Introduction to Gromov Hyperbolic Spaces

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Certificate of Examination

This is to certify that the dissertation titled **Introduction to Gromov Hyperbolic Spaces** submitted by **Mr. Vaishnav Dilip** (Reg. No. MS16047) for the partial fulfilment of BS-MS dual degree programme of the Institute, has been examined by the thesis committee duly appointed by the Institute. The committee finds the work done by the candidate satisfactory and recommends that the report be accepted.

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Declaration

The work presented in this dissertation has been carried out by me under the guidance of Dr. Soma Maity at the Indian Institute of Science Education and Research Mohali.

This work has not been submitted in part or in full for a degree, a diploma, or a fellowship to any other university or institute. Whenever contributions of others are involved, every effort is made to indicate this clearly, with due acknowledgement of collaborative research and discussions. This thesis is a bonafide record of original work done by me and all sources listed within have been detailed in the bibliography.

Vaishnav Dilip
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In my capacity as the supervisor of the candidate's project work, I certify that the above statements by the candidate are true to the best of my knowledge.

Dr. Soma Maity
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Contents

Introduction

Gromov hyperbolic spaces or δ-hyperbolic spaces are metric spaces satisfying certain metric relations between the points depending on a non-negative real number δ. The definition, introduced by Eliyahu Rips, generalizes the metric properties of classical hyperbolic geometric and that of trees. The field was further developed thoroughly by Mikhael Gromov. Hyperbolicity is very useful to the study of certain infinite groups called the hyperbolic groups.

Chapter 0-3 contains some of the basic concepts required to go through this thesis. Chapter 0 looks at the definition of metric spaces and associated notions like the metric graphs, length metric and valuation metric. In Chapter 1, we look at what Cayley Graphs are. They are a very important object required in the study of Hyperbolic groups. In Chapter 2, we look at cell complexes and apply Van Kampen's theorem to find the fundamental group of a space to which 2-cells have been attached. In Chapter 3, we look at the Hopf Rinow theorem and the Arzela Ascoli theorem. These also prove to be an important tool in our later studies.

From Chapter 4, we look at Gromov hyperbolic spaces. In Chapter 4, we look at the definition of Gromov hyperbolic spaces and some equivalent conditions to check for Gromov hyperbolicity. We also look at the hyperbolicity of the subspaces and its implications. Since examples form an important part of understanding any new concept, we look at examples of Hyperbolic spaces in Chapter 5. Geodesic triangles in hyperbolic spaces have some important properties such as δ-slimness and δ-thinness, which are very often used as equivalent formulations of hyperbolicity. These properties are gone through Chapter 6.

Next, we look at the boundary of hyperbolic spaces. Chapter 7 introduces the boundary of hyperbolic spaces using sequences. In the next chapter, Chapter 8, we inspect the boundary as a set of rays. We look at what quasi-geodesics are and the stability of quasi-geodesics. These concepts aid us in proving the lemma on the visibility of the boundary of hyperbolic space.

In Chapter 9, we prove that if X_1 and X_2 are two geodesic metric spaces with X_2 hyperbolic and $f : X_1 \to X_2$ is a quasi-isometry, then X_1 is also hyperbolic, and f induces an embedding ∂f from ∂X_1 to ∂X_2. Towards the end of this chapter, we show that 0-hyperbolic geodesic spaces are trees.

We introduce hyperbolic groups in Chapter 10. Let Γ be a finitely generated group and G be a finite system of generators of Γ. We say that Γ is δ-hyperbolic relative to the system of generators G if it is δ-hyperbolic with respect to the associated word metric. We later prove in this chapter that the hyperbolicity of the group doesn't depend on the choice of system of generators. We see some examples of hyperbolic groups in Chapter 11. We prove that if X is a proper geodesic space, and Γ is the group of isometries of X acting properly discontinuous on this space, and such that the action is cocompact, then Γ is hyperbolic if and only if X is and additionally we have a canonical homeomorphism $\partial\Gamma \to \partial X$. This theorem provides us with numerous examples of hyperbolic groups.

The final chapter in this thesis, Chapter 12, introduces the three models of hyperbolic spaces and shows the equivalence of these models.

Chapter 0

Basic Notions

For this chapter, I have referred to *Metric Spaces of Non-Positive Curvature by Bridson & Haefliger*[Mar99], *Abstract Algebra by David S.Dummit & Richard M.Foote* [Dav04] and *Topology by James Munkres* [Mun03]

0.1 Metric Spaces

Definition 0.1.1. Let X be a set. A metric on X is a real valued function $d : X \times X \to \mathbb{R}$ satisfying the following, for all $x, y, z \in X$:

- Positivite Definite : $d(x, y) \geq 0$ and $d(x, x) = 0$ iff $x = y$.
- Symmetry: $d(x, y) = d(y, x)$.
- Triangle Inequality: $d(x, z) \leq d(x, y) + d(y, z)$.

We refer to $d(x, y)$ as the distance between the points x and y.

Definition 0.1.2. A metric space is said to be complete if every Cauchy sequence in it converges.

Definition 0.1.3. If Y is a subset of X, then the restriction of d to $Y \times Y$ is called an induced metric on Y.

Given $x \in X$ and $r > 0$, the open ball of radius r about x, denoted by $B(x, r)$, is the set $\{y \in X | d(x, y) < r\}$ and the closed ball is denoted by $\overline{B}(x, r)$. Associated with d, one has a topology with basis set as the set of open balls $B(x, r)$.

Definition 0.1.4. The metric space is said to be proper if, in this topology, for every $x \in X$ and for every $r > 0$, the closed ball $\overline{B}(x, r)$ is compact.

Definition 0.1.5. A metric space (X, d) is said to be separable if there exists a countable dense subset S of X.

Definition 0.1.6. Given two metric spaces (X, d_X) and (Y, d_Y), where d_X denotes the metric on the set X, and d_Y is the metric on set Y, a function $f : X \to Y$ is called Lipschitz continuous if there exists a real constant $K \geq 0$ such that, for all x_1 and x_2 in X,

$$d_Y(f(x_1), f(x_2)) \leq K d_X(x_1, x_2). \tag{1}$$

Any such K is referred to as a Lipschitz constant for the function f.

Definition 0.1.7. An isometry from one metric space (X, d) to another (X', d') is a bijection $f : X \to X'$ such that $d'(f(x), f(y)) = d(x, y)$ for all $x, y \in X$. If such a map exists then, (X, d) is said to be isometric to (X', d').

Example 0.1.8. Euclidean metric d on $\mathbb{R}^n$

$$d(x, y) := (\sum_{i=1}^{n} |x_i - y_i|^2)^2$$

where $x = (x_1, ..., x_n)$ and $y = (y_1, ..., y_n)$. $(\mathbb{R}^n, d)$ will be denoted as $\mathbb{E}^n$.

Definition 0.1.9 (Hausdorff distance)**.** Let X and Y be two non-empty subsets of a metric space (M, d). We define their Hausdorff distance $d_H(X, Y)$ by

$$d_H(X, Y) = \max\left\{ \sup_{x \in X} \inf_{y \in Y} d(x, y), \sup_{y \in Y} \inf_{x \in X} d(x, y) \right\}.$$

Equivalently,

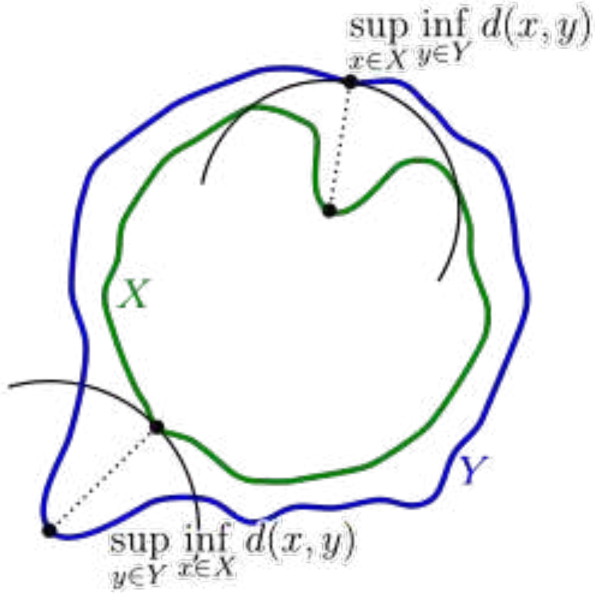
$$d_H(X, Y) = \inf\{\varepsilon \geq 0\,;\ X \subseteq Y_\varepsilon \text{ and } Y \subseteq X_\varepsilon\},$$

where

$$X_\varepsilon := \bigcup_{x \in X} \{z \in M\,;\ d(z, x) \leq \varepsilon\},$$

that is, the set of all points within ε of the set X (sometimes called the ε -fattening of X or a generalized ball of radius ε around X.

Informally, two sets are close in the Hausdorff distance if every point of either set is close to some point of the other set. The Hausdorff distance is the longest distance, you can be forced to travel by an adversary who chooses a point in one of the two sets, from where you then must travel to the other set. In other words, it



is the greatest of all the distances from a point in one set to the closest point in the other set.

0.2 Geodesics

Definition 0.2.1. Let (X, d) be a metric space. A geodesic path joining $x \in X$ to $y \in X$ is a map c from a closed interval $[0, l] \subset \mathbb{R}$ to X such that $c(0) = x, c(l) = y$ and $d(c(t), c(t')) = |t-t'|$ for all $t, t' \in [0, l]$ (in particular, $l = d(x, y)$). If $c(0) = x$, then c is said to issue from x. The image α of c is called a geodesic segment with endpoints x and y.(There is a $1 - 1$ correspondence between geodesic paths in X and pairs (α, x), where α is a geodesic segment in X and x is an endpoint of α.)

Definition 0.2.2. Let $I \subseteq \mathbb{R}$ be an interval. A map $c : I \to X$ is said to be a linearly reparameterized geodesic or a constant speed geodesic, if there exists a constant λ such that $d(c(t), c(t')) = \lambda|t - t'|$ for all $t, t' \in I$.

Definition 0.2.3. A geodesic ray in X is a map $c : [0, \infty) \to X$ such that $d(c(t), c(t')) = |t-t'|$ for all $t, t' \geq 0$. A geodesic line in X is a map $c : \mathbb{R} \to X$ such that $d(c(t), c(t')) = |t - t'|$ for all $t, t' \in \mathbb{R}$.

Definition 0.2.4. A local geodesic in X is a map c from an interval $I \subseteq \mathbb{R}$ to X with the property that for every $t \in I$ there exists $\epsilon > 0$ such that $d(c(t'), c(t'')) = |t' - t''|$ for all $t', t'' \in I$ with $|t - t'| + |t - t''| \leq \epsilon$.

Definition 0.2.5. (X, d) is said to be a geodesic metric space (or, more briefly, a geodesic space) if every two points in X are joined by a geodesic. We say that (X, d) is uniquely geodesic if there is exactly one geodesic joining x to y, for all $x, y \in X$.

Definition 0.2.6. Given $r > 0$, a metric space (X, d) said to be $r-$geodesic if for every pair of points $x, y \in X$ with $d(x, y) < r$ there is a geodesic joining x to y. And X is said to be $r-$uniquely geodesic if there is a unique geodesic segment joining each such pair of points x and y.

Definition 0.2.7. A subset C of a metric space (X, d) is said to be convex if every pair of points $x, y \in C$ can be joined by a geodesic in X and the image of every such geodesic is contained in C. If this condition holds for all points $x, y \in C$ with $d(x, y) < r$, then C is said to be $r-$convex.

0.3 Metric Graphs

Intuitively speaking, metric graphs are the spaces that one obtains by taking a connected graph (i.e., a connected 1-dimensional CW-complex), metrizing the individual edges of the graph as bounded intervals of the real line, and then defining the distance between two points to be the infimum of the lengths of paths joining them, where "length" is measured using the chosen metrics on the edges. It does not take long to realize that if one is not careful about the way in which the metrics on the edges are chosen, then various unpleasant pathologies can arise. Before considering these, we give a more precise formulation of the above.

Definition 0.3.1. A combinatorial graph $\mathcal{G}$ consists of two (possibly infinite) sets, $\mathcal{V}$ (the vertices) and $\mathcal{E}$ (the edges), together with two maps, $\partial_0 : \mathcal{E} \to \mathcal{V}$ and $\partial_1 : \mathcal{E} \to \mathcal{V}$ (the endpoint maps). We assume that V is the union of the images of ∂_0 and ∂_1. One associates to $\mathcal{G}$ the set $X_{\mathcal{G}}$ (more briefly, X) that is obtained by taking the quotient of $\mathcal{E} \times [0, 1]$ by the equivalence relation generated by $(e, i) \sim (e', i')$ if $\partial_i(e) = \partial_{i'}(e')$, where $e, e' \in \mathcal{E}$ and $i, i' \in \{0, 1\}$.

Let $p : \mathcal{E} \times [0, 1] \to X$ be the quotient map. We identify $\mathcal{V}$ with the image in X of $\mathcal{E} \times \{0, 1\}$. For each $e \in \mathcal{E}$, let $f_e : [0, 1] \to X$ denote the map that sends $t \in [0, 1]$ to $p(e, t)$. Note that f_e is injective on $(0, 1)$. If $f_e(0) = f_e(1)$, the edge e is a called a loop. To define a metric on X, one first specifies a map

$$\lambda : \mathcal{E} \to (0, \infty)$$

associating a length $\lambda(e)$ to each edge e. A piecewise linear path is a map $c : [0, 1] \to X$ for which there is a partition $0 = t_0 \leq t_1 \leq ... \leq t_n = 1$ such that each is of the form $f_{e_i} \circ c_i$, where $e_i \in \mathcal{E}$ and c_i is an affine map from $[t_i, t_{i+1}]$ into $[0,1]$. We say that c joins x to y if $c(0) = x$ and $c(1) = y$.

Definition 0.3.2. The length of a path c along the graph is defined to be $l(c) = \sum_{i=0}^{n-1} l(c_i)$, where $l(c_i) = \lambda(e_i)|c_i(t_i) - c_{i+1}(t_{i+1})|$.

We assume that X is connected, i.e. any two points are joined by such a path.

Definition 0.3.3. We define a pseudometric $d : X \times X \to [0, \infty]$ by setting $d(x,y)$ equal to the infimum of the length of piecewise linear paths joining x to y. The space X with its pseudometric d is called a metric graph.

For any edge e, the distance between $p(e, 1/2)$ and $\partial_i(e)$ is $\lambda(e)/2$.

Definition 0.3.4. A combinatorial graph $\mathcal{G}$ is called a tree if the corresponding metric graph X where all the edges have length one is connected and simply connected.

Definition 0.3.5. The Cayley graph $\mathcal{C}_{\mathcal{A}}(\Gamma)$ of a group Γ with respect to a generating set $\mathcal{A}$ is the metric graph whose vertices are in 1-1 correspondence with the elements of Γ and which has an edge (labelled a) of length one joining γ to γa for each $\gamma \in \Gamma$ and $a \in \mathcal{A}$.
In the notation of (0.3.1), $\mathcal{V} = \Gamma$, $\mathcal{E} = \{(\gamma, a) | \gamma \in \Gamma, a \in \mathcal{A}\}$, $\partial_0(\gamma, a) = \gamma$, $(\gamma, a) = \gamma a$, and $\lambda : \mathcal{E} \to [0, \infty)$ is the constant function 1.

