# Study of Littlewood-Richardson coefficients and an application

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# Certificate of Examination

This is to certify that the dissertation titled "Study of Littlewood-Richardson coefficients and an application" submitted by Mr. Sachin C S (Reg. No. MS15150) for the partial fulfilment of BS-MS dual degree programme of the Institute, has been examined by the thesis committee duly appointed by the Institute. The committee finds the work done by the candidate satisfactory and recommends that the report be accepted.

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Dated: May 29, 2021

# Declaration

The work in this dissertation has been carried out by me under the guidance of Dr. Tanusree Khandai at the Indian Institute of Science Education and Research Mohali.

This work has not been submitted in part or in full for a degree, a diploma, or a fellowship to any other university or institute. Whenever contributions of others are involved, every effort is made to indicate this clearly, with due acknowledgement of collaborative research and discussions. This thesis is a bonafide record of original work done by me and all sources listed within have been detailed in the bibliography.

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Dated: May 29, 2021

In my capacity as the supervisor of the candidate's project work, I certify that the above statements by the candidate are true to the best of my knowledge.

Tanusree Khandai

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# Contents

# Abstract

Symmetric functions arise in several branches of mathematics such as Combinatorics, Representation of symmetric groups and Algebraic geometry. Schur polynomials are a family of symmetric polynomials that are indexed by partitions of positive integers. These polynomials span the space of symmetric polynomials and their products can again be written as a linear combination of Schur polynomials with non-negative integer coefficients known as Littlewood-Richardson coefficients.

In this thesis, we begin with the study of the combinatorics of Young tableaux, the words associated to them,the plactic monoid and the tableau ring. In the subsequent chapters we discuss the Robinson-Schensted correspondence and its applications,introduces the Schur polynomials and LR coefficients and discuss three combinatorial models that help compute the LR coefficients. Finally in chapter 4 we apply the combinatorial methods studied to give an elementary proof of a result on Schur positivity in a very special case.

# Chapter 1

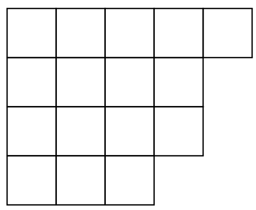
# The Tableau Ring

In this chapter, we fix the basic notations of a combinatorial object called young tableaux

## 1.1 Notations

### 1.1.1 Young diagram

A Young diagram is a collection of boxes, arranged in left-justified rows, with a weakly decreasing number of boxes in each row. Listing the number of boxes in each row gives a partition of the integer n which is the total number of boxes in the diagram. Also conversely every partition of n corresponds to a Young diagram.
For example , the partition of 16 into 5+4+4+3 corresponds to the Young diagram.



We usually denote a partition by $\lambda = (\lambda_1,....,\lambda_l)$ , a sequence of weakly decreasing positive integers. We usually identify a partition $\lambda$ with the corresponding diagram.
Any way of putting a positive integer in each box of a Young diagram will be called a filling of the diagram.

### 1.1.2 Young tableaux

A Young tableau, or simply a tableau is, a filling that is

1. weakly increasing across each row

2. strictly increasing down each column

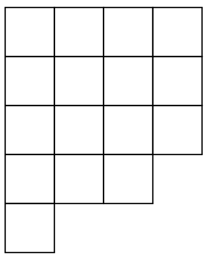
We say that $\lambda$ is the shape of tableau. For example

| | | | | |
|---|---|---|---|---|
| 1 | 2 | 2 | 3 | 5 |
| 2 | 3 | 4 | 4 | |
| 3 | 4 | 5 | 6 | |
| 5 | 5 | 6 | | |

is a tableau of shape (5,4,4,3).
A standard tableau is a tableau in which the entries are the numbers from 1 to n each occurring once. For example

| | | | | |
|---|---|---|---|---|
| 1 | 3 | 7 | 12 | 15 |
| 2 | 5 | 10 | 13 | |
| 4 | 8 | 11 | 16 | |
| 6 | 9 | 14 | | |

The entries of tableaux can be taken from any totally ordered set, but we usually take positive integers.
Flipping a Young diagram over its main diagonal gives the conjugate diagram. The conjugate of $\lambda$ will be denoted here by $\tilde{\lambda}$. For example, the partition in the above examples (5,4,4,3) has the conjugate diagram (4,4,4,3,1)



Any numbering T of a diagram determines a numbering of the conjugate, called the transpose, and denoted $T^{\tau}$. The transpose of a standard tableau is a standard tableau, but the transpose of a tableau need not be a tableau.

### 1.1.3 Schur polynomial

Associated to each partition $\lambda$ and integer m such that $\lambda$ has at most m rows, there is an important symmetric polynomial $s_{\lambda}(x_1, ...., x_m)$ called Schur polynomial.
These polynomials can be defined in the following way. To any numbering T of a Young diagram we have a monomial, denoted $x^T$, which is the product of the variables $x_i$ corresponding to the i's that occur in T. For the very first example of tableau, this monomial is

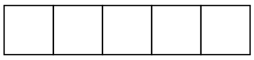
$x_1x_2^3x_3^3x_4^3x_5^4x_6^2$. Formally,

$$x^T = \prod_{i=1}^{m}(x_i)^{number\ of\ times\ i\ occur\ in\ T}$$

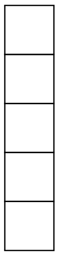
The Schur polynomial $s_\lambda(x_1, ...., x_m)$ is the sum

$$s_\lambda(x_1, ...., x_m) = \sum x^T$$

of all monomials coming from tableaux T of shape $\lambda$ using the numbers from 1 to m. These polynomials are symmetric in the variables $x_1, ...., x_m$, and they form an additive basis for the ring of symmetric polynomials. we will prove this results later. We mention here two special cases of Schur polynomial.
The Young diagram of $\lambda$ = (n) has n boxes in a row



The Schur polynomial for this parition is $n^{th}$ complete symmetric polynomial, which is the sum of all distinct monomials of degree n in the variables $x_1, ...., x_m$; this is usually denoted $h_n(x_1, ...., x_m)$. For the other extreme case i.e., $\lambda$ = ($1^n$), the Young diagram is



The corresponding Schur polynomial is the $n^{th}$ elementary symmetric polynomial, which is the sum of all monomials $x_{i1}x_{i2}....x_{in}$ for all strictly increasing sequences $1 \leq i_1 < i_2 < ... < i_n \leq m$ and is denoted $e_n(x_1, ...., x_m)$.

### 1.1.4 Skew tableau

A partition $\mu = (\mu_1, \mu_2....)$is said to be contained in a partition $\lambda = (\lambda_1, \lambda_2....)$ if $\mu_i \leq \lambda_i$ for all i. A skew diagram is the diagram obtained by removing a smaller young diagram from a larger one that contains it. The resulting skew shape is denoted $\lambda/\mu$. A skew tableau is a filling of the boxes of a skew diagram with positive integers, weakly increasing in rows and strictly increasing in columns. For example, if $\lambda$ =(4,4,3,1) and $\mu$ =(3,3,1), then the following is a skew tableau on $\lambda/\mu$

$$\begin{array}{cccc} & & & 2 \\ & & & 4 \\ & 1 & 3 & \\ 2 & & & \end{array}$$

The set $\{1, 2, ...., m\}$ of first m positive integers is denoted [m].

## 1.2 Calculus of tableaux

There are two basic operations on tableaux called the Schensted-bumping operation and Schutzenberger sliding operation. Majority of combinatorial properties of tableaux can be deduced from these two operations. In this section we define these two operations and claim that these two operations can be used to define a multiplication on the set of all tableaux.

### 1.2.1 Row bumping or Row insertion

The algorithm for this process is as follows:- We take a tableau T and a positive integer x. first we check x against the entries of first row of T, if x is at least as large as the last entry of the first row we simply place x in a new box at the end of the first row of T. the resulting diagram is clearly a tableaux. Now,if that is not the case, that is there is an entry in the first row of T which is strictly larger than x then we bump the left-most entry which is strictly larger than x and place x in that box. Now the bumped entry comes to the second row and the above process is repeated, this process is continued until bumped entry can be put at the end the row it is bumped into, or until it is bumped out at the bottom, where it forms a new row with one entry.

So what happens here is we take a tableau T and a positive integer x and inserts this positive integer into this tableau to get a new tableaux denoted $T \longleftarrow x$ which has one additional box than T and with some rearrangement so as to preserve the properties of a tableau.

The weakly increasing property of rows of a tableau is clearly preserved. Now here we check that the strictly increasing property of tableau is also preserved. Let z be the entry directly below y of two successive rows in a tableau ($y < z$),let x bumps the entry y in that row ($x < y$), so now y can only bump entries to the left of or the box containing z in the next row, thus the entry lying above the new position of y is no larger than x, so is strictly smaller than y. Hence the strictly increasing property across column of a tableau is preserved.

**Example 1.** *Consider the row insertion of 2 into the given tableau*

| | | | | |
|---|---|---|---|---|
| 1 | 2 | 2 | 3 | 5 |
| 2 | 3 | 4 | 4 | |
| 3 | 4 | 5 | 6 | |
| 5 | 5 | 6 | | |

*First 2 comes and bumps 3 from the first row, this bumped 3 bumps the left-most 4 of the second row, which then bumps the 5 of third row, which then bumps 6 of the fourth row and finally this 6 is placed in a new box in a new row. So the resulting tableau will look like:*

| | | | | |
|---|---|---|---|---|
| 1 | 2 | 2 | 2 | 5 |
| 2 | 3 | 3 | 4 | |
| 3 | 4 | 4 | 6 | |
| 5 | 5 | 5 | | |
| 6 | | | | |

Note that the above operation of row bumping is invertible. That is given the resulting tableau of row bumping, together with the position of new added box, we can retrieve the initial tableau and the positive integer inserted just by reversing the above process.
A row bumping $T \longleftarrow x$ defines a collection R of boxes of the tableau $T \longleftarrow x$ consisting of all the bumped boxes in the row insertion in sequence together with the box where the last bumped element lands. This is called the bumping route. Also the box which is in $T \longleftarrow x$ but not in T is called the new box of the row bumping.
For example the bumping route of the previous example consists of the shaded boxes in the given figure, and the new box is the box containing 6 in the last row.

| | | | | |
|---|---|---|---|---|
| 1 | 2 | 2 | **2** | 5 |
| 2 | 3 | **3** | 4 | |
| 3 | 4 | **4** | 6 | |
| 5 | 5 | **5** | | |
| **6** | | | | |

It is clear from the process of bumping that a bumping route has at most one box in each of several successive rows, starting at the top. We say that route $R$ is strictly left( respectively weakly left) of a route $R'$, if for each row which contains a box of $R'$, $R$ has a box which is left of( respectively left or equal to) the box in $R'$.
Now using these terminologies we can introduce a new lemma about the bumping process which states the consequence of two successive row-insertions. This lemma relates the relative size of elements inserted to the position of newly added boxes

**Lemma 1.2.1.** *Row Bumping Lemma: Consider two successive row insertions, first row inserting $x$ in a tableau T and then row inserting $x'$ in the resulting tableau $T \longleftarrow x$, giving rise to two routes $R$ and $R'$, and two new boxes $B$ and $B'$.*

- *If $x \leq x'$, then $R$ is strictly left of $R'$, and $B$ is strictly left of and weakly below $B'$.*
- *If $x > x'$, then $R'$ is weakly left of $R$ and $B'$ is weakly left of and strictly below $B$.*

*Proof.* Suppose $x \leq x'$, $x$ bumps an element $y$ from the first row of T. now when $x'$ comes to bump an element $y'$ of the first row of $T \longleftarrow x$, it can only bump an element which is strictly to the right of box containing $x$ since $x \leq x'$. In particular $y \leq y'$. This argument continues from row to row. The route for $R$ cannot stop above that of $R'$ since any bumping route cannot end in the middle of a row, now if $R'$ stops first, the route for $R$ never moves to the right, hence the box $B$ must be strictly left of and weakly below $B'$.
Now to prove the second part of the lemma, suppose $x > x'$. let $x$ bumps the entry $y$ from the first row , now when $x'$ comes to bumps the entries of the first row of $T \longleftarrow x$ it can only bump the entries at or to the left of the box where $x$ is bumped since $x > x'$ so in particular we have $y > y'$. The same argument can be repeated on successive rows. Here the route $R'$ must continue at least one row below that of $R$. This proves the lemma. □

**Proposition 1.2.2.** *Let T be a tableau of shape $\lambda$, and let*

$$U = ((T \longleftarrow x_1) \longleftarrow x_2).... \longleftarrow x_p,$$

*for some $x_1, x_2, ..., x_p$. Let $\mu$ be the shape of U. If $x_1 \leq x_2 \leq .... \leq x_p$, then no two of the boxes in $\mu/\lambda$ are in the same column. Conversely, suppose U is a tableau on a shape $\mu$, and $\lambda$ a young diagram contained in $\mu$, with p boxes in $\mu/\lambda$. If no two boxes in $\mu/\lambda$ are in the same column, then there is a unique tableau T of shape $\lambda$, and unique $x_1 \leq x_2 \leq .... \leq x_p$ such that $U = ((T \longleftarrow x_1) \longleftarrow x_2).... \longleftarrow x_p$,.*

*Proof.* The first claim of the proposition is a direct consequence of the Row Bumping lemma. So we need only prove the converse. Here we have the diagram $\mu/\lambda$ with no two boxes in the same column. So we will do the reverse row bumping on U using the boxes of $\mu/\lambda$, starting from the right-most box and going to the left. Let T be the resulting tableau and $x_p, ...., x_1$ the elements bumped out in order. Row Bumping lemma ensures the condition $x_1 \leq x_2 \leq .... \leq x_p$ . This proves the converse. □

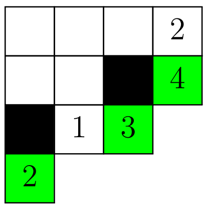
This bumping process can be used to define product on a given pair of tableaux T and U. The product T*U is defined in the following way. First of all we row insert the first element of the last row of U into T then the next element in the same row and so on till all the elements of the last row of U is inserted into T. Now repeat the same process with the second last row U from left to right in order and move up U until all the rows are exhausted. This will give a new tableaux and is said to be the product of T and U denoted T*U.

