{\partial x^k} &= \sum_{i=1}^n \Gamma^i_{jk} \frac{\partial}{\partial x^i} \\
\therefore \nabla_X Y &= \sum_{i=1}^n x^i (\sum_{j=1}^n \frac{\partial}{\partial x^i}(y^j)\frac{\partial}{\partial x^j} + \sum_{j=1}^n (y_j \nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j})) \\
&= \sum_{i=1}^n x^i (\sum_{j=1}^n \frac{\partial}{\partial x^i}(y^j)\frac{\partial}{\partial x^j} + \sum_{j=1}^n (y_j \sum_{k=1}^n \Gamma^k_{ij} \frac{\partial}{\partial x^k})) \\
\text{reindexing we have} &= \sum_{i=1}^n x^i \sum_{j=1}^n \frac{\partial}{\partial x^i}(y^j)\frac{\partial}{\partial x^j} + \sum_{j=1}^n (y_j \sum_{k=1}^n \Gamma^k_{ij} \frac{\partial}{\partial x^k}) \\
&= \sum_{i=1}^n \sum_{j=1}^n x^i \frac{\partial}{\partial x^i}(y^j)\frac{\partial}{\partial x^j} + \sum_{j=1}^n (y^j x^i \sum_{k=1}^n \Gamma^k_{ij} \frac{\partial}{\partial x^j}) \\
&= \sum_{j=1}^n (\sum_{i=1}^n x^i \frac{\partial}{\partial x^i}(y^j) + \sum_{j=1}^n (y^j x^i \sum_{k=1}^n \Gamma^k_{ij})) \frac{\partial}{\partial x^j} \\
&= \sum_{j=1}^n (X(y^j) + \sum_{j=1}^n (y^j x^i \sum_{k=1}^n \Gamma^k_{ij})) \frac{\partial}{\partial x^j}
\end{aligned}$$

This formula shows that $(\nabla_X Y)_p$ depends only on $x^i(p), y^i(p)$ and $(X \cdot y^i)(p)$. Moreover, $x^i i(p)$ and $y^i(p)$ depends only on X_p and Y_p, and $(X \cdot y^i)(p) = \frac{d}{dt} y^j(c(t))_{|t=0}$ depends only on the values of y^i or Y along a curve c whose tangent vector at $p = c(0)$ is X_p. □

Definition 2.3.2. *Consider a curve $c\colon I \to M$, where I is the unit interval and M is a smooth manifold. If V is a vector field defined along the differentiable curve $c\colon I \to M$ with $\dot{c} \neq 0$, its* **covariant derivative** *along c is the vector field defined* *along c given by*

$$\frac{DV}{dt}(t):\ = \nabla_{\dot{c}(t)}V = (\nabla_X Y)_{c(t)} \tag{2.2}$$

for any vector fields $X,Y \in \mathscr{X}(M)$ such that $X_{c(t)} = \dot{c}(t)$ and $Y_{c(s)} = V(s)$ with $s \in (t-\varepsilon, t+\varepsilon)$ for some $\varepsilon > 0$.

Note that for a curve $c\colon I \to M$ avector field defined along a differentiable curve is a differentiable map $V\colon I \to TM$ such that $V(t) \in T_{c(t)}M$ $\forall t \in I$. Also, proposition 2.3.1 tells us that $(\nabla_X Y)_(c(t))$ does not depend on the choice of X,Y, so in local coordinates we have $x^i(t):\ = x^i(c(t))$ and

$$V(t) = \sum_{i=1}^{n} V^i(t)\left(\frac{\partial}{\partial x^i}\right)_{c(t)},$$

then

$$\frac{DV}{dt}(t) = \sum_{i=1}^{n}\left(\dot{V}^i(t) + \sum_{j,k=1}^{n}\Gamma_{ij}^{k}(c(t))\dot{x}^j(t)V^k(t)\right)\left(\frac{\partial}{\partial x^i}_{c(t)}\right) \tag{2.3}$$

Definition 2.3.3. *A vector field V defined along a differentiable curve $c\colon I \to M$ is said to be* **parallel** *along c if*

$$\frac{DV}{dt}(t) = 0$$

for all $t \in I$. The curve c is called a **geodesic** *of the connection ∇ if $\dot{c}$ is parallel along c , i.e if*

$$\frac{D\dot{c}}{dt}(t) = 0\ \forall t \in I.$$

In local coordinates $x\colon W \to \mathbb{R}^n$, the condition for V to be parallel along c is clear if we let equation 2.3 = 0, i.e

$$\sum_{i=1}^{n}\left(\dot{V}^i(t) + \sum_{j,k=1}^{n}\Gamma_{jk}^{i}(c(t))\dot{x}^j(t)V^k(t)\right)$$

for each i, this represent a system of first-order linear ODE's for the components of V. We take it for granted that using Picard-Lindelof theorem, together with the global existence theorem for linear ODE's, given a curve $c\colon I \to M$, a point

$p \in c(I)$ and a vector $v \in T_pM$, there exists a unique vector field $V : I \to TM$ parallel along c such that $V(0) = v$, this is called the **parallel transport** of v along c.

Also geodesic equations are

$$\ddot{x}^i + \sum_{j,k=1}^{n} \Gamma^i_{jk}(c(t))\dot{x}^j(t)\dot{x}^k(t) \ (i = 1,...,n)$$

2.4 Levi-Civita connection

In the case of Riemannian manifold, there is a particular choice of connection, called the **Levi** – **Civita Connection**, with special geometric properties.

Definition 2.4.1. *A connection ∇ in a Riemannian manifold $(M, \langle .,. \rangle)$is said to be compatible with the metric if*

$$X \cdot \langle Y, Z \rangle = \langle \nabla_X Y, Z \rangle + \langle Y, \nabla_X Z \rangle$$

for all $X, Y, Z \in \mathscr{X}(M)$.

If ∇ is compatible with the metric, then the inner product of two vectors fields V_1 and V_2, parallel along a curve, is constant along the curve :

$$\frac{d}{dt}\langle V_1(t), V_2(t) \rangle = \langle \nabla_{\dot{c(t)}} V_1(t), V_2(t) \rangle + \langle V_1(t), \nabla_{\dot{c(t)}} V_2(t) \rangle = 0.$$

In particular, parallel transport preserves lengths of vectors and angles between vectors. Therefore, if $c \colon I \to M$ is ageodesic, then $\|\dot{c(t)}\| = k$ is constant. If $a \in I$, the length s of the geodesic between a and t is

$$s = \int_a^t \|\dot{c}(v)\| dv = \int_a^t k dv = k(t-a).$$

Theorem 2.4.1 (Levi-Civita)**.** *If $(M, \langle .,. \rangle)$ is a Riemannian manifold then there exists a unique connection ∇ on M which is symmetric and compatible with $\langle .,. \rangle$. In local coordinates $(x^1, x^2, \ldots, x^n)$, the christoffel symbols for this connection are*

$$\Gamma^i_{jk} = \frac{1}{2}\sum_{l=1}^{n} g^{il}\left(\frac{\partial g_{kl}}{\partial x^j} + \frac{\partial g_{jl}}{\partial x^k} + \frac{\partial g_{jk}}{\partial x^l}\right)$$

where $(g^{ij}) = (g_{ij})^{-1}$.

2.5 Minimising properties of geodesics

Let M be a differentiable manifold with an affine connection ∇. Given a point $p \in M$ and a tangent vector $v \in T_pM$, there exists a unique geodesic $c_v\colon I \to M$, defined on a maximal open interval $I \subset \mathbb{R}$, such that $0 \in I, c_v(0) = p$ and $\dot{c_v}(0) = v$. Consider now the curve $\gamma\colon J \to M$ defined by $\gamma(t) = c_v(at)$, where $a \in \mathbb{R}$ and J is the inverse image of I by the map $t \to at$. So,

$$\dot{\gamma}(t) = a\dot{c_v}(at), \tag{2.4}$$

and consequently

$$\nabla_{\dot{\gamma}}\dot{\gamma} = \nabla_{a\dot{c_v}}(a\dot{c_v}) = a^2\nabla_{\dot{c_v}}\dot{c_v} = 0 \tag{2.5}$$

Thus γ is also a geodesic. Since $\gamma(0) = c_v(0) = p$ and $\dot{\gamma}(0) = a\dot{c_v}(0) = av$, we see that γ is the unique geodesic with initial velocity $av \in T_pM$(that is, $\gamma = c_{av}$). Therefore, we have $c_{av}(t) = c_v(at)\forall t \in I$. This property is referred to as the **homogeneity** of geodesics. Observe that one can make the interval J arbitrarily large by making a sufficiently small. If $1 \in I$, we define $exp_p(v)$ for v in some open neighbourhood U of the origin in T_pM. The map $exp_p\colon U \subset T_pM \to M$ thus obtained is called the **exponential map** at p.