0.4 The Length of a Curve

Let X be a metric space. For us, a curve or a path in X is a continuous map c from a compact interval $[a,b] \subset \mathbb{R}$ to X. We say that c joins the point $c(a)$ to the point $c(b)$. If $c_1 : [a_1, b_1] \to X$ and $c_2 : [a_2, b_2] \to X$ are two paths such that $c_1(b_1) = c_2(a_2)$, their concatenation is the path $c : [a_1, b_1 + b_2 - a_2] \to X$ defined by $c(t) = c_1(t)$ if $t \in [a_1, b_1]$ and $c(t) = c_2(t + a_2 - b_1)$ if $t \in [b_1, b_1 + b_2 - a_2]$. More generally, the concatenation of a finite sequence of paths $c_i : [a_i, b_i] \to X$, with $c_i(b_i) = c_{i+1}(a_{i+1})$ for $i = 1, 2, ..., n-1$, is defined inductively by concatenating $c_1, ..., c_{n-1}$ and then concatenating the result with c_n.

Definition 0.4.1. Let X be a metric space. The length $l(c)$ of a curve $c : [a,b] \to X$ is

$$l(c) = \sup_{a = t_0 \leq t_1 \leq \ldots \leq t_n = b} \sum_{i=0}^{n-1} d(c(t_i), c(t_{i+1})),$$

where the supremum is taken over all possible partitions(no bound on n) with $a = t_0 \leq t_1 \leq .... \leq t_n = b$.

The length of c is either a non-negative number, or it is infinite. The curve c is said to be rectifiable if its length is finite.

Proposition 0.4.2. *Let (X,d) be a metric space and let $c:[a,b]\to X$ be a path.*

(1) $l(c)\geq d(c(a),c(b))$, and $l(c)=0$ if and only if c is a constant map

(2) If ϕ is a weakly monotonic map from an interval $[a',b']$ onto $[a,b]$, then $l(c)=l(c\circ\phi)$.

(3) Additivity: If c is the concatenation of two paths c_1 and c_2, then $l(c)=l(c_1)+l(c_2)$.

(4) The reverse path $\overline{c}:[a,b]\to X$ defined by $\overline{c}(t)=c(b+a-t)$ satisfies $l(\overline{c})=l(c)$.

(5) If c is rectifiable of length l, then the function $\lambda:[a,b]\to[0,l]$ defined by $\lambda(t)=l(c|_{[a,t]})$ is a continuous weakly monotonic function.

(6) Reparameterization by arc length: If c and λ are as in (5), then there is a unique path $\tilde{c}:[0,1]\to X$ such that

$$\tilde{c}\circ\lambda=c \text{ and } l(\tilde{c}|_{[0,t]})=t.$$

(7) Lower seimcontinuity: Let (c_n) be a sequence of paths $[a,b]\to X$ converging uniformly to a path c. If c is rectifiable, then for every $\epsilon>0$, there exists an integer $N(\epsilon)$ such that

$$l(c)\leq l(c_n)+\epsilon$$

whenever $n>N(\epsilon)$.

Proof.

(1) Since by definition, $l(c)$ is the supremum over all partions, $l(c)$ is greater than any one partion ($n=1$). Hence $l(c)\geq d(c(a),c(b))$.
If $l(c)=0$, all sums of distances are 0. Hence $d(c(t_i),c(t_{i+1}))=0$. Hence c is a constant map.
Conversly, if c is a constant map, $d(c(t_i),c(t_{i+1}))=0$ for all $i\in\mathbb{Z}$.

(2) Since a weakly monotonic map is injective and ϕ is surjective as well, we see that c and $c\circ\phi$ are isometric. Hence $l(c)=l(c\circ\phi)$.

(3)

$$\begin{aligned} l(c) &= \sup_{a=t_0\leq t_1\leq \ldots \leq t_n=b} \sum_{i=0}^{n-1} d(c(t_i), c(t_{i+1})) \\ &\leq \sup_{a=t_0\leq t_1\leq \ldots \leq t_n=l} \sum_{i=0}^{n-1} d(c(t_i), c(t_{i+1})) + \sup_{l=t_0\leq t_1\leq \ldots \leq t_n=b} \sum_{i=0}^{n-1} d(c(t_i), c(t_{i+1})) \\ &\leq l(c_1) + l(c_2) \end{aligned}$$

Now

$$l(c_1) + \sum_{i=0}^{n-1} d(c(t_i), c(t_{i+1})) \leq l(c)$$

Now taking the supremum over the partitions of the domain of c_2, we see that

$$l(c_1) + l(c_2) \leq l(c)$$

Hence $l(c) = l(c_1) + l(c_2)$.

(4) From (2) we see that $\overline{c}(t) = c(b + a - t) = c \circ \phi$ where $\phi(t) = b + a - t$. Hence $l(\overline{c}) = l(c)$.

(5) Property (3) reduces the proof of (5) to showing that, given $\epsilon > 0$, one can partition $[a, b]$ into finitely many subintervals so that the length of c restricted to each of these subintervals is at most ϵ. To see that this can be done, we first use the uniform continuity of the map $c : [a, b] \to X$ to choose $\delta > 0$ such that $d(c(t), c(t')) < \delta/2$ for all $t, t' \in [a, b]$ with $|t - t'| < \delta$. Since $l(c)$ is finite, we can find a partition $a = t_0 < t_1 < \ldots < t_k = b$ such that

$$\sum_{i=0}^{k-1} d(c(t_i), c(t_{i+1})) > l(c) - \epsilon/2.$$

Taking a refinement of this partition if necessary, we may assume that $|t_i - t_{i+1}| < \delta$ for $i = 0, ., k - 1$, and hence $d(c(t_i), c(t_{i+1})) < \epsilon/2$. But $l(c|_{[t_i,t_{i+1}]}) \geq d(c(t_i), c(t_{i+1}))$, and $l(c) = \sum l(c|_{[t_i,t_{i+1}]})$ by (3). Hence

$$l(c) = \sum_{i=0}^{k-1} l(c|_{[t_i,t_{i+1}]}) \geq \sum_{i=0}^{k-1} d(c(t_i), c(t_{i+1})) > l(c) - \epsilon/2,$$

with each summand in the first sum no less than the corresponding summand in the second sum. Hence, for all i we have $l(c|_{[t_i,t_{i+1}]}) - d(c(t_i), c(t_{i+1})) \leq \epsilon/2$, and in particular $l(c|_{[t_i,t_{i+1}]}) < \epsilon$.

(6) Follows from (5) and (2).

(7) Choose $a = t_0 < t_1 < .. < t_k = b$ such that

$$l(c) \le \sum_{i=0}^{k-1} d(c(t_i), c(t_{i+1})) + \epsilon/2.$$

Then we choose $N(\epsilon)$ big enough to ensure that $d(c(t), c(t_n)) < \epsilon/4k$ for all $n > N(\epsilon)$ and all $t \in [a, b]$. By the triangle inequality, $d(c(t_i), c(t_{i+1})) \le 2\epsilon/4k + d(c_n(t_i), c_n(t_{i+1}))$. Hence

$$l(c) \le k\frac{\epsilon}{2k} + \sum_{i=0}^{k-1} d(c_n(t_i), c_n(t_{i+1})) + \epsilon/2 \le \epsilon + l(c_n).$$

□

Definition 0.4.3. A path $c : [a, b] \to X$ is said to be parameterized proportional to arc length if the map λ defined above in 0.4.2(5) is linear.

0.5 Length Metric

Definition 0.5.1. Let (X, d) be a metric space. d is said to be a length metric (otherwise known as an inner metric) if the distance between every pair of points $x, y \in X$ is equal to the infimum of the length of rectifiable curves joining them. (If there are no such curves, then $d(x, y) = \infty$.) If d is a length metric, then (X, d) is called a length space.

Proposition 0.5.2. *Let (X, d) be a metric space, and let $\overline{d} : X \times X \to [0, \infty]$ be the map which assigns to each pair of points $x, y \in X$ the infimum of the lengths of rectifiable curves which join them. (If there are no such curves then $d(x, y) = \infty$.)*

(1) $\overline{d}$ is a metric.

(2) $\overline{d}(x, y) \ge d(x, y)$ for all $x, y \in X$.

(3) If $c : [a, b] \to X$ is continuous with respect to the topology induced by $\overline{d}$, then it is continuous with respect to the topology induced by d.(The converse is false in general.)

(4) *If a map* $c : [a, b] \to X$ *is rectifiable curve in* (X, d)*, then it is a continuous and rectifiable curve in* $(X, \overline{d})$.

(5) *The length of a curve* $c : [a, b] \to X$ *in* $(X, \overline{d})$ *is the same as its length in* (X, d).

(6) $\overline{\overline{d}} = \overline{d}$.

Proof.

(1) $\overline{d}$ is a metric:

  (i) $\overline{d}(x, y) \geq 0$ follows from the fact that length of rectifiable curves are greater than zero always.
  $\overline{d}(x, y) = 0$ implies that $d(x, y) = 0$. Hence $x = y$.
  Conversely, if $x = y$, then $d(x, y) = 0$, Hence $\overline{d}(x, y) = 0$.

  (ii) $\overline{d}(x, y) = \inf l(c) = \inf l(\overline{c}) = \overline{d}(y, x)$.

  (iii) Let x, y, z be fixed, and let $\epsilon > 0$. Then, by the definition of infimum, there exists a path γ_1 from x to y with $L(\gamma_1) < \overline{d}(x, y) + \epsilon/2$, and there exists a path γ_2 from y to z with $L(\gamma_2) < \overline{d}(y, z) + \epsilon/2$. Let $\gamma = \gamma_1\gamma_2$ be the path γ_1 followed by γ_2 from x to z. Then $\overline{d}(x, z) \leq L(\gamma) = L(\gamma_1) + L(\gamma_2) < \overline{d}(x, y) + \overline{d}(y, z) + \epsilon$. Thus for all $\epsilon > 0$ we have $\overline{d}(x, z) < \overline{d}(x, y) + \overline{d}(y, z) + \epsilon$,thus $\overline{d}(x, z) \leq \overline{d}(x, y) + \overline{d}(y, z)$.

(2) $l(c) \geq d(x, y)$ for all c joining x to y. Hence $\overline{d}(x, y) \geq d(x, y)$.

(3) d induces a finer topology compared to $\overline{d}$. Hence (3) holds.

(4) Follows from 0.4.2(5)

(5) Let $c : [a, b] \to X$ be a path which has length $l(c)$ with respect to the metric d, and length $\overline{l}(c)$ with respect to the metric $\overline{d}$. On the one hand, we have that $\overline{l}(c) \geq l(c)$, by (2), and on the other hand

$$\overline{l}(c) = \sup_{a=t_0 \leq t_1 \leq \ldots \leq t_k = b} \sum_{i=0}^{k-1} \overline{d}(c(t_i), c(t_{i+1})) \leq \sup_{a=t_0 \leq t_1 \leq \ldots \leq t_k = b} \sum_{i=0}^{k-1} l(c|_{[t_{i-1}, t_i]}) = l(c).$$

Hence $\overline{l}(c) = l(c)$.

(6) Follows from (4) and (5).

∎

Definition 0.5.3. Let (X, d) be a metric space. The map $\overline{d}$ defined in (3.2) is called the length metric (or inner metric) associated to d, and $(X, \overline{d})$ is called the length space associated to (X, d). Note that $\overline{d} = d$ if and only if (X, d) is a length space. The induced length metric on a subset $Y \subseteq X$ is the length metric associated to the restriction of d to $Y \times Y$ (which in general will not be the same as the restriction to $Y \times Y$ of d).

Example 0.5.4. Consider the set of rational numbers $\mathbb{Q}$ with the usual metric d induced from $\mathbb{R}$. In the associated length metric d, the distance between every pair of distinct points of $\mathbb{Q}$ is infinite. Hence d induces the discrete topology on $\mathbb{Q}$.

0.6 Valuation Metric

Definition 0.6.1. A discrete valuation on a field K is a function $v : K^{\times} \to \mathbb{Z}$ satisfying:

(i) v is surjective.

(ii) $v(xy) = v(x) + v(y)$ for all $x, y \in K^{\times}$.

(iii) $v(x + y) \geq min\{v(x), v(y)\}$ for all $x, y \in K^{\times}$ with $x + y \neq 0$.

The subring $\{x \in K | v(x) \geq 0\} \cup \{0\}$ is called the valuation ring of v.

The valuation v is often extended to all of K by defining $v(0) = +\infty$, in which case (ii) and (iii) hold for all $a, b \in K$.

Definition 0.6.2. A discrete valuation v on a field K defines an associated metric, d_v, on K as follows: fix any real number $\beta > 1$ (the actual value of β does not matter for verifying the axioms of a metric), and for all $a, b \in K$ define

$$d_v(a, b) := \|a - b\|_v \text{ where } \|a\|_v = \beta^{-v(a)}$$

and where we set $d_v(a, a) = 0$.

We now check that the metric defined in this manner is indeed a metric.

Proposition 0.6.3.

(i) $d_v(a, b) \geq 0$, *with equality holding if and only if* $a = b$.

(ii) $d_v(a, b) = d_v(b, a)$ *i.e.* d_v *is symmetric.*

(iii) $d_v(a,b) \leq d_v(a,c) + d_v(c,b)$, *for all* $a, b, c \in K$, *i.e.* d_v *satisfies the triangle inequality.*

Proof.
Observe that $v(1) = v(-1) = 0$. Now $v(-a) = v(a) + v(-1) = v(a)$. Hence, $v(a-b) = min\{v(a), v(b)\}$.

(i) $d_v(a,b) = \beta^{-v(a-b)} \geq 0$ since $\beta > 1$. Now if $d_v(a,b) = 0$, we see that $\beta^{-v(a-b)} = 0$, which implies $v(a-b) = \infty$. Hence $a = b$. The other way follows from the definition.

(ii) $v(a-b) = v(b-a)$. Hence $d_v(a,b) = d_v(b,a)$.

(iii) $v(a-b) \geq min\{v(a-c), v(c-b)\}$. Hence

$$\beta^{-v(a-b)} \leq \beta^{-min\{v(a-c),v(c-b)\}} \leq \beta^{-v(a-c)} + \beta^{-v(c-b)}.$$

Hence $d_v(a,b) \leq d_v(a,c) + d_v(c,b)$ for all $a, b, c \in K$.

□

Proposition 0.6.4. *The valuation metric defined in the above manner satisfies the following inequality*

$$d_v(a,b) \leq max\{d_v(a,c), d_v(c,b)\} \text{ for all } a, b, c \in K.$$

Proof.

$$v(a-b) \geq min\{v(a-c), v(c-b)\}$$
$$\beta^{-v(a-b)} \leq \beta^{-min\{v(a-c),v(c-b)\}}$$
$$\beta^{-v(a-b)} \leq \beta^{max\{-v(a-c),-v(c-b)\}}$$
$$d_v(a,b) \leq max\{d_v(a,c), d_v(c,b)\}$$

□

Chapter 1

Cayley Graphs

In this chapter, we see how graph theory and algebra interplay. Specifically, we are going to develop Cayley Graphs and Schrier Diagrams. We would also see some examples later. For this chapter, I have referred to *Cayley Graphs* by *Padraic Bartlett* [Bar].