**Proposition 1.2.3.** *This product operation makes the set of all tableaux into an associative monoid. The empty tableau is a unit in this monoid.*

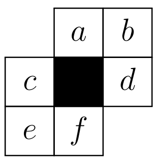
We will prove this proposition later

### 1.2.2 Sliding; jeu de taquin

This is an another operation on skew tableaux,which can also be used to construct a product between tableaux.
An inside corner of a skew diagram $\lambda/\mu$ is a box in the deleted diagram $\mu$ such that the boxes below and to the right are not in $\mu$. In the given example the shaded boxes with black are the inside corners for $\lambda$ = (4,4,3,1) and $\mu$ = (3,3,1)




Similarly an outside corner is a box in the diagram $\lambda$ such that boxes neither to the right or below it is in $\lambda$. In the above example the boxes shaded with green are outside corners.
Note that a skew diagram can arise out of more than one choice of $\lambda$ and $\mu$.
The procedure for sliding operation is as follows; Given a skew tableau and an inside corner, it slides smaller of its two neighbours one to the right or below into the empty box. In case both neighbours are the same the one below is preferred. This induce a new hole(empty box) in the skew diagram. Now the above process is repeated to this empty box and is continued till the hole becomes an outside corner. Since in the process of sliding the box which is added is an inside corner and box which is removed is an outside corner The shape of the diagram remains a skew diagram. Now we are gonna check the weakly increasing and strictly increasing property of skew tableau in both horizontal and vertical slide.
Lets first check the horizontal slide.




let $d < f$, then $d$ slides into the hole and hole shifts to the box where $d$ initially was, then the diagram look like;

$$\begin{array}{ccc} & a & b \\ c & d & \blacksquare \\ e & f & \end{array}$$

Since $a \leq b < d$ and $d < f$ we have $a < d < f$ as required. Hence horizontal case is verified.
Now let $f \leq d$ then $f$ slides into the hole and hole slides into the position of $f$, then the diagram looks like;

$$\begin{array}{ccc} & a & b \\ c & f & d \\ e & \blacksquare & \end{array}$$

Since $c < e \leq f$ and $f \leq d$ we have $c \leq f \leq d$ as required. So vertical case also verified.
Similar to the bumping process the sliding process is also invertible.
Given any skew tableau S, take any inside corner and do this sliding process, take the resulting skew tableau and an inside corner and do the same process and continue this process until there is no more inside corners. The result is a tableau. This tableau is called a rectification of S and the whole process is called the jeu de taquin.

**Proposition 1.2.4.** *Starting with a given skew tableau, all choices of inside corners lead to the same rectified tableau. that is, it is independent of the sequence of inside corners chosen.*

We will prove this proposition also later.
Now lets define the product on a pair of tableaux using sliding. Given any two tableau T and U construct a skew tableau $T * U$ by taking a rectangle of empty squares with same number of columns as T and same number of rows as U, and place U to the right and T below this rectangle. For example,

$$T * U = \begin{array}{ccccc} \square & \square & \square & 2 & 5 \\ \square & \square & \square & 4 & \\ \square & \square & \square & 6 & \\ 5 & 5 & 5 & & \\ 6 & & & & \end{array}$$

The product of T and U is defined to be Rect($T * U$), which is unique assuming the previous proposition.

**Proposition 1.2.5.** *The product defined using the bumping and sliding agrees.*

We will prove this proposition too later.

## 1.3 Words; The plactic monoid

Here we introduce the notion of words, which helps us to represent a tableau as a sequence of positive integers. This will be crucial in proving the propositions stated earlier.
Any sequence of positive integers is defined to be a word. Given two words $w$ and $w'$, $w*w'$ is the juxtaposition of words $w$ and $w'$
Let $w$ = 2223 and $w'$ = 3431, then $w * w'$ = 22233431.
Now we define the word of a tableau T by reading the entries of T from left to right and bottom to top in order. For example the tableau

| | | | | |
|---|---|---|---|---|
| 1 | 2 | 2 | 3 | 5 |
| 2 | 3 | 4 | 4 | |
| 3 | 4 | 5 | 6 | |
| 5 | 5 | 6 | | |

has word 5563456234412235. Usually the word of a tableau T is denoted by $w(T)$. Conversely given the word of a tableau $w(T)$, we can recover the tableau T. For this we first breakdown the given word into pieces, that is whenever an entry is strictly larger than the preceding entry we put a break after the former. This partitions the word into pieces. Here each piece is a row of the tableau T. For the above word it looks like $556|3456|2344|12235$, Now we stack each piece over the preceding piece in a left-justified manner. The result is the tableau T.
Here note that every word cannot come from a tableau. For a word to come from a tableau, the pieces must have weakly increasing length and when stacked up , the columns should have strictly increasing property. Also note that any word can arise out of a skew tableau and different skew tableaux can have the same word which is not true in the case of tableaux.

Now we are going to understand what the bumping process does to the word of a tableau. And this will help us in relating the word of a product of two tableaux to the words of its factors. Suppose an element $x$ is inserted into a row. We first factor this row as $ux'v$ where $u$ and $v$ are weakly decreasing words, $x'$ is a integer, such that each integer in $u$ is no larger than $x$ and $x'$ is strictly larger than $x$. Bumping process replaces $x'$ by $x$ , so the word $ux'v$ becomes $uxv$ and $x'$ is bumped into the next row. So the basic algorithm is

$$(ux'v)*x \longrightarrow x'uxv \qquad \text{if } u \leq x < x' \leq v$$

For example row insertion of 2 into the tableau with word $(44)(234)(1225)$ can be expressed as follows;
$(44)(234)(1225)*2 \mapsto (44)(234)5(1222) \mapsto (44)(2345)(1222)$

Bumping process can be explained in the language of a computer program, In the process, the word is broken down into atomic pieces as described earlier. This helps to reveal its inner structure and will be crucial to the proofs of the propositions stated earlier. The procedure is as follows;
When we row insert an element x into a tableau T. We first check x against the last entry of first row of T. If it is not larger than x, place x in a new box at the end of first row. Now in case the last entry z is larger than x and also the entry y before if it is larger than x, we shift x one step to the left and repeat the process.The steps can be listed as follows;
Lets factor the first row into $ux'v$, where the relations of u,$x'$,v are same as above. then,

$$\begin{array}{lll} ux'v_1...v_{q-1}v_qx & \mapsto & ux'v_1...v_{q-1}xv_q \quad (x < v_{q-1} \leq v_q) \\ . & \mapsto & ux'v_1...xv_{q-1}v_q \quad (x < v_{q-2} \leq v_{q-1}) \\ ..... & \mapsto & ux'v_1xv_2...v_{q-1}v_q \quad (x < v_1 \leq v_2) \\ . & \mapsto & ux'xv_1...v_{q-1}v_q \quad (x < x' \leq v_1) \end{array}$$

In each of the above steps the basic transformation is

$$yzx \quad \mapsto \quad yxz \quad \text{if } x < y \leq z \qquad (K')$$

Let us continue the above process with $x$ bumping $x'$ and $x'$ successively moving to left;

$$\begin{array}{lll} u_1...u_{p-1}v_px'xv & \mapsto & u_1...u_{p-1}x'u_pxv \quad (u_p \leq x < x') \\ . & \mapsto & u_1...x'u_{p-1}u_pxv \quad (u_{p-1} \leq u_p < x') \\ ..... & \mapsto & u_1x'u_2u_3...u_pxv \quad (u_2 \leq u_3 < x') \\ . & \mapsto & x'u_1u_2...u_pxv \quad (u_1 \leq u_2 < x') \end{array}$$

Each of these transformations is governed by the rule

$$xzy \quad \mapsto \quad zxy \quad \text{if } x \leq y < z \qquad (K'')$$

An elementary Knuth transformation on a word applies one of the transformations ($K'$) or ($K''$), or their inverses, to three consecutive letters in the word.
Two words are said to be Knuth equivalent if each can be changed to another by a sequence of elementary Knuth transformations and denoted by $w \equiv w'$. All of the above discussion proves the following proposition.

**Proposition 1.3.1.** *For any tableau T and a positive integer x,*

$$w(T \longleftarrow x) \equiv w(T) * x$$

*That is, word of a tableau T after inserting x is Knuth equivalent to the word of T juxtaposed with x.*

**Corollary 1.3.1.1.** *If T*U is the product of two tableaux T and U, constructed by row-inserting the word of U into T, then*

$$w(T * U) \equiv w(T) * w(U).$$

Corollary follows since the first construction of the product T*U of two tableaux was by successively row inserting the letters of the word of U into T.
But the fact that sliding procedure preserves the Knuth equivalence of the words of skew tableau is not so obvious. We are gonna prove that the Knuth equivalence class of a word is unchanged by each step in a slide. One thing to noted here is that in each step of sliding the configuration may not be a skew tableau, but rather a skew tableau with a hole in it. However the word of such configuration is also defined by reading the entries from left to right and bottom to top.
In case of a horizontal slide, the word itself is not changed. Therefore the claim is evident.
Now let us see what happens in a vertical slide. Consider the general case of the given tableau,

| | | | | | | |
|---|---|---|---|---|---|---|
| $u_1$ | .... | $u_p$ | $\blacksquare$ | $y_1$ | .... | $y_q$ |
| $v_1$ | .... | $v_p$ | $x$ | $z_1$ | .... | $z_q$ |

changing to

| | | | | | | |
|---|---|---|---|---|---|---|
| $u_1$ | .... | $u_p$ | $x$ | $y_1$ | .... | $y_q$ |
| $v_1$ | .... | $v_p$ | $\blacksquare$ | $z_1$ | .... | $z_q$ |

where $u_i$'s, $v_i$'s, $y_j$'s, and $z_j$'s are weakly increasing sequences, $u_i < v_i$ and $y_j < z_j$ for all i and j, also $v_p \leq x \leq y_1$.
Let $u = u_1....u_p$ , $v = v_1....v_p$ , $y = y_1....y_q$ $z = z_1.....z_q$.
Given this we must prove that

$$vxzuy \equiv vzuxy \tag{1.1}$$

We will prove this using induction on the value of p. When p=0, above equation becomes $xzy \equiv zxy$. On expanding we have

$$xz_1....z_qy_1....y_q \equiv z_1....z_qxy_1....y_q$$

Consider the left hand side of the above equation, If $y_1$ is inserted in a row with entries x,$z_1....z_q$, then the entry $z_1$ is bumped out of the row. We know that row-insertion respects Knuth equivalence by proposition(1.3.1). Therefore we have the following,

$$xz_1....z_qy_1 \equiv z_1xy_1z_2...z_q$$

Now row-insertion of $y_2$ into the row with entries x,$y_1$,$z_2$,....,$z_q$ bumps the entry $z_2$ and hence have the following,

$$xy_1z_2...z_qy_2 \equiv z_2xy_1y_2z_3....z_q$$

Continuing this process of row-insertion till $y_q$ we have

$$xz_1....z_qy_1....y_q \equiv z_1....z_qxy_1....y_q$$

which is as required.
Now let $p \geq 1$ and assume equation (1.1) for smaller p. Set
$u' = u_2....u_p, \quad v' = v_2...v_p$
As usual we start with the left hand side of equation(1.1) i.e, $vxzuy = v_1v'xzu_1u'y$.
Row inserting $u_1$ in the row with word $v_1v'xz$ bumps $v_1$, giving $v_1v'xzu_1 \equiv v_1u_1v'xz$ by proposition(1.3.1). Which implies

$$v_1v'xzu_1u'y \equiv v_1u_1v'xzu'y$$

Assumed equation for p-1 gives $v'xzu'y \equiv v'zu'xy$, so we have,

$$v_1u_1v'xzu'y \equiv v_1u_1v'zu'xy$$

Finally, row-inserting $u_1$ in the row with word $v_1v'z$ bumps $v_1$, giving the equivalence $v_1v'zu_1 \equiv v_1u_1v'z$. hence it follows

$$v_1u_1v'zu'xy \equiv v_1v'zu_1u'xy = vzuxy$$

So we have proved for p assuming for p-1.
This completes the proof of the following proposition:

**Proposition 1.3.2.** *If one skew tableau can be obtained from another skew tableau by a sequence of slides, then their words are Knuth equivalent.*

Now We are going to state an important result which will form the base for the proofs of the propositions stated earlier.