Proposition 2.5.1. *There exists an open set $U \subset T_pM$ containing the origin such that $exp_p\colon U \to M$ is a diffeomorphism onto some open set $V \subset M$ containing p(called a* **normal neighbourhood***).*

Proof. We assume the fact that "the exponential map is differentiable". If $v \in T_pM$ is such that $exp_p(v)$ is defined, we have , by homogeneity, that $exp_p(tv) = c_{tv}(1) = c_v(t)$. Consequently,

$$(dexp_p)_0 v = \frac{d}{dt}exp_p(tv)_{|t=0} = \frac{d}{dt}c_v(t)_{|t=0} = v.$$

We conclude that $(dexp_p)_0)\colon T_o(T_pM) \approxeq T_pM \to T_pM$ is the identity map. By the inverse function theorem, exp_p is then a diffeomorphism of some open neighbourhood U of $0 \in T_pM$ onto some open set $V \subset M$ containing $p = exp_p(0)$. $\square$

Example. *Consider the Levi-civita connection in S^2 with the standard metric, and let $p \in S^2$. Then $exp_p(v)$ is well defined for all $v \in T_pS^2$, but it is not a diffeomorphism, as it is clearly not injective. However, its restriction to the open ball $B_\pi(0) \subset T_pS^2$ is a diffeomorphis, onto $S^2 - p$.*

Now let (M, $\langle .,. \rangle$) be a Riemannian manifold and ∇ its levi-civita connection. Since $\langle .,. \rangle$ defines an inner product in T_pM, we can think of T_pM as the Euclidean $n-$ space $\mathbb{R}^n$. Let E be the vector field defioned on T_pM $\{0\}$ by

$$E_v = \frac{v}{||v||},$$

and define $X := (exp_p)_* E$ on V $\{p\}$, where $V \subset M$ is a normal neighbourhood.

We have

$$\begin{aligned} X_{exp_p(v)} &= (dexp_p)_v)E_v = \frac{d}{dt} exp_p \left(v + t\frac{v}{||v||} \right)_{t=0} \\ &= \frac{d}{dt} c_v \left(1 + \frac{t}{||v||} \right)_{t=0} = \frac{1}{||v||} \dot{c}_v(1). \end{aligned}$$

Since $||\dot{c}_v(1)|| = ||\dot{c}_v(0)|| = ||v||$, we see that $X_{exp_p}(v)$ is the unit tangent vector to the geodesic c_v. In particular, X must satisfy

$$\nabla_X X = 0.$$

For each $\varepsilon > 0$ such that $\overline{B_\varepsilon(0)} \subset U := {exp_p}^{-1}(V)$, we define the **normal ball** with center p and radius ε as the open set $B_\varepsilon(p) := exp_p(B_\varepsilon(0))$, and the **normal sphere** of radius ε centered at p as the compact submanifold $S_\varepsilon(p) := exp_p(\partial B_\varepsilon(0))$. We will now prove that X is, and hence the geodesics through p are, orthogonal to normal spheres. For that, we choose a local parametrization $\phi : W \subset \mathbb{R}^{n-1} \to S^{n-1} \subset T_pM$, and use it to define a parametrisation $\tilde{\phi}$: $(0, +\infty) \times W \subset \mathbb{R}^{n-1} \to T_pM$ through

$$\tilde{\phi}(r, \theta^1, \ldots, \theta^{n-1}) = r\phi(\theta^1, \ldots, \theta^{n-1})$$

(hence $(r, \theta^1, \ldots, \theta^{n-1})$ are spherical coordinates on T_pM).
Note that

$$\frac{\partial}{\partial r} = E,$$

since,

$$E_{\tilde{\phi}}(r, \theta) = E_{r\phi(\theta)} = \phi(\theta) = \frac{\partial \tilde{\phi}}{\partial r}(r, \theta),$$

and so

$$X = (exp_p)_* \frac{\partial}{\partial r}. \tag{2.6}$$

Since $\frac{\partial}{\partial \theta^i}$ is tangent to r, where r= ε, the vector fields

$$Y_i := (exp_p)_* \frac{\partial}{\partial \theta^i} \tag{2.7}$$

are tangent to $S_\varepsilon(p)$. Notice also that $\|\frac{\partial}{\partial \theta^i}\| = \|\frac{\partial \tilde{\phi}}{\partial \theta^i}\| = r\|\frac{\partial \phi}{\partial \theta^i}\|$ is proportional to r, and consequently $\frac{\partial}{\partial \theta^i} \to 0$ as $r \to 0$, implying that $(Y_i)_q \to 0_p$ as $q \to p$. Since exp_p is a local diffeomorphism, the vector fields X and Y_i are linearly independent at each point. Also,

$$[X_i, Y_i] = \left[(exp_p)_* \frac{\partial}{\partial r}, (exp_p)_* \frac{\partial}{\partial \theta^i}\right] = (exp_p)_* \left[\frac{\partial}{\partial r}, \frac{\partial}{\partial \theta^i}\right] = 0$$

or, since the Levi-Civita connection is symmetric,

$$\nabla_X Y_i = \nabla_{Y_i} X.$$

To prove that X is orthogonal to the normal spheres $S_\varepsilon(p)$, we show that $X$is orthogonal to each of the vector fields Y_i. In fact, since $\nabla_X X = 0$ and $\|X\| = 1$, we have

$$X \cdot \langle X, Y_i \rangle = \langle \nabla_X X, Y_i \rangle + \langle X, \nabla_X Y_i \rangle = \langle X, \nabla_{Y\,i} X \rangle = \frac{1}{2} Y_i \cdot \langle X, X \rangle = 0$$

and hence $\langle X, Y_i \rangle$ is constant along each geodesic through p. Consequently,

$$\langle X, Y_i \rangle (exp_p v) = \langle X_{exp_p v}, (Y_i)_{exp_p v} \rangle = \lim_{t \to 0} \langle X_{exp_p v}, (Y_i)_{exp_p v} \rangle = 0$$

(as $\|X\| = 1$ and $(Y_i)_q \to 0_p$ as $q \to p$), and so every geodesic through p is orthogonal to all normal spheres centered at p.

The current result helps us in deducing the following proposition.

Definition 2.5.1. *A normal neighbourhood $V \subset M$ is called a* **totally normal neighbourhood** *if $\exists\ \varepsilon > 0$ such that $V \subset B_\varepsilon(p)\ \forall p \in V$.*

2.6 Hopf-Rinow theorem

Definition 2.6.1. *A Riemannian manifold (M, $\langle .,. \rangle$) is said to be* **geodesically complete** *if, $\forall p \in M$, the map exp_p is defined in T_pM.*

Theorem 2.6.1 (Hopf-Rinow theorem)**.** *Let (M, $\langle .,. \rangle$) be a connected Riemannian manifold and $p \in M$. The following assertions are equivalent* :

1. *M is geodesically complete,*
2. *(M,d) is a complete metric space,*
3. *exp_p is defined in T_pM.*

Moreover, if (M, $\langle .,. \rangle$) is geodesically complete then $\forall q \in M$ $\exists$ a geodesic c connecting p to q with l(c)=d(p,q).

For proof refer [5]

Chapter 3

Fundamental theorem of surface theory

3.1 Introduction

In this chapter the discussion will be mainly about certain properties of immersed manifolds that will help us in proving few important theorems like Gauss Theorema Egregium, Synge's Inequality, Weingarten's equations and Codazzi- Mainardi Equations. All these will help us in proving the Fundamental theorem on Surface theory by *Bonnet*.

3.2 Theorema Egregium

Few theorems on immersed manifolds will prepare us for *Gauss Theorema Egregium* .

An *immersion* i is a smooth map from $M^n \to N^m$, where M, N are smooth manifolds, where $M \subseteq N$, such that rank of i is *'n'*.

Remark. *If N is endowed with a Riemannian metric (N, $\langle\rangle$), then M has the induced Riemannian metric (M, $i_*\langle\rangle$),*

Theorem 3.2.1. *Let $i : M \to N$ be an immersion. Suppose N has a Riemannian connection ∇' (N, $\langle\rangle$), and M has the induced Riemannian connection ∇ (M, $i_*\langle\rangle$), then if p is a point in a neighbourhood U of M, X is a vectorfield $\in M_p$, and Y is a vector field which is everywhere tangent to M then ,*

$$\nabla_{X_p} Y = \top(\nabla'_{X_p} Y)$$

Definition 3.2.1. *Given a manifold M with a point p, there is a well defined tensor field 's', such that* $s : M_p \times M_p \to M_p^{\perp}$ *for each* $p \in M$*, such that*

$$s(X_p, Y_p) = \perp(\nabla'_{X_p} Y)$$

for any vector field Y extending Y_p*.*

Theorem 3.2.2. *The tensor s is symmetric.*

Proof. Let X and Y be extensions of $X_p, Y_p \in M_p$ to all of N which are tangent to M at all points of M. Then

$$\begin{aligned}\perp(\nabla'_{X_p} Y) - \perp(\nabla'_{Y_p} X) &= \perp(\nabla'_{X_p} Y - \nabla'_{Y_p} X) \\ &= \perp(\nabla'_X Y(p) - \nabla'_Y X(p)) \\ &= \perp([X,Y](p))\end{aligned}$$

Since $[X,Y]$ is tangent to M at all points of M $\perp([X,Y](p)) = 0$, hence 's' is symmetric tensor. □

Combining the above two theorems we have the following decomposition

$$\nabla'_{X_p} Y = \perp \nabla'_{X_p} Y + \top \nabla'_{X_p} Y \tag{3.1}$$

which can be further written as

$$\boxed{\begin{array}{l}\text{The Gauss Formulas :} \\ \quad \nabla'_{X_p} Y = \nabla_{X_p} Y \ + \quad s(X_p, Y_p),\end{array}}$$

