Advantage of Quantum Superposition in Entanglement Harvesting

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Certificate of Examination

This is to certify that the dissertation titled **" Advantage of Quantum Superposition in Entanglement Harvesting"** submitted by **Mr. Upayan Roy** (Reg. No. MS16041) for the partial fulfilment of BS-MS dual degree programme of the Institute, has been examined by the thesis committee duly appointed by the Institute. The committee finds the work done by the candidate satisfactory and recommends that the report be accepted.

Dr. Kinjalk Lochan Dr. Sandeep K. Goyal Dr. Manabendra Nath Bera

(supervisor)

Dated: 9^{th} May 2021

Declaration

The work presented in this dissertation has been carried out by me under the guidance of Dr. Manabendra Nath Bera at the Indian Institute of Science Education and Research Mohali.

This work has not been submitted in part or in full for a degree, a diploma,or a fellowship to any other university or institute. Whenever contributions of others are involved, every effort is made to indicate this clearly, with due acknowledgement of collaborative research and discussions. This thesis is a bonafide record of original work done by me and all sources listed within have been detailed in the bibliography.

Upayan Roy

(candidate)

Dated: 9$^{\text{th}}$ May 2021

In my capacity as the supervisor of the candidate's project work, I certify that the above statements by the candidate are true to the best of my knowledge.

Dr. Manabendra Nath Bera
(Supervisor)

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Abstract

The study of vacuum entanglement is boosted with idea of entanglement harvesting (EH), transferring the entanglement into a simpler system with well known measures of entanglement. We will see that trapped ion chains provide a practical model for studying EH. Recent studies have shown the quantum advantage of temporal superposition used to enhance the efficiency of EH. We will incorporate superposition in the ion trap model, to try to come up with an experimentally viable study of applying quantum superposition in EH.

Chapter 1

Introduction

Quantum entanglement is probably the most fundamental characteristics of quantum mechanics. Right from being noticed in context of "spooky action at a distance" by Einstein Rosen and Podolsky (EPR) in 1935 (Einstein 35), its peculiarity has caught the attention of physicists for generations (Horodecki 09). Bell in 1964 (Bell 64) came up with his inequality, claiming that two subsystems can have stronger correlations than can be predicted from any local classical theory. Over the years we have found ways to use entanglement as a *resource* in tasks like cryptography (Ekert 91) and teleportation (Bennett 93).

In a famous proposal by Summers and Werner in 1985 (Summers 85), they claimed that the vacuum state of a quantum field is entangled. They came up with a mathematically rigorous proof in a set of two papers in 1987 (Summers 87) to validate their claim. What they showed is, two space-like separated regions in the space-time having a quantum field, will have some non-local quantum correlations. In the context of Algebraic Quantum Field Theory (AQFT), this means that, there exists some operators in the C^*-algebra corresponding to the two regions, with which we can show a Bell-like inequality violation. We will explain it in detail in chapter 2.

Hence, the vacuum state of a quantum field is a huge reservoir of this entanglement which can be used as a resource. Also, it is now found that a proper understanding of the entanglement can unravel mysteries about the global space-time structure like curvature (Steeg 09) and topology (Martín-Martínez 16). The concept of field entanglement is also used in frontiers of fundamental physics like AdS-CFT correspondence in string theory. Ryu-Takayanagi in 2006 (Ryu 06) conjectured that there is a relation between the entanglement entropy of a quantum field with a property of a higher dimensional AdS space. This again indicates that the entanglement has a role to play in the global geometry of space-time. In a recent work by Brown *et. al.* (Brown 21), they were able to replicate the behaviour of wormhole using an entangled quantum system. In this paper they have talked about using a *trapped ion chain* as an experimentally realizable quantum system. We will also use a system of ion traps in our study (chapter 4). In condensed matter studies have shown that, the ground state entanglement characterizes the quantum phases of matter (Filippone 11) . So all these examples show us how vital is the understanding of vacuum entanglement in various fields.

The problem with the study of entanglement in quantum fields is that we have a very limited understanding of the measures of entanglement for such systems. However, we know some good measures for 2×2 and 2×3 dimensional systems like Negativity (Peres 96) (Horodecki 96) and Concurrence(Wootters 98). So, we can "swap" the entanglement present in a larger dimensional system into some 2×2 system, by locally coupling with it, at two different locations. Subsequently we can measure the entanglement "taken up" by the simpler system using these well defined entanglement measures.

This idea led to the works of Valentini (Valentini 91) and Reznik *et. al.* (Reznik 03) respectively on Electromagnetic (EM) fields and real scalar fields. Valentini could show that if two initially uncorrelated atoms interact locally with the EM field vacuum at two different locations, they get entangled after some time even if the interaction events remain space-like separated. The internal electronic levels of the atoms form the 2×2 system in this case, into which the EM vacuum entanglement gets *harvested*. Reznik obtained similar results using Unruh-DeWitt (Smith 19) detectors in real scalar fields. Unruh-DeWitt detectors are again a pair of systems with 2 energy levels, which can locally couple with a quantum field. In Figure 1.1 we depict the quantum field as a discrete chain of balls, to which two detectors are locally coupled. The detectors get entangled after coupling for some finite time even if the coupling events remain causally separated. The source of the entanglement is the entanglement already present in the vacuum field itself as any other possibility of interaction is ruled out by causal separation. So, this serves as the lower limit of the amount of entanglement present in the vacuum.

In section 3.2 we will talk about the work of Reznik in detail. We will also talk about an interesting concept of *temporal superposition* which would be used to enhance the efficiency of entanglement harvesting (Henderson 20). Recent works (Foo 21) have showed us that this concept of superposition has very powerful effect in better entanglement harvesting in various types of space-times. It provides us with a better tool to study the quantum field entanglements that depend on space-time properties.

However, the issue with entanglement harvesting is, it requires very intricate detector-field coupling, which is not experimentally viable. In chapter 4 we will discuss about

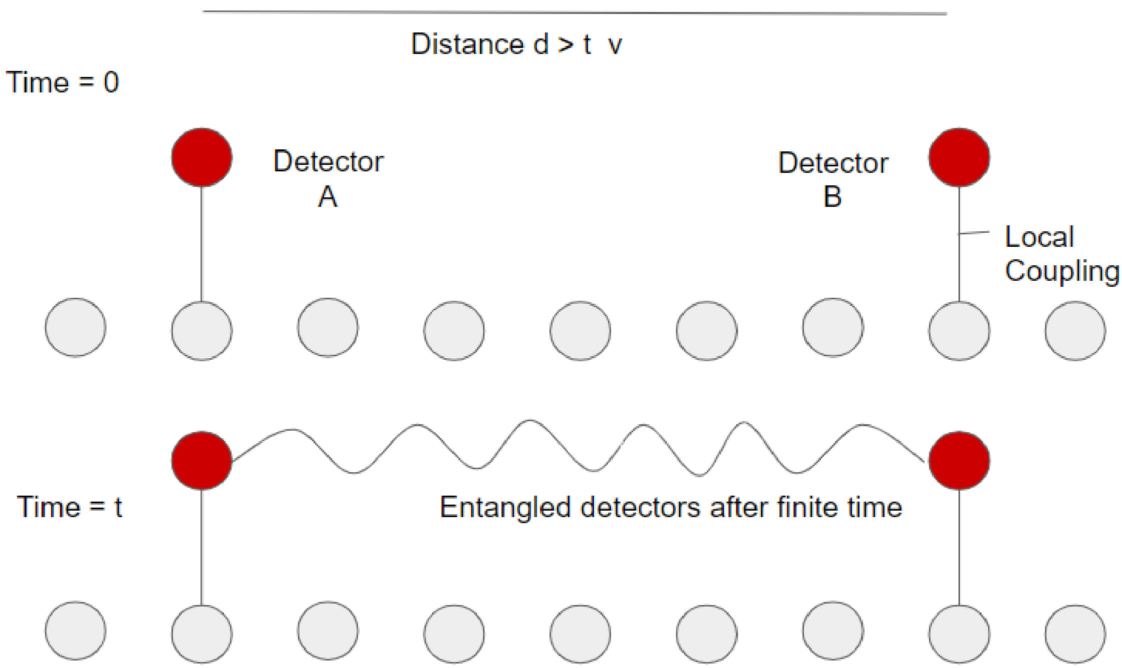


Figure 1.1: Two detectors A and B are coupled locally at two far away locations in space-time having a quantum field in vacuum state. After some finite time t, we observe that the detectors get entangled. It happens even if t is less than the time required for any physical information to travel the distance d from A to B

a simple system of trapped ion chain which can be thought of as a discrete analogue of an one-dimensional quantum field. The chain of ions form a system of n-dimensional quantum harmonic oscillator, with a ground state analogous to the quantum field vacuum. Each ions have their internal electronic structure from which a couple of energy levels can be coupled with the vibrational ground state of the harmonic oscillator using lasers. This way we can possibly "harvest" the entanglement into the internal levels. (Retzker 05)

It would interesting to see the effect of superposition in ion trap entanglement harvesting. Firstly, because it is something that is experimentally viable. Secondly, it is much easier to study the nature of entanglement in a system of discrete modes, like the harmonic oscillator than a continuous mode quantum field. We will see in chapter 5 that with a certain interaction Hamiltonian, it is only possible to extract entanglement when we use superposition. We will try to find a condition for choosing the coupling constants, such that we get a maximum entanglement harvesting.

We will see that using the superposition we are able to harvest up to 15% of the initial entanglement present in the ground state of a two-ion in a chain system. This indicates a clear quantum advantage of incorporating superposition in entanglement harvesting in the ion trap model.

Chapter 2

Vacuum Entanglement

As we mentioned in the introduction (chapter 1), Summers and Werner showed that the vacuum state of a quantum field is entangled (Summers 85)(Summers 87). In this chapter we try to understand what it is meant for a quantum field to be entangled. In section 2.1 we will discuss about the claim by Einstein, Rosen and Podolsky in 1935, which was the first attempt (Einstein 35) to mathematically formulate the strange non-classical behaviour of entanglement. It remained a mystery till Bell tried to validate the claim in 1964 but instead ended up showing that nature in the formulation of quantum physics do behave counter-intuitively (Bell 64).

Summers and Werner extended the argument to quantum fields by using the formulation of Algebraic Quantum Field Theory (AQFT) as we will see section 2.3. They used a generalized Bell's inequality in the formulation of *order unit spaces* (section 2.2) and then in C^*-algebraic framework. We will try to get the essence of the idea without going too much into the mathematical rigour. We will try to introduce the mathematical definitions as and when required.

2.1 Bell's Inequality

The most peculiar aspect of quantum mechanics is the claim that the physical properties of a system to be measured do not exist independent of observation. In 1935 Einstein, Rosen and Podolsky came up with a paper "Can Quantum-Mechanical Description of Physical Reality be Considered Complete?" (Einstein 35), to question the completeness of the quantum theory. They argued that there exists some "hidden variables" that we overlook in the theory.

2.1.1 Classical case

Let us look at what we should expect in a classical scenario (M. Nielsen 00). Charlie (C) prepares an ensemble of systems (say many copies of a system of two particles) and sends one each to Alice and Bob. Alice and Bob measure two properties each (Q, R and S, T) on the parts of the system they received. Both Alice and Bob randomly chooses between measurements Q or R and S or T respectively just before measuring. They perform the measurements in such a way that they remain causally separated, i.e., the events of measurements remain space-like separated. So, the events in the process (of which among the two measurements to be preformed and then performing it) of one party remain space-like separated from that of the other party. The outcomes of the measurements are either 1 or -1, so $Q, R, S, T \in \{-1, 1\}$. It is illustrated in Figure 2.1.

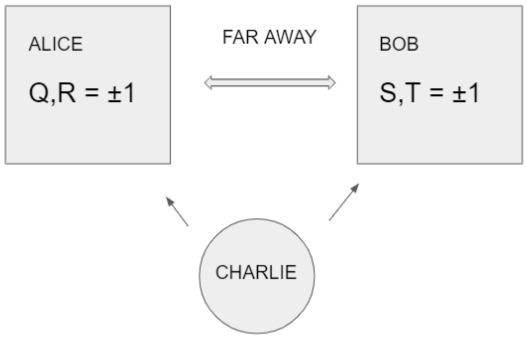


Figure 2.1: The figure represents the labs of Alice and Bob as squares that are “far away” such that the measurements remain space-like separated. Q and R are measured by Alice while S and T are measured by Bob.

We will be particularly interested in the quantity

$$\begin{aligned} & QS + RS + RT - QT \\ & = (Q + R)S + (R - Q)T = \pm 2 \end{aligned} \tag{2.1}$$

as $R, Q = \pm 1$, either of $(Q + R)$ or $(R - Q)$ vanishes and the other is ± 2. Classically we expect that the systems will have a certain probability of being in a particular state depending on the preparation. So we can have a classical probability distribution as a function of the values of the quantities that are measured. Using such a probability distribution we can get the expectation value of the combination,

$$\begin{aligned} & \mathbf{E}(QS + RS + RT - QT) \qquad (2.2) \\ = & \sum_{QRST} p(Q, R, S, T)(QS + RS + RT - QT) \end{aligned}$$

$$\leq \sum_{QRST} p(Q,R,S,T) \times 2 = 2$$

So,

$$\mathbf{E}(QS) + \mathbf{E}(RS) + \mathbf{E}(RT) - \mathbf{E}(QT) \leq 2 \tag{2.3}$$

. This is the Bell or CHSH inequality (Clauser 69).

2.1.2 The Quantum Case

Now, we consider a quantum system prepared in the following two-qubit state,

$$|\phi\rangle = \frac{|01\rangle - |10\rangle}{\sqrt{2}} \tag{2.4}$$

The two qubits are separated and sent to Alice and Bob respectively. Let the observables we encountered before in Equation 2.1 be,

$$Q = Z_1, \quad R = X_1, \quad S = \frac{-Z_2 - X_2}{\sqrt{2}}, \quad T = \frac{Z_2 - X_2}{\sqrt{2}} \tag{2.5}$$

where Z_i and X_i are the σ_z and σ_x operators respectively of individual qubits. We can show that,

$$\langle QS\rangle + \langle RS\rangle + \langle RT\rangle - \langle QT\rangle = 2\sqrt{2} \tag{2.6}$$

where, the angled brackets indicate the expectation values of the operators in the given state (Equation 2.4). Hence the inequality is clearly violated. This is not possible for any classical theory that has the following assumptions,

1. **Realism:** Physical properties take values independent of observation.

2. **Locality:** Bob's measurements do not affect that of Alice.

The violation indicates that one of the two (or both of) the assumptions does not hold true under the quantum theory.

2.2 Bell's inequality in Order Unit Spaces

We can generalize the idea that an inequality like the Bell's inequality can characterize the non-classicality present in a theory. We will follow the line of arguments given in the 1987 paper by Summers and Werner (Summers 87). We will have to look at a few basic mathematical definitions to formulate such a generalization.

- **Order Unit Spaces** are vector spaces with a defined ordering and containing a unit element. It is given as $(\mathscr{A}, \geq, \mathbf{1})$. So, we arbitrarily assign a relation, $\mathrm{A} > \mathrm{B} \forall \mathrm{A}, \mathrm{B} \in \mathscr{A}$.

- **Convex Cone** is defined as $\mathscr{A}_+ \equiv \{a \in \mathscr{A} | a \geq \mathbf{0}\}$ such that,

  1. $\forall r > 0$, $r \in \mathbb{R}$, $r\mathscr{A}_+ \subseteq \mathscr{A}_+$, hence it is closed under a scaling with a positive element in the field over which the vector space is defined.
  2. $\mathscr{A}_+ + \mathscr{A}_+ \subseteq \mathscr{A}_+$, hence it is also closed under vector addition.

Notice, we can use $\mathscr{A}_+$ to order $\mathscr{A}$. To do so, we will say, for a,$b \in \mathscr{A}$, if $(a - b) \in \mathscr{A}_+$, then $a \geq b$.

In a general quantum theory we need to define two main objects. The *preparation* or the states in which we prepare our system and *measurement* or the operators with which we bring about a change in the system. In a more mathematically rigorous sense we have the following definitions.

- **Measurements** with finite possible outcomes are represented as the set $\{a_i\}$, $a_i \in \mathscr{A}_+$, $\sum_i a_i = \mathbf{1}$, where $\mathscr{A}_+ \equiv \{a^* a | a \in \mathscr{A}\}$. If we look closely, $\sum_i a_i = \mathbf{1}$ is basically the Kraus' theorem for a complete set of Kraus operators.

- **Preparations** are the normalized linear functionals on $\mathscr{A}$ that maps each element in $\mathscr{A}$ to some number. If, ω is a state on $\mathscr{A}$, then $\omega_i = \omega(a_i)$ is the probability of obtaining result i. We will see that in our familiar notion we can write $\omega_\rho(A) = \mathrm{Tr}(\rho A)$, where ρ is the state and A is the operator.

Now the stage is set to define a generalized Bell's inequality in this context. In order to do so we need to define something known as the correlation duality. **Correlation duality** consists of two order unit spaces $\mathscr{A}$, $\mathscr{B}$ and a mapping $\hat{p} : \mathscr{A} \times \mathscr{B} \to \mathbb{R}$, such that, $\hat{p}(a, b) \geq 0$ and $\hat{p}(\mathbf{1}_\mathscr{A}, \mathbf{1}_\mathscr{B}) = 1$, where $a \in \mathscr{A}$ and $b \in \mathscr{B}$. $\mathbf{1}_\mathscr{A}$ and $\mathbf{1}_\mathscr{B}$ represent the unit elements in $\mathscr{A}$ and $\mathscr{B}$ respectively.

Let us consider measurements that admits two outcomes (and hence consisting of only two elements), corresponding to $\{a_+, a_-\} \subset \mathscr{A}$, where we can see a_+,$a_- \geq \mathbf{0}$, $a_+ + a_- = \mathbf{1}$. It can be mapped to elements like $-\mathbf{1} \leq a \leq \mathbf{1}$ using $a_\pm = \frac{1}{2}(\mathbf{1} \pm a)$. We will look at two such measurements each at **A** and **B**. $a_1, a_2 \in \mathscr{A}$ and $b_1, b_2 \in \mathscr{B}$. We have the following theorem,

Theorem 1 $\chi = \frac{1}{2} \left| \hat{p}(a_1, (b_1 + b_2)) + \hat{p}(a_2, (b_1 - b_2)) \right|$

1. $\chi \leq 2$

2. *(a) If* $\mathscr{A}$ *is Hermitian part of* C^* *algebra then* $\chi \leq \sqrt{2}$

   (b) If $\chi = \sqrt{2}$, $\forall a \in \mathscr{A}$
   $\omega([a_i, a]) = 0, \omega(a_i^2 a) = \omega(a), \omega((a_1 a_2 + a_2 a_1)a) = 0$

3. $\chi \leq 1$ *if,*

   (a) $\mathscr{A}$ *is classical(Abelian).*

   (b) ω *is pure on* $\mathscr{A}$

   (c) $\exists\ \xi_\alpha \in \mathscr{A}^*$, $\eta_\alpha \in \mathscr{B}^*$, *such that,* $\forall a \in \mathscr{A}, b \in \mathscr{B}$
   $\hat{p}(a, b) = \sum \lambda_\alpha \xi_\alpha(a) \eta_\alpha(b)$.

We will not go into the detailed proof of the theorem which is present in full rigor in the paper of Summers and Werner (1987). If we look closely, we can make some interesting observations.

Notice that $\chi = \frac{1}{2} \left|\hat{p}(a_1, (b_1 + b_2)) + \hat{p}(a_2, (b_1 - b_2))\right|$ has the same form as the left hand side of the CHSH inequality in Equation 2.3. The right hand side would be 4 instead of $2\sqrt{2}$ as we have a $1/2$ factor on the left. So point 1, is a general inequality theorem.

We get the CHSH inequality in the a more specialized order unit space known as the C^* algebra. If we construct our theory such that the measurements are represented by a C^* algebra, then we get our familiar quantum theory in a generalized form. Before moving on, let us have a brief mathematical description of C^* algebra.

An algebra is a vector space with an additional bilinear mapping in the form of vector multiplication $* : \mathscr{A} \times \mathscr{A} \to \mathscr{A}$ or $\mathbf{A} * \mathbf{B} \in \mathscr{A}\ \forall\ \mathbf{A}, \mathbf{B} \in \mathscr{A}$. A C^* algebra is a special kind of algebra with the following properties,

1. We can define an involution $\mathbf{A}^*$ for every $\mathbf{A} \in \mathscr{C}$ such that, $\mathbf{A}^{**} = (\mathbf{A}^*)^* = \mathbf{A}$, where $\mathscr{C}$ is the C^* algebra. The involution is the generalized hermitian conjugation for the operators.

2. For all $\mathbf{A}, \mathbf{B} \in \mathscr{C}$ we have $(\mathbf{A} + \mathbf{B})^* = \mathbf{A}^* + \mathbf{B}^*$ and, $(\mathbf{AB})^* = \mathbf{B}^*\mathbf{A}^*$

3. $(c\mathbf{A})^* = \bar{c}(\mathbf{A})^*$ for all c in the complex field over which the algebra is defined. $\bar{c}$ is the complex conjugate of c.

4. $\|\mathbf{A}^*\mathbf{A}\| = \|\mathbf{A}\|\|\mathbf{A}^*\|$ which is the most important property of the C^* algebra.

Coming back to the theorem, if $\omega(x) = 0$ implies, $x = 0, \forall x \in \mathscr{A}$, then from the second part (2(b)), $a_i^2 = 1$ and $((a_1a_2 + a_2a_1)) = 0$. When $\mathscr{A}$ is the Hermitian part of C^* algebra, i.e., when $\mathbf{A}^* = \mathbf{A}$, and ω is restricted to 2×2 matrix algebra $M_2(\mathbb{C})$, the elements a_1, a_2 and $a_3 \equiv -(i/2)[a_1, a_2]$ form the Pauli spin matrices. We know that the observation of Bell's inequality violation was experimentally realized using spin qubit systems (Aspect 81), which are basically some 2×2 systems. Hence, this theory has a direct experimental realization in the form of this special case.

The point 3 is all about the situations in which we do not have an entanglement. 3a) is the classical Abelian case which follows the Bell's inequality. In 3b) purity is a situation when we can decompose the mapping $\hat{p}$ into a product of local mappings from individual subalgebras. So, $\hat{p}(a, b) = \omega(a)\eta(b)$ where, $a \in \mathscr{A}$ and $b \in \mathscr{B}$, $\omega : \mathscr{A} \to \mathbb{R}$ and $\eta : \mathscr{B} \to \mathbb{R}$. Finally, 3c) is the most general case in which we decompose the global mapping as, $\hat{p}(a, b) = \sum \lambda_\alpha \xi_\alpha(a)\eta_\alpha(b)$. Here $\mathscr{A}^*$ and $\mathscr{B}^*$ are the dual spaces. or the space of mappings from $\mathscr{A}$ and $\mathscr{B}$ to $\mathbb{R}$ which are basically the local state spaces.

The *maximal Bell correlation* $\beta(\hat{p}, \mathscr{A}, \mathscr{B})$ is defined as:

$$\beta(\hat{p}, \mathscr{A}, \mathscr{B}) \equiv \frac{1}{2}\mathsf{sup}(\hat{p}(a_1, b_1) + \hat{p}(a_1, b_2) + \hat{p}(a_2, b_1) - \hat{p}(a_2, b_2)) \tag{2.7}$$

where supremum is over all $a_i \in \mathscr{A}, b_j \in \mathscr{B}$ and, $-\mathbf{1}_{\mathscr{A}} \leq a_i \leq \mathbf{1}_{\mathscr{A}}$, $-\mathbf{1}_{\mathscr{B}} \leq b_i \leq \mathbf{1}_{\mathscr{B}}$ So, $\beta(\hat{p}, \mathscr{A}, \mathscr{B}) = 1$ means all possible quadruple satisfy Bell's inequality. If we have measurements that violate the classical inequality as we did in subsection 2.1.2, we have reached the quantum realm, which cannot be explained by any local classical theory. We have experimentally demonstrated (Aspect 81) that we can have such situations, implying that nature does behave quantum mechanically.

For, quantum theories, in C^* algebraic framework, the right hand side is $\sqrt{2}$, which is an equivalent inequality. Again if we find measurements for which even that inequality is violated, we say that the observation cannot be explained by the postulates of standard quantum mechanics. We have not yet experimentally violated this inequality, which implies that the quantum theory is safe so far.

Let us now try to understand how we study a quantum field in the framework of C^* algebra, and what are the equivalent correlation duality that we can use to form a Bell like inequality.

2.3 Vacuum Entanglement in context of Algebraic QFT

In the 1950's and 60's people were trying to come up with a proper mathematical structure for studying quantum fields. While Wightman (Streater 64) tried to focus on the field, Haag, Kastler, and Araki (Araki 99)(Haag 64) considered the observables as the primary objects for the theory. They claimed that the field states emerge naturally from the mathematical description of the space of observables. We will be looking at the postulates formulated by Haag, Kastler, and Araki in the form of Algebraic Quantum Field theory in the following subsection.

2.3.1 Algebraic Quantum Field Theory

In AQFT we assign a C^* algebra $\mathscr{A}(\mathscr{O})$ to each space-time region, $\mathscr{O} \subset \mathbb{R}^4$. So, each possible subset or location of the Minkowski space-time has an associated local C^* algebra, representing the observables in that region. These local C^* algebras will be satisfying the following conditions,

1. **Isotony:** If $\mathscr{O}_1 \subseteq \mathscr{O}_2$ then $\mathscr{A}(\mathscr{O}_1) \subseteq \mathscr{A}(\mathscr{O}_2)$. We say $\mathscr{A}(\mathscr{O})$ are subalgebras of the global algebra C^* algebra $\mathscr{A}$ generated by $\cup_{\mathscr{O}\subset\mathbb{R}^4}\mathscr{A}(\mathscr{O})$.

   So, for any location within a larger location we have a smaller subalgebra for it. We can think of any location as a smaller location within the entire space-time, and hence all local algebras are some sub-algebra of the global algebra.

2. **Poincaré covariance:** $\exists\{a_\lambda|\lambda \in \mathscr{P}\}$ where $\mathscr{P}$ is the Poincaré group, a_λ are automorphisms on $\mathscr{A}$ such that, $a_\lambda(\mathscr{A}(\mathscr{O})) = \mathscr{A}(\mathscr{O}_\lambda)$ where $\mathscr{O}_\lambda = \lambda(\mathscr{O})$

   This postulate makes sure that we have a corresponding notion of translations and Lorentz transformations in the algebra of operators. The algebra associated with the transformed space-time is equivalent to an unique automorphism of that algebra. So, the algebra is mapped to another algebra in accordance to the transformation in the spacetime. Here λ represents the space-time Poincaré transformation while a_λ is the corresponding automorphism in the algebra.

3. **Locality:** If $\mathscr{O}_1$ and $\mathscr{O}_2$ are space-like separated, then $[\mathscr{A}(\mathscr{O}_1), \mathscr{A}(\mathscr{O}_2)] = 0$

   This postulates incorporates the Einstein's causality in the algebra of the observables. When the two locations $\mathscr{O}_1$ and $\mathscr{O}_2$ are space-like separated, we expect the operators associated with these two regions to commute.

4. **Physical Representation:** There is a representation π of $\mathscr{A}$ on Hilbert space $\mathscr{H}$ such that, there is a unitary representation $U(\mathscr{P})$ of the Poincaré group, where,

   (a) $U(\lambda)\pi(A)U(\lambda)^{-1} = \pi(a_\lambda(A)), \forall A \in \mathscr{A}$

   (b) Generators of translation group satisfy, $(P_0^2 - P_1^2 - P_2^2 - P_3^2) \geq 0$

   There is a Gelfand–Naimark–Segal (GNS) representation of a C^* algebra on a Hilbert space $\mathscr{H}$, that maps each element of the algebra to some bounded operators on $\mathscr{H}$ (Arveson 76). Here the mapping is the π. So we can find an $\xi \in \mathscr{H}$ such that $\omega(a) = \langle \pi(a)\xi, \xi \rangle$. We saw in postulate 2 that, there is a automorphism a_λ corresponding to some Poincaré transformation λ. For such automorphisms we have a corresponding unitary operation that transforms the operator acting on the Hilbert space over which the algebra is represented. 4b) is basically the 4-momentum conservation under Poincaré transformation, as the translations must be generated by some momentum operators P_i.

Now, we move on to the description of the vacuum state and the correlation duality in this picture of algebraic QFT.

2.3.2 The Vacuum state and inequality violation

The two local algebras that we required to form a correlation duality in order unit spaces, are just the sub-algebras corresponding to the two space-like separated regions $\mathscr{O}_1$ and $\mathscr{O}_2$. The global state ϕ_0 will be the mapping $\hat{p}$. So in our case the correlation duality will be of the form, $(\phi_0, \mathscr{A}(\mathscr{O}_1), \mathscr{A}(\mathscr{O}_2))$ where $\mathscr{O}_1$ and $\mathscr{O}_2$ are space-like separated regions.

It can be shown that there exists a unique vacuum vector $\Omega \in \mathscr{H}$. Ω is the vacuum state, which means, it is translation invariant. So the vacuum state is $\phi_0(A) = \langle \Omega, \pi(A)\Omega \rangle, A \in \mathscr{A}$.

There is a very important property called the clustering property that emerges in this formalism.

Clustering property of the vacuum state: If $\mathscr{O}_1, \mathscr{O}_2$ are bounded subsets of space-time and $a \in \mathbb{R}^4$ is some space-like vector,

$$\lim_{t \to \infty} \phi_0(U(ta)AU(ta)^{-1}B) = \phi_0(A)\phi_0(B) \tag{2.8}$$

for any $A \in \mathscr{A}(\mathscr{O}_1)$ and $B \in \mathscr{A}(\mathscr{O}_2)$. With the works of Fredenhagen (Fredenhagen 85) and Araki (Araki 62) it can be stated that the clustering behaves like R^{-2} for massless and e^{-mR} for massive fields, where R is roughly the distance between $\mathscr{O}_1$ and $\mathscr{O}_2$. This property implies that as the two space-time locations are translated infinitely far from each other, they can be written in a product form and hence they will no longer be violating the classical inequality.

In the next chapter we will try to see how we can experimentally realize the fact that the quantum field vacuum is entangled.

Chapter 3

Entanglement Harvesting

As we mentioned in the introduction, we do not have a good understanding of how to quantify the entanglement in an infinite dimensional system like the quantum field. Hence to experimentally demonstrate that the quantum field is indeed entangled, we need to first transfer the entanglement in a simpler (2×2) system and then measure it using Negativity (Peres 96) or Concurrence (Wootters 98). In the following section, we will briefly discuss about the work of Reznik in 2003 (Reznik 03), in which he used a Unruh-DeWitt detector to "probe" a real scalar quantum field.

3.1 Entanglement Harvesting using Unruh-DeWitt detectors

In this paper titled "Entanglement from the vacuum", Reznik considers a massless relativistic scalar field $\phi(\vec{x}, t)$ in 3 spatial dimensions. The field is locally coupled with two Unruh-DeWitt detectors in two distant regions. A Unruh-DeWitt detector is basically a system with two discrete energy levels with an energy gap Ω.

The interaction hamiltonian used to couple the field and the detectors is,

$$\begin{aligned} H_{int} &= H_A + H_B \\ &= \epsilon_A(\tau)(e^{-i\Omega\tau}\sigma_A^+ + e^{+i\Omega\tau}\sigma_A^-) \otimes \phi(x_A(\tau), t) \\ &+ \epsilon_B(\tau')(e^{-i\Omega\tau'}\sigma_B^+ + e^{+i\Omega\tau'}\sigma_B^-) \otimes \phi(x_B(\tau'), t) \end{aligned} \tag{3.1}$$

where τ and τ' are the proper times of the detectors A and B respectively. $\epsilon_A(\tau)$ and $\epsilon_B(\tau')$ are the coupling parameters which are active(non-zero) at some parts of the space-time trajectories of the detectors. The couplings are switched on in such a way that the parts of the trajectories in which they couple remain space-like separated. σ_A^+, σ_A^- and σ_B^+, σ_B^- are the raising and lowering operators of the Detectors A and B respectively. Ω is the energy gap between the two energy levels of the detectors and ϕ is the quantum field operator.

The trajectories are chosen as follows,

$$\begin{aligned} x_A &= -L/2\cosh(2\tau/L) \\ t_A &= L/2\sinh(2\tau/L) \end{aligned} \tag{3.2}$$

$$\begin{aligned} x_B &= L/2\cosh(2\tau'/L) \\ t_B &= L/2\sinh(2\tau'/L) \end{aligned} \tag{3.3}$$

This choice of trajectories makes sure that they remain causally separated. Hence we can now write the unitary evolution operator as a product, as the local Hamiltonians H_A and H_B will commute.

$$\begin{aligned} U &= e^{-i\int\int(H_A(\tau)+H_B(\tau'))d\tau d\tau'} \\ &= e^{-i\int H_A(\tau)d\tau} \otimes e^{-i\int H_B(\tau')d\tau'} \\ &= \left(\mathbf{I}_A - i\int H_A(\tau)d\tau + \frac{1}{2}\left(-i\int H_A(\tau)d\tau\right)^2 + O(\epsilon_A^3)\right) \otimes \\ &\quad \left(\mathbf{I}_B - i\int H_B(\tau')d\tau' + \frac{1}{2}\left(-i\int H_B(\tau')d\tau'\right)^2 + O(\epsilon_B^3)\right) \end{aligned} \tag{3.4}$$

Now, both the detectors and the field are initially assumed to be in their respective ground and vacuum states. $|\Psi_i\rangle = |{\downarrow_A}\rangle |{\downarrow_B}\rangle |0\rangle$. Now, evolving with the unitary operator we get,

$$\begin{aligned} |\Psi_f\rangle &= \Big[(\mathbf{1} - \Phi_A^-\Phi_A^+ + \Phi_B^-\Phi_B^+)|{\downarrow\downarrow}\rangle - \Phi_A^+\,\Phi_B^+|{\uparrow\uparrow}\rangle \\ &- \; i\Phi_A^+\mathbf{1}_B|{\uparrow\downarrow}\rangle - i\mathbf{1}_A\Phi_B^+|{\downarrow\uparrow}\rangle\Big]|0\rangle + O(\epsilon_i^3) \end{aligned} \tag{3.5}$$

where we have,

$$\Phi_i^\pm = \int d\tau\epsilon_i(\tau)e^{\pm i\Omega\tau}\phi(x_i(\tau),t) \tag{3.6}$$

$(i = A, B)$. Note that we have only expanded the unitary operators upto second order in the coupling parameters as they are very small.

The final state after tracing over the field degrees of freedom has the following form.

$$\rho = \begin{pmatrix} 1-C & 0 & 0 & -\langle X_{AB}|0\rangle \\ 0 & |E_A|^2 & \langle E_B|E_A\rangle & 0 \\ 0 & \langle E_A|E_B\rangle & |E_B|^2 & 0 \\ -\langle 0|X_{AB}\rangle & 0 & 0 & |X_{AB}|^2 \end{pmatrix} \tag{3.7}$$

where, $|E_A\rangle \equiv \Phi_A^+ |0\rangle$, $|E_B\rangle \equiv \Phi_B^+ |0\rangle$, $|X_{AB}\rangle \equiv \Phi_A^+\Phi_B^+ |0\rangle$, $C = 2\mathrm{Re}\,\langle 0| \mathrm{T}(\Phi_A^-\Phi_A^+ + \Phi_B^-\Phi_B^+) |0\rangle$ and $|X_{AB}|^2 = \langle X_{AB}|X_{AB}\rangle$. We have written the density matrix in the basis $\{|\downarrow\downarrow\rangle, |\downarrow\uparrow\rangle, |\uparrow\downarrow\rangle, |\uparrow\uparrow\rangle\}$.

The condition for non separability (Peres 96)(Horodecki 96) for the density matrix in Equation 3.7 is given as,

$$\begin{aligned} |\langle 0|X_{AB}\rangle|^2 &> |E_A|^2|E_B|^2 \qquad \text{and} \\ |\langle E_B|E_A\rangle|^2 &> |X_{AB}|^2 \end{aligned} \tag{3.8}$$

Now, we will be calculating the terms. We have,

$$|E_A|^2 = \int d\tau_A \int d\tau_A' e^{-i\Omega(\tau_A'-\tau_A)} D^+(A', A) \tag{3.9}$$

and

$$\langle 0|X_{AB}\rangle = \int d\tau_A \int d\tau_B e^{i\Omega(\tau_A+\tau_B)} D^+(A, B) \tag{3.10}$$

where we have the Wightman function given as, $D^+(x', x) = \langle 0| \phi(x', t')\phi(x, t)) |0\rangle = -\frac{1}{4\pi^2}((t'-t-i\epsilon)^2 - (\vec{x}' - \vec{x})^2)$

Now, if we put in the trajectories we considered in Equation 3.2 and Equation 3.3,

$$D^+(A', A) = -\frac{1}{4\pi^2 L^2 \sinh^2[(\tau_A' - \tau_A - i\epsilon)/L]} \tag{3.11}$$

$$D^+(A, B) = \frac{1}{4\pi^2 L^2 \cosh^2[(\tau_B + \tau_A - i\epsilon)/L]} \tag{3.12}$$

we can put in these functions and calculate the integrals. We get,

$$\frac{|\langle 0|X_{AB}\rangle|}{|E_A|^2} = e^{\pi\Omega L/2} \tag{3.13}$$

and hence, it does satisfy the condition of non-separability. Hence we conclude that the final density matrix in Equation 3.7, did have some entanglement starting from a separable local ground state. We could successfully 'transfer' some of the entanglement into the detector pair system.

Now, entanglement harvesting depends on the configuration and internal properties of the detectors(Pozas-Kerstjens 17). It depends on the

- energy gap of the two levels of the detectors,
- coupling constants or the strength of the detector field coupling,
- spacial separation of the two detectors, and
- switching profile, or the time for which the detector-field coupling remains active.

In the paper by Pozas-Kerstjens *et. al.* (Pozas-Kerstjens 17), they have come up with an interesting no-go theorem, for a special kind of switching profile.

Theorem 2 No-go theorem *Entanglement Harvesting is not possible if the detectors have a Dirac-delta window function.*

In other words, when the detector-field coupling is switched on and off in an infinitesimally short time interval, we see no harvested entanglement. We will see that this theorem will be challenged by using the novel idea of *Temporal Superposition* and *Indefinite Causal Ordering.* (Henderson 20). We will discuss about this in the following section.

3.2 Temporal Superposition and Indefinite Causal Order in Entanglement Harvesting

The most radical claim of Einstein's Relativity is, the causal relations of events are dynamic in nature. They depend on the metric, which is the solution of the Einstein field equations given a matter-energy distribution. In order to incorporate such an idea into the quantum framework, it was found necessary to consider quantum events with no fixed order (Hardy 09). It is a completely new and unexplored area as most of quantum mechanics is done assuming some fixed arrangement of events in time.

Shannon's theory (Shannon 48) of quantum communication, assumes a definite configuration of channels. It was later found (Aharonov 90) that in general however, we can have a quantum superposition of these channels. This superposition can also be in the order of these channels in time. The temporal order can be controlled by a control qubit, the superposition of which can lead to an indefinite causal ordering. This setup was termed as the quantum Switch (Chiribella 13a).

It was shown by the works of Chiribella *et. al.* (Chiribella 13b) that indefinite ordering can be used to perform some unique tasks. It is impossible to accomplish such computations with the same number of definitely ordered gates. More recently, Procopio *et. al* (Procopio 15) experimentally realized the quantum switch.

In the paper by Henderson *et. al.* (Henderson 20), it was found that, using *quantum controlled superposition* of a pair of co-moving Unruh DeWitt detectors we can have a more efficient entanglement harvesting. It is found we get a violation of the no-go theorem we got in Theorem 2.

We will have a couple of co-moving Unruh DeWitt detectors in a real scalar field. The interaction Hamiltonian is given as

$$H_I(t) = \sum_{D=A,B} \sum_{i=0,1} \lambda_D \chi_{D,i}(t) \left(e^{i\Omega_D t}\sigma_+ + e^{-i\Omega_D t}\sigma_-\right) \otimes \phi\left(\mathbf{x_D(t)}\right) \otimes |i\rangle_C \langle i| \tag{3.14}$$

where, λ_D, $\chi_{D,i}$, ϕ are the coupling constants, switching functions, and scalar field operator respectively. The $\sigma_\pm$ and Ω_D are the raising/lowering ladder operators and the energy gap of the two-level detectors. System C is the control qubit that determines the switching functions of the detectors A and B.

There are four terms in the Hamiltonian, two each for the two detectors A and B. $\chi_{D,0}(t)$ are chosen when the control qubit is in $|0\rangle$ state and $\chi_{D,1}(t)$ are chosen when the control qubit is in $|1\rangle$ state. Now, we can choose the functions $\chi_{D,i}(t)$'s to depend on time in a certain way as we will discuss in the following. Things get interesting when we have a superposition in the control qubit.

As we have a common rest frame for the two detectors, lets assume in the frame we divide the space-time into equal time space-like slices. There are a couple of distinct cases that we are considering.

- **Past-Future case:** When $\text{supp}(\chi_{A,0}) = \text{supp}(\chi_{B,0}) \leq \text{supp}(\chi_{A,1}) = \text{supp}(\chi_{B,1})$, the two detectors are switched on simultaneously but in superposition of two different time slices.

  Here supp is the support of a function, i.e., the subset of the domain (here its the time) in which they are non-zero. If we look at the interaction hamiltonian Equation 3.14, in this case we have both the $\chi_{D,0}(t)$'s and $\chi_{D,1}(t)$'s being activated at some particular time say t_0 and t_1 respectively. In Figure 3.1(left), we have this phenomenon of simultaneous activation of the two detectors at either of the two different time slices. When the control qubit is in a superposition, we get a superposition of the coupled unitary evolution (corresponding to the interaction hamiltonian) at two different times t_0 and t_1.

- **Cause-Effect case:** When $\text{supp}(\chi_{A,0}) = \text{supp}(\chi_{B,1}) \leq \text{supp}(\chi_{A,1}) = \text{supp}(\chi_{B,0})$, one detector is switched on in the causal past of another, but in superposition of causal ordering.

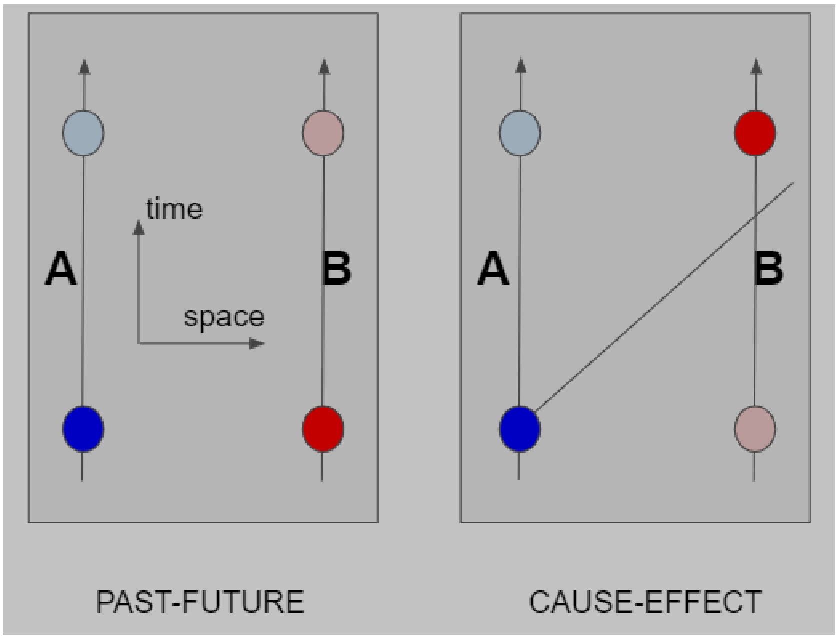


Figure 3.1: The Past-Future case and the Cause-Effect case, with the faded and brightened colors in separate branch of superposition. In the past-future image, both A and B are switched on simultaneously at either of the two times. The 45° line in cause-effect image indicates the trajectory of light from A to B, showing that A is in the causal past of B in one branch of superposition, while B is in the causal past of A in the other.(Henderson(2018))

Figure 3.1(right) we have, a situation in which $\text{supp}(\chi_{A,0}(t)) \leq \text{supp}(\chi_{B,0}(t))$ when control is in $|0\rangle$ and so we have the coupling at A taking place before that at B. Also, $\text{supp}(\chi_{A,1}(t)) \geq \text{supp}(\chi_{B,1}(t))$ when control is in $|1\rangle$ in which B is before A. Now, again if the control qubit is in a superposition, we get a superposition in the unitaries with different causal orders of the events.

3.3 Evolving the system

Initially both the detectors and the field are in their respective ground and vacuum states. The control qubit is in $|+\rangle_C = \frac{1}{\sqrt{2}}(|0\rangle_C + |1\rangle_C)$ which is the superposition state.

The density matrix corresponding to the initial state of the entire system ABC and the field is,

$$\rho_0 = |0\rangle_A \langle 0| \otimes |0\rangle_B \langle 0| \otimes |+\rangle_C \langle +| \otimes |0\rangle_F \langle 0| \tag{3.15}$$

Again we expand the unitary operators upto the second order in coupling parameter.

$$\begin{aligned} U &= \mathbb{I} - i\int_{-\infty}^{\infty} dt H_I(t) \\ &- \int_{-\infty}^{\infty} dt \int_{-\infty}^{t} dt' H_I(t) H_I(t') + \mathscr{O}\left(\lambda^3\right) \\ &= \mathbb{I} + U^{(1)} + U^{(2)} + \mathscr{O}\left(\lambda^3\right) \end{aligned} \tag{3.16}$$

Evolving the initial state and tracing over the field we get,

$$\rho_{ABC} = Tr_F\left[\rho_0 + U^{(1)}\rho_0 U^{(1)\dagger} + U^{(2)}\rho_0 + \rho_0 U^{(2)\dagger}\right] + \mathscr{O}(\lambda^4)$$
$$= \frac{1}{2}\sum_{i,j}\begin{bmatrix} (1+Y_{ii}+Yjj) & 0 & 0 & \mathscr{M}_{jj}^* \\ 0 & P_{B,ij} & \mathscr{L}_{AB,ji}^* & 0 \\ 0 & \mathscr{L}_{AB,ij} & P_{A,ij} & 0 \\ \mathscr{M}_{ii} & 0 & 0 & 0 \end{bmatrix} \otimes |i\rangle_C \langle j| \tag{3.17}$$

where the $U^{(1)}$ and $U^{(2)}$ are the first and the second order terms of the Dyson series expansion of the unitary operator corresponding to the Hamiltonian given in Equation 3.14.

The various terms appearing in the final density matrix are given as:

- $P_{D,ij}$: Local terms of correlation at the same time when $i = j$ and non-local correlation at different times when $i \neq j$ both for the same detector.

$$P_{D,ij} = \lambda_D^2 \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} dt' \chi_{D,i}(t)\chi_{D,j}(t')\langle\phi(\mathbf{x_D}(\mathbf{t'}))\phi(\mathbf{x_D}(\mathbf{t}))\rangle e^{i\Omega_D(t-t')} \tag{3.18}$$

- The Y_{ii}'s are simply related, $Y_{ii} + Y_{ii}^* = -(P_{A,ii} + P_{B,ii})$

- $\mathscr{M}$: Non-local field correlations between different detectors, responsible for **entanglement harvesting**

$$\mathscr{M} = -\lambda_A\lambda_B \sum_{D\neq D'} \int_{-\infty}^{\infty} dt \int_{-\infty}^{t} dt' \chi_{D,i}(t)\chi_{D',i}(t') \langle\phi(\mathbf{x_D}(\mathbf{t'}))\phi(\mathbf{x_{D'}}(\mathbf{t}))\rangle e^{i\Omega_D(t)} e^{i\Omega_{D'}(t')} \tag{3.19}$$

- $\mathscr{L}_{AB}$:

$$\mathscr{L}_{AB,ij} = \lambda_A\lambda_B \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} dt' \chi_{A,i}(t)\chi_{B,j}(t') \langle\phi(\mathbf{x_B}(\mathbf{t'}))\phi(\mathbf{x_A}(\mathbf{t}))\rangle e^{i\Omega_A(t)} e^{-i\Omega_B(t')} \tag{3.20}$$

Tracing over the qubit C in Equation 3.17 will give us in the classical results that we got in Equation 3.7.

$$\rho_{AB}^{(tr)} = \begin{bmatrix} 1-(P_A^{(tr)}+P_B^{(tr)}) & 0 & 0 & \mathscr{M}^* \\ 0 & P_B^{(tr)} & \mathscr{L}_{AB}^{(tr)*} & 0 \\ 0 & \mathscr{L}_{AB}^{(tr)} & P_A^{(tr)} & 0 \\ \mathscr{M} & 0 & 0 & 0 \end{bmatrix} \tag{3.21}$$

Instead if the Control qubit is measured in $|+\rangle$ basis, some interference terms are retained.

$$\rho_{AB}^{(+)} = \begin{bmatrix} 1-(P_A^{(+)}+P_B^{(+)}) & 0 & 0 & \mathscr{M}^* \\ 0 & P_B^{(+)} & \mathscr{L}_{AB}^{(+)*} & 0 \\ 0 & \mathscr{L}_{AB}^{(+)} & P_A^{(+)} & 0 \\ \mathscr{M} & 0 & 0 & 0 \end{bmatrix} \tag{3.22}$$

where we find terms like $P_{D,i\neq j}$ and $\mathscr{L}_{AB,i\neq j}$, containing two-point correlation functions between non-local events of the activation of same or different detectors in superposition.

Now that we have put the entanglement in the detectors, we will again measure it using standard measures.

3.4 Measuring Entanglement

In order to quantify entanglement harvesting we need a measure of entanglement for a two-qubit system in some general mixed state. Hill and Wootters in 1997 (Hill 97) (further developed by Wooters in 1998 (Wootters 98)) came up with the idea of **concurrence** which is a simple and general measure of entanglement for two-qubit systems. For the density matrix of Equation 3.21 and Equation 3.22, It is given as,

$$\mathscr{C}^{(+/tr)} = 2max\left\{0, |\mathscr{M}| - \sqrt{P_A^{(+/tr)} P_B^{(+/tr)}}\right\} \tag{3.23}$$

It can be shown that $|\mathscr{M}| < \sqrt{P_A^{(tr)} P_B^{(tr)}}$ and so $\mathscr{C}^{(tr)} = 0$, which is the **no-go theorem** (Theorem 2). Now as $P_D^{(+)} \leq P_D^{(tr)}$, we may get a situation in which $\mathscr{C}^{(+)}$ does not vanish. So, by introducing the superposition we have opened up the possibility of violating the no-go theorem. In the next section we will look at some plots to see in which situations we get the violation.

3.5 Results

In Figure 3.2 concurrence is plotted as a function of spatial separation s and time difference T between the two branches of superposition in Past-Future scenario. It demonstrates the violation of no-go theorem, i.e., $|\mathscr{M}| - P_D^{(+)} > 0$, in $2+1$ dimensions. Non-perturbative violation of no-go theorem is shown in the other two plots, with (*centre*) being the Cause-Effect scenario ($T_{A,0} \leq T_{B,1} \leq T_{A,1} \leq T_{B,0}$) and the (*right*) the Past-Future scenario ($T_{A,0} \leq T_{B,0} \leq T_{A,1} \leq T_{B,1}$).

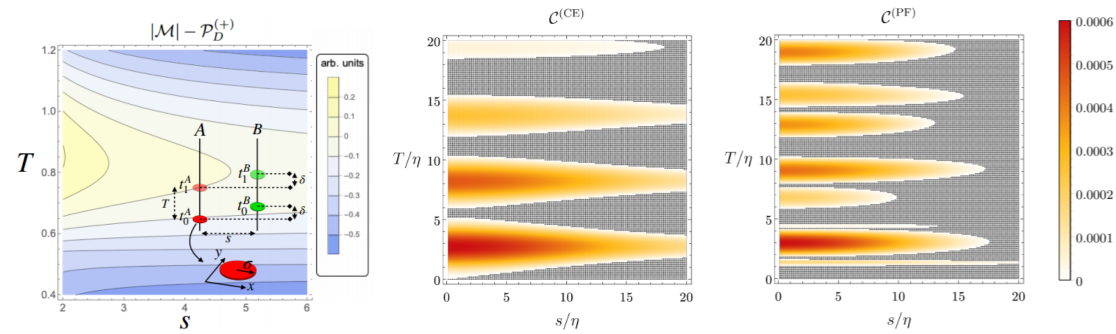


Figure 3.2: Harvested entanglement for different space-time separation of the detectors. (Henderson 20)

The effect on entanglement harvesting can also be seen for point-like detectors. To avoid divergences a different window function is used, which is $\cos(2t/\eta)$, for $-\pi/4 \leq t/\eta \leq \pi/4$. In Figure 3.3 the concurrence is plotted as function of space-time separation again, for classical mixture (*right*), and Cause-Effect superposition (*left*). The green line marks the region of space-like separation. The harvesting efficiency increases, with space-like harvesting only possible for the superposition case, for the given choice of parameters.

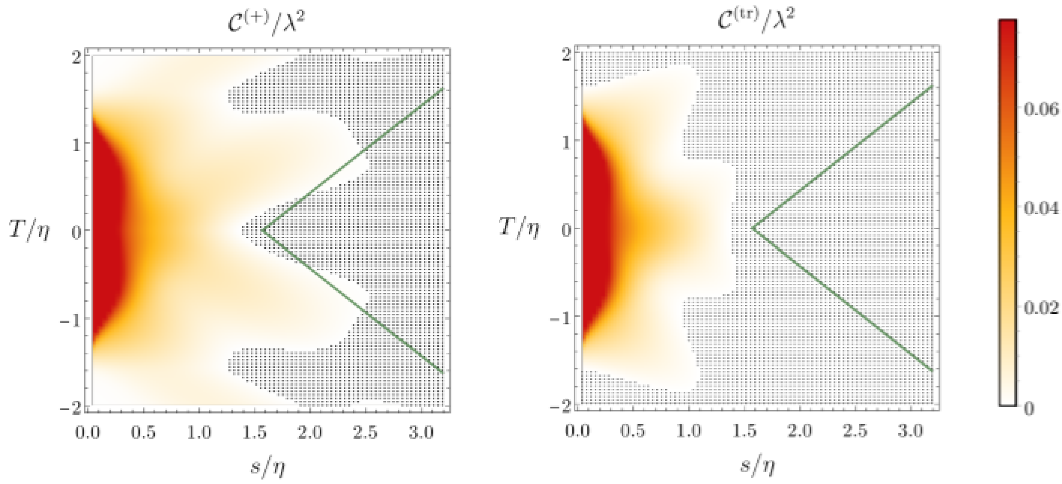


Figure 3.3: Difference between the classical case (*right*), and the temporal superposition case (*left*) indicating improvement in harvesting efficiency. Also small parts of space-like separated region (enclosed within the green lines) showing some non-zero harvesting shows that the source of entanglement is exclusively the vacuum state itself. (Henderson 20)

Hence we see, that we get a clear violation of the no-go theorem. We get a completely new scenario when we introduce the superposition, which cannot be obtained otherwise. We also see a clear enhancement of efficiency in entanglement harvesting using temporal superposition. However, the process of entanglement harvesting involves very precise detector-field coupling. So, in the next chapter we will try to look at an experimentally realizable system of studying entanglement harvesting involving trapped ion chains.

Chapter 4

Trapped Ion Chain

The issue with entanglement harvesting is that it requires very precise detector-field coupling. So, we will look at a simpler model which has similar properties of a scalar quantum field. We will see how an entanglement harvesting is done in a system of trapped ions in a chain.

Wolfgang Paul came up with a technique to trap ions (Paul 90), for which he shared the 1989 Nobel Prize with H.G. Dehmelt. The ions are placed in a setup illustrated in Figure 4.1(left). There are some rod electrodes arranged around and axis, with some time varying electric potential. The nature of the potential is illustrated in Figure 4.1(right). The potential well spins around the z-axis at some angular velocity ω. The mathematical form of the potential is,

$$\phi = \frac{U_0 + V_0 \cos(\Omega t)}{2r_0^2}(x^2 - y^2) \tag{4.1}$$

This makes sure that the ions are trapped along the z-axis. Here Ω is the angular velocity with which the potential well is rotating about the z-axis, while the U_0 and V_0 are electrostatic potentials applied using the rods. r_0 is the distance of the rods from the z-axis. We will consider that the potential is very deep compared to that along z-axis, and hence we would neglect any oscillation in the x and y direction.

In the z direction an electrostatic field is used to create an approximately harmonic potential, using some ring electrodes (as illustrated in Figure 4.1(left)). The ions also interact among themselves due to their respective charges through electrostatic interaction. So, in total we have the system in a potential as follows,

$$V = \sum_{i=1}^{N} \left(\frac{m\omega_z^2}{2} z_i^2 + \sum_{j=i+1}^{N} \frac{q^2}{4\pi\epsilon_0} \frac{1}{|z_i - z_j|} \right) \tag{4.2}$$

where we have the harmonic oscillator term with frequency ω_z, and mass of each oscillator

m. q in the coulomb potential term is the charge of each ion, while the z_i's represent the position of each ion on the z-axis.

Cirac and Zoller in 1995 (Cirac 95) , proposed that using such a linearly trapped cold ion chain we can model a quantum computer. We can identify a couple of internal levels in the electronic structure of the ions, and use them as a two level qubit system in each ion. A laser can be used to couple the internal levels with the phonon modes of the harmonic chain of ions. We will look at it in detail in section 4.2.

Before that let us look at an interesting result we can derive about the vacuum state which has a *Gaussian nature*, as we will see in the following section.

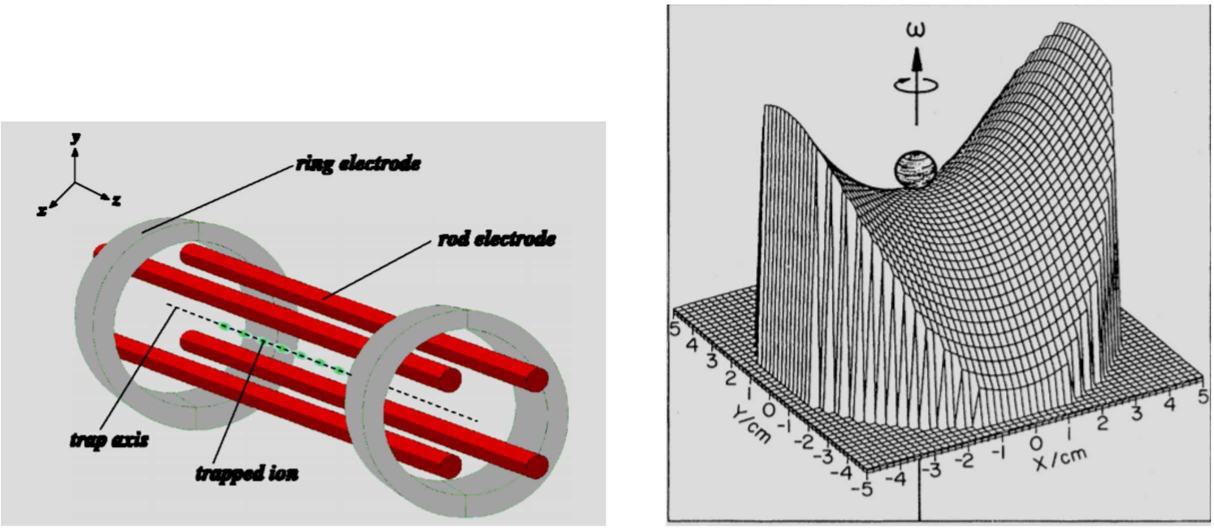


Figure 4.1: (*left*): The setup used to apply the desired potential to trap the ions in a linear chain.(Sasura 02) ((*right*): The Paul trap potential (Paul 90)

4.1 Gaussian State Entanglement

Although we have a fairly unclear notion of entanglement measures in a general infinite dimensional system, we do have a way to analyse it in some special cases. In this section we will discuss about a special kind of infinite dimensional states called Gaussian states.

The ground state of a system with quadratic canonical operators in the Hamiltonian (like the harmonic oscillator) is Gaussian (discussed in more detail in (Schuch 06)). Which means that the phase space distribution function corresponding to the state has a Gaussian structure.

For Gaussian states we have an interesting result by Botero and Reznik (Botero 03). They show that a pure Gaussian state can be decomposed, in a bipartite division, into a product of *two mode squeezed states*(Arvind 95) and local ground states of remaining modes.

$$|\psi\rangle = \left|\tilde{\psi}_1\right\rangle_{\tilde{A}_1\tilde{B}_1} \left|\tilde{\psi}_2\right\rangle_{\tilde{A}_2\tilde{B}_2} \cdot\cdot \left|\tilde{\psi}_s\right\rangle_{\tilde{A}_s\tilde{B}_s} |0\rangle_{\tilde{A}_F} |0\rangle_{\tilde{B}_F} \tag{4.3}$$

for some $s \leq min(m,n)$ and $\left|\tilde{\psi}_k\right\rangle_{\tilde{A}_k\tilde{B}_k}$ being two mode squeezed states

$$\left|\tilde{\psi}_k\right\rangle_{\tilde{A}_k\tilde{B}_k} = \frac{1}{\sqrt{Z_k}} \sum_n e^{\beta_k n/2} |n\rangle_{\tilde{A}_k} |n\rangle_{\tilde{B}_k} \tag{4.4}$$

and $|0\rangle_{\tilde{A}_F}$ and $|0\rangle_{\tilde{B}_F}$ are local ground states of remaining modes. We have illustrated it pictorially in Figure 4.2. The linear chain is divided into a bipartite division of A(red) and B(blue). We can think of each balls representing a mode. Some balls on one side are paired with another on the other side. These pairs form two mode squeezed states. The entanglement of the system in this particular bipartite division, will hence be the sum of the entanglements in each of these pairs of modes. We will try to go through the argument leading to this in the following. The full mathematical details can be found in (Botero 03).

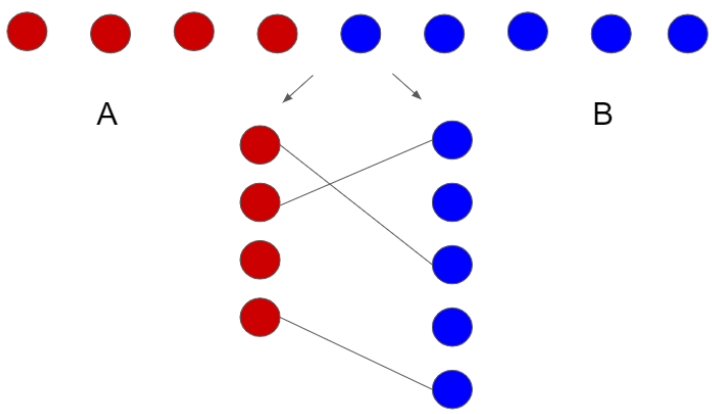


Figure 4.2: A multimode system is divided into two parts, A and B. According to the results of Botero-Reznik (Botero 03) we can take some local symplectic transformation in A and B to decompose the state into several 2 mode entangled pairs, with each mode in the pair coming from different parts. The rest of the modes remain in local vacuum state. The total entanglement of the system is the sum of the entanglements of the pairs.

For gaussian states, entanglement entropy can be expressed in terms of *symplectic eigenvalues*. We write the phase space variables in a vector form as

$$r = (\hat{q}_1, .., \hat{q}_n, \hat{p}_1, ..\hat{p}_n)^T \tag{4.5}$$

where the $\hat{q}_i$ and $\hat{p}_i$ are the position and momentum operators of the ith oscillator.

Symplectic eigenvalues are the absolute values of the eigenvalues of $M = \Gamma\Omega$, where,

$$\Gamma_{ij} = tr[\rho r_i r_j] \tag{4.6}$$

is the covariance matrix and Ω symplectic form. The covariance matrix can be written in this simple form for gaussian states as the expectation values of position and the momenta can be made to vanish by taking local displacements, which would not affect the entanglement as they are local operations.

According to Williamson's theorem (Williamson 36), any real symmetric matrix can be diagonalized by some symplectic transformation. Hence we can diagonalize the covariance matrix by finding a suitable symplectic transformation $\Gamma' = S\Gamma S^T$. The eigenvalues of Γ' and Γ would not be same in general. We can show that the matrix $M = \Gamma\Omega$ undergoes an eigenvalue preserving similarity transformation when we take a symplectic transform of Γ. Also the absolute values of the eigenvalues of M' are the eigenvalues of Γ', and hence we can use them directly and not worry about finding a diagonalizing symplectic transform.

It can be shown that for pure gaussian state the symplectic eigenvalues $\sigma = 1/2$ (Botero 03). In our bipartite division if we perform local symplectic transformations, on covariance matrices of each part, to get to the Williamson normal form, we would see that both parts will have equal number of modes for which $\sigma \geq 1/2$ and hence in mixed form. These modes will pair up to form several 1×1 mode entangled pairs giving us the result we got in Equation 4.3.

We can calculate the entanglement entropy by using the symplectic eigenvalues we get from one of the two parts. The entanglement entropy is given as,

$$\begin{aligned} S(\rho'_A) = \sum_{i=1}^{s} &[(\sigma'_i + 1/2)\log_2(\sigma'_i + 1/2) \\ &- (\sigma'_i - 1/2)\log_2(\sigma'_i - 1/2)] \end{aligned} \tag{4.7}$$

Where the σ'_i's are the symplectic eigenvalues of one part. The total entanglement is the sum of the entanglements coming from each 1×1 mode entangled pair.

In case of 2 ions in the chain we can label the ions as A and B. Now as the ground state is gaussian we should be able to write it in the form of a two mode squeezed state.

$$|vac\rangle = \sqrt{1 - e^{-2\beta}} \sum_n e^{-\beta n} |n\rangle_A |n\rangle_B \tag{4.8}$$

$$e^{-\beta} = \sqrt{(\lambda - 1/2)/(\lambda + 1/2)} \tag{4.9}$$

λ is the the symplectic eigenvalue given as $\lambda = (1/4)[\sqrt{\nu_0/\nu_1} + \sqrt{\nu_1/\nu_0}]$, ν_0 and ν_1 being *collective* and *breathing* mode frequencies (Retzker 05). (Note, in Equation 4.8 we have $e^{-\beta n}$ without a 1/2 factor in the exponential as in Equation 4.4. The factor is adjusted by taking the square root in the RHS of Equation 4.9 which is not there in (Botero 03)) We get this results by writing the Hamiltonian in the decoupled normal mode basis and calculating the corresponding covariance matrix. We can use the formula given in (4.7) to calculate the entanglement of the ground state of such a system.

In the next section, we will continue discussing about the ion trap system in more detail. We will try to harvest the entanglement present in the vibrational ground state into the internal electronic levels of the ions.

4.2 Coupling Internal Levels with Phonon Modes

The ion chain is cooled using techniques of laser cooling to get it to the vibrational ground state (Cirac 92). This ground state of the ion chain is the analogue of the entangled vacuum state of a quantum field.

A laser will be used to couple the internal electronic levels of the k'th ion in the chain with the vibrational modes, as illustrated in Figure 4.3. A Jaynes–Cummings (Jaynes 63) interaction Hamiltonian can be set up using the laser. Jaynes-Cummings model (JCM) mainly talks about the interaction of two level systems with bosonic fields, in which generally a photon field is used by coupling a microwave cavity with the two-level system. In the paper by Cirac *et. al.* (Cirac 93) they have proposed the use of phonon field as the bosonic field. The interaction hamiltonian that we get if a standing electromagnetic wave is put on a trapped ion (Cirac 92) , is of the same form as the JCM interaction Hamiltonian. It is of the following form,

$$H_{int}^{(k)} = \Omega(t) \left(e^{-i\phi}\sigma_+^{(k)} + e^{i\phi}\sigma_-^{(k)} \right) \hat{x}_k \tag{4.10}$$

(Retzker 05) where ϕ and $\hat{x}_k$ is the laser phase and displacement operator of k'th ion. Here $\Omega(t)$ is the Rabi frequency which can be identified as the coupling constant, indicating the strength of the interaction between phonon mode and the internal levels. It is dependent on time in the way we choose to put the laser pulses.

The laser is directed on two such ions (A and B in Figure 4.3) chosen from among the chain of them. The idea is to let the internal levels couple with the phonon (vibrational) modes, kept in the ground state, for a finite duration of time and "pick up" or harvest the entanglement present in it. The interaction will only be switched on for a short duration of time so that the phonon wave cannot travel from one ion to the other in that time.

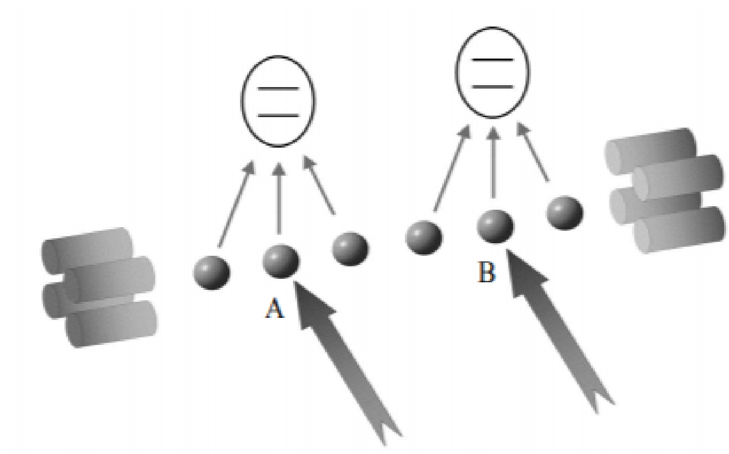


Figure 4.3: Laser is used to couple phonon modes with internal levels of ions A and B (Retzker 05) chosen from the chain of ions in the trap. In the internal electronic structure, 2 energy levels are considered, which are represented by the two lines.

This is done to eliminate the possibility of any classical correlations appearing in the final state of the 2×2 dimensional internal level system of the two ions we chose.

Going back to the case of two ions in the trap, we notice that $e^{-\beta}$ is small. hence, most entanglement is in the first two terms of Equation 4.8, i.e., we say $|vac\rangle \approx |0\rangle |0\rangle + e^{-\beta} |1\rangle |1\rangle$. We want to "swap" this vacuum state into the internal levels which are initially both kept in the ground state. As we know good measures of entanglement for 2×2 systems like Negativity (Peres 96) and Concurrence (Wootters 98) we can measure the entanglement of the internal levels much like what is done in case of entanglement harvesting.

$$|vac\rangle |\downarrow\rangle |\downarrow\rangle \xrightarrow{\text{swap}} |\chi\rangle \left[|\downarrow\rangle |\downarrow\rangle + e^{-\beta} |\uparrow\rangle |\uparrow\rangle \right] \tag{4.11}$$

Where $|\chi\rangle$ is the final state of the harmonic oscillator after the interaction. We see that the first two levels of the vacuum has been swapped into the internal levels.

We will use unitary swap operations, $e^{i\pi/4(\tilde{\sigma}_x\sigma_x + \tilde{\sigma}_y\sigma_y)}$ to make the swap. We can notice that $\tilde{\sigma}_x$ and $\tilde{\sigma}_y$ can be approximated by x and p operator. So we want a series of unitary operations of the form $V(\alpha) = e^{i\alpha\sigma_x x}$ and $W(\beta) = e^{i\beta\sigma_y p}$ to perform the swap. To obtain $V(\alpha)$, laser with $\phi = 0$ is sent for $T \ll 1/\nu_0$ such that $\int \Omega(t)dt = \alpha$. For $W(\beta)$ two laser beams at $\phi = \pi/2$ are used at a gap of $dt = \tau$, each having the effect of $V'(\beta) = exp(i\beta\sigma_y x)$

$$V'_{t=\tau}(\beta')V'_{t=0}(\beta') = exp(-i\beta\sigma_y(x + p\tau/m) + O(\nu^2\tau^2))exp(i\beta\sigma_y x) \tag{4.12}$$

Taking the limit $\nu^2\tau^2 \ll 1$ and setting $\beta = \beta'\tau/m$ we effectively obtain $W(\beta)$. Optimizing the entanglement of the internal levels over the free coupling parameters we can find the required condition on the pulses to get an efficient swap.

In the paper (Retzker 05) they claimed to have successfully transferred 97% of the entanglement in the ground state (calculated using the symplectic eigenvalues) into the internal level system. We can extend this idea further to n ions in a chain and find the entanglement of the ground state. (Retzker 05).

So, we saw a very practical and experimentally viable method of studying vacuum entanglement using ion traps. In the following chapter, we will try to explicitly calculate the entanglement transferred. We will see that the entanglement cannot be transferred (or harvested) using a simple unitary evolution constructed using the interaction hamiltonian in Equation 4.10. But, we do get an entanglement harvesting using a superposition.

Chapter 5

Temporal Superposition and Harvesting Entanglement in Ion Trap

In the previous chapter we discussed about ion traps being an experimentally viable system to study entanglement harvesting. In the paper (Retzker 05) they used a complicated series of evolution operators to successfully swap the ground state into the internal levels. Let us see what we get if we construct the unitary out of the simple interaction hamiltonian in Equation 4.10. Next, we will put in the superposition of unitaries to see the effect of the superposition.

5.1 Without Superposition

As we saw in the previous chapter, most of the entanglement is present in the first two levels of the two-mode squeezed state as $e^{-\beta}$ is small. It would be informative to consider only 2-levels $\{|0\rangle, |1\rangle\}$, as a "toy model" for the 2 trapped ion harmonic oscillator (HO) system.

$$|\psi_i\rangle = \left(1 + e^{-2\beta}\right)^{-1/2} \left(|00\rangle + e^{-\beta}|11\rangle\right)_{HO} |{\downarrow\downarrow}\rangle_{int} \tag{5.1}$$

Where we can consider β to be any positive real number. $|ii\rangle = |i\rangle_A \otimes |i\rangle_B$, $i \in \{0, 1\}$ are the 2×2 dimensional harmonic oscillator part. The internal levels (indicated by *int*) are kept in the ground state as earlier. As we can see the value of β determines the amount of entanglement present initially in the HO system. The interaction Hamiltonian coupling the vibrational modes with the internal levels of the ion is as we saw earlier in Equation 4.10,

$$H_{I,k} = \Omega(t)\left(e^{-i\phi}\sigma_k^+ + e^{i\phi}\sigma_k^-\right)\hat{x}_k \tag{5.2}$$

where again $\Omega(t)$ is the time dependent coupling constant. ϕ is the laser phase while $\sigma_k^\pm$ and $\hat{x}_k$ are the internal level raising-lowering operators and the displacement operator of the k^{th} ion respectively.

$$\sigma^+ = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \qquad \sigma^- = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \tag{5.3}$$

$$\hat{x} = \frac{i}{\sqrt{2m}}(|1\rangle\langle 0| + |0\rangle\langle 1|) \tag{5.4}$$

So in the matrix form the interaction Hamiltonian looks like,

$$H_{I,D} = \frac{i\Omega(t)}{\sqrt{2m}}\begin{pmatrix} 0 & e^{i\phi} \\ e^{-i\phi} & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{5.5}$$

in $\{|\downarrow\rangle, |\uparrow\rangle\} \otimes \{|0\rangle, |1\rangle\}$ basis. D is the index for the modes (which are the ions in the previous case) which would be A for one mode and B for the other.

The unitary evolution operator constructed out of the interaction hamiltonian given in Equation 5.2 is,

$$\begin{aligned} U_D &= exp(-iH_{D,I}\tilde{a}) \\ &= \sum_{p=0}^{\infty}\left[\frac{1}{(2p)!}\left(\frac{\tilde{a}}{\sqrt{2m}}\right)^{2p}(-1)^p \mathbf{I}_{int} \otimes \mathbf{I}_{HO}\right. \\ &\quad \left. + \frac{1}{(2p+1)!}\left(\frac{\tilde{a}}{\sqrt{2m}}\right)^{2p+1}(-1)^p \begin{pmatrix} 0 & e^{i\phi} \\ e^{-i\phi} & 0 \end{pmatrix}_{int} \otimes \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}_{HO}\right] \end{aligned} \tag{5.7}$$

where $D = \{A, B\}$ and $\tilde{a} = \int_{-\infty}^{\infty} \Omega dt$. The summations form the sin and cos series,

$$U_D = \left[\cos(a)(\mathbf{I}_{int} \otimes \mathbf{I}_{HO}) + \sin(a)\begin{pmatrix} 0 & e^{i\phi} \\ e^{-i\phi} & 0 \end{pmatrix}_{int} \otimes \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}_{HO}\right] \tag{5.8}$$

where we have put $\tilde{a}/\sqrt{2m} = a$.

Now this is the operator acting on one ion. We have to put a similar operation on the other ion as well. So the total operation is,

$$\begin{aligned} U_B \otimes U_A &= \cos^2(a)\left[(\mathbf{I}_A \otimes \mathbf{I}_B)_{int} \otimes (\mathbf{I}_A \otimes \mathbf{I}_B)_{HO}\right] \\ &+ \cos(a)\sin(a)\left[\left\{\mathbf{I}_A \otimes \begin{pmatrix} 0 & e^{i\phi} \\ e^{-i\phi} & 0 \end{pmatrix}\right\}_{int} \otimes \left\{\mathbf{I}_A \otimes \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\right\}_{HO}\right] \\ &+ \sin(a)\cos(a)\left[\left\{\begin{pmatrix} 0 & e^{i\phi} \\ e^{-i\phi} & 0 \end{pmatrix} \otimes \mathbf{I}_B\right\}_{int} \otimes \left\{\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \otimes \mathbf{I}_B\right\}_{HO}\right] \\ &+ \sin^2(a)\left[\left\{\begin{pmatrix} 0 & e^{i\phi} \\ e^{-i\phi} & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & e^{i\phi} \\ e^{-i\phi} & 0 \end{pmatrix}\right\}_{int} \otimes \left\{\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\right\}_{HO}\right] \end{aligned} \tag{5.9}$$

We have the following theorem,

Theorem 3 *Using the unitary evolution operator given in Equation 5.9 we do not get any entanglement harvesting.*

To go about proving the theorem we will have to evolve the system using the unitary evolution operator (Equation 5.9).

$|\psi_f\rangle = U_A \otimes U_B |\psi_i\rangle$ becomes,

$$\begin{aligned} |\psi_f\rangle &= \left(1+e^{-2\beta}\right)^{-1/2}\left[\cos^2(a)\left\{|00\rangle + e^{-\beta}|11\rangle\right\}|\downarrow\downarrow\rangle\right. \\ &+ \cos(a)\sin(a)\left\{(|01\rangle - e^{-\beta}|10\rangle)e^{-i\phi}|\downarrow\uparrow\rangle + (|10\rangle - e^{-\beta}|01\rangle)e^{-i\phi}|\uparrow\downarrow\rangle\right\} \\ &+ \left.\sin^2(a)\left\{|11\rangle + e^{-\beta}|00\rangle\right\}e^{-2i\phi}|\uparrow\uparrow\rangle\right] \end{aligned} \tag{5.10}$$

Now forming the density matrix and taking the trace over the harmonic oscillator part $\rho_{int} = \mathrm{Tr}_{HO}(|\psi_f\rangle\langle\psi_f|)$ we get,

$$\rho_{int} = \begin{pmatrix} c^4 & 0 & 0 & \frac{2e^{-\beta}s^2c^2e^{2i\phi}}{1+e^{-2\beta}} \\ 0 & s^2c^2 & -\frac{2e^{-\beta}s^2c^2}{1+e^{-2\beta}} & 0 \\ 0 & -\frac{2e^{-\beta}s^2c^2}{1+e^{-2\beta}} & s^2c^2 & 0 \\ \frac{2e^{-\beta}s^2c^2e^{-2i\phi}}{1+e^{-2\beta}} & 0 & 0 & s^4 \end{pmatrix} \tag{5.11}$$

where $c = \cos(a)$ and $s = \sin(a)$

The entanglement monotone that we are going to use here is Concurrence (Wootters 98). The Concurrence for a density matrix in the form

$$\rho = \begin{pmatrix} \rho_{11} & 0 & 0 & \rho_{14} \\ 0 & \rho_{22} & \rho_{23} & 0 \\ 0 & \rho_{23}^* & \rho_{33} & 0 \\ \rho_{14}^* & 0 & 0 & \rho_{44} \end{pmatrix} \tag{5.12}$$

is given (Henderson 20) as

$$C = 2\max\{|\rho_{14}| - \sqrt{\rho_{22}\rho_{33}}, 0\} \tag{5.13}$$

For the density matrix that we got the condition for positive Concurrence turns out to be,

$$\sin^2(a)\cos^2(a)\left(\frac{2e^{-\beta}}{1+e^{-2\beta}}\right) > \sin^2(a)\cos^2(a) \tag{5.14}$$

which can only be achieved if,

$$(1-e^{-\beta})^2 < 0 \tag{5.15}$$

which is not possible for any real values of β. Hence in this scheme we do not get any entanglement harvesting in the internal levels. Hence Theorem 3 is proved.

5.2 Incorporating Superposition

Now we put in a superposition in the evolution to see whether Theorem 3 still holds.

For incorporating superposition we will have to introduce the control qubit. The time dependent coupling parameter $\Omega(t)$ will depend on the control qubit state. The interaction Hamiltonian is,

$$H_{I,k} = \sum_{i=0}^{1} \Omega_i(t)\left(e^{-i\phi}\sigma_k^+ + e^{i\phi}\sigma_k^-\right)\hat{x}_k \otimes |i\rangle\langle i|_C \tag{5.16}$$

where C represents the control qubit part of the state.

So the unitary time evolution operator is

$$U = (U_0 \otimes |0\rangle\langle 0|_C + U_1 \otimes |1\rangle\langle 1|_C) \tag{5.17}$$

where U_0 and U_1 are the unitaries formed with coupling parameter Ω_0 and Ω_1 respectively. This would make sure that U_0(U_1) will act if the initial state of control qubit is in $|0\rangle$($|1\rangle$). To incorporate the superposition in evolution, we will put the initial control qubit in a superposition.

$$|\psi_i\rangle = \left(1+e^{-2\beta}\right)^{-1/2}\left(|00\rangle + e^{-\beta}|11\rangle\right)_{HO}|\downarrow\downarrow\rangle_{int} \otimes \left(\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)_C\right) \tag{5.18}$$

So now the final state is,

$$|\psi_f\rangle = \frac{1}{\sqrt{2}}(|\psi_{f,0}\rangle \otimes |0\rangle + |\psi_{f,1}\rangle \otimes |1\rangle) \tag{5.19}$$

Finally we will measure the final state in $\left(\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)_C\right)$ state of the control qubit. So at the end the state (without the control part) has four terms.

$$|\psi_f\rangle\langle\psi_f| = \frac{1}{4}\left[|\psi_{f,0}\rangle\langle\psi_{f,0}| + |\psi_{f,0}\rangle\langle\psi_{f,1}| + |\psi_{f,1}\rangle\langle\psi_{f,0}| + |\psi_{f,1}\rangle\langle\psi_{f,1}|\right] \tag{5.20}$$

The terms are as follows,

$$\mathrm{Tr}_{HO}(|\psi_{f,i}\rangle\langle\psi_{f,i}|) = \begin{pmatrix} c_i^4 & 0 & 0 & \frac{2e^{-\beta}s_i^2c_i^2e^{2i\phi}}{1+e^{-2\beta}} \\ 0 & s_i^2c_i^2 & -\frac{2e^{-\beta}s_i^2c_i^2}{1+e^{-2\beta}} & 0 \\ 0 & -\frac{2e^{-\beta}s_i^2c_i^2}{1+e^{-2\beta}} & s_i^2c_i^2 & 0 \\ \frac{2e^{-\beta}s_i^2c_i^2e^{-2i\phi}}{1+e^{-2\beta}} & 0 & 0 & s_i^4 \end{pmatrix}$$

where $c_i = \cos(a_i)$ and $s_i = \sin(a_i)$ for $i = \{0, 1\}$. For the cross terms, $\mathrm{Tr}_{HO}(|\psi_{f,i}\rangle\langle\psi_{f,j}|) =$

$$\begin{pmatrix} c_i^2c_j^2 & 0 & 0 & \frac{2e^{-\beta}s_j^2c_i^2e^{2i\phi}}{1+e^{-2\beta}} \\ 0 & s_ic_is_jc_j & -\frac{2e^{-\beta}s_ic_is_jc_j}{1+e^{-2\beta}} & 0 \\ 0 & -\frac{2e^{-\beta}s_ic_is_jc_j}{1+e^{-2\beta}} & s_ic_is_jc_j & 0 \\ \frac{2e^{-\beta}s_i^2c_j^2e^{-2i\phi}}{1+e^{-2\beta}} & 0 & 0 & s_i^2s_j^2 \end{pmatrix} \tag{5.21}$$

$s_k = \sin(a_k)$ and $c_k = \cos(a_k)$, $k \in \{i, j\}$ where $i \neq j$. So, here the condition for concurrence to be positive ($|\rho_{14}| - \sqrt{\rho_{22}\rho_{33}} > 0$) is,

$$(\sin^2(a_0) + \sin^2(a_1))(\cos^2(a_0) + \cos^2(a_1))\left(\frac{2e^{-\beta}}{1 + e^{-2\beta}}\right) > (\sin(a_0)\cos(a_0) + \sin(a_1)\cos(a_1))^2 \tag{5.22}$$

which is

$$\left(\frac{2e^{-\beta}}{1 + e^{-2\beta}}\right) > \kappa \tag{5.23}$$

where,

$$\kappa = \frac{(\sin(a_0)\cos(a_0) + \sin(a_1)\cos(a_1))^2}{(\sin^2(a_0) + \sin^2(a_1))(\cos^2(a_0) + \cos^2(a_1))} \tag{5.24}$$

In Figure 5.1 we have plotted the inequality considering both κ and β to be positive real numbers. The boundary of the plot is at $\left(\frac{2e^{-\beta}}{1+e^{-2\beta}}\right) = \kappa$ and the green area is the values of κ and β for which we get a positive concurrence. Hence, we have clearly violated Theorem 3 using the superposition.

Figure 5.1: The κ vs β inequality is plotted to demonstrate that we do get a region where it is possible to get non-zero entanglement unlike the case with no superposition.

The initial state of the Harmonic Oscillator part was $\left(1+e^{-2\beta}\right)^{-1/2}\left(|00\rangle + e^{-\beta}|11\rangle\right)_{HO}$. The Concurrence for a pure state like,

$$|\psi\rangle = a\,|00\rangle + b\,|01\rangle + c\,|10\rangle + d\,|11\rangle \tag{5.25}$$

is given as,

$$C = 2|ad - bc| \tag{5.26}$$

(An-Min 03). So our initial concurrence is,

$$C_i = \left(\frac{2e^{-\beta}}{1+e^{-2\beta}}\right) \tag{5.27}$$

For $\beta \approx 1.99$, $C_i \approx 0.266$ e-bits. We can analyse the expression of Concurrence we got in the internal levels as a function of the a_0 and a_1 for the extrema points,

$$\begin{aligned} C(a_0, a_1) &= (1/2)[(\sin^2(a_0) + \sin^2(a_1))(\cos^2(a_0) + \cos^2(a_1))\left(\frac{2e^{-\beta}}{1+e^{-2\beta}}\right) \\ &- (\sin(a_0)\cos(a_0) + \sin(a_1)\cos(a_1))^2] \end{aligned} \tag{5.28}$$

We can see that when $a_1 = \pi/2 + a_0$, we get an extrema point, with final concurrence,

$$C_f = \left(\frac{e^{-\beta}}{1+e^{-2\beta}}\right) = (1/2)C_i \tag{5.29}$$

So we see that, at maxima we get back half of the total entanglement present in the initial 2-level Harmonic Oscillator state. When however $a_0 = a_1$, we get back to the case similar to no superposition, with zero final entanglement. In our analysis we never considered the coupling to be weak, and hence the condition for maximum entanglement harvesting can be achieved.

We can plot the final Concurrence as a function of the parameters a_0 and a_1 (see Figure 5.2). We can see that the 45° line from origin corresponding to $a_0 = a_1$ has no concurrence. The 45° from $a_0 = 0$, $a_1 = \pm\pi/2$ has the maximum concurrence of ≈ 0.134 e-bits. As we see this is half of the initial concurrence.

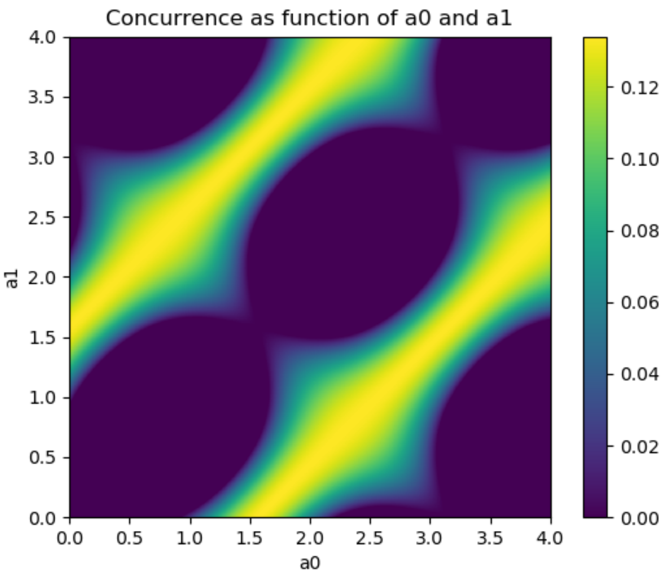


Figure 5.2: The concurrence is plotted as a function of the parameters a_0 and a_1. The color gradient as indicated in the color bar is used to represent the value of concurrences. This plot is for the value $\beta \approx 1.99$ which is what we expect for 2 ions in a chain.

5.2.1 Extending to Higher Dimensions

We can extend the analysis to higher dimensions of Harmonic Oscillator states by considering the corresponding $\hat{x}$ operators. The $\hat{x}$ operator for n^{th} dimension is,

$$\hat{x}_n = \frac{i}{\sqrt{2m}} \begin{pmatrix} 0 & -1 & 0 & 0 & \dots & 0 \\ 1 & 0 & -\sqrt{2} & 0 & \dots & 0 \\ 0 & \sqrt{2} & 0 & -\sqrt{3} & \dots & 0 \\ 0 & 0 & \sqrt{3} & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & -\sqrt{n-1} \\ 0 & 0 & 0 & 0 & \sqrt{n-1} & 0 \end{pmatrix} \tag{5.30}$$

The initial state of the Harmonic Oscillator part will be

$$|\psi_i\rangle_{HO,n} = \left(\sum_{k=0}^{n-1} e^{-2\beta k} \right)^{-1/2} \left(\sum_{k=0}^{n-1} e^{-\beta k} |kk\rangle \right) \tag{5.31}$$

Using these we can do the same calculation as we did in the previous section with the 2-level Harmonic Oscillator. Doing the calculations numerically, we got some final internal level concurrence distribution for different values of a_0 and a_1. The plots are given in Figure 5.3. We can observe that the entanglement harvesting is not as efficient when we consider more levels in the Harmonic Oscillator.

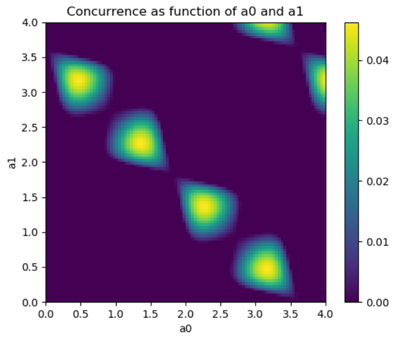
In order to compare how much of the entanglement we transfer from the Harmonic Oscillator to the entanglement we use the measure Negativity (Peres 96). Negativity of a state ρ is given as (Vidal 02),

$$\mathcal{N}(\rho) \equiv \frac{||\rho^{T_A}||_1 - 1}{2} \tag{5.32}$$

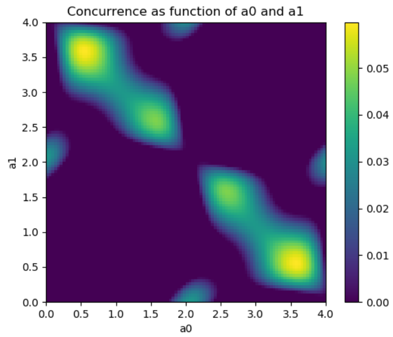
where $||\rho||_1 = \mathrm{Tr}(\sqrt{\rho^\dagger \rho})$ is the trace norm, and ρ^{T_A} is the partial transpose of the matrix ρ. For our state (Equation 5.31), we get,

$$||\rho^{T_A}||_1 = \frac{\left(\sum_{k=0}^{n-1} e^{-\beta k}\right)^2}{\left(\sum_{k=0}^{n-1} e^{-2\beta k}\right)} \tag{5.33}$$

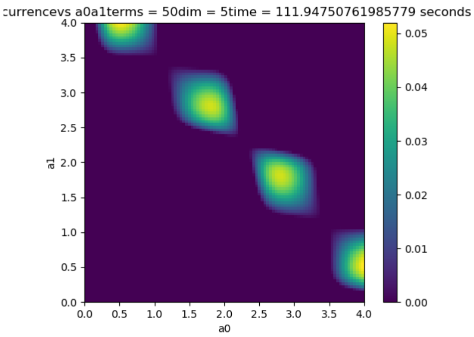
We can clearly see that for 2-level system, (2×2) the negativity is just half of the concurrence. This is also true for the the final internal level matrix which is in the form of Equation 5.12. So, we can compare the initial and the final harvested entanglement with optimum a_0 and a_1 setting for $\beta \approx 1.99$. We have tabulated the values of initial negativity and final negativity in Table 5.1. Also in Figure 5.4 we can see the same data plotted.



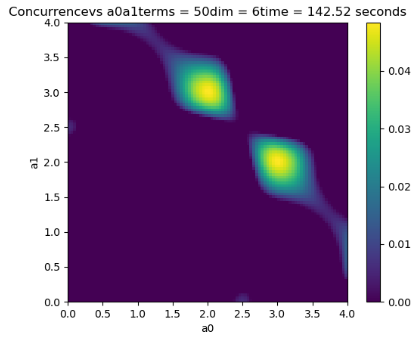
(a) 3D: Maxima is = 0.046 e-bits, $a_0 = 3.173$, $a_1 = 0.454$



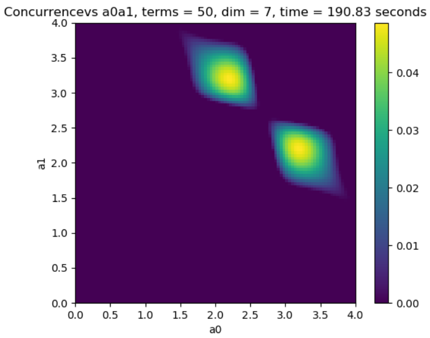
(b) 4D: Maxima is = 0.060 e-bits, $a_0 = 0.539$, $a_1 = 3.592$



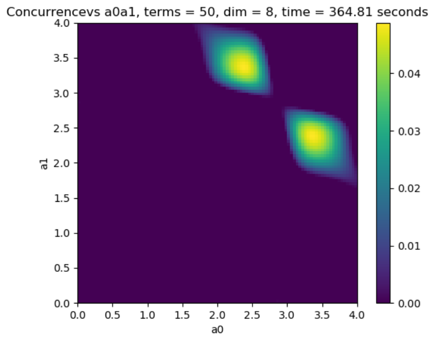
(c) 5D: Maxima is = 0.054 e-bits, $a_0 = 0.504$, $a_1 = 4.084$



(d) 6D: Maxima is = 0.049 e-bits, $a_0 = 2.013$, $a_1 = 3.013$



(e) 7D: Maxima is = 0.049 e-bits, $a_0 = 2.201$, $a_1 = 3.201$



(f) 8D: Maxima is = 0.049 e-bits, $a_0 = 2.380$, $a_1 = 3.380$

Figure 5.3: Concurrence of the internal levels when we consider the Harmonic Oscillator systems with different dimensions. Also, the maximum concurrence and one of the points of maxima are found for each dimension.

The initial negativity converges to the value 0.158 while the final negativity remains around 0.024 (for $\beta \approx 1.99$). We can see from Equation 5.33 that for $n \to \infty$ the initial state negativity formula is,

$$\mathscr{N}_\infty = \frac{\frac{(1-e^{-2\beta})}{(1-e^{-\beta})^2} - 1}{2} = \frac{e^{-\beta} - e^{-2\beta}}{(1-e^{-\beta})^2} \tag{5.34}$$

For $\beta \approx 1.99$, $\mathscr{N}_\infty \approx 0.158$. So, after 5 dimensions the negativity converges to the value to the number of decimal places considered here. So, to our approximation we have already converged to the initial and final negativity for infinite dimensional case. Hence, we conclude that with superposition we can obtain upto $\approx 15\%$ of the initial entanglement in the internal levels.

dim	N_i	N_f
2	0.134	0.067
3	0.155	0.023
4	0.157	0.030
5	0.158	0.027
6	0.158	0.024
7	0.158	0.024
8	0.158	0.024

Table 5.1: Initial Negativity N_i, and Negativity harvested optimally N_f. for different dimensions (Hilbert space dimensions of the states) of Harmonic Oscillator state considered for $\beta \approx 1.99$.

So, we see that we are able to harvest entanglement using the superposition which was not possible without it. Hence we establish that superposition in evolution has a clear advantage in the process of entanglement harvesting.

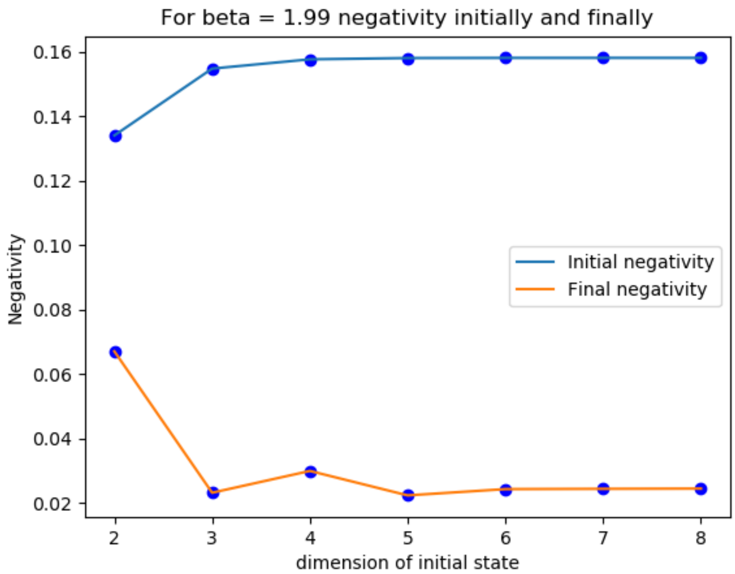


Figure 5.4: Initial negativity and final (optimally harvested entanglement in the internal level system) negativity is plotted as a function of the dimension of the initial state Hilbert space considered for $\beta \approx 1.99$.

Conclusion

We talked about the peculiar nature of the the quantum field vacuum which is entangled. In order to study and experimentally validate such a claim we introduced the idea of entanglement harvesting. The enormous resource of entanglement present naturally in the vacuum can find a lot of potential applications if we can successfully harvest it. The vacuum entanglement is intricately dependent on the structure of space-time itself. Hence, a viable protocol of studying it opens up opportunities to unravel the mysteries of the fundamental properties of nature.

We looked at the novel idea of applying temporal superposition in entanglement harvesting. There is a clear enhancement in the efficiency of entanglement harvesting. We are able to harvest entanglement in situations which proved to be impossible to harvest from without superposition. Hence, we see a quantum advantage.

However it is experimentally challenging to study entanglement harvesting in quantum fields. So, we looked at a practical model of trapped ion chains to make the study experimentally viable. Constructing a Jaynes-Cummings interaction hamiltonian that couples the vibrational modes with the internal electronic levels of the ions, we swapped the vibrational mode and the internal level states, using a chain of unitary evolution. We got the entanglement present in the ground state of the vibrational modes, which is analogous to the quantum field vacuum state, into the internal levels.

However, we saw that with a simplistic unitary evolution, we do not get any entanglement harvesting. Once we apply the superposition again in ion trap model, however, we get an efficient entanglement harvesting. This again demonstrates the quantum advantage that we have when we apply the superposition in entanglement harvesting. Building from the "toy model" we could see that, considering the entire infinite dimensional Hilbert space, we can harvest at most $\approx 15\%$ of the initial entanglement present in the ground state of the Harmonic oscillator.

Future outlook

We considered a very simplistic model to demonstrate the advantage of superposition in entanglement harvesting. The next step would be to apply this in an n-mode system, having n ions in the chain. This would give us some prediction, which can be experimentally verified to establish the result. We can also study the relation between harvested entanglement and the distance of the ions if we have the detailed calculation of n ions case.

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