**Theorem 1.3.3.** *Every word is Knuth equivalent to the word of a unique tableau.*

*Proof.* Given an arbitrary word say $w = x_1x_2...x_r$, Construct a tableau by the following procedure; Consider a single box tableau with $x_1$ as its entry, now row insert successively $x_2$,$x_3$,....$x_r$. that is

$$((...((\boxed{\mathrm{x}_1} \leftarrow x_2) \dot{\leftarrow} ..) \leftarrow x_r)$$

This gives us a tableau. By proposition (1.3.1) word of this tableau is Knuth equivalent to the word $w$. We call this the canonical procedure for constructing a tableau whose word is Knuth equivalent to a given word, and we denote the resulting tableau by $P(w)$. The uniqueness claim in the theorem is not evident. This will require a new idea and will be proved later. □

Now for the time being we assume the theorem and draw a few consequences, including the proofs of the three propositions (1.2.3, 1.2.4, 1.2.5) stated earlier. From proposition(1.3.2) and the theorem it follows;

**Corollary 1.3.3.1.** *The rectification of a skew tableau $S$ is the unique tableau whose word is Knuth equivalent to the word of $S$. If $S$ and $S'$ are skew tableaux, then Rect($S$) = Rect($S'$) if and only if $w(S) \equiv w(S')$.*

Theorem (1.3.3) can be used to define a third product T*U of two tableaux. Define T*U to be the unique tableau whose word is Knuth equivalent to the word $w(T) * w(U)$, where the product of two words is defined simply by writing one after the other.

**Corollary 1.3.3.2.** *The three constructions of the product of two tableaux agree.*

*Proof.* It is enough to show that each of the first two constructions produces a product T*U with the property that $w(T * U) = w(T) * w(U)$. For the construction using bumping this follows from the corollary (1.3.1.1) and for the construction using sliding it follows from the proposition (1.3.2). □

Our next task is to prove the uniqueness claim of Theorem(1.3.3). Once we have proved this uniqueness, then from the theory we constructed till now we have proof for the propositions(1.2.3, 1.2.4, 1.2.5). As a first step towards this task we are going to introduce the concept of increasing sequences of a word.

## 1.4 Increasing sequences

Given a word $w$, define the following:
Let $L(w, 1)$ be the length of a longest weakly increasing sequence of $w$ and let $L(w, 2)$ be the largest number that can be realized as the sum of the length of two disjoint weakly increasing sequences of $w$. Similarly for any $k \in \mathbb{N}$, let $L(w, k)$ be the largest possible number that can be obtained as the sum of the length of k disjoint weakly increasing sequences of $w$. To illustrate this concept, let

$$w = 1121322132.$$

Since $111223$ is a weakly increasing sequence of $w$ and no sequence in $w$ of length $> 6$ is a weakly increasing sequence we have $L(w, 1) = 6$.
Clearly the whole word $w$ cannot be written as a sum of two disjoint weakly increasing sequences of $w$, therefore $L(w, 2) \neq 10$. Since $111222$ and $233$ form a disjoint weakly increasing sequences of $w$ we have $L(w, 2) = 9$.
So it immediately follows that $L(w, 3) = 10$.
Also note that for any $k > 3$, $L(w, k) = 10$ ( by considering zero sequences).

**Proposition 1.4.1.** *Let $T$ be a tableau on the shape $\lambda = (\lambda_1, ..., \lambda_m)$ and $w$ be it's word. Then we have,*

$$L(w, k) = \lambda_1 + \lambda_2 + ... + \lambda_m \qquad \textit{for all } k \geq 1$$

*Proof.* Assume we are given a weakly increasing sequence $w_1$ of $w$. Suppose we have two entries from the same column of T in the sequence $w_1$. This would contradict the weakly increasing property of $w_1$, which means that there is at most one entry from each column of T for any weakly increasing sequence of $w$. Since the word of the first of T forms a weakly increasing sequence we have

$$L(w, 1) = \text{number of columns of T} = \lambda_1$$

Let $w'_1$ = word of first row of T
$w'_2$= word of second row of T
.
.
$w'_k$ = word of k$^{th}$ row of T.
By the definition of tableau each $w'_i$ is a weakly increasing sequence and for any $k \geq 1$ the sum of the length of these sequences is $\lambda_1 + \lambda_2 + ... + \lambda_k$.
Now if we take any k disjoint weakly increasing sequences from $w$. Since any weakly increasing can contain at most one entry from each column of T, we can replace each of

these k disjoint sequences by sequences from the first k rows of T with the same number of boxes. This proves our claim.

□

**Proposition 1.4.2.** *Let $w$ and $w'$ be two Knuth equivalent words, then*

$$L(w,k) = L(w',k) \qquad \textit{for all k.}$$

*Proof.* The words $w$ and $w'$ and Knuth equivalent implies that $w'$ can be obtained from $w$ by a sequence of elementary Knuth transformations and vice versa. So it suffices to show that $L(w,k) = L(w',k)$ when $w$ and $w'$ are two sides of an elementary Knuth transformations. The two possible cases of this are the following,

(i) one $u \cdot yxz \cdot v \equiv u \cdot yzx \cdot v \qquad (x < y \leq z)$

(ii) $u \cdot xzy \cdot v \equiv u \cdot zxy \cdot v \qquad (x \leq y < z)$

Let $M$ be a collection of disjoint weakly increasing sequence of $w'$, then clearly the same collection $M$ forms a disjoint weakly increasing for $w$ also in both cases (1) and (2). Therefore we have

$$L(w,k) \geq L(w',k) \qquad (1)$$

Now need to prove the other direction. That is,$L(w',k) \geq L(w,k)$. This is not trivial as before, so we will prove this in cases.

<u>Case1</u>

Let $M$ be a collection of disjoint weakly increasing sequences of $w$. Suppose no sequence in this collection $M$ have both x and z occurring simultaneously, then the same collection $M$ forms a disjoint weakly increasing sequence for $w'$.

<u>Case2</u>

Suppose there exist a sequence in $M$ with both x and z occurring simultaneously, let us assume this sequence to be $u_1 \cdot xz \cdot v_1$ then the same sequence won't be a weakly increasing sequence for $w'$.To prove this case we divide case into two subcases.

<u>subcase 1</u>

Suppose no sequence in $M$ uses the entry y, then the sequence $u_1 \cdot yz \cdot v_1$ forms a weakly increasing sequence with the same number of entries in case (i) and $u_1 \cdot xy \cdot v_1$ in case (ii) for $w'$. Remaining all other sequences $M$ unchanged, we have a new collection $M'$ with same total number of entries of $M$.

<u>subcase 2</u>

Suppose some sequence in $M$ uses the entry y, that is, we also have a sequence $u_2 \cdot y \cdot v_2$

in $M$. Here the sequences $u_2 \cdot yz \cdot v_1$ and $u_1 \cdot x \cdot v_2$ forms a disjoint weakly increasing sequence for case (i) and $u_1 \cdot xy \cdot v_2$ and $u_2 \cdot z \cdot v_1$ in case (ii) for $w'$. Similar to subcase 1, Remaining all other sequences of $M$ unchanged we have a new collection $M'$ with same total number of entries as that of $M$.

So in short, given a collection of disjoint sequences of $w$ we were able to construct a collection $M'$ for $w'$ with the same total number of entries. This proves

$$L(w', k) \geq L(w, k) \qquad (2)$$

Combining (1) and (2) we have,

$$L(w', k) = L(w, k) \qquad \text{for all k,}$$

This completes the proof.

□

REMARK: Proposition(1.5.1) , (1.5.2) together implies that for each word in a equivalence class, when we do the canonical procedure of constructing a tableau, each of the resulting tableau will have the same shape.

**Proposition 1.4.3.** *If $w$ and $w'$ are two Knuth equivalent words. Let $w_0$ and $w_0'$ are the words obtained by removing $l$ largest and $m$ smallest letters from each, then $w_0$ and $w_0'$ are Knuth equivalent words*

*Proof.* It suffices to check whether removing the largest letter from two Knuth equivalent words gives two equivalent words then all the claims of the proposition will follow by induction and symmetry. Here it is sufficient to check the proposition holds in cases (i) and (ii) of the proof of proposition(1.5.2).

Case 1

If the letter removed from the words is not one among x,y and z, then the Knuth equivalence of the resulting words is obvious.

Case 2

Now if the entry removed is one among x,y and z, then the entry removed being largest it should be z, in which case the resulting words are same.

This proves the proposition.

□

Now We have enough tools to prove the uniqueness of Theorem(1.3.3).

**Proposition 1.4.4.** *Given a tableau $T$ and its word $w(T)$. If a word $w$ is Knuth equivalent to $w(T)$, then $w$ uniquely determines $T$.*

*Proof.* To prove this we use induction on the length of the word $w$. For a word of length 1 the statement clearly holds. As a consequence of proposition (1.5.1),(1.5.2) the shape $\lambda$ of T is determined by $w$.

$$\lambda_k = L(w,k) - L(w,k-1)$$

Suppose y is the largest letter in $w$. Let $T_0$ be the tableau obtained by removing the right most occurrence of y from $T$ and let $w_0$ be the word obtained by removing the right most occurrence of y from $w$. Clearly then $w(T)_0 = w(T_0)$. By proposition 3 we have $w_0$ Knuth equivalent to $w(T_0)$. Therefore by induction $T_0$ is the unique tableau whose word is Knuth equivalent to $w_0$. Since we know the shape of $T_0$ and $T$, the only possibility of $T$ is that it is obtained from $T_0$ by placing y in the box which is there in the Young diagram of $T$ but not that of $T_0$. $\square$

Now since we have proved the uniqueness of Theorem(1.3.3), the proofs for propositions(1.2.3, 1.2.4, 1.2.5).

## 1.5 Tableau ring

Let F be the free monoid consisting of words from the alphabet $[m]$ and let R be the equivalence relation generated by the Knuth relations (K') and (K"). Let $M = F/R$ be monoid consisting of equivalence class of words, where empty word $\phi$ is the unit. Since for $w \equiv w'$ and $v \equiv v'$ we have $w \cdot v \equiv w \cdot v' \equiv w' \cdot v'$ and therefore the operation is well defined. This moniod M is called the Plactic monoid. This plactic monid M is isomorphic to monoid of tableaux we defined earlier, where each tableau T is mapped to the equivalence class of the word $w(T)$.
Now given a monoid we have an associated group ring by taking the formal linear combinations. For the monoid of tableaux this ring is called the tableau ring $R_{[m]}$. Note that this is an associative but not commutative ring.

Summary of the chapter

- Introduced the notion of Young tableaux.
- Introduced two fundamental operations on tableaux and used each to define product on the set of tableaux.
- Defined the notion of words and used it to define a third product on set all tableaux.
- Proved each product agree with each other.
- Under each product the set of all tableaux forms an associative monoid.
- Finally using this monoid defined the the tableau ring $R_{[m]}$

# Chapter 2

# Robinson Schensted Knuth Correspondence And Applications

The row bumping algorithm can be used to give a remarkable one-to-one correspondence between matrices with nonnegative integer entries and pairs of tableaux of the same shape, known as the Robinson Schensted-Knuth correspondence(R-S-K).

## 2.1 The Correspondence

**Definition 1.** *We say a two-rowed array* $\omega = \begin{pmatrix} u_1 & u_2 & \dots & u_r \\ v_1 & v_2 & \dots & v_r \end{pmatrix}$ *is in lexicographic order if the following equations hold*

$$u_1 \le u_2 \le \cdots \le u_r \tag{2.1}$$

$$v_{k-1} \le v_k, \ \ if \ u_{k-1} = u_k \tag{2.2}$$

*This is the ordering on pairs* $\binom{u}{v}$, *with the top entry taking precedence:* $\binom{u}{v} \le \binom{u'}{v'}$, *if* $u < u'$, *or if* $u = u'$ *and* $v \le v'$.

**Example 2.** $\begin{pmatrix} 1 & 1 & 2 & 3 \\ 1 & 2 & 2 & 1 \end{pmatrix}$ *is an example for lexicographic ordering of a two rowed array while* $\begin{pmatrix} 1 & 1 & 2 & 3 \\ 2 & 1 & 2 & 1 \end{pmatrix}$ *is not.*

Given any two rowed array $\omega = \begin{pmatrix} u_1 & u_2 & \dots & u_r \\ v_1 & v_2 & \dots & v_r \end{pmatrix}$ that is in lexicographic order, we can construct a pair of tableaux (P,Q) with the same shape by the following procedure.

Start with $P_1 = \boxed{\mathrm{v}_1}$ and $Q_1 = \boxed{\mathrm{u}_1}$. To construct $(P_k, Q_k)$ from $(P_{k-1}, Q_{k-1})$, row insert $v_k$ in $P_{k-1}$, getting $P_k$; add a box to $Q_{k-1}$ in the position of the new box in $P_k$ and place $u_k$ in this box to get $Q_k$. The (P,Q) is the last in a sequence of pairs $(P_k, Q_k)$, $1 \leq k \leq r$. We have seen that each $P_k$ is a tableau. To prove inductively that each $Q_k$ is a tableau, we need to check that when we place an entry $u_k$ under an entry $u_i$ of $Q_{(k-1)}$, then $u_k > u_i$. Suppose not, they must be equal by (2.1) of Definition 1, then by (2.2) of Definition 1 we have $v_i \leq v_{i+1} \leq ... \leq v_k$. So by the row bumping lemma, the added boxes going from $P_i$ to $P_k$ should be in different columns, which is a contradiction. This implies $u_k > u_i$.
Now we look at the reverse process, that is given an arbitrary ordered pair of tableaux (P,Q) of same shape, how to get back the two rowed array that is in lexicographic order.