Theorem 3.2.3 (Theorema Egregium)**.** *Let M be isometrically immersed in N, and let R and* R' *denote curvature tensors of M and N respectively. Then for all* $X_p, Y_p, Z_p, W_p \in M_p$ *we have*

$$\langle R'(X_p,Y_p)Z_p, W_p\rangle = \langle R(X_p,Y_p)Z_p, W_p\rangle + \langle s(X_p,Z_p), s(Y_p,W_p)\rangle - \langle s(Y_p,Z_p), s(X_p,W_p)\rangle$$

Proof. Extend X_p, Y_p, Z_p, W_p to vector fields X, Y, Z, W which are tangent along M. Then Gauss formulas yield

$$\begin{aligned}(a) \ \ \nabla'_X(\nabla'_Y Z) &= \nabla'_X(\nabla_Y Z) + \nabla'_X(s(Y,Z)) \\ &= \nabla_X(\nabla_Y Z) + s(X, \nabla_Y Z) + \nabla'_X(s(Y,Z)\end{aligned}$$

and similarly

$$(b)\ \ \nabla'_Y(\nabla'_X Z) = \nabla_Y(\nabla_X Z) + s(Y, \nabla_X Z) + \nabla'_Y(s(X,Z)$$

as well as

$$(c)\quad \nabla'_{[X,Y]}Z = \nabla_{[X,Y]}Z + s([X,Y],Z)$$

Recall the formula for curvature tensor,

$$R'(X,Y)Z = \nabla'_X\nabla'_Y Z - \nabla'_Y\nabla'_X Z - \nabla'_{[X,Y]}Z,$$

now substitute (a)(,b),(c) into this formula, we get

$$\begin{aligned}
\langle R'(X,Y)Z,W\rangle &= \langle \nabla'_X\nabla'_Y Z - \nabla'_Y\nabla'_X Z - \nabla'_{[X,Y]}Z, W\rangle \\
&= \langle \nabla'_X\nabla'_Y Z, W\rangle - \langle \nabla'_Y\nabla'_X Z, W\rangle - \langle \nabla'_{[X,Y]}Z, W\rangle \\
applying\ Gauss\ formulas &= \{\langle \nabla_X(\nabla_Y Z + s(X,\nabla y Z) + \nabla'_X s(Y,Z)), W\rangle \\
&\quad - \{\langle \nabla_Y(\nabla_X Z), W\rangle + \langle s(Y,\nabla_X Z), W\rangle \\
&\quad + \langle \nabla'_Y s(X,Z), W\rangle\} - \{\langle \nabla_{[X,Y]}Z, W\rangle + \langle s([X,Y],Z), W\rangle\}\} \\
&\because \{1\} - \langle s(,), W\rangle = 0 \text{ the equation will be reduced as follows} \\
&= \{\langle \nabla_X(\nabla_Y Z), W\rangle + \langle \nabla'_X s(Y,Z)), W\rangle \\
&\quad - \langle \nabla_Y(\nabla_X Z), W\rangle\, \langle \nabla'_Y s(X,Z), W\rangle - \langle \nabla_{[X,Y]}Z, W\rangle\} \\
&= \langle R(X,Y)Z,W\rangle + \langle \nabla'_X(s(Y,Z)), W\rangle - \langle \nabla'_Y s(X,Z), W\rangle - \{*\} \\
&\{1\} \implies X(\langle s(Y,Z), W) = 0 \\
&\text{by the property of Riemannian metric}\langle,\rangle \\
&\therefore \langle \nabla'_X s(Y,Z), W\rangle + \langle s(Y,Z), \nabla'_X W\rangle = 0 - \{2\} \\
applying\ Gauss\ formulas\ ,\ \{2\} &= \langle \nabla'_X s(Y,Z), W\rangle + \langle s(Y,Z), \nabla_X W + s(X,W)\rangle \\
&= \langle \nabla'_X s(Y,Z), W\rangle + \langle s(Y,Z), s(X,W)\rangle \\
&\because s(Y,Z) = \perp(\nabla'_{Y_p} Z), \langle s(Y,Z), \nabla_X W\rangle = 0 \\
&\therefore \{2\} \implies \langle \nabla'_X s(Y,Z), W\rangle = -\langle s(Y,Z), s(X,W)\rangle \\
hence\ \{*\} &= \langle R(X,Y)Z,W\rangle - \langle s(Y,Z), s(X,W)\rangle - \langle s(X,Z), s(Y,W)\rangle
\end{aligned}$$

□

3.3 Synge's inequality

Recall that if $P \subset M_p$ is a 2-dimensional subspace of M_p, we define the *sectional curvature K(P)* as

$$K(P) = \langle R(X,Y)Y, X \rangle$$

for orthonormal $X, Y \in P$. Since we had studied Gauss Theorema Egregium(G.T.E) , we observe an inequality which is a corollary to G.T.E that relates the sectional curvatures of a 2-dimensional subspace in an immersed and ambient manifold.

Corollary 3.3.0.1. *Let M be isometrically immersed in N, and let $\gamma : [a,b] \to M$ be a curve in M which is a geodesic in N. Then for all 2-dimensional $P \subset M_{\gamma(t)}$ with $\gamma'(t) \in P$ we have*

$$K(P) \leqslant K'(P)$$

In particular if M is a surface then for all $p = \gamma(t)$ we have

$$K(M_p) \leqslant K'(M_p)$$

Proof. Let γ be parametrized by arclength. Let $X_p = \gamma'(t)$ and let $Y_p \in P$ be a unit vector perpendicular to X_p. Appplying Gauss's equation with $Z_p = Y_p$ and $W_p = X_p$, we have

$$K'(P) = K(P) + \langle s(X_p, Y_p), s(X_p, Y_p) \rangle - \langle s(Y_p, Y_p), s(X_p, X_p) \rangle$$

If we let X be the vectorfield $X(t) = \gamma'(t)$ along γ, then X is parallel along γ, so we have $0 = \nabla'_X X \implies \perp(\nabla'_X X)(p) = s(X_p, Y_p) = 0$
hence, the desired inequality holds. □

Remark. *For the case of surface equality holds for all $p = \gamma(t)$ if and only if $M_{\gamma(t)}$ is parallel along γ, in the sense that pertains to N.*

From now on we consider the specific situation where M is a **hypersurface** in N, that is *a submanifold of codimension 1*.

Remark. *If M is a hypersurface of N then,*

1. *$\exists$ unit normal vector field for M on neighbourhood U of a point $p \in M$*
2. *$\langle \nu, \nu \rangle = 1$ & $\nu(q) \in {M_q}^{\perp}$ $\forall q \in U$*

Theorem 3.3.1. *Let M be a hypersurface in N nd let ν be a unit normal field on a neighborhood of p in M.*

(a) $\forall X_p \in M_p$ *we have*

$$\nabla'_{X_p} v \in M_p$$

(b) *If Y is a vector field tangent along M, then we have*

$$\boxed{\begin{array}{l}\textit{The Weingarten Equations}\\ \langle \nabla'_{X_p} v, Y_p \rangle = -\langle v, \nabla'_{X_p} Y \rangle \quad = -\langle v, s(X_p, Y_p) \rangle\end{array}}$$

(c) *Consequently,*

$$\langle \nabla'_{X_p} v, Y_p \rangle = \langle X_p, \nabla'_{Y_p} v \rangle$$

Proof. (a)

$$\begin{aligned} \langle v, v \rangle &= 1 \\ \therefore X_p(\langle v, v \rangle) &= 0 \\ \therefore \langle \nabla'_{X_p} v, v \rangle + \langle v, \nabla'_{X_p} v \rangle &= 0 \\ \implies \langle \nabla'_{X_p} v, v \rangle &= 0 \\ i.e \nabla'_{X_p} v \perp v \implies \nabla'_{X_p} v \in M_p \end{aligned}$$

(b)

$$\begin{aligned} \langle v, Y_p \rangle &= 0 \\ X_p \langle v, Y_p \rangle &= \langle \nabla'_{X_p} v, Y_p \rangle + \langle v, \nabla'_{X_p} Y_p \rangle \\ &\implies \langle \nabla'_{X_p} v, Y_p \rangle = -\langle v, \nabla'_{X_p} Y_p \rangle \\ \text{Gauss equations} &= \langle v, \top(\nabla' X_p Y_p) + \perp(\nabla'_{X_p} Y_p) \rangle \\ \text{observe that } &\because v \perp M_p \& \top(\nabla'_{X_p} Y_p) = \nabla_{X_p} Y_p \\ \text{so } \langle v, \nabla_{X_p} Y_p \rangle &= 0 \\ \implies \langle v, \nabla'_{X_p} Y_p \rangle &= \langle v, \perp(\nabla'_{X_p} Y_p) \rangle \\ \text{i.e} \langle v, \nabla'_{X_p} Y_p \rangle &= \langle v, s(X_p, Y_p) \rangle \end{aligned}$$

Hence $\langle \nabla'_{X_p} v, Y_p \rangle = -\langle v, \nabla'_{X_p} Y \rangle = -\langle v, s(X_p, Y_p) \rangle$

$\square$

Corollary 3.3.1.1. *Let M^n be a hypersurface in $\mathbb{R}^{n+1}$ and let v be a unit normal field on a neighbourhood of p in M. Then for all $X_p, Y_p \in M_p$ we have*

$$s(X_p, Y_p) = II(X_p, Y_p) \cdot v(p)$$

where $II(X_p, Y_p) \cdot v(p)$ is the second fundamental form of M defined for the choice v of unit normal field, namely

$$II(X_p, Y_p) = -\langle dv(X_p), Y_p \rangle$$

Proof. Observe that

$$\nabla'_{X_p} v = [X_p(v)]_p = [dv(X_p)]_p = dv(X_p),$$

Remark. *Since v can be seen as a map $v : M \to S^{n-1} \subset \mathbb{R}^{n+1}$, we have the vector valued differential form $dv : M_p \to \mathbb{R}^{n+1}$, and $dv(X_p) \in \mathbb{R}^{n+1}$ should be moved back to a parallel vector in M_p, equivalently, $dv(X_p)$ denotes $v_*(X_p) \in S^{n-1}_{v(p)}$ moved back to a parallel vector in M_p.*

So according to Theorem 3.3.1

$$\begin{aligned}
\langle v, s(X_p, Y_p) \rangle &= -\langle \nabla'_{X_p} v, Y_p \rangle \\
&= -\langle dv(X_p), Y_p \rangle \\
&= II(X_p, Y_p) - (*) \\
\text{also we know } \; s(X_p, Y_p) &= \perp(\nabla'_{X_p} Y_p) \\
&\implies s(X_p, Y_p) = k' v \text{ where } k' \text{is scalar} \\
\text{hence to find } s(X_p, Y_p) & \text{ we take inner product with v} \\
\therefore \langle s(X_p, Y_p), v \rangle &= \langle k' v, v \rangle = k' \\
\text{we know that by}(*), \langle s(X_p, Y_p), v \rangle &= II(X_p, Y_p) \\
\text{hence } k' &= II(X_p, Y_p) \\
\text{and} \therefore s(X_p, Y_p) &= II(X_p, Y_p) \cdot v(p)
\end{aligned}$$

□

Theorem 3.3.2. *Let M be a hypersurface in N, and let v be a unit normal field on a neighbourhood of p in M, with corresponding II. Then for all $X_p, Y_p, Z_p \in M_p$ we have :*

$$\boxed{\begin{array}{l} \textit{The Codazzi-Mainardi Equations} \\ \langle R'(X_p,Y_p)Z_p, v(p)\rangle = (\nabla_{X_p} II(Y_p,Z_p) \quad \text{-}(\nabla_{Y_p} II(X_p,Z_p) \end{array}}$$

Proof. Recall the equations derived in the proof of Theorem 3.2.3

$$\begin{aligned}
(1)\ \ \nabla'_X(\nabla'_Y Z) &= \nabla'_X(\nabla_Y Z) + \nabla'_X(s(Y,Z)) \\
&= \nabla_X(\nabla_Y Z) + s(X,\nabla_Y Z) + \nabla'_X(s(Y,Z) \\
(2)\ \ \nabla'_Y(\nabla'_X Z) &= \nabla_Y(\nabla_X Z) + s(Y,\nabla_X Z) + \nabla'_Y(s(X,Z) \\
(3)\quad \nabla'_{[X,Y]} Z &= \nabla_{[X,Y]} Z + s([X,Y],Z) \\
&= \nabla_{[X,Y]} Z + s(\nabla_X Y,Z) - s(\nabla_Y X,Z)
\end{aligned}$$

This shows that the normal component of $R'(X,Y)Z$ is given by

$$\begin{aligned}
&\text{normal component of} R'(X,Y)Z = \\
&\qquad [\perp \nabla'_X(s(Y,Z) - s(\nabla_X Y,Z) - s(\nabla_Y X,Z)] \\
&\qquad - [\perp \nabla'_Y(s(X,Z) - s(\nabla_Y X,Z) - s(X,\nabla_Y Z)]
\end{aligned}$$

Also, as

$$s(Y,Z) = II(Y,Z)\cdot(v),$$

we have,

$$\nabla'_X(s(Y,Z) = X(II(Y,Z))\cdot v + II(Y,Z)\cdot \nabla'_X v$$

so by Theorem 3.3.1 $\nabla'_X v \in M_p \implies \langle II(Y,Z)\cdot \nabla'_X v, v\rangle = 0$

$$\therefore \langle \nabla'_X(s(Y,Z), v\rangle = X(II(Y,Z))\cdot v$$

hence

$$\begin{aligned}
\langle R'(X,Y)Z, v\rangle &= [X(II(Y,Z)) - II(\nabla_X Y,Z) - II(Y,\nabla_X Z)] \\
&\quad - [Y(II(X,Z))] - II(\nabla_Y X,Z) - II(X,\nabla_Y Z)
\end{aligned}$$

□

A particular case to observe this is when our ambient space N has constant curvature K_0.

Corollary 3.3.2.1. *Let N have constant curvature K_0. Then for isometrically immersed in N we have* :

$$\boxed{\begin{array}{l} \textit{Gauss's Equations:} \\ \langle R(X_p,Y_p)Z_p,W_p\rangle + \quad \langle s(X_p,Z_p),s(Y_p,W_p)\rangle - \quad \langle s(Y_p,Z_p),s(X_p,W_p)\rangle \\ = K_0[\langle X_p,W_p\rangle\cdot\langle Y_p,Z_p\rangle \quad -\langle X_p,Z_p\rangle\cdot\langle Y_p,W_p\rangle] \end{array}}$$

And if M is a hypersurface we have :

$$\boxed{\begin{array}{l} \textit{The Codazzi-Mainardi Equations:} \\ (\nabla_{X_p})II(Y_p,Z_p) = (\nabla_{Y_p}II)(X_p,Z_p) \end{array}}$$

3.4 Fundamental theorem of surface theory(Bonnet-1867)

From the theory of curves , we know that κ and τ formed a complete set of invariants for a curve up to translations and rotations (elements of $SO(3)$), by showing that they were a complete set of invariants up to rotation for the function $s \to (\mathbf{t}(s),\mathbf{n}(s),\mathbf{b}(s)$, this was accomplished using the Serret-Frenet formulas, which are differential equations for $(\mathbf{t},\mathbf{n},\mathbf{b})$, involving only κ and τ. In this theorem we will observe an analogous situation for surfaces and see whether every immersion $f : U \to \mathbb{R}^3$ is described completely by the corresponding g_{ij} and l_{ij}, which are "invariant under proper Euclidean motion".

In the case of surfaces, we have the three vectors (f_1, f_2, N), where if f is an immersion, $f : U \to \mathbb{R}^3$ (for $U \subset \mathbb{R}^2$ open), then

$$f_1(s,t) = \frac{\partial f}{\partial s}(s,t),$$

$$f_2(s,t) = \frac{\partial f}{\partial t}(s,t)$$

and

$$N = \frac{f_1 \times f_2}{|f_1 \times f_2|}$$

Introducing this new g^{kj} notation is really helpful, where

$$\sum_k g_{ik}g^{kj} = \delta_i^j$$

First we try to express the derivatives of each of f_1, f_2, N as linear combinations of these three vectors.

We want to write

$$f_{ik} = \sum_{h=1}^{2} A_{ik}^{h} f_h + B_{ik} N$$

To find A_{ik}^{h} we take innerproduct of f_{ik} with f_j.

$$\begin{aligned}
\text{We know that} \langle f_i, f_j \rangle &= g_{ij} \\
\text{differentiating, } \langle f_{ik}, f_j \rangle + \langle f_i, f_{jk} \rangle &= \frac{\partial}{\partial k}(g_{ij}) = g_{ij,k} \ (k = 1,2) \\
&= \langle \nabla_{\frac{\partial}{\partial x_k}} f_i, f_j \rangle + \langle f_i, \nabla_{\frac{\partial}{\partial x_k}} f_j \rangle \\
\text{similarly} \langle f_{ji}, f_k \rangle + \langle f_j, f_{ki} \rangle &= g_{jk,i} \\
\langle f_{kj}, f_i \rangle + \langle f_k, f_{li} \rangle &= g_{ki,j}
\end{aligned}$$

adding the first two and subtracting the third we have,

$$\begin{aligned}
\langle f_{ik}, f_j \rangle &= \frac{1}{2}(g_{ij,k} + g_{jk,i} - g_{ki,j}) \\
&= [ik, j],
\end{aligned}$$

where $[ik, j]$ is the Christoffel symbol for the metric $I_f = f^*\langle , \rangle$ on U with respect to the standard coordinate system (s,t) on $\mathbb{R}^2$, so we have,

$$[ik, j] = \langle f_i, f_j \rangle = \sum_{h=1}^{2} A_{ik}^{h} g_{hj}$$

To explicitly find A_{ik}^{h}, take inner product with g^{ij} for each i, so,

$$A_{ik}^{\rho} = \sum_{j=1}^{2} g^{\rho j}[ik, j] = \Gamma_{ik}^{\rho}$$

Recall that $\sum_{\rho} g_{i\rho} g^{\rho j} = \delta_i^j$

Similarly to find B_{ik} we take inner-product with N, i.e

$$B_{ik} = \langle f_{ik}, N \rangle = -\langle N_i, f_k \rangle = l_{ik}$$

Thus,

$$f_{ik} = \sum_{h=1}^{2} \Gamma_{ik}^{h} f_h + l_{ik} N \ \{\text{The Gauss Formulas}\} \tag{3.2}$$

Observe that in the equation 3.2 we are expressing the covariant derivative of $\frac{\partial}{\partial x_i}$ as its tangential component (f_h) and normal component (N), which is similar to how we expressed connection on the ambient manifold as its tangential and normal component in the equation 3.1.