1.1 Free Groups, Generating Sets, Presented Groups and Cosets

Definition 1.1.1. The free group on n generators $a_1, a_2, ..., a_n$ denoted

$$\langle a_1, ..., a_n \rangle$$

is the following group:

- The elements of the group are all of the strings of the form

$$a_{i_1}^{k_1} a_{i_2}^{k_2} .... a_{i_l}^{k_l}$$

  where the indices $i_1, i_2, ..., i_l$ are all valid indices for the $a_1, ..., a_n$ and $k_1, ..., k_l$ are all integers.

- We also have identity element e in the group, which corresponds to the empty string that contains no elements.

- Given two strings s_1, s_2, we can concatenate these two strings into the word $s_1 s_2$ by simply writing the string that consists of the string s_1 followed by the string s_2.

- Whenever we have a^k in a string, we think of this as being $\overbrace{a.a......a}^{\text{k copies}}$, i.e. k copies of a. If we have multiple consecutive strings of a's, we can combine them together into one such a^k: for example, the word a^3aa^2 is the same thing as the word a^6.

- Finally, if we ever have an aa^{-1} or an $a^{-1}a$ occurring next to each other in a string, we can simply replace this pairing with the empty string, e.

Example 1.1.2. The free group on two generators $\langle a, b\rangle$ contains strings like

$$a, a^2, b, ab, a^2ba, a^6b^4a^{-2}b^3a^1$$

We concatenate strings simply by placing one after another and reducing the terms pairwise.

$$a^2b^{-2}a^3ba^3.a^{-3}b^{-1}a^1b^3 = a^2b^{-2}a^3ba^3a^{-3}b^{-1}a^1b^3 = a^2b^{-2}a^4b^3$$

This is a group as concatenation is associative; the empty string e is clearly an identity, and we can invert any word by simply reversing it and switching the signs on the k_i's

$$a_{i_1}^{k_1}.a_{i_2}^{k_2}...a_{i_n}^{k_n}.a_{i_n}^{-k_n}...a_{i_2}^{-k_2}.a_{i_1}^{-k_1} = e$$

Definition 1.1.3. Given a group G, we say that it is generated by some collection of elements $a_1, ..., a_n \in G$ if we can write any element in G as some combination of the elements $a_1, ..., a_n$ and their inverses.

Remark. Some groups have multiple different sets of generators: i.e. $\langle Z, +\rangle$ is generated both by the single element $\{1\}$ and also by the pair of elements $\{2, 3\}$.

In our above discussion, we have primarily defined groups by giving a set and an operation on that set. There are other ways of defining a group as shown below.

Definition 1.1.4. A group presentation is a collection of n generators $a_1, ..., a_n$ and m words $R_1, ...R_m$ from the free group $\langle a_1, ...a_n\rangle$, which we write as

$$\langle a_1, ...a_n | R_1, ...R_m\rangle.$$

We associate this presentation with the group defined as follows:

- Start off with the free group $\langle a_1, ..., a_n\rangle$.

- Now, declare that within this free group, the words $R_1, ...R_m$are all equal to the empty string: i.e. if we have any words that contain some R_i as a substring, we can simply "delete" this R_i from the word

Example 1.1.5. Consider the group with the presentation

$$\langle a|a^n\rangle.$$

This is the collection of all words written with one symbol a, where we regard $a^n = e$.

$$\langle a|a^n\rangle = \{e, a, a^2, a^3, ..., a^{n-1}\}$$

Often, we will give a group a presentation of the form:

$$\langle a_1, ..., a_n|R_1 = R_2, R_3 = R_4, ..., ...R_{m-1} = R_m\rangle$$

Definition 1.1.6. Suppose that G is a group, $s \in G$ is some element of G and H is a subgroup of G. We define the right coset of H corresponding to s as the set

$$Hs = \{hs|h \in H\}.$$

From now, we will simply call these objects cosets.

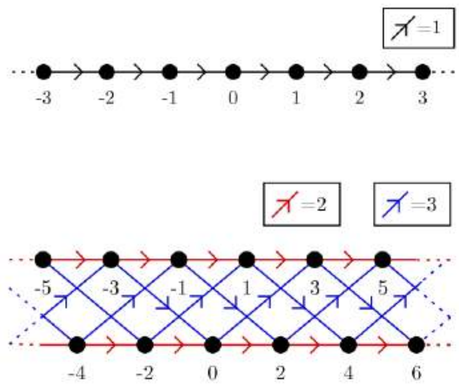
1.2 Cayley Graphs

Definition 1.2.1. Take any group A along with a generating set S. We define the Cayley graph $G_{A,S}$ associated to A as the following directed graph:

- Vertices: the vertices of G_A are precisely the elements of A.
- Edges: for two vertices x, y, create the oriented edge (x, y) if and only if there is some generator $s \in S$ such that $x.s = y$. If this happens, we decorate the edge (x, y) with this generator s so that we can keep track of how we have formed our connections.

Example 1.2.2. The integers $\mathbb{Z}$ with generator 1 has the following simple Cayley graph. For each vertex, there are two edges: one to the successor of the integer and one to its predecessor.

Example 1.2.3. The integers $\mathbb{Z}$ with the generating set $\{2, 3\}$ has the following Cayley graph: Again, our vertices are integers. Now since there are two generators,



one could add the generators or subtract the generator from the integer. Hence each node is connected to four other nodes.

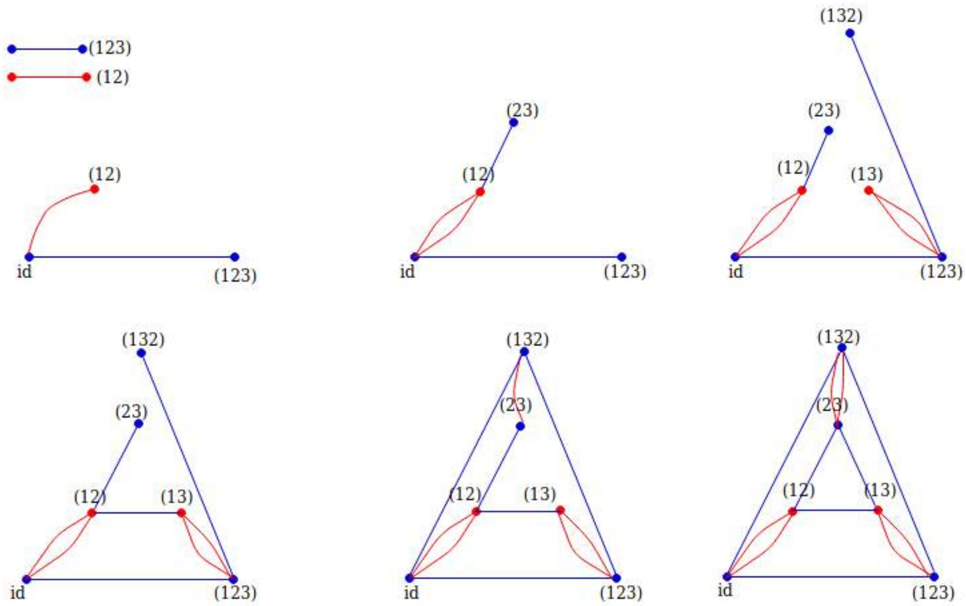
Notice that the above Cayley graphs are different. Hence the Cayley graph for the same group could differ depending on the generator set used.

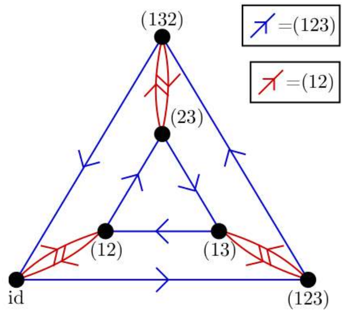
Example 1.2.4. Cayley graph of S_3 with generators (12) and (123) :
We start with id and find the elements connected to it, namely (12) and (123). We go on recursively find the Cayley graph.

$$(12)(12) = id \quad (123)(123) = (132)$$
$$(123)(123)(123) = id \quad (123)(12) = (13)$$
$$(13)(12) = (123) \quad (132)(12) = (23)$$
$$(23)(12) = (132) \quad (12)(123) = (23)$$
$$(23)(123) = (13) \quad (13)(123) = (12)$$

Example 1.2.5. Cayley graph of $\langle a, b | a^3 = b^2 = (ab)^2 = id \rangle$: As before again, we begin with the identity element and draw our graph by recursively finding the edges of each new element. We first draw the identity element. Then we find the elements connected to identity through the corresponding generator edges.

$$id.b = b$$
$$id.a = a$$





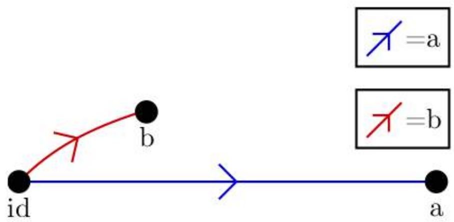
Now we go to vertices b and a and find the elements connected to them.

$$b.b = id$$

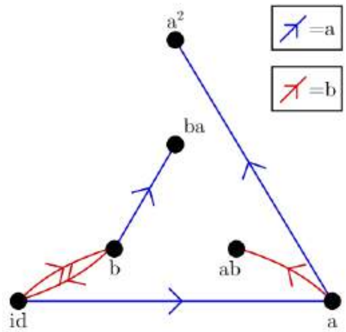
$$b.a = ba$$

$$a.b = ab$$

$$a.a = a^2$$

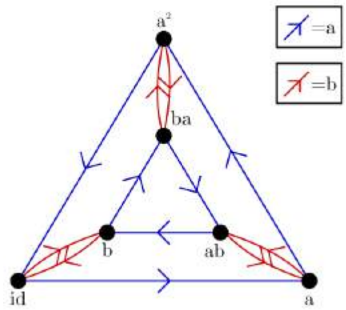


Now we go to the vertices ba, ab and a^2. Here we need to use the words which are



defined to be identity.

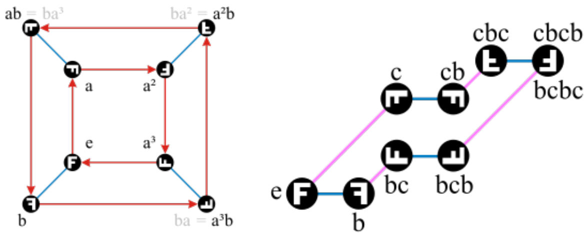
$$
\begin{aligned}
ba.b &= bab = a^3bab = a^2(a^3 = abab = 1)\\
ba.a &= baa = ba^2 = ab(abab = 1)\\
ab.b &= abb = a(b^2 = 1)\\
ab.a &= aba = b(abab = 1)\\
a^2.a &= a^3 = id\\
a^2.b &= ba(abab = 1)
\end{aligned}
$$



Remark.

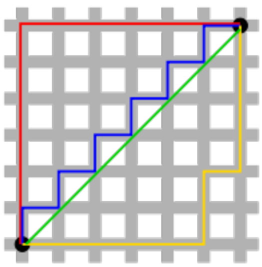
- Same group with two different generator sets have different Cayley graphs. Eg: $\mathbb{Z}$ with generating sets $\{1\}$ and $\{2,3\}$, $\mathbf{D_4}$ with following presentations: $\langle a, b | a^4 = b^2 = e, ab = ba^3 \rangle$ and $\langle b, c | b^2 = c^2 = e, bcbc = cbcb \rangle$

- Formally, for a given choice of generators, one has the word metric (the natural distance on the Cayley graph), which determines a metric space. The word



(a) (b)

metric is a metric on G, assigning to any two elements g, h of G a distance $d(g, h)$ that measures how efficiently their difference $g^{-1}h$ can be expressed as a word whose letters come from a generating set for the group. The word metric on G is very closely related to the Cayley graph of G: the word metric measures the length of the shortest path in the Cayley graph between two elements of G.
Eg: The group of integers $\mathbb{Z}$ is generated by the set $\{-1, +1\}$. The integer -3 can be expressed as $-1-1-1+1-1$, a word of length 5 in these generators. But the word that expresses -3 most efficiently is $-1-1-1$, a word of length 3. The distance between 0 and -3 in the word metric is therefore equal to 3. More generally, the distance between two integers m and n in the word metric is equal to $|m-n|$ because the shortest word representing the difference $m-n$ has a length equal to $|m-n|$.



1.3 Schreier Graphs

Definition 1.3.1. Take a group G, a subgroup H of G and some collection of elements S that(along with the elements in H) generate G. We create the Schreier diagram corresponding to this collection of information as follows:

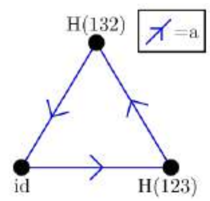
- Vertices: the various right cosets of H in G.
- Edges: connect two cosets K, L with an edge if and only if there is some elements $s \in S$ such that $Ks = L$.

In this sense, a Cayley graph is simply a Schreier diagram where we set $H = \{id\}$.

Example 1.3.2. Lets take $G = S_3$ as before, with $H = \{id, (12)\}$ and generating set $S = \{(123)\}$. Now we have three cosets for H:

$$H.(12) = H.id = \{id, (12)\}$$
$$H.(13) = H.(132) = \{(13), (132)\}$$
$$H.(23) = H.(123) = \{(23), (123)\}$$

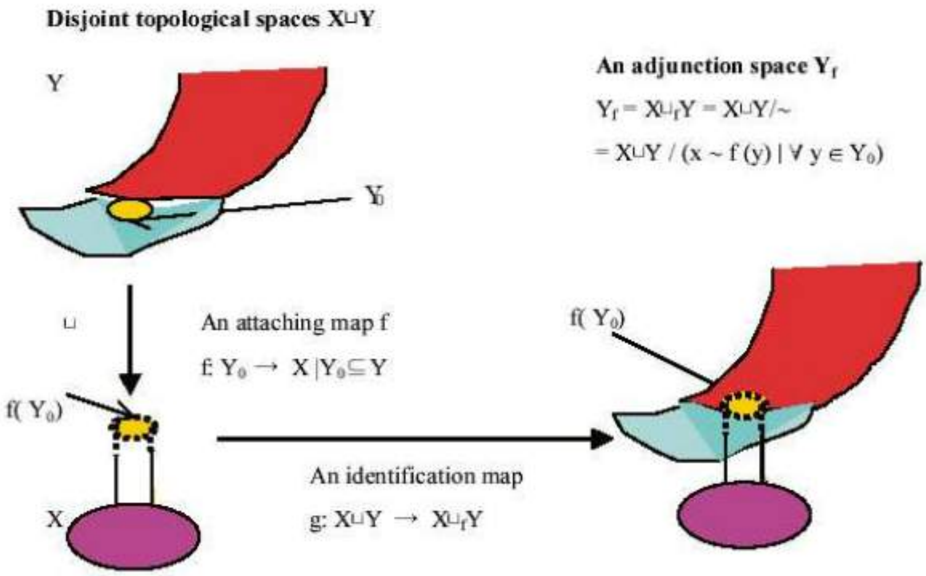
This gives us a fairly simple Schreier diagram



Chapter 2

Cell Complexes

In the last chapter, we saw how groups have a graph associated with them. In this chapter, we see how every group has associated with it a 2-dimensional cell complex. We begin by seeing what cell complexes are. I have referred to *Algebraic Topology* by *Allen Hatcher* [Hat02] for this chapter.



Definition 2.0.1. Let X and Y be topological spaces, and let A be a subspace of Y. Let $f : A \to X$ be a continuous map (called the attaching map). One forms the adjunction space $X \cup_f Y$ (sometimes also written as $X +_f Y$) by taking the disjoint union of X and Y and identifying a with $f(a)$ for all $a \in A$. Formally,

$$X \cup_f Y = (X \amalg Y)/\sim$$

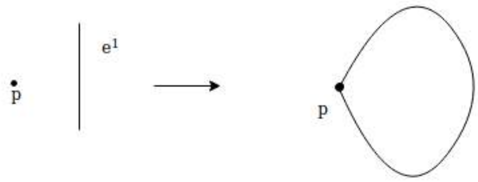
where the $\sim$ equivalence relation is generated by $a \sim f(a)$ for all a in A, and the quotient is given the quotient topology. As a set, $X \cup_f Y$ consists of the disjoint union of X and $(Y - A)$. Intuitively, one may think of Y as being glued onto X via the map f.

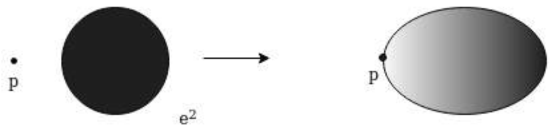
Definition 2.0.2. A k-cell is a k-dimensional disc.

$$e^k = \{x \in \mathbf{R}^k : |x| \leq 1\}.$$

Definition 2.0.3. A CW complex is a space built out of smaller spaces, iteratively attaching cells. Attaching a k-cell to another space X means, intuitively, forming the union of X and e^k, where we glue the boundary of e^k to X.

Example 2.0.4. Let X be a single point p and attach a 1-cell $e^1 = [-1, 1]$ to X so that the two endpoints attach at the point p. The result is a circle. Alternatively, one could attach a 2-cell to X by collapsing its boundary circle to p; the result is a 2-sphere.

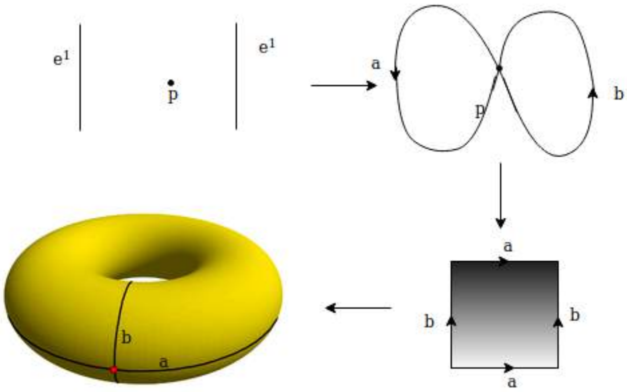


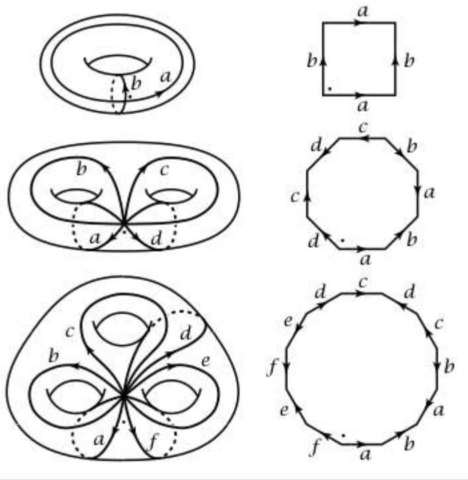


Example 2.0.5. You could attach several cells. For example, attaching two 1-cells to a single point yields the figure 8. The 2-torus is built by attaching a square to the figure 8. Since the square is topologically a disc, this is a 2-cell attachment. The boundary of the square (disc) is attached in a more interesting way than the previous examples: its boundary runs along the two loops, a, b, of the figure 8 in the order $b^{-1}a^{-1}ba$. Similarly, one could construct toruses of higher genuses.

2.1 Constructing Spaces

A natural generalization to construct cell complexes using cells is as below:





(i) Start with a discrete set X^0, whose points are regarded as 0-cells.

(ii) Inductively, form the n skeleton X^n from X^{n-1} by attaching n-cells e^n_α via maps $\phi_\alpha : S^{n-1} \to X^{n-1}$. This means that X^n is the quotient space of the disjoint union $X^{n-1} \amalg_\alpha D^n_\alpha$ of X^{n-1} with a collection of n-disks D^n_α under the identifications $x \sim \phi_\alpha(x)$ for $x \in \partial D^n_\alpha$. Thus as a set, $X^n = X^{n-1} \amalg_\alpha e^n_\alpha$ where each e^n_α is an open n-disk.

(iii) One can either stop this inductive process at a finite stage, setting $X = X^n$ for some $n < \infty$, or one can continue indefinitely, setting $X = \cup_n X^n$. In the latter case, X is given the weak topology: A set $A \subset X$ is open (or closed) if and only if $A \cap X^n$ is open (or closed) in X^n for each n.

If $X = X^n$ for some n, then X is said to be finite-dimensional, and the smallest such n is the dimension of X, the maximum dimension of cells of X.

2.2 Van Kampen's Theorem

Theorem 2.2.1. *If X is the union of path-connected open sets A_α each containing the basepoint $x_0 \in X$ and if each intersection $A_\alpha \cap A_\beta$ is path-connected, then the homomorphism $\Phi : *_\alpha \pi_1(A_\alpha) \to \pi_1(X)$ is surjective. If in addition each intersection $A_\alpha \cap A_\beta \cap A_\gamma$ is path-connected, then the kernel of Φ is the normal subgroup N generated by all elements of the form $i_{\alpha\beta}(\omega)i_{\beta\alpha}(\omega)^{-1}$ for $\omega \in \pi_1(A_\alpha \cap A_\beta)$, and so Φ induces an isomorphism $\pi_1(X) \approx *_\alpha \pi_1(A_\alpha)/N$.*

2.3 Application of Van Kampen's Theorem to Cell Complexes

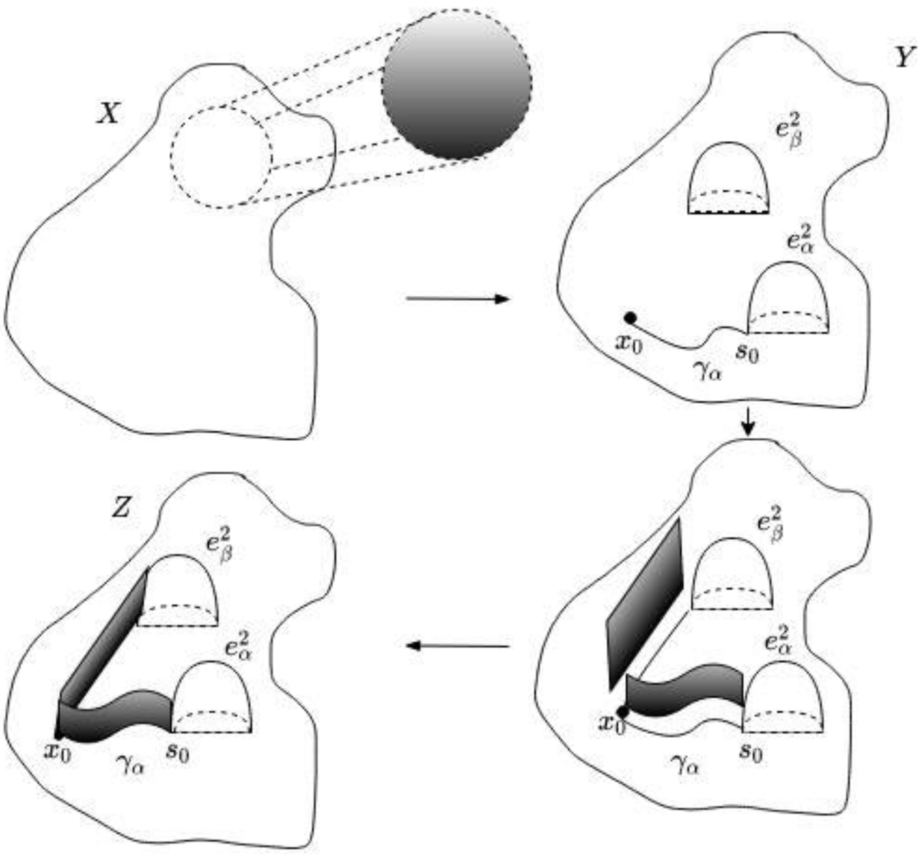
We examine how the fundamental group of a space is affected by attaching 2-cells to the space.

Suppose we attach a collection of 2-cells e^2_α to a path connected space X via maps $\varphi_\alpha : S^1 \to X$, producing a space Y. If s_0 is a basepoint of S^1, then φ_α determines a loop at $\varphi_\alpha(s_0)$ that we shall call φ_α, even though technically loops are maps $I \to X$ rather than $S^1 \to X$. For different α 's the basepoints $\varphi_\alpha(s_0)$ of these loops φ_α may not all coincide. To remedy this, choose a basepoint $x_0 \in X$ and a path γ_α in X from x_0 to $\varphi_\alpha(s_0)$ for each α. Then $\gamma_\alpha\varphi_\alpha\overline{\gamma_\alpha}$ is a loop at x_0. This loop may not be nullhomotopic in X, but it will certainly be nullhomotopic after the cell e^2_α is attached. Thus the normal subgroup $N \subset \pi_1(X, x_0)$ generated by all the loops $\gamma_\alpha\varphi_\alpha\overline{\gamma_\alpha}$ for varying α lies in the kernel of the map $\pi_1(X, x_0) \to \pi_1(Y, x_0)$ induced by the inclusion $X \hookrightarrow Y$.

Proposition 2.3.1. *The inclusion $X \hookrightarrow Y$ induces a surjection $\pi_1(X, x_0) \to \pi_1(Y, x_0)$ whose kernel is N. Thus $\pi_1(Y) \approx \pi_1(X)/N$.*

The kernel N is independent of the choice of the paths γ_α, but this can also be seen directly: If we replace γ_α by another path η_α having the same endpoints, then $\gamma_\alpha\varphi_\alpha\overline{\gamma_\alpha}$ changes to $\eta_\alpha\varphi_\alpha\overline{\eta_\alpha} = (\eta_\alpha\overline{\gamma_\alpha})\gamma_\alpha\varphi_\alpha\overline{\gamma_\alpha}(\gamma_\alpha\overline{\eta_\alpha})$, so $\gamma_\alpha\varphi_\alpha\overline{\gamma_\alpha}$ and $\eta_\alpha\varphi_\alpha\overline{\eta_\alpha}$ define conjugate elements of $\pi_1(X, x_0)$.

Proof. Let us expand Y to a slightly larger space Z that deformation retracts onto Y and is more convenient for applying van Kampen's theorem. The space Z is obtained from Y by attaching rectangular strips $S_\alpha = I \times I$, with the lower edge $I \times \{0\}$ attached along γ_α, the right edge $\{1\} \times I$ attached along an arc $\beta_\alpha : I \to Y$ whose



origin is at $\phi_\alpha(s_0)$ in e^2_α, and all the left edges $\{0\} \times I$ of the different strips identified together. The top edges of the strips are not attached to anything, and this allows us to deformation retract Z onto Y. So the space Z is of the form

$$Z = \frac{Y \amalg (\cup_\alpha(\gamma_\alpha(I) \times I))}{(\gamma_\alpha(s), 0) \sim \gamma_\alpha(s), (\gamma_\alpha(1), t) \sim \beta_\alpha(t)}$$

To show that Z deformation retracts onto Y, consider the map $r : Z \to Y$ as follows:

$$r(z) = \begin{cases} id_Y(z) & z \in Y \\ \gamma_\alpha(s_1) & z = [(\gamma_\alpha(s_1), s_2)], \text{ with } s_1 \neq 1 \text{ and } s_2 \neq 0. \end{cases}$$

r is well-defined since $r[(\gamma_\alpha(s_1), 0)] = \gamma_\alpha(s_1) = z = id_Y(\gamma_\alpha(s_1))$, for $z = [(\gamma_\alpha(s_1), 0)]$ and also, $r[(\gamma_\alpha(0), s_2)] = \gamma_\alpha(0) = x_0 = \gamma_\beta(0) = r[(\gamma_\beta(0), s_2)]$, for $[(\gamma_\alpha(0), s_2)] = [(\gamma_\beta(0), s_2)]$.$r[(\gamma_\alpha(1), t)] = [(\gamma_\alpha(1), t)] = [\beta_\alpha(t)] = r[\beta_\alpha(t)]$ since $\beta_\alpha(t) \in Y$. r is continuous since assuming that W is an open subset of Y, then $r^{-1}(W) = W \cup_\alpha (\gamma_\alpha^{-1}(W) \times I)$ is open as union of open sets.

Consider the inclusion $i : Y \to Z$, we have $r \circ i(y) = r([y]) = [y] = y = id_Y(y)$ by definition of the map r. So $r \circ i = id_Y$ We have

$$i \circ r(z) = \begin{cases} id_Y(z) & z \in Y \\ (\gamma_\alpha(s_1), 0) & otherwise \end{cases}$$

Consider the map $H : Z \times I \to Z$ defined by

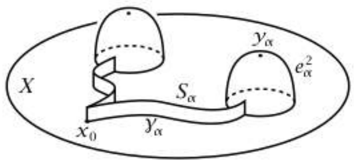
$$H(z,t) = \begin{cases} [id_Y(z)] & z \in Y \\ [(\ \gamma_\alpha(s_1), ts_2)] & otherwise z = (\gamma_\alpha(s_1), s_2) \end{cases}$$

H is well-defined since $H([(\gamma_\alpha(s_1), 0)], t) = [\gamma_\alpha(s_1)] = z = [id_Y(\gamma_\alpha(s_1))]$ by construction of Z, for $z = [(\gamma_\alpha(s_1), 0)]$ and also, $H([(\gamma_\alpha(0), s_2)], t) = [(\gamma_\alpha(0), ts_2)] = [(x_0, ts_2)] = [(\gamma_\beta(0), ts_2)] = H([(\gamma_\beta(0), s_2)], t)$, for $[(\gamma_\alpha(0), s_2)] = [(\gamma_\beta(0), s_2)]$. H is continuous since assume O is an open subset of Z, then $H^{-1}(O) = (O \cup_\alpha (\gamma_\alpha^{-1}(O) \times I)) \times I$ is open as the cartesian product of the segment I and union of open set.

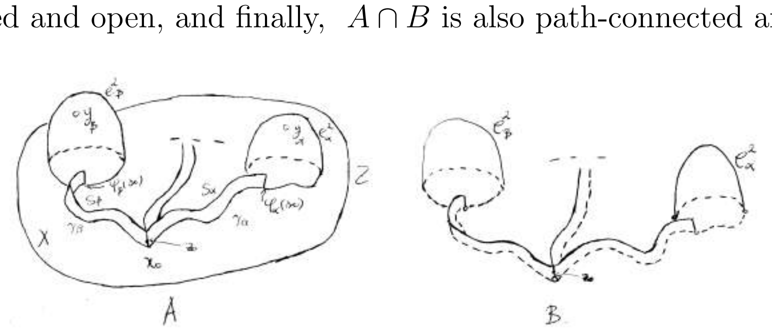
$$H(z,0) = \begin{cases} id_Y(z) & z \in Y \\ [(\ \gamma_\alpha(s_1), 0)] & otherwise \end{cases}$$

$$H(z,1) = \begin{cases} id_Y(z) & z \in Y \\ [(\ \gamma_\alpha(s_1), s_2)] & otherwise \end{cases}$$

Hence $H(z,0) = i \circ r(z)$ and $H(z,1) = id_Z(z)$. Hence we have that $\pi_1(Y) \cong \pi_1(Z)$. In each cell e_α^2, choose a point y_α not in the arc along which S_α is attached.



Let $A = Z - \cup_\alpha y_\alpha$ and let $B = Z - X = \cup_\alpha(e_\alpha^2 \cup S_\alpha) - \gamma_\alpha\varphi_\alpha\overline{\gamma_\alpha}(I)$. We have $A \cap B = (Z - X) - \cup_\alpha y_\alpha = \cup_\alpha(e_\alpha^2 \cup S_\alpha) - (\gamma_\alpha \cup \gamma_\alpha\varphi_\alpha\overline{\gamma_\alpha}(I))$. Hence, A is path-connected and open since $A_C = \cup_\alpha Y_\alpha$ is closed as a finite set, B is also path-connected and open, and finally, $A \cap B$ is also path-connected and open.



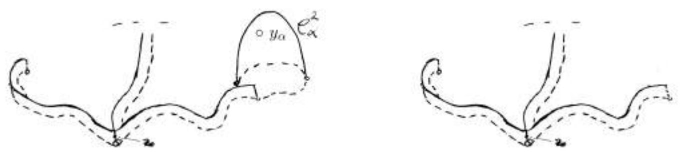
Since $x_0 \notin A \cap B$, to apply van Kampen's theorem to the cover $\{A, B\}$ of Z, we need to have a new basepoint, say $z_0 \in A \cap B$. That point z_0 is chosen close to x_0 on the segment where all the bands S_α intersect.
Now, the hypothesis of van Kampen's theorem are verified. Then:

(i) the map $\Phi : \pi_1(A, z_0) * \pi_1(B, z_0) \to \pi_1(Z, z_0)$ is surjective.

(ii) Its kernel $ker\Phi = N \lhd \pi_1(A, z_0) * \pi_1(B, z_0)$ generated by $i_1^*([\delta_\alpha])i_2^*([\delta_\alpha]^{-1}) \in \pi_1(A, z_0) * \pi_1(B, z_0), \forall[\delta_\alpha] \in \pi_1(A \cap B)$ implies $N = \langle\langle[\delta_\alpha]_\alpha\rangle\rangle$.

(i) and (ii) imply that $\pi_1(Z, z_0) \cong (\pi_1(A, z_0) * \pi_1(B, z_0))/N$. But B is contractible. Hence $\pi_1(B, z_0) = 1$. So we have

$$\pi_1(Z, z_0) \cong \pi_1(A, z_0)/N.$$



So it remains only to see that $\pi_1(A \cap B)$ is generated by the loops $\gamma_\alpha\varphi_\alpha\overline{\gamma_\alpha}$, or rather by loops in $A \cap B$ homotopic to these loops. If this is shown, we are done since A_α deformation retracts onto a circle in $e_\alpha^2 - y_\alpha$, we have $\pi_1(A_\alpha) \approx \mathbb{Z}$ generated by a loop homotopic to $\gamma_\alpha\varphi_\alpha\overline{\gamma_\alpha}$, and the result follows.
Here, for each α, we consider the open subsets $A_\alpha = A \cap B - \cup_{\beta\neq\alpha}e_\alpha^2$ and we apply van Kampen's theorem on the cover $\{A_\alpha|\alpha\}$ of $A \cap B$. We have that A_α's and $\cap_\alpha A_\alpha$ are open and path-connected, also $z_0 \in \cap_\alpha A_\alpha$. Applying van Kampen's theorem, we have the following:

(i) the map $\Phi : *_\alpha\pi_1(A_\alpha, z_0) \to \pi_1(A \cap B, z_0)$ is surjective.

(ii) Its kernel $ker\Phi = N \lhd *_\alpha\pi_1(A_\alpha, z_0)$ generated by $i_1^*([\lambda_\alpha])i_2^*([\lambda_\alpha]^{-1}) \in *_\alpha\pi_1(A_\alpha, z_0), \forall[\lambda_\alpha] \in \pi_1(\cap_\alpha A_\alpha)$ but $\cap_\alpha A_\alpha$ is contractible, that is, $\pi_1(\cap_\alpha A_\alpha, z_0) = 1$ which implies $N = 1$.

The points (i) and (ii) imply that $\pi_1(A \cap B, z_0) \cong *_\alpha\pi_1(A_\alpha, z_0)$. But each A_α deformation retracts onto the circle S^1 in $e_\alpha^2 - y_\alpha$ due to the holes created by the withdrawal of points y_α, that is, $\pi_1(A_\alpha, z_0) = \mathbb{Z}$ for each α, so we have $\pi_1(A \cap B, z_0) \cong *_\alpha\mathbb{Z}$.

We have that $A = \frac{(Y - \cup_\alpha y_\alpha) \amalg (\cup_\alpha(\gamma_\alpha(I) \times I))}{(\gamma_\alpha(s),0) \sim \gamma_\alpha(s)}$ deformation retracts onto X as in the case of Z and Y.

We have $\pi_1(X, x_0) \cong \pi_1(X, x_1) \cong \pi_1(A, x_1) \cong \pi_1(A, z_0)$ and $\pi_1(Y, z_0) \cong \pi_1(Y, x_0)$ since the fundamental group is independent of the choice of the basepoint for path-connected spaces. Therefore we have the following result:

$$\pi_1(Y, x_0) \cong \pi_1(X, x_0)/N, \text{ with } N = \langle\langle \{[\gamma_\alpha \varphi_\alpha \gamma_\alpha]\}_\alpha \rangle\rangle$$

where $[\gamma_\alpha \varphi_\alpha \gamma_\alpha] = \Psi([\delta_\alpha]), \Psi : \pi_1(Z, z_0) \to \pi_1(Z, x_0)$ is the basepoint-change isomorphism. □

Corollary 2.3.2. Fundamental group of an orientable surface M_g of genus g.

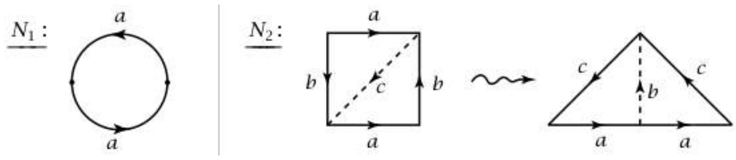
Proof. The orientable surface M_g of genus g has a cell structure with one 0-cell, $2g$ 1-cells, and one 2-cell. The 1-skeleton is a wedge sum of $2g$ circles, with fundamental group free on $2g$ generators. The 2-cell is attached along the loop given by the product of the commutators of these generators, say $[a_1, b_1], ..., [a_g, b_g]$. Therefore

$$\pi_1(M_g) \approx \langle a_1, b_1, ..., a_g, b_g | [a_1, b_1]...[a_g, b_g] \rangle$$

where $\langle g_\alpha | r_\beta \rangle$ denotes the group with generators g_α and relators r_β, in other words, the free group on the generators g_α modulo the normal subgroup generated by the words r_β in these generators. □

Nonorientable surfaces can be treated in the same way. If we attach a 2-cell to the wedge sum of g circles by the word $a_1^2....a_g^2$, we obtain a nonorientable surface N_g. For example, N_1 is the projective plane $\mathbb{RP}^2$, the quotient of D^2 with antipodal points of ∂D^2 identified. And N_2 is the Klein bottle, though the more usual representation of the Klein bottle is as a square with opposite sides identified via the word $aba^{-1}b$. If one cuts the square along a diagonal and reassembles the resulting two triangles as shown in the figure, one obtains the other representation as a square with sides identified via the word a^2c^2. By the Proposition 2.3.1, $\pi_1(N_g) \approx \langle a_1, ..., a_g | a_1^2.....a_g^2 \rangle$. This abelianizes to the direct sum of $\mathbb{Z}_2$ with $g-1$ copies of $\mathbb{Z}$ since in the abelianization, we can rechoose the generators to be $a_1, ..., a_{g-1}$ and $a_1 + ... + a_g$, with $2(a_1 + ... + a_g) = 0$. Hence N_g is not homotopy equivalent to N_h if $g \neq h$, nor is N_g homotopy equivalent to any orientable surface M_h.

Corollary 2.3.3. The surface M_g is not homeomorphic, or even homotopy equivalent, to M_h if $g \neq h$.



Proof. The abelianization of $\pi_1(M_g)$ is the direct sum of $2g$ copies of $\mathbb{Z}$. So if $M_g' \simeq M_h$ then $\pi_1(M_g) \approx \pi_1(M_h)$, hence the abelianizations of these groups are isomorphic, which implies $g = h$. □

Corollary 2.3.4. For every group G there is a 2-dimensional cell complex X_G with $\pi_1(X_G) \approx G$.

Proof. Choose a presentation $G = \langle g_\alpha | r_\beta \rangle$. This exists since every group is a quotient of a free group, so the g_α's can be taken to be the generators of this free group with the r_β's generators of the kernel of the map from the free group to G. Now construct X_G from $\bigvee_\alpha S^1_\alpha$ by attaching 2-cells e^2_β by the loops specified by the words r_β. □

Chapter 3

Hopf-Rinow Theorem

Definition 3.0.1. Let X be a topological space. If (x_n) is a sequence of points of X, and if

$$n_1 < n_2 < ... < n_i < ...$$

is an increasing sequence of positive integers, then the sequence (y_i) defined by setting $y_i = x_{n_i}$ is called a subsequence of the sequence (x_n). The space X is said to be sequentially compact if every sequence of points of X has a convergent subsequence.

Definition 3.0.2. Let (X, d) and (Y, d') be two metric spaces, and F a family of functions from X to Y.
The family F is equicontinuous at a point $x_0 \in X$ if for every $\epsilon > 0$, there exists a $\delta > 0$ such that $d'(f(x_0), f(x)) < \epsilon$ for all $f \in F$ and all x such that $d(x_0, x) < \delta$.
The family is pointwise equicontinuous if it is equicontinuous at each point of X.
The family F is uniformly equicontinuous if for every $\epsilon > 0$, there exists a $\delta > 0$ such that $d'(f(x_1), f(x_2)) < \epsilon$ for all $f \in F$ and all $x_1, x_2 \in X$ such that $d(x_1, x_2) < \delta$.

Definition 3.0.3. Let X be a topological space. X is called locally compact if every point x of X has a compact neighbourhood, i.e., there exists an open set U and a compact set K, such that $x \in U \subseteq K$.

Definition 3.0.4. Suppose E is a set and $(f_n)_{n\in\mathbb{N}}$ is a sequence of real-valued functions on it. We say the sequence $(f_n)_{n\in\mathbb{N}}$ is uniformly convergent on E with limit $f : E \to \mathbb{R}$ if for every $\epsilon > 0$, there exists a natural number N such that for all $n \geq N$ and all $x \in E$

$$|f_n(x) - f(x)| < \epsilon.$$

Lemma 3.0.5. *Every closed subspace of a compact space is compact.*

Lemma 3.0.6. *The image of a compact space under a continuous map is compact.*

Lemma 3.0.7. *Let X be a metrizable space. Then X is compact if and only if X is sequentially compact.*

Definition 3.0.8. A metric space (X, d) is said to be proper if, for every $x \in X$ and every $r > 0$, the closed ball $\overline{B}(x, r)$ is compact.

3.1 Hopf-Rinow Theorem

Proposition 3.1.1. *Let (X, d) be a metric space and $c : [a, b] \to X$ be a path. Let (c_n) be a sequence of paths $[a, b] \to X$ converging uniformly to a path c. If c is rectifiable, then for every $\epsilon > 0$, there exists an integer $N(\epsilon)$ such that*

$$l(c) \leq l(c_n) + \epsilon$$

whenever $n > N(\epsilon)$.

Proof. Choose $a = t_0 < t_1 < ... < t_k = b$ such that

$$l(c) \leq \sum_{i=0}^{k-1} d(c(t_i), c(t_{i+1})) + \epsilon/2$$

Then we choose $N(\epsilon)$ big enough to ensure that $d(c(t), c_n(t)) < \epsilon/4k$ for all $n > N(\epsilon)$ and all $t \in [a, b]$(by uniform convergence). By the triangle inequality,

$$\begin{aligned} d(c(t_i), c(t_{i+1}) &\leq d(c(t_i), c_n(t_i)) + d(c_n(t_i), c_n(t_{i+1})) + d(c(t_{i+1}), c_n(t_{i+1})) \\ &\leq 2\epsilon/4k + d(c_n(t_i), c_n(t_{i+1})). \end{aligned}$$

Hence

$$l(c) \leq k\epsilon/2k + \sum_{i=0}^{k-1} d(c_n(t_i), c_n(t_{i+1})) + \epsilon/2 \leq \epsilon + l(c_n)$$

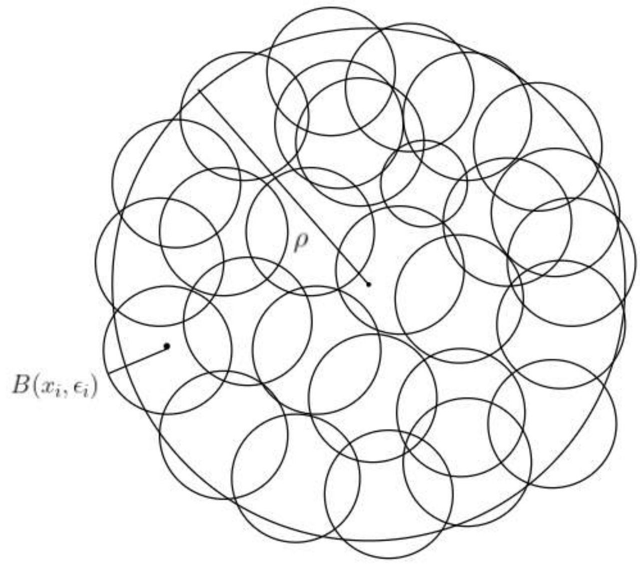
□

Theorem 3.1.2 (Hopf-Rinow Theorem)**.** *Let X be a length space. If X is complete and locally compact, then*

(1) Every closed bounded subset of X is compact.

(2) X is a geodesic space (i.e. joining two points of X, there exists a minimizing geodesic).

Proof.
For (1), it suffices to prove that closed balls about a fixed point $a \in X$ are compact. This follows from the fact that every closed bounded set is contained in a closed ball.
Given $r > 0$, we denote by $\overline{B}(r) = \{x \in X | d(a,x) \leq r\}$ the closed ball with centre a and radius r. Consider the set of non-negative numbers A such that $\overline{B}(\rho)$ is compact for every $\rho \in A$.



$0 \in A$ since singleton sets are compact. We claim that A is both open and closed.

Because X is assumed to be locally compact, the interval contains a neighbourhood of 0. To see that it contains a neighbourhood of each of its other points, we fix $\rho \geq 0$ such that $\overline{B}(\rho)$ is compact, and use the local compactness of X to cover $\overline{B}(\rho)$ with finitely many balls $B(x_i, \epsilon_i)$ such that each $\overline{B}(x_i, \epsilon_i)$ is compact. There is a strictly positive lower bound, say, 2δ on the distance from any point in $\overline{B}(\rho)$ to the closed set $X - \cup B(x_i, \epsilon_i)$. This follows from the fact that the distance funtion d is a continuous function and $d(., X - \cup B(x_i, \epsilon_i)) : \overline{B}(\rho) \to \mathbb{R}$ is a function from a compact set. Hence $\overline{B}(\rho + \delta)$ is a closed subset of the compact set $\cup \overline{B}(x_i, \epsilon_i)$ and thus compact. Hence A is open (follows from Lemma 3.0.5 and Lemma 3.0.6).

It remains to prove that if $\overline{B}(r)$ is compact for all $r < \rho$ then $\overline{B}(\rho)$ is compact i.e. A is closed. It suffices to show that every sequence of points $x_n \in \overline{B}(\rho)$ such that $d(a, x_n)$ converges to ρ has a convergent subsequence (sequentially compact). We fix such a sequence (x_n).

Let ϵ_p be a sequence of positive numbers tending to 0. For each p and each n, we can find a point y_n^p such that

$$d(a, y_n^p) < \rho - \epsilon_p/2 \text{ and } d(y_n^p, x_n) \leq \epsilon_p$$

(To see this, one chooses a path c of length smaller than $d(a, x_n) + \epsilon_p/2$ joining a to x_n, and then chooses a convenient point y_n^p on this path). For each p, the points y_n^p are contained in the compact ball $\overline{B}(\rho - \epsilon_p/2)$. Hence we can extract from $(y_n^1)_{n\in\mathbb{N}}$ a convergent subsequence $(y_{n_k^1}^1)_{k\in\mathbb{N}}$; from the sequence $(y_{n_k^1}^2)_{k\in\mathbb{N}}$, we can then extract a convergent subsequence, and so on. Eventually, by a diagonal process, we obtain a sequence of integers $(n_k)_{k\in\mathbb{N}}$ such that the sequence $(y_{n_k}^p)_{k\in\mathbb{N}}$ converges for all p. We claim that the corresponding sequence $(x_{n_k})_{k\in\mathbb{N}}$ is Cauchy. Indeed, for a given $\epsilon > 0$, we can choose p such that $\epsilon_p < \epsilon/3$ and then use the fact that the sequence $y_{n_k}^p$ is convergent (hence Cauchy) to see that for large k, k', we have

$$d(y_{n_k}^p, y_{n_{k'}}^p) < \epsilon/3,$$

and hence

$$\begin{aligned} d(x_{n_k}, x_{n_{k'}}) &< d(x_{n_k}, y_{n_k}^p) + d(y_{n_k}^p, y_{n_{k'}}^p) + d(y_{n_{k'}}^p, x_{n_{k'}}) \\ &< \epsilon_p + \epsilon/3 + \epsilon_p \\ &< \epsilon. \end{aligned}$$

We shall now prove (2). Let a and b be distinct points of X. For every integer $n > 1$, there is a path $c_n : [0,1] \to X$, parameterized proportional to its arc length, such that $l(c_n) < d(a,b) + 1/n$. Such a family of paths $[c_n]$ is equicontinuous; indeed forall $t, t' \in [0,1]$ we have:

$$|t - t'| = \frac{l(c_n|_{[t,t']})}{l(c_n)} \geq \frac{d(c_n(t), c_n(t'))}{d(a,b) + 1}$$

hence $d(c_n(t), c_n(t')) < \epsilon$ if $|t - t'| < \frac{\epsilon}{d(a,b)+1}$. The image of each path c_n is contained in the compact set $\overline{B}(2d(a,b))$. By the Arzelà-Ascoli theorem (see below)($[0,1]$ is a separable metric space and X is a compact) there is a subsequence of the $(c_n)_{n\in\mathbb{N}}$ converging uniformly to a path $c : [0,1] \to X$. Finally, by Proposition 3.1.1, we have

$$l(c) \leq \liminf l(c_{k_n}) = d(a,b).$$

But $l(c) \geq d(a,b)$, so in fact $l(c) = d(a,b)$, and therefore c is a linearly reparameterized geodesic joining a to b. ☐

Corollary 3.1.3. A length space is proper if and only if it is complete and locally compact.

3.2 Arzela Ascoli Theorem

Lemma 3.2.1. *If $f : Y \to X$ is a uniformly continuous map between two metric spaces and (x_n) is a Cauchy sequence in Y, then $(f(x_n))$ is a Cauchy sequence in X.*

Proof. Since (x_n) is a Cauchy sequence, we have that for every $\delta > 0$, $\exists N \in \mathbb{N}$ such that for $m, n \geq N$,

$$d_X(x_n, x_m) < \delta$$

We also have that f is a uniformly continuous map. Hence we get that

$$d_Y(f(x_n), f(x_m)) < \epsilon$$

for all $n, m > N$. Thus $(f(x_n))$ is Cauchy as well. ☐

Lemma 3.2.2. *Let X and Y be metric spaces, $S \subseteq Y$, and $f : S \to X$ be uniformly continuous. If two sequences (x_n) and (y_n) in S converge to the same limit in Y and if the sequence $f(x_n)$ converges, then the sequence $f(y_n)$ converges and $\lim f(x_n) = \lim f(y_n)$.*

Proof. Consider the interlaced sequence z_n

$$z_n = \begin{cases} x_{\frac{n+1}{2}} & \text{if n is odd} \\ y_{n/2} & \text{if n is even} \end{cases}$$

By hypothesis, there exists a point a in Y such that $x_n \to a$ and $y_n \to a$. It is easy to see that the "interlaced" sequence (z_n) also converges to a : :since (x_n) converges a, there exists $N \in \mathbb{N}$ such that $d(x_n, a) < \epsilon$ for all $n > N$. Similarly we have $d(y_n, a) < \epsilon$ for all $n > M$ where $M \in \mathbb{N}$. Now choosing $A = \max\{M, N\}$, we see that $d(z_n, a) < \epsilon$ for all $n > A$. Hence (z_n) is Cauchy in Y and therefore in S (follows from the fact that all convergent sequences are Cauchy), and by lemma above (applied to S), the sequence $f(z_n)$ is Cauchy in X. The sequence $f(x_n)$ is a subsequence of $f(z_n)$ and is, by hypothesis, convergent (follows from the fact that if a subsequence of a Cauchy sequence is convergent, the sequence itself is convergent). Therefore, the sequence $f(z_n)$ converges and

$$\lim f(x_n) = \lim f(z_n) = \lim f(y_n).$$

$\square$

Lemma 3.2.3. *Let X and Y be metric spaces, S a subset of Y, and $f : S \to X$. If f is uniformly continuous and X is complete, then there exists a unique continuous extension of f to $\overline{S}$.*

Proof. Define $g : \overline{S} \to X$ by $g(a) = \lim f(y_n)$ where (y_n) is a sequence in S converging to a.
First, we show that g is well defined. To do this, we show that $\lim f(y_n)$ does exist (follows from Lemma 3.2.1), and the value assigned to g at a does not depend on the particular sequence (y_n) chosen, i.e., if $x_n \to a$ and $y_n \to a$, then $\lim f(x_n) = \lim f(y_n)$ (follows from Lemma 3.2.2).
Now g is an extension of f, since when $y \in S$ and if $y_n \to y$, then $\lim f(y_n) = f(y) = g(y)$.
Now we show the uniqueness of the extension. Let h and g be two extensions of the same function f. Now consider the set $B = \{y \in \overline{S} | h(y) = g(y)\}$. Observe that $S \subseteq B$. We show that this set is closed. This would imply that $B = \overline{S}$.
X is a Hausdorff space then the set $D = \{(x, x) \in X \times X, x \in X\}$ is closed in $X \times X$. Now let $p : \overline{S} \to X \times X$ be the map defined by $p(y) = (h(y), g(y))$ it is a continuous map, and $B = p^{-1}(D)$ is closed. $\square$

Remark. Observe that every function in a family of uniformly equicontinuous functions is uniformly continuous.

Lemma 3.2.4 (Arzelia-Ascoli)**.** *If X is a compact metric space and Y is a separable metric space, then every sequence of uniformly equicontinuous maps $f_n : Y \to X$ has a subsequence that converges(uniformly on compact subsets) to a continuous map $f : Y \to X$.*

Proof. We first fix a countable dense set $Q = \{q_1, q_2, ...\}$ in Y. Then we use the compactness of X to choose a subsequence $f_{n(1)}$ such that the sequence of points $f_{n(1)}(q_1)$ converges in X as $n \to \infty$; we call the limit point $f(q_1)$. We then pass to a further subsequence $f_{n(2)}$ to ensure that $f_{n(2)}(q_2)$ converges to some point $f(q_2)$, as $n \to \infty$. Proceeding by recursion on k, we pass to further subsequences $f_{n(k)}$ so that as $n(k)$ tends to infinity, $f_{n(k)}(q_j)$ converges to $f(q_j)$ for all $j \leq k$. The diagonal subsequence $f_{n(n)}$ has the property that $\lim\limits_{n\to\infty} f_{n(n)}(q) = f(q)$ for all $q \in Q$.

By uniform equicontinuity, for every $\epsilon > 0$ there exists $\delta > 0$ such that if $d(y, y') < \delta$ then $d(f_n(y), f_n(y')) < \epsilon$ for all n. So taking the limit, $d(f(q), f(q')) \leq \epsilon$ for all $q, q' \in Q$ with $d(q, q') < \delta$. Since X is compact (hence complete) it follows that f has a unique continuous extension $Y \to X$, which again satisfies this inequality (from Lemma 3.2.1 ,Lemma 3.2.2 and Lemma 3.2.3 above).

It remains to show that the convergence of $f_{n(n)}$ to f is uniform on compact subsets. Given $\epsilon > 0$ we choose δ as above. Given a compact subset $C \subseteq Y$ we fix $N > 0$ so that for every $y \in C$ there exists $j(y) < N$ with $d(y, q_{j(y)}) < \delta$, then we fix M sufficiently large so that $d(f_{n(n)}(q_j), f(q_j)) < \epsilon$ for all $n > M$ and all $j < N$. Then, for all $y \in C$ and all $n > M$ we have:

$$\begin{aligned} d(f(y), f_{n(n)}(y)) &\leq d(f(y), f(q_{j(y)})) + d(f(q_{j(y)}), f_{n(n)}(q_{j(y)})) + d(f_{n(n)}(q_{j(y)}), f_{n(n)}(y)) \\ &\leq 3\epsilon. \end{aligned}$$

Thus $f_{n(n)} \to f$ uniformly on compact sets as $n \to \infty$. ☐

Corollary 3.2.5. If X is a compact metric space and if $c_n : [0, 1] \to X$ is a sequence of linearly reparameterized geodesics, then there exists a linearly reparameterized geodesic $c : [0, 1] \to X$ and a subsequence $c_{n(i)}$ such that $c_{n(i)} \to c$ uniformly as $n(i) \to \infty$.

Proof. For every $n \in N$ and all $t, t' \in [0, 1]$, we have $d(c_n(t)c_n(t')) \leq D|t - t'|$, where D is the diameter of X. Thus, the c_n are equicontinuous. Now by the Arzela Ascoli theorem above, we are done. Hence uniform limit of geodesics is a geodesic. ☐

Lemma 3.2.6. *Every sequence of equicontinuous maps $\{f_n\}$ which converges pointwise on a compact set to f, converges uniformly to f.*

Proof. Let $\epsilon > 0$. By equicontinuity, there exists a $\delta > 0$ such that for all n,

$$d(x, y) < \delta \implies d(f_n(x), f_n(y)) < \epsilon/3.$$

Since K is compact, there exist finitely many points $\{p_1, , p_m\}$ such that

$$K \subset \cup_{k=1}^{m} B_\delta(p_k).$$

Now by pointwise convergence, we see that for every $\epsilon > 0$, there exists $N_k \in \mathbb{N}$ such that

$$d(f_n(p_k), f(p_k)) < \epsilon$$

when $n > N_k$. Let $N = \max\{N_1, , N_m\}$. Now for any $x \in K$, there exists p_k such that $d(x, p_k) < \delta$. Then for $n > N$,

$$\begin{aligned} d(f_n(x), f(x)) &< d(f_n(x), f_n(p_k)) + d(f_n(p_k), f(p_k)) + d(f(p_k), f(x)) \\ &< \epsilon/3 + \epsilon/3 + \epsilon/3 \\ &< \epsilon. \end{aligned}$$

Hence proved. □

Lemma 3.2.7. *Let x and y be points in a proper geodesic metric space X. Suppose that there is a unique geodesic segment joining x to y in X; Let $c : [0,1] \to X$ be a linear parameterization of this segment. Let $c_n : [0,1] \to X$ be linearly reparameterized geodesics in X, and suppose that the sequence of points $c_n(0)$ and $c_n(1)$ converge to x and y, respectively. Then $c_n \to c$ uniformly.*

Proof. We fix $R > 0$ so that the image of each of the paths c_n lies in the (compact) closed ball of radius R about x. If the sequence c_n did not converge to c pointwise, then there would exist $\epsilon > 0, t_0 \in (0,1)$ and an infinite sequence c_{n_i} such that $d(c_{n_i}(t_0), c(t_0)) \geq \epsilon$ for all n_i. The previous corollary would yield a subsequence of the c_{n_i} converging uniformly to a linearly reparameterized geodesic $c' : [0,1] \to X$ joining $x = \lim c_{n_i}(0)$ to $y = \lim c_{n_i}(1)$. But since $d(c'(t_0), c(t_0)) \geq \epsilon$, this would contradict the uniqueness of c.
Thus $c_n \to c$ pointwise. Using the fact that c and c_n are geodesic, it is easy to see that the convergence must be uniform. □

Chapter 4

Hyperbolic Metric Spaces

In mathematics, a hyperbolic metric space is a metric space satisfying certain metric relations (depending quantitatively on a nonnegative real number δ) between points. The definition, introduced by Mikhael Gromov, generalizes the metric properties of classical hyperbolic geometry and trees. Hyperbolicity is a large-scale property and is very useful to the study of certain infinite groups called (Gromov-)hyperbolic group

4.1 Gromov Product

Let (X, x_0) be a pointed metric space. Then the following notations are adopted:

- $|x - y| = d(x, y)$ denotes the distance between the points x and y $\forall x, y \in (X, x_0)$.
- $|x| = |x|_{x_0} = d(x, x_0)$.

We define the Gromov product $(x.y)$ based at the point x_0 to be

$$(x.y)_{x_0} = \frac{1}{2}(d(x, x_0) + d(y, x_0) - d(x, y))$$

With this definition in hand, we can show the following.

Proposition 4.1.1. *Prove the following:*

(i) $(x.y) = (y.x)$

(ii) $(x.x) = d(x, x_0)$

(iii) $(x.x_0) = 0$

(iv) Triangle inequality

$$0 \leq (x.y) \leq min(d(x,x_0), d(y,x_0))$$

Proof. (i)

$$(x.y) = \frac{1}{2}(d(x,x_0) + d(y,x_0) - d(x,y))$$
$$(y.x) = \frac{1}{2}(d(y,x_0) + d(x,x_0) - d(y,x))$$

(ii) $(x.x) = \frac{1}{2}(d(x,x_0) + d(x,x_0) - d(x,x)) = d(x,x_0)$

(iii) $(x.x_0) = \frac{1}{2}(d(x,x_0) + d(x_0,x_0) - d(x,x_0)) = 0$

(iv)

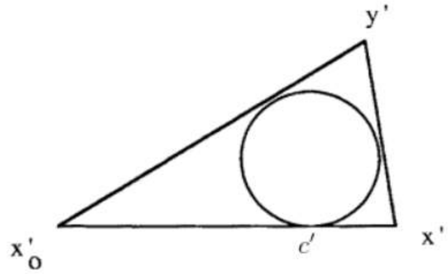
$$\begin{aligned}
|d(x,x_0) - d(y,x_0)| &\leq d(x,y) \leq d(x,x_0) + d(y,x_0) \\
\Rightarrow -d(x,x_0) - d(y,x_0) &\leq -d(x,y) \leq -|d(x,x_0) - d(y,x_0)| \\
\Rightarrow d(x,x_0) + d(y,x_0) - d(x,x_0) - d(y,x_0) &\leq d(x,x_0) + d(y,x_0) - d(x,y) \\
&\leq d(x,x_0) + d(y,x_0) - |d(x,x_0) - d(y,x_0)| \\
\Rightarrow 0 \leq (x.y) &\leq min(d(x,x_0), d(y,x_0))
\end{aligned}$$

□

We can calculate $(x.y)$ in the following manner. Consider the triangle $[x'_0, x', y']$ in the Euclidean plane which realizes the metric space $\{x_0, x, y\}$

$$d(x'_0, x') = d(x_0, x) \quad , \quad d(x'_0, y') = d(x_0, y) \text{ and } d(x', y') = d(x, y)$$

Let c' be the point of contact of the circle inscribed in the triangle $[x'_0, x', y']$ with



a side emanating from x'_0. Then $(x.y)$ is the distance between the points x'_0 and c'

(Remark: In the above construction, the Euclidean plane can be replaced by the Hyperbolic plane $\mathbb{H}^2$ or even by a sphere of fairly large radius).
Now we look at how the change in basepoint affects the Gromov product.

Proposition 4.1.2. $|(x.y)_{x_0} - (x.y)_{x_1}| \leq d(x_0, x_1)$

Proof.

$$\begin{aligned}|(x.y)_{x_0} - (x.y)_{x_1}| &= |\frac{1}{2}(d(x,x_0) + d(y,x_0) - d(x,y)) - \frac{1}{2}(d(x,x_1) + d(y,x_1) - d(x,y))| \\ &= \frac{1}{2}|d(x,x_0) + d(y,x_0) - d(x,x_1) - d(y,x_1)|\end{aligned}$$

Now using triangle inequality twice,

$$\begin{aligned}|(x.y)_{x_0} - (x.y)_{x_1}| &\leq \frac{1}{2}|d(x_0,x_1) + d(x_0,x_1)| \\ &\leq d(x_0,x_1)\end{aligned}$$

□

4.2 Hyperbolicity

Definition 4.2.1. Let $\delta \geq 0$. Then we say that the pointed metric space (X, x_0) is δ-hyperbolic when

$$(x.y) \geq min((x.z), (y.z)) - \delta$$

for all $x, y, z \in X$.

Let us study the effect of changing the base point on the hyperbolicity condition above.

Lemma 4.2.2. *If (X, x_0) is δ-hyperbolic, then (X, x_1) is δ'-hyperbolic where $\delta' = \delta + 2d(x_0, x_1)$.*

Proof. Since (X, x_0) is δ-hyperbolic,

$$(x.y)_{x_0} \geq min((x.z)_{x_0}, (y.z)_{x_0}) - \delta.$$

Now from the inequality in Proposition 4.1.2, we have that,

$$-d(x_0, x_1) \leq (x.y)_{x_0} - (x.y)_{x_1} \leq d(x_0, x_1)$$
$$\Rightarrow (x.y)_{x_1} - d(x_0, x_1)| \leq (x.y)_{x_0} \leq (x.y)_{x_1} + d(x_0, x_1).$$

Now putting this in the hyperbolicity inequality, we get

$$(x.y)_{x_1} + d(x_0, x_1) \geq min(((x.z)_{x_1} - d(x_0, x_1)), ((y.z)_{x_1} - d(x_0, x_1))) - \delta$$
$$\Rightarrow (x.y)_{x_1} \geq min((x.z)_{x_1}, (y.z)_{x_1}) - \delta - 2d(x_0, x_1)$$

Hence $\delta' = \delta + 2d(x_0, x_1)$ □

As a consequence we get that if (X, x_0) is δ-hyperbolic, then (X, x_1) is $\delta'-$ hyperbolic with $\delta' = \delta + 2d(x_0, x_1)$.

Lemma 4.2.3. *If (X, x_0) is δ-hyperbolic, then*

$$(x.y) + (z.t) \geq min((x.z) + (y.t), (x.t) + (y.z)) - 2\delta \tag{4.1}$$

for all $x, y, z \in X$.

Proof. From the hyperbolicity inequality, we get

$$(x.y) \geq min((x.t), (y.t)) - \delta \tag{4.2}$$
$$(z.t) \geq min((z.x), (x.t)) - \delta \tag{4.3}$$
$$(x.y) \geq min((x.z), (z.y)) - \delta \tag{4.4}$$
$$(z.t) \geq min((z.y), (y.t)) - \delta \tag{4.5}$$

Case I When $(y.t)$ is the largest. Choose (1) and (4). Adding these, we get that

$$(x.y) + (z.t) \geq (x.t) + (z.y) - 2\delta$$

Case II When $(x.t)$ is the largest. Choose (1) and (2). Adding these, we get that

$$(x.y) + (z.t) \geq (y.t) + (z.x) - 2\delta$$

And similarly for the other cases as well. Hence we conclude that

$$(x.y) + (z.t) \geq min((x.t) + (z.y), (y.t) + (z.x)) - 2\delta$$

□

In the beginning, we saw how the change in basepoint affects the hyperbolicity inequality. Now we show a stronger version of the same below.

Proposition 4.2.4. *If (X, x_0) is δ-hyperbolic, then (X, x_1) is 2δ-hyperbolic for all points $x_1 \in X$.*

Proof. To prove the proposition, we use the lemma above. Add to both sides of the equation,

$$(d(x,t) + d(y,t) + d(z,t) - d(x,x_0) - d(y,x_0) - d(z,x_0) - d(t,x_0))/2$$

Then we get

$$(x.y)_t \geq min((z.y)_t, (x.z)_t) - 2\delta$$

Hence Proved. □

Now, we define what we mean by δ-hyperbolic spaces and hyperbolic spaces.

Definition 4.2.5. A metric space X is said to be δ-hyperbolic if (X, x_0) is δ-hyperbolic for all $x_0 \in X$. We say that X is hyperbolic if X is δ-hyperbolic for some $\delta \geq 0$.

The Proposition 4.2.4 shows that if there exists $x_0 \in X$ for which (X, x_0) is δ-hyperbolic, then X is 2δ-hyperbolic with respect to some other point.

4.3 Subsets of δ-Hyperbolic Spaces

Let X be a metric space and $Y \subset X$. We say that Y is bounded if the function $dist(., Y)$ is bounded on X. Now, we are interested to see when would hyperbolicity of a subset of a metric space imply the hyperbolicity of the metric space.

Proposition 4.3.1. *Let X be a metric space and $Y \subset X$. If X is δ-hyperbolic, then Y is δ-hyperbolic. Conversely, if Y is δ-hyperbolic and bounded, then X is δ'-hyperbolic where $\delta' = \delta + 6\eta$ with $\eta = \sup_{x \in X} dist(x, Y)$.*

Proof. Proving one way is obvious. Let us look at the converse: Let $x, y, z, b \in X$ be given and assume that $\eta = \sup_{x \in X} d(x, Y) < \infty$. Fix $x', y', z', b' \in Y$ such that $d(x, x'), d(y, y'), d(z, z'), d(b, b')$ are all $\leq \eta$. Using the triangle inequality, one can show that

$$\begin{aligned}(x'.y')_{b'} &\leq (x.y)_b + 3\eta \\ (x.z)_b &\leq (x'.z')_{b'} + 3\eta \\ (y.z)_b &\leq (y'.z')_{b'} + 3\eta.\end{aligned}$$

Then combining the above with the fact that Y is δ-hyperbolic, we get

$$\begin{aligned}(x.y)_b &\leq (x'.y')_{b'} - 3\eta \\ &\leq min\{(x'.z')_{b'}, (y'.z')_{b'}\} - \delta - 3\eta \\ &\leq min\{(x.z)_b - 3\eta, (y.z)_b - 3\eta\} - \delta - 3\eta \\ &= min\{(x.z)_b, (y.z)_b\} - \delta - 6\eta.\end{aligned}$$

So X is δ'-hyperbolic with $\delta' = \delta + 6\eta$. □

4.4 Four-Point Condition

In previous sections, we have defined hyperbolicity and looked at how the change in basepoint affects the condition. In this section we look at an important reformulation of the hyperbolicity condition(refer 4.2.1), the four-point condition. Later we look into its geometrical interpretation as well.

Proposition 4.4.1. *The metric space X is said to be δ-hyperbolic if and only if*

$$d(x,y) + d(z,t) \leq max(d(x,z) + d(y,t), d(x,t) + d(y,z)) + 2\delta, \tag{4.6}$$

for all $x, y, z, t \in X$.

Proof.
$(\Leftarrow)$

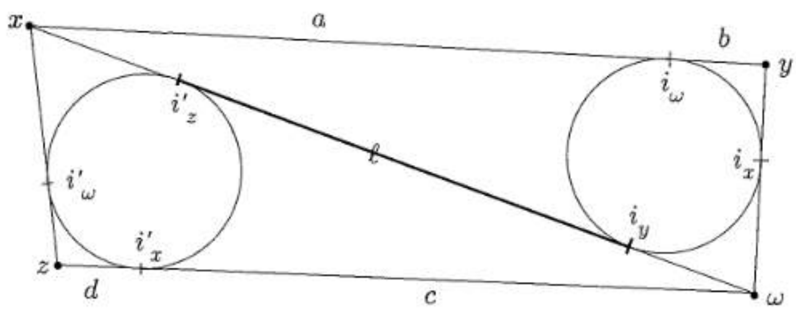
$$\begin{aligned}d(x,y) + d(z,t) &\leq max(d(x,z) + d(y,t), d(x,t) + d(y,z)) + 2\delta \\ d(x,y) - d(x,t) - d(y,t) &\leq max(d(x,z) - d(x,t) - d(z,t), d(y,z) - d(y,t) - d(z,t)) + 2\delta \\ -2(x.y)_t &\leq max(-2(x.z)_t, -2(y.z)_t) + 2\delta \\ (x.y)_t &\geq min((x.z)_t, (y.z)_t) - \delta\end{aligned}$$

$(\Rightarrow)$

Since X is δ-hyperbolic,

$$
\begin{aligned}
(x.y)_t &\geq min((x.z)_t, (y.z)_t) - \delta \\
d(x,t) + d(y,t) - d(x,y) &\geq min(d(x,t) + d(z,t) - d(x,z), d(y,t) + d(z,t) - d(y,z)) - 2\delta \\
-d(x,y) - d(z,t) &\geq min(-d(x,z) - d(y,t), -d(y,z) - d(x,t)) - 2\delta \\
d(x,y) + d(z,t) &\leq max(d(x,z) + d(y,t), d(y,z) + d(x,t)) + 2\delta
\end{aligned}
$$

□



The geometry behind this equality becomes apparent if we think of w, x, y, z as the vertices of a tetrahedron; $d(x,y) + d(z,w), d(x,z) + d(y,w)$ and $d(x,w) + d(y,z)$ correspond to the sums of the lengths of the three opposite pairs of edges. With this picture in mind, we call these three sums the pair sizes of $[w, x, y, z]$. The inequality 4.6 states that if we list the pair sizes in increasing order, say $S < M < L$, then $L - M < 2\delta$.
Suppose $S = d(x,z) + d(y,w), M = d(x,y) + d(z,w)$ and $L = d(x,w) + d(y,z)$. In terms of comparison triangles, the inequality $S \leq M$ means that by choosing adjoining comparison triangles $\overline{\triangle}(x,w,y)$ and $\overline{\triangle}(x,w,z)$ in $\mathbb{E}^2$, we obtain the configuration shown in the figure below, with $l \geq 0$. For convenience, we have omitted the overbars in labelling the comparison points. We shall examine the inequality $L - M < 2\delta$ in terms of this comparison figure.

Chapter 5

First Examples of Hyperbolic Spaces

In the last chapter, we introduced the notion of δ-hyperbolicity. In this chapter, we look at some examples of hyperbolic spaces. Towards the end, we look at δ-ultrametricity.

5.1 Examples of Hyperbolic Spaces

5.1.1 Bounded Metric Spaces

Proposition 5.1.1. *Any bounded metric space X is δ-hyperbolic for $\delta = diam(X)$.*

Proof. We know that $(x.y) \leq D = diam(X) \ \ \forall x, y \in X$. Hence, $min((x.z), (y.z)) \leq D$ for $z \in X$. Hence the hyperbolicity inequality holds with $\delta = D$.

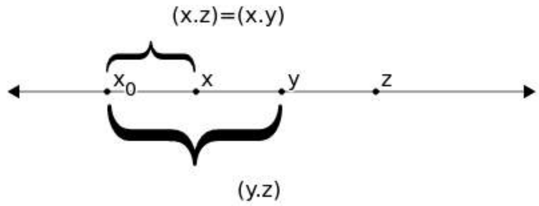
$$(x.y) \geq min((x.z), (y.z)) - D$$

□

5.1.2 The Real Line

The real line is 0-hyperbolic. Indeed, we notice that in $\mathbb{R}, (x.y)$ is the distance of the point x_0 from the line segment $[x, y]$.

Proposition 5.1.2. *Real line is 0-hyperbolic*



Proof. In $\mathbb{R}$, we have that

$$(x.y) = \frac{1}{2}(d(x, x_0) + d(y, x_0) - d(x, y)) = d(x, x_0)$$

Now similarly, we get that $(y.z) = d(y, x_0)$ and $(x.z) = d(x, x_0)$. Hence we see that

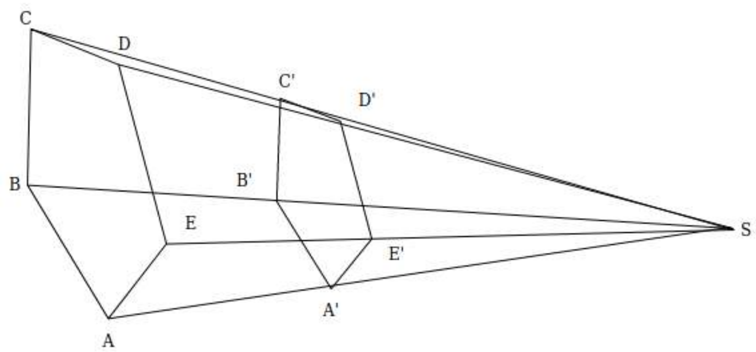
$$(x.y) = d(x, x_0) = min((x.z), (y.z))$$

Hence $\mathbb{R}$ is 0-hyperbolic. □

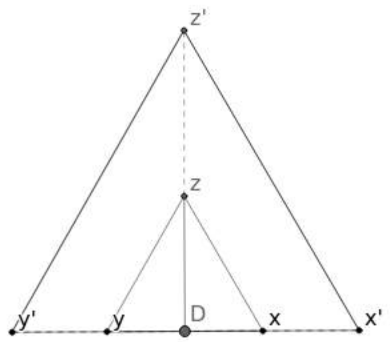
Definition 5.1.3. A homothety (or homothecy, or homogeneous dilation) is a transformation of an affine space determined by a point S called its centre, and a nonzero number λ called its ratio, which sends

$$M \to S + \lambda\overrightarrow{SM}.$$

In other words, it fixes S and sends each M to another point N such that the segment SN is on the same line as SM but scaled by a factor λ.



Proposition 5.1.4. $\mathbb{R}^n$ *is not hyperbolic.*



Proof. Now we have

$$(x.y) = \frac{1}{2}(d(x, x_0) + d(y, x_0) - d(x, y)) = 0$$
$$(x.z) = \frac{1}{2}(d(x, x_0) + d(z, x_0) - d(x, z)) \neq 0$$
$$(y.z) = \frac{1}{2}(d(y, x_0) + d(z, x_0) - d(y, z)) \neq 0$$

Hence $min((x.z), (y.z)) = r > 0$. Now for $\mathbb{R}^n$ to be hyperbolic, it should be δ-hyperbolic for some δ. But observe that if we consider a homothetic transformation based at x_0 and ratio λ, we get that $0 \geq \lambda r - \delta$, which is not true when λ is very large. □

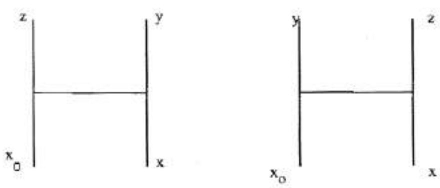
5.1.3 Real Trees

Proposition 5.1.5. *All real trees are* 0*-hyperbolic.*

Proof. We first observe that the product $(x.y)$ is equal to $d(x_0, [x, y])$.

$$(x.y) = \frac{1}{2}(d(x, x_0) + d(y, x_0) - d(x, y)) = d(x_0, [x, y])$$

Now to complete the proof of the proposition, observe in that the subtree of T



form the meeting point of the geodesics connecting two points of $\{x_0, x, y, z\}$. Observe that in both the cases,

$$(x.y) \geq min((x.z), (y.z))$$

Hence real trees are 0-hyperbolic. □

5.2 δ-ultrametric Spaces

A metric space X is said to satisfy δ-ultrametric inequality if

$$d(x, y) \leq max(d(x, z), d(y, z)) + \delta \text{ for all } x, y, z \in X.$$

Proposition 5.2.1. *All metric spaces which satisfy δ-ultrametric inequality are δ-hyperbolic.*

Proof. δ-ultrametric inequality gives

$$d(x, y) \leq max(d(x, z), d(y, z)) + \delta \quad (5.1)$$

$$d(x, y) \leq max(d(x, t), d(y, t)) + \delta \quad (5.2)$$

$$d(z, t) \leq max(d(z, x), d(t, x)) + \delta \quad (5.3)$$

$$d(z, t) \leq max(d(z, y), d(t, y)) + \delta \quad (5.4)$$

Now let us assume that $d(y, t)$ is the minimum. Hence from equations(2.2) and (2.4) we get that

$$d(x, y) + d(z, t) \leq d(x, t) + d(z, y) + 2\delta$$

Now let us assume that $d(x, t)$ is the minimum. Hence from equations(2.2) and (2.3), we get that

$$d(x, y) + d(z, t) \leq d(y, t) + d(z, x) + 2\delta$$

Hence,

$$d(x, y) + d(z, t) \leq max(d(x, t) + d(z, y), d(y, t) + d(z, x)) + 2\delta.$$

This gives the δ-hyperbolicity inequality (refer to Proposition 4.4.1) □

Example 5.2.2.

1) If K is a field provided with a valuation v taking real values(for example, the field of p-adic numbers $\mathbb{Q}_p$ provided with p-adic valuation). Then $d(x, y) = e^{-v(x-y)}$ defines a metric on K satisfies the 0-ultrametric inequality. Hence (K, d) is 0-hyperbolic(refer Proposition 0.6.4).

2) If (X, d) is a metric space with some metric. We obtain a new metric on X given by

$$d'(x, y) = log(1 + d(x, y)).$$

The above metric satisfies the δ-ultrametric inequality with $\delta = log2$. Indeed, we have:

$$d(x, y) \leq d(x, z) + d(y, z) \leq 2max(d(x, z), d(y, z)),$$

which gives

$$d'(x, y) \leq max(d'(x, z), d'(y, z)) + log2.$$

Thus we deduce that (X, d') is δ-hyperbolic with $\delta = log2$.

Remark. The real line is hyperbolic but does not satisfy the ultrametric inequality.

Chapter 6

Hyperbolicity and Geodesic Triangles

6.1 Geodesic Spaces

Definition 6.1.1. A metric space X is said to be a geodesic space if two points in the space can be connected by a geodesic segment.

A metric space (X, d) is termed a length-metric space if the distance between any two points in it equals the infimum of the lengths of all the paths joining them.

$$d_i(x, y) := \inf L(\sigma)$$

where the infimum is taken over all rectifiable curves $\sigma : [0, 1] \to X$ from x to y, i.e., $\sigma(0) = x$, $\sigma(1) = y$. Here, the length of a path is defined as the supremum, over all partitions of the unit interval, of the sums of distances between the images of endpoints of each part.

$$L(\sigma) = \sup_{a=t_0<....<t_n=b} \sum_{i=0}^{n-1} d(\sigma(t_i), \sigma(t_{i+1}))$$

A geodesic metric space is a metric space if given any two points, there exists a path between them whose length equals the distance between the points. Hence all geodesic spaces are length metric spaces.

Example.

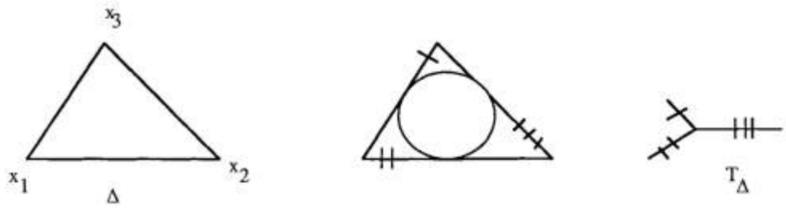
- $\mathbb{R}^2$ without the origin is a length space which is not geodesic.
- All real trees are geodesic.

- According to the Hopf-Rinow theorem, all complete Riemannian manifolds are geodesic.
- All complete locally compact length spaces are geodesic(Hopf-Rinow theorem).

In a geodesic space, we denote a geodesic segment between the points x and y by $[x, y]$. (Note that $[x, y]$ is not necessarily unique).

6.2 Geodesic Triangles

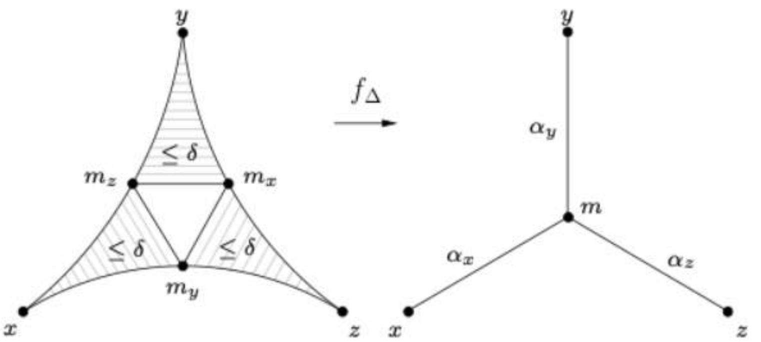
In a metric space, a geodesic triangle is the meeting of three geodesic segments $[x, y], [y, z]$ and $[z, x]$; we denote such a triangle as $[x, y, z]$. The geodesic segments form the sides of the geodesic triangle and the extremities of these segments form the vertices.
Given a geodesic triangle $[x_1, x_2, x_3]$, we associate with it a tripod denoted as T_Δ. The tripod T_Δ is the metric space obtained from the edges of the Euclidean comparison triangle $\Delta' = [x', y', z']$ by identifying the line segments emanating from a vertex of the triangle and touching the inscribed circle at a single point and doing this identification for all the sides. There is an important application for $f_\Delta : \Delta \to T_\Delta$, which we will see later. Note here that the restriction of f_Δ on each side is an isometry.



6.3 Hyperbolicity and Thinness of geodesic triangles

Definition 6.3.1. A geodesic triangle Δ is said to be δ-thin if two points of Δ which are on the same image by f_Δ are always at a distance $\leq \delta$ from one another.

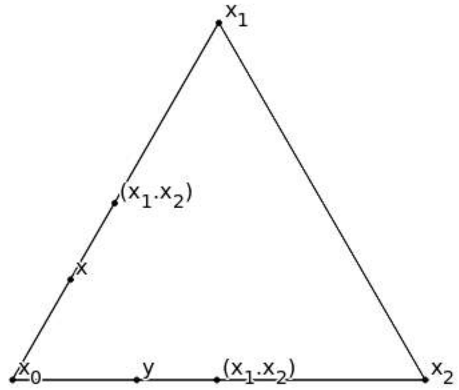
Proposition 6.3.2. *Let X be a geodesic metric space. Then we have the following implications:*



1) If X is δ-hyperbolic, then all geodesic triangles of X are 4δ-thin.

2) If all the geodesic triangles of X are δ-thin, then X is δ-hyperbolic.

Proof.



1) Let $\Delta = [x_0, x_1, x_2]$ be the geodesic triangle and let $x \in [x_0, x_1], y \in [x_0, x_2]$ having the same image by f_Δ. Hence let $t = d(x, x_0) = d(y, x_0) \leq (x_1.x_2)$. With base point x_0, the δ-hyperbolicity condition is:

$$(x.y) \geq min((x.x_2), (x_2.y)) - \delta$$

Applying the hyperbolicity condition again on $(x.x_2)$, we get

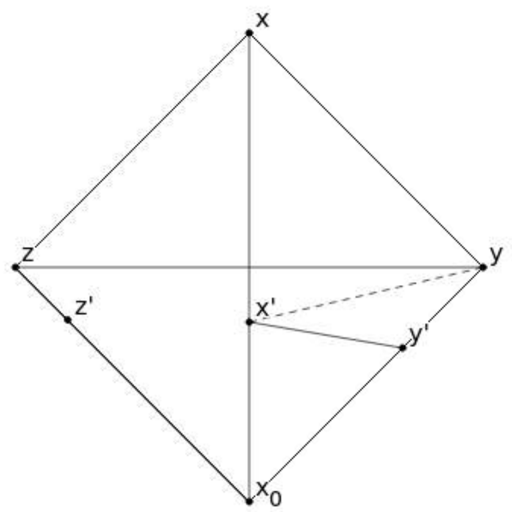
$$(x.y) \geq min((x.x_1), (x_1.x_2), (x_2.y)) - 2\delta$$

Now $(x.x_1) = \frac{1}{2}(d(x, x_0) + d(x_1, x_0) - d(x, x_1)) = t \leq (x_1.x_2)$, since these points lie on the same geodesic segment. Similarly $(y.x_2) = t \leq (x_1.x_2)$.

Hence

$$(x.y) = \frac{1}{2}(d(x, x_0) + d(y, x_0) - d(x, y)) = t - \frac{1}{2}(d(x, y)) \geq t - 2\delta,$$

hence $d(x, y) \leq 4\delta$, which shows that the geodesic triangles of X are 4δ-thin.



2) Now for the second implication. Let x_0 be the base point and $x, y, z \in X$. Consider the geodesic triangles $[x_0, x, y]$, $[x_0, x, z]$ and $[x_0, y, z]$ and let x', y' and z' be points respectively in the segments $[x_0, x]$, $[x_0, y]$ and $[x_0, z]$ such that

$$d(x', x_0) = d(y', x_0) = d(z', x_0) = min((x.z), (y.z))$$

(Notice that $min((x.z), (y.z)) \leq min(d(x, x_0), d(y, x_0), d(z, x_0))$.) Since the triangles $[x_0, x, z]$ and $[x_0, y, z]$ are δ-thin, we have that $d(x', z') \leq \delta$ and $d(y', z') \leq \delta$ which gives according to the triangle inequality

$$d(x', y') \leq d(x', z') + d(z', y') \leq 2\delta \tag{6.1}$$

Now applying the triangle inequality multiple times, we get that

$$d(x, y) \leq d(x, x') + d(x', y') + d(y, y'),$$

$$\text{then } d(x, y) \leq d(x, x_0) + d(y, x_0) - 2min((x.z), (y.z)) + d(x', y')$$

which gives on using 6.1

$$(x.y) \geq min((x.z),(y.z)) - \delta$$

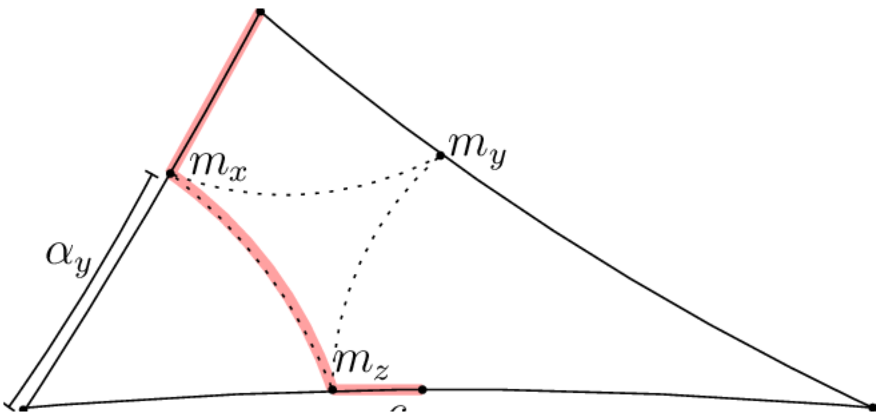
□

6.4 Hyperbolicity and Internal Size of Geodesic Triangles

Definition 6.4.1. We define internal point of a geodesic triