Thermodynamics of Indefinite Causal Orders using Quantum Switch

Samyak Pratyush Prasad(MS16044)

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Certificate of Examination

This is to certify that the dissertation titled “Thermodynamics of Indefinite Causal Orders using Quantum Switch” submitted by Samyak Pratyush Prasad (Reg. No. MS16044) for the partial fulfilment of BS-MS dual degree program of the Institute, has been examined by the thesis committee duly appointed by the Institute. The committee finds the work done by the candidate satisfactory and recommends that the report be accepted.

Dr. Kinjalk Lochan	Dr. Sandeep K. Goyal	Dr. Manabendra Nath Bera (Local Supervisor)	Dr. Alexia Auffèves (External Supervisor)	Dr. Cyril Branciard (External Supervisor)

Dated: May 13, 2021

Declaration

The work presented in this dissertation has been carried out by me under the guidance of Dr. Manabendra Nath Bera (local supervisor), Dr. Alexia Auffèves (external supervisor) and Dr. Cyril Branciard (external supervisor) at the Indian Institute of Science Education and Research Mohali. This work has not been submitted in part or in full for a degree, a diploma, or a fellowship to any other university or institute. Whenever contributions of others are involved, every effort is made to indicate this clearly, with due acknowledgement of collaborative research and discussions. This thesis is a bonafide record of original work done by me and all sources listed within have been detailed in the bibliography.

Samyak Pratyush Prasad
(Candidate)

Dated: May 13, 2021

In my capacity as the supervisor of the candidate's project work, I certify that the above statements by the candidate are true to the best of my knowledge.

Dr. Manabendra Nath Bera
(Local Supervisor)

Dr. Alexia Auffèves
(External Supervisor)

Dr. Cyril Branciard
(External Supervisor)

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Abstract

The phenomenon of Indefinite Causal Orders is known to show many strange behaviours such as perfect information communication through noisy channels and activation of free energy of thermal states using thermal baths at the same temperature. The quantum switch can be used to create these strange correlations and has been used to construct advantageous devices such as a quantum refrigerator. Throughout this thesis, we study the thermodynamics of indefinite causal orders with the help of the quantum switch. We will review this phenomenon and analyze the advantages by studying every party involved in the creation of indefinite causal orders. We find that the thermodynamics of quantum measurements play a key role in any advantages seen. We then find bounds and display the role of information and correlations in the energetics of the phenomenon. Finally, we study a stochastic approach to thermodynamics using quasi-probabilities and fluctuation theorems which probe the fluctuations of observables. Here, we separate the "classical" and "quantum" parts of an evolution which relates to incompatibility of measurement of observables. This can be used to separate and study the truly quantum part of the quantum switch setup and hence, probe into the physics of indefinite causal orders.

Chapter 1

Introduction

Information is Physical.

— **Rolf Landauer**

1.1 Motivation and Problem Statement

Thermodynamics of quantum systems and quantum information theory are known to have many parallels between them. Thermodynamics adds additional constraints to the framework of quantum information theory by considering energetics and hence, the Hamiltonian of evolving quantum systems. Recent advancements in the rapidly growing field of quantum information theory has shown that a superposition can be created, not only between quantum states, but between the causal orders of channel application and moreover, can be utilized. The quantum switch [GGK+18] is a device through which different correlations produced by indefiniteness in causal orders [OCB12] can be used to show several advantages in quantum communication and signal transfer [CBB+21, PDE+20], though without the violation of any causal bounds. This recently proposed setup has already been experimentally realized [GGK+18]. Taking advantage of the similarity between quantum information and quantum thermodynamics, recently thermodynamical advantages of using the quantum switch have been proposed [FV20, GAP20] which directly use the calculations of the quantum information paradigm. However, these advantages have not been analyzed in detail and there seems to be a few inconsistencies. In this project, we wish to investigate these inconsistencies, extend the analysis to more general settings and, ask whether thermodynamic advantage is possible through the use of the quantum switch. Furthermore, we extend the analysis to stochastic thermodynamics and derive a fluctuation theorem for the switch. This will help us analyze the truly quantum behaviour of the quantum switch and furthermore, help us figure out the factors that contribute to it.

Two papers have recently been submitted which introduce a quantum refrigeration cycle [FV20] and show free-energy increment through thermal maps [GAP20] using the quantum switch. These papers use thermalization channels in the quantum switch to study and utilize the free-energy advantage received by the system. In order to derive an advantage they rely on using a post-selected state of the system corresponding to a certain measurement of the control. The underlying assumption in both articles is that the increase in free-energy of the post-measurement state of the system is due to energy flowing out of the thermal baths used in the thermalization channels. However, this assumption is implicitly taken and further, they don't analyze the energy due to the measurement of the control. Recent papers such as Ref. [PS20] analyze the use of post-selection and show that channels relying on post-selection can be used to extract unbounded amount of work from the point of view of the working system. Hence, a detailed analysis of the baths is needed to understand if the quantum switch uses the baths to derive an advantage or just the post-selection on the measurement results of the control. To completely analyze the thermodynamic advantage we need to ask where the energy is coming from and further how much energy each bath contributes as compared to the energy due to the measurement of the control. It is possible that all the energetic advantage is due to the measurement of the control in which case a thermodynamic advantage of using the baths is not seen. Hence, for a thermodynamic advantage it is important to check if the baths contribute any energy and then use this energy in an advantageous cycle.

Another aim of this project is to investigate the thermodynamics of the quantum switch by analyzing fluctuations in the work done using the switch. However, the concept of work has yet to be clearly defined. In stochastic thermodynamics, work is a stochastic quantity and has several definitions [BCG+18] and hence, so does fluctuations in its distributions. Use of quasiprobability distribution is one way that was introduced by Allahverdyan [All14] to address the work distribution for non-equilibrium processes. This was used to derive a fluctuation theorem for such processes. To study the quantum aspects of the thermodynamics, it is important to study the fluctuation theorems, which helps quantify the effect of quantum coherence. Hence, it is crucial to derive quasiprobability distributions and fluctuation theorems for the quantum switch to analyze the truly quantum effects of it.

1.2 Thesis Structure

To analyze the above questions we split the project into two parts. For the first part we analyze the energetics of the system and the two baths placed in the quantum switch. We then study its thermodynamics advantage and where its source. In the second part, we study quasiprobability distribution and use this to derive a fluctuation theorem for the quantum switch. We then, quantify the truly quantum phenomenon taking place in the quantum switch using this fluctuation theorem.

1.2.1 Energy Analysis

Usually, analysis of the interactions between a system and a thermal bath is done by focusing on the system. This is done because the bath is a macroscopic body with uncountable degrees of freedom. Thus, by making considerations such as Markovian dynamics, master equations for processes such as thermalization have be written which depict the evolution of a system in the presence of a bath. For thermalization, the solutions of the master equations asymptotically approach the thermal state. The total dynamics can be considered by using a completely positive and trace-preserving (CPTP) map which takes the initial state of the system to a final state (the thermal state in case of thermalization). However, both descriptions assume that any energy change of the system is due to heat coming from or going into the bath - the heat is implicit and can't be shown by considering the dynamics of the bath. The implicit assumption is correct when describing just a system and a bath, but in presence of an external system which creates quantum correlations in the application of different baths on the system, the measurement of the control plays an important role and implicit assumptions may no longer be straightforwardly correct. Hence, to begin with the energy analysis we propose to use an effective system to model the thermal baths and the interaction. In doing so, we would be able to analyze the heat flow between the system and the baths explicitly by considering changes in the energy of the effective bath system. More precisely, we want to find an effective system for modelling the bath and an interaction, whose reduced dynamics on the system is given by the same CPTP map (Kraus decomposition) resulting from the thermalization dynamics of the master equation. We first propose the use of thermal qubits to model the baths by showing that the Kraus sum for the interaction with the system will be the same as when master equations are used to get to it. We then add the effective model to the switch and analyze the energy changes of the thermal baths to answer the question: "how much energy do each of the baths contribute in the switch as compared to the measurement of the control?" We will then utilize the energy from the baths, if possible, to implement a thermodynamic protocol to depict the advantage.

1.2.2 Fluctuation Theorem

It is accepted that the evolution of quantum systems is governed by the Schrodinger's equation [SC95], which is essentially a unitary operation on the system. These unitary operations can be revered in general. But then, if all evolution is reversible, i.e., physical laws are time-symmetric, why are processes like thermalization irreversible? Stochastic thermodynamics [DC19] studies interaction of an environment with a system by focussing on the trajectory of the system state which are perturbed by interactions with the environment. Thermodynamic quantities such as heat and work in this theory are studied as stochastic variables. This helps study out-of-equilibrium phenomenon and leads to interesting relations known as Fluctuation theorems which link equilibrium and non-equilibrium quantities for both classical and quantum processes [Jar97, Cro98]. These fluctuation theorems have been replicated using the two-point measurement (TPM) approach. However, there are several difficulties when coherence in the energy basis is involved [BCG+18, DC19]. We will approach the question of the fluctuation theorem for the quantum switch using an extension of the TPM approach. This will lead to quasiprobability distributions and hence, a fluctuation relation which we then analyze. We will attempt to separate the quantum and classical contributions to this fluctuation theorem and analyze the truly quantum phenomenon due to the quantum switch.

1.3 Preliminaries

Before starting the discussion on the quantum switch, we will revise some key concepts from quantum information theory and quantum thermodynamics. This will be useful in understanding the thermodynamic phenomenon involved in using in-

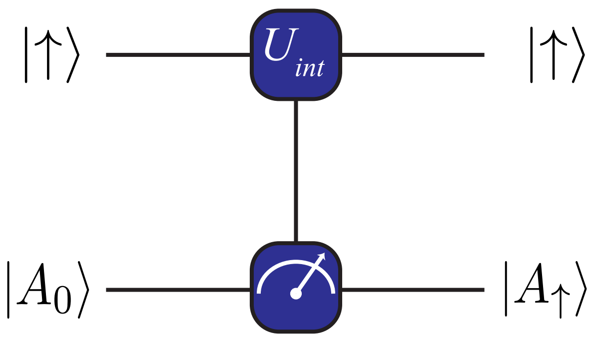


Figure 1.1: *Quantum system interacting with a measurement device. The measurement device is modelled as a quantum system.* $|A_0\rangle$ *is the ready state of the measurement device which couples with the system spin(here) state through an interaction unitary* U_{int}*. The resulting state of the measurement device (*$|A_\uparrow\rangle$*) is then observed to get the spin state (*$|\uparrow\rangle$*). Observation of measurement device collapses the system state.*

definite causal orders. We will cover the bare minimum in this section. For a more detailed and mathematical description, readers are referred to [HZ11].

1.3.1 Generalized Measurements

Measurements will play a central role throughout this thesis. Hence, it is crucial to understand how measurements are mathematically expressed in quantum mechanics.

When a property of a macroscopic system is being measured, such as current produced by it in a circuit, a measuring device is used, like an ammeter. Similarly to measure a property of a quantum system, a macroscopic device is used. Such a device is known as a pointer or a meter (ancilla is also used). The meter has a set of distinguishable states. These states are analogous to the needle of say the ammeter pointing to a number. We can only know the current with the accuracy of the width of the pointer. For the meter, these distinguishable states are known as the pointer states. By treating the meter as a macroscopic quantum system, we consider the pointer states to be a set of distinguishable quantum states, i.e., those that can have superpositions and quantum correlations. When a property of a quantum system is measured, the experimentalist does not look at the quantum system directly (for obvious reasons), they look at the meter's pointer state. This pointer state gives information of some property (or observable) of the quantum system. According to the seminal work of Dr. John von Neumann, the eigenvalues of the observables can be correlated to the pointer states for the information acquisition by the meter. As all closed quantum systems are governed by unitary evolution, the interaction between the system and the meter is also a unitary interaction which correlates the system and the meter. When this meter is measured, this correlation is collapsed, and the value of the meter, gives the information about the system observable being measured. For example, let us consider the measurement of the spin of a system in the σ_Z basis, with eigenstates $\{|\uparrow\rangle, |\downarrow\rangle\}$. We prepare the system in the state $|\psi\rangle = \left(\frac{|\uparrow\rangle+|\downarrow\rangle}{\sqrt{2}}\right)$. The meter is initially prepared in the pointer state $|A_0\rangle$. When the interaction unitary U couples them, the resultant state is,

$$|A_0\rangle |\psi\rangle \xrightarrow{U} \frac{|A_\uparrow\rangle |\uparrow\rangle + |A_\downarrow\rangle |\downarrow\rangle}{\sqrt{2}}.$$

As you can see, the $|\uparrow\rangle$ ($|\downarrow\rangle$) spin state of the system is correlated with the $A_\uparrow$ ($A_\downarrow$) state of the meter. Hence, when we look at the meter, we collapse the meter state and measure the state of the system in the σ_Z basis. This simulates a projective measurement to say state $|\uparrow\rangle$ of the system with probability,

$$p_\uparrow = \mathrm{Tr}_{SA}\left[(|A_\uparrow\rangle \langle A_\uparrow| \otimes \mathbb{I})\, U\, |A_0\psi\rangle \langle \psi A_0|\, U^\dagger\right].$$

In general, this scheme can be followed for any observable by expanding the system in the basis of the observable and coupling it with a meter. Furthermore, the measurement of the meter need not be in the basis of the coupling. In this case, the measurement on the system will not be projective but is modelled by a set of positive-operator valued measures (POVMs). These POVMs are represented in literature by operators $\{E_i = M_i^\dagger M_i\}_i$ which act only on the system and give with probability p_i, the post-measurement state of the system ρ_i. These are given by,

$$\begin{aligned} \rho_i &= \frac{M_i \rho M_i^\dagger}{\mathrm{Tr}\left[E_i \rho\right]}, \\ p_i &= \mathrm{Tr}\left[E_i \rho\right]. \end{aligned}$$

These POVM elements are complete $\sum_i E_i = \mathbb{I}$, and describe the most general kind of measurement possible on the system using a meter. Generalized measurements are an indispensable tool in study of quantum open system. They form the basis for making quantum trajectories and studying irreversibility [Elo17]. We will use them now to model channels and later to study thermodynamic properties of measurements.

1.3.2 Channels and Operations

As we have seen till now, all the processes for closed quantum systems is governed by some unitary evolution. Such an evolution preserves information, i.e., there is a way to go back to the initial state from the final state [Cro08]. In general however, there is always some loss of information. Such loss of information is modelled by some interaction of a system with an environment. By coupling with an environment, but not being able to measure it, the state of the system after the coupling in never completely known. Hence, the information about the system is reduced. From the point of view of the system, the evolution is no longer unitary, but a map from some physical state to some other physical state. On the bloch sphere, unitary can be visualized as rotations. This is explained by the fact that they take pure states to pure states, i.e., evolution is restricted on the bloch surface for pure states. They don't reduce the radius. However, loss of information can be thought of as reduction in bloch sphere radius, where the zero information or the completely mixed state has zero radius. To describe a general evolution which encodes the reduction in bloch radius too, we need channels and operations.

Channels are completely-positive trace preserving (CPTP) maps while, operations are completely-positive trace non-increasing maps. Throughout, we will focus only on CPTP maps. [1] Any map $\mathcal{E}$ that describes a channel acting on density operators (ρ) adheres to the following properties [Pre18]:-

- Linear: $\mathcal{E}(\alpha\rho_1 + \beta\rho_2) = \alpha\mathcal{E}(\rho_1) + \beta\mathcal{E}(\rho_2)$.
- Preserves Positivity: $\mathcal{E}(\rho) \geq 0 \implies \mathcal{E}(\rho) = \mathcal{E}(\rho)^\dagger \;\forall\rho = \rho^\dagger$.
- Preserves Trace: $\mathrm{Tr}[\mathcal{E}(\rho)] = \mathrm{Tr}[\rho]$.
- Completely Positive: The extension map $\mathcal{E}\otimes\mathbb{I}$ also preserves positivity, where $\mathbb{I}$ is the identity operator on the environment of ρ.

The first three properties are straight-forward and say that channels map valid states to valid states. However, the last point says that, if we look at both the map acting on a state and the environment of the state, the total state must remain positive after the operation. Even though, the extension map translates to acting the channel on the system only and doing nothing on the environment, some maps may make the total system + environment state non-positive. An example of such a map is the partial trace [NC10].

As we have noted, non-unitary channels may lead to some loss in information. Such loss in information can be modelled by coupling some environment with the system $\rho \in H^S$ through some global unitary U_{SE} and then not looking at the environment. This leads to a system state $\mathcal{E}(\rho)$. If we model the environment by a meter with states $\{|e_i\rangle\langle e_i|\}_i \in H^E$ and initialize it in state $|e_0\rangle\langle e_0|$, we can get the operation on the system. Hence, a channel action of $\mathcal{E}$ can be modelled by,

$$\mathcal{E}(\rho) = \mathrm{Tr}_E\left[U_{SE}\left(\rho\otimes|e_0\rangle\langle e_0|\right)U_{SE}^\dagger\right] = \sum_i \langle e_i|U_{SE}|e_0\rangle\rho\langle e_0|U_{SE}^\dagger|e_i\rangle = \sum_i M_i\rho M_i^\dagger, \tag{1.1}$$

where, $\{M_i\}_i$ are known as Kraus operators of the quantum channel. These give the POVM operators $M_i^\dagger M_i = E_i$ and hence, are trace preserving as $\sum_i M_i^\dagger M_i = \mathbb{I}$. It is important to note that channels don't have a unique set of Kraus operators, but several sets which are unitarily related. We won't go into more details about channels here.

Channels are crucial to our study as, the quantum switch and measurements are essentially channels which we will use throughout. Moreover, thermodynamic processes like thermalization are also channels. So before we study the thermodynamics of the quantum switch, let us revisit some useful concepts in quantum thermodynamics.

1.3.3 The Gibb's state

To understand the thermodynamics in the quantum regime, let us follow the discussions in [DC19]. When a system which interacts with a large thermal bath or reservoir at temperature T (inverse temperature β), it equilibrates to that temperature.

[1] The trace non-increasing may come from losing some statistics of some quantum systems while making measurements. Hence, as all the statistics is not present, the probabilities don't sum to one and hence, the trace can becomes less than one.

For an initial phase space distribution, with $\Gamma = (\vec{q}, \vec{p})$ being the microstates, this equilibrium distribution is known as the Boltzmann-Gibbs equilibrium state which is,

$$p_{\lambda}^{eq}(\Gamma) = \frac{1}{Z_{\lambda}} \exp\left(-\beta H(\Gamma; \lambda)\right),$$

where, $H(\Gamma; \lambda)$ denotes the Hamiltonian of the system parameterized with λ. The partition function (Z_{λ}) and the Free energy (F_{λ}) are given by,

$$\begin{aligned} Z_{\lambda} &= \int d\Gamma \exp\left(-\beta H(\Gamma; \lambda)\right), \\ F_{\lambda} &= -\frac{1}{\beta} \ln\left[Z_{\lambda}\right]. \end{aligned}$$

Identically, for a quantum state with Hamiltonian H, this state is known as the Gibb's state and is given by,

$$\zeta^{\beta} = \frac{\exp\left[-\beta H\right]}{Z}, \tag{1.2}$$

$$Z = \mathrm{Tr}\left[\exp\left[-\beta H\right]\right], \tag{1.3}$$

$$F = -\frac{1}{\beta} \ln\left[Z\right]. \tag{1.4}$$

Such a state has the lowest free energy, i.e., using a thermal bath at temperature β, no useful energy can be extracted from it. It is interesting to note that any qubit state can be written as the Gibb's state in its eigenbasis basis and at some particular temperature. Its entropy can be written as,

$$S\left[\zeta^{\beta}\right] = -\mathrm{Tr}\left[\zeta^{\beta} \ln\left(\zeta^{\beta}\right)\right] = \beta\left[E - F\right],$$

where, the energy of the state is $E = \mathrm{Tr}\left[H\zeta^{\beta}\right]$. For isothermal processes,

$$\begin{aligned} dS &= \beta\left(\mathrm{Tr}\left[H d\zeta^{\beta}\right] + \left(\mathrm{Tr}\left[\zeta^{\beta} dH\right] - dF\right)\right) = \beta \mathrm{Tr}\left[H d\zeta^{\beta}\right] \\ \implies dE &= dQ + dW = \mathrm{Tr}\left[H d\zeta^{\beta}\right] + \mathrm{Tr}\left[\zeta^{\beta} dH\right]. \end{aligned}$$

Heat dQ is identified as the change of internal energy associated with a change of entropy, and the work dW is identified as change of internal energy due to the change of the Hamiltonian of the system. Such an identification is true for the Gibb's state however, is not true in general. Coherence in the energy basis leads to an informational energetic cost. To begin to understand this, let us discuss the Maxwell's Demon paradox and its resolution.

1.3.4 Information Acquisition and Erasure

"Information is physical". This was an insight by Rolf Landauer that led to the discovery of $k_B T \ln 2$ as the fundamental energy cost of information erasure [Ben82, DC19] and the resolution of a paradox known as the Maxwell's Demon. To understand how information is physical, let us consider the details and the resolution of the paradox.

Maxwell's Demon is a thought experiment where there exists an intelligent demon who can observe a box containing an ideal gas at equilibrium with the environment. The demon inserts a partition in this box with a controllable door. The demon opens the door when a fast particle from the right tries to go left (say if the velocity is greater than a certain v) and a slow particle from the left tries to go right. The operation of this door is considered to have no cost (the door is frictionless). Thus, the demon is able to create a temperature difference from an equilibrium state, and can hence, activate free energy without providing any energy. This shows that the working of the demon is against the second law of thermodynamics. This paradox was resolved by considering the acquisition and storage of information. Here, the demon *measures* the velocity of each gas particle and does a controlled operation depending on this state. To do such an operation, the demon must store the information in a memory and then use it. This memory then needs to be reset. Bennett showed that the reset of this physical memory has a fundamental cost of $k_B T \ln 2$ and by considering this energy cost, the second law can be restored. The minimum amount of energy is required to reset the minimum amount of information, i.e., one bit. Hence, information acquisition has a fundamental cost and is considered physical.

However, information erasure, though important, is not the only way a measurement may contribute to some apparent increase of a system's energy. The measurement apparatus, that the system interacts with, may also exchange energy with the system. This can be due to the light pulses that are used in the measurement process, etc. A description of origin of such a *quantum heat* is given in [Elo17]. We will discuss information erasure and interaction with an external source during measurement in section 3.3. But first, let us understand the central theme of the thesis, i.e., indefinite causal orders.

Chapter 2

Indefinite Causal Order and the Quantum Switch

Well! I've often seen a cat without a grin, but a grin without a cat! It's the most curious thing I ever saw in my life!

— **Alice**
Alice's Adventure in Wonderland

Experiments in quantum physics can lead to many counter-intuitive phenomenon. Just like Alice saw the grin of the Cheshire cat without its body, optical experiments have been interpreted to separate the polarization of a photon from the photon itself [APRS13]. Moreover, such experiments can superpose two operations happening on a single system at two distinct location, i.e., make them appear to happen simultaneously. These experiments have been implemented on interferometers which have been studied in depth using quantum optics [GK04], and been understood as a kind of quantum interference [CSMS15]. In this chapter, we take this a step further. We study an interferometer setup known as the quantum switch, which superposes the causal orders of two distinct quantum operations. In all these experiments, there are two crucial ingredients that are needed to reveal these counter-intuitive phenomenon: quantum interference and quantum entanglement. We will see that, to observe the superposition of causal orders, we will need at least two distinct systems with a global operation between them, and then a final measurement. The entanglement due to the global operation followed by a partial measurement will display a behaviour which can never reveal the order of occurrence of two processes that happen on a system, i.e., they create an indefiniteness in causal orders.

We will begin by studying the basic concepts of causal structures and causal orders, following which, we will try to understand how the quantum switch creates an indefiniteness in causal orders. Next, we will cover the mathematical details of the switch and compare it with operations in a superposition of path. We will conclude the chapter by reviewing some advantages that have been found using the quantum switch both in information theory and thermodynamics. In particular, we will show that the free energy of the system increases after passing through the setup.

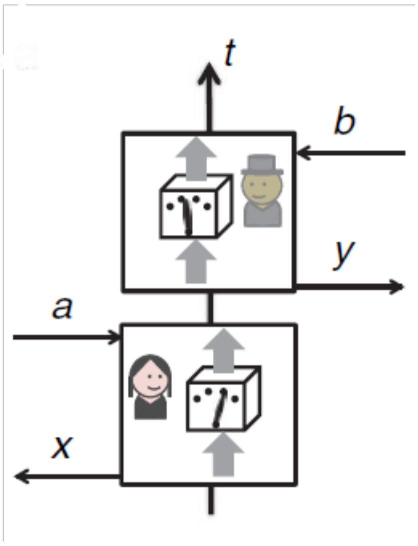


Figure 2.1: *Alice in the causal past of Bob (A $\preceq$ B): Time t increases upwards. Alice receives a package at a time strictly before Bob. Alice and Bob need to make a guess (x and y respectively) about the bit with the other (a and b). Alice can signal her bit a to Bob, but Bob can't signal his bit b to Alice as the package passes through her laboratory at an earlier time. [OCB12]*

2.1 Causal Structure and Causal Order

To understand causal structure and causal order, we will start with a simple example in line with [OCB12]. Imagine that there are two parties, Alice and Bob (as usual), in *closed laboratories*. Closed laboratories imply that Bob can't meddle with Alice's laboratory while she does her experiments and vice-versa. We will make this more precise soon. They decide to play a game where, Alice freely chooses a random bit a (this choice is not influenced by any event in the past) and similarly, Bob chooses b. They then make guesses, x and y respectively, of the other's bit. The game is to correctly guess the other party's bit. Before the guess is made, a package containing a system (any system that can be localized into a package) is passed between Alice and Bob once. Their laboratories are opened only to receive or send the package. They are free to do any operation on the system in the package. Let us assume that Alice receives the package first as in Fig. 2.1. Now, Alice is free to encode her bit a into the system which Bob can then find out when he receives the package. Then he can make the correct guess $y = a$. This implies that Alice can signal some information to Bob. Such a signal creates a statistical correlation between a freely chosen random variable with Alice and a random variable with Bob at a later time. *Freely chosen variables* can only be correlated with variables are a later time, all correlations with past variables are ignored. Now, notice that Alice can't make the guess x with certainty as Bob can't send his bit b to her, as only one package is passed between them once. This implies that because Bob can't send Alice any information, he can't create correlations with her variables, and hence, he can't signal to Alice. Notice also that a party can signal to the other only if the package passes through their laboratory at an earlier time. Hence, the events at Alice's laboratory (A) must be in the past of the events at Bob's laboratory (B) for Alice to be able to signal to

Bob and vice-versa. The ability to signal forms the basis of a causal structure. In this situation, the events A are said to be in the *causal past* of events B or identically B is in the *causal future* of A and so, A was able to signal to Bob. This is denoted by A $\preceq$ B.

The above scenario is an example of a causal structure with closed laboratories. Formally, a causal structure is a set of events locations(such as A and B) which have the partial ordering relation $\preceq$ between them. $\preceq$ creates causal orders between events and event locations. As seen above, if A $\preceq$ B, then the party at A can signal to the party at B by creating correlations between freely chosen variables at A and random variables at B which can then be observed. But signalling from B to A is not possible. Because the laboratories are closed, any freely chosen random variable at one laboratory cannot affect the choices in the other laboratory without a signal. That is, the agents can't meddle with each other's choices without explicit signalling (which occurs in the example only when the package is sent to the other party). The partial order $\preceq$ satisfies three conditions: (a) A $\preceq$ A; (b) if A $\preceq$ B and B $\preceq$ C, then A $\preceq$ C; (c) if A $\preceq$ B and B $\preceq$ A, then A $=$ B. The condition (c) says that, A and B can't be in each other's causal past: there are no causal loops. By A $\npreceq$ B, we denote the situation where A is not in the causal past of B. Interestingly both relations A $\npreceq$ B and B $\npreceq$ A can hold simultaneously (A $\npreceq\nsucceq$ B), as we will show in the next section.

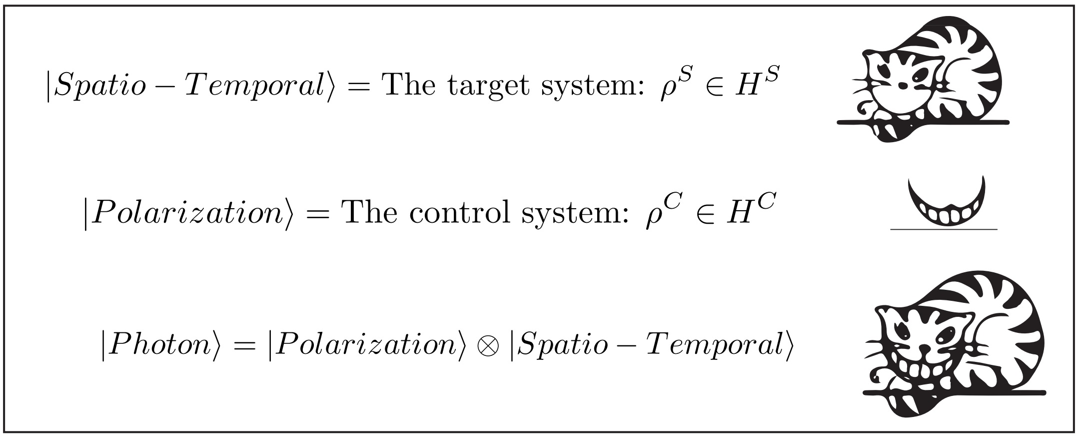
2.2 Creating Indefinite Causal Order

In the previous section, we discussed causal structure and causal ordering using partial ordering between event locations $\preceq$. In what follows, we will continue with the assumption of closed laboratories in a causal structure which can have freely chosen random variables. These laboratories will be placed inside an interferometric device known as the *quantum switch*. The quantum switch(or switch) creates a superposition between the causal orders of the operations in the laboratories. In essence, the causal order of the operations becomes indefinite and A $\npreceq$ B and B $\npreceq$ A holds simultaneously (A $\npreceq\nsucceq$ B). It is important to note that this will be done at an expense of another quantum system which is measured to create this indefiniteness. But before going into how the quantum switch creates indefinite causal orders(ICOs), let us first understand the key components that are needed to create the switch.

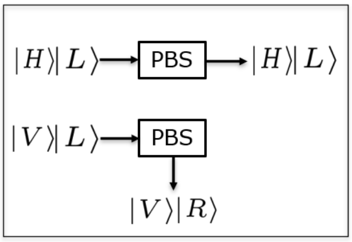
2.2.1 Ingredients for the Quantum Switch

The quantum switch is an interferometric device, as mentioned earlier. The system (like the package in section 2.1) that is operated on is of course light or photon. The laboratories, or in a more information theoretic language-the channels (we will call these channels **A** and **B** in general, unless specified), are some optical setups which manipulate the photon passing through them; for example, optical cavities [HR06]. The causal orders between **A** and **B** are considered. The main ingredients to create the indefiniteness in the causal orders of **A** and **B** are:

- Two quantum systems as internal degrees of freedom of a single photon mode:
  - Target system: $\rho^S \in H^S = |\text{Spatio-Temporal modes}\rangle$



(a) *Modes of Photon*



(b) *Polarization Beam Splitter Working: $|L\rangle$ and $|R\rangle$ are the same spatial modes of the photon but in different directions. $|H\rangle$ and $|V\rangle$ are the polarization modes of the photon*

Figure 2.2: *The components of the quantum switch and the causal order of the channel action*

  - Control system: $\rho^C \in H^C = |\text{Polarization modes}\rangle$
  - $|\text{Photon}\rangle = |\text{Spatio-Temporal modes}\rangle \otimes |\text{Polarization modes}\rangle$

- Polarization Beam Splitter (PBS)

- Two channels **A** and **B**, which act only on the target system:
  A / B: $H^S \to H^S$

Modes of Photon

The spatio-temporal modes are continuous modes that refer to the spatial and temporal distribution of the photon wave-packet. There are many examples of these such as Gaussian modes, Hermite-Gaussian modes, etc. In addition to these modes, photons have a polarization degree of freedom which can be manipulated independently of the spatio-temporal modes. A photon has two orthogonal polarization states. Here, we will take the orthonormal basis of the polarization modes as the horizontal $|H\rangle$ and the vertical $|V\rangle$ polarization. We refer the readers to optics books such as the one by Gerry and Knight [GK04] for a more detailed description of the modes of a photon. We will now describe the action of the polarization beam splitter.

Polarization Beam Splitter

The polarization beam splitter (PBS) works by reflecting or transmitting an incoming beam of light based on its polarization. As shown in Fig. 2.2 the horizontally polarized photon $|H\rangle$ goes straight through the PBS, while vertically polarized photon $|V\rangle$ gets reflected. $|R\rangle$ and $|L\rangle$ denote the spacial modes of the photons differing int the direction of propagation. Interestingly, when the polarization of the input photon is in a superposition of the two, such as the $|+\rangle = \left(\frac{|H\rangle+|V\rangle}{\sqrt{2}}\right)$, the paths of the direction of propagation is superposed with the respective amplitudes of $|H\rangle$ and $|V\rangle$. As we will see next, this will help create the superposition in the causal orders.

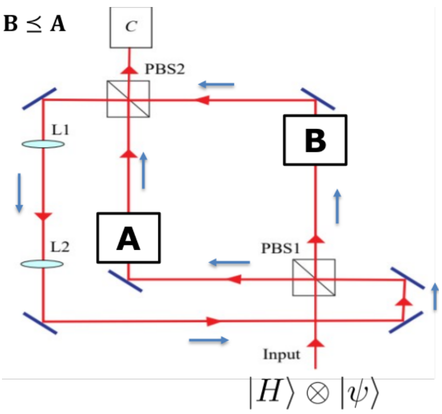
All the components are summarized in Fig. 2.2. Now that we have all the ingredients necessary for making the quantum switch, let us describe the action of the switch without further ado.

2.2.2 Switch Action

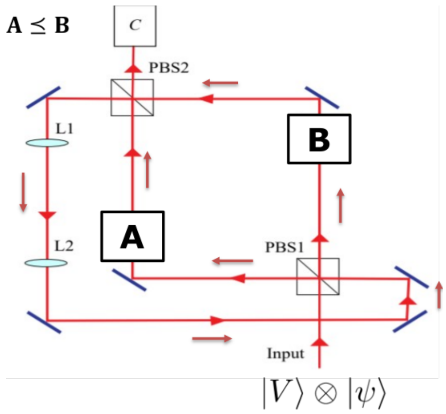
We begin with an arbitrary system state ρ^S encoded in the spatio-temporal modes of a photon and, let us prepare the control in the horizontally polarized $|H\rangle\langle H|$ state. The state of the photon is hence, $\left(\rho^S \otimes \rho^C\right) = \left(\rho^S \otimes |H\rangle\langle H|\right)$. As you can see from the Fig. 2.3, the photon with $|H\rangle$ polarization goes through all the PBS. The photon's spatio-temporal mode ρ^S is hence, acted upon by channel **B** before **A**, i.e., this situation depicts $\mathbf{B} \preceq \mathbf{A}$. Next, let us prepare the initial state such that the control is in the vertically polarized $|V\rangle\langle V|$ state. Again, seeing the action of the PBS, you can verify that this preparation leads to $\mathbf{A} \preceq \mathbf{B}$. As you can see, it is the path of the photon that is coupled to the causal order which is hence, coupled to the polarization modes, or equivalently, the control. The control in this way selects the causal order of the application of the channels in the switch. Therefore, if we measure the control after the switch operation in the $\{|H\rangle, |V\rangle\}$ basis, we can determine which causal order occurred. We can also determine their probability of occurrence, if the control is prepared in a probabilistic mixture of $|H\rangle\langle H|$ and $|V\rangle\langle V|$.

To create a superposition in the causal orders is now straight forward; we just create a superposition in the control. So, let us prepare the control in the $|+\rangle\langle +|$ state. We see that this is a situation where both causal orders appear to occur simultaneously. However, if we measure the control at the end in the $\{|H\rangle, |V\rangle\}$ basis, we will not see that the effect on the system is such that, either of the two definite causal orders occurred. Hence, to see any interference effect, we must measure the control in the $\{|+\rangle, |-\rangle\}$ basis. This leads to very interesting situations. Note that after the switch action, the system state gets correlated with the control state. As we have seen in section 1.3.1, measurement of a correlated ancilla affects the system state as well. The measurement of the control in the $\{|+\rangle, |-\rangle\}$ basis does not give any information about the causal order. Hence, after the collapse of the control in this basis, the modified system state too does not show a state that comes out of the action of the channels in any definite causal order, but has some interference terms (we will see this in the next section). Hence, the channels act on the system as if $\mathbf{A} \npreceq \nsucceq \mathbf{B}$. Hence, indefinite causal order of channel action of **A** and **B** has been created.

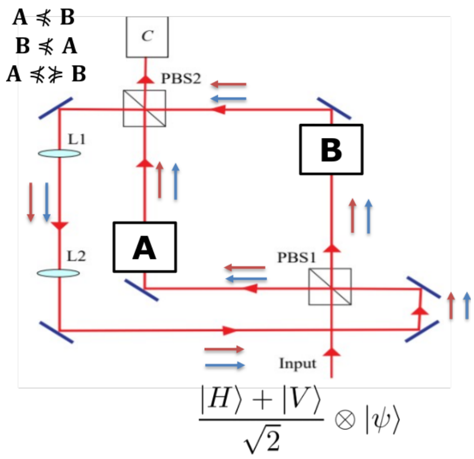
To understand and study the implications of this behaviour, let us look at the



(a) *B acts before A*



(b) *A acts before B*



(c) *The causal order of **A** and **B** are indefinite*

Figure 2.3: *The Quantum Switch with channels **A** and **B** [GGK+18]. When the polarization (control) is prepared in the $|+\rangle\langle+|$ state and measured in the $|\pm\rangle\langle\pm|$ basis, the causal orders are indefinite and **A** $\npreceq$ **B** and **B** $\npreceq$ **A** hold simultaneously.*

switch as a channel acting on the control + system bipartite state. For this let us first compute its Kraus operators.

2.2.3 Kraus Operators for the Quantum Switch

We have already studied that the action of any channel can be expressed in terms of Kraus operators in section 1.3.2. Here, we will find the Kraus operators for the Quantum Switch(QS) in terms of the Kraus operators of the channels **A** and **B** that are a part of it. To do this we take the global unitary of the switch as $\left(U^{SCEF}\right)$ which acts on the system $\left(\rho \in H^S\right)$, control $\left(\sigma \in H^C\right)$ and two environments, one corresponding to each channel. Let's take the two environments as $\{|e_i\rangle\langle e_i|\}_i \in H^E$ which is used with channel **A** and $\{|f_j\rangle\langle f_j|\}_j \in H^F$ which is used in channel **B**. The superscripts denote the Hilbert space being considered. We prepare the environment as $(|e_0\rangle\langle e_0| \otimes |f_0\rangle\langle f_0|)$. We take the unitaries corresponding to the channels **A** and **B** as U_A^{SE} and U_B^{SF} respectively.

The global unitary for the quantum switch is,

$$U^{SCEF} = \left(|0\rangle\langle 0| \otimes U_A^{SE} U_B^{SF} + |1\rangle\langle 1| \otimes U_B^{SF} U_A^{SE}\right). \tag{2.1}$$

The action of the switch unitary is given by,

$$\begin{aligned}U^{SCEF}\left(\sigma \otimes \rho \otimes |e_0\rangle\langle e_0| \otimes |f_0\rangle\langle f_0|\right)\left(U^{SCEF}\right)^\dagger = &\left(|0\rangle\langle 0| \otimes U_A^{SE} U_B^{SF} + |1\rangle\langle 1| \otimes U_B^{SF} U_A^{SE}\right)\\ &.\left(\sigma \otimes \rho \otimes |e_0\rangle\langle e_0| \otimes |f_0\rangle\langle f_0|\right)\\ &.\left(|0\rangle\langle 0| \otimes \left(U_A^{SE} U_B^{SF}\right)^\dagger + |1\rangle\langle 1| \otimes \left(U_B^{SF} U_A^{SE}\right)^\dagger\right).\end{aligned}$$

The Kraus operators for the individual channels are:

$$A_i = \langle e_i| U_A^{SE} |e_0\rangle .$$

$$B_j = \langle f_j| U_B^{SF} |f_0\rangle .$$

As shown in section 1.3.2, we trace over the environments of the channel to get the Kraus operators. Hence, to get the Kraus operators of the switch, we trace over both the environments **A** and **B**,

$$\begin{aligned}&\mathrm{Tr}_{EF}\left(U^{SCEF}\left(\sigma \otimes \rho \otimes |e_0\rangle\langle e_0| \otimes |f_0\rangle\langle f_0|\right)\left(U^{SCEF}\right)^\dagger\right)\\ &= \sum_{i,j}\left(|0\rangle\langle 0| \otimes \langle e_i| U_A^{SE} |e_0\rangle \langle f_j| U_B^{SF} |f_0\rangle + |1\rangle\langle 1| \otimes \langle f_j| U_B^{SF} |f_0\rangle \langle e_i| U_A^{SE} |e_0\rangle\right)\\ &.\left(\sigma\otimes\rho\right)\left(|0\rangle\langle 0| \otimes \langle f_0| \left(U_B^{SF}\right)^\dagger |f_j\rangle \langle e_0| \left(U_A^{SE}\right)^\dagger |e_i\rangle + |1\rangle\langle 1| \otimes \langle e_0| \left(U_A^{SE}\right)^\dagger |e_i\rangle \langle f_0| \left(U_B^{SF}\right)^\dagger |f_j\rangle\right)\\ &= \sum_{i,j}\left(|0\rangle\langle 0| \otimes A_i B_j + |1\rangle\langle 1| \otimes B_j A_i\right)\left(\sigma\otimes\rho\right)\left(|0\rangle\langle 0| \otimes B_j^\dagger A_i^\dagger + |1\rangle\langle 1| \otimes A_i^\dagger B_j^\dagger\right).\end{aligned}$$

Hence, the Kraus operators of the Quantum Switch are,

$$W_{ij} = |0\rangle\langle 0| \otimes A_i B_j + |1\rangle\langle 1| \otimes B_j A_i. \tag{2.2}$$

For consistency, let's check the completeness relation,

$$\sum_{i,j} W_{ij}^\dagger W_{ij} = \mathbb{I}^C \otimes \mathbb{I}^S. \tag{2.3}$$

Hence, the action of the switch channel is given by,

$$S\left(\sigma\otimes\rho\right) = \sum_{i,j} W_{ij}\left(\sigma\otimes\rho\right) W_{ij}^\dagger. \tag{2.4}$$

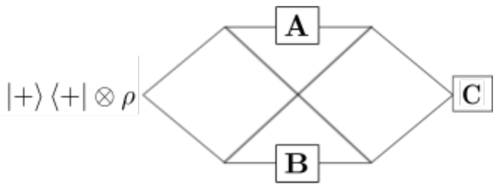
As you can see, the diagonal elements of the control $|0\rangle\langle 0|$ and $|1\rangle\langle 1|$ correspond to the channel action of $\mathbf{B} \preceq \mathbf{A}$ and $\mathbf{A} \preceq \mathbf{B}$ respectively. On the other hand, the switch creates some interference terms between the causal orders of the channel action on the off-diagonal elements of the control. Hence, to see any unique effect of the switch, the control shouldn't be measured in the diagonal basis of the control. It is worth noting that, when the control is traced over, i.e., it is not considered we have,

$$\mathrm{Tr}_C\left(S\left(\sigma\otimes\rho\right)\right)=\mathrm{Tr}\left(\sum_{i,j}W_{ij}\left(\sigma\otimes\rho\right)W_{ij}^{\dagger}\right)=\sum_{i,j}p_0^C\left(A_iB_j\rho B_j^{\dagger}A_i^{\dagger}\right)+\sum_{i,j}p_1^C\left(B_jA_i\rho A_i^{\dagger}B_j^{\dagger}\right)$$
$$=p_0^C C_{\mathbf{A}}\left(C_{\mathbf{B}}\left(\rho\right)\right)+p_1^C C_{\mathbf{B}}\left(C_{\mathbf{A}}\left(\rho\right)\right),$$

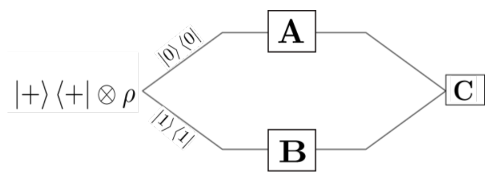
where, $\{p_i^C\}$ are the diagonal elements of the control, $C_{\mathbf{A/B}}$ are the channel maps corresponding to **A**/**B**. This in effect is like tossing a biased coin which selects $\mathbf{A}\preceq\mathbf{B}$ or $\mathbf{B}\preceq\mathbf{A}$ based on the result. Hence, to get any interesting effect of the switch, one must *measure* the control in the off-diagonal basis with respect to the basis of the control that couples to the definite causal orders. As we will see, any advantage of the switch comes from such a measurement.

2.2.4 Superposition of Path vs Quantum Switch

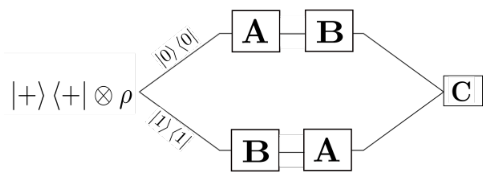
2.2.5 Switch Action



(a) *A schematic representation of the Quantum Switch.*



(b) *A schematic representation of the superposition of paths of two channels* ***A*** *and* ***B***.



(c) *A schematic representation of the superposition of paths of four channels with two copies of* ***A*** *and* ***B*** *each.*

Figure 2.4: *The quantum switch creates* $\mathbf{A}\npreceq\npreceq\mathbf{B}$ *using two environments while in the superposition of paths case, a superposition of channel operation* $\mathbf{A}\preceq\mathbf{B}$ *and* $\mathbf{B}\preceq\mathbf{A}$ *is created using four different environments within the different channels.* ***C*** *represents control measurement at the end.*

Before studying the advantages of the quantum switch, let us first compare it with a seemingly simpler setup. Instead of creating a superposition in causal orders, let us create a superposition in path (SP), i.e., a superposition in the operation of the two channels **A** and **B**. As can be seen in Fig. 2.4, the control state $|0\rangle\langle 0|$ corresponds to the action of channel **A** and $|1\rangle\langle 1|$ corresponds to the action of channel **B**. To contrast the operation of the SP and the switch, let us first find the Kraus operators of SP.

As usual we take the control to be $\sigma \in H^C$ and the target system to be $\rho \in H^S$, the two environments as $\{|e_i\rangle\langle e_i|\}_i \in H^E$ and $\{|f_j\rangle\langle f_j|\}_j \in H^F$. We prepare the environment as $(|e_0\rangle\langle e_0| \otimes |f_0\rangle\langle f_0|)$. The action of the global unitary is now,

$$\begin{aligned}U^{SCEF}\left(\sigma \otimes \rho \otimes |e_0\rangle\langle e_0| \otimes |f_0\rangle\langle f_0|\right)\left(U^{SCEF}\right)^{\dagger} = &\left(|0\rangle\langle 0| \otimes U^{SE}_{\mathbf{A}} + |1\rangle\langle 1| \otimes U^{SF}_{\mathbf{B}}\right)\\ &.\left(\sigma \otimes \rho \otimes |e_0\rangle\langle e_0| \otimes |f_0\rangle\langle f_0|\right)\\ &.\left(|0\rangle\langle 0| \otimes \left(U^{SE}_{\mathbf{A}}\right)^{\dagger} + |1\rangle\langle 1| \otimes \left(U^{SF}_{\mathbf{B}}\right)^{\dagger}\right).\end{aligned}$$

Now, we trace over both the environments to get the Kraus operators acting on the system + control state

$$\begin{aligned}&\mathrm{Tr}_{E,F}\left(U^{SCEF}\left(\sigma \otimes \rho \otimes |e_0\rangle\langle e_0| \otimes |f_0\rangle\langle f_0|\right)\left(U^{SCEF}\right)^{\dagger}\right)\\ &\quad= \sum_{i,j}\left(|0\rangle\langle 0| \otimes \delta_{j,0}\langle e_i| U^{SE}_{\mathbf{A}}|e_0\rangle + |1\rangle\langle 1| \otimes \delta_{i,0}\langle f_j| U^{SF}_{\mathbf{B}}|f_0\rangle\right)\\ &\quad= (\sigma \otimes \rho)\left(|0\rangle\langle 0| \otimes \langle e_0| \left(U^{SE}_{\mathbf{A}}\right)^{\dagger}|e_i\rangle\delta_{j,0} + |1\rangle\langle 1| \otimes \langle f_0|\left(U^{SF}_{\mathbf{B}}\right)^{\dagger}|f_j\rangle\delta_{i,0}\right),\end{aligned}$$

where, now

$$\begin{aligned}A_i &= \langle e_i| U^{SE}_{\mathbf{A}}|e_0\rangle.\\ B_j &= \langle f_j| U^{SF}_{\mathbf{B}}|f_0\rangle.\end{aligned}$$

and hence,

$$\begin{aligned}&\mathrm{Tr}_{E,F}\left(U^{SCEF}\left(\sigma \otimes \rho \otimes |e_0\rangle\langle e_0| \otimes |f_0\rangle\langle f_0|\right)\left(U^{SCEF}\right)^{\dagger}\right)\\ &= \sum_{i,j}\left(|0\rangle\langle 0| \otimes \delta_{j,0}A_i + |1\rangle\langle 1| \otimes \delta_{i,0}B_j\right)(\sigma \otimes \rho)\left(|0\rangle\langle 0| \otimes \delta_{j,0}\left(A_i\right)^{\dagger} + |1\rangle\langle 1| \otimes \delta_{i,0}\left(B_j\right)^{\dagger}\right).\end{aligned}$$

The Kraus operators for the SP are,

$$Y_{ij} = \left(|0\rangle\langle 0| \otimes \delta_{j,0}A_i + |1\rangle\langle 1| \otimes \delta_{i,0}B_j\right). \tag{2.5}$$

There are delta functions in the Kraus operators which raises a flag. For consistency, let's check the completeness relation:

$$\sum_{i,j} Y^{\dagger}_{ij}Y_{ij} = \mathbb{I}^C \otimes \mathbb{I}^S. \tag{2.6}$$

Hence, the action of the SP channel is given by,

$$I\left(\sigma \otimes \rho\right) = \sum_{i,j} Y_{ij}\left(\sigma \otimes \rho\right)Y^{\dagger}_{ij}. \tag{2.7}$$

Let us expand and check the "interesting" off-diagonal terms of the control,

$$
\begin{aligned}
I(\sigma\otimes\rho) &= \sum_{i,j} Y_{ij}(\sigma\otimes\rho)Y_{ij}^{\dagger}\\
&= \sum_{i,j}\left(|0\rangle\langle 0|\otimes\delta_{j,0}A_i + |1\rangle\langle 1|\otimes\delta_{i,0}B_j\right)(\sigma\otimes\rho)\left(|0\rangle\langle 0|\otimes\delta_{j,0}\left(A_i\right)^{\dagger} + |1\rangle\langle 1|\otimes\delta_{i,0}\left(B_j\right)^{\dagger}\right)\\
&= \sum_{i,j}\left(\langle 0|\sigma|0\rangle\,|0\rangle\langle 0|\otimes\delta_{j,0}\delta_{j,0}A_i\rho A_i^{\dagger}\right) + \sum_{i,j}\left(\langle 1|\sigma|1\rangle\,|1\rangle\langle 1|\otimes\delta_{i,0}\delta_{i,0}B_j\rho B_j^{\dagger}\right)\\
&+ \sum_{i,j}\left(\langle 0|\sigma|1\rangle\,|0\rangle\langle 1|\otimes\delta_{i,0}\delta_{j,0}A_i\rho B_j^{\dagger}\right) + \sum_{i,j}\left(\langle 1|\sigma|0\rangle\,|1\rangle\langle 0|\otimes\delta_{i,0}\delta_{j,0}B_j\rho A_i^{\dagger}\right)\\
&= \langle 0|\sigma|0\rangle\,|0\rangle\langle 0|\otimes C_{\mathbf{A}}(\rho) + \langle 1|\sigma|1\rangle\,|1\rangle\langle 1|\otimes C_{\mathbf{B}}(\rho)\\
&+ \left(\langle 0|\sigma|1\rangle\,|0\rangle\langle 1|\otimes A_0\rho B_0^{\dagger}\right) + \left(\langle 1|\sigma|0\rangle\,|1\rangle\langle 0|\otimes B_0\rho A_0^{\dagger}\right). \qquad (2.8)
\end{aligned}
$$

In contrast to the quantum switch we see that the off-diagonal terms depend on a particular Kraus operator, here the 0^{th} due to the delta function $\delta_{i,0}$ and $\delta_{j,0}$ we encountered earlier in the Kraus operator $Y_{i,j}$. However, this delta appears because of our arbitrary choice of environment as $|e_0\rangle\langle e_0|$ and $|f_0\rangle\langle f_0|$. We could have instead chosen the environments to be $|e_x\rangle\langle e_x|$ and $|f_y\rangle\langle f_y|$ $(\{x,y\}\in[0,d-1])$ which would give us $\delta_{i,x}$ and $\delta_{j,y}$ leading to A_x and B_y on the off-diagonal terms. This indicates that the outcome of the SP for the system + control depends on the particular choice of the environment, i.e., it is implementation dependent. [1] The SP is not a proper channel unlike the switch. Moreover, notice that the SP has a quantum circuit representation. The quantum switch does not, as the quantum circuit has time moving to the right [NC10]. The switch has two different causal orders of gates superposed which can't be written in a circuit representation. Nevertheless, it may be possible to derive the final state of the switch from the action of a quantum circuit. A more detailed analysis of the SP can be found in the publication by Abbott et al. [AWH+20].

The above has made it clear that the quantum switch is operationally different from the superposition of path scenario, even if the superposition of path tries to simulate superposition in causal orders as in Fig. 2.4. Let us now study some examples where the switch has shown some kind of advantage.

2.3 Advantages of the Quantum Switch

Now, that we understand (more or less) how the quantum switch operates, let us discuss some of the possible applications of the switch. Chiribella et al. [CBB+21] have shown that the ICO created by the switch can lead to perfect information transfer even over noisy channels. Moreover, groups such as Felce et al. [FV20] and Guha et al. [GAP20] have shown some very counter intuitive results using ICO where, the temperature of a quantum system is raised using baths at the same

[1]There may be cases where Kraus operators can be constructed for SP which is not implementation dependent. This may correspond to the choice of environments which are correlated with each other. However, we have assumed closed laboratories and hence, the environments can't be correlated.

temperature of the system. The thermodynamics of the latter phenomenon will form the crux of the thesis. But first let's study these advantages of the switch. To keep the explanations short and simple we will take the example of qubits throughout this section.

2.3.1 Information Transfer using the Quantum Switch

The quantum switch can be used to transfer complete quantum information over two noisy channels. Noisy channels are channels like the depolarizing or the dephasing channel [NC10, Pre18] where you lose some information about the input state after passing through it. To understand this, and to realize why the quantum switch provides a definite advantage, let us look at the action of the entanglement-breaking channel.

The Entanglement-Breaking channel - losing quantum information

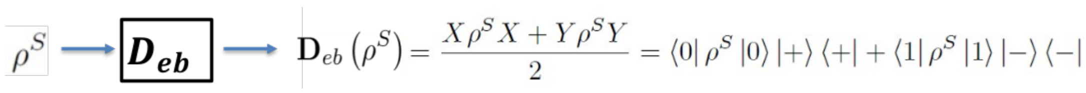


Figure 2.5: *Action of the entanglement-breaking channel with bit flip X and simultaneous bit and phase flip Y.*

When talking about qubits, there are three types of errors that we can talk about,

- Bit flip: $|0\rangle \to |1\rangle$ or $|1\rangle \to |0\rangle$. This transformation can be done by the Pauli matrix $\sigma_X \equiv X$.
- Phase flip: $|0\rangle \to |0\rangle$ or $|1\rangle \to -|1\rangle$. This transformation can be done by the Pauli matrix $\sigma_Z \equiv Z$.
- Simultaneous Bit and Phase flip: $|0\rangle \to i|1\rangle$ or $|1\rangle \to -i|0\rangle$. This transformation can be done by the Pauli matrix $\sigma_Y \equiv Y$.

A depolarizing channel passes a qubit entering it non-erroneously with a probability p_0 and imparts takes it to the completely mixed state $(\mathbb{I}/2)$ with probability $(1-p_0)$. This corresponds to imparting an error on it with particular probabilities p_X, p_Y and p_Z corresponding the action of the error operators X, Y and Z respectively. The action of the depolarizing channel on a state ρ is hence given by,

$$\mathbf{D}(\rho) = p_0\rho + p_X X\rho X + p_Y Y\rho Y + p_Z Z\rho Z.$$

Notice also that this is the Kraus decomposition of the depolarizing channel (refer section 1.3.2). When $p_0 = p_Z = 0$ and $p_X = p_Y = \frac{1}{2}$, then,

$$\mathbf{D}_{eb}(\rho) = \frac{1}{2}(X\rho X + Y\rho Y). \tag{2.9}$$

This is an entanglement-breaking channel. Notice that $\mathbf{D}_{eb}(\rho)$ is such that it has no off-diagonal elements, it is a diagonal matrix. Hence, any quantum information and correlation that is present in the computational basis is lost. That is, if there was superposition in the computational basis, or if the system was entangled with another system, this information is lost after the system passes through $\mathbf{D}_{eb}$. It can be further shown that no combination of the $\mathbf{D}_{eb}$ channels can lead to quantum information transfer.

To see the advantage of ICO, let us place two $\mathbf{D}_{eb}$ channels in the switch. We will see that we can get back the initial state ρ without any losses!

Channels in ICO do not lose!

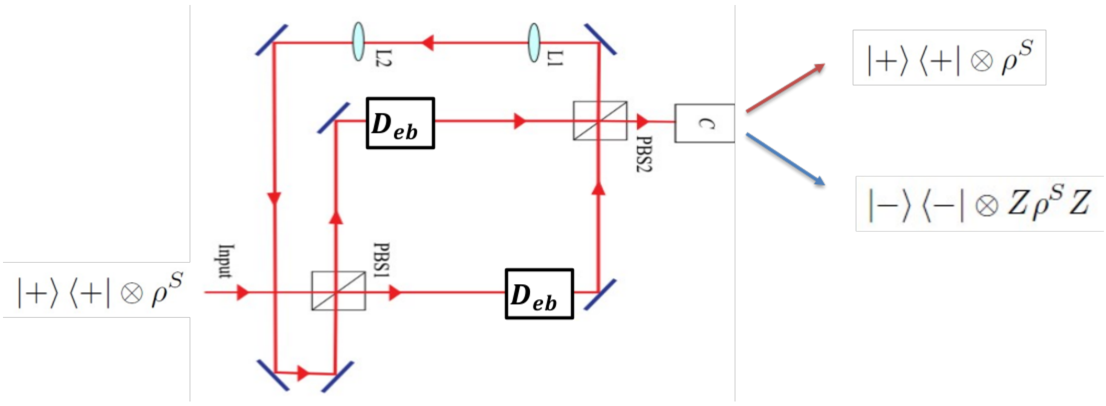


Figure 2.6: *Two entanglement breaking channels ($\mathbf{D}_{eb}$) in the quantum switch [GGK+18]. The two outputs are for the initial state ρ^S for control measurement outcome $|+\rangle\langle+|$ and a rotated initial state $Z\rho^S Z$ for control measurement outcome $|-\rangle\langle-|$.*

In section 2.2.3 we found the Kraus operators of the quantum switch (2.2). To get the action of the two entanglement-breaking channels in the switch, we just replace the channel Kraus operators with that of $\mathbf{D}_{eb}$, i.e., $\frac{1}{2}X$ and $\frac{1}{2}Y$. We initialize the control in the $|+\rangle\langle+|$ state and take the system in some state ρ. Now using equation (2.4), we get

$$S\left(|+\rangle\langle+|\otimes\rho\right)=\frac{1}{2}|+\rangle\langle+|\otimes\rho+\frac{1}{2}|-\rangle\langle-|\otimes Z\rho Z.$$

As you can see, if we now measure the control in the $|\pm\rangle\langle\pm|$ basis, for the $|+\rangle\langle+|$ state we get ρ, and for the $|-\rangle\langle-|$ state we get $Z\rho Z$. Given that the measurement of the control results in state $|-\rangle\langle-|$ we can get back the state ρ by just applying the unitary transformation Z to $Z\rho Z$. Hence, we can successfully transfer a state ρ through two entanglement-breaking channels in a quantum switch without loss of any information.

2.3.2 Thermodynamic "Advantage" of the Quantum Switch

As we have just seen, the ICO created by the quantum switch can lead to some counter-intuitive phenomenon. Let us jump right into another one which is even more so. The quantum switch can heat up a system, at say a temperature T, by thermalizing with thermal baths or heat reservoirs at the same temperature T. At first glance, this seems like it would violate fundamental laws of physics such as the second law of thermodynamics. The publications [FV20] and [GAP20], show that it does not. The rest of this thesis is devoted to analysing this phenomenon and asking if there is truly some thermodynamic advantage and if so why?

As a first step, let us review the advantage as shown by Felce et al. [FV20] where they make a quantum refrigeration cycle using thermalization in ICO. As you may have guessed, the channel that will be used in the switch here, is the thermalization channel. So, let us study this channel and understand how it acts.

Thermalization as a Pin Map

In section 1.3.3, we learnt that any thermalization process for a quantum system, leads to a state known as the Gibb's state, i.e., the state with zero free energy with respect to the bath. It can be straightforwardly shown that any two level quantum system can be written as a Gibb's state, albeit in different basis. If we fix the basis and the temperature(inverse temperature β), the thermalization process leads to the same Gibb's state (ζ^β) regardless of the system being thermalized. When viewed as a channel, thermalization channel ($\mathbf{C}_\beta$) is a map which takes the whole state space to a single state ζ^β. Any channel which takes the whole state space to a single state is a Pin map, and hence, the thermalization channel is a Pin map. Such a channel, is in essence a completely depolarizing channel, where you lose all information of the initial state. A completely depolarizing channel maps to a completely mixed state and can be modelled using unitary operations randomly acting on the input state [HZ11]. Hence, we can write the completely mixed state as,

$$\frac{1}{2}\mathbb{I} = \frac{1}{2}\mathrm{Tr}\left(\rho\right)\mathbb{I} = \frac{1}{4}\sum_{i=1}^{4} U_i \rho U_i^\dagger.$$

To get the Kraus operators of the $\mathbf{C}_\beta$ we just write,

$$\boldsymbol{C}_\beta\left(\rho\right) = \zeta^\beta \mathbb{I} = \mathrm{Tr}\left(\rho\right)\zeta^\beta\mathbb{I} = \frac{1}{2}\sum_{i=1}^{4}\sqrt{\zeta^\beta}U_i\rho U_i^\dagger\left(\sqrt{\zeta^\beta}\right)^\dagger, \tag{2.10}$$

and hence, the Kraus operators are,

$$K_i^\beta = \sqrt{\frac{\zeta^\beta}{2}}U_i. \tag{2.11}$$

This can also be seen from the unitary basis expansion of the operators. Now, that we know the action of the thermalization channel, let us put two of them in the switch, where both maps the input to the same temperature Gibb's state. This models thermalization at the same temperature.

Cooling using ICO

We now place two thermalization channels with baths at the same temperature (inverse temperature β) and prepare the system in the Gibb's state (ζ^β) which is

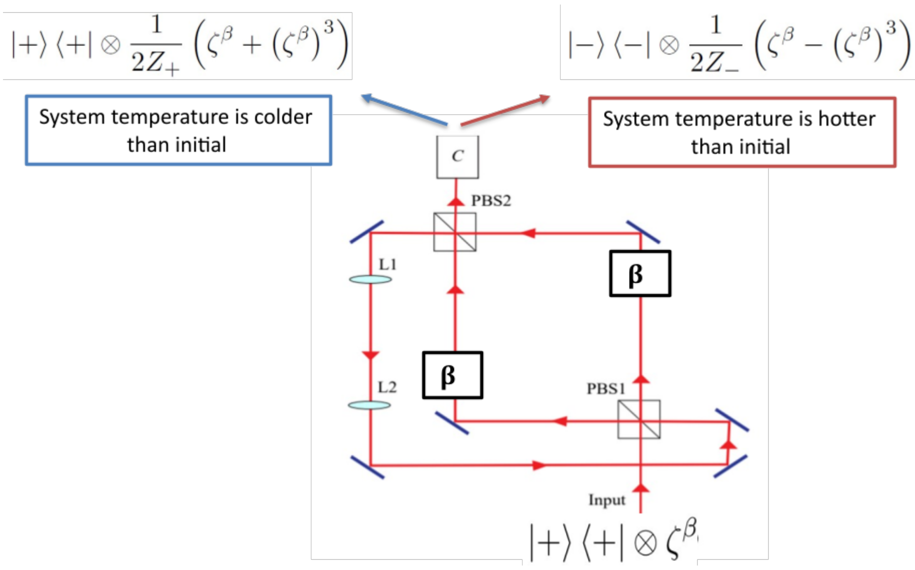


Figure 2.7: *Quantum refrigeration cycle using ICO [GGK+18]. Control and system are prepared in state $|+\rangle\langle+|\otimes\zeta^\beta$. For the $|+\rangle\langle+|$ measurement of the control, the system classically thermalizes with the cold baths in the switch. While for $|-\rangle\langle-|$ measurement of the control, the system classically thermalizes with some other hotter bath and releases heat into it. Hence, this makes a refrigeration cycle.*

also at the same temperature (β). We prepare the control in the $|+\rangle\langle+|$ state as usual. The action of the switch gives,

$$S\left(|+\rangle\langle+|,\zeta^\beta\right)=\frac{1}{2}\left(|0\rangle\langle0|+|1\rangle\langle1|\right)\otimes\zeta^\beta+\frac{1}{2}\left(|0\rangle\langle1|+|1\rangle\langle0|\right)\otimes\left(\zeta^\beta\right)^3. \tag{2.12}$$

The details of the calculations can be found in the appendix A.1. When the control is measured in the $|\pm\rangle\langle\pm|$ basis, we get,

$$\zeta^S_{f\pm}=\frac{1}{Z_\pm}\left(\frac{1}{2}\zeta^\beta\pm\frac{1}{2}\left(\zeta^\beta\right)^3\right), \tag{2.13}$$

where, the partition function is,

$$Z_\pm=\frac{1}{2}\pm\frac{1}{2}\mathrm{Tr}\left[\left(\zeta^\beta\right)^3\right].$$

It can be shown that the temperature of the system is lower than initial if $|+\rangle\langle+|$ is measured on the control , while it is higher if $|-\rangle\langle-|$ measured. As we can see, the free energy of the system is non-zero for the two cases. The publications [FV20, GAP20] claim that this increase in free energy comes from the thermal baths that are used in the switch. Hence, when $|-\rangle\langle-|$ is measured, the system heats up and the bath cools down. As can be seen from Fig. 2.7, this effect is used as a refrigerator.

When $|-\rangle\langle-|$ is measured, the system takes heat from the two baths that are a part of the switch and heats up to a temperature which is higher than the temperature of another "hot" bath. The system then classically thermalizes with this hot bath giving heat to it, to come back to the initial state. Whereas, when $|+\rangle\langle+|$ is measured, the system cools down, giving heat to the baths in the switch. This is then thermalized with the baths part of the switch again, so that the initial configuration is recovered and the ICO can be created again with a different control initialized in the $|+\rangle\langle+|$ state. Repeating the entire process, leads to the cooling down of the baths in the switch, and hence, a refrigeration cycle.

The second law is not affected due to the Maxwell's demon. Notice, that the thermalization process after the measurement of the control is dependent on the measurement result. Hence, it is a feedback cycle where some register is being reset (see section 1.3.4). The work cost of erasure is given by,

$$\Delta W_E = \frac{1}{\beta_R} S\left(\rho_R\right) = \frac{1}{\beta_R} \sum_i p_i \ln p_i,$$

where, $i \in \{+,-\}$, β_R is the inverse temperature of the resetting reservoir, and $p_\pm$ is the probability of the measurement result of the control given by,

$$p_\pm = \mathrm{Tr}\left[\frac{1}{2}\zeta^\beta \pm \frac{1}{2}\left(\zeta^\beta\right)^3\right].$$

This leads to the energy cost for a positive cycle (till $|-\rangle\langle-|$ is measured), which is $\Delta W^{cycle} = \frac{1}{p_-\beta_R}\sum_\pm p_\pm \ln p_\pm$ as the average the number of measurements performed per cycle is $\frac{1}{p_-}$. The efficiency of the cycle is given by,

$$\eta = -\frac{\text{Heat Transfer from cold reservoir}}{\Delta W^{cycle}}.$$

This energy cost then amounts to reducing the efficiency to less than the Carnot efficiency. This ensures that the second law is not violated.

2.4 Conclusions and Inferences

In this chapter, we studied how the quantum switch creates an effect known as Indefinite Causal Order between operation of channels. We studied the advantages of using this setup for quantum communication and quantum refrigeration (or activation of free energy). For the communication task, we saw that two depolarizing channels in ICO, can send the complete initial state for any measurement of the control. Whereas, for the refrigeration cycle, we saw that a particular measurement of the control leads to a positive cycle. It was claimed that heat transfer occurred after the measurement of the control between the system and the baths, but this was never proved.

From these examples, it is important to note that the measurement of the control is a key step in gaining any advantage. In the refrigeration cycle, only after the measurement step energy came into the system. But from where? We have never looked at the energy transfer process. Also here, we considered the control to be freely available. But what if the control was a costly resource? We will try to answer these questions in the subsequent chapters and hence, analyse the thermodynamics of the quantum switch.

Chapter 3

Probing the Thermal Baths

The concept of 'measurement' becomes so fuzzy on reflection that it is quite surprising to have it appearing in physical theory at the most fundamental level ... does not any analysis of measurement require concepts more fundamental than measurement? And should not the fundamental theory be about these more fundamental concepts?

— **John Stewart Bell**

The concept of measurement in quantum mechanics continues to elude physicists throughout the world. Mathematically, we have seen in section 1.3.1 that statistics of quantum measurement results can be reproduced using positive operator valued measures acting on the initial state. Recently, Shrapnel et al. [SCM18] took this a step further combined the Born rule and the collapse postulate into a unified theory described by process matrices. Physically, the measurement of a quantum state has several implications. Measurements can be used to model dissipative evolution like decoherence and other processes when the quantum system is open to the environment [BP02]. It can be used for quantum cryptography and quantum key distribution tasks [SP00]. Measurements play a role in error correction in quantum communication tasks [HZ11, Wil, NC10]. Even in section 2.3, we saw that there was always a measurement of the control involved to get some advantage of the quantum switch. Thermodynamically, it is fascinating to understand that work can be extracted from quantum measurements and the Maxwell's demon can be used to run engines [EHMHA17].

It is clear that quantum measurement and collapse are very important in studying any quantum phenomenon. In this chapter, we will analyse the claimed thermodynamic advantage of the quantum switch as in section 2.3. We ask several question to probe this advantage. First, how much energy does each bath contribute towards the activation of free energy of the target system. Second, is there another source of energy that has been unaccounted? If there is, what is the fraction of energy coming the baths and the other source . We will see that (SPOILER ALERT!!!) the energy does not come from the baths, but only from the measurement of the control. We prove this by modelling the thermalization as a SWAP operation and later rigorously prove that any free energy activation/benefit is considered only as the work benefit

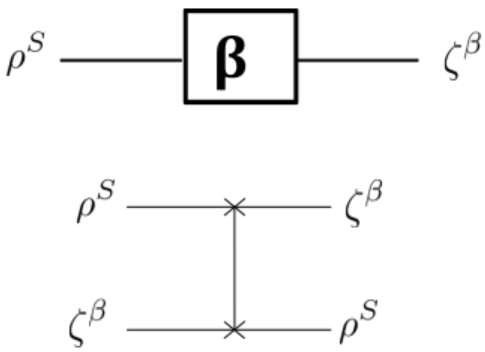


Figure 3.1: *Thermalization as a SWAP gate.*

due to the measurement.

Let us start by discussing the first question, i.e., how much energy does each bath contribute. To do this, we will change the temperatures of the bath and the system and analyze the effect of this. Then we will calculate the energy exchange between the baths and the system. But to probe the energy exchanges, let us first model the thermalization process as an energy preserving SWAP operation.

3.1 Thermalization as SWAP Operation

We have already studied the thermalization channel in section 2.3. However, we studied its effect on the target system only. In practise, thermalization occurs due to some coupling between the system and some thermal bath. Due to the macroscopic nature of the bath, the process of thermalization is not exactly known and is under debate [BCG+18, DC19]. There are several different approaches to it through quantum stochastic thermodynamics, collisional models and resource theories [BCG+18, GHR+16, RDCPL19].

Instead of going into the exact process of thermalization, we will simply model a global unitary U which will act between a target qubit $\rho \in \mathbb{H}^T$ and a thermal state $\zeta^\beta \in \mathbb{H}^B$ modelling the bath. Taking the thermal state for modelling a macroscopic bath works as, *on an average*, if we take out a qubit from such a bath to interact with, it will be a thermal state. So the statistics that we get at the end can be thought of as coming from the interactions of the system with copies of the thermal state taken from the same macroscopic bath.

For modelling thermalization, this global unitary should follow certain rules so that the interaction between the system and the bath is closed and the energy exchanges are correct:

- It must thermalize the qubit to the temperature of the thermal state
- It must change the bath state so that the final state of the bath shows whether heat has flown either out or into it, depending on the state of the target qubit.
- It must be energy preserving, i.e., it must commute with the total Hamiltonian of the target and the bath states ($H = H^T + H^B$) so that it does not pump in any energy: $[U, H] = 0$.

- It must not create or destroy any correlations for the two states.

Here, we take the target Hamiltonian as $H^T = \Delta |1\rangle \langle 1|$ and the Hamiltonian of the thermal state also as $H^B = \Delta |1\rangle \langle 1|$. An operation that fits the above constraints is the unitary SWAP gate (U_{SWAP}) [HZ11]. As shown in Fig. 3.1, the SWAP gate is a two input operation which SWAPs the input states $\rho_1 \otimes \rho_2 \in \mathbb{H}_1 \otimes \mathbb{H}_2$, to give $\rho_2 \otimes \rho_1 \in \mathbb{H}_1 \otimes \mathbb{H}_2$. Mathematically, this is an operation which maps the state ρ_1 to an isomorphic state ρ_2 in $\mathbb{H}_1$ and vice versa for ρ_2. Hence, its action on the target system and the bath is given by,

$$U^{\beta}_{SWAP}\left(\rho \otimes \zeta^{\beta}\right)\left(U^{\beta}_{SWAP}\right)^{\dagger} = \left(\zeta^{\beta} \otimes \rho\right). \tag{3.1}$$

For qubits, the SWAP gate is given by,

$$U_{SWAP} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{3.2}$$

As one can easily check, $[U^{\beta}_{SWAP}, H] = 0$ and it does not create or destroy any correlations. Looking at the energy change of the target and the bath, we see that,

$$\Delta E_T = \left(\text{Tr}\left[H^T \zeta^{\beta}\right] - \text{Tr}\left[H^T \rho\right]\right) = -\left(\text{Tr}\left[H^B \rho\right] - \text{Tr}\left[H^B \zeta^{\beta}\right]\right) = -\Delta E_B.$$

This shows that the target system thermalizes and the energy either goes into or out of the bath. Hence, the SWAP gate is a good model of thermalization for our purposes. The SWAP gate has been used to model thermalization in the quantum switch and has been experimentally implemented too [FV20, NZX+20]. Now that we have the tools to study the energetics of the bath, let us move on to understanding the effect of changing the temperatures of the two baths and the initial state.

3.2 Different Baths in ICO

The rest of this chapter will follow the protocol given below for the switch action and the subsequent analysis. By the end of this, we will have an idea of the energetics that takes place in the switch, at least for the SWAP gate model of thermalization. The analytical analysis for this is given in the appendix A.1. Following this we will study the work benefit of the switch.

3.2.1 Changing the Bath Temperature

The Protocol:

- The initial state of the system to be analyzed is taken to be a thermal state at inverse temperature β_S, i.e., ζ^{β_S}.

- There are two baths/reservoirs or equivalently two thermalizing channels that are part of the quantum switch, one hot and one cold, at inverse temperatures β_H and β_C respectively ($\beta_C > \beta_H$).
- The states of the baths are taken to be ζ^{β_H} for the hot bath and ζ^{β_C} for the cold bath.
- The control of the quantum switch is initialized in state $|+\rangle\langle+|$ unless specified.
- We will first measure the control in the $|\pm\rangle\langle\pm|$ basis (unless specified) and analyze the final system inverse temperature for different values of initial system and bath temperatures.
- Then we will calculate the local state of the baths and the system and analyze the energetics.
- Then we will see what happens when we measure the control in different directions. This direction will be given by,

$$\Psi(\theta,\varphi) = \cos\left(\frac{\theta}{2}\right)|0\rangle + e^{i\varphi}\sin\left(\frac{\theta}{2}\right)|1\rangle . \tag{3.3}$$

  We will change the θ and ϕ angles.

As there are various parameters, let us plot the final system temperature against varying bath temperature. This may give us an idea of how the relative temperatures of the bath affects the system temperature. An analytical analysis of this is done in the appendix A.1. In what follow, we have numerically plotted the results for different cases of the bath temperatures and control measurement results.

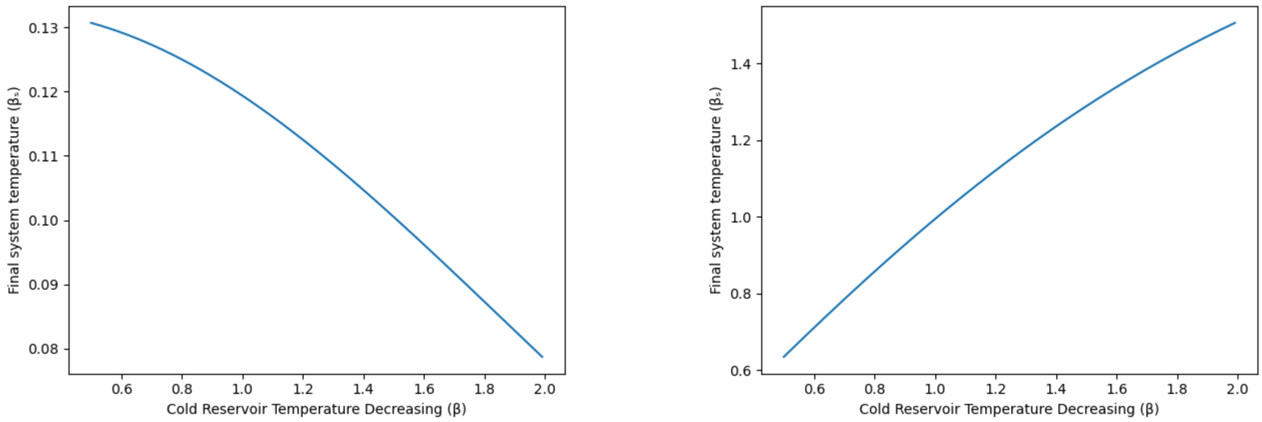
Some points to note are:-

- ☐ All values of $k_B T$ are of the order of the energy gap between say, two atomic levels, i.e., $k_B T \sim \hbar\omega = \Delta$. This places us in the quantum regime.
- ☐ We are only dealing with qubits.
- ☐ The action of the switch on the state of the system (ρ) and control ($|+\rangle\langle+|$), before measurement of the control is

$$S\left((|+\rangle\langle+|)\otimes\rho\right) = \frac{1}{2}|0\rangle\langle 0|\otimes\zeta^{\beta_C} + \frac{1}{2}|1\rangle\langle 1|\otimes\zeta^{\beta_H} + \frac{1}{2}\left(|0\rangle\langle 1|\otimes\zeta^{\beta_C}\rho\zeta^{\beta_H} + |1\rangle\langle 0|\otimes\zeta^{\beta_H}\rho\zeta^{\beta_C}\right). \tag{3.4}$$

Plots and analysis for varying bath temperatures:

- For a $|-\rangle$ measurement of the control: If $\beta_S = \beta_C$,i.e., we prepare the state at the temperature of the cooler reservoir, we will always get a final state temperature above the hotter reservoir temperature all the time.
- For a $|-\rangle$ measurement of the control: If $\beta_H = \beta_S$, i.e., if the initial state of the system is prepared at the higher temperature, we can still get a higher final temperature for appropriately chosen cold reservoir temperature β_C.



(a) *Control Measurement:* $|-\rangle$

(b) *Control Measurement:* $|+\rangle$

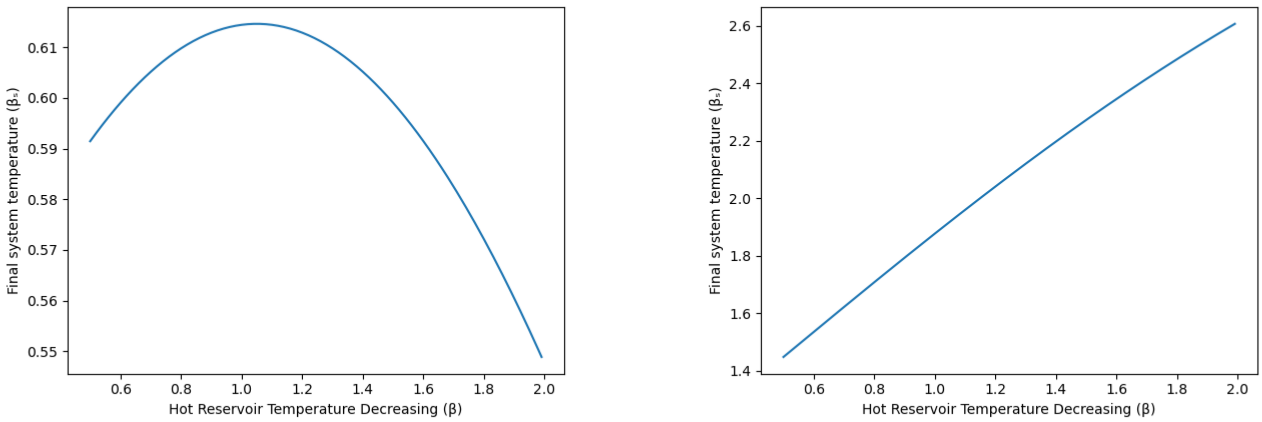
Figure 3.2: $0.5 < \beta_C = \beta_S < 2$, $\beta_H = 0.4$

- For a $|+\rangle$ measurement of the control: If $\beta_S = \beta_C$,i.e., we prepare the state at the temperature of the cooler reservoir, we can still get a lower final temperature for appropriately chosen cold reservoir temperature β_C.
- For a $|+\rangle$ measurement of the control: If $\beta_S = \beta_H$,i.e., we prepare the state at the temperature of the cooler reservoir, we can still get a higher final temperature(hotter than the hot reservoir) for appropriately chosen hot reservoir temperature β_H.
- For a $|-\rangle$ measurement of the control: If $\beta_S = \beta_C$,i.e., we prepare the state at the temperature of the cooler reservoir, we can get population inversion only when the hotter reservoir has a thermal state which has negative temperature.
- For a $|-\rangle$ measurement of the control: If $\beta_S > \beta_C$,i.e., we prepare the state at the temperature colder than that of the cold reservoir, we can get population inversion for appropriately chosen hot reservoir temperature β_H.

After this analysis it is still unclear "how much" energy each of the reservoirs contribute. So, let us now look at the state of the baths and calculate the energy change for each state due to the switch operation.

3.2.2 Looking at the Bath States

We want to analyze the energy change after measurement of the control for the system going into the QS and the thermal baths. For this we use the model of the thermalizing channel as a SWAP gate $\left(U^{\beta}_{SWAP}\right)$ which acts between the system state$\left(\rho \in \mathbb{H}^{S}\right)$ and a thermal state $\left(\zeta^{\beta}\right)$. Here, β_H and β_C will denote the inverse temperatures of the hot and cold baths $\left(\zeta^{\beta_H}\right)$ and $\left(\zeta^{\beta_C}\right)$ respectively. We initialize the control in the state $\sigma \in \mathbb{H}^{C}$. The superscript corresponds to the thermal bath that the operator will act on. For example, $U^{\beta_H}_{SWAP}$ acts between the system and the hot bath.



(a) *Control Measurement:* $|-\rangle$ **(b)** *Control Measurement:* $|+\rangle$

Figure 3.3: $0.5 < \beta_H = \beta_S < 2$, $\beta_C = 2.1$

The initial state we start with is: $\left(\sigma \otimes \rho \otimes \zeta^{\beta_H} \otimes \zeta^{\beta_C}\right)$. The Hamiltonian for the system and the baths is taken to be $H = \Delta |1\rangle \langle 1|$, and we assign zero Hamiltonian to the control for these calculations. The complete details of all the calculations are given in the appendix A.2.

The operator for the quantum switch with the $U^{\beta_H}_{SWAP}$ and $U^{\beta_C}_{SWAP}$ is:

$$W = |0\rangle \langle 0| \otimes U^{\beta_H}_{SWAP} U^{\beta_C}_{SWAP} + |1\rangle \langle 1| \otimes U^{\beta_C}_{SWAP} U^{\beta_H}_{SWAP}.$$

which is nothing but a unitary operation. The final state after the four party switch interaction is given by,

$$\begin{aligned} S(\sigma \otimes \rho \otimes \zeta^{\beta_H} \otimes \zeta^{\beta_C}) = W(\sigma \otimes \rho \otimes \zeta^{\beta_H} \otimes \zeta^{\beta_C})W^\dagger \\ =& \sigma_{00} |0\rangle \langle 0| \otimes \left(\zeta^{\beta_H} \otimes \zeta^{\beta_C} \otimes \rho\right) \\ &+ \sigma_{10} |1\rangle \langle 0| \otimes U^{\beta_C}_{SWAP} U^{\beta_H}_{SWAP} \left(\rho \otimes \zeta^{\beta_H} \otimes \zeta^{\beta_C}\right) \left(U^{\beta_C}_{SWAP}\right)^\dagger \left(U^{\beta_H}_{SWAP}\right)^\dagger \\ &+ \sigma_{01} |0\rangle \langle 1| \otimes U^{\beta_H}_{SWAP} U^{\beta_C}_{SWAP} \left(\rho \otimes \zeta^{\beta_H} \otimes \zeta^{\beta_C}\right) \left(U^{\beta_H}_{SWAP}\right)^\dagger \left(U^{\beta_C}_{SWAP}\right)^\dagger \\ &+ \sigma_{11} |1\rangle \langle 1| \otimes \left(\zeta^{\beta_C} \otimes \rho \otimes \zeta^{\beta_H}\right). \end{aligned}$$

As you can clearly see, if we trace over the control and system, we will get a bath states to be ρ depending on the causal order. This agrees with the fact that the measurement of the control is needed to see an interference effect. However, let us measure the control in the $|\pm\rangle \langle\pm|$ basis and check the effect on the local bath states. We will take the system to be in the thermal state ζ^{β_S}. The state of the hot bath after the $|\pm\rangle \langle\pm|$ measurement of the control is,

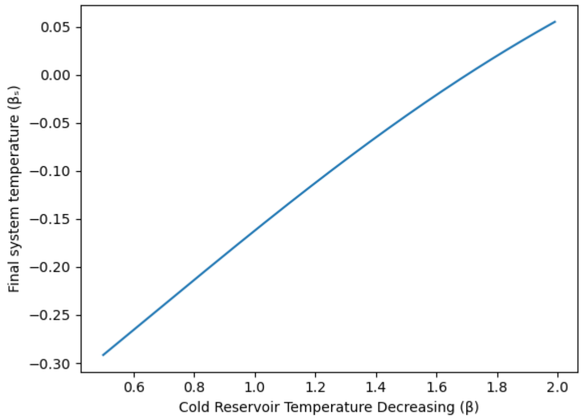


Figure 3.4: *Population inversion:* $0.5 < \beta_C < 2 < \beta_S = 2.1$*,* $\beta_H = 0.4$*, Control Measurement:* $|-\rangle$

$$\zeta^H_{f\pm} = \frac{\langle \pm | \operatorname{Tr}_{S,C}\left(S(|+\rangle\langle +| \otimes \zeta^{\beta_S} \otimes \zeta^{\beta_H} \otimes \zeta^{\beta_C})\right)|\pm\rangle}{\operatorname{Tr}\left(\langle \pm | \operatorname{Tr}_{S,C}\left(S(|+\rangle\langle +| \otimes \zeta^{\beta_S} \otimes \zeta^{\beta_H} \otimes \zeta^{\beta_C})\right)|\pm\rangle\right)}$$
$$= \frac{1}{Z^H_\pm}\left(\frac{1}{4}\left(\zeta^{\beta_S}\right) + \frac{1}{4}\left(\zeta^{\beta_C}\right) \pm \frac{1}{2}\left(\zeta^{\beta_S}\zeta^{\beta_H}\zeta^{\beta_C}\right)\right), \quad (3.5)$$

where, the partition function is,

$$Z^H_\pm = \operatorname{Tr}\left(\frac{1}{4}\left(\zeta^{\beta_S}\right) + \frac{1}{4}\left(\zeta^{\beta_C}\right) \pm \frac{1}{2}\left(\zeta^{\beta_S}\zeta^{\beta_H}\zeta^{\beta_C}\right)\right) = \frac{1}{2} \pm \frac{1}{2}\operatorname{Tr}\left(\zeta^{\beta_S}\zeta^{\beta_H}\zeta^{\beta_C}\right)$$
$$= \frac{1}{2} \pm \left(\frac{p^{\beta_S}p^{\beta_H}p^{\beta_C}}{2} + \frac{(1-p^{\beta_S})(1-p^{\beta_H})(1-p^{\beta_C})}{2}\right).$$

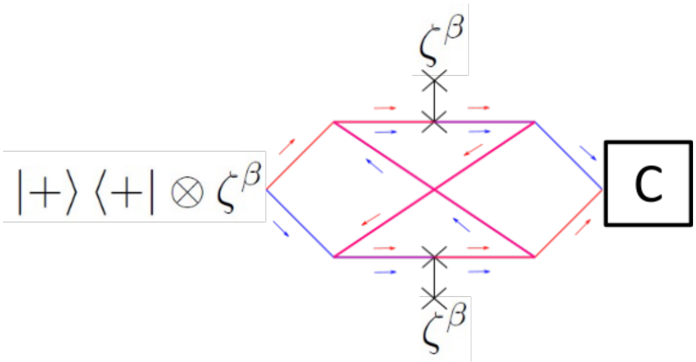
The state of the cold bath is,

$$\zeta^C_{f\pm} = \frac{\langle \pm | \operatorname{Tr}_{S,H}\left(S(|+\rangle\langle +| \otimes \zeta^{\beta_S} \otimes \zeta^{\beta_H} \otimes \zeta^{\beta_C})\right)|\pm\rangle}{\operatorname{Tr}\left(\langle \pm | \operatorname{Tr}_{S,H}\left(S(|+\rangle\langle +| \otimes \zeta^{\beta_S} \otimes \zeta^{\beta_H} \otimes \zeta^{\beta_C})\right)|\pm\rangle\right)}$$
$$= \frac{1}{Z^C_\pm}\left(\frac{1}{4}\left(\zeta^{\beta_S}\right) + \frac{1}{4}\left(\zeta^{\beta_H}\right) \pm \frac{1}{2}\left(\zeta^{\beta_S}\zeta^{\beta_C}\zeta^{\beta_H}\right)\right), \quad (3.6)$$

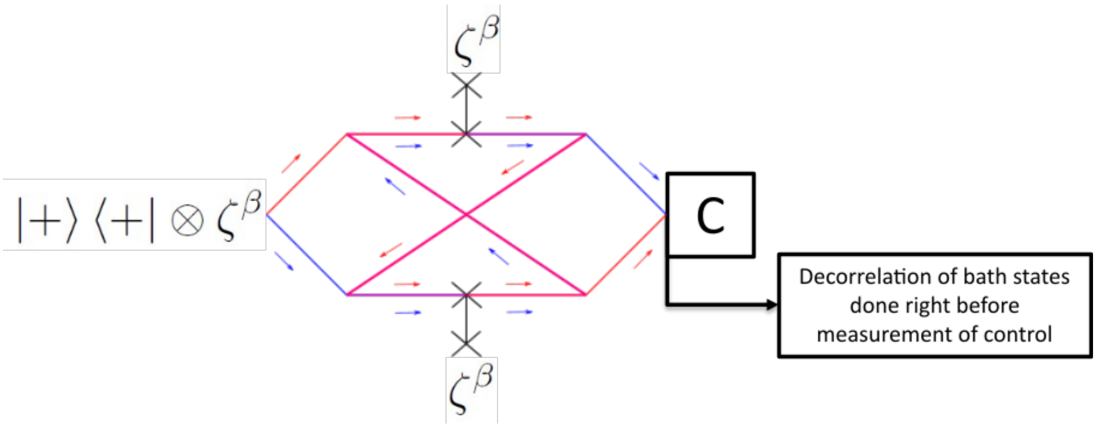
where, the partition function is,

$$Z^C_\pm = \operatorname{Tr}\left(\frac{1}{4}\left(\zeta^{\beta_S}\right) + \frac{1}{4}\left(\zeta^{\beta_H}\right) \pm \frac{1}{2}\left(\zeta^{\beta_S}\zeta^{\beta_C}\zeta^{\beta_H}\right)\right) = \frac{1}{2} \pm \frac{1}{2}\operatorname{Tr}\left(\zeta^{\beta_S}\zeta^{\beta_C}\zeta^{\beta_H}\right)$$
$$= \frac{1}{2} \pm \left(\frac{p^{\beta_S}p^{\beta_C}p^{\beta_H}}{2} + \frac{(1-p^{\beta_S})(1-p^{\beta_C})(1-p^{\beta_H})}{2}\right).$$

Now if we start the system and the two baths at the same temperature as in section 2.3, we see that they all collapse to the same state given by (2.13)! This



(a) *Thermalization using SWAP in the quantum switch.*



(b) *Decorrelating the bath states before the measurement of the control.*

Figure 3.5: *Schematic representation of thermalization SWAP model in the quantum switch*

looks like, depending on the measurement of the control, energy is either going out of all three systems or flowing into all three systems. Hence, there seems to be some inconsistency. In appendix A.2, we see that even if we decorrelate the bath states before the measurement of the control as in Fig. 3.5 we see that the final system state does not change. Hence, the energy must not be coming from the baths at all. It looks like the measurement of the control is providing all the energy. So let us first analyze the effect of the measurement of the control in different directions.

3.2.3 Measurement of the Control in Different Directions

Let us now see the effect of measuring the control in different directions as in (3.3). In Fig. 3.6 we see that,

- $\theta = \frac{\pi}{2} \implies |-\rangle$ measurement of control.
- $\theta = \pi \implies |1\rangle$ measurement of control, gives action of hot reservoir channel after cold reservoir channel, i.e., final state is at inverse temperature β_H.
- Here also we see population inversion, even though the system is prepared at temperature of cold reservoir(β_C) if the control is measured along $\theta = \frac{3\pi}{4}$

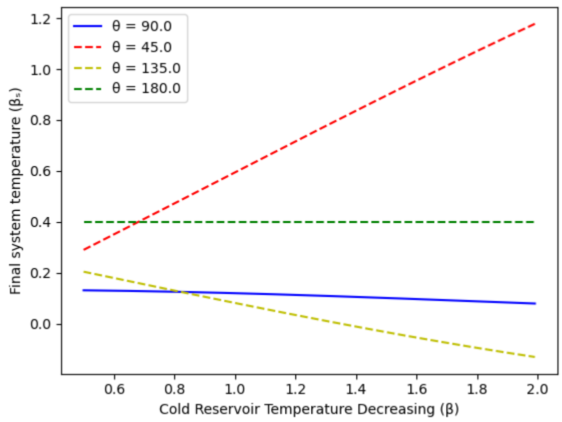


Figure 3.6: *Different control measurement angles* $\left(\Psi(\theta,\varphi) = cos\left(\frac{\theta}{2}\right)|0\rangle - sin\left(\frac{\theta}{2}\right)|1\rangle\right)$*:* $0.5 < \beta_C = \beta_S < 2$ *,* $\beta_H = 0.4$*;* $\varphi = \pi$

direction and the temperature of the hot and cold reservoirs are relatively far.

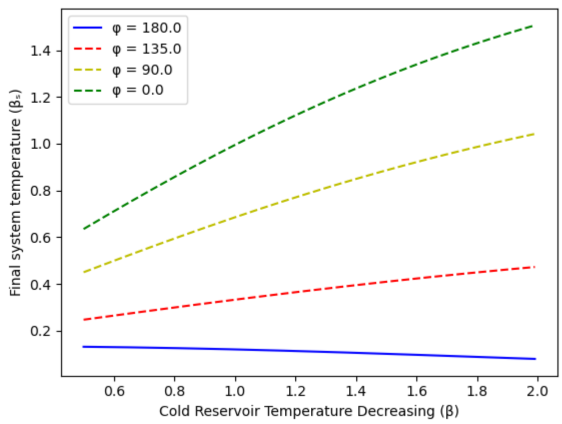


Figure 3.7: *Different control measurement angles* $\left(\Psi(\theta,\varphi) = \frac{1}{\sqrt{2}}|0\rangle + \frac{e^{i\varphi}}{\sqrt{2}}|1\rangle\right)$*:* $0.5 < \beta_C = \beta_S < 2$ *,* $\beta_H = 0.4$*;* $\theta = \frac{\pi}{2}$

In Fig. 3.7 we see that,

- $\varphi = \pi \implies |-\rangle$ measurement of control.
- $\varphi = 0 \implies |+\rangle$ measurement of control.
- The phase of the control gives a continuous change from the hottest $|-\rangle$ measurement to the coldest $|+\rangle$ measurement.

We see that θ gives a more drastic change in the final system temperature over the phase φ. It is very clear that the way an experimentalist measures the control has a drastic effect on the final system temperature. It is hence, this measurement that is controlling the energetics between the system, the baths and the control qubit.

In appendix A.2, we use this thermalization model and calculate the energy due to the measurement of the control (Δ_M) to be,

$$\Delta_M = \Delta_S = \left(\frac{1-2Z}{2Z}\right) \mathrm{Tr}\left(H\zeta^{\beta}\right) \pm \frac{1}{2Z}\mathrm{Tr}\left(H\left(\zeta^{\beta}\right)^3\right),$$

for the simple case where the initial temperatures are the same. We see that it is equal to the energy change of the system (Δ_S). Hence, at least for this model, the energy seems to come only from the measurement. Now, let us rigorously prove that, using the switch channel if there is any activation in free energy for the target system, it is due to energy coming from the measurement of the control.

3.3 Work Benefit of the Switch Channel

Measurement of a system makes the dynamics of the system open. As we have seen in section 1.3.1, we always need a measurement device which correlates with the system and makes the measurement of the desired property. We have also seen that storing this measurement result also has an associated work cost. To describe the energetics of the switch along with the measurement of the control, we will need to take into account all the parties that take part in it. Using this, we need to understand what it means to extract energy using a channel, i.e., the *work benefit* of the channel.

In this section, we will follow the ideas of Purves et al. [PS20]. We first, briefly review the concepts used from this article, and then find the work benefit of the quantum switch over a classical switch, i.e., over the case where the occurrence of **A** $\preceq$ **B** and **B** $\preceq$ **A** are conditioned by the result of a classical coin toss. This will give us a handle on classifying the advantage or disadvantage of the quantum switch. Then we will study the work benefit of measurement channels and prove that the activation of free energy is only due to the work benefit of the control measurement channel.

3.3.1 Work Benefit of Channels

Let us fix the notation and the setup for what follows next. We prepare a system $\rho^S \in \mathbb{H}^S$ with its intrinsic Hamiltonian H^S and a control $\rho^C \in \mathbb{H}^C$ with Hamiltonian H^C. We start with $\rho^T = \rho^C \otimes \rho^S$ and call it the "target" with Hamiltonian for it being $H_T = H^C \otimes \mathbb{I} + \mathbb{I} \otimes H^S$. The subsystem is denoted by the superscript letter. We have a reservoir at temperature $T(\beta^{-1})$ containing thermal states $\tau^B \in \mathbb{H}^{\mathrm{B}}$ with Hamiltonian H^B. A channel C which acts on the target. A subscript such as C_{Sw} denotes the kind of channel, here it being the switch channel. By $S(\rho)$ we denote the Von Neumann entropy of ρ, by $F(\rho)$, its free energy and the diagonal matrix of ρ, $\sum_s \langle s|\,\rho\,|s\rangle\,|s\rangle\,\langle s|$ by $\Delta(\rho)$. Now, the work benefit of a channel is the maximum amount of energy that can be extracted using it. Hence, as we will see, it is maximized over all states that it can act on. For the purposes of definition, we do not restrict to the case of the switch channel. Hence, the target state and the channel could be arbitrary. To define the work benefit of a channel, we follow the protocol given below,

(a) Target and bath are initialized in uncorrelated state: $\rho = \rho^T \otimes \tau^B$, for which we don't assign any work benefit/cost: $W_a = 0$.

(b) A channel is applied only on the target state: $\sigma = C[\rho^T] \otimes \tau^B$. The work benefit of this operation is $W_b = \text{Tr}\left(H_T \rho^T - H_T C[\rho^T]\right)$.

(c) Unitary interaction is applied between the target and the bath to reset the state of the target: $\sigma' = U \sigma U^\dagger = U\left(C[\rho^T] \otimes \tau^B\right) U^\dagger$ such that $\text{Tr}_B\left(\sigma'\right) = \text{Tr}_B\left(U\left(C[\rho^T] \otimes \tau^B\right) U^\dagger\right) = \rho^T = \rho^C \otimes \rho^S$. This unitary commutes with all Hamiltonians and hence, does not contribute to any energy. For this we see that the work benefit is bounded by the change in free energy of the target: $W_c \leq \Delta F_T = F\left(\sigma^T\right) - F\left(\rho^T\right)$ (refer appendix B.1.2).

(d) Work benefit is got by optimizing over all initial states:

$$W_C^{Benefit} = \max_{\rho^T}\left\{W_a + W_b + W_c\right\} = \max_{\rho^T} \beta^{-1}\left(S\left(\rho^T\right) - S\left(C\left(\rho^T\right)\right)\right). \tag{3.7}$$

All the calculations of this section for the case of the switch are detailed in appendix B. Before we go on to comparing the quantum and the classical switch, let us consider the cost of resetting the ancillae(environment) that are used in the implementation of each channel. These ancillae also need to be reset to their initial state.

Resetting the Environment

Any channel is simulated using an ancilla $\left(\rho^E\right)$ and a global unitary U^{TE}. We give this ancilla a zero Hamiltonian. So, we have,

$$C\left(\rho^T\right) = \text{Tr}_E\left(U^{TE}\left(\rho^T \otimes \rho^E\right)\left(U^{TE}\right)^\dagger\right) = \text{Tr}_E\left(\xi\right).$$

For this unitary interaction we have,

$$S\left(\rho^T \otimes \rho^E\right) = S\left(\rho^T\right) + S\left(\rho^E\right) = S\left(\xi\right).$$

Using subadditivity of entropy [Wit20] we have,

$$\begin{aligned} S\left(\rho^T\right) + S\left(\rho^E\right) = S\left(\xi\right) &\leq S\left(\xi^T\right) + S\left(\xi^E\right) \\ \implies 0 \leq& \Delta S_T + \Delta S_E. \end{aligned}$$

Reset is done using a thermal state, hence, the work benefit of reset of the ancilla is bounded by the change in free energy,

$$W_{reset}^{Benefit} \leq -\beta^{-1}\Delta S_E. \tag{3.8}$$

Hence, the bound for work benefit for the complete cycle is:

$$W_C^{Benefit} + W_{reset}^{Benefit} \leq 0. \tag{3.9}$$

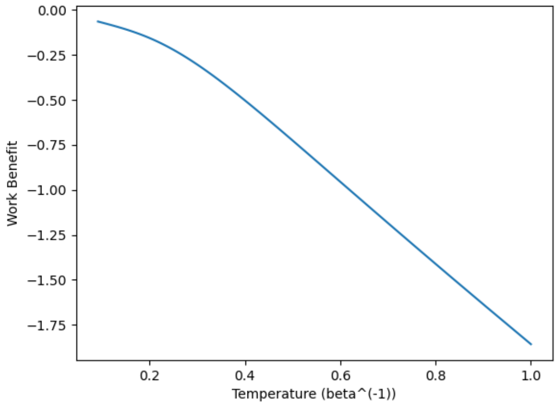


Figure 3.8: *Work Benefit of the classical vs the quantum switch. The control is initialized the $|+\rangle\langle+|$ state.The baths and system are initialized in the thermal state ζ^{β}. As $W_{Cl}^{Benefit} = 0$ and $W_{Sw}^{Benefit} < 0$, the classical switch seems to do better than the quantum switch in this example.*

However, for the case of the switch, there are two ancillae or environments, ρ^E and ρ^F. From B.1, we see that,

$$W_{Sw}^{Benefit} + W_{reset}^{Benefit} \leq -\beta^{-1}\left(\Delta S_T + \Delta S_E + \Delta S_F\right) \leq -\beta^{-1}\left(I\left(\xi^{TF}\right) + I\left(\xi^{EF}\right)\right), \tag{3.10}$$

where, the mutual information I is a positive quantity. Hence, we see that, due to correlations created, the switch seems to give a worse bound for work benefit, if the environment is taken into account.

3.3.2 Classical vs the Quantum Switch

Let us imagine that the cost of reset is zero. This can be due to thermalization with some environment, due to which, the experimentalist need not provide any energy. So now, we will compare the work benefit of the classical switch and the quantum switch. The calculations are detailed in appendix B.2. The work benefit of the classical switch channel is given by,

$$W_{Cl}^{Benefit} = \beta^{-1}\left(S\left[\rho^S\right] - S\left[\rho_{00}^C C_A\left(C_B\left(\rho^S\right)\right) + \rho_{11}^C C_B\left(C_A\left(\rho^S\right)\right)\right] + I\left[\sigma_{Cl}^{CS}\right]\right). \tag{3.11}$$

The work benefit of the quantum switch channel is given by,

$$W_{Sw}^{Benefit} = W_{Cl}^{Benefit} + \beta^{-1}\left(\Delta \boldsymbol{C}\left[\rho^C\right] + I\left[\sigma^{CS}\right] - I\left[\sigma_{Cl}^{CS}\right]\right), \tag{3.12}$$

where, $\Delta \boldsymbol{C}$ is the change in the coherence function, defined by $\boldsymbol{C}[\rho] = (S[\Delta[\rho]] - S[\rho])$. Now, that we can relate the two work benefit, let us take the particular example of the baths in the switch as in section 2.3. We don't maximize over all states in this case, but we calculate the work benefit for this scenario while using the classical or the quantum switch. This is given in Fig. 3.8. We see that, the work benefit of the quantum switch is negative while that of the classical switch is zero. This indicates that the classical switch may be better for work extraction. This may be due to creation and consumption of "quantum" correlations in the quantum switch operation which is not done by the classical switch.

3.3.3 Work Benefit of Measurement Channel

For the measurement channel, the same protocol as before is followed, but here, conditional unitaries with probability p_i are applied to reset the target using the bath in step (c) depending on the output state after step (b). Here, the work benefit of step (b) and step (c) change as:

$$\begin{aligned} W_b = \mathrm{Tr}\left(H_T\rho^T - \sum_i p_i H_T C_i[\rho^T]\right) = \mathrm{Tr}\left(H_T\rho^T - \sum_i p_i H_T \sigma_i^T\right), \\ W_c \leq \Delta F_T = \sum_i p_i F\left(\sigma_i^T\right) - F\left(\rho^T\right). \end{aligned} \tag{3.13}$$

Work benefit for the measurement channel is:

$$W_M^{Benefit} = \max_{\rho^T}\left\{W_a + W_b + W_c\right\} = \max_{\rho^T}\beta^{-1}\left(S\left(\rho^T\right) - \sum_i p_i S\left(\frac{C_i\left(\rho^T\right)}{p_i}\right)\right). \tag{3.14}$$

Free energy activation in the switch

Finally, let us apply this to the protocol of the refrigeration cycle in section 2.3. Here, we check the work benefit for the switch measurement channel. The initial state is,

$$\rho^T = \left|+\right\rangle\left\langle+\right| \otimes \zeta^\beta.$$

The probabilities are,

$$p_\pm = \mathrm{Tr}\left(\frac{\zeta^\beta \pm \left(\zeta^\beta\right)^3}{2}\right) = Z_\pm.$$

The action of the measurement channel is given by,

$$C_\pm\left(\left|+\right\rangle\left\langle+\right| \otimes \zeta^\beta\right) = \left|\pm\right\rangle\left\langle\pm\right| \otimes \frac{\zeta^\beta \pm \left(\zeta^\beta\right)^3}{2} = \left|\pm\right\rangle\left\langle\pm\right| \otimes Z_\pm \zeta^{\beta^\pm}.$$

Now, we have,

$$S\left(\frac{C_\pm\left(\left|+\right\rangle\left\langle+\right| \otimes \zeta^\beta\right)}{p_\pm}\right) = S\left(\zeta^{\beta^\pm}\right),$$

and so,

$$W_M^{Benefit} = \beta^{-1}\left(\sum_{\pm} p_\pm S\left(\zeta^\beta\right) - \sum_{\pm} p_\pm S\left(\zeta^{\beta^\pm}\right)\right)$$
$$= \sum_{\pm} p_\pm\left[(E_\beta - E_{\beta^\pm}) - (F_\beta - F_{\beta^\pm})\right] = \sum_{\pm} p_\pm\left[\Delta E_\pm - \Delta F_\pm\right],$$

$$\langle \Delta F\rangle_\pm = -W_M^{Benefit}. \tag{3.15}$$

This is true for thermalizations at different temperatures also. This shows that the activation in free energy that we get from the switch channel is due to the work cost of measurement of the control only. We see that when the control measurement results are conditionally used (like in the refrigeration cycle) the activation in free energy is due to the interaction with the thermal qubit used in the measurement channel. The baths in the switch do not contribute to this, and hence, do not extract or release heat from or to the system. Physically this energy can be thought of to come from some energy source (light pulses, etc.) that are used to make the measurement.

3.4 Conclusions and Inferences

In this chapter, we analysed the energetic of all the quantum systems that take part in a thermodynamic protocol in an ICO. We saw that it is the measurement of the control that is providing the energy which activates the free energy of the thermal state that is used in the switch. We proved this, by modelling the thermalization process in the switch as SWAP gates which showed that the baths did not provide any energy. Also, we calculated the work benefit of measurement channel for the switch and saw that, it was this that provided the energy. Hence, the results such as in [FV20, NZX+20] or section 2.3 should questioned and further investigated. Moreover, we saw that the creation of quantum correlations in the switch, makes the work extraction worse as the work benefit of the classical switch is greater than the quantum switch, at least for our example.

In what follows, we will depart from our example of the thermal operations in ICO, and study the thermodynamics of the switch in a different manner. We will use ideas of stochastic thermodynamics and find fluctuation theorems for the switch. This will help us analyze what are the truly quantum thermodynamic phenomenon that take place in the quantum switch.

Chapter 4

Fluctuations in the Quantum Switch

The law that entropy always increases holds, I think, the supreme position among the laws of Nature. If someone points out to you that your pet theory of the universe is in disagreement with Maxwell's equations - then so much the worse for Maxwell's equations. If it is found to be contradicted by observation - well, these experimentalists do bungle things sometimes. But if your theory is found to be against the Second Law of Thermodynamics I can give you no hope; there is nothing for it to collapse in deepest humiliation.

— **Sir Arthur Stanley Eddington**
New Pathways in Science

The second law of thermodynamics has proved universal and continues to do so. Even in the quantum domain where certain entropies can become negative, and fluctuations of observables can exceed their average values, it survives. Stochastic thermodynamics lays the foundations to make this possible. By changing traditionally averaged over thermodynamics quantities such as heat and work to stochastic variables, it is able to probe non-equilibrium processes at a microscopic scale through certain fluctuation theorems. These fluctuation theorems give rise to generalized second laws of thermodynamics beyond averages, and to new ways of measuring changes in free energy; the most famous of them being the Jarzynski equality [Jar97] given by,

$$\left\langle e^{-\beta(W-\Delta F)}\right\rangle = 1, \tag{4.1}$$

where, W is the stochastic work done during a non-equilibrium process and ΔF is the change in equilibrium free energy. The second law follows directly from Jensen's inequality. In the classical regime, the averaging is over phase space trajectories. Readers are directed to [DC19] for a short review on stochastic thermodynamics.

In the quantum regime, there is no unique notion of trajectories. Hence, there are several different approaches to studying non-equilibrium phenomenon in this regime [BCG+18]. In this chapter, we will study two point trajectories constructed by energy measurements and construct corresponding work distributions. Using

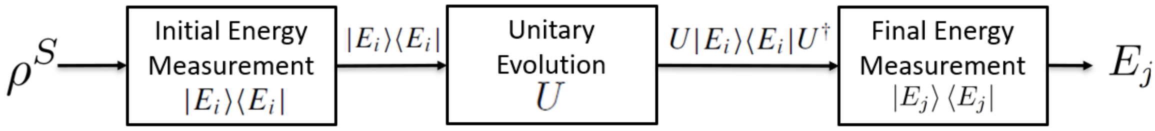


Figure 4.1: *The Two-Point Measurement protocol.*

this we will come to a fluctuation theorem. As we will see, this fluctuation theorem will let us probe into quantum phenomenon and give us information about the incompatibility of observables. Finally with this probe, we will peek at the truly quantum phenomenon in the quantum switch operation. But first let us study how to approach work in this theory as a stochastic quantity or a random variable.

4.1 Work as a Stochastic Variable

Let us consider a simple scenario where we have a quantum system $\rho \in \mathbb{H}^S$ with an initial Hamiltonian $H(0)$ which evolves under the action of a unitary U. Moreover, let us also consider that the Hamiltonian evolution is either time independent or cyclic, i.e., if U operates for time τ, then $H(0) = H(\tau)$, in the Schrodinger picture. The final state of the system is of course, $U\rho U^\dagger$. Our problem is to find how much work is done due to this unitary driving of the system.

One of the first and simplest approach to answer this was the two-point measurement(TPM) protocol. To study this we will now move into the Heisenberg picture, where the operators evolve rather than the state. Let us assume the initial Hamiltonian is given by $H = \sum_i E_i |E_i\rangle\langle E_i|$ and the final Hamiltonian is given by $U^\dagger H U = H' = \sum_j E_j |E_j'\rangle\langle E_j'|$. As shown in Fig. 4.1 this protocol makes two projective energy measurements, first of E_i with $|E_i\rangle\langle E_i|$, then after the unitary of E_j with $|E_j'\rangle\langle E_j'|$. The work value is hence, defined as,

$$w_{ji} = E_j - E_i. \tag{4.2}$$

As measurements are intrinsically probabilistic in the quantum mechanics, the work value becomes a random variable. Using this work value we can construct the TPM probability distribution p_{ji}^{TPM} and further the work distribution $p^{TPM}(w)$,

$$\begin{aligned} p_{ji}^{TPM} &= \langle E_i|\rho|E_i\rangle |\langle E_j|U|E_i\rangle|^2, \\ p^{TPM}(w) &= \sum_{i,j} p_{ji}^{TPM}\delta(w - w_{ji}). \end{aligned} \tag{4.3}$$

This approach of defining work has several advantages. It is similar to the classical definition of work and can reproduce the classical Jarzynski equality (4.1) for initial thermal states, which seems to validate the protocol. However, this approach has several crucial disadvantages too. First notice that, the average work calculated using any definition of work value should be equal to $\langle W\rangle = \mathrm{Tr}[(H' - H)\rho]$, which is just the average energy change. This is because the evolution is closed and no other party except ρ is involved. However, if the average work is calculated using (4.3), this does not equal $\langle W\rangle$ in general. Moreover, the first measurement destroys

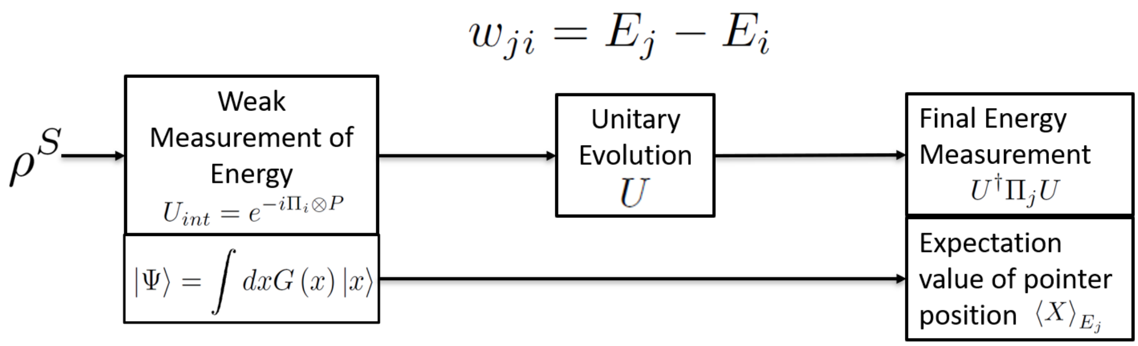


Figure 4.2: *The Modified TPM protocol with weak measurement and post-selection.*

all the quantum coherence in the energy basis of the initial state. As we can see, the issue with the TPM protocol is measurement. As we have seen in the previous chapters, measurements make the dynamics of the system open and leads to loss of information due to some coupling with an ancillary (environment, measurement apparatus, etc); it can even provide energy to the system being measured. Hence, even though it is useful for the initial Gibb's state, the TPM protocol is not useful to define work values when there is quantum coherence[1] in the initial state. It is evident that another approach is needed to define work values, one that involves non-invasive work measurements. This can be done using weak measurements as we will see in the next section. We will derive a quasiprobability distribution for the work values. For a concise review of weak measurements, readers are referred to [TC13].

4.2 The Margenau-Hill Distribution

In this section, we will follow the publication by Lostaglio [Los18]. This modifies the TPM protocol to account for the invasive nature of the projective measurement using a weak-measurement scheme. The details of all calculations of this section are given in appendix C.1. The protocol is as follows:

- A measuring apparatus is initialized in the state $|\Psi\rangle = \int dxG(x)|x\rangle$, where $G(x)$ is a Gaussian state with some width(standard deviation) σ.
- The energy of the system ρ is weakly measured by coupling it to the apparatus using the unitary $U_{int} = e^{-i\Pi_i \otimes P}$, where P is the momentum operator which translates the pointer states of the apparatus when measuring the energy.
- The unitary evolution U is performed on ρ and a final projective measurement is done to post-select the energy E_j.
- The expectation value of the position of the apparatus $\langle X \rangle_{E_j}$ is now measured, where, the subscript depicts post-selection.

[1] From here onwards, when we talk about coherence, it will always mean superposition in the energy basis, unless specified.

Fig. 4.2 summarizes the protocol. In the strong measurement regime, where the width of $G(x)$ goes to zero ($\sigma \to 0$), we have,

$$\langle X \rangle_{E_j} \to \frac{\text{Tr}\left(\rho \left|E_i\right\rangle \left\langle E_i\right|\right) \text{Tr}\left(U^\dagger \left|E_j\right\rangle \left\langle E_j\right| U \left|E_i\right\rangle \left\langle E_i\right|\right)}{\text{Tr}\left(\rho U^\dagger \left|E_j\right\rangle \left\langle E_j\right| U\right)} = \frac{\text{Tr}\left(\rho \Pi_i\right) \text{Tr}\left(U^\dagger \Pi_j U \Pi_i\right)}{\text{Tr}\left(\rho U^\dagger \Pi_j U\right)} = \frac{p_{ji}^{TPM}}{p_f\left(E_j\right)}, \tag{4.4}$$

where, $p_f(E_j)$ is the probability of the final energy measurement being E_j. Notice that $p_{ji}^{TPM} = \langle X \rangle_{E_j} p_f(E_j)$ and hence, $\langle X \rangle_{E_j}$ can be interpreted as the conditional probability of first energy measurement being E_i given that the final energy measurement is E_j. Now in the weak-measurement regime where, the width of the Gaussian $G(x)$ goes to infinity ($\sigma \to \infty$), we have,

$$\langle X \rangle_{E_j} \to \frac{\text{ReTr}\left[\rho \Pi_i U^\dagger \Pi_j U\right]}{\text{Tr}\left[\rho U^\dagger \Pi_j U\right]} = \frac{p_{ji}^{MH}}{p_f\left(E_j\right)}, \tag{4.5}$$

where, p_{ji}^{MH} is the Margenau-Hill(MH) Distribution. Again, interpreting $\langle X \rangle_{E_j}$ as the conditional probability, p_{ji}^{MH} is interpreted as the joint probability of the initial measurement being E_i and the final measurement being E_j in the weak-measurement regime. Here, again the work values are identified as 4.2. But now, the distribution for work is given by,

$$p^{MH}(w) = \sum_{i,j} p_{ji}^{MH} \delta\left(w - w_{ji}\right). \tag{4.6}$$

There are a few points to note here. First, p^{MH} is not a probability distribution. This distribution shows the following properties:

- Normalization condition: $\sum_{i,j} p_{ji}^{MH} = 1$,
- Marginal distributions: $\sum_i p_{ji}^{MH} = \text{Tr}\left[\rho U^\dagger \Pi_j U\right] = p_f(E_j)$, and $\sum_j p_{ji}^{MH} = \text{Tr}\left[\rho \Pi_i\right] = p_0(E_i)$,
- It can be negative for certain states.

The MH distribution shows properties of a probability distribution, but it can be negative for certain states. Hence, it is a quasi-probability distribution. Due to this, the work distribution $p^{MH}(w)$ can also be negative in certain regions. Nevertheless, this work distribution gives the correct average energy and also, it reproduces the TPM work distribution for non-coherent initial states, i.e.,

$$p^{MH}(w) = p^{TPM}(w) \, \forall \rho \text{ such that } [\rho, H] = 0.$$

Hence, this quasiprobability distribution can also reproduce results such as the Jarzynski equality. The quasiprobability distributions are widely studied not only because they reproduce standard results, but they offer something new. It has been shown that the negativity of the distribution in fact indicates a purely non-classical phenomenon taking place, a phenomenon which cannot be replicated using classical systems and operations [Spe08]. [2] We can rewrite the MH distribution into a part that corresponds to a probability distribution and a 'quantum' part due to

[2] Non-negativity indicates non-contextuality, i.e., non-negative distributions can be simulated using statistics produced by classical systems and operations, as detailed in Spekkens' paper.

which the negativity may arise. As shown in the publication by Johansen [Joh07], p_{ji}^{MH} can be rewritten as,

$$\mathrm{ReTr}\left[\rho E_i \Pi_j\right] = \mathrm{Tr}\left[\rho \Pi_i U^\dagger \Pi_j U \Pi_i\right] + \frac{1}{2}\mathrm{Tr}\left[\left(\rho - \rho_i'\right) U^\dagger \Pi_j U\right], \tag{4.7}$$

where,

$$\rho_i' = \Pi_i \rho \Pi_i + \left(\mathbb{I} - \Pi_i\right)\rho\left(\mathbb{I} - \Pi_i\right). \tag{4.8}$$

As you can see, $\mathrm{Tr}\left[\rho \Pi_i U^\dagger \Pi_j U \Pi_i\right]$ is a probability distribution of measuring energy E_i then E_j after the unitary evolution U. This is like the 'classical' part of the distribution as it is always positive. $\frac{1}{2}\mathrm{Tr}\left[\left(\rho - \rho_i'\right) U^\dagger \Pi_j U\right]$ leads to the negativity in the distribution. ρ_i' can be viewed as a post-measurement state for a two-outcome measurement, where when E_i is measured with Π_i, the outcome is 1 and when any other energy is measured with any other projector orthogonal to Π_i, i.e. $(\mathbb{I} - \Pi_i)$, the outcome is 0. The non-classical aspect of the distribution is also noted in the kind of fluctuation theorem that arises from it. By using this distribution to average over the exponentiated work values, we get a number that is not 1 as in (4.1), but a higher or a lower number Υ. This gives the fluctuations due to presence of coherence in the system and/or evolution. We will use equation (4.7) to separate the non-classical part of the fluctuations from Υ.

But first, without further ado, let us derive the fluctuation theorem containing this interesting Υ term.

4.3 Fluctuation Theorem

The derivation of the fluctuation theorem for the MH distribution was first done by Allahverdyan [All14] as shown below,

$$\begin{aligned}\left\langle e^{-\beta w_{ji}}\right\rangle = \sum_{i,j} p_{ij}^{MH} e^{-\beta(E_j - E_i)} &= \mathrm{Re}\sum_{i,j}\mathrm{Tr}\left[\rho e^{\beta E_i}\Pi_i e^{-\beta E_j} U^\dagger \Pi_j U\right] \\ &= \mathrm{Re}\sum_{i,j}\mathrm{Tr}\left[\rho Z e^{\beta E_i}\Pi_i Z^{-1} e^{-\beta E_j} U^\dagger \Pi_j U\right] \\ &= \mathrm{ReTr}\left[\rho\left(\frac{e^{\beta H}}{Z}\right)^{-1} U^\dagger \frac{e^{-\beta H}}{Z} U\right] = \mathrm{ReTr}\left[\rho\rho_{eq}^{-1} U^\dagger \rho_{eq} U\right] = \Upsilon,\end{aligned} \tag{4.9}$$

where, ρ_{eq} is the Gibb's state for the initial Hamiltonian H. Allahverdyan analyzes this fluctuation quantity Υ in [All14]. One direct observation is that, the term is equal to 1 when the initial state is ρ_{eq}, which gives (4.1) as expected.

In this chapter, we take a different approach to analyze the fluctuation theorem. As we saw earlier, equation (4.7) can be used to separate the 'classical' and the 'quantum' parts. We apply this to Υ and get,

$$\begin{aligned}\Upsilon &= \sum_{i,j} e^{-\beta(E_j-E_i)}\mathrm{Tr}\left[\rho\Pi_i U^\dagger \Pi_j U\right] = \Upsilon_d + \Upsilon_c \\ &= \sum_{i,j} e^{-\beta(E_j-E_i)}\mathrm{Tr}\left[\Pi_i\rho\Pi_i U^\dagger \Pi_j U\right] + \sum_{i,j} e^{-\beta(E_j-E_i)}\frac{1}{2}\mathrm{Tr}\left[\left(\rho-\rho_i'\right)U^\dagger\Pi_j U\right]. \qquad (4.10)\end{aligned}$$

Now we see something interesting. As detailed in appendix C.2, if we split the density matrix (ρ) into the diagonal (ρ_d) and the off-diagonal part (ρ_c), Υ_d is non-zero only for ρ_d and Υ_c is non-zero only for ρ_c. Hence, we write,

$$\Upsilon_d = \sum_{i,j} e^{-\beta(E_j-E_i)}\mathrm{Tr}\left[\Pi_i\rho_d\Pi_i U^\dagger \Pi_j U\right], \qquad (4.11)$$

and,

$$\Upsilon_c = \frac{1}{2}\sum_{i,j} e^{-\beta(E_j-E_i)}\mathrm{Tr}\left[\left(\rho_c-\rho_i'\right)U^\dagger\Pi_j U\right]. \qquad (4.12)$$

We see that $\Upsilon_d = 1$ when, $[U,H]=0$ and/or the initial state is ρ_{eq}. For both these cases, $\Upsilon_c = 0$ which implies $\Upsilon = 1$. Hence, to see deviations from (4.1), the unitary must generate coherence in the initial Hamiltonian basis and the initial state should not be in equilibrium. Now to analyze these terms, let us restrict our attention on qubits. We will now express the classical and quantum parts in terms of parameters on the Bloch sphere and look at a simple example.

4.3.1 Analysis of Coherence on the Bloch Sphere

All calculations in this section are detailed in appendix C.3.For the case of a qubit, as usual we take the Hamiltonian as $H = \Delta\left|1\right\rangle\left\langle 1\right|$. For the qubit case, we have just two projectors $\Pi_i \in \{\left|0\right\rangle\left\langle 0\right|, \left|1\right\rangle\left\langle 1\right|\}$. We define the transition probability between the two states as,

$$\xi = \left|\left\langle i\left|U\right|j\right\rangle\right|^2 = \left|\left\langle j\left|U\right|i\right\rangle\right|^2, \qquad (4.13)$$

for $i \neq j$. The classical term Υ_d can be written as,

$$\Upsilon_d = 1+(\chi-1)\,\xi = 1+\left(\frac{2r_\beta\left(r_\beta-r_z\right)}{\left(1-r_\beta^2\right)}\right)\xi = 1+\left(\frac{2r_\beta^2}{1-r_\beta^2}\right)\xi - \left(\frac{2r_z r_\beta}{1-r_\beta^2}\right)\xi, \qquad (4.14)$$

where r_z is the magnitude σ_Z component of the radius of the initial state on the Bloch sphere, r_β is the magnitude of the radius of the Gibb's state ρ_{eq} and,

$$\chi - 1 = \frac{2r_\beta\left(r_\beta-r_z\right)}{\left(1-r_\beta^2\right)}.$$

For these, we have the bounds,

$$\sqrt{1-r_z^2}-1 \leq (\chi-1) < \infty,$$

$$0 \leq \xi \leq 1,$$

$$e^{-\beta\Delta E} \leq \Upsilon_d < \infty \implies -\beta\Delta E \leq \ln\Upsilon_d < \infty.$$

As you can see, this term can become arbitrarily large, but can never become negative. But as we will see Υ can become negative. Any negativity in it comes from Υ_c and hence, due to quantum coherence. To express this in terms of Bloch sphere parameters, we will need to express the unitary U as,

$$U = \begin{bmatrix} e^{i\delta}\cos\phi & -e^{-i\gamma}\sin\phi \\ e^{i\gamma}\sin\phi & e^{-i\delta}\cos\phi \end{bmatrix},$$

where, δ, γ and ϕ are parameters that depend on the specific U [NC10]. We can express ρ_c as,

$$\rho_c = \frac{1}{2}\left(r_x\sigma_X + r_y\sigma_Y\right) = \frac{1}{2}r\sin\left(\theta\right)\left(\sigma_X\cos\varphi + \sigma_Y\sin\varphi\right),$$

where, r is the magnitude of the Bloch radius of ρ, θ is the angle made by this Bloch radius with the axis of σ_Z and φ is its azimuthal angle. We write $\alpha = \delta + \gamma$ which is again a parameter of U. Using these we get,

$$\Upsilon_c = \left(\frac{-rr_\beta}{1-r_\beta^2}\right)\sin\left(\theta\right)\cos\left(\alpha-\varphi\right)\sin\left(2\phi\right). \tag{4.15}$$

We have already seen that Υ_c is non-zero only if the initial state has coherence. Moreover, for classical dynamics which is given by either identity or a bit flip, $\Upsilon_c = 0$ as in this case, $\phi = \frac{\pi}{2}$ and $(\alpha - \varphi) = 0$ as $\gamma = \frac{\pi}{2}$ and we choose $\delta = \left(\varphi - \frac{\pi}{2}\right)$. Therefore, the dynamics also should generate coherence for Υ_c to be non-zero. Let us now look at a simple example to get an intuition about the quantities.

Example

We will take the example of the qubit being in the $|+\rangle = \left(\frac{|0\rangle+|1\rangle}{\sqrt{2}}\right)$ state and the Gibb's state radius as $r_\beta = \frac{1}{3}$. We will look at Υ for different values of α and ξ (or ϕ). For this we have,

$$\Upsilon_c = \left(\frac{-rr_\beta}{1-r_\beta^2}\right)\sin\left(\theta\right)\sin\left(2\phi\right)\cos\left(\alpha-\varphi\right) = \left(\frac{-rr_\beta}{1-r_\beta^2}\right)\sin\left(\theta\right)\left(\sqrt{\xi\left(1-\xi\right)}\right)\cos\left(\alpha-\varphi\right)$$

where,

- $r = 1$ and $r_\beta = \frac{1}{3}$
- $\theta = \frac{\pi}{2}$ and $\varphi = 0$
- $\Upsilon_c = \left(\frac{-rr_\beta}{1-r_\beta^2}\right)\sin\left(\theta\right)\left(\sqrt{\xi\left(1-\xi\right)}\right)\cos\left(\alpha-\varphi\right) = -\frac{3}{8}\sin\left(2\phi\right)\cos\left(\alpha\right)$
  $= -\frac{3}{4}\left(\sqrt{\xi\left(1-\xi\right)}\right)\cos\left(\alpha\right)$

- $\Upsilon_d = 1 + \left(\frac{2r_\beta\left(r_\beta - r\cos\theta\right)}{\left(1-r_\beta^2\right)} - 1\right)\sin^2\phi = 1 + \left(\frac{2r_\beta\left(r_\beta - r\cos\theta\right)}{\left(1-r_\beta^2\right)} - 1\right)\xi = 1 + \left(\frac{1}{4} - 1\right)\xi = 1 - \frac{3}{4}\xi$
- $\Upsilon = 1 - \frac{3}{4}\left(\xi + \sqrt{\xi\left(1-\xi\right)}\right)\cos\left(\alpha\right)$

For this case, $\Upsilon \geq 0$ always but Υ_c can be negative. Υ_c shows more sensitive quantity than Υ for the measuring the contribution of quantum coherence to the work fluctuations during a unitary evolution.

4.4 Summary

Let us summarize the results till now. We started with using the Margenau-Hill distribution to get,

$$\Upsilon = \sum_{i,j} e^{-\beta(E_j - E_i)}\text{Tr}\left[\rho\Pi_i U^\dagger \Pi_j U\right]$$

$$\Upsilon_d = \sum_{i,j} e^{-\beta(E_j - E_i)}\text{Tr}\left[\Pi_i \rho_d \Pi_i U^\dagger \Pi_j U\right]$$

$$\Upsilon_c = \sum_{i,j} e^{-\beta(E_j - E_i)}\frac{1}{2}\text{Tr}\left[\left(\rho_c - \rho_i'\right) U^\dagger \Pi_j U\right]$$

We expressed them for qubits on a Bloch sphere as:

$$\Upsilon_d = 1 + \left(\chi - 1\right)\xi = 1 + \left(\frac{2r_\beta\left(r_\beta - r\cos\theta\right)}{\left(1-r_\beta^2\right)} - 1\right)\sin^2\phi = 1 + \left(\frac{2r_\beta\left(r_\beta - r\cos\theta\right)}{\left(1-r_\beta^2\right)} - 1\right)\xi$$

$$\Upsilon_c = \left(\frac{-rr_\beta}{1-r_\beta^2}\right)\left(\sqrt{\xi\left(1-\xi\right)}\right)\sin\left(\theta\right)\cos\left(\alpha - \varphi\right)$$

$$\begin{aligned}\Upsilon &= 1 + \sin^2\phi\left(\frac{2r_\beta\left(r_\beta - r\cos\theta\right)}{\left(1-r_\beta^2\right)} - 1\right) - \left(\frac{rr_\beta}{1-r_\beta^2}\right)\sin\theta\cos\left(\alpha-\varphi\right)\sin2\phi \\ &= 1 + \left(\frac{2r_\beta\left(r_\beta - r\cos\theta\right)}{\left(1-r_\beta^2\right)} - 1\right)\xi - \left(\frac{rr_\beta}{1-r_\beta^2}\right)\left(\sqrt{\xi\left(1-\xi\right)}\right)\sin\theta\cos\left(\alpha-\varphi\right)\end{aligned}$$

where, the Bloch vector is $\vec{r} = (r\sin\theta\cos\varphi, r\sin\theta\sin\varphi, r\cos\theta)$. For $\Upsilon = \Upsilon_d + \Upsilon_c$ we noticed (Subscript cl corresponds to the classical case):-

- $\rho_{cl}, U_{cl} \implies \Upsilon \geq 0, \Upsilon_c = 0,$
- $\rho_{cl}, U \implies \Upsilon \geq 0, \Upsilon_c = 0,$
- $\rho, U_{cl} \implies \Upsilon \geq 0, \Upsilon_c = 0,$
- $\rho, U \implies \Upsilon \geq 0$ or $\Upsilon \leq 0, \Upsilon_c \neq 0$: For the $\left|+\right\rangle\left\langle+\right|$ state we saw that, $\Upsilon \geq 0$ for all α and ξ, even if $\Upsilon_c \leq 0$.

Υ_c seems to give a more sensitive signature of the incompatibility between the initial projectors$\{\Pi_i\}_i$ and the final projectors$\left\{U^\dagger \Pi_j U\right\}_j$.

Chapter 5

Conclusion

> *This proposal, like all proposals for quantum computation, relies on speculative technology, does not in its current form take into account all possible sources of noise, unreliability and manufacturing error, and probably will not work.*
>
> — **Rolf Landauer**

Rolf Landauer suggested that every publication on quantum computers should have the above quote as a footnote [Llo99]. Even though Landauer was one of the pioneers of the field, he was one of the biggest critics. However now, Landauer's spirit may be happy as this footnote no longer seems to be required. Experiments have been able to replicate quantum phenomenon and many companies like IBM have made quantum computers which can manipulate nearly 70 qubits. The central setup of this thesis, the quantum switch, has been implemented, the unusual phenomenon have been experimentally observed [NZX+20, GGK+18]. Creation of superposition of higher number of causal orders has also been studied and proposed [PDE+20]. However, as we have seen throughout the thesis, study of every party in the setup is important and may have been ignored in some publications. The publications such as [NZX+20, FV20], did not look at the bath states, even though they too used the SWAP gate in the quantum switch for simulating thermalization. Let us now summarize all the results and look at the future work.

5.1 Summary and Inferences

Below we list out and summarize very briefly the most important inferences from the thesis:-

- The quantum switch creates a phenomenon of indefinite causal order of channels. This effect is observed only in the coherent part the control in the basis coupling with the causal orders.

- The quantum switch can lead to interesting effects like communication through noisy channels, and activation of free energy for thermal states using thermal baths at the same temperature. It can even lead to extreme cases like population inversion.

- The analysis of the SWAP gate setup for thermalization, showed that the energy of the baths and the system increases or decreases together due to the measurement.

- The activation of free energy is due to the measurement of the control alone. This was confirmed by studying the measurement channel and the thermal bit used in it to make the measurement.

- The Margenau-Hill distribution can be used to study the fluctuations in process which involve coherence in the energy basis. It leads to a fluctuation theorem which gives access to studying the coherent behaviour.

- The negative part of the Margenau-Hill distribution gives rise to fluctuations which are only present when the initial state has coherence and the evolution also generates coherence. This will be useful to study the coherence contribution in the quantum switch.

As can be inferred from above there is a lot of scope of research remaining to understand the thermodynamics of indefinite causal orders. Let us end this thesis by summarizing what we think is the future work left to be done.

5.2 Future Work

Below we list out and summarize the future work that continues the work of this thesis:-

- The coherence term of the fluctuations Υ_c needs to be studied for the case of the quantum switch. Here, first the MH distribution either needs to be generalized for channels, or, for a global unitary which acts on a four-party system (the system under study, the control, the two environments of the channels).

- Using this fluctuation term, it may be possible to prove more strongly that the thermal baths do not provide any energy to the system in the thermalization setup in the switch, for the activation of free energy.

- The fluctuation term Υ needs to be studied in context of weak measurements and weak values. This may give us a more concrete interpretation of the term.

- A geometric interpretation could be found for the fluctuation term Υ and be related with quantities such as the geometric phase and/or the Bargmann invariants.

- Finally, an experimental setup needs to be proposed to probe the fluctuations and verify the results. The experiment could require separation of the control and system degrees of freedom into two separate physical systems (presently they are the internal degrees of freedom of a single photon as seen in section 2.2.1). This may be done by using a quantum dot and a photon as the system and control respectively.

We end the thesis here. Feel free to contact me with queries/suggestions :)
Fin...?

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Appendix A

Thermalization in the Quantum Switch

Here, we will detail the calculations for the switch with two thermal baths at different temperatures. This includes the analytical calculations for the final temperature of the system and calculations of the final states after the action of the switch and measurement of the control.

A.1 Analysis of the System after Switch Operation

Let $\rho \in \mathbb{H}^S \equiv$ a qubit system and $\sigma \in \mathbb{H}^C \equiv$ control for the quantum switch(QS). The Hamiltonian of the system is $H = \Delta |1\rangle \langle 1|$. Let $\{U_i\}$ be a set of orthogonal unitary operators in $d * d = 2 * 2$ dimensional operator space. We take two different thermalizing channels ($\mathcal{E}_x$, $x \in \{1, 2\}$, $\beta_1 > \beta_2$) which give a thermal state for any input state. We have the following definitions for the qubit case:

$$\beta = \frac{1}{\Delta} \log \left(\frac{p}{1-p} \right),$$

$$\zeta^{\beta_x} = \frac{1}{Z_x} \begin{bmatrix} 1 & 0 \\ 0 & e^{-\beta_x \Delta} \end{bmatrix} \quad A_x = \sqrt{\zeta^{\beta_x}},$$

$$\mathcal{E}_x(\rho) = \frac{1}{d} \sum_i^{d^2} A_x U_i \rho U_i^\dagger A_x^\dagger,$$

$$Z_x = 1 + e^{-\beta_x \Delta}.$$

Kraus operators for $\mathcal{E}_x \equiv K_i^x = \sqrt{\frac{1}{d}} A_x U_i$.

Kraus operators for the switch $\equiv W_{ij} = |0\rangle \langle 0| \otimes K_i^1 K_j^2 + |1\rangle \langle 1| \otimes K_j^2 K_i^1$.

State of Control $\equiv \sigma = \alpha |0\rangle \langle 0| + (1 - \alpha) |1\rangle \langle 1| + \sqrt{\alpha(1-\alpha)} (|0\rangle \langle 1| + |1\rangle \langle 0|)$.

State after action of switch is:

$$S(\sigma\otimes\rho)=\sum_{i,j}\left(|0\rangle\langle 0|\otimes K_i^1K_j^2+|1\rangle\langle 1|\otimes K_j^2K_i^1\right)(\sigma\otimes\rho)\left(|0\rangle\langle 0|\otimes K_j^{2\dagger}K_i^{1\dagger}+|1\rangle\langle 1|\otimes K_i^{1\dagger}K_j^{2\dagger}\right)$$
$$=\sum_{i,j}\Bigg(\alpha|0\rangle\langle 0|\otimes K_i^1K_j^2\rho K_j^{2\dagger}K_i^{1\dagger}+(1-\alpha)|1\rangle\langle 1|\otimes K_j^2K_i^1\rho K_i^{1\dagger}K_j^{2\dagger}$$
$$+\sqrt{\alpha(1-\alpha)}\left(|0\rangle\langle 1|\otimes K_i^1K_j^2\rho K_i^{1\dagger}K_j^{2\dagger}+|1\rangle\langle 0|\otimes K_j^2K_i^1\rho K_j^{2\dagger}K_i^{1\dagger}\right)\Bigg)$$
$$=\Bigg(\alpha|0\rangle\langle 0|\otimes\zeta^{\beta_1}+(1-\alpha)|1\rangle\langle 1|\otimes\zeta^{\beta_2}$$
$$+\sqrt{\alpha(1-\alpha)}\left(|0\rangle\langle 1|\otimes\frac{1}{d}\sum_i A_1\mathrm{Tr}\left(U_i^\dagger A_1^\dagger\rho\right)U_i\zeta^{\beta_2}+|1\rangle\langle 0|\otimes\frac{1}{d}\sum_j A_2\mathrm{Tr}\left(U_j^\dagger A_2^\dagger\rho\right)U_j\zeta^{\beta_1}\right)\Bigg).$$

A $(d*d)$ matrix can be expanded in a unitary basis as,

$$\rho=\frac{1}{d}\sum_i^{d^2}\mathrm{Tr}\left(U_i^\dagger\rho\right)U_i.$$

Using this we can write,

$$S(\sigma\otimes\rho)=\alpha|0\rangle\langle 0|\otimes\zeta^{\beta_1}+(1-\alpha)|1\rangle\langle 1|\otimes\zeta^{\beta_2}+\sqrt{\alpha(1-\alpha)}\left(|0\rangle\langle 1|\otimes\zeta^{\beta_1}\rho\zeta^{\beta_2}+|1\rangle\langle 0|\otimes\zeta^{\beta_2}\rho\zeta^{\beta_1}\right). \tag{A.1}$$

If the temperatures were equal as in section 2.3 we get,

$$S\left(|+\rangle\langle +|,\zeta^\beta\right)=\frac{1}{2}\left(|0\rangle\langle 0|+|1\rangle\langle 1|\right)\otimes\zeta^\beta+\frac{1}{2}\left(|0\rangle\langle 1|+|1\rangle\langle 0|\right)\otimes\left(\zeta^\beta\right)^3. \tag{A.2}$$

Now, we measure the control qubit in the $\{|\pm\rangle\}$ basis. We get,

$$\langle\pm|S(\sigma\otimes\rho)|\pm\rangle=\rho'_\pm=\frac{\alpha}{2}\zeta^{\beta_1}+\frac{(1-\alpha)}{2}\zeta^{\beta_2}\pm\frac{\sqrt{\alpha(1-\alpha)}}{2}\left(\zeta^{\beta_1}\rho\zeta^{\beta_2}+\zeta^{\beta_2}\rho\zeta^{\beta_1}\right).$$

The normalized state at the end is,

$$\rho_f^\pm=\left(\frac{\alpha}{2}\zeta^{\beta_1}+\frac{(1-\alpha)}{2}\zeta^{\beta_2}\pm\frac{\sqrt{\alpha(1-\alpha)}}{2}\left(\zeta^{\beta_1}\rho\zeta^{\beta_2}+\zeta^{\beta_2}\rho\zeta^{\beta_1}\right)\right)/\mathrm{Tr}(\rho'_\pm).$$

Let us take the input state as ζ^{β_1} which is at a lower temperature.

$$\rho'_\pm=\frac{\alpha}{2}\zeta^{\beta_1}+\frac{(1-\alpha)}{2}\zeta^{\beta_2}\pm\sqrt{\alpha(1-\alpha)}\left(\zeta^{\beta_1}\right)^2\zeta^{\beta_2}.$$

$$\rho_f^{\pm} = \left(\frac{\alpha}{2}\zeta^{\beta_1} + \frac{(1-\alpha)}{2}\zeta^{\beta_2} \pm \sqrt{\alpha(1-\alpha)}\left(\zeta^{\beta_1}\right)^2\zeta^{\beta_2}\right)/\mathrm{Tr}(\rho_{\pm}^{'})$$
$$= \begin{bmatrix} \frac{\alpha Z_1 Z_2}{2Z_1^2 Z_2} + \frac{(1-\alpha)Z_1^2}{2Z_1^2 Z_2} \pm \frac{2\sqrt{\alpha(1-\alpha)}}{2Z_1^2 Z_2} & 0 \\ 0 & \frac{\alpha e^{-\beta_1\Delta}}{2Z_1} + \frac{(1-\alpha)e^{-\beta_2\Delta}}{2Z_2} \pm \frac{\sqrt{\alpha(1-\alpha)}e^{-(2\beta_1+\beta_2)\Delta}}{Z_1^2 Z_2} \end{bmatrix}/\mathrm{Tr}(\rho_{\pm}^{'}).$$

To focus on the effect of the temperature of the thermal baths, without loss of generality, we take $\alpha = \frac{1}{2}$,

$$\rho_f^{\pm} = \frac{Z_1 Z_2 + Z_1^2 \pm 2}{(4Z_1^2 Z_2)\,\mathrm{Tr}(\rho_{\pm}^{'})}\begin{bmatrix} 1 & 0 \\ 0 & \frac{e^{-\beta_1\Delta}Z_1Z_2 + e^{-\beta_2\Delta}Z_1^2 \pm 2e^{-(2\beta_1+\beta_2)\Delta}}{Z_1Z_2+Z_1^2\pm 2}. \end{bmatrix}.$$

To get the temperature of this state (β):

$$\frac{e^{-\beta_1\Delta}Z_1Z_2 + e^{-\beta_2\Delta}Z_1^2 \pm 2e^{-(2\beta_1+\beta_2)\Delta}}{Z_1Z_2+Z_1^2\pm 2} = e^{-\beta\Delta}.$$

When $\left|-\right\rangle\left\langle-\right|$ state of the control is measured we have,

$$e^{-\beta\Delta} = \frac{1 + e^{(\beta_1-\beta_2)\Delta} + e^{-\beta_1\Delta} + 3e^{-\beta_2\Delta}}{3e^{-\beta_1\Delta} + e^{-\beta_2\Delta} + e^{-(\beta_1+\beta_2)\Delta} + e^{-2\beta_1\Delta}}e^{-\beta_1\Delta},$$

and when $\left|+\right\rangle\left\langle+\right|$ state of the control is measured we have,

$$e^{-\beta\Delta} = \frac{1 + e^{(\beta_1-\beta_2)\Delta} + e^{-\beta_1\Delta} + 3e^{-\beta_2\Delta} + 4e^{-(\beta_1+\beta_2)\Delta}}{4 + 3e^{-\beta_1\Delta} + e^{-\beta_2\Delta} + e^{-(\beta_1+\beta_2)\Delta} + e^{-2\beta_1\Delta}}e^{-\beta_1\Delta}.$$

Let $r_x = e^{-\beta_x\Delta}$, for $x \in \{1,2\}$ then we have,

$$\beta_1 > \beta_2 \implies -\beta_1 < -\beta_2 \implies e^{-\beta_1} < e^{-\beta_2} \implies r_1 < r_2 \implies T_1 < T_2.$$

Let's figure out when the coefficient of $e^{-\beta_1\Delta}$, in the $\left|-\right\rangle\left\langle-\right|$ measurement of control, is greater than one,

$$\frac{1 + e^{(\beta_1-\beta_2)\Delta} + e^{-\beta_1\Delta} + 3e^{-\beta_2\Delta}}{3e^{-\beta_1\Delta} + e^{-\beta_2\Delta} + e^{-(\beta_1+\beta_2)\Delta} + e^{-2\beta_1\Delta}} > 1,$$
$$\implies 0 > 2\left(e^{-\beta_1\Delta} - e^{-\beta_2\Delta}\right) + \left(e^{-(\beta_1+\beta_2)\Delta} - e^{(\beta_1-\beta_2)\Delta}\right) + e^{-2\beta_1\Delta} - 1,$$
$$\implies 0 > 2\left(r_1 - r_2\right) + \left(r_1 r_2 - \frac{r_2}{r_1}\right) + r_1^2 - 1,$$
$$\implies 0 > r_1^3 + r_1^2\left(2 + r_2\right) - r_1\left(1 + 2r_2\right) - r_2.$$

But also we need,

$$r_1 < r_2.$$

From the graph (I took $r_2 = 0.1$) we can also see that, if $r_2 < r_1$, i.e., if the initial state of the system is prepared at the higher temperature, we can still get a higher final temperature.

$$\frac{e^{-\beta_1\Delta}+e^{-\beta_2\Delta}+e^{-2\beta_1\Delta}+3e^{-(\beta_1+\beta_2)\Delta}}{3e^{-\beta_1\Delta}+e^{-\beta_2\Delta}+e^{-(\beta_1+\beta_2)\Delta}+e^{-2\beta_1\Delta}}=\frac{r_1+r_2+r_1^2+3r_1r_2}{3r_1+r_2+r_1r_2+r_1^2}=\frac{r_1r_2^{-1}+1+r_1^2r_2^{-1}+3r_1}{3r_1+r_2+r_1r_2+r_1^2}r_2.$$

Let's figure out when the coefficient of $e^{-\beta_2\Delta}$ in the $\left|-\right\rangle\left\langle-\right|$ measurement is greater than one,

$$\begin{aligned}&\frac{r_1r_2^{-1}+1+r_1^2r_2^{-1}+3r_1}{3r_1+r_2+r_1r_2+r_1^2}>1\\ &\implies 0>(r_2-1)(r_2+r_1).\end{aligned}$$

This immediately gives the relations,

$$-r_1>r_2 \text{ and } r_2>1,$$

which are never satisfied, or

$$1>r_2 \text{ and } -r_1<r_2,$$

which are always satisfied. This means that to get a temperature higher than the hotter reservoir, $e^{-\beta_2\Delta}<1$ which is always satisfied, and hence, we will get a temperature above the hotter reservoir all the time if we prepare the state at the temperature of the cooler reservoir.

Now let's figure out when the coefficient of $e^{-\beta_1\Delta}$ in the $\left|+\right\rangle\left\langle+\right|$ measurement is lesser than one,

$$\begin{aligned}&\frac{1+e^{(\beta_1-\beta_2)\Delta}+e^{-\beta_1\Delta}+3e^{-\beta_2\Delta}+4e^{-(\beta_1+\beta_2)\Delta}}{4+3e^{-\beta_1\Delta}+e^{-\beta_2\Delta}+e^{-(\beta_1+\beta_2)\Delta}+e^{-2\beta_1\Delta}}<1\\ \implies&\frac{1+r_2r_1^{-1}+r_1+3r_2+4r_1r_2}{4+3r_1+r_2+r_1r_2+r_1^2}<1\\ \implies& 0<r_1^3+r_1^2(2-3r_2)+r_1(3-2r_2)-r_2.\end{aligned}$$

But also we need,

$$r_1<r_2.$$

From the graph (I took $r_2=0.1$) we can also see that for a range of values for r_1 we do get a lower final temperature.

$$\begin{aligned}\frac{e^{-\beta_1\Delta}+e^{-\beta_2\Delta}+e^{-2\beta_1\Delta}+3e^{-(\beta_1+\beta_2)\Delta}+4e^{-(2\beta_1+\beta_2)\Delta}}{4+3e^{-\beta_1\Delta}+e^{-\beta_2\Delta}+e^{-(\beta_1+\beta_2)\Delta}+e^{-2\beta_1\Delta}}&=\frac{r_1+r_2+r_1^2+3r_1r_2+4r_1^2r_2}{4+3r_1+r_2+r_1r_2+r_1^2}\\ &=\frac{r_1r_2^{-1}+1+r_1^2r_2^{-1}+3r_1+4r_1^2}{4+3r_1+r_2+r_1r_2+r_1^2}r_2.\end{aligned}$$

Now let's figure out when the coefficient of $e^{-\beta_2\Delta}$ in the $\left|+\right\rangle\left\langle+\right|$ measurement is lesser than one,

$$\frac{r_1 r_2^{-1} + 1 + r_1^2 r_2^{-1} + 3r_1 + 4r_1^2}{4 + 3r_1 + r_2 + r_1 r_2 + r_1^2} > 1$$
$$\implies 0 > r_2^2 + 3r_2\left(1 - r_1\right) - r_1.$$

But also we need,

$$r_1 < r_2.$$

The graph shows that we can also see that for suitably chosen r_1 and r_2 we do get a higher final temperature if the system is prepared at the colder temperature. For population inversion the term in the $|-\rangle$ outcome should be greater than 1,

$$\frac{e^{-\beta_1\Delta} + e^{-\beta_2\Delta} + e^{-2\beta_1\Delta} + 3e^{-(\beta_1+\beta_2)\Delta}}{3e^{-\beta_1\Delta} + e^{-\beta_2\Delta} + e^{-(\beta_1+\beta_2)\Delta} + e^{-2\beta_1\Delta}} > 1$$
$$\implies r_1\left(r_2 - 1\right) > 0.$$

which give the relations,

$$r_1 > 0 \text{ and } r_2 > 1.$$

or,

$$r_1 < 0 \text{ and } r_2 < 1.$$

These show that if $|-\rangle$ is measured and initial state is at the temperature of the cold reservoir then, we can get population inversion only when the hotter reservoir has a thermal state which has negative temperature.

A.2 Energy Analysis for Thermalization in ICO

There are 2 thermalizing channels that are a part of the quantum switch(QS). One is a hot bath and the other is a cold bath. We want to analyze the energy change after measurement of the control for the system going into the QS and the thermal baths. For this we model the thermalizing channel as a swap gate $\left(U^{\beta}_{SWAP}\right)$ which acts between the system state$\left(\rho \in \mathbb{H}^{\mathrm{S}}\right)$ and a thermal state $\left(\zeta^{\beta}\right)$. Here, β_H and β_C will denote the inverse temperatures of the hot and cold baths $\left(\zeta^{\beta_H}\right)$ and $\left(\zeta^{\beta_C}\right)$ respectively. We initialize the control in the state $\sigma \in \mathbb{H}^{\mathrm{C}}$. The superscript corresponds to the thermal bath that the operator will act on. For example, $U^{\beta_H}_{SWAP}$ acts between the system and the hot bath.

The initial state we start with is: $\left(\sigma \otimes \rho \otimes \zeta^{\beta_H} \otimes \zeta^{\beta_C}\right)$. The Hamiltonian for the system and the baths is taken to be $\mathbf{H} = \Delta |1\rangle\langle 1|$, and we assign zero Hamiltonian to the control for these calculations.

The operator for the quantum switch with the $U^{\beta_H}_{SWAP}$ and $U^{\beta_C}_{SWAP}$ is:

$$W = |0\rangle\langle 0| \otimes U^{\beta_H}_{SWAP} U^{\beta_C}_{SWAP} + |1\rangle\langle 1| \otimes U^{\beta_C}_{SWAP} U^{\beta_H}_{SWAP}.$$

A.2.1 Change in temperature/heat for the baths with SWAP operations for Thermalization

We will first calculate the state at the end of the quantum switch after the measurement of the control and then get the states of the baths individually (Partial trace). Using this we will be able to calculate the heat change for the system and the baths.

Final state at the end of the switch

$$
\begin{aligned}
S(\sigma\otimes\rho\otimes\zeta^{\beta_H}\otimes\zeta^{\beta_C}) &= W(\sigma\otimes\rho\otimes\zeta^{\beta_H}\otimes\zeta^{\beta_C})W^\dagger \\
&= \left(|0\rangle\langle 0|\otimes U^{\beta_H}_{SWAP}U^{\beta_C}_{SWAP} + |1\rangle\langle 1|\otimes U^{\beta_C}_{SWAP}U^{\beta_H}_{SWAP}\right)\left(\sigma\otimes\rho\otimes\zeta^{\beta_H}\otimes\zeta^{\beta_C}\right) \\
&\quad \cdot\left(|0\rangle\langle 0|\otimes\left(U^{\beta_C}_{SWAP}\right)^\dagger\left(U^{\beta_H}_{SWAP}\right)^\dagger + |1\rangle\langle 1|\otimes\left(U^{\beta_H}_{SWAP}\right)^\dagger\left(U^{\beta_C}_{SWAP}\right)^\dagger\right) \\
&= \langle 0|\sigma|0\rangle|0\rangle\langle 0|\otimes\left(\zeta^{\beta_H}\otimes\zeta^{\beta_C}\otimes\rho\right) \\
&+ \langle 1|\sigma|0\rangle|1\rangle\langle 0|\otimes U^{\beta_C}_{SWAP}U^{\beta_H}_{SWAP}\left(\rho\otimes\zeta^{\beta_H}\otimes\zeta^{\beta_C}\right)\left(U^{\beta_C}_{SWAP}\right)^\dagger\left(U^{\beta_H}_{SWAP}\right)^\dagger \\
&+ \langle 0|\sigma|1\rangle|0\rangle\langle 1|\otimes U^{\beta_H}_{SWAP}U^{\beta_C}_{SWAP}\left(\rho\otimes\zeta^{\beta_H}\otimes\zeta^{\beta_C}\right)\left(U^{\beta_H}_{SWAP}\right)^\dagger\left(U^{\beta_C}_{SWAP}\right)^\dagger \\
&+ \langle 1|\sigma|1\rangle|1\rangle\langle 1|\otimes\left(\zeta^{\beta_C}\otimes\rho\otimes\zeta^{\beta_H}\right).
\end{aligned}
$$

To calculate the off-diagonal terms (off-diagonal from the point of view of the control) we need the unitaries of the SWAP gate. The SWAP gate is:

$$U^{\beta_C}_{SWAP} = \sum_{m,n}|n\rangle\langle m|\otimes\mathbb{I}^{\beta_H}\otimes|m\rangle\langle n|.$$

$$U^{\beta_H}_{SWAP} = \sum_{m,n}|n\rangle\langle m|\otimes|m\rangle\langle n|\otimes\mathbb{I}^{\beta_C}.$$

Now to evaluate the off-diagonal terms:

$$
\begin{aligned}
\left(U^{\beta_C}_{SWAP}U^{\beta_H}_{SWAP}\right)\left(\rho\otimes\zeta^{\beta_H}\otimes\zeta^{\beta_C}\right)\left(U^{\beta_H}_{SWAP}U^{\beta_C}_{SWAP}\right)^\dagger &= \left(\sum_{q,m,n}|q\rangle\langle m|\otimes|m\rangle\langle n|\otimes|n\rangle\langle q|\right) \\
&\quad \cdot\left(\rho\otimes\zeta^{\beta_H}\otimes\zeta^{\beta_C}\right) \\
&\quad \cdot\left(\sum_{r,s,p}|p\rangle\langle s|\otimes|s\rangle\langle r|\otimes|r\rangle\langle p|\right)
\end{aligned}
$$

$$= \sum_{q,m,n,r,s,p}\langle m|\rho|p\rangle|q\rangle\langle s|\otimes\langle n|\zeta^{\beta_H}|s\rangle|m\rangle\langle r|\otimes\langle q|\zeta^{\beta_C}|r\rangle|n\rangle\langle p|.$$

The off-diagonal terms are just Hermitian conjugates of each other, so,

$$\left(U^{\beta_H}_{SWAP}U^{\beta_C}_{SWAP}\right)\left(\rho\otimes\zeta^{\beta_H}\otimes\zeta^{\beta_C}\right)\left(U^{\beta_C}_{SWAP}U^{\beta_H}_{SWAP}\right)^{\dagger} = \sum_{q,m,n,r,s,p}\left(\langle p|\,\rho\,|m\rangle\,|s\rangle\,\langle q|\right)\otimes\left(\langle s|\,\zeta^{\beta_H}\,|n\rangle\,|r\rangle\,\langle m\right.$$
$$\otimes\left(\langle r|\,\zeta^{\beta_C}\,|q\rangle\,|p\rangle\,\langle n|\right).$$

Now that we have the final state after the action of the switch, let's get the state of the thermal baths after measurement of the control.

Hot bath state

Partial trace over system and cold bath for off-diagonal terms:

$$\mathrm{Tr}_{S,C}\left(\left(U^{\beta_C}_{SWAP}U^{\beta_H}_{SWAP}\right)\left(\rho\otimes\zeta^{\beta_H}\otimes\zeta^{\beta_C}\right)\left(U^{\beta_H}_{SWAP}U^{\beta_C}_{SWAP}\right)^{\dagger}\right)$$
$$=\sum_{q,m,r,p}\langle m|\,\rho\,|p\rangle\,\langle p|\,\zeta^{\beta_H}\,|q\rangle\,\langle q|\,\zeta^{\beta_C}\,|r\rangle\,|m\rangle\,\langle r| = \rho\zeta^{\beta_H}\zeta^{\beta_C},$$

and,

$$\mathrm{Tr}_{S,C}\left(\left(U^{\beta_H}_{SWAP}U^{\beta_C}_{SWAP}\right)\left(\rho\otimes\zeta^{\beta_H}\otimes\zeta^{\beta_C}\right)\left(U^{\beta_C}_{SWAP}U^{\beta_H}_{SWAP}\right)^{\dagger}\right)$$
$$=\sum_{q,m,r,p}\langle r|\,\zeta^{\beta_C}\,|q\rangle\,\langle q|\,\zeta^{\beta_H}\,|p\rangle\,\langle p|\,\rho\,|m\rangle\,|r\rangle\,\langle m| = \zeta^{\beta_C}\zeta^{\beta_H}\rho.$$

Now the final state of the control and the hot bath is:

$$\mathrm{Tr}_{S,C}\left(S(\sigma\otimes\rho\otimes\zeta^{\beta_H}\otimes\zeta^{\beta_C})\right) = \langle 0|\,\sigma\,|0\rangle\,|0\rangle\,\langle 0|\otimes\left(\zeta^{\beta_C}\right) + \langle 1|\,\sigma\,|1\rangle\,|1\rangle\,\langle 1|\otimes(\rho)$$
$$+\langle 1|\,\sigma\,|0\rangle\,|1\rangle\,\langle 0|\otimes\left(\rho\zeta^{\beta_H}\zeta^{\beta_C}\right) + \langle 0|\,\sigma\,|1\rangle\,|0\rangle\,\langle 1|\otimes\left(\zeta^{\beta_C}\zeta^{\beta_H}\rho\right).$$

Let's take the control in the $|+\rangle\langle+|$ state and measure along $|\pm\rangle\langle\pm|$ states.

$$\langle\pm|\,\mathrm{Tr}_{S,C}\left(S(|+\rangle\langle+|\otimes\rho\otimes\zeta^{\beta_H}\otimes\zeta^{\beta_C})\right)|\pm\rangle = \frac{1}{4}\left(\zeta^{\beta_C}\right)+\frac{1}{4}(\rho)\pm\frac{1}{4}\left(\rho\zeta^{\beta_H}\zeta^{\beta_C}+\zeta^{\beta_C}\zeta^{\beta_H}\rho\right).$$

Let's take the initial state of the system to also be a thermal state$\left(\zeta^{\beta_S}\right)$, then the final state of the hot bath is:

$$\zeta^H_{f\pm} = \frac{\langle\pm|\,\mathrm{Tr}_{S,C}\left(S(|+\rangle\langle+|\otimes\zeta^{\beta_S}\otimes\zeta^{\beta_H}\otimes\zeta^{\beta_C})\right)|\pm\rangle}{\mathrm{Tr}\left(\langle\pm|\,\mathrm{Tr}_{S,C}\left(S(|+\rangle\langle+|\otimes\zeta^{\beta_S}\otimes\zeta^{\beta_H}\otimes\zeta^{\beta_C})\right)|\pm\rangle\right)}$$
$$=\frac{1}{Z}\left(\frac{1}{4}\left(\zeta^{\beta_S}\right)+\frac{1}{4}\left(\zeta^{\beta_C}\right)\pm\frac{1}{2}\left(\zeta^{\beta_S}\zeta^{\beta_H}\zeta^{\beta_C}\right)\right). \quad (A.3)$$

$$
\begin{aligned}
Z = \mathrm{Tr}\left(\frac{1}{4}\left(\zeta^{\beta_S}\right) + \frac{1}{4}\left(\zeta^{\beta_C}\right) \pm \frac{1}{2}\left(\zeta^{\beta_S}\zeta^{\beta_H}\zeta^{\beta_C}\right)\right) = \frac{1}{2} \pm \frac{1}{2}\mathrm{Tr}\left(\zeta^{\beta_S}\zeta^{\beta_H}\zeta^{\beta_C}\right) \\
= \frac{1}{2} \pm \left(\frac{p^{\beta_S}p^{\beta_H}p^{\beta_C}}{2} + \frac{(1-p^{\beta_S})(1-p^{\beta_H})(1-p^{\beta_C})}{2}\right).
\end{aligned}
$$

Cold bath state

Following the calculations of the hot bath state, the final state of the control and the cold bath is:

$$
\begin{aligned}
\mathrm{Tr}_{S,H}\left(S(\sigma \otimes \rho \otimes \zeta^{\beta_H} \otimes \zeta^{\beta_C})\right) = \langle 0|\,\sigma\,|0\rangle\,|0\rangle\,\langle 0| \otimes (\rho) + \langle 1|\,\sigma\,|1\rangle\,|1\rangle\,\langle 1| \otimes \left(\zeta^{\beta_H}\right) \\
+ \langle 1|\,\sigma\,|0\rangle\,|1\rangle\,\langle 0| \otimes \left(\zeta^{\beta_H}\zeta^{\beta_C}\rho\right) + \langle 0|\,\sigma\,|1\rangle\,|0\rangle\,\langle 1| \otimes \left(\rho\zeta^{\beta_C}\zeta^{\beta_H}\right).
\end{aligned}
$$

Let's again take the control in the $|+\rangle\langle +|$ state and measure along $|\pm\rangle\langle \pm|$ states.

$$
\langle \pm|\,\mathrm{Tr}_{S,H}\left(S(|+\rangle\langle +| \otimes \rho \otimes \zeta^{\beta_H} \otimes \zeta^{\beta_C})\right)|\pm\rangle = \frac{1}{4}\left(\zeta^{\beta_H}\right) + \frac{1}{4}(\rho) \pm \frac{1}{4}\left(\zeta^{\beta_H}\zeta^{\beta_C}\rho + \rho\zeta^{\beta_C}\zeta^{\beta_H}\right).
$$

Again, taking the system to be in a thermal state:

$$
\begin{aligned}
\zeta^{C}_{f\pm} = \frac{\langle \pm|\,\mathrm{Tr}_{S,H}\left(S(|+\rangle\langle +| \otimes \zeta^{\beta_S} \otimes \zeta^{\beta_H} \otimes \zeta^{\beta_C})\right)|\pm\rangle}{\mathrm{Tr}\left(\langle \pm|\,\mathrm{Tr}_{S,H}\left(S(|+\rangle\langle +| \otimes \zeta^{\beta_S} \otimes \zeta^{\beta_H} \otimes \zeta^{\beta_C})\right)|\pm\rangle\right)} \\
= \frac{1}{Z}\left(\frac{1}{4}\left(\zeta^{\beta_S}\right) + \frac{1}{4}\left(\zeta^{\beta_H}\right) \pm \frac{1}{2}\left(\zeta^{\beta_S}\zeta^{\beta_C}\zeta^{\beta_H}\right)\right). \quad \text{(A.4)}
\end{aligned}
$$

$$
\begin{aligned}
Z = \mathrm{Tr}\left(\frac{1}{4}\left(\zeta^{\beta_S}\right) + \frac{1}{4}\left(\zeta^{\beta_H}\right) \pm \frac{1}{2}\left(\zeta^{\beta_S}\zeta^{\beta_C}\zeta^{\beta_H}\right)\right) = \frac{1}{2} \pm \frac{1}{2}\mathrm{Tr}\left(\zeta^{\beta_S}\zeta^{\beta_C}\zeta^{\beta_H}\right) \\
= \frac{1}{2} \pm \left(\frac{p^{\beta_S}p^{\beta_C}p^{\beta_H}}{2} + \frac{(1-p^{\beta_S})(1-p^{\beta_C})(1-p^{\beta_H})}{2}\right).
\end{aligned}
$$

Heat change after measurement

Heat change in the hot bath after measurement is given by:

$$
\mathrm{Tr}\left(H\zeta^{H}_{f\pm}\right) - \mathrm{Tr}\left(H\zeta^{\beta_H}\right) = \Delta\left(\frac{(1-p^{\beta_C}) + (1-p^{\beta_S})}{4Z} \pm \frac{(1-p^{\beta_H})(1-p^{\beta_C})(1-p^{\beta_S})}{2Z} - (1-p^{\beta_H})\right).
$$

Heat change in the cold bath after measurement is given by:

$$
\mathrm{Tr}\left(H\zeta^{C}_{f\pm}\right) - \mathrm{Tr}\left(H\zeta^{\beta_C}\right) = \Delta\left(\frac{(1-p^{\beta_H}) + (1-p^{\beta_S})}{4Z} \pm \frac{(1-p^{\beta_H})(1-p^{\beta_C})(1-p^{\beta_S})}{2Z} - (1-p^{\beta_C})\right).
$$

Heat change in the system after measurement is given by:

$$
\mathrm{Tr}\left(H\zeta^{S}_{f\pm}\right) - \mathrm{Tr}\left(H\zeta^{\beta_S}\right) = \Delta\left(\frac{(1-p^{\beta_H}) + (1-p^{\beta_C})}{4Z} \pm \frac{(1-p^{\beta_H})(1-p^{\beta_C})(1-p^{\beta_S})}{2Z} - (1-p^{\beta_S})\right).
$$

Total heat change in the baths:

$$\begin{aligned}&\text{Tr}\left(H\zeta^{H}_{f\pm}\right)-\text{Tr}\left(H\zeta^{\beta_H}\right)+\text{Tr}\left(H\zeta^{C}_{f\pm}\right)-\text{Tr}\left(H\zeta^{\beta_C}\right)\\ &=\Delta[\frac{2(1-p^{\beta_S})+(1-p^{\beta_C})+(1-p^{\beta_H})}{4Z}\pm\frac{(1-p^{\beta_H})(1-p^{\beta_C})(1-p^{\beta_S})}{Z}-(1-p^{\beta_H})-(1-p^{\beta_C})].\end{aligned}$$

Heat unaccounted for:

$$\begin{aligned}\Delta_? = \big(\text{Tr}\left(H\zeta^{S}_{f\pm}\right)-\text{Tr}\left(H\zeta^{\beta_S}\right)\big)+\big(\text{Tr}\left(H\zeta^{H}_{f\pm}\right)-\text{Tr}\left(H\zeta^{\beta_H}\right)+\text{Tr}\left(H\zeta^{C}_{f\pm}\right)-\text{Tr}\left(H\zeta^{\beta_C}\right)\big)\\ =\Delta\Bigg[\pm\frac{3(1-p^{\beta_H})(1-p^{\beta_C})(1-p^{\beta_S})}{2Z}\\ +\frac{(1-p^{\beta_H})+(1-p^{\beta_C})+(1-p^{\beta_S})}{2Z}-(1-p^{\beta_H})-(1-p^{\beta_C})-(1-p^{\beta_S})\Bigg].\end{aligned}$$

When the system and the baths start at the same state then the heat change for all three seems to be the same as the final states for all three are the same! There seems to be some source of energy that is unaccounted. This might come from the work cost of measurement.

A.2.2 Comparing previous calculations to this one

Let's verify the state of the system at the end, just to check if the procedure is correct. Partial trace over the bath states gives the off-diagonal terms as:

$$\begin{aligned}&\text{Tr}_{\zeta^H,\zeta^C}\left(\left(U^{\beta_H}_{SWAP}U^{\beta_C}_{SWAP}\right)\left(\rho\otimes\zeta^{\beta_H}\otimes\zeta^{\beta_C}\right)\left(U^{\beta_C}_{SWAP}U^{\beta_H}_{SWAP}\right)^{\dagger}\right)\\ &\qquad=\sum_{q,m,n,s}\langle s|\,\zeta^{\beta_H}\,|n\rangle\,\langle n|\,\rho\,|m\rangle\,\langle m|\,\zeta^{\beta_C}\,|q\rangle\,|s\rangle\,\langle q| = \zeta^{\beta_H}\rho\zeta^{\beta_C},\end{aligned}$$

and,

$$\begin{aligned}&\text{Tr}_{\zeta^H,\zeta^C}\left(\left(U^{\beta_C}_{SWAP}U^{\beta_H}_{SWAP}\right)\left(\rho\otimes\zeta^{\beta_H}\otimes\zeta^{\beta_C}\right)\left(U^{\beta_H}_{SWAP}U^{\beta_C}_{SWAP}\right)^{\dagger}\right)\\ &\qquad=\sum_{q,m,n,s}\langle q|\,\zeta^{\beta_C}\,|m\rangle\,\langle m|\,\rho\,|n\rangle\,\langle n|\,\zeta^{\beta_H}\,|s\rangle\,|q\rangle\,\langle s| = \zeta^{\beta_C}\rho\zeta^{\beta_H}.\end{aligned}$$

The measurement of the control in the $|\pm\rangle\langle\pm|$ basis gives,

$$\langle\pm|\,\text{Tr}_{\zeta^H,\zeta^C}\left(S(|+\rangle\langle+|\otimes\rho\otimes\zeta^{\beta_H}\otimes\zeta^{\beta_C})\right)|\pm\rangle=\frac{1}{4}\left(\zeta^{\beta_H}\right)+\frac{1}{4}\left(\zeta^{\beta_C}\right)\pm\frac{1}{4}\left(\zeta^{\beta_H}\rho\zeta^{\beta_C}+\zeta^{\beta_C}\rho\zeta^{\beta_H}\right).$$

If the system is a thermal state then the final state is,

$$\zeta^{S}_{f\pm}=\frac{1}{Z}\left(\frac{1}{4}\left(\zeta^{\beta_H}\right)+\frac{1}{4}\left(\zeta^{\beta_C}\right)\pm\frac{1}{2}\left(\zeta^{\beta_H}\zeta^{\beta_S}\zeta^{\beta_C}\right)\right).$$

Which is what we should get. So the calculations seem correct and at least doesn't change the analysis of the system.

A.2.3 Non-Selective Measurement on baths after SWAP Operation

The problem with the previous calculations seems to be that the system remains correlated with the bath after the application of thermalization (SWAP operation). To remove these correlation we will now do an unselected measurement on the bath states and check if the heat change is affected. The Kraus operators of the QS are now:

$$W_{ij} = |0\rangle\langle 0| \otimes \left(\Pi_i^{\beta_H} U_{SWAP}^{\beta_H}\right)\left(\Pi_j^{\beta_C} U_{SWAP}^{\beta_C}\right) + |1\rangle\langle 1| \otimes \left(\Pi_j^{\beta_C} U_{SWAP}^{\beta_C}\right)\left(\Pi_i^{\beta_H} U_{SWAP}^{\beta_H}\right).$$

The final state after the switch action is:

$$\begin{aligned}
S'(\sigma\otimes\rho\otimes\zeta^{\beta_H}\otimes\zeta^{\beta_C}) &= \sum_{i,j} W_{ij}(\sigma\otimes\rho\otimes\zeta^{\beta_H}\otimes\zeta^{\beta_C})W_{ij}^\dagger \\
&= \langle 0|\sigma|0\rangle\,|0\rangle\langle 0| \otimes \left(\zeta^{\beta_H}\otimes\zeta^{\beta_C}\otimes \mathrm{Diag}\,(\rho)\right) \\
+\sum_{i,j}\langle 1|\sigma|0\rangle\,|1\rangle\langle 0| &\otimes \left(\Pi_j^{\beta_C} U_{SWAP}^{\beta_C}\right)\left(\Pi_i^{\beta_H} U_{SWAP}^{\beta_H}\right)\left(\rho\otimes\zeta^{\beta_H}\otimes\zeta^{\beta_C}\right)\left(\Pi_j^{\beta_C} U_{SWAP}^{\beta_C}\right)^\dagger\left(\Pi_i^{\beta_H} U_{SWAP}^{\beta_H}\right)^\dagger \\
+\sum_{i,j}\langle 0|\sigma|1\rangle\,|0\rangle\langle 1| &\otimes \left(\Pi_i^{\beta_H} U_{SWAP}^{\beta_H}\right)\left(\Pi_j^{\beta_C} U_{SWAP}^{\beta_C}\right)\left(\rho\otimes\zeta^{\beta_H}\otimes\zeta^{\beta_C}\right)\left(\Pi_i^{\beta_H} U_{SWAP}^{\beta_H}\right)^\dagger\left(\Pi_j^{\beta_C} U_{SWAP}^{\beta_C}\right)^\dagger \\
&\quad + \langle 1|\sigma|1\rangle\,|1\rangle\langle 1| \otimes \left(\zeta^{\beta_C}\otimes \mathrm{Diag}\,(\rho)\otimes\zeta^{\beta_H}\right).
\end{aligned}$$

where, $\mathrm{Diag}(M)$ is a diagonal matrix with diagonal entries of M. As before, we evaluate the off-diagonal terms and get,

$$\begin{aligned}
&\left(\Pi_j^{\beta_C} U_{SWAP}^{\beta_C}\right)\left(\Pi_i^{\beta_H} U_{SWAP}^{\beta_H}\right)\left(\rho\otimes\zeta^{\beta_H}\otimes\zeta^{\beta_C}\right)\left(\Pi_j^{\beta_C} U_{SWAP}^{\beta_C}\right)^\dagger\left(\Pi_i^{\beta_H} U_{SWAP}^{\beta_H}\right)^\dagger \\
&\quad = \sum_{q,m,n,r,s,p} \langle m|\rho|p\rangle\,|q\rangle\langle s| \otimes \langle n|\zeta^{\beta_H}|s\rangle\,\Pi_i^{\beta_H}|m\rangle\langle r|\Pi_i^{\beta_H} \otimes \langle q|\zeta^{\beta_C}|r\rangle\,\Pi_j^{\beta_C}|n\rangle\langle p|\Pi_j^{\beta_C} \\
&\quad = \langle i|\rho|j\rangle\,|i\rangle\langle j| \otimes \langle j|\zeta^{\beta_H}|j\rangle\,|i\rangle\langle i| \otimes \langle i|\zeta^{\beta_C}|i\rangle\,|j\rangle\langle j|.
\end{aligned}$$

Hot bath state

Partial trace over system and cold bath for off-diagonal terms:

$$\begin{aligned}
&\sum_{i,j} \mathrm{Tr}_{S,C}\left(\Pi_j^{\beta_C} U_{SWAP}^{\beta_C}\right)\left(\Pi_i^{\beta_H} U_{SWAP}^{\beta_H}\right)\left(\rho\otimes\zeta^{\beta_H}\otimes\zeta^{\beta_C}\right)\left(\Pi_j^{\beta_C} U_{SWAP}^{\beta_C}\right)^\dagger\left(\Pi_i^{\beta_H} U_{SWAP}^{\beta_H}\right)^\dagger \\
&\quad = \sum_i \langle i|\rho|i\rangle\langle i|\zeta^{\beta_H}|i\rangle\langle i|\zeta^{\beta_C}|i\rangle\,|i\rangle\langle i| \\
&\quad = \mathrm{Diag}\left(\rho\zeta^{\beta_H}\zeta^{\beta_C}\right).
\end{aligned}$$

Similarly,

$$\sum_{i,j} \mathrm{Tr}_{S,C}\left(\left(\Pi_i^{\beta_H} U_{SWAP}^{\beta_H}\right)\left(\Pi_j^{\beta_C} U_{SWAP}^{\beta_C}\right)\left(\rho\otimes\zeta^{\beta_H}\otimes\zeta^{\beta_C}\right)\left(\Pi_i^{\beta_H} U_{SWAP}^{\beta_H}\right)^\dagger\left(\Pi_j^{\beta_C} U_{SWAP}^{\beta_C}\right)^\dagger\right)$$
$$= \mathrm{Diag}\left(\zeta^{\beta_C}\zeta^{\beta_H}\rho\right).$$

Again taking the system to be a thermal state ζ^{β_S}, after measurement the state of the hot bath is hence,

$$\zeta_{f\pm}^{H} = \frac{\langle\pm|\,\mathrm{Tr}_{S,C}\left(S'(|+\rangle\langle+|\otimes\zeta^{\beta_S}\otimes\zeta^{\beta_H}\otimes\zeta^{\beta_C})\right)|\pm\rangle}{\mathrm{Tr}\left(\langle\pm|\,\mathrm{Tr}_{S,C}\left(S'(|+\rangle\langle+|\otimes\zeta^{\beta_S}\otimes\zeta^{\beta_H}\otimes\zeta^{\beta_C})\right)|\pm\rangle\right)}$$
$$= \frac{1}{Z}\left(\frac{1}{4}\left(\zeta^{\beta_S}\right)+\frac{1}{4}\left(\zeta^{\beta_C}\right)\pm\frac{1}{2}\left(\zeta^{\beta_S}\zeta^{\beta_H}\zeta^{\beta_C}\right)\right).$$

Hence, as we can see, the state at the end remains the same. So the calculations won't change and we still have a discrepancy in heat.

A.2.4 Complete Decorrelation of bath

In the previous calculations, we saw that after each SWAP operation, the bath still has classical correlations with the system and control. So let's try decorrelating the bath after the switch has operated but the control has not been measured. The final state after the switch operation is:

$$\begin{aligned}
S(\sigma\otimes\rho\otimes\zeta^{\beta_H}\otimes\zeta^{\beta_C}) = W(\sigma\otimes\rho\otimes\zeta^{\beta_H}\otimes\zeta^{\beta_C})W^\dagger \\
=&\sigma_{00}\,|0\rangle\langle0|\otimes\left(\zeta^{\beta_H}\otimes\zeta^{\beta_C}\otimes\rho\right)\\
&+\sigma_{10}\,|1\rangle\langle0|\otimes\sum_{q,m,n,p}\langle m|\,\rho\,|p\rangle\,|q\rangle\langle n|\otimes\langle n|\,\zeta^{\beta_H}\,|n\rangle\,|m\rangle\langle q|\otimes\langle q|\,\zeta^{\beta_C}\,|q\rangle\,|n\rangle\langle p|\\
&+\sigma_{01}\,|0\rangle\langle1|\otimes\sum_{q,m,n,p}\langle p|\,\rho\,|m\rangle\,|n\rangle\langle q|\otimes\langle n|\,\zeta^{\beta_H}\,|n\rangle\,|q\rangle\langle m|\otimes\langle q|\,\zeta^{\beta_C}\,|q\rangle\,|p\rangle\langle n|\\
&+\sigma_{11}\,|1\rangle\langle1|\otimes\left(\zeta^{\beta_C}\otimes\rho\otimes\zeta^{\beta_H}\right).
\end{aligned}$$

The local bath states are:

$$\begin{aligned}
\zeta_f^{\beta_H} &= \sigma_{00}\zeta^{\beta_C}+\sigma_{11}\rho,\\
\zeta_f^{\beta_C} &= \sigma_{00}\rho+\sigma_{11}\zeta^{\beta_H}.
\end{aligned}$$

The system + control state is:

$$\begin{aligned}
\rho_f^{S,C} =&\sigma_{00}\,|0\rangle\langle0|\otimes\left(\zeta^{\beta_H}\right)\\
&+\sigma_{10}\,|1\rangle\langle0|\otimes\zeta^{\beta_C}\rho\zeta^{\beta_H}\\
&+\sigma_{01}\,|0\rangle\langle1|\otimes\zeta^{\beta_H}\rho\zeta^{\beta_C}\\
&+\sigma_{11}\,|1\rangle\langle1|\otimes\left(\zeta^{\beta_C}\right).
\end{aligned}$$

The decorrelated state is hence:

$$\begin{aligned}S'(\sigma\otimes\rho\otimes\zeta^{\beta_H}\otimes\zeta^{\beta_C}) = \sigma_{00}\left|0\right\rangle\left\langle 0\right|\otimes\zeta^{\beta_H}\otimes\zeta_f^{\beta_H}\otimes\zeta_f^{\beta_C}\\ +\sigma_{10}\left|1\right\rangle\left\langle 0\right|\otimes\zeta^{\beta_C}\rho\zeta^{\beta_H}\otimes\zeta_f^{\beta_H}\otimes\zeta_f^{\beta_C}\\ +\sigma_{01}\left|0\right\rangle\left\langle 1\right|\otimes\zeta^{\beta_H}\rho\zeta^{\beta_C}\otimes\zeta_f^{\beta_H}\otimes\zeta_f^{\beta_C}\\ +\sigma_{11}\left|1\right\rangle\left\langle 1\right|\otimes\zeta^{\beta_C}\otimes\zeta_f^{\beta_H}\otimes\zeta_f^{\beta_C}.\end{aligned}$$

If the control is prepared in the $\left|+\right\rangle\left\langle +\right|$ state and the states(temperatures) of the system and the baths are taken to be same initially $\left(\zeta^{\beta}\right)$ and the control is measured in the $\left|\pm\right\rangle\left\langle \pm\right|$ basis then:

$$\rho_{f\pm}^S = \frac{1}{2Z}\left(\zeta^\beta\pm\left(\zeta^\beta\right)^3\right),$$
$$Z = \frac{1}{2}\pm\text{Tr}\left(\left(\zeta^\beta\right)^3\right),$$
$$\zeta_{f\pm}^H = \zeta_{f\pm}^C = \zeta^\beta.$$

<u>**Energy from measurement of control system**</u>

The energy change of the system due to measurement of control is:

$$\begin{aligned}\Delta_S &= \text{Tr}\left(H\rho_{f\pm}^S\right) - \text{Tr}\left(H\zeta^\beta\right) = \frac{1}{2Z}\text{Tr}\left(H\zeta^\beta\right)\pm\frac{1}{2Z}\text{Tr}\left(H\left(\zeta^\beta\right)^3\right) - \text{Tr}\left(H\zeta^\beta\right)\\ &= \left(\frac{1-2Z}{2Z}\right)\text{Tr}\left(H\zeta^\beta\right)\pm\frac{1}{2Z}\text{Tr}\left(H\left(\zeta^\beta\right)^3\right).\end{aligned}$$

The energy change of the bath due to measurement of control is:

$$\Delta_{H,C} = 0.$$

Therefore, energy change due of the baths is zero. Again, all the energy seems to be coming from the measurement but this time the expressions match. Hence, the energy due to the measurement is:

$$\Delta_M = \Delta_S = \left(\frac{1-2Z}{2Z}\right)\text{Tr}\left(H\zeta^\beta\right)\pm\frac{1}{2Z}\text{Tr}\left(H\left(\zeta^\beta\right)^3\right).$$

Appendix B

Work Benefit of Channels

Here, we derive the results for the work benefit of the quantum switch channel. We start with proving that the second law holds for the described protocol. We then use the protocol to compare the quantum switch with the classical switch channel. Finally, we work out the example from the text.

B.1 Analysis for the Quantum Switch

B.1.1 Notations and Setup

To study the work benefit of the quantum switch(QS), we will need the following ingredients:

1. A system ρ^S in Hilbert space $\mathbb{H}^S$ and Hamiltonian H^S and a control ρ^C in Hilbert space $\mathbb{H}^C$ and Hamiltonian H^C. We start with $\rho^T = \rho^C \otimes \rho^S$ and call it the "target" with Hamiltonian for it being $H_T = H^C \otimes \mathbb{I} + \mathbb{I} \otimes H^S$. The subsystem will be denoted by the superscript letter.

2. A reservoir at temperature $T(\beta^{-1})$ containing thermal states τ^B in Hilbert space $\mathbb{H}^{\mathrm{B}}$ and Hamiltonian H^B.

3. A channel C which acts on the target. The subscript such as C_{Sw} denotes the kind of channel, here it being the switch channel.

4. $S\left(\rho\right)$ denotes the Von Neumann entropy of ρ and $F\left(\rho\right)$ is its free energy.

5. $\Delta\left(\rho\right)$ denotes the diagonal matrix $\sum_s \langle s|\, \rho\, |s\rangle\, |s\rangle\, \langle s|$.

6. To define work benefit, we follow the protocol:

   (a) Target and bath are initialized in uncorrelated state: $\rho = \rho^T \otimes \tau^B$, for which we don't assign any work benefit/cost: $W_a = 0$.

   (b) A channel is applied only on the target state: $\sigma = C[\rho^T] \otimes \tau^B$. For the work benefit of the QS we will apply C_{Sw}, so: $\sigma = C_{Sw}[\rho^T] \otimes \tau^B$. The work benefit of this operation is $W_b = \mathrm{Tr}\left(H_T\rho^T - H_T C_{Sw}[\rho^T]\right)$.

(c) Unitary interaction is applied between the target and the bath to reset the state of the target: $\sigma' = U\sigma U^\dagger = U\left(C_{Sw}[\rho^T] \otimes \tau^B\right) U^\dagger$ such that $\text{Tr}_B\left(\sigma'\right) = \text{Tr}_B\left(U\left(C_{Sw}[\rho^T] \otimes \tau^B\right) U^\dagger\right) = \rho^T = \rho^C \otimes \rho^S$. This unitary commutes with all Hamiltonians and hence, does not contribute to any energy. For this we see that the work benefit is bounded by the change in free energy of the target: $W_c \leq \Delta F_T = F\left(\sigma^T\right) - F\left(\rho^T\right)$. This bound can be reached following an optimal protocol [SSP14].

(d) Work benefit is got by optimizing over all initial states and protocols:

$$W_{Sw}^{Benefit} = \text{max}_{\rho^T}\left\{W_a + W_b + W_c\right\} = \text{max}_{\rho^T}\beta^{-1}\left(S\left(\rho^T\right) - S\left(C_{Sw}\left(\rho^T\right)\right)\right). \tag{B.1}$$

We will do a lot of playing around with this equation for the quantum switch case, and compare it with the classical switch to see if there is a benefit. Later we will also consider conditional work benefit due to the measurement of the control. But before that, we will prove the second law for this protocol and, we also consider the cost of resetting the ancillae that are used in the implementation of each channel. These ancillae also need to be reset to their initial state.

B.1.2 Proof of Second Law

Here, we prove the second law for the protocol. We start with a target system ρ^T and an uncorrelated bath state τ^B. A unitary transformation acts between them such that $\rho' = U\left(\rho^T \otimes \tau^B\right) U^\dagger$. As the system and bath are initially correlated and they undergo a unitary interaction, we have

$$\begin{aligned} S\left[\rho^T \otimes \tau^B\right] &= S\left[\rho^T\right] + S\left[\tau^B\right] = S\left[\rho'\right] \\ &= S\left[U\left(\rho^T \otimes \tau^B\right) U^\dagger\right] \leq S\left[\text{Tr}_B\left(\rho'\right)\right] + S\left[\text{Tr}_T\left(\rho'\right)\right] = S\left[\rho'^T\right] + S\left[\rho'^B\right] \\ &\implies \Delta S_T + \Delta S_B \geq 0. \end{aligned} \tag{B.2}$$

For such a protocol, we take the change in internal energy as the average change in energy of the system, the heat as the average change in energy of the baths, and the work is implicitly defined to adhere to the first law, i.e.,

$$\Delta U = \text{Tr}\left[H^S \rho'^T\right] - \text{Tr}\left[H^S \rho^T\right],$$

$$Q = \text{Tr}\left[H^B \rho'^B\right] - \text{Tr}\left[H^B \rho^B\right].$$

The first law is given by,

$$\Delta U + Q + W = 0. \tag{B.3}$$

Using this and the above inequality, we get,

$$\Delta F_T + \Delta F_B + W \leq 0.$$

As the thermal state is a state of minimum free energy, we get,

$$W \leq -\Delta F_T.$$

In [SSP14] we see that, one can reach the bound for individual quantum systems, upto a second order of approximation.

B.1.3 Work benefit of ancilla resetting for the Quantum Switch

The quantum switch channel is simulated using two ancillae $\left(\rho^E \otimes \rho^F\right)$, one for each channel in it:

$$C_{Sw}\left(\rho^T\right) = \mathrm{Tr}_{EF}\left(U_{sw}^{TEF}\left(\rho^T \otimes \rho^E \otimes \rho^F\right)\left(U_{sw}^{TEF}\right)^\dagger\right) = \mathrm{Tr}_{EF}\left(\xi\right).$$

For this unitary interaction we have,

$$S\left(\rho^T \otimes \rho^E \otimes \rho^F\right) = S\left(\rho^T\right) + S\left(\rho^E\right) + S\left(\rho^F\right) = S\left(\xi\right).$$

Using subadditivity of entropy we have,

$$\begin{aligned} S\left(\rho^T\right) + S\left(\rho^E\right) + S\left(\rho^F\right) = S\left(\xi\right) \leq S\left(\xi^{TF}\right) + S\left(\xi^{EF}\right) - S\left(\xi^F\right) \\ 0 \leq & S\left(\xi^{TF}\right) - S\left(\rho^T\right) + S\left(\xi^{EF}\right) - S\left(\rho^E\right) - S\left(\xi^F\right) - S\left(\rho^F\right). \end{aligned}$$

From the definition of mutual information, we have,

$$S\left(\xi^{AB}\right) = S\left(\xi^A\right) + S\left(\xi^B\right) - I\left(\xi^{AB}\right),$$

so,

$$\begin{aligned} & 0 \leq S\left(\xi^T\right) + S\left(\xi^F\right) - I\left(\xi^{TF}\right) \\ & -S\left(\rho^T\right) + S\left(\xi^E\right) + S\left(\xi^F\right) - I\left(\xi^{EF}\right) - S\left(\rho^E\right) - S\left(\xi^F\right) - S\left(\rho^F\right) \\ \implies & I\left(\xi^{TF}\right) + I\left(\xi^{EF}\right) \leq \left[S\left(\xi^T\right) - S\left(\rho^T\right)\right] + \left[S\left(\xi^E\right) - S\left(\rho^E\right)\right] + \left[S\left(\xi^F\right) - S\left(\rho^F\right)\right] \\ \implies & I\left(\xi^{TF}\right) + I\left(\xi^{EF}\right) \leq \Delta S_T + \Delta S_E + \Delta S_F. \end{aligned}$$

Reset is done using a thermal state, hence, the work benefit of reset of the ancillae is bounded by the change in free energy:

$$W_{reset}^{Benefit} \leq -\beta^{-1}\left(\Delta S_E + \Delta S_F\right). \tag{B.4}$$

Hence, the bound for work benefit for the complete cycle is:

$$W_{Sw}^{Benefit} + W_{reset}^{Benefit} \leq -\beta^{-1}\left(\Delta S_T + \Delta S_E + \Delta S_F\right) \leq -\beta^{-1}\left(I\left(\xi^{TF}\right) + I\left(\xi^{EF}\right)\right). \tag{B.5}$$

As we know, the mutual information is a positive quantity. Hence, this seems to show that the work benefit of using the quantum switch $\left(W_{Sw}^{Benefit} + W_{reset}^{Benefit}\right)$ is always negative and never be optimal, i.e., $\left(W_{Sw}^{Benefit} + W_{reset}^{Benefit} = 0\right)$, unless the mutual information become zero, which shouldn't be possible for the quantum switch or even superposition of paths.

Exact value of work benefit including reset

Here, we will use a tri-partite mutual information, defined as:

$$I\left(\xi^{TEF}\right) = S\left(\xi^{T}\right) + S\left(\xi^{E}\right) + S\left(\xi^{F}\right) - S\left(\xi^{TE}\right) - S\left(\xi^{TF}\right) - S\left(\xi^{EF}\right) - S\left(\xi\right).$$

Instead of using the subadditivity of entropy to get a bound, we write:

$$\begin{aligned}
S\left(\xi\right) =& S\left(\rho^{T}\right) + S\left(\rho^{E}\right) + S\left(\rho^{F}\right) \\
=& S\left(\xi^{T}\right) + S\left(\xi^{E}\right) + S\left(\xi^{F}\right) - S\left(\xi^{TE}\right) - S\left(\xi^{TF}\right) - S\left(\xi^{EF}\right) - I\left(\xi^{TEF}\right) \\
\implies & I\left(\xi^{TEF}\right) + S\left(\xi^{TE}\right) + S\left(\xi^{TF}\right) + S\left(\xi^{EF}\right) = \Delta S_T + \Delta S_E + \Delta S_F.
\end{aligned}$$

As the work benefit of reset of the ancillae is bounded by the change in free energy we have,

$$W_{reset}^{Benefit} \leq -\beta^{-1}\left(\Delta S_E + \Delta S_F\right),$$

and hence,

$$W_{Sw}^{Benefit} + W_{reset}^{Benefit} \leq -\beta^{-1}\left(I^{T:E:F} + S\left(\xi^{TE}\right) + S\left(\xi^{TF}\right) + S\left(\xi^{EF}\right)\right),$$

where, we reach this bound for the optimized reset protocol and hence,

$$W_{Sw}^{Benefit} + W_{reset}^{Benefit} = -\beta^{-1}\left(I^{T:E:F} + S\left(\xi^{TE}\right) + S\left(\xi^{TF}\right) + S\left(\xi^{EF}\right)\right).$$

B.2 Work Benefit for Classical vs Quantum Switch

The work benefit of the classical switch is (Maximization will be assumed wherever necessary):

$$\begin{aligned}
W_{Cl}^{Benefit} &= \beta^{-1}\left(S\left(\rho_{cl}^{T}\right) - S\left(C_{Cl}\left(\rho_{cl}^{T}\right)\right)\right) = \beta^{-1}\left(S\left(\Delta\left(\rho^{C}\right)\right) + S\left(\rho^{S}\right)\right) \\
&\quad -\beta^{-1} S\left(\rho_{00}^{C}\left|0\right\rangle\left\langle 0\right| \otimes \sum_{ij} A_i B_j \rho^S B_j^{\dagger} A_i^{\dagger} + \rho_{11}^{C}\left|1\right\rangle\left\langle 1\right| \otimes \sum_{ij} B_j A_i \rho^S A_i^{\dagger} B_j^{\dagger}\right) \\
&= \beta^{-1}\left(S\left[\rho^{S}\right] - S\left[\rho_{00}^{C} C_A\left(C_B\left(\rho^{S}\right)\right) + \rho_{11}^{C} C_B\left(C_A\left(\rho^{S}\right)\right)\right] + I\left[\sigma_{Cl}^{CS}\right]\right). \qquad \text{(B.6)}
\end{aligned}$$

where, σ_{Cl}^{CS} is the final state of the system and control after the operation of the classical switch unitary.

The work benefit for the quantum switch is:

$$
\begin{aligned}
W_{Sw}^{Benefit} &= \beta^{-1}\left(S\left(\rho^{T}\right) - S\left(C_{Sw}\left(\rho^{T}\right)\right)\right) = \beta^{-1}\left(S\left(\rho^{C}\right) + S\left(\rho^{S}\right)\right) \\
&-\beta^{-1} S\Bigg(\rho_{00}^{C}\left|0\right\rangle\left\langle 0\right| \otimes \sum_{ij} A_i B_j \rho^S B_j^{\dagger} A_i^{\dagger} + \rho_{11}^{C}\left|1\right\rangle\left\langle 1\right| \otimes \sum_{ij} B_j A_i \rho^S A_i^{\dagger} B_j^{\dagger} \\
&+\rho_{01}^{C}\left|0\right\rangle\left\langle 1\right| \otimes \sum_{ij} A_i B_j \rho^S A_i^{\dagger} B_j^{\dagger} + \rho_{10}^{C}\left|1\right\rangle\left\langle 0\right| \otimes \sum_{ij} B_j A_i \rho^S B_j^{\dagger} A_i^{\dagger}\Bigg) \\
&= \beta^{-1}\left(S\left[\rho^{C}\right] + S\left[\rho^{S}\right]\right) - \beta^{-1} S\left[\rho_{00}^{C} C_A\left(C_B\left(\rho^{S}\right)\right) + \rho_{11}^{C} C_B\left(C_A\left(\rho^{S}\right)\right)\right] \\
&-\beta^{-1} S\Bigg[\Delta\left(\rho^{C}\right) + \rho_{01}^{C}\mathrm{Tr}\left(\sum_{ij} A_i B_j \rho^S A_i^{\dagger} B_j^{\dagger}\right)\left|0\right\rangle\left\langle 1\right| \\
&+\rho_{10}^{C}\mathrm{Tr}\left(\sum_{ij} B_j A_i \rho^S B_j^{\dagger} A_i^{\dagger}\right)\left|1\right\rangle\left\langle 0\right|\Bigg] + \beta^{-1} I\left[\sigma^{CS}\right]\Bigg) \\
&= W_{Cl}^{Benefit} + \beta^{-1}\left(S\left[\rho^{C}\right] - S\left[\chi^{C}\right] + I\left[\sigma^{CS}\right] - I\left[\sigma_{Cl}^{CS}\right]\right),
\end{aligned}
\tag{B.7}
$$

where, σ^{CS} is the final state of the system and control after the operation of the quantum switch unitary, and the final state of the control system is,

$$
\chi^{C} = \Delta\left(\rho^{C}\right) + \rho_{01}^{C}\mathrm{Tr}\left(\sum_{ij} A_i B_j \rho^S A_i^{\dagger} B_j^{\dagger}\right)\left|0\right\rangle\left\langle 1\right| + \rho_{10}^{C}\mathrm{Tr}\left(\sum_{ij} B_j A_i \rho^S B_j^{\dagger} A_i^{\dagger}\right)\left|1\right\rangle\left\langle 0\right|.
$$

Now, we use the coherence function, defined as,

$$
\boldsymbol{C}\left[\chi\right] = \min_{\sigma\epsilon I_r} D\left(\chi||\sigma\right),
$$

where, I_r is the set of incoherent states in the basis of χ and $D\left(\chi||\sigma\right)$ is the relative entropy (Kullback-Leibler divergence). Now, using Lagrange multipliers we find the minimum to be at $\sigma = \Delta\left[\chi\right]$, hence,

$$
\boldsymbol{C}\left[\chi^{C}\right] = S\left[\Delta\left[\chi^{C}\right]\right] - S\left[\chi^{C}\right],
$$

$$
W_{Sw}^{Benefit} = W_{Cl}^{Benefit} + \beta^{-1}\left(S\left[\rho^{C}\right] - S\left[\Delta\left[\chi^{C}\right]\right] + \boldsymbol{C}\left[\chi^{C}\right] + I\left[\sigma^{CS}\right] - I\left[\sigma_{Cl}^{CS}\right]\right).
$$

Notice that $\Delta\left[\rho^{C}\right] = \Delta\left[\chi^{C}\right]$, so,

$$
\begin{aligned}
W_{Sw}^{Benefit} &= W_{Cl}^{Benefit} + \beta^{-1}\left(S\left[\rho^{C}\right] - S\left[\Delta\left(\rho^{C}\right)\right] + \boldsymbol{C}\left[\chi^{C}\right] + I\left[\sigma^{CS}\right] - I\left[\sigma_{Cl}^{CS}\right]\right) \\
&= W_{Cl}^{Benefit} + \beta^{-1}\left(\boldsymbol{C}\left[\chi^{C}\right] - \boldsymbol{C}\left[\rho^{C}\right] + I\left[\sigma^{CS}\right] - I\left[\sigma_{Cl}^{CS}\right]\right).
\end{aligned}
$$

Therefore,

$$
W_{Sw}^{Benefit} = W_{Cl}^{Benefit} + \beta^{-1}\left(\Delta\boldsymbol{C}\left[\rho^{C}\right] + I\left[\sigma^{CS}\right] - I\left[\sigma_{Cl}^{CS}\right]\right). \tag{B.8}
$$

Example

We will take the simple case of a qubit system with the control in the $|+\rangle\langle+|$ state. We shall put two thermalization channels (to same temperature β) in the switch and initiate the system in the thermal state ζ^β with the same temperature β, where, $A=\sqrt{\zeta^\beta}$. The final state of the control is,

$$\begin{aligned}\chi^C =& \rho^C_{00}|0\rangle\langle 0| + \rho^C_{11}|1\rangle\langle 1| \\ &+\frac{1}{d^2}\text{Tr}\left(\sum_{ij} AU_iAU_j\zeta^\beta U_i^\dagger AU_j^\dagger A\right)\rho^C_{01}|0\rangle\langle 1| + \frac{1}{d^2}\text{Tr}\left(\sum_{ij} AU_jAU_i\zeta^\beta U_j^\dagger AU_i^\dagger A\right)\rho^C_{10}|1\rangle\langle 0| \\ =&\frac{1}{2}\left(\mathbb{I}+\text{Tr}\left(\left(\zeta^\beta\right)^3\right)(|0\rangle\langle 1|+|1\rangle\langle 0|)\right) = \frac{1}{2}\left(\mathbb{I}+\text{Tr}\left(\left(\zeta^\beta\right)^3\right)(\sigma_X)\right).\end{aligned}$$

The entropy of the thermal state can be written as,

$$S\left[\zeta^\beta\right] = -\text{Tr}\left[\zeta^\beta \ln\left(\zeta^\beta\right)\right] = -\text{Tr}\left[\frac{e^{-\beta H}}{Z}\ln\left(\frac{e^{-\beta H}}{Z}\right)\right] = \beta\text{Tr}\left[H\zeta^\beta\right] + \ln(Z)\,\text{Tr}\left[\zeta^\beta\right] = \beta\left(E_{\zeta^\beta} - F_{\zeta^\beta}\right),$$

and,

$$S\left(\Delta\left(\rho^C\right)\right) = S\left(\Delta\left(\chi^C\right)\right) = \log(2).$$

The coherence functions are,

$$\boldsymbol{C}\left(\rho^C\right) = \boldsymbol{C}(|+\rangle\langle+|) = S[\Delta[|+\rangle\langle+|]] - S[|+\rangle\langle+|] = \log(2),$$

$$\begin{aligned}\boldsymbol{C}\left(\chi^C\right) = \boldsymbol{C}\left(\frac{1}{2}\left(\mathbb{I}+\text{Tr}\left(\left(\zeta^\beta\right)^3\right)(\sigma_X)\right)\right) &= S\left[\frac{1}{2}(\mathbb{I})\right] - S\left[\frac{1}{2}(\mathbb{I}+a\sigma_X)\right] \\ &=\log(2) + \left(\frac{1+a}{2}\right)\log\left(\frac{1+a}{2}\right) + \left(\frac{1-a}{2}\right)\log\left(\frac{1-a}{2}\right),\end{aligned}$$

where,

$$a = \text{Tr}\left(\left(\zeta^\beta\right)^3\right).$$

So the change in coherence function and the mutual information are,

$$\Delta\boldsymbol{C}\left[\rho^C\right] = \boldsymbol{C}\left[\chi^C\right] - \boldsymbol{C}\left[\rho^C\right] = \left(\frac{1+a}{2}\right)\log\left(\frac{1+a}{2}\right) + \left(\frac{1-a}{2}\right)\log\left(\frac{1-a}{2}\right),$$

$$I\left[\sigma^{CS}_{Cl}\right] = 0.$$

The work benefit for the classical case is,

$$W^{Benefit}_{Cl} = 0.$$

The mutual information can be written as,

$$
\begin{aligned}
I\left[\sigma^{CS}\right] &= S\left[\zeta^{\beta}\right] + S\left[\frac{1}{2}\left(\mathbb{I} + a\sigma_X\right)\right] - S\left(\frac{1}{2}\left(\mathbb{I}\otimes\zeta^{\beta} + \sigma_X \otimes \left(\zeta^{\beta}\right)^3\right)\right) \\
&= S\left[\zeta^{\beta}\right] - S\left(\frac{1}{2}\left(\mathbb{I}\otimes\zeta^{\beta} + \sigma_X \otimes \left(\zeta^{\beta}\right)^3\right)\right) - \left(\frac{1+a}{2}\right)\log\left(\frac{1+a}{2}\right) - \left(\frac{1-a}{2}\right)\log\left(\frac{1-a}{2}\right).
\end{aligned}
$$

We simplify this term by term. So, we have,

$$
\begin{aligned}
& S\left(\frac{1}{2}\left(\mathbb{I}\otimes\zeta^{\beta} + \sigma_X \otimes \left(\zeta^{\beta}\right)^3\right)\right) \\
&= -\mathrm{Tr}\left[\left(\frac{1}{2}\left(\mathbb{I}\otimes\zeta^{\beta} + \sigma_X \otimes \left(\zeta^{\beta}\right)^3\right)\right)\ln\left(\left(\frac{\mathbb{I}\otimes\zeta^{\beta}}{2}\right)\left(\mathbb{I}\otimes\mathbb{I} + \sigma_X\otimes\left(\zeta^{\beta}\right)^2\right)\right)\right] \\
&= -\mathrm{Tr}\left[\left(\frac{1}{2}\left(\mathbb{I}\otimes\zeta^{\beta} + \sigma_X \otimes \left(\zeta^{\beta}\right)^3\right)\right)\left(\ln\left(\frac{\mathbb{I}\otimes\zeta^{\beta}}{2}\right) + \ln\left(\mathbb{I}\otimes\mathbb{I} + \sigma_X\otimes\left(\zeta^{\beta}\right)^2\right)\right)\right] \\
&= -\mathrm{Tr}\left[\left(\frac{\mathbb{I}\otimes\zeta^{\beta}}{2}\right)\ln\left(\frac{\mathbb{I}\otimes\zeta^{\beta}}{2}\right)\right] - \mathrm{Tr}\left[\left(\frac{\sigma_X\otimes\left(\zeta^{\beta}\right)^3}{2}\right)\ln\left(\frac{\mathbb{I}\otimes\zeta^{\beta}}{2}\right)\right] \\
& -\mathrm{Tr}\left[\left(\frac{\mathbb{I}\otimes\zeta^{\beta}}{2}\right)\ln\left(\mathbb{I}\otimes\mathbb{I} + \sigma_X\otimes\left(\zeta^{\beta}\right)^2\right)\right] - \mathrm{Tr}\left[\left(\frac{\sigma_X\otimes\left(\zeta^{\beta}\right)^3}{2}\right)\ln\left(\mathbb{I}\otimes\mathbb{I} + \sigma_X\otimes\left(\zeta^{\beta}\right)^2\right)\right].
\end{aligned}
$$

Now the terms,

$$
-\mathrm{Tr}\left[\left(\frac{\mathbb{I}\otimes\zeta^{\beta}}{2}\right)\ln\left(\frac{\mathbb{I}\otimes\zeta^{\beta}}{2}\right)\right] = \log(2) + S\left[\zeta^{\beta}\right] = \log(2) + \beta\left(E_{\zeta^{\beta}} - F_{\zeta^{\beta}}\right),
$$

and,

$$
-\mathrm{Tr}\left[\left(\frac{\sigma_X\otimes\left(\zeta^{\beta}\right)^3}{2}\right)\ln\left(\frac{\mathbb{I}\otimes\zeta^{\beta}}{2}\right)\right] = -\frac{1}{2}\mathrm{Tr}\left[\left(\sigma_X\ln\left(\frac{\mathbb{I}}{2}\right)\otimes\left(\zeta^{\beta}\right)^3 + \sigma_X\otimes\left(\zeta^{\beta}\right)^3\ln\left(\zeta^{\beta}\right)\right)\right] = 0.
$$

Hence, the work benefit of the quantum switch for this case is,

$$
W_{Sw}^{Benefit} = \beta^{-1}\left(-\ln(2) + \mathrm{Tr}\left[\left(\frac{\mathbb{I}\otimes\zeta^{\beta}}{2}\right)\left(\mathbb{I}\otimes\mathbb{I} + \sigma_X\otimes\left(\zeta^{\beta}\right)^2\right)\ln\left(\mathbb{I}\otimes\mathbb{I} + \sigma_X\otimes\left(\zeta^{\beta}\right)^2\right)\right]\right). \tag{B.9}
$$

We use this for the numerical analysis for comparing the quantum switch and the classical switch. We see that this becomes negative, indicating that the classical switch is better for work extraction. This may be due to creation and consumption of quantum correlations in the quantum switch operation.

Appendix C

Quasiprobabilites and Fluctuation Theorem

C.1 Approaching Quasiprobability Distribution

Here, we will discuss the weak measurement of work following the publication by Lostaglio [Los18]. We apply the weak measurement and modify the two-point measurement scheme to measure the energy of the system ρ. We will change the notation a little here. The initial Hamiltonian will be $H = \sum_i \epsilon_i |i\rangle\langle i| = \sum_i \epsilon_i E_i$ with $E_i = |i\rangle\langle i|$ being the initial energy projectors and the final Hamiltonian in the Heisenberg picture will be $U^\dagger H U = \sum_j \epsilon_j \Pi_j$ where, $\Pi_j = U^\dagger |j\rangle\langle j| U$ are the final energy projectors in the Heisenberg picture. With this let us now get the MH distribution from the modified TPM protocol.

C.1.1 Modified Two-Point Measurement Scheme

Following the protocol in 4.2, we initialize the measurement apparatus (M) in the state:

$$|\Psi\rangle = \int dx G(x) |x\rangle ,$$

where, $G(x)$ is a Gaussian distribution, centered at 0 and with standard deviation s given by,

$$G(x) = \left(\pi s^2\right)^{-1/4} \exp\left[\frac{-x^2}{2s^2}\right] .$$

The interaction unitary U_{int} is now applied to couple ρ and M,

$$U_{int} = e^{-iE_i \otimes P} = \left(E_i \otimes e^{-iP} + E_i^\perp \otimes \mathbb{I}\right) ,$$

we define,

$$\begin{aligned}\rho^{11} &= E_i \rho E_i, \\ \rho^{01} &= E_i^{\perp} \rho E_i, \\ \rho^{10} &= E_i \rho E_i^{\perp}, \\ \rho^{00} &= E_i^{\perp} \rho E_i^{\perp},\end{aligned}$$

where, $E_i^{\perp}$ projectors onto the orthogonal subspace of E_i, and hence,

$$U_{int}\left(\rho \otimes |\Psi\rangle\langle\Psi|\right) U_{int}^{\dagger} = \sum_{k,k'=0}^{1} \rho^{kk'} \otimes \int_{-\infty}^{+\infty} dx dy G\left(x-k\right) G\left(y-k'\right) |x\rangle\langle y| .$$

Unselective projective measurement of $\{|x\rangle\langle x|\}$ on the measurement device followed by the unitary driving U is given by:

$$\begin{aligned}\text{Tr}_M\left[U U_{int}\left(\rho \otimes |\Psi\rangle\langle\Psi|\right) U_{int}^{\dagger} U^{\dagger}\right] &= \sum_{k,k'=0}^{1} U\rho^{kk'} U^{\dagger} \otimes \int_{-\infty}^{+\infty} dz \int\int_{-\infty}^{+\infty} dx dy G\left(x-k\right) G\left(y-k'\right) \delta_{x,z}\delta_{y,z} \\ &= \sum_{k,k'=0}^{1} U\rho^{kk'} U^{\dagger} \otimes \int_{-\infty}^{+\infty} dz G\left(z-k\right) G\left(z-k'\right).\end{aligned}$$

This unitary is followed by a final energy measurement. The expectation value of the device position, upon post-selecting outcome ϵ_j in the final energy measurement is:

$$\langle X\rangle_{\epsilon_j} = \frac{\text{Tr}\left(\left(|\tilde{j}\rangle\langle\tilde{j}| \otimes X\right) U_{int}\left(\rho \otimes |\Psi\rangle\langle\Psi|\right) U_{int}^{\dagger}\right)}{q_j} = \frac{\text{Tr}\left(X\sigma_j\right)}{q_j},$$

where,

$$\begin{aligned}\left|\tilde{j}\right\rangle &= U^{\dagger}\left|j\right\rangle, \\ \sigma_j &= \left\langle\tilde{j}\right| U_{int}\left(\rho \otimes |\Psi\rangle\langle\Psi|\right) U_{int}^{\dagger}\left|\tilde{j}\right\rangle,\end{aligned}$$

and q_j is the normalization given by,

$$
\begin{aligned}
q_j = \text{Tr}\left(\sigma_j\right) = \int dz \sum_{k,k'=0}^{1} \langle\tilde{j}|\, \rho^{kk'} \,|\tilde{j}\rangle \int_{-\infty}^{+\infty} dxdyG\left(x-k\right)G\left(y-k'\right)\delta_{xz}\delta_{yz} \\
&= \sum_{k,k'=0}^{1} \langle\tilde{j}|\, \rho^{kk'} \,|\tilde{j}\rangle \int_{-\infty}^{+\infty} dzG\left(z-k\right)G\left(z-k'\right) \\
&= \langle\tilde{j}|\, \rho^{00} \,|\tilde{j}\rangle \int_{-\infty}^{+\infty} dz \left|G\left(z\right)\right|^2 \\
&+ \langle\tilde{j}|\, \rho^{11} \,|\tilde{j}\rangle \int_{-\infty}^{+\infty} dz \left|G\left(z-1\right)\right|^2 \\
&+ \langle\tilde{j}|\, \rho^{01} \,|\tilde{j}\rangle \int_{-\infty}^{+\infty} dzG\left(z\right)G\left(z-1\right) \\
&+ \langle\tilde{j}|\, \rho^{10} \,|\tilde{j}\rangle \int_{-\infty}^{+\infty} dzG\left(z-1\right)G\left(z\right) \\
&= \langle\tilde{j}|\, \rho^{00} \,|\tilde{j}\rangle + \langle\tilde{j}|\, \rho^{11} \,|\tilde{j}\rangle \\
&+ 2\text{Re}\,\langle\tilde{j}|\, \rho^{01} \,|\tilde{j}\rangle \left(\pi s^2\right)^{-1/2}\left(e^{-1/2s^2}\right)\int_{-\infty}^{+\infty} dz\text{exp}\left[\frac{-\left(z^2-z\right)}{s^2}\right] \\
&= \langle\tilde{j}|\, \rho^{00} \,|\tilde{j}\rangle + \langle\tilde{j}|\, \rho^{11} \,|\tilde{j}\rangle + 2\text{Re}\,\langle\tilde{j}|\, \rho^{01} \,|\tilde{j}\rangle \left(e^{-1/\left(4s^2\right)}\right).
\end{aligned}
$$

Finally we get,

$$
\langle X\rangle_{\epsilon_j} = \frac{\langle\tilde{j}|\, \rho^{11} \,|\tilde{j}\rangle + e^{-1/\left(4s^2\right)}\text{Re}\,\langle\tilde{j}|\, \rho^{10} \,|\tilde{j}\rangle}{\langle\tilde{j}|\, \rho \,|\tilde{j}\rangle - 2\left(1-e^{-1/(4s^2)}\right)\text{Re}\,\langle\tilde{j}|\, \rho^{10} \,|\tilde{j}\rangle}.
$$

Now in the strong measurement regime $s \to 0$:

$$
\langle X\rangle_{\epsilon_j} = \frac{\langle\tilde{j}|\, \rho^{11} \,|\tilde{j}\rangle}{q_j} = \frac{\text{Tr}\left(UE_i\rho E_iU^\dagger\,|j\rangle\,\langle j|\right)}{q_j} = \frac{p_{tpm}\left(w|P\right)}{p_f\left(\epsilon_j\right)}. \tag{C.1}
$$

As we have seen, this leads to the interpretation of $\langle X\rangle_{\epsilon_j}$ as the probability of ϵ_i being the initial energy measurement given that ϵ_j is the second measured energy value. In the weak measurement regime $s \to \infty$:

$$
\begin{aligned}
\langle X\rangle_{\epsilon_j} &= \frac{\langle\tilde{j}|\, \rho^{11} \,|\tilde{j}\rangle + \text{Re}\,\langle\tilde{j}|\, \rho^{10} \,|\tilde{j}\rangle}{\langle\tilde{j}|\, \rho \,|\tilde{j}\rangle} = \text{Re}\frac{\langle\tilde{j}|\, E_i\rho\left(E_i+E_i^\perp\right) |\tilde{j}\rangle}{\langle\tilde{j}|\, \rho \,|\tilde{j}\rangle} \\
&=\text{Re}\frac{\langle\tilde{j}|\, E_i\rho\left(\mathbb{I}\right) |\tilde{j}\rangle}{\langle\tilde{j}|\, \rho \,|\tilde{j}\rangle} = \text{Re}\frac{\text{Tr}\left(\rho E_i\Pi_j\right)}{\text{Tr}\left(\rho\Pi_j\right)} = \frac{p_{ji}^{MH}}{p_f\left(\epsilon_j\right)}
\end{aligned} \tag{C.2}
$$

Hence, we get the MH distribution from the modified TPM protocol.

C.1.2 Method of Best-Possible Estimate

Another method to get to this distribution is to try to find the best possible estimate of the initial energy measurement from the final energy similar to Allahverdyan's work [All14]. We need the definitions: $H_I = H$, $H_F = U^\dagger H U$ and $\Pi_j = |\tilde{j}\rangle\langle\tilde{j}|$. To get the best possible estimate, we define a function f on H_F which we optimize to get the best estimate. We will see that the MH distribution will arise out of this optimization. We have the function:

$$f\left(H_F\right) = \sum_j f_j \Pi_j$$

We optimize this to get f_j. To optimize we use the statistical deviation [Hal04] between the two operators defined as:

$$\text{Tr}\left[\rho\left(f\left(H_F\right) - H_I\right)^2\right]. \tag{C.3}$$

Using differentiation, we can minimize this to get,

$$f_j = \text{Re}\left(\frac{\text{Tr}\left[\Pi_j \rho H_I\right]}{\text{Tr}\left[\Pi_j \rho\right]}\right) = \sum_i \epsilon_i\left(0\right) p_{i|j}$$

where, $p_{i|j}$ is the probability of measuring ϵ_i given ϵ_j was measured in the second measurement. Hence,

$$\text{Tr}\left[\Pi_j\rho\right] p_{i|j} = p_j p_{i|j} = \text{ReTr}\left[\rho E_i \Pi_j\right] = p_{ji}^{MH} \tag{C.4}$$

C.2 Getting Coherent part from Negativity of Margenau-Hill Distribution

Now that we have seen how the MH distribution is derived, let us see look at how the negativity of this distribution affects the fluctuation term Υ. The Margenau-Hill distribution can be re-written as:

$$\begin{aligned} \text{ReTr}\left[\rho E_i \Pi_j\right] &= \text{Tr}\left[\rho E_i \Pi_j E_i\right] + \frac{1}{2}\text{Tr}\left[\left(\rho - \rho_i'\right)\Pi_j\right], \\ \rho_i' &= E_i \rho E_i + \left(\mathbb{I} - E_i\right)\rho\left(\mathbb{I} - E_i\right). \end{aligned} \tag{C.5}$$

The second terms gives the non-classical part. Now let us look at $\Upsilon$and substitute the re-written expression ($p_{ij} \equiv p_{ji}^{MH}$):

$$\begin{aligned} \Upsilon &= \sum_{i,j} p_{ij} e^{-\beta(\epsilon_j - \epsilon_i)} = \text{Re}\sum_{i,j} e^{-\beta(\epsilon_j-\epsilon_i)}\text{Tr}\left[\rho E_i \Pi_j\right] \\ &= \sum_{i,j} e^{-\beta(\epsilon_j-\epsilon_i)}\text{Tr}\left[\rho E_i \Pi_j E_i\right] + \sum_{i,j} e^{-\beta(\epsilon_j-\epsilon_i)}\frac{1}{2}\text{Tr}\left[\left(\rho - \rho'\right)\Pi_j\right] \\ &= \Upsilon_d + \Upsilon_c. \end{aligned}$$

The classical part of Υ is

$$\Upsilon_d = \sum_{i,j} e^{-\beta(\epsilon_j-\epsilon_i)} \mathrm{Tr}\left[E_i \rho E_i \Pi_j\right] = \mathrm{Tr}\left[\left(\rho^{eq}\right)^{-\frac{1}{2}} \rho \left(\rho^{eq}\right)^{-\frac{1}{2}} U^\dagger \rho^{eq} U\right] = \Upsilon - \Upsilon_c.$$

Therefore we have (ρ^{eq} is the Gibb's state),

$$\Upsilon = \frac{1}{2}\mathrm{Tr}\left[\left[\left(\rho^{eq}\right)^{-1}, \rho\right]_+ U^\dagger \rho^{eq} U\right] = \mathrm{ReTr}\left[\rho \left(\rho^{eq}\right)^{-1} U^\dagger \rho^{eq} U\right], \tag{C.6}$$

$$\Upsilon_d = \sum_{i,j} e^{-\beta(\epsilon_j-\epsilon_i)} \mathrm{Tr}\left[\rho E_i \Pi_j E_i\right] = \mathrm{Tr}\left[\left(\rho^{eq}\right)^{-\frac{1}{2}} \rho \left(\rho^{eq}\right)^{-\frac{1}{2}} U^\dagger \rho^{eq} U\right], \tag{C.7}$$

$$\Upsilon_c = \sum_{i,j} e^{-\beta(\epsilon_j-\epsilon_i)} \frac{1}{2}\mathrm{Tr}\left[\left(\rho - \rho_i'\right)\Pi_j\right] = \frac{1}{2}\mathrm{Tr}\left[\left[\left(\rho^{eq}\right)^{-1}, \rho\right]_+ U^\dagger \rho^{eq} U\right] - \mathrm{Tr}\left[\left(\rho^{eq}\right)^{-\frac{1}{2}} \rho \left(\rho^{eq}\right)^{-\frac{1}{2}} U^\dagger \rho^{eq} U\right]. \tag{C.8}$$

C.3 Analysing Υ

Here we analyze $\Upsilon = \Upsilon_d + \Upsilon_c$. First, let's find out when the non-classical part is 0:

$$\begin{aligned}
\Upsilon_c = \frac{1}{2}\mathrm{Tr}\left[\left[\left(\rho^{eq}\right)^{-1}, \rho\right]_+ U^\dagger \rho^{eq} U\right] - \mathrm{Tr}\left[\left(\rho^{eq}\right)^{-\frac{1}{2}} \rho \left(\rho^{eq}\right)^{-\frac{1}{2}} U^\dagger \rho^{eq} U\right] = 0 \\
\implies \mathrm{Tr}\left[\left\{\frac{1}{2}\left[\left(\rho^{eq}\right)^{-1}, \rho\right]_+ - \left(\rho^{eq}\right)^{-\frac{1}{2}} \rho \left(\rho^{eq}\right)^{-\frac{1}{2}}\right\} U^\dagger \rho^{eq} U\right] = 0.
\end{aligned}$$

One trivial way, this is true is when,

$$\begin{aligned}
\frac{1}{2}\left[\left(\rho^{eq}\right)^{-1}, \rho\right]_+ = \left(\rho^{eq}\right)^{-\frac{1}{2}} \rho \left(\rho^{eq}\right)^{-\frac{1}{2}} \\
\implies \left(\rho^{eq}\right)^{-1} \rho + \rho \left(\rho^{eq}\right)^{-1} = 2\left(\rho^{eq}\right)^{-\frac{1}{2}} \rho \left(\rho^{eq}\right)^{-\frac{1}{2}} \\
\implies \left(\rho \left(\rho^{eq}\right)^{\frac{1}{2}} - \left(\rho^{eq}\right)^{\frac{1}{2}} \rho\right)\left(\rho^{eq}\right)^{\frac{1}{2}} - \left(\rho^{eq}\right)^{\frac{1}{2}}\left(\rho \left(\rho^{eq}\right)^{\frac{1}{2}} - \left(\rho^{eq}\right)^{\frac{1}{2}} \rho\right) = 0 \\
\implies \left[\left[\rho, \left(\rho^{eq}\right)^{\frac{1}{2}}\right]_-, \left(\rho^{eq}\right)^{\frac{1}{2}}\right]_- = 0.
\end{aligned} \tag{C.9}$$

So, if $[\rho, H]_- = 0$, then $\Upsilon_c = 0$. With this trivial result, let us move on to find the bounds of Υ for different cases of initial states and unitary evolution. From here on, we will primarily focus on the case where, ρ is a qubit.

C.3.1 Non-coherent(Diagonal) Initial States

Let's start with the case where, $\rho = \rho_d$, i.e., it is diagonal in the initial energy basis. As we saw previously $\Upsilon_c = 0$ for any ρ_d.

$$\Upsilon = \Upsilon_d = \sum_{i,j} e^{-\beta(\epsilon_j-\epsilon_i)} \mathrm{Tr}\left[E_i \rho_d E_i \Pi_j\right] = \mathrm{Tr}\left[\left(\rho^{eq}\right)^{-\frac{1}{2}} \rho_d \left(\rho^{eq}\right)^{-\frac{1}{2}} U^\dagger \rho^{eq} U\right] = \mathrm{Tr}\left[\rho_d \left(\rho^{eq}\right)^{-1} U^\dagger \rho^{eq} U\right].$$

Using Jensen's inequality:

$$e^{-\beta \mathrm{Tr}[\rho_d(H'-H)]} \leq \sum_{i,j} \mathrm{Tr}\left[E_i \rho_d E_i \Pi_j\right] e^{-\beta(\epsilon_j-\epsilon_i)}$$

$$\implies e^{-\beta \Delta E} \leq \Upsilon_d \implies -\beta \Delta E \leq \ln \Upsilon^{Cl}. \tag{C.10}$$

$\Upsilon_d = 1$ when:

$$\sum_{i,j} e^{-\beta(\epsilon_j-\epsilon_i)} \mathrm{Tr}\left[E_i \rho_d E_i \Pi_j\right] = 1 = \mathrm{Tr}\left[\rho_d \left(\rho^{eq}\right)^{-1} U^\dagger \rho^{eq} U\right] = \sum_{i,j} p_i e^{-\beta(\epsilon_j-\epsilon_i)} \left|\langle i\left|U\right|j\rangle\right|^2.$$

This happens when:

$$\rho_d = \rho^{eq}, \tag{C.11}$$

that is, the initial state is the thermal state. Also when:

$$[U, H]_- = 0, \tag{C.12}$$

that is, the evolution does not provide any energy and does not create coherence in the energy levels. Even for non-coherent initial states, the evolution can result in $\Upsilon < 1$ or $\Upsilon > 1$, as a result of the shift from the equilibrium state or the generation of coherence by the unitary evolution. Let us take the example of a qubit, and study the effect of coherence: Let,

$$\xi = \left|\langle i\left|U\right|j\rangle\right|^2 = \left|\langle j\left|U\right|i\rangle\right|^2, \tag{C.13}$$

for $i \neq j$, This denotes the transition probabilities. So,

$$1 - \xi = \left|\langle i\left|U\right|i\rangle\right|^2,$$

and,

$$0 \leq \xi \leq 1. \tag{C.14}$$

Hence, for qubits we have:

$$\Upsilon_d = \sum_{i,j} p_i e^{-\beta(\epsilon_j-\epsilon_i)} \left|\langle i\left|U\right|j\rangle\right|^2 = \sum_{i\neq j} p_i e^{-\beta(\epsilon_j-\epsilon_i)} \xi + \sum_i p_i \left|\langle i\left|U\right|i\rangle\right|^2 = \left(\sum_{i\neq j} p_i e^{-\beta(\epsilon_j-\epsilon_i)} - 1\right) \xi + 1$$
$$= 1 + (\chi - 1)\,\xi. \tag{C.15}$$

χ denotes the shift of the initial diagonal state from the equilibrium state. It is one when $\rho_d = \rho^{eq}$. while, ξ gives the transition probabilities and an idea of the generation of coherence. For simplicity, let us take $\epsilon_0 = 0, \epsilon_1 = \Delta$. We have,

$$\chi - 1 = \sum_{i\neq j} p_i e^{-\beta(\epsilon_j-\epsilon_i)} - 1.$$

We make the following re-definitions,

$$e^{-\beta\Delta} = \frac{1-r_\beta}{1+r_\beta} \implies e^{\beta\Delta} = \frac{1+r_\beta}{1-r_\beta},$$

$$p_0 = \frac{1+r_z}{2}, p_1 = \frac{1-r_z}{2}.$$

where, r_β is the magnitude of the radius of the Gibb's state on the Bloch sphere and r_z is the magnitude of the σ_Z axis projection of the radius of the initial state ρ.

$$\chi - 1 = \left(\frac{1+r_z}{2}\right)\frac{1-r_\beta}{1+r_\beta} + \left(\frac{1-r_z}{2}\right)\frac{1+r_\beta}{1-r_\beta} - 1 = \frac{1-r_\beta+r_z-r_zr_\beta}{2\left(1+r_\beta\right)} + \frac{1+r_\beta-r_z-r_zr_\beta}{2\left(1-r_\beta\right)} - 1$$
$$= \frac{1-r_\beta+r_z-r_zr_\beta}{2\left(1+r_\beta\right)} + \frac{-1+3r_\beta-r_z-r_zr_\beta}{2\left(1-r_\beta\right)} = \frac{-2r_zr_\beta+2r_\beta^2}{\left(1-r_\beta^2\right)} = \frac{2r_\beta\left(r_\beta-r_z\right)}{\left(1-r_\beta^2\right)}$$

$$\implies \chi - 1 = \frac{2r_\beta\left(r_\beta-r_z\right)}{\left(1-r_\beta^2\right)}. \tag{C.16}$$

Therefore,

$$\Upsilon_d = 1 + (\chi-1)\,\xi = 1 + \left(\frac{2r_\beta\left(r_\beta-r_z\right)}{\left(1-r_\beta^2\right)}\right)\xi = 1 + \left(\frac{2r_\beta^2}{1-r_\beta^2}\right)\xi - \left(\frac{2r_zr_\beta}{1-r_\beta^2}\right)\xi \tag{C.17}$$

The maxima of $(\chi-1)$ can approach positive infinity, while the minima is $\sqrt{1-r_z^2}-1$ at $r_\beta = \frac{r_z}{1+\sqrt{1-r_z^2}}$. This has been found using wolframalpha. So, in summary we have,

$$\sqrt{1-r_z^2} - 1 \leq (\chi-1) < \infty, \tag{C.18}$$

$$0 \leq \xi \leq 1, \tag{C.19}$$

$$e^{-\beta\Delta E} \leq \Upsilon^{Cl} < \infty \implies -\beta\Delta E \leq \ln\Upsilon^{Cl} < \infty. \tag{C.20}$$

Notice:-

- $\Upsilon_d = 1$, when $\xi = 0$ or $r_z = r_\beta$.
- $\Upsilon_d < 1$, when $\xi \neq 0$ or $r_z > r_\beta$, i.e., the initial state is more pure than the equilibrium state.
- $\Upsilon_d > 1$, when $\xi \neq 0$ or $r_z < r_\beta$, i.e., the initial state is more mixed than the equilibrium state.

In effect, any value of $\Upsilon \neq 1$ tells that there is some coherence generation due to the unitary for diagonal, non-equilibrium initial states. The purity of the initial state makes $\Upsilon < 1$ or $\Upsilon > 1$. So, coherence generation is necessary for $\Upsilon \neq 1$ for diagonal initial states but not sufficient as we have to guarantee that the initial state is not the equilibrium state.

C.3.2 Coherent Initial States

Any state ρ can be written as $\rho = \rho_d + \rho_c$ where, ρ_c has only the off-diagonal terms, or the "coherent part" of the state. We have already seen that $\Upsilon_c = 0$ for ρ_d and we have studied Υ_d for ρ_d. Now, we study the effect of ρ_c on Υ. $\Upsilon_d = 0$ for ρ_c of course as $E_i \rho_c E_i = 0$. We only need to study Υ_c.

Effect of coherence on Υ

$$
\begin{aligned}
\Upsilon_c = \Upsilon_c^{NCl} &= \sum_{i,j} e^{-\beta(\epsilon_j - \epsilon_i)} \frac{1}{2} \mathrm{Tr}\left[\left(\rho_c - \rho_i^{c\prime}\right) \Pi_j\right] = \frac{1}{2} \mathrm{Tr}\left[\left[\left(\rho^{eq}\right)^{-1}, \rho_c\right]_+ U^\dagger \rho^{eq} U\right] \\
&= \frac{1}{2} \sum_{i,j} e^{-\beta(\epsilon_j - \epsilon_i)} \mathrm{Tr}\left[\rho_c \Pi_j\right] - \frac{1}{2} \sum_{i,j} e^{-\beta(\epsilon_j - \epsilon_i)} \mathrm{Tr}\left[\left(\mathbb{I} - E_i\right) \rho_c \left(\mathbb{I} - E_i\right) \Pi_j\right],
\end{aligned}
$$

where,

$$
\rho_i^{c\prime} = \left(\mathbb{I} - E_i\right) \rho_c \left(\mathbb{I} - E_i\right),
$$

First notice, $\Upsilon_c = 0$ when, $[U, H] = 0$. Therefore, whenever there is no generation of coherence because of the evolution, we have $\Upsilon = 1$ which agrees with everything done previously. For the case of qubits,

$$
\left(\mathbb{I} - E_i\right) = E_i^\perp,
$$

Therefore,

$$
\mathrm{Tr}\left[\left(\mathbb{I} - E_i\right) \rho_c \left(\mathbb{I} - E_i\right) \Pi_j\right] = 0,
$$

and so,

$$
\begin{aligned}
\Upsilon_c &= \frac{1}{2} \sum_{i,j} e^{-\beta(\epsilon_j - \epsilon_i)} \mathrm{Tr}\left[\rho_c \Pi_j\right] \\
&= \frac{1}{2} \sum_{i,j,k \neq l} c_{kl} e^{-\beta(\epsilon_j - \epsilon_i)} \mathrm{Tr}\left[|k\rangle \langle l| U^\dagger |j\rangle \langle j| U\right] \\
&= \frac{1}{2} \sum_{i,j,k \neq l,m} c_{kl} e^{-\beta(\epsilon_j - \epsilon_i)} \delta_{km} \langle l| U^\dagger |j\rangle \langle j| U |m\rangle \\
&= \frac{1}{2} \sum_{i,j,k \neq l} c_{kl} e^{-\beta(\epsilon_j - \epsilon_i)} \langle l| U^\dagger |j\rangle \langle j| U |k\rangle \\
&= \frac{1}{2} \sum_{i} e^{\beta \epsilon_i} \sum_{j,k \neq l} c_{kl} e^{-\beta \epsilon_j} \langle l| U^\dagger |j\rangle \langle j| U |k\rangle \\
&= \frac{1}{2} \sum_{i} e^{\beta \epsilon_i} \left(\sum_{j \neq k} c_{jk} e^{-\beta \epsilon_j} \langle k| U^\dagger |j\rangle \langle j| U |j\rangle + \sum_{j \neq k} c_{kj} e^{-\beta \epsilon_j} \langle j| U^\dagger |j\rangle \langle j| U |k\rangle \right) \\
&= \sum_{i} e^{\beta \epsilon_i} \mathrm{Re} \left(\sum_{j \neq k} c_{jk} e^{-\beta \epsilon_j} \langle k| U^\dagger |j\rangle \langle j| U |j\rangle \right).
\end{aligned}
$$

For qubits, the unitary matrix is:

$$U = \begin{bmatrix} e^{i\delta}\text{cos}\phi & -e^{-i\gamma}\text{sin}\phi \\ e^{i\gamma}\text{sin}\phi & e^{-i\delta}\text{cos}\phi \end{bmatrix},$$

and so,

$$\begin{aligned}\text{Re}\left(\sum_{j\neq k} c_{jk}e^{-\beta\epsilon_j}\langle k|\,U^\dagger\,|j\rangle\,\langle j|\,U\,|j\rangle\right) &= \text{Re}\left(-c_{01}e^{i(\gamma+\delta)}\text{cos}\phi\text{sin}\phi + c_{10}e^{-\beta\Delta}e^{-i(\gamma+\delta)}\text{cos}\phi\text{sin}\phi\right)\\ &= \frac{\text{Re}\left(-ce^{i(\gamma+\delta)} + \overline{c}e^{-\beta\Delta}e^{-i(\gamma+\delta)}\right)}{2}\text{sin}2\phi\\ &= \frac{\text{Re}\left(-ce^{i\alpha} + \overline{c}e^{-\beta\Delta}e^{-i\alpha}\right)}{2}\text{sin}2\phi,\end{aligned}$$

where, $\gamma + \delta = \alpha$. For states on the Bloch sphere:

$$\rho_c = \frac{1}{2}\left(r_x\sigma_x + r_y\sigma_y\right) = \frac{1}{2}r\text{sin}\left(\theta\right)\left(\sigma_x\text{cos}\varphi + \sigma_y\text{sin}\varphi\right),$$

where, r is the bloch sphere radius, θ is the angle made by the bloch vector r with the Z axis, and φ is the angle made by the projection of the bloch vector on the XY plane with the X axis.

$$c = \frac{1}{2}r\text{sin}\left(\theta\right)e^{-i\varphi},$$

substituting we get,

$$\begin{aligned}\frac{\text{Re}\left(-ce^{i\alpha} + \overline{c}e^{-\beta\Delta}e^{-i\alpha}\right)}{2}\text{sin}\left(2\phi\right) &= \frac{\text{Re}\left(-e^{i(\alpha-\varphi)} + e^{-\beta\Delta}e^{-i(\alpha-\varphi)}\right)}{4}r\text{sin}\left(\theta\right)\text{sin}\left(2\phi\right)\\ &= \left(\frac{e^{-\beta\Delta}-1}{4}\right)r\text{sin}\left(\theta\right)\text{cos}\left(\alpha-\varphi\right)\text{sin}\left(2\phi\right).\end{aligned}$$

Using $e^{-\beta\Delta} = \frac{1-r_\beta}{1+r_\beta}$, we get,

$$\left(\frac{e^{-\beta\Delta}-1}{4}\right)r\text{sin}\left(\theta\right)\text{cos}\left(\alpha-\varphi\right)\text{sin}\left(2\phi\right) = \left(\frac{-rr_\beta}{2\left(1+r_\beta\right)}\right)\text{sin}\left(\theta\right)\text{cos}\left(\alpha-\varphi\right)\text{sin}\left(2\phi\right),$$

$$\Upsilon_c = \left(\frac{-rr_\beta}{1-r_\beta^2}\right)\text{sin}\left(\theta\right)\text{cos}\left(\alpha-\varphi\right)\text{sin}\left(2\phi\right). \tag{C.21}$$

We have already seen that Υ_c is non-zero only if the initial state has coherence. Moreover, for classical dynamics which is given by either identity or a bit flip, $\Upsilon_c = 0$ as in this case, $\phi = \frac{\pi}{2}$ and $(\alpha - \varphi) = 0$ as $\gamma = \frac{\pi}{2}$ and we choose $\delta = \left(\varphi - \frac{\pi}{2}\right)$. Therefore, the dynamics also should generate coherence for Υ_c to be non-zero.

Negativity of Υ_c and Υ

Here, we find where Υ_c becomes negative and where Υ becomes negative. Let $r_{xy} = r\sin(\theta)$. First note, $\Upsilon_c \leq 0$ if,

$$\cos(\alpha - \varphi)\sin 2\phi \geq 0,$$

as,

$$\left(\frac{-r_{xy}r_\beta}{1 - r_\beta^2}\right) \leq 0.$$

This happens when,

$$\{\cos(\alpha - \varphi) \geq 0\} \quad \text{and} \quad \{\sin 2\phi \geq 0\},$$

i.e.,

$$\left\{-\frac{\pi}{2} + 2n\pi + \varphi \leq \alpha \leq \frac{\pi}{2} + 2n\pi + \varphi\right\} \quad \text{and} \quad \left\{n\pi \leq \phi \leq \frac{\pi}{2} + n\pi\right\}, \tag{C.22}$$

or when,

$$\{\cos(\alpha - \varphi) \leq 0\} \quad \text{and} \quad \{\sin 2\phi \leq 0\},$$

i.e.,

$$\left\{\frac{\pi}{2} + 2n\pi + \varphi \leq \alpha \leq \frac{3\pi}{2} + 2n\pi + \varphi\right\} \quad \text{and} \quad \left\{-\frac{\pi}{2} + n\pi \leq \phi \leq n\pi\right\}, \tag{C.23}$$

Now Υ is negative in parts of these regions. Take $r_z = r\cos\theta$. To analyze this, first notice that $\xi = \sin^2\phi$. So,

$$\Upsilon_d = 1 + (\chi - 1)\xi = 1 + \left(\frac{2r_\beta(r_\beta - r_z)}{(1 - r_\beta^2)} - 1\right)\sin^2\phi,$$

and,

$$\Upsilon = \Upsilon_d + \Upsilon_c = 1 + \sin^2\phi\left(\frac{2r_\beta(r_\beta - r_z)}{(1 - r_\beta^2)} - 1\right) - \left(\frac{r_{xy}r_\beta}{1 - r_\beta^2}\right)\cos(\alpha - \varphi)\sin 2\phi. \tag{C.24}$$

First, Υ_c is most negative in terms of r, θ and r_β when, $\cos(\alpha - \varphi)\sin 2\phi = 1$. Let us take, $\phi = \frac{\pi}{4}$ and $\alpha - \varphi = 0$ or $\phi = \frac{3\pi}{4}$ and $\alpha - \varphi = \pi$. So, for Υ to be negative, we need:

$$\frac{1}{2} + \frac{\left(r_\beta^2 - r_\beta r(\cos\theta + \sin\theta)\right)}{(1 - r_\beta^2)} < 0$$

$$\implies \frac{1 + r_\beta^2}{2r_\beta} < r(\cos\theta + \sin\theta).$$

which is possible. In general we want:

$$\Upsilon = 1 + \sin^2\phi\left(\frac{2r_\beta\left(r_\beta - r_z\right)}{\left(1 - r_\beta^2\right)} - 1\right) - \left(\frac{r_{xy}r_\beta}{1 - r_\beta^2}\right)\cos\left(\alpha - \varphi\right)\sin 2\phi \leq 0$$

$$\implies r_\beta^2\left(1 - 3\xi\right) + 2r_\beta r\left(\sin\theta\cos\left(\alpha - \varphi\right)\sqrt{\xi\left(1 - \xi\right)} + \xi\cos\theta\right) - \left(1 - \xi\right) \geq 0.$$

Let us try to find the minimum value of Υ with respect to the process dependent quantities, i.e., α and ϕ:

$$\frac{\partial\Upsilon}{\partial\alpha} = \left(\frac{r_{xy}r_\beta}{1 - r_\beta^2}\right)\sin\left(\alpha - \varphi\right)\sin 2\phi = 0,$$

which is satisfied when: $\alpha = \varphi + n\pi$ or $\phi = n\pi$.

$$\frac{\partial\Upsilon}{\partial\phi} = \sin 2\phi\left(\frac{2r_\beta\left(r_\beta - r_z\right)}{\left(1 - r_\beta^2\right)} - 1\right) - \left(\frac{2r_{xy}r_\beta}{1 - r_\beta^2}\right)\cos\left(\alpha - \varphi\right)\cos 2\phi = 0,$$

$$\implies \sin 2\phi\left(\frac{2r_\beta\left(r_\beta - r_z\right)}{\left(1 - r_\beta^2\right)} - 1\right) = \left(\frac{2r_{xy}r_\beta}{1 - r_\beta^2}\right)\cos\left(\alpha - \varphi\right)\cos 2\phi.$$

When $\phi = n\pi$, this condition is satisfied when: $\alpha = \varphi + n\pi + \frac{\pi}{2}$. When $\alpha = \varphi + n\pi$, this condition is satisfied when:

$$\phi = \frac{1}{2}\arctan\left[\frac{\left(\pm 2r_{xy}r_\beta\right)}{\left(3r_\beta^2 - 2r_z r_\beta - 1\right)}\right].$$

For this we have,

$$\Upsilon_c = -\left(\frac{r_{xy}r_\beta}{1 - r_\beta^2}\right)\cos\left(\alpha - \varphi\right)\sin 2\phi = -\left(\frac{r_{xy}r_\beta}{1 - r_\beta^2}\right)\sin\left\{\arctan\left(\frac{\pm 2r_{xy}r_\beta}{3r_\beta^2 - 2r_z r_\beta - 1}\right)\right\},$$

$$\sin\{\arctan\left(x\right)\} = \pm\frac{x}{\sqrt{x^2 + 1}},$$

$$\Upsilon_c = \pm\left(\frac{r_{xy}r_\beta}{1 - r_\beta^2}\right)\frac{\left(\frac{2r_{xy}r_\beta}{3r_\beta^2 - 2r_z r_\beta - 1}\right)}{\sqrt{\left(\frac{\pm 2r_{xy}r_\beta}{3r_\beta^2 - 2r_z r_\beta - 1}\right)^2 + 1}},$$

which are the extrema values of Υ_c. Next let us summarize and conclude the results for analysis Υ.

C.4 Summary

We had:

$$\Upsilon = \sum_{i,j} e^{-\beta(\epsilon_j - \epsilon_i)}\mathrm{Tr}\left[\rho E_i \Pi_j\right],$$

$$\Upsilon_d = \sum_{i,j} e^{-\beta(\epsilon_j-\epsilon_i)}\text{Tr}\left[E_i\rho_d E_i\Pi_j\right],$$

$$\Upsilon_c = \sum_{i,j} e^{-\beta(\epsilon_j-\epsilon_i)}\frac{1}{2}\text{Tr}\left[\left(\rho_c-\rho_c'\right)\Pi_j\right],$$

We expressed them for qubits on a Bloch sphere as:

$$\Upsilon_d = 1+(\chi-1)\,\xi = 1+\left(\frac{2r_\beta\left(r_\beta-r\cos\theta\right)}{\left(1-r_\beta^2\right)}-1\right)\sin^2\phi = 1+\left(\frac{2r_\beta\left(r_\beta-r\cos\theta\right)}{\left(1-r_\beta^2\right)}-1\right)\xi,$$

$$\Upsilon_c = \left(\frac{-rr_\beta}{1-r_\beta^2}\right)\left(\sqrt{\xi\left(1-\xi\right)}\right)\sin\left(\theta\right)\cos\left(\alpha-\varphi\right),$$

$$\begin{aligned}\Upsilon &= 1+\sin^2\phi\left(\frac{2r_\beta\left(r_\beta-r\cos\theta\right)}{\left(1-r_\beta^2\right)}-1\right)-\left(\frac{rr_\beta}{1-r_\beta^2}\right)\sin\theta\cos\left(\alpha-\varphi\right)\sin2\phi\\ &=1+\left(\frac{2r_\beta\left(r_\beta-r\cos\theta\right)}{\left(1-r_\beta^2\right)}-1\right)\xi-\left(\frac{rr_\beta}{1-r_\beta^2}\right)\left(\sqrt{\xi\left(1-\xi\right)}\right)\sin\theta\cos\left(\alpha-\varphi\right),\end{aligned}$$

where, the Bloch vector is $\vec{r}=(r\sin\theta\cos\varphi, r\sin\theta\sin\varphi, r\cos\theta)$. For $\Upsilon=\Upsilon_d+\Upsilon_c$ we notice:-

- $\rho_{cl}, U_{cl} \implies \Upsilon \geq 0,\ \Upsilon_c = 0$
- $\rho_{cl}, U \implies \Upsilon \geq 0,\ \Upsilon_c = 0$
- $\rho, U_{cl} \implies \Upsilon \geq 0,\ \Upsilon_c = 0$
- $\rho, U \implies \Upsilon \geq 0$ or $\Upsilon \leq 0,\ \Upsilon_c \neq 0$: For the $\left|+\right\rangle\left\langle+\right|$ state we saw that, $\Upsilon \geq 0$ for all α and ξ, even if $\Upsilon_c \leq 0$

So, Υ_c gives a signature of the incompatibility between the initial projectors$\{E_i\}_i$ and the final projectors$\{\Pi_j\}_j$.

Q.E.D.