Next to express N_i in terms of f_1, f_2, N, we write N_i as,

$$N_i = \sum_{h=1}^{2} C_i^h f_h + 0.N$$

just as we did above,

$$l_{ij} = \langle -N_i, f_j \rangle = -\sum_{h=1}^{2} C_i^h g_{hj},$$

and consequently

$$C_i^\rho = -\sum_{j=1}^{2} g^{\rho j} l_{ij}$$

Introducing new symbols l_i^h we can write,

$$N_i = -\sum h = 1^2 \left(\sum_{j=1}^{n} g_{hj} l_{ij} \right) f_h = -\sum_{h=1}^{2} l_i^h f_h \text{ \{The Weingarten Equations\}} \tag{3.3}$$

The following theorem is useful in proving the fundamental theorem of surfaces.

Theorem 3.4.1. *Let $U \times V \subset \mathbb{R}^m \times \mathbb{R}^n$ be open, where U is a neighbourhood of $0 \in \mathbb{R}^m$, and let $f_i \colon U \times V \subset \mathbb{R}^n$ be C^∞ functions, for $i = 1, \ldots, m$. Then for every $x \in V$, there is at most one function*

$$\alpha \colon W \to V,$$

defined in a neighbourhood W of O in $\mathbb{R}^m$, satisfying

$$\begin{aligned} \alpha(0) &= x \\ \frac{\partial \alpha}{\partial t^j}(t) &= f_j(t, \alpha(t)) \forall t \in W \end{aligned}$$

Remark. *Any two functions α_1, α_2, as mentioned above, defined on W_1, W_2, agree on the component of $W_1 \cap W_2$ which contains 0.*

Theorem 3.4.2 (Fundamental theorem of Surface theory, (Bonnet- 1867))**.** *Let $U \subset \mathbb{R}^2$ be a convex open set containing (0,0)*

1. *Let $f, \overline{f} : U \to \mathbb{R}^3$ be two immersions, and define*

$$g_{ij} = \langle f_i, f_j \rangle \qquad \overline{g_{ij}} = \langle \overline{f_i}, \overline{f_j} \rangle$$
$$N = \frac{f_1 \times f_2}{\sqrt{g_{11}g_{22} - g{12}^2}} \qquad \overline{N} = \frac{\overline{f_1 \times f_2}}{\sqrt{\overline{g}_{11}\overline{g}_{22} - \overline{g}{12}^2}}$$
$$l_{ij} = \langle -N_i, f_j \rangle = \langle -N, f_{ij} \rangle \qquad \overline{l}_{ij} = \langle -\overline{N}_i, \overline{f}_j \rangle = \langle -\overline{N}, \overline{f}_{ij} \rangle$$

   Suppose that $g_{ij} = \overline{g}_{ij}$ and $g_{ij} = \overline{g}_{ij}$ on U. Then there is a proper Euclidean motion A such that $\overline{f} = A \circ f$

2. *Let g_{ij} and g_{ij} (i,j =1,2) be functions on U which satisfy*

   i. *$g_{ij} = g_{ji}$ and $l_{ij} = l_{ji}$, and (g_{ij}) is positive definite on U, so that we can define corresponding g^{ij} and Γ_{ij}^k*

   ii. *Gauss's Equation :*

$$\begin{aligned} l_{11}l_{22} - (l_{12})^2 &= R_{1212} \\ &= \sum_{\rho=1}^{2} g_{1\rho}(\Gamma^\rho)_{22,1} - (\Gamma^\rho)_{21,2} + \sum_{h=1}^{2}(\Gamma_{22}^h\Gamma_{h1}^\rho - \Gamma_{21}^h\Gamma_{h2}^\rho) \end{aligned}$$

   iii. *The Codazzi- Mainardi Equations :*

$$l_{12,1} - l_{11,2} + \sum_{h=1}^{2}\Gamma_{12}^h l_{h1} - \sum_{h=1}^{2}\Gamma_{11}^h l_{h2} = 0$$

$$l_{22,1} - l_{21,2} + \sum_{h=1}^{2}\Gamma_{22}^h l_{h1} - \sum_{h=1}^{2}\Gamma_{21}^h l_{h2} = 0$$

Then there is an immersion $f : U \to \mathbb{R}^3$ such that

$$g_{ij} = \langle f_i, f_j \rangle$$
$$l_{ij} = \langle -N_i, f_j \rangle = \langle N, f_{ji} \rangle, \text{for } N = \frac{f_1 \times f_2}{\sqrt{g_{11}g_{22} - g_{12}{}^2}}$$

Proof. Spivak follows a convenient notation to prove this theorem, let's follow the same

$$\begin{array}{ccc} \mathbf{v_1} = f_1 & \mathbf{v_2} = f_2 & \mathbf{v_3} = N \\ \overline{\mathbf{v}}_1 = \overline{f}_1 & \overline{\mathbf{v}}_2 = \overline{f}_2 & \overline{\mathbf{v}}_3 = \overline{N} \end{array}$$

First choose a roatation $B \in \mathrm{SO}(3)$ such that

$$B(v_\alpha(0,0)) = \overline{v}_\alpha(0,0) \quad \alpha = 1,2,3.$$

This is possible because $g_{ij(0)} = \overline{g}_{ij}(0)$ for $i, j = 1,2$ and because the two triples of vectors $(\mathbf{v}_1(0,0), \mathbf{v}_2(0,0), \mathbf{v}_3(0,0))$ and $(\overline{\mathbf{v}}_1(0,0), \overline{\mathbf{v}}_2(0,0), \overline{\mathbf{v}}_3(0,0))$ are both positively oriented, with the third perpendicular to the first two. If we let $\tilde{f} = B \circ f$, then it is easy to see that

$$\tilde{g}_{ij} = g_{ij} = \overline{g}_{ij}$$

$$\overline{\mathbf{v}}_3 = B \circ \mathbf{v}_3$$

$$\tilde{l}_{ij} = l_{ij} = \overline{l}_{ij}$$

We claim that the maps

$$(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3), (\overline{\mathbf{v}}_1, \overline{\mathbf{v}}_2, \tilde{\mathbf{v}}_3) \colon U \to \mathbb{R}^3$$

which are equal at zero are infact equal at everywhere. Recall that Gauss formulas and Weingarten equations give

$$\mathbf{v}_{i,k}(s,t) = \sum_{h=1}^{2} \Gamma_{ik}^{h}(s,t)\mathbf{v}_h(s,t) + l_{ik}(s,t)\mathbf{v}_3(s,t) \quad i = 1,2 \tag{3.4}$$

$$\mathbf{v}_{3,k}(s,t) = -\sum_{h=1}^{2} \left(\sum_{j=1}^{2} g^{hj}(s,t) l_{kj}(s,t) \right) \mathbf{v}_h(s,t) \tag{3.5}$$

for the $\mathbf{v}_\alpha$, while for the $\overline{\mathbf{v}}_\alpha$ we obtain the corresponding equations with $\tilde{\Gamma}_{ik}^{h}$, $\tilde{l}_{ik}$ and $\tilde{g}^{hj}$. But $\tilde{l}_{ik} = \overline{l}_{ik}$, and since $\overline{g}_{ij} = \tilde{g}_{ij}$ we also have $\overline{g}^{hj} = \tilde{g}^{ij}$ and $\overline{\Gamma}_{ik}^{h} = \tilde{\Gamma}_{ik}^{h}$. So the two maps $(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$ and $(\overline{\mathbf{v}}_1, \overline{\mathbf{v}}_2, \tilde{\mathbf{v}}_3)$ satisfy the same equations, and is equal at $(0,0)$, hence they must be equal on U by Theorem 3.4.1. So this implies that $\overline{f}$ and $\tilde{f} = B \circ f$ have the same partial derivatives, and therefore differ by a constant vector. Consequently there is a translation $T : \mathbb{R}^3 \to \mathbb{R}^3$ with $\overline{f} = T \circ \tilde{f} = (T \circ B) \circ f$.

To prove (2), we again use Theorem 3.4.1 to conclude that equations 3.4 and 3.5 written in terms of the given l_{ij} and g_{ij}, has a solution $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \colon U \to \mathbb{R}^3$ with any desired initial conditions. Moreover the functions $\mathbf{v}_\alpha$ can be defined on all of U because the equations 3.4 and 3.5 are linear. Since (g_{ij}) is positive definite at $(0,0)$ there is a solution for which the following conditions are satisfied at (s,t)= (0,0)

$$
\begin{array}{lll}
(a) & \langle \mathbf{v_i}(s,t), \mathbf{v_j}(s,t) \rangle = g_{ij}(s,t) & i,j = 1,2 \\
(b) & \langle \mathbf{v_i}(s,t), \mathbf{v_3}(s,t) \rangle = 0 & i = 1,2 \\
(c) & |\mathbf{v_3}(s,t)| = 1 &
\end{array}
$$

$(d)(\mathbf{v_1}(s,t), \mathbf{v_2}(s,t), \mathbf{v_3}(s,t))$is positively oriented.

We will show that conditions (a)-(d) actually hold at all points of U.

The equations 3.4 and 3.5 for $\mathbf{v_1}(s,t)\mathbf{v_2}(s,t)\mathbf{v_3}(s,t)$ gives the equations

$$
\begin{aligned}
(A)\langle \mathbf{v_i}, \mathbf{v_j} \rangle_k &= \langle \mathbf{v_{i,k}}, \mathbf{v_j} \rangle + \langle \mathbf{v_{i,k}}, \mathbf{v_{j,k}} \rangle \\
&= \sum_{h=1}^{2} \Gamma_{ik}^{h} \langle \mathbf{v_h}, \mathbf{v_j} \rangle + \sum_{h=1}^{2} \Gamma_{jk}^{h} \langle \mathbf{v_h}, \mathbf{v_i} \rangle + l_{ik} \langle \mathbf{v_3}, \mathbf{v_i} \rangle + l_{jk} \langle \mathbf{v_3}, \mathbf{v_j} \rangle
\end{aligned}
$$

for i,j=1,2, as well as

$$
\begin{aligned}
(B)\langle \mathbf{v_i}, \mathbf{v_3} \rangle_k &= \langle \mathbf{v_{i,k}}, \mathbf{v_3} \rangle + \langle \mathbf{v_i}, \mathbf{v_{3,k}} \rangle \\
&= l_{ik} - \sum_{h=1}^{2} \left(\sum_{j=1}^{2} g^{hj} l_{kj} \right) \langle \mathbf{v_i}, \mathbf{v_h} \rangle
\end{aligned}
$$

and

$$
(C) \qquad \langle \mathbf{v_3}, \mathbf{v_3} \rangle_k = 2 \langle \mathbf{v_3}, \mathbf{k}, \mathbf{v_3} \rangle = 0
$$

[Equations (A)-(C) all hold for k=1,2.]

But we also have

$$
\begin{aligned}
g_{ij,k} &= [ik, j] + [jk, i] \\
&= \sum_{h=1}^{2} \Gamma_{ik}^{h} g_{hj} + \sum_{h=1}^{2} \Gamma_{jk}^{h} g_{hi}.
\end{aligned}
$$

This shows that the set of equations (A)-(C) are satisfied both by

the set of functions : $\langle \mathbf{v_i}, \mathbf{v_j}, \rangle (j = 1,2), \langle \mathbf{v_3}, \mathbf{v_1}, \rangle, \langle \mathbf{v_3}, \mathbf{v_2}, \rangle, \langle \mathbf{v_3}, \mathbf{v_3}, \rangle$

and by

the set of functions : $g_{ij}(j = 1,2), 0 \quad ,0 \quad , \quad 1$

Moreover we chose the $\mathbf{v_i}$ so that these two collections of functions have the same value at (0,0). I t follows that they have the same values on all of U. In other words equations (a)- (c) hold on all of U.Moreover, (a) and (b)[and non singularity of (g_{ij})] imply that $(\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3})$ are always linearly independent. So condition (d)at (0,0) implies (d) everywhere.

We now claim that there is a function $f \colon U \to \mathbb{R}^3$ satisfying $f_i = \mathbf{v_i}$. In order to prove this, we just show that $\mathbf{v_{i,j}} = \mathbf{v}_{j,i}$. But this follows from equations 3.4 and 3.5, by symmetry of the Γ^h_{ik} and l_{ik}. We now have $\langle f_i, f_j \rangle = g_{ij}$ by (a). Moreover, (b)-(d) then show that $\mathbf{v_3} = n$, consequently,

$$\langle f_{ij}, n \rangle = \langle \mathbf{v_{i,j}}, \mathbf{v_3} \rangle = l_{ij}$$

by 3.4 and 3.5 together with (b) and (c). ◻

Theorem 3.4.3 (Hadamard)**.** *(1)If M is a convex surface in $\mathbb{R}^3$, then $K(p) \geqslant 0$ for all $p \in M$.*

(2) Let M be a compact connected 2-manifold, and $f \colon M \to \mathbb{R}^3$ an immersion with $K(p) > 0$ for all $p \in M$. Then

1. *The manifold M is orientable, and the normal map $N \colon M \to S^2 \subset \mathbb{R}^3$ is a diffeomorphism,*

2. *The map $f \colon M \to \mathbb{R}^3$ is an imbedding, and $f(M)$ is convex.*

Chapter 4

Surfaces of constant curvature

In this chapter we will determine precisey which surfaces in $\mathbb{R}^3$ can be obtained by isometrically immersing complete manifolds with constant curvature $K > 0, K = 0, K < 0$.

4.1 Hilbert's lemma

Theorem 4.1.1. *Let X be a C^∞ vector field on M with $X(p) \neq 0$. Then there is a coordinate system (x,U) around p such that*

$$X = \frac{\partial}{\partial x^1} \qquad\qquad \textit{on } U.$$

Proof. Assume $M = \mathbb{R}^n$ for a convenient coordinate system say$(t^1, t^2, ..., t^n)$ and $p = 0 \in \mathbb{R}^n$. Moreover, we can assume that $X(0) = \frac{\partial}{\partial t^1}|_0$. The idea of proof is that in a neighbourhood of 0 there is a unique integral curve through each point $(0, a^2, \ldots, a^n)$, if q lies on the integral curve through this point, we will use $a^2, \ldots, a^n$ as the last $n-1$ coordinates of q and the time interval it takes the curve to get to q as the first coordinate. To do this, let X generate ϕ_t and consider the map $\mathscr{X}$ defined on a neighbourhood of 0 in $\mathbb{R}^n$ by

$$\mathscr{X}(a^1, a^2, \ldots, a^n) = \phi_{a^1}(0, a^2, \ldots, a^n).$$

We compute that for $a = (a^1, a^2, \ldots, a^n)$,

$$\mathscr{X}_* \left(\frac{\partial}{\partial t^1}\bigg| \right)_a (f) = \left(\frac{\partial}{\partial t^1}\bigg| \right)_a (f \circ \mathscr{X})$$

$$= \lim_{h \to 0} \frac{1}{h}[f(\mathscr{X}(a^1 + h, a^2, \ldots a^n)) - f(\mathscr{X}(a))]$$

$$= \lim_{h \to 0} \frac{1}{h}[f(\phi_{a^1+h}(0, a^2, \ldots, a^n)) - f(\mathscr{X}(a))]$$

$$= \lim_{h \to 0} \frac{1}{h}[f(\phi_h(\mathscr{X}(a))) - f(\mathscr{X}(a))]$$

$$= (Xf)(\mathscr{X}(a))$$

Moreover, for $i > 1$ we can at least compute

$$\mathscr{X}_* \left(\frac{\partial}{\partial t^1}\bigg| \right)_0 (f) = \left(\frac{\partial}{\partial t^1}\bigg| \right)_0 (f \circ \mathscr{X})$$

$$= \lim_{h \to 0} \frac{1}{h}[f(\mathscr{X}(0, \ldots, h, \ldots, 0)) - f(0)]$$

$$= \lim_{h \to 0} \frac{1}{h}[f(0, \ldots, h, \ldots, 0) - f(0)]$$

$$= \frac{\partial f}{\partial t^1}\bigg|_0$$

Since $X(0) = \partial/\partial t^1|_0$ by assumption, this shows that $\mathscr{X}_{*0} = I$ is non-singular. Hence $x = \mathscr{X}^{-1}$ may be used as a coordinate system in a neighbourhood of 0. This is the desired coordinate system, for it is easy to see that the equation $\mathscr{X}_*(\partial/\partial t^1) = X \circ \mathscr{X}$, which we have just proved, is equivalent to $X = \partial/\partial t^1$. □

Proposition 4.1.1. *Let X_1, X_2 be linearly independant vector fields in a neighbourhood of a point p in a 2-dimensional manifold M. Then there is an imbedding $f\colon U \to M$, where $U \subset \mathbb{R}^2$ is open and $p \in f(U)$, whose i^{th} parameter lines lie along the integral curves of X_i.*

Proof. Assume that $p = 0 \in \mathbb{R}^2$, and that $X_i(0) = (e_i)_0$. Every point q in a sufficiently small neighbourhood of 0 is on a unique integral curve of X_1 through a point $(0, x^2(q))$ which we proved in Theorem 4.1.1. Similarly , q is on a unique integral curve of X_2 through a point $(x^1(q), 0)$.

The map $q \to (x^1(q), x^2(q))$ is C^∞, with Jacobian equal to I at 0. Its inverse, in a sufficiently small neighbourhood of 0, is the required diffeomorphism. □

Remark. *Let p be a point on a surface M in $\mathbb{R}^3$.*

1. *If p is not an umbilic point, then there is an imbedding $f\colon U \to M$, with $p \in f(U)$, whose parameter curves are lines of curvature.*

2. *If $K(p) < 0$, then there is an imbedding $f\colon U \to M$, with $p \in f(U)$, whose parameter curves are asymptotic curves.*

Lemma 4.1.2 (Hilbert)**.** *Let M be a surface immersed in $\mathbb{R}^3$, and let $p \in M$ be a non- umbilic point. Let $k_1 \geqslant k_2$ be the two principal curvatures on M and suppose that k_1 has a local maximum at p, and k_2 has a local maximum at p. Then $K(P) \leqslant$* 0.

Proof. According to the given remark above, we can choose an imbedding $f\colon U \to M$, with $p \in f(U)$, whose coordinate lines are the lines of curvature. Then Gauss's equation and the Codazzi-Mainardi equations become

$$K = -\frac{1}{2\sqrt{EG}}\left[\left(\frac{E_2}{\sqrt{EG}}\right)_2 + \left(\frac{G_1}{\sqrt{EG}}\right)_1\right] \tag{1}$$

$$l_2 = \frac{E_2}{2}\left(\frac{l}{E} + \frac{n}{G}\right) = \frac{E_2}{2}(k_1 + k_2) \tag{2}$$

$$n_1 = \frac{G_1}{2}\left(\frac{l}{E} + \frac{n}{G}\right) = \frac{G_1}{2}(k_1 + k_2) \tag{3}$$

the second equality's in (2) and (3) follow from the fact that

$$l = k_1 E, \qquad n = k_2 G$$

Moreover, differentiation of these last two equations yields

$$l_2 = \frac{\partial k_1}{\partial t}E + k_1 E_2, \qquad n_1 = \frac{\partial k_2}{\partial s}G + k_2 G_1.$$

The functions k_i are differentiable near p, since the functions H and K are differentiable, and $k_i = H +_- \sqrt{H^2 - K}$, where $H^2 - K > 0$ in a neighbourhood of the non-umbilic point p. Together with (2)and (3) we then have

$$E_2 = -\frac{2E}{K_1 - K_2}.\frac{\partial K_1}{\partial t} \tag{2'}$$

$$G_1 = -\frac{2G}{K_1 - K_2}.\frac{\partial K_2}{\partial s}. \tag{3'}$$

Substituting (2') and (3') into (1) gives

$$(1') \qquad K = -\frac{1}{2EG}[-\frac{2E}{K_1 - K_2}.\frac{\partial^2 K_1}{\partial t^2} + \frac{2G}{K_1 - K_2}.\frac{\partial^2 K_2}{\partial s^2}]$$
$$+ \text{something continuous}.\frac{\partial K_1}{\partial t}$$
$$+ \text{something continuous}.\frac{\partial K_2}{\partial s}$$

Since k_1 has a local maximum at p, and k_2 a local minimum, we have

$$\frac{\partial K_1}{\partial t}(p) = \frac{\partial K_2}{\partial s}(p) = 0, \qquad \frac{\partial^2 K_1}{\partial t^2}(p) \leqslant 0, \qquad \frac{\partial^2 K_2}{\partial s^2}(p) \geqslant 0$$

Together with (1') this shows that $K(p) \leqslant O$. ◻

Theorem 4.1.3. *If M is a compact connected surface in $\mathbb{R}^3$ with constant curvature $K > 0$, then M is a sphere.*

Proof. Let $k_1 \geqslant k_2$ be the principal curvatures on M, and let p be a point where k_1 achieves its maximum. Then $k_2 = K/k_1$ has its minimum at p. If we had $k_1(p) > k_2(p)$, so that p was not an umbilic, then the lemma would imply that $K(p) \leqslant 0$, a contradiction. Hence $k_1(p) = k_2(p)$. Moreover, for any point $q \in M$ we then have

$$k_1(p) \geqslant k_1(q) \geqslant k_2(q) \geqslant k_2(p) = k_1(p),$$

so also $k_1(q) = k_2(q)$. Thus all points of M are umbilics ◻

Theorem 4.1.4. *If M is a compact connected surface in $\mathbb{R}^3$, with K everywhere$>$ 0, and constant mean curvature H, then M is a sphere.*

Refer [4] for proofs.

Lemma 4.1.5. *Let M be a 2-dimensional immersed submanifold of $\mathbb{R}^3$ with constant curvature $K > 0$. Then for every point $p \in M$ there is a diffeomorphism*

$$g\colon (-\varepsilon, \varepsilon) \times (-\varepsilon, \varepsilon) \to M,$$
$$g(0,0) = p$$

whose parameter curves are asymptotic curves parametrized by arclength.

For any 2-dimensional Riemannian manifold M. an immersion $g\colon (a,b)\times(c,d)\to M$ is called **Tschebyscheff net** if all parameter curves are parametrized by arclength.If we think of the domain $(a,b)\times(c,d)$ as a piece of cloth woven from fibres parallel to the axes, then the immersion g doesn't stretch any fibres. So if we can find Tschebyscheff nets around each point then our surface could be outfitted in a tight fitting cloth. Previous lemma(4.1.5) shows that this can always be done on a submanifold of $\mathbb{R}^3$ with constant negative curvature.

Lemma 4.1.6. *Let M be a 2-dimensional Riemannian manifold and $g\colon (a,b)\times(c,d)\to M$ is a Tschebyscheff net. Define $\omega\colon (a,b)\times(c,d)\to\mathbb{R}$ as follows : $\omega(s_0,t_0)$ is the unique number with $0<\omega(s_0,t_0)<\pi$ such that $\omega(s_0,t_0)$ is an angle between*

$$\left.\frac{dg(s,t_0)}{ds}\right|_{s=s_0} \qquad \textit{and} \qquad \left.\frac{dg(s_0,t)}{dt}\right|_{t=t_0}$$

Then ω satisfies the differential equation

$$\frac{\partial^2\omega}{\partial s\partial t}=(-K)\sin\omega.$$

Now we can prove the theorem, which still requires quite a bit of argument. We will use the term **asymptotic Tschebyscheff net** for a Tschebyscheff net of the sort discussed in Lemma 4.1.5, with all parameter curves being asymptotic curves.

Theorem 4.1.7. *A complete surface M with constant curvature $K=-1$ cannot be immersed in $\mathbb{R}^3$.*

Proof. The proof depends on establishing two facts

(a) Suppose that M could be immersed in $\mathbb{R}^3$. Then there would be a Tschebyscheff net $f\colon \mathbb{R}^2\to M$, from the whole plane to M, and the function ω, defined on all of $\mathbb{R}^2$, which gives the angle between the first and second parameter lines would satisfy

$$\frac{\partial^2\omega}{\partial s\partial t}=\sin\omega, \qquad 0<\omega<\pi$$

(b) There is no function $\omega\colon \mathbb{R}^2\to\mathbb{R}$ satisfying

$$\frac{\partial^2\omega}{\partial s\partial t}=C\sin\omega, \qquad 0<\omega<\pi$$

where $C>0$ is any constant.

$\square$

Chapter 5

The Gauss Bonnet Theorem

In this chapter we will look at *Cartan's Structure equations*, a powerful computational method which employs differential forms to calculate the curvature. These equations will be further used to prove the *Gauss – Bonnet theorem*, relating the curvature of a compact surface to its topology. This theorem gives a simple example of how the curvature of a complete Riemannian manifold can constrain its topology.

5.1 Cartan's structure equations

Recall. *Let* Π *be a 2-dimensional subspace of* T_pM *and let* X_p, Y_p *be two linearly independent elements of* Π. *Then the* **sectional curvature** *of* Π *is defined as*

$$K(\Pi) = -\frac{R(X_p, Y_p, X_p, Y_p)}{\|X_p\|^2\|Y_p\|^2 - \langle X_p, Y_p\rangle^2}$$

Note that $\|X_p\|^2\|Y_p\|^2 - \langle X_p, Y_p\rangle^2$ is the square of the area of the parallelogram in T_pM spanned by X_p, Y_p, also observe that the definition of sectional curvature does not depend on the choice of the linearly independent vectors X_p, Y_p. We will now see that understanding the sectional curvature of every section of T_pM completely determines the curvature tensor on this space.

Proposition 5.1.1. *The Riemannian curvature tensor at p is uniquely determined by the values of the sectional curvatures of sections(i.e, 2-dimensional subspaces of* T_pM*).*

For proof please refer [5] A Riemannian manifold is called **isotropic at a point** $p \in M$ if its sectional curvature is a constant K_p for every section $\Pi \subset T_pM$. Moreover, it is called **isotropic** if it is isotropic at all points. Note that every 2-dimensional manifold is trivially isotropic. Its sectional curvature $K(p) := K_p$ is called the **Gauss curvature**.

Remark. *As we will see later, the Gauss curvature measures how much the local geometry of the surface differs from the geometry of the Euclidean plane. For instance, its integral over a disk D on the surface gives the angle by which a vector is rotated when parallel-transported around the boundary of D. Alternatively, its integral over the interior of a geodesic triangle $\triangle$ is equal to the difference between the sum of the inner angles of $\triangle$ and π*

Now we will reformulate the properties of the Levi-Civita connection and of the Riemannian curvature tensor in terms of differential forms.

A **field of frames** $\{X_1 \dots X_n\}$, is a set of n vector fields that, at each point p of V, form a basis for T_pM. Then we consider a **field of dual coframes**, that is, 1-forms $\{\omega^1 \dots \omega^n\}$ on V such that $\omega^i(X_j) = \delta_{ij}$. From the properties of a connection, in order to define $\nabla_X Y$ we just have to establish the values of

$$\nabla_{X_i} X_j = \sum_{k=1}^{n} \Gamma_{ij}^k X_k,$$

where Γ_{ij}^k is defined as the kth component of the vector field $\nabla_{X_i} X_j$ on the basis $\{X_i\}_{i=1}^n$. Given the values of the Γ_{ij}^k on V, we can define 1-forms $\omega_j^k (j,k = 1,\dots n)$ in the following way :

$$\omega_j^k := \sum_{i=1}^{n} \Gamma_{ij}^k \omega^i$$

Conversely, given these forms, we can obtain the values of Γ_{ij}^k through

$$\Gamma_{ij}^k = \omega_j^k(X_i).$$

The connection is then completely determined from these forms :

given two vector fields $X = \sum_{i=1}^n a^i X_i$ and $Y = \sum_{i=1}^n b^i X_i$, we have

$$\begin{aligned}
\nabla_{X_i} X_j = \nabla_{\sum_{i=1}^n a^i X_i} X_j = \sum_{i=1}^n a^i \nabla_{X_i} X_j = \sum_{i,k=1}^n a^i \Gamma_{ij}^k X_k \\
= \sum_{i,k=1}^n a^i \omega_j^k (X_i) X_k = \sum_{k=1}^n \omega_j^k (X) X_k
\end{aligned} \tag{5.1}$$

and hence

$$\begin{aligned}
\nabla_X Y = \nabla_X \left(\sum_{i=1}^n b^i X_i \right) = \sum_{i=1}^n \big((X \cdot b^i) X_i + b^i \nabla_X X_i \big) \\
= \sum_{j=1}^n \Big((X \cdot b^j) + b^i \omega_i^j (X) \Big) X_j
\end{aligned} \tag{5.2}$$

Note that the values of the forms ω_j^k at X are the components of $\nabla_X X_j$ relative to the field of frames, that is,

$$\omega_j^i (X) = \omega^i (\nabla_x X_j). \tag{5.3}$$

The ω_j^k are called the **connection forms**.

Theorem 5.1.1 (Cartan's)**.** *Let V be an open subset of a Riemannian manifold M on which we have defined a field of frames $\{X_1, \dots X_n\}$. Let $\{\omega_1, \dots \omega_n\}$ be the corresponding field of coframes. Then the connection forms of the Levi-Civita connection are the unique solution of the equations*

1. $d\omega^i = \sum_{j=1}^n \omega^j \wedge \omega_i^j$,

2. $dg_{ij} = \sum_{k=1}^n (g_{kj} \omega_i^k + g_{ki} \omega_j^k)$,

where $g_{ij} = \langle X_i, X_j \rangle$

Proof. Proof is clearly mentioned in [5], so we skip it. □

In addition to connection forms, we can also define **curvature forms**. For this again we consider an open subset V of M where we have a field of frames $\{X_1, \dots X_n\}$(hence a corresponding field of dual coframes $\{\omega^1, \dots \omega^n\}$). We then define 2 forms $\Omega_k^l (k, l = 1, \dots, n)$ by :

$$\Omega_k^l(X,Y) := \omega^l(R(X,Y)X_k),$$

for all vector fields X,Y in V (i.e $R(X,Y)X_k = \sum_{l=1}^n \Omega_k^l(X,Y)X_l$). Using the basis $\{\omega^i \wedge \omega^j\}_{i<j}$ for 2-forms, we have :

$$\Omega_k^l = \sum_{i<j} \Omega_k^l(X_i,X_j)\omega^i \wedge \omega^j = \sum_{i<j} \omega^l(R(X_i,X_j)X_k)\omega \wedge \omega^j \quad = \sum_{i<j} R_{ijk}^l \omega^i \wedge \omega^j = \frac{1}{2}\sum_{i,j=1}^n R_{ijk}^l \omega^i \wedge \omega^j,$$

where the R_{ijk}^l are the coefficients of the curvature relative to these frames

$$R(X_i,X_j)X_k = \sum_{l=1}^n R_{ijk}^l X_l$$

The curvature forms satisfy the following equation.

Proposition 5.1.2. *In the above notation,*
3. $\quad \Omega_i^j = d\omega_i^j - \sum_{k=1}^n \omega_i^k \wedge \omega_k^j$, *for every i,j = 1, ... n.*

Equations $1,2$ and 3 are known as the **Cartan′s structure equations**. These equations are listed below

1. $d\omega^i = \sum_{j=1}^n \omega^j \wedge \omega_i^j$
2. $dg_{ij} = \sum_{k=1}^n (g_{kj}\omega_i^k + g_{ki}\omega_j^k)$,
3. $\Omega_i^j = d\omega_i^j - \sum_{k=1}^n \omega_i^k \wedge \omega_k^j$,

where $\omega^i(X_j) = \delta_{ij}$, $\omega_j^k = \sum_{i=1}^n \Gamma_{ij}^k \omega^i$ and $\Omega_i^j = \sum_{k<l} R_{kli}^j \omega^k \wedge \omega^l$.

Example. *For a field of orthonormal frames in* $\mathbb{R}^n$ *with Euclidean metric, the curvature forms must vanish (as R=O), and we obtain the following structure equations* :

1. $d\omega^i = \sum_{j=1}^n \omega^j \wedge \omega_i^j$

2. $\omega_i^j + \omega_j^i = 0$,

3. $d\omega_i^j = \sum_{k=1}^n \omega_i^k \wedge \omega_k^j$,

Proposition 5.1.3. *If* $\{E_1,E_2\}$ *and* $\{F_1,F_2\}$ *have the same orientation then, denoting by* ϖ_1^2 *and* ϖ_1^2 *the corresponding connection forms, we have* $\varpi_1^2 - \omega_1^2 = \sigma$, *where* $\sigma := adb - bda$.*(where* $a,b\colon V \to \mathbb{R}$ *are such that* $a^2+b^2=1$*)*

5.2 Gauss - Bonnet theorem

Using Cartan's structure equations we will now prove the **Gauss** – **Bonnet Theorem**. Let M be a compact, oriented, 2-dimensional manifold and X a vector field o M.

Definition 5.2.1. *A point $p \in M$ is said to be a* **singular point** *of X if $X_p = 0$. A singular point is said to be an* **isolated singularity** *if there exists a neighbourhood $V \subset M$ of p such that p is the only singular point of X in V.*

Since M is compact, if all the singularities of X are isolated then they are infinite in number. To each isolated singularity $p \in V$ of $X \in \mathscr{X}(M)$ one can associate an integer number, called the **index** of X at p, as follows :

1. fix a Riemannian metric in M
2. choose a positively oriented orthonormal frame $\{F_1, F_2\}$ defined on $V \backslash \{p\}$, such that

   $$F_1 = \frac{X}{\|X\|}$$

   let $\{\overline{\omega}^1, \{\overline{\omega}^2\}$ be the dual coframe and let $\{\overline{\omega}_1{}^2$ be the corresponding connection form
3. possibly shrinking V, choose a positively oriented orthonormal frame $\{E_1, E_2\}$, defined on V, with dual coframe $\{\omega^1, \omega^2\}$ and connection form ω_1^2
4. take a neighbourhood D of p in V, homeomorphic to a disc, with smooth boundary ∂D, endowed with the induced orientation, and define the index I_p of X at p as

   $$2\pi I_p = \int_{\partial D} \sigma,$$

   where $\sigma := \overline{\omega}_1^2 - \omega_1^2$ is the form in Proposition 5.1.3

Here σ satisfies $\sigma = d\theta$, where θ is the angle between E_1 and F_1. Hence I_p must be an integer. Next one should check the well definedness of I_p, and also show that I_p does not depend on the choice of Riemannian metric.

Theorem 5.2.1 (Gauss- Bonnet)**.** *Let M be a compact, oriented, 2-dimensional manifold and let X be a vector field in M with isolated singularities $p_1 \dots p_k$. Then*

$$\int_M K = 2\pi \sum_{i=1}^{k} I_{p_i}$$

for any Riemannian metric on M, where K is the Gauss curvature.

Proof. Consider the positively oriented orthonormal frame $\{F_1, F_2\}$, with

$$F_1 = \frac{X}{\|X\|}$$

defined on $M \ \cup_{i=1}^{k} \{p_i\}$, with dual coframe $\{\overline{\varpi}^1, \overline{\varpi}^2\}$ and connection form $\overline{\varpi}_1^2$. For $r > 0$ sufficiently small, we take $B_i \coloneqq B_r(p_i)$ such that $\overline{B_i} \cap \overline{B_j} = \phi$ for $i \neq j$ and note that

$$\int_{M \setminus \cup_{i=1}^{k} B_i} K = \int_{M \setminus \cup_{i=1}^{k} B_i} K \overline{\varpi}^1 \wedge \overline{\varpi}^2 = -\int_{M \setminus \cup_{i=1}^{k} B_i} K d\overline{\varpi_1}^2$$
$$\int_{\cup_{i=1}^{k} \partial B_i} \overline{\varpi_1}^2 = \sum_{i=1}^{k} \int_{\partial B_i} \overline{\varpi_1}^2,$$

where the ∂B_i have the orientation induced by the orientation of B_i. Taking the limit as $r \to 0$ one obtains

$$\int_M K = 2\pi \sum_{i=1}^{k} I_{p_i}.$$

□

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