Given an arbitrary ordered pair of tableaux (P,Q) of same shape, one can perform the reverse row bumping process, to get a sequence of pairs of tableaux

$$(\mathrm{P}\ ,\ \mathrm{Q}) = (P_r, Q_r),\ (P_{r-1}, Q_{r-1})\ , \ldots ,(P_1, Q_1),$$

with the two tableaux in each pair having the same shape, and each having one fewer box than the preceding. To construct $(P_{k-1}, Q_{k-1})$ from $(P_k, Q_k)$, one finds the box that in which $Q_k$ has the largest entry; if there are several equal entries, the box that is farthest to the right is selected. $Q_{k-1}$ is the result of simply removing the entry of this box from $Q_k$. and $P_{k-1}$ is the result of performing the reverse row-insertion to $P_k$ using this box. Let $u_k$ be the entry removed from $Q_k$, and let $v_k$ be the entry that is bumped from the top row of $P_k$.
One gets from this a two rowed array $\omega = \begin{pmatrix} u_1 & u_2 & \ldots & u_r \\ v_1 & v_2 & \ldots & v_r \end{pmatrix}$. By the very nature of construction it follows that $u_i$'s are in weakly increasing order. That is,

$$u_1 \leq u_2 \leq \cdots \leq u_r$$

Now if $u_{k-1} = u_k$, then by case(1) of Row bumping lemma, the entry $v_k$ removed from $P_k$ is atleast as large as the entry $v_{k-1}$ removed from $P_{k-1}$ in the next step.

$$\implies v_{k-1} \leq v_k \quad \text{if} \quad u_{k-1} = u_k$$

Therefore the two rowed array obtained in the above procedure is in lexicographic ordering.

**Definition 2.** *We call a two rowed-array a word if it's top row consists of entries 1 to r in order. In addition if the bottom row consists of distinct entries of* $[r]$*, then such arrays are termed permutations.*

What we have done till here in this chapter amounts to the following important result.

**Theorem 2.1.1.** <u>*RSK Theorem*</u>

*The above operations set up a one-to-one correspondence between two-rowed lexicographic arrays $\omega$ and ordered pairs of tableaux (P,Q) with same shape. This is called the Robinson-Schensted-Knuth (R-S-K) correspondence.*

A couple of corollaries that can be implied by the above theorem.

**Corollary 2.1.1.1.** *$\omega$ is a word if and only if $Q$ is a standard tableau. This special case of R-S-K is called Robinson-Schensted correspondence.*

**Corollary 2.1.1.2.** *$\omega$ is a permutation if and only if $P$ and $Q$ are standard tableaux. This special case is called Robinson-Schensted correspondence.*

Given a lexicographic array we can associate it with an m*n matrix A (where m and n is the largest entry in the first and second row of the array respectively) whose (i,j) entry is the number of times $\binom{i}{j}$ occurs in the array. For example the array $\begin{pmatrix} 1 & 1 & 1 & 2 \\ 1 & 2 & 2 & 2 \end{pmatrix}$, has the matrix $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$

The R-S-K correspondence is then a correspondence between matrices A with nonneagative integer entries and ordered pairs (P,Q) of tableaux of the same shape. If A is an m*n matrix, then P has entries in [n] and Q has entries in [m].

### 2.1.1 Symmetry Theorem

**Theorem 2.1.2.** *If an array $\omega = \begin{pmatrix} u_1 & u_2 & \dots & u_r \\ v_1 & v_2 & \dots & v_r \end{pmatrix}$ corresponds to the pair of tableaux (P,Q), then the array $\omega' = \begin{pmatrix} v_1 & v_2 & \dots & v_r \\ u_1 & u_2 & \dots & u_r \end{pmatrix}$ corresponds to the pair (Q,P).*

In terms of matrix , turning an array upside down corresponds to taking the transpose of the matrix. The symmetry theorem then says that if the matrix A corresponds to the the tableau pair (P,Q), then the transpose $A^\tau$ corresponds to (Q,P). In particular symmetric matrices correspond to the pairs of the form (P,P). This implies that involutions in the symmetric group $S_n$ correspond to pairs (P,P) with P a standard tableau with n boxes; so there is a one-to-one correspondence between involutions and standard tableaux.

## 2.2 Applications of R-S-K correspondence

The R-S-K correspondence mentioned in the above section can be used find solutions to some counting problems.

### 2.2.1 Combinatorial identities

Let $f_\lambda$ denote the number of standard tableaux on the shape $\lambda$.
By R-S-K correspondence a pair of standard tableaux of same shape with n boxes in each corresponds to an $n{*}n$ permutation matrix and conversely any $n{*}n$ permutation matrix corresponds to a pair of standard tableaux of same shape with n boxes in each. Also we know that the number of $n{*}n$ permutation matrix is $n!$. Therefore we have the identity,

$$\textstyle\sum_{\lambda \,\vdash\, n} f_\lambda^2 = n! \qquad (1)$$

Let $d_\lambda(m)$ be the number of tableaux on shape $\lambda$ with entries from $[m]$. Then by a similar argument we have the following identity,

$$\textstyle\sum_{\lambda \,\vdash\, n} d_\lambda(m) f_\lambda = m^n \qquad (2)$$

REMARK: In both (1) and (2) we reduced the problem of counting the tableaux into a problem of counting matrices which is in general easier.
The number of involutions of a symmetric group $S_n$ is given by,

$$\textstyle\sum_{k=0}^{[n/2]} n!/(n-2k)!\; 2^k\; k!$$

We have already shown that the pairs (P,P) of standard tableaux of n boxes are in one-to-one correspondence with the involutions of symmetric group $S_n$. Therefore we have,

$$\textstyle\sum_{\lambda \,\vdash\, n} f_\lambda = \sum_{k=0}^{[n/2]} n!/(n-2k)!\; 2^k\; k!$$

A third application is the following,
By R-S-K correspondence we have a bijection between set of all pair of tableaux of same shape and set of all $m{*}n$ matrices with non-negative entries, where n is the highest entry in P and m is the highest entry in Q.
Under this correspondence, let A be the matrix associated to the pair (P,Q). Let us denote the product of the monomials P and Q by $x^P y^Q$, then we have,

$$x^P y^Q = \textstyle\prod_{i=1}^{n} \prod_{j=1}^{m} (x_j\; y_i)^{a(i,j)}$$

[That is we are able to write the product of the monomials of P and Q using the entries of their associated matrix A]
Then it follows;

$$\textstyle\sum_{A \in M_{m*n}} \prod_{i=1}^{n} \prod_{j=1}^{m} (x_j\; y_i)^{a(i,j)} = \sum_\lambda s_\lambda(x_1, \ldots, x_n) s_\lambda(y_1, \ldots, y_m)$$

Using the summation formula of a geometric progression on LHS of the above equation gives

$$\prod_{i=1}^{n} \prod_{j=1}^{m} 1/(1 - x_i\, y_j) = \sum_{\lambda} s_{\lambda}(x_1, ..., x_n) s_{\lambda}(y_1, ..., y_m)$$

where the summation is over all partitions $\lambda$. This is a formula given by Cauchy and Littlewood.

An another important application will be introduced in the next chapter to show a result.

Summary of the chapter

- Using the row bumping algorithm introduced a remarkable one-to-one correspondence between the pair of tableaux of same shape and matrices with non-negative integer entries called R-S-K correspondence
- Stated symmetry theorem
- Gave three applications of R-S-K correspondence in counting problems.

# Chapter 3

# Littlewood Richardson coefficients

It is well known that there exists a one-to-one correspondence between irreducible finite dimensional representations of $S_n$ and the partitions of n. Moreover if $\lambda, \mu$ and $\nu$ are integer partitions with at most n parts, and $S^\lambda$, $S^\mu$ and $S^\nu$ the corresponding irreducible representation of $S_n$, then the Littlewood-Richardson(LR) coefficient $c^\lambda_{\mu\nu}$ gives the multiplicity of $S^\nu$ in the tensor product of $S^\lambda \otimes S^\mu$.In representation, the study of LR coefficient is therefore important. In this chapter we review certain combinatorial methods to compute them.

## 3.1 Ring of symmetric polynomials

Let $x_1, x_2, ..., x_n$ be independent indeterminates. We know that the symmetric group $S_n$ acts on the ring $\mathbb{Z}[x_1, .., x_n]$ by permuting the variables $x_i$. An $f \in \mathbb{Z}[x_1, .., x_n]$ is said to be a symmetric polynomial, if it is invariant under the action of $S_n$. The collection of all such symmetric polynomials form a subring of $\mathbb{Z}[x_1, .., x_n]$ and is denoted by $\Lambda_n$. We shall write,

$$\Lambda_n = \mathbb{Z}[x_1, .., x_n]^{S_n}$$

For any $g \in \Lambda_n$, we have;

$$g = \textstyle\sum_{r\geq 0} g^{(r)}$$

where $g^{(r)}$ is the homogeneous symmetric polynomial of degree r. This means that $\Lambda_n$ is a graded ring,

$$\Lambda_n = \textstyle\bigoplus_{r\geq 0} \Lambda_n^r$$

where $\Lambda_n^r$ is the additive group of homogeneous symmetric polynomials of degree r in variables $x_1, .., x_n$.

If we add m more indeterminates, we have

$$\Lambda_{n+i} = \mathbb{Z}[x_1, .., x_{n+i}]^{S_{n+i}} \quad \text{for } 1 \leq i \leq m$$

Now by sending $x_{n+1}, x_{n+2}, ..., x_{n+i}$ to zero, we have a surjective homomorphism of graded rings,

$$\Lambda_{n+i} \longrightarrow \Lambda_n$$

Note that the mapping $\Lambda^r_{n+i} \longrightarrow \Lambda^r_n$ is surjective for all $r \geq 0$ and is bijective if $r \leq n$. Let,

$$\Lambda^r = \varprojlim \Lambda^r_n$$

By definition of inverse limit, each element of $\Lambda^r$ is a sequence $(f_n)_{n\geq 0}$ where $(f_n) \in \Lambda^r_n$ for each n and $f_n$ is obtained from $f_{n+1}$ by setting $x_{n+1} = 0$. Also for each $r \geq 0$, let

$$\Lambda = \bigoplus\nolimits_{r\geq 0} \Lambda^r$$

The elements of $\Lambda$ are not called polynomials anymore, traditionally they are called symmetric functions.

### 3.1.1 Monomial symmetric polynomial

For an integer n, let $p(n)$ be the set of all partitions of n. Now define,
$p_m(n) = \{\lambda = (\lambda_1, \lambda_2, ...) \in p(n) | \lambda_i = 0 \text{ for } i > m.\}$

Associated to each partition $\lambda = (\lambda_1, \lambda_2, ..) \in p_m(n)$ consider the polynomial obtained by taking sum of all monomials obtained by permuting all the variables of the monomial $x_1^{\lambda_1} x_2^{\lambda_2} ... x_m^{\lambda_m}$. Clearly by the very definition this polynomial is symmetric and is called the monomial symmetric polynomial .It is denoted by $m_\lambda$.

**Proposition 3.1.1.** *The set* $M^n_m = \{m_\lambda : \lambda \in p_m(n)\}$ *is a* $\mathbb{Z}$ *basis for* $\Lambda^n_m$

*Proof.* Let $p(x) \in \Lambda^n_m$, and suppose that with respect to the lexicographic ordering on $p(n), \lambda = (\lambda_1, ..., \lambda_m) \in p_m(n)$ is the maximal element such that the coefficient $b_\lambda$ of $x_1^{\lambda_1} x_2^{\lambda_2} ... x_m^{\lambda_m}$ in $p(x)$ is non-zero. Since $p(x)$ is a symmetric polynomial this implies that $p(x) - b_\lambda m_\lambda \in \Lambda^n_m$ and if $\mu = (\mu_1, ..., \mu_m) \in p_m(n)$ is such that the coefficient $b_\mu$ of $x_1^{\mu_1} x_2^{\mu_2} ... x_m^{\mu_m}$ in $p(x) - b_\lambda m_\lambda$ is non-zero, then $\mu \leq \nu$ in the lexicographic ordering. Since the partitions of an integer n under lexicographic ordering is a total ordering we can repeat the same process by taking the next maximal element in the ordering. Since the polynomial $p(x)$ contain only finitely many terms the above process will terminate at some point. This implies that the set $\{m_\lambda : \lambda \in p_m(n)\}$ spans $\Lambda^m_n$.
Now we will check whether this set is linearly independent. For if $\sum a_\lambda m_\lambda = 0$ with $\lambda$ maximal and $a_\lambda \neq 0$. This implies that the coefficient of $x^\lambda$ is $a_\lambda \neq 0$. This is contradiction since on the R.H.S of $\sum a_\lambda m_\lambda = 0$ the coefficient of $x^\lambda$ is zero. This proves the set we have is indeed a linearly independent one. This proves the claim of the proposition. □

### 3.1.2 Schur polynomials

In chapter 1 we have defined the tableau ring $R_{[m]}$. Given this, we have a have a canonical ring homomorphism from the ring $R_{[m]}$ to the ring of polynomials over integers ,i.e $\mathbb{Z}[x_1, x_2, ..., x_m]$, where each tableau T is mapped to its corresponding monomial.
The Schur polynomials are defined as follows; Given a partition $\lambda$, define $S_\lambda[m] = S_\lambda$ to be an element of the ring $R_{[m]}$ obtained as the sum of all tableaux T of shape $\lambda$ with entries from $[m]$. The image of $S_\lambda$ under this homomorphism in the polynomial ring is the Schur polynomial $s_\lambda(x_1, ..., x_m)$.
Note that since in the tableau ring $R_{[m]}, T{\cdot}U = Rect(T{*}U)$. Using the ring homomorphism $R_{[m]} \to \mathbb{Z}[x_1, x_2, ..., x_m]$, the product of T and U must be mapped to $x^{Rect(T*U)}$. Hence via the homomorphism,

$$x^T \cdot x^U = x^{Rect(T*U)}$$

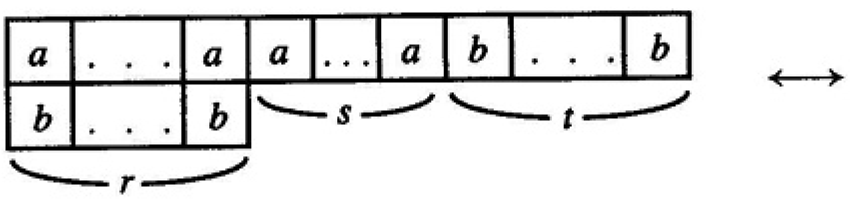
which implies,

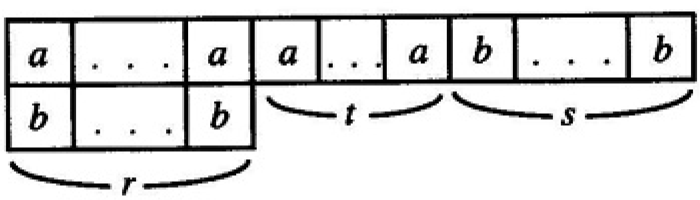
$$s_\mu \cdot s_\nu = \textstyle\sum_{T\in\Pi\mu, U\in\Pi\nu} x^{Rect[T:U]}$$

where $\Pi\mu$ (resp. $\Pi\nu$) denote the set of all tableaux T (resp. $U$) of shape $\mu$(resp.$\nu$).
Now we are introducing a proposition which is an application of R-S-K correspondence from which the fact Schur polynomials are symmetric follows.

**Proposition 3.1.2.** *The number of tableaux on a given shape $\lambda$ with $m_1 1's$ $m_2 2's$,...,$m_n n's$ is same as the number of tableaux on $\lambda$ with $m_{\sigma(1)} 1's$ $m_{\sigma(2)} 2's$,...,$m_{\sigma(n)} n's$ for any $\sigma \in S_n$.*

*Proof.* First consider the case where the tableaux consists of 2 distinct entries. In this case there is at most one tableaux possible. In the case where there is no tableau the proposition immediately follows. Now for the case where there is one tableau we have the following correspondence,





Therefore the proposition holds in this case. Now let $\lambda$ be a arbitrary shape and P be a fixed tableau on this shape, then by R-S-K correspondence between the pair of tableaux (P,Q) and matrices with non negative integers, the proposition translates into proving the following two sets have the same cardinality.

$$D = \{A : P(A) = P; \text{ with row sums } m_1, m_2, ..., m_n\}$$

$$E = \{A : P(A) = P; \text{ with row sums } m_{\sigma(1)}, m_{\sigma(2)}, ..., m_{\sigma(n)}\}$$

Since the transpositions $\sigma$ of k and k+1 where $1 \leq k \leq n$ generate the symmetric group $S_n$, so we need only prove the result when $\sigma$ is a transposition.
Let A be a set in D,

$$\text{A} = \begin{bmatrix} F \\ G \\ H \end{bmatrix}$$

where F is the first k-1 rows, G next two and H the rest.
We know that P(A) is constructed by taking the corresponding two rowed lexicographic array of A and doing the canonical procedure of tableau construction in the bottom row. Therefore it follows that,

$$\text{P(A)} = \text{P(F)} \cdot \text{P(G)} \cdot \text{P(H)}$$

We have one-to-one correspondence between matrix G having row sums $m_k$ and $m_{k+1}$ with matrix G' having row sums $m_{k+1}$ and $m_k$ and with P(G) = P(G') by the case considered first when translated into the language of matrices. Then

$$\text{A'} = \begin{bmatrix} F \\ G' \\ H \end{bmatrix}$$

is the corresponding matrix in E. This proves the proposition. □

Since $x^T = x_1^{m_1} x_2^{m_2} ... x_n^{m_n}$, we have the following.

**Corollary 3.1.2.1.** *Schur polynomials are symmetric polynomials.*

Now we claim the following result.

**Proposition 3.1.3.** *The set $S_m^n = \{s_\lambda : \lambda \in p_m(n)\}$ forms a $\mathbb{Z}$ basis for $\Lambda_m^n$.*

*Proof.* Observe that the cardinality of $S_m^n$ is same as the cardinality of $M_m^n$, So it is sufficient to check that $S_m^n$ spans $\Lambda_m^n$ over $\mathbb{Z}$. Notice that since $x^\lambda$ is also the leading monomial in $s_\lambda(x)$, the same proof as in proposition (3.1.1) shows that $S_m^n$ spans $\Lambda_m^n$ over $\mathbb{Z}$ .This proves the proposition. □

Now since we have ,

$$\Lambda_m = \bigoplus_{n\geq 0} \Lambda_m^n$$

$s_\lambda$ forms a basis for $\Lambda_m$ over $\mathbb{Z}$ as well, for $\lambda \in p(n)$ for some n.

## 3.2 Littlewood-Richardson coefficient

As a consequence of the proposition (3.1.3) we havee the following;

**Definition 3.** *Let $\mu$ and $\nu$ be two partitions and let $s_\mu$ and $s_\nu$ be the corresponding Schur polynomials. Then we have the following expression for the product of two Schur polynomials $s_\mu$ and $s_\nu$.*

$$s_\mu \cdot s_\nu = \sum_\lambda c_{\mu\nu}^\lambda s_\lambda \quad \textit{where } c_{\mu\nu}^\lambda \in \mathbb{Z} \qquad \textit{(1)}$$

*The coefficients $c_{\mu\nu}^\lambda$ arising in the above expression are defined to be the LR coefficients.*

Note that given three partitions $\mu, \nu$ and $\lambda$ we have a unique $c_{\mu\nu}^\lambda$ value. Clearly $c_{\mu\nu}^\lambda = 0$ if $|\lambda| \neq |\mu| + |\nu|$.

Let $\Pi_\lambda$ ={ set of all tableaux of shape $\lambda$.}
As observed earlier,

$$s_\mu \cdot s_\nu = \sum_{T\in\Pi\mu, U\in\Pi\nu} x^{Rect(T*U)} \qquad (2)$$

Comparing expressions (1) and (2) observe that if $V_0 = Rect(T*U)$ for some $V_0$ of Shape $\lambda$, T of shape $\mu$ and U of shape $\nu$ and there exist $c_{\mu\nu}^\lambda$ pairs of (T',U') such that $Rect(T'*U') = V_0$, then coefficient of $x^{Rect(T*U)}$ in (2) will be $c_{\mu\nu}^\lambda$. On the other hand coefficient of $x^{V_0}$ in $s_\lambda$ is 1. Therefore comparing the expressions (1) and (2) we see that

$$c_{\mu\nu}^\lambda = \text{cardinality of } R(\mu, \nu, V_0)$$

where $R(\mu, \nu, V_0)$ = { [T,U]: T is a tableau on $\mu$, U is a tableau on $\nu$ and $Rect(T*U) = V_0$} for any $V_0$ of shape $\lambda$.
Note that $c_{\mu\nu}^\lambda$ is independent of the choice of $V_0$ of shape $\lambda$.

Now let us define an another set,
For any tableau $U_0$ with shape $\nu$, let

$S(\lambda/\mu, U_0)$ = { skew tableaux S on $\lambda/\mu$ : Rect(S) = $U_0$ }

**Proposition 3.2.1.** *For any tableaux $U_0$ on $\nu$ and $V_0$ on $\lambda$, there is a canonical one to one correspondence*

$$R(\mu, \nu, V_0) \Longleftrightarrow S(\lambda/\mu, U_0)$$

This proposition is a consequence of Robinson Schensted correspondence stated in chapter 2.Now we will assume this result and drew a couple of corollaries.

**Corollary 3.2.1.1.** *The set $R(\nu, \mu, V_0)$ also have cardinality $c^{\lambda}_{\mu\nu}$. That is,*

$$c^{\lambda}_{\mu\nu} = c^{\lambda}_{\nu\mu}$$

REMARK: This is an important symmetry of LR coefficent $c^{\lambda}_{\mu\nu}$ and will be recalled later to prove a special case of a conjecture.

**Corollary 3.2.1.2.** *Let $s_{\lambda/\mu}$ and $s_\nu$ be the Schur polynomials then we have,*

$$s_{\lambda/\mu} = \sum_{\nu} c^{\lambda}_{\mu\nu} s_\nu$$

This follows since the cardinalities of the sets in proposition (3.3.1) is $c^{\lambda}_{\mu\nu}$.

## 3.3 Combinatorial interpretations of Littlewood-Richardson number

In this section we introduce three combinatorial interpretations of Littlewood-Richardson number. Even though they seem to be different we will show that each is related to another and are equivalent in some sense. This methods of computing the Littlewood-Richardson number was discussed in [**1**].

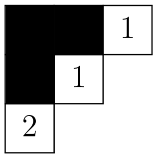
### 3.3.1 Littlewood-Richardson Rule

**Definition 4.** *A word $w = x_1, ...., x_m$ is called a Yamanouchi word if when read backwards starting at $x_m$ to any letter in $w$, number of times the integer k appear is greater than or equal to number of times the integer k+1 appear ,for any positive integer.*

For example the word 12132211 is a Yamanouchi word but 11$\underline{232211}$ is not since the last 6 letters contain more $2's$ than $1's$.

**Definition 5.** *A skew tableau S is said to be a Littlewood-Richardson skew tableau if its word is a Yamanouchi word.*

For example,



is a Littlewood-Richardson skew tableau on the shape (3,2,1)/(2,1) with word 211.
The following result gives a combinatorial interpretation of LR coefficients.

**Theorem 3.3.1.** *The number of Littlewood-Richardson skew tableau of shape $\lambda/\mu$ with content $\nu$ is equal to the LR number $c^{\lambda}_{\mu\nu}$.*

Now we define some definitions and lemma necessary to prove the theorem.

**Lemma 3.3.2.** *If $w$ and $w'$ are Knuth equivalent words, then $w$ is a Yamanouchi word if and only if $w'$ is a Yamanouchi word.*

*Proof.* It suffices to check the lemma on the two elementary Knuth transformations defined earlier, since any two Knuth equivalent words is a sequence of these two elementary Knuth transformations.
Case 1: Suppose
$w = uxzyv \mapsto uzxyv = w'$      with $x \leq y < z$
We need to check under this transformation, the fact that the number of times k appear is greater than or equal to the number of times k+1 appear ,is preserved as we read from right to left , for any positive integer k. If $x < y < z$ there is no change, so the only non trivial case is when $x = y = k$ and $z = k + 1$. For either of the words to be a Yamanouchi, notice that the number of times k appear in $v$ should be greater than or equal to the number of times k+1 appear in $v$. In this case both words $xzyv$ and $zxyv$ are Yamanouchi words. Hence case 1 is proved.

Case 2: Suppose
$w = uyxzv \mapsto uyzxv = w'$      with $x < y \leq z$
Here the non trivial case is when $x = k$ and $y = z = k+1$. Notice that neither of the words are Yamanouchi unless the number of k's in $v$ is strictly larger than the number (k+1)'s , and if this is the case, both words $yxzv$ and $yzxv$ will have atleast as many k's as (k+1)'s, and hence will be Yamanouchi words. This proves the lemma. □

For any partition $\nu$ define the tableau $U(\nu)$ to be the tableau of shape $\nu$ whose $i^{th}$ row consists entirely of the integer $i$. For example,

$$U(\nu) = \begin{array}{|c|c|c|} \hline 1 & 1 & 1 \\ \hline 2 & 2 & 2 \\ \hline 3 & 3 & \multicolumn{1}{c}{} \\ \cline{1-2} \end{array} \qquad\qquad \text{for } \nu = (3,3,2)$$

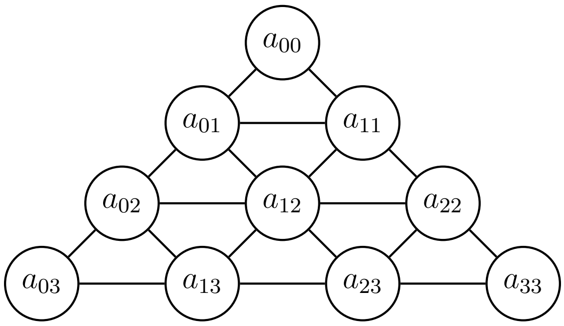
**Lemma 3.3.3.** *A skew tableau $S$ is a Littlewood-Richardson skew tableau of content $\nu$ if and only if its rectification is the tableau $U(\nu)$.*

*Proof.* Since the only LR tableau on a shape $\nu$ is $U_\nu$ due the Yamanouchi word condition. This lemma immediately follows from Lemma(3.4.2). $\square$

Now combining the Lemmas (3.4.2),(3.4.3) with proposition (3.3.1) we have the proof for Theorem (3.4.1).The statement of Theorem(3.4.1) is called the Littlewood Richardson rule.

### 3.3.2 Littlewood-Richardson triangles

For a positive integer k, a hive graph $\Delta_k$ of size k is a graph in the plane with $\binom{k+2}{2}$ nodes arranged in a triangular grid containing $k^2$ small equilateral triangles. For k = 3 the hive graph $\Delta_3$ is of the following form,




Let $T_k$ be the $\binom{k+2}{k}$ - 1 dimensional R vector space spanned by the set $\{A = (a_{ij})_{0\leq i\leq j\leq k} : a_{00} = 0\}$.
Now our task is to code each Littlewood-Richardson tableau as an element of $T_k$ satisfying certain conditions.

**Definition 6.** *A vector $A \in T_k$ is said to be a LR triangle of size k if it satisfies the following inequalities.*

1. $a_{ij} \geq 0$*, for all* $1 \leq i < j < k$ *(P).*
2. $\sum_{p=0}^{i-1} a_{pj} \geq \sum_{p=0}^{i} a_{p(j+1)}$*, for all* $1 \leq i \leq j < k$ *(CS)*
3. $\sum_{q=i}^{j} a_{iq} \geq \sum_{q=i+1}^{j+1} a_{(i+1)q}$*, for all* $1 \leq i \leq j < k$ *(LR)*

REMARK: The inequality,

$$\sum_{p=0}^{j} a_{pj} \geq \sum_{p=0}^{j+1} a_{p(j+1)}, \qquad \text{for } 1 \leq j < k$$

follows from (CS) and (LR) with $i = j$.

**Definition 7.** *Cone of a vector space is a subset such that all its positive scalar multiples are contained within the set.*

Take the collection of all LR triangles in $T_k$ and denote it by $LR_k$.
Let $A \in LR_k$ and $\alpha$ be a positive scalar in R. Clearly each $\alpha a_{ij}$ is $\geq 0$ for $1 \leq i < j < k$. Hence (P) holds. Multiplication by a positive scalar clearly preserves the the (CS) and (LR) property. Therefore $\alpha A \in LR_k$. This implies that $LR_k$ is a cone of $T_k$ and is called the LR cone.

Let $D_k = \{\lambda = (\lambda_1, ..., \lambda_k) \in \mathbb{R}^k : \lambda_1 \geq \lambda_2 \geq ... \geq \lambda_k\}$ and for $\lambda \in D_k$, let $|\lambda| = \sum_{i=1}^{k} \lambda_i$. Notice that if $\lambda \in P_k(n)$ for some positive integer n and $k \leq n$ then $\lambda \in D_k$.
Given $A = (a_{ij}) \in LR_k$,

- $\mu_j = a_{0j}$, for $1 \leq j < k$ (B1)
- $\lambda_j = \sum_{p=0}^{j} a_{pj}$, for $1 \leq j < k$ (B2)
- $\nu_i = \sum_{q=i}^{k} a_{iq}$, for $1 \leq i < k$ (B3)

Since $A \in LR_k$,

$$\textstyle\sum_{p=0}^{j} a_{pj} \geq \sum_{p=0}^{j+1} a_{p(j+1)}, \qquad \text{for } 1 \leq j < k$$

$\implies \lambda_1 \geq \lambda_2 \geq \lambda_3$.
Since,

$$a_{0j} \geq a_{0(j+1)} + a_{1(j+1)} \qquad \text{for all } j \text{ and } a_{ij} \geq 0.$$

$\implies \mu_j \geq \mu_{j+1}$ for all $j$.
Further,

$$\textstyle\sum_{q=i}^{j} a_{iq} \geq \sum_{q=i+1}^{j+1} a_{(i+1)q}, \text{ for all } 1 \leq i \leq j < k$$

$\implies \nu_i \geq \nu_{i+1}$.

That is, for $A \in LR_k$, the associated k tuples $\lambda$,$\mu$ and $\nu$ are in $D_k$ and clearly

$$|\lambda| = |\mu| + |\nu|.$$

An element $A \in LR_k$ is said to be of type $(\lambda, \mu, \nu)$ if $A$ satisfies the conditions B1,B2,B3. Let

$$LR_k(\lambda, \mu, \nu) = \{A \in LR_k : A \text{ is of type } (\lambda, \mu, \nu)\}$$

Conversely, given a triple of partitions $\lambda, \mu, \nu \in D_k$ with $|\lambda| = |\mu| + |\nu|$ and T a LR skew tableau of shape $\lambda/\mu$ with content $\nu$, let $A_T = (a_{ij}) \in T_k$ be defined as follows:

- $a_{00} = 0, a_{0j} = \mu_j \qquad$ for $1 \leq j \leq k$

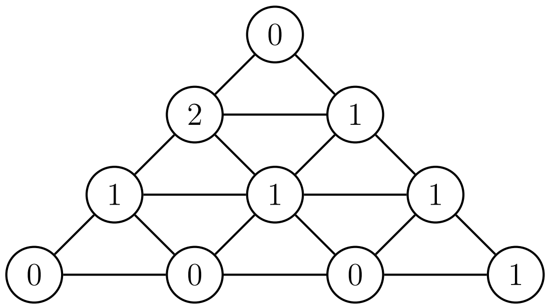
- $a_{ij}$ equal the number of $i's$ in the j$^{th}$ row $j$ of T  for $1 \leq i \leq j \leq k$

It is straightforward to check that $A_T \in LR_k(\lambda, \mu, \nu)$.

**Example 3.** *Let $\lambda = (3, 3, 1), \mu = (2, 1)$ and $\nu = (2, 1, 1)$ and let T be the LR skew tableau as follows,*

| | | |
|---|---|---|
| | | 1 |
| | 1 | 2 |
| 3 | | |

*Here k =3 and dim $T_3$ = 10-1 =9,*
*then by definition $A_T$ is,*




*Clearly $A_T \in LR_3(\lambda, \mu, \nu)$.*

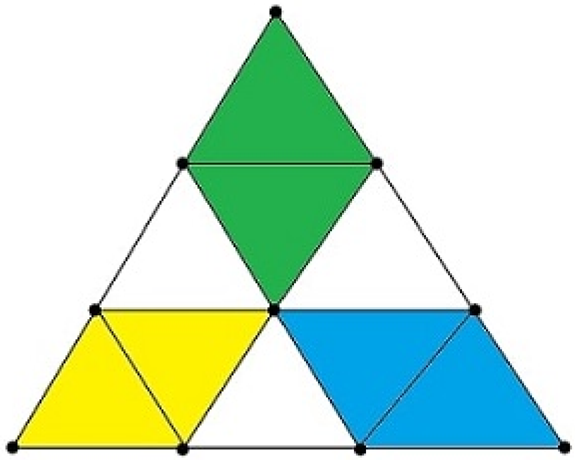
**Theorem 3.3.4.** *Let $\lambda, \mu$,$\nu \in D_k$ be partitions such that $\mid \lambda \mid = \mid \mu \mid + \mid \nu \mid$. Then there exist a bijective correspondence between non- negative integer points of $LR_k(\lambda, \mu, \nu)$ and $S(\lambda/\mu, U(\nu))$ where $U(\nu)$ is as defined in section(3.3.1) and*

$$S(\lambda/\mu, U(\nu) = \{ \textit{ skew tableaux of shape } \lambda/\mu : Rect(S) = U(\nu)\}$$

*Proof.* Let $T \in S(\lambda/\mu, U(\nu))$, then content of $T$ is $\nu$. Let $A_T$ be the corresponding element in $T_k$. $A_T$ satisfies (P) since all its entries are non-negative integers by definition. The strictly increasing property across column of T account for $A_T$ satisfying (CS). And finally the fact $T$ is a LR skew tableau forces $A_T$ to satisfy (LR). Thus $A_T$ clearly belongs to $LR_k$ and in particular it belongs to the convex polytope $LR_k(\lambda, \mu, \nu)$.
Conversely, for any Littlewood-Richardson triangle $A = (a_{ij})$ in $LR_k(\lambda, \mu, \nu)$ with integer entries, define a tableau $T_A$ of shape $\lambda/\mu$ in the following way ; place $a_{ij}$ $i's$ in the $j^{th}$ row, for each $i$ and $j$, in weakly decreasing order. The result will be a Littlewood-Richardson tableau of shape $\lambda/\mu$ with content $\nu$. Hence by lemma(3.3.3) its rectification is $U(\nu)$. Both constructions are inverse of each other. This establishes the bijection. Hence the theorem proved. □

### 3.3.3 Hives

This is another combinatorial interpretation of the Littlewood-Richardson number. Here once again we consider $\Delta_k$, the hive graph of size k. We know that there are $k^2$ equilateral triangles in the triangular grid of $\Delta_k$. Each of the two adjacent triangles in this grid combine to form an rhombus with a pair of opposite edges having obtuse angle and the other pair being acute. There will be three types of rhombus in the figure. One tilted to left(blue), one vertical(green) and one tilted to right(yellow).

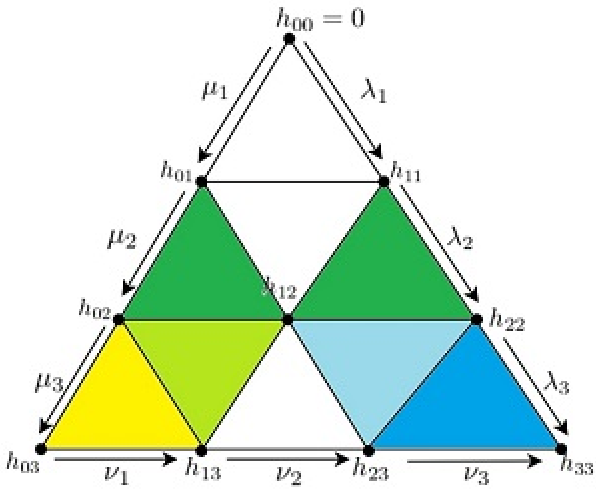


**Definition 8.** *A hive of size k is a labeling $H = (h_{ij})_{0\leq i\leq j\leq k}$ of the nodes of $\Delta_k$ with real numbers such that the sum of the labels of the obtuse nodes is greater than or equal to sum of the labels of acute nodes, for each rhombus in $\Delta_k$. That is we say $H = (h_{ij})$ is a hive of size k if it satisfies the following inequalities.*

- $h_{ij} - h_{i(j-1)} \geq h_{(i-1)j} - h_{(i-1)(j-1)}$, *for* $1 \leq i < j \leq k$ *(R)*
- $h_{(i-1)j} - h_{(i-1)(j-1)} \geq h_{i(j+1)} - h_{ij}$, *for* $1 \leq i \leq j \leq k$ *(V)*
- $h_{ij} - h_{(i-1)j} \geq h_{(i+1)(j+1)} - h_{i(j+1)}$, *for* $1 \leq i \leq j \leq k$ *(L)*

Let $H_k$ be the cone of all hives of size k, with $h_{00} = 0$. We refer to this cone as hive cone. Given $H = (h_{ij}) \in H_k$, set

- $\mu_j = h_{0j} - h_{0(j-1)}$, for $1 \leq j \leq k$ (C1)
- $\lambda_j = h_{jj} - h_{(j-1)(j-1)}$, for $1 \leq j \leq k$ (C2)
- $\nu_i = h_{ik} - h_{(i-1)k}$, for $1 \leq i \leq k$ (C3)




Note,

$$\begin{aligned} &\mu_2 = h_{02} - h_{01} \geq h_{13} - h_{12} &&\text{by (V),} \\ &h_{13} - h_{12} \geq h_{03} - h_{02} = \mu_3 &&\text{by (R)} \\ &\implies \mu_2 \geq h_{13} - h_{12} \geq \mu_3 \end{aligned}$$

Or in general,

$$\mu_j \geq h_{(j-1)(j+1)} - h_{(j-1)j} \geq \mu_{j+1}$$

Symmetrically,
Since $\lambda_j = h_{jj} - h_{(j-1)(j-1)}$ and,

$$\begin{aligned} &\lambda_2 = h_{22} - h_{11} \geq h_{23} - h_{12} &&\text{by (V),} \\ &h_{23} - h_{12} \geq h_{33} - h_{22} = \lambda_3 &&\text{by (L)} \\ &\implies \lambda_2 \geq h_{33} - h_{12} \geq \lambda_3 \end{aligned}$$

Or in general,

$$\lambda_j \geq h_{j(j+1)} - h_{(j-1)j} \geq \lambda_{j+1}$$

Finally using we have,

$$\begin{aligned} &\nu_1 = h_{13} - h_{03} \geq h_{12} - h_{02} &&\text{by (R),} \\ &h_{12} - h_{02} \geq h_{23} - h_{13} = \nu_2 &&\text{by (L)} \\ &\implies \nu_1 \geq h_{12} - h_{02} \geq \nu_2 \end{aligned}$$

Or in general,

$$\nu_i \geq h_{i(i+1)} - h_{(i-1)(i+1)} \geq \nu_{i+1}$$

This implies that the vectors $\lambda = (\lambda_1, ...., \lambda_k)$ , $\mu = (\mu_1, ...., \mu_k)$ and $\nu = (\nu_1, ...., \nu_k)$ are in $D_k$ and also $\mid \lambda \mid = \mid \mu \mid + \mid \nu \mid$.
Given $\lambda, \mu, \nu \in D_k$ with $\mid \lambda \mid = \mid \mu \mid + \mid \nu \mid$, a hive $H \in H_k$ is said to be of type $(\lambda, \mu, \nu)$ if the labels $h_{ij}$'s of $H$ satisfy the conditions C1,C2 and C3. Let,

$$H_k(\lambda, \mu, \nu) = \{H \in H_k : H \text{ is of type } (\lambda, \mu, \nu)\}$$

For any positive integer k, we define a linear map $\Phi_k : T_k \rightarrow T_k$ as follows:- $\Phi_k(a_{ij}) = h_{ij}$, where

$$h_{ij} = \sum_{p=0}^{i} \sum_{q=p}^{j} a_{pq}$$

**Theorem 3.3.5.** *The map $\Phi_k$ maps $LR_k$ bijectively onto $H_k$ and for $\lambda, \mu, \nu \in D_k$ with $|\lambda| = |\mu| + |\nu|$, $\Phi_k$ maps $LR_k(\lambda, \mu, \nu)$ onto $H_k(\lambda, \mu, \nu)$.*

*Proof.* Let $E_{ij}$ be the canonical basis of $T_k$, that is $E_{ij} = (e^{ij}_{pq})$, where

$$e^{ij}_{pq} = \{, 1, \quad \text{if p= i and q=j;} \\ 0, \quad \text{otherwise.}\}$$

Now we order this basis elements in lexicographic order of the subindices, that is'

$$\{E_{01}, E_{02}, ..., E_{0k}, E_{11}, ...E_{1k}, ..., E_{kk}\}$$

According to the definition of the map $\phi_k$.

$$\Phi_k(a_{ij}) = h_{ij}$$

where $h_{ij} = a_{01} + ... + a_{0j} + a_{11} + ... + a_{1j} + ...... + a_{ii} + ... + a_{ij}$

Hence under the map $\phi_k$, we have

$$\phi_k(E_{ij}) = E_{ij} + E_{i(j+1)} + ...... + E_{kk}.$$

implying that with respect to the basis $\{E_{01}, E_{02}, ..., E_{0k}, E_{11}, ...E_{1k}, ..., E_{kk}\}$ the matrix of $\Phi_k$ is lower triangular with 1's on the diagonal.Therefore the determinant is one, which implies the map is volume preserving. Now since in defining the map we have only used the operation of addition on real numbers, we have all the integer points in $T_k$ mapped to some other integer points in $T_k$ bijectively, that is $Z^{\binom{k+2}{2}-1}$ bijectively onto $Z^{\binom{k+2}{2}-1}$. Now define the inverse of $\Phi_k$ by $\Phi_k^{-1}(h_{ij}) = (a_{ij})$ where

$$a_{ij} = \begin{cases} h_{ij} - h_{i(j-1)} - h_{(i-1)j} + h_{(i-1)(j-1)}, \text{ if } 1 \leq i < j \leq k. \\ h_{0j} - h_{0(j-1)}, \text{ if } i = 0 \text{ and } 1 \leq j \leq k. \\ h_{jj} - h_{(j-1)j}, \text{ if } 1 \leq i = j \leq k \ \} \end{cases} .$$

Let $(a_{ij}) \in LR_k$ and $(h_{ij}) = \Phi_k(a_{ij})$, then we have,

$$h_{st} - h_{s(t-1)} = \sum_{p=0}^{s} a_{pt}$$
and
$$h_{(s+1)t} - h_{st} = \sum_{q=s+1}^{t} a_{(s+1)q}$$

for $0 \leq s < t \leq k$. Using these identities we can check the $(a_{ij})$ defined above satisfies (P),(CS), or (LR), respectively if and only if $(h_{ij})$ satisfies (R),(V), or (L) respectively. Hence $\Phi_k(LR_k) = H_k$, It is clear that $(a_{ij})$ and $(h_{ij})$ have same type; therefore $\Phi_k(LR_k(\lambda, \mu, \nu)) = H_k(\lambda, \mu, \nu))$ for all $\lambda, \mu, \nu \in D_k$. Hence proved the theorem.

□

Now we drew some corollaries.

**Corollary 3.3.5.1.** *The number of integer points in $H_k(\lambda, \mu, \nu))$ is $c^{\lambda}_{\mu\nu}$ for all $\lambda, \mu, \nu \in D_k$ with non negative integer coefficients.*

**Corollary 3.3.5.2.** *Vol($H_k(\lambda, \mu, \nu)$) = Vol($LR_k(\lambda, \mu, \nu)$), for all $\lambda, \mu, \nu \in D_k$.*

In short we have introduced the Littlewood-Richardson number in the context of multiplication of elements of the tableau ring $R_{[m]}$. Then we gave three methods to compute the Littlewood-Richardson number. Now we have enough theory to understand statement of the conjecture in question and sufficient tools to prove it at least for lower values of k.

Chapter Summary

- Introduced the ring of symmetric polynomials.
- Mentioned two important family of symmetric polynomials called monomial symmetric polynomials and Schur polynomials.
- Proved each forms a basis for symmetric polynomials over $\mathbb{Z}$.
- Introduced LR coefficients in this context.
- Finally gave 3 combinatorial methods to compute the LR coefficients.(LR rule, LR triangles,Hives)

# Chapter 4

# An application of the combinatorial interpretation of LR coefficients

A symmetric function is said to be Schur positive if it has positive integer coefficients when written as a linear combination of Schur functions. A question in this area that has generated a lot of interest is when are expressions of the kind

$$s_\mu s_\nu - s_\lambda s_\rho \qquad (1)$$

is Schur positive.
It was conjectured by Fomin-Fulton-Li and Poon in 2003 that an expression like (1) holds whenever $(\mu, \nu, \lambda, \rho)$ is a 4 tuple of partition satisfying certain conditions. This conjecture was partially proved by Bergeron and Mcnamara in 2004 and in complete generality by Lam,Postinikov and Pylyavskyy in 2007. Here using the method of LR triangles we prove the mentioned result in some special case.

## 4.1 Statement of FFLP conjecture

Given two partitions $\mu = (\mu_1, \mu_2, ....)$ and $\nu = (\nu_1, \nu_2, ...)$. Let $\mu \cup \nu = (\sigma_1, \sigma_2, ....)$ be the partition obtained by arranging all parts of $\mu$ and $\nu$ in the weakly decreasing sequence. Let

$$\sigma^o_{\mu\cup\nu} = (\sigma_1, \sigma_3, \sigma_5, ...)$$
$$\sigma^e_{\mu\cup\nu} = (\sigma_2, \sigma_4, \sigma_6, ...)$$

then

$$s_{\sigma^o_{\mu\cup\nu}} s_{\sigma^e_{\mu\cup\nu}} - s_\mu s_\nu \qquad \text{is Schur positive.}$$

In other words, it was conjectured in FFLP that given two partitions $\mu$,$\nu$,

$$c^\lambda_{\mu\nu} \leq c^\lambda_{\sigma^o_{\mu\cup\nu}\sigma^e_{\mu\cup\nu}}$$

for all partitions $\lambda$.

## 4.2 Proof for the conjecture

Here we are proving the conjecture for special cases when k =2,3, where k is maximum number of parts of the partitions involved.

### 4.2.1 Proof for k=2

*Proof.* CLAIM: Let $\lambda, \mu, \nu$ be partitions with at most 2 parts,then

$$c^{\lambda}_{\mu^*\nu^*} \geq c^{\lambda}_{\mu\nu}$$

**Proposition 4.2.1.** *Let $\lambda, \mu, \nu$ be partitions with at most 2 parts,then*

$$c^{\lambda}_{\mu\nu} \leq 1$$

*Proof.* We know that $c^{\lambda}_{\mu\nu}$ is the number of Littlewood-Richardson tableau on the shape $\lambda/\mu$ with content $\nu$.
Also note that for k=2, we can put only 1's and 2's in the tableau. For a LR tableau with shape $\lambda/\mu$ and content $\nu$, all the entries of first row are forced to be one so as to preserve the reverse lattice property of LR tableau.
Now if $\nu_2 > 0$ ( that is the number of 2's is at least 1), then all the 2's are forced to be placed towards the end of the second row with no 1 in between them so as to preserve the weakly increasing property across row of a tableau. Therefore automatically all the remaining 1's are placed in the second row and to the left of 2's of the tableau.
So in short, this forces LR tableaux to have exactly one configuration, if there exist any. This implies that there is at most one LR tableau on the shape $\lambda/\mu$ with content $\nu$. i.e $c^{\lambda}_{\mu\nu} \leq 1$, that is $c^{\lambda}_{\mu\nu} \in \{0, 1\}$

□

<u>CASE 1</u>
If $c^{\lambda}_{\mu\nu}$ = 0, For some $\lambda, \mu, \nu$
then clearly $c^{\lambda}_{\mu^*\nu^*}$ is either 0 or 1. Therefore in both cases,

$$c^{\lambda}_{\mu^*\nu^*} \geq c^{\lambda}_{\mu\nu}$$

<u>CASE 2</u>
If $c^{\lambda}_{\mu\nu}$ = 1 and $(\mu^*, \nu^*) = (\mu, \nu)$ for some $\lambda, \mu, \nu$
then $c^{\lambda}_{\mu^*\nu^*} = c^{\lambda}_{\mu\nu}$, Therefore,

$$c^{\lambda}_{\mu^*\nu^*} \geq c^{\lambda}_{\mu\nu}$$

So the only case remaining to be proved is when $c^{\lambda}_{\mu\nu}$ = 1 and $(\mu^*, \nu^*) \neq (\mu, \nu)$. This will exhaust all the possible cases. To prove this case we introduce the following proposition.

**Proposition 4.2.2.** *Let $\lambda, \mu, \nu$ be three partitions with k=2, then $c^{\lambda}_{\mu\nu}$ = 1 if and only if the partitions satisfies the following conditions.*

- $|\lambda| = |\mu| + |\nu|$
- *$\lambda$ contains both $\mu$ and $\nu$.*
- $\lambda_2 - \mu_1 \leq \nu_2 \leq \lambda_1 - \mu_1 \leq \nu_1 \leq \lambda_1 - \mu_2$ *and* $\nu_2 \leq \lambda_2 - \mu_2$

*Proof.* We know that $c^{\lambda}_{\mu\nu}$ is the number of ways in which a tableau of shape $\lambda$ can be written as a product of tableaux of shape $\mu$ and $\nu$, where the product is defined using bumping. In that case , we have $c^{\lambda}_{\mu\nu}$ = 0, unless $|\lambda| = |\mu| + |\nu|$ or $\lambda$ contains both $\mu$ and $\nu$. This implies that if $c^{\lambda}_{\mu\nu}$ = 1, then $|\lambda| = |\mu| + |\nu|$ and $\lambda$ contains both $\mu$ and $\nu$.
Now if $c^{\lambda}_{\mu\nu}$ = 1, then the following inequalities hold,

- $\lambda_2 - \mu_1 \leq \nu_2$, Since the strictly increasing property down a column need to satisfied by an LR tableau.
- $\nu_2 \leq \lambda_2 - \mu_2$, Since two can only appear in the second row of the LR tableau.
- $\nu_2 \leq \lambda_1 - \mu_1$, otherwise, it would violate the reverse lattice property of the LR tableau.
- $\lambda_1 - \mu_1 \leq \nu_1$, Since all the entries of the first row of a LR tableau are 1 to preserve the reverse lattice property.
- $\nu_1 \leq \lambda_1 - \mu_2$, Clearly the upper bound for $\nu_1$ is $\lambda_1 - \mu_2$.

Combining all the above inequalities, we get
$\lambda_2 - \mu_1 \leq \nu_2 \leq \ \leq \lambda_1 - \mu_1 \leq \nu_1 \leq \lambda_1 - \mu_2$ and $\nu_2 \leq \lambda_2 - \mu_2$.
Therefore we have proved one direction.
Conversely, If we are given partitions $\lambda, \mu, \nu$ satisfying the properties mentioned above, then we can clearly construct a LR tableau of shape $\lambda/\mu$ and content $\nu$. It is almost trivial so we are skipping the proof. Therefore the reverse direction is also proved, which establishes the proof of the proposition. □

Now using this proposition we will prove the remaining case.
CASE 3

If $c^{\lambda}_{\mu\nu}$ = 1 and $(\mu^*, \nu^*) \neq (\mu, \nu)$ for some $\lambda, \mu, \nu$.
So by the above proposition we have,

- $|\lambda| = |\mu| + |\nu|$

- $\lambda$ contains both $\mu$ and $\nu$.
- $\lambda_2 - \mu_1 \leq \ \nu_2 \ \leq \ \lambda_1 - \mu_1 \ \leq \ \nu_1 \ \leq \ \lambda_1 - \mu_2$ and $\nu_2 \leq \lambda_2 - \mu_2$

We need to show that $c^{\lambda}_{\mu^*\nu^*} = 1$.
So by the above proposition it is enough to show the following

- $\mid \lambda \mid = \mid \mu^* \mid + \mid \nu^* \mid$
- $\lambda$ contains both $\mu^*$ and $\nu^*$.
- $\lambda_2 - \mu_1^* \leq \ \nu_2^* \ \leq \ \lambda_1 - \mu_1^* \ \leq \ \nu_1^* \ \leq \ \lambda_1 - \mu_2^*$ and $\nu_2^* \leq \lambda_2 - \mu_2^*$

Since $\mid \mu^* \mid + \mid \nu^* \mid = \mid \mu \mid + \mid \nu \mid$, We have $\mid \lambda \mid = \mid \mu^* \mid + \mid \nu^* \mid$.
Given $(\mu, \nu)$ which not a fixed point there is exactly 4 possible cases of weakly decreasing sequences,( that is 4 possible choices of $(\mu^*, \nu^*)$.) which are

1. $\mu_1\mu_2\nu_1\nu_2 \implies \mu^* = (\mu_1, \nu_1)$ and $\nu^* = (\mu_2, \nu_2)$
2. $\mu_1\nu_1\nu_2\mu_2 \implies \mu^* = (\mu_1, \nu_2)$ and $\nu^* = (\nu_1, \mu_2)$
3. $\nu_1\nu_2\mu_1\mu_2 \implies \mu^* = (\nu_1, \mu_1)$ and $\nu^* = (\nu_2, \mu_2)$
4. $\nu_1\mu_1\mu_2\nu_2 \implies \mu^* = (\nu_1, \mu_2)$ and $\nu^* = (\mu_1, \nu_2)$

It is enough to verify the conditions for the sub cases (1) and (2), since the other two follow by symmetry.
<u>SUB CASE 1</u>
$\mu_1\mu_2\nu_1\nu_2 \implies \mu^* = (\mu_1, \nu_1)$ and $\nu^* = (\mu_2, \nu_2)$ ,Also note that $\mu_1 \geq \mu_2 \geq \nu_1 \geq \nu_2$
First we will check that $\mu^*$ and $\nu^*$ is contained in $\lambda$.

$$\mu_1^* = \mu_1 \leq \lambda_1$$
$$\mu_2^* = \nu_1 \leq \mu_2 \leq \lambda_2$$

Therefore $\mu^*$ is contained in $\lambda$
By a similar argument we can show that $\nu^*$ is contained in $\lambda$.
Now we are left to check whether $\lambda, \mu^*$ and $\nu^*$ satisfies the following inequalities,

$$\lambda_2 - \mu_1 \leq \ \nu_2 \ \leq \ \lambda_1 - \mu_1 \ \leq \ \mu_2 \ \leq \ \lambda_1 - \nu_1 \text{ and } \ \nu_2 \leq \lambda_2 - \nu_1$$

Note that we have the following inequalities

$$\lambda_2 - \mu_1 \leq \ \nu_2 \ \leq \ \lambda_1 - \mu_1 \ \leq \ \nu_1 \ \leq \ \lambda_1 - \mu_2 \text{ and } \nu_2 \leq \lambda_2 - \mu_2$$

Therefore the inequalities

$$\lambda_2 - \mu_1 \leq \ \nu_2$$
$$\text{and } \nu_2 \ \leq \ \lambda_1 - \mu_1$$

immediately follows.
To prove the inequality $\lambda_1 - \mu_1 \ \leq \ \mu_2$ we suppose

$$\lambda_1 - \mu_1 \ > \ \mu_2$$
$$\implies \nu_1 \geq \lambda_1 - \mu_1 \ > \ \mu_2$$
$$\implies \nu_1 > \mu_2$$
which is a contradiction.
Therefore $\lambda_1 - \mu_1 \ \leq \ \mu_2$

Since we have $\nu_1 \leq \lambda_1 - \mu_2$, it follows that $\mu_2 \leq \lambda_1 - \nu_1$.
Finally we need to check the inequality $\nu_2 \leq \lambda_2 - \nu_1$

For this suppose $\nu_2 > \lambda_2 - \nu_1$,
$$\implies \nu_2 > \lambda_2 - \nu_1 \geq \lambda_2 - \mu_2$$
$$\implies \nu_2 > \lambda_2 - \mu_2$$
This is a contradiction.
Therefore $\nu_2 \leq \lambda_2 - \nu_1$.

Combining all the above inequalities we have

$$\lambda_2 - \mu_1 \leq \ \nu_2 \ \leq \ \lambda_1 - \mu_1 \ \leq \ \mu_2 \ \leq \ \lambda_1 - \nu_1 \text{ and } \ \nu_2 \leq \lambda_2 - \nu_1$$

So, In short $\lambda, \mu^*, \nu^*$ satisfies the three properties of the proposition. And therefore $c^{\lambda}_{\mu^*\nu^*}$=1.
This implies,

$$c^{\lambda}_{\mu^*\nu^*} \geq c^{\lambda}_{\mu\nu}$$

<u>SUB CASE 2</u>
$\mu_1\nu_1\nu_2\mu_2 \implies \mu^* = (\mu_1, \nu_2)$ and $\nu^* = (\nu_1, \mu_2)$ ,Also note that $\mu_1 \geq \nu_1 \geq \nu_2 \geq \mu_2$
Since,

$$\mu_1^* = \mu_1 \leq \lambda_1,$$
$$\mu_2^* = \nu_2 \leq \lambda_2$$
$$\implies \mu^* \text{ is contained in } \lambda$$

By a similar argument $\nu^*$ is also contained in $\lambda$.
So we are left to prove the inequalities

$$\lambda_2 - \mu_1 \leq \ \mu_2 \ \leq \ \lambda_1 - \mu_1 \ \leq \ \nu_1 \ \leq \ \lambda_1 - \nu_2 \text{ and } \ \mu_2 \leq \lambda_2 - \nu_2.$$

For this, first suppose

$$\begin{gathered}
\lambda_2 - \mu_1 > \mu_2 \\
\implies \lambda_2 - \mu_2 > \mu_1 \\
\implies \nu_1 \geq \lambda_2 - \mu_2 > \mu_1 \\
\implies \nu_1 > \mu_1 \\
\text{which is a contradiction.} \\
\text{Therefore } \lambda_2 - \mu_1 \leq \mu_2
\end{gathered}$$

Now suppose,

$$\begin{gathered}
\mu_2 > \lambda_2 - \nu_2 \\
\implies \nu_2 > \lambda_2 - \mu_2 \\
\text{which itself is a contradiction.} \\
\text{Therefore } \mu_2 \leq \lambda_2 - \nu_2.
\end{gathered}$$

Now suppose,

$$\begin{gathered}
\mu_2 > \lambda_1 - \mu_1 \\
\implies \nu_2 \geq \mu_2 > \lambda_1 - \mu_1 \\
\implies \nu_2 > \lambda_1 - \mu_1 \\
\text{This is a contradiction.} \\
\text{Hence } \mu_2 \leq \lambda_1 - \mu_1
\end{gathered}$$

The inequality $\lambda_1 - \mu_1 \leq \nu_1$ follows immediately.
To prove the final inequality, suppose,

$$\begin{gathered}
\nu_1 > \lambda_1 - \nu_2 \\
\implies \nu_2 > \lambda_1 - \nu_1 \geq \lambda_1 - \mu_1 \\
\implies \nu_2 > \lambda_1 - \mu_1 \\
\text{This is a contradiction.} \\
\text{Therefore } \nu_1 \leq \lambda_1 - \nu_2
\end{gathered}$$

Combining all the above inequalities we have,

$$\lambda_2 - \mu_1 \leq \ \mu_2 \ \leq \ \lambda_1 - \mu_1 \ \leq \ \nu_1 \ \leq \ \lambda_1 - \nu_2 \text{ and } \ \mu_2 \leq \lambda_2 - \nu_2.$$

Again in this sub case the partitions $\lambda, \mu^*, \nu^*$ satisfy the three properties of the proposition. Therefore $c^{\lambda}_{\mu^*\nu^*}$=1., which implies,

$$c^{\lambda}_{\mu^*\nu^*} \geq c^{\lambda}_{\mu\nu}$$

These 3 cases exhaust all the possibilities and in each case we have, $c^{\lambda}_{\mu^*\nu^*} \geq c^{\lambda}_{\mu\nu}$.
Therefore For the partitions $\lambda, \mu, \nu$ with at most 2 parts we have,

$$c^{\lambda}_{\mu^*\nu^*} \geq c^{\lambda}_{\mu\nu}$$

This completes the proof for k =2.

□

### 4.2.2 Proof for k =3

We have a different method of proof in this context, here we use the combinatorial interpretation of LR coefficients, in particular the LR triangles which we discussed in the previous chapter.

*Proof.* As for k=2 , the case when $c^{\lambda}_{\mu\nu}$ is zero and $(\mu, \nu)$ is a fixed point the claim follows immediately. So the non-trivial case is when $(\mu, \nu)$ is not a fixed point. So will give a proof for this case.
First we will give a general idea of the proof. We know that $c^{\lambda}_{\mu\nu}$ is the number of non-negative integer points in the convex polytope $LR_k(\lambda, \mu, \nu)$ inside the cone $LR_k$ of the underlying vector space $T_3$. Therefore $c^{\lambda}_{\mu^*\nu^*}$ is the number of non-negative integer points in the convex polytope $LR_k(\lambda, \mu^*, \nu^*)$ inside the cone $LR_k$ of the underlying vector space $T_3$.Now If we can construct a linear operator $\Phi$ on $T_3$ such that when $\Phi$ restricted to the convex polytope $LR_k(\lambda, \mu, \nu)$ is mapped injectively into the convex polytope $LR_k(\lambda, \mu^*, \nu^*)$, also under the map $\Phi$, the non-negative integer points of $LR_k(\lambda, \mu, \nu)$ should be mapped to non-negative integer points of $LR_k(\lambda, \mu^*, \nu^*)$, then we can claim that,

$$c^{\lambda}_{\mu^*\nu^*} \geq c^{\lambda}_{\mu\nu}$$

So here we will introduce a candidate for such a map when k=3,
For any A = $(a_{ij}) \in T_3$, we have an associated $\lambda, \mu, \nu \in R^3$ defined as follows,

- $\mu = (a_{01}, a_{02}, a_{03})$
- $\nu = (a_{11} + a_{12} + a_{13}, a_{22} + a_{23}, a_{33})$

- $\lambda = (a_{01} + a_{11}, a_{02} + a_{12} + a_{22}, a_{03} + a_{13} + a_{23} + a_{33})$

Now let $\sigma = \mu \cup \nu = (\sigma_1, \sigma_2, \sigma_3...)$ be the partition obtained by rearranging all parts of $\mu$ and $\nu$ in weakly decreasing sequence. Since real numbers forms a totally ordered set, this weakly decreasing sequence is unique. That is given a $\mu$ and $\nu$, there is exactly one decreasing sequence consisting of its components.
Let call the map we define to be $\Phi : T_3 \to T_3$, and for any $(a_{ij}) \in T_3$,
let $\Phi((a_{ij})) = (b_{ij})$
First of all note that

$$c^{\lambda}_{\mu\nu} \geq c^{\lambda}_{\nu\mu}$$

Hence there exist a bijection between the sets $LR_k(\lambda, \mu, \nu)$ and $LR_k(\lambda, \nu, \mu)$.
Since $\mu_1 \geq \mu_i$, for all i and
$\nu_1 \geq \nu_i$, for all i
Therefore $\sigma_1 = \mu_1$ or $\nu_1$.
If $\sigma_1 = \mu_1$, then for $A \in LR_k(\lambda, \mu, \nu)$ we show that there exist $B \in LR_k(\lambda, \mu^*, \nu^*)$ such that $\Phi(A) = B$. And if $\sigma_1 = \nu_1$, then for $A \in LR_k(\lambda, \nu, \mu)$ we show that there exist $B \in LR_k(\lambda, \nu^*, \mu^*)$ such that $\Phi(A) = B$.
Therefore in the rest of the proof we assume without loss of generality that $\sigma_1 = a_{01} = \mu_1$.
Now to make sure that the image of the convex polytope $LR_k(\lambda, \mu, \nu)$ is contained within the convex polytope $LR_k(\lambda, \mu^*, \nu^*)$ under the map $\Phi$, we impose the following conditions on $\Phi((a_{ij})) = (b_{ij})$.

- $b_{01} = \sigma_1 = a_{01}, \quad b_{02} = \sigma_3, \quad b_{03} = \sigma_5, \quad b_{33} = \sigma_6$
- $b_{11} + b_{12} + b_{13} = \sigma_2$
- $b_{22} + b_{23} = \sigma_4$
- $b_{01} + b_{11} = \lambda_1$
- $b_{02} + b_{12} + b_{22} = \lambda_2$
- $b_{03} + b_{13} + b_{23} + b_{33} = \lambda_3$

Note that the above mentioned conditions are necessary but not sufficient for the map to satisfy the above property. We need also to show that each $(b_{ij}) \in LR_k$, that is each $(b_{ij})$ satisfies the three properties of the LR cones. Now define each $b_{ij}$,'s as follows

- $b_{01} = a_{01}$
- $b_{11} = a_{11}$

- $b_{02} = \sigma_3$
- $b_{03} = \sigma_5$
- $b_{13} = max\{0, \sigma_5 + a_{13} - (\mu_2 - \sigma_3) - (\sigma_5 - \mu_3)\}$
- $b_{12} = max\{0, \sigma_2 - a_{11} - b_{13}\}$
- $b_{22} = max\{0, \lambda_2 - b_{12} - b_{02}\}$
- $b_{23} = max\{0, \sigma_4 - b_{22}.\}$
- $b_{33} = \sigma_6$

Now we defined all the $b_{ij}$'s for k = 3 case, It is trivial to check the map $\Phi$ is indeed linear. The $b_{ij}$'s are precisely defined in such a way that they satisfy the three properties (P),(CS) and (LR) of the LR cones. So now we have proved that under the map $\Phi$ the image of the convex polytope $LR_k(\lambda, \mu, \nu)$ is contained in the convex polytope $LR_k(\lambda, \mu^*, \nu^*)$.
Now are task is to show that the map $\Phi$ is injective when restricted to the convex polytope $LR_k(\lambda, \mu, \nu)$. This immediately follows from the fact that each element $(a_{ij})$ of $LR_k(\lambda, \mu, \nu)$ is uniquely determined by the values of $a_{13}$, that is different elements of $LR_k(\lambda, \mu, \nu)$ has different value of $a_{13}$. Since the map we defined depends on the value of $a_{13}$, each element of $LR_k(\lambda, \mu, \nu)$ has a unique image in $LR_k(\lambda, \mu^*, \nu^*)$.
Now the only remaining thing to be checked is that the non negative integer points of $LR_k(\lambda, \mu, \nu)$ is mapped to the non-negative integer points of $LR_k(\lambda, \mu^*, \nu^*)$. Since the map which we defined contains only operations of addition and subtraction of integers this immediately follows.
This proves our claim which is,

$$c^{\lambda}_{\mu^*\nu^*} \geq c^{\lambda}_{\mu\nu}$$

$\square$

The case for k $\geq$4 involves a lot of computations. So we think it requires some kind of computer programming. So for the time being we are not working on that.We are looking forward to work on this later.

Chapter Summary

- Introduced the statement of FFLP conjecture
- Gave an elementary proof of the conjecture for k=2 case.
- proved the case for k=3 using the LR triangle method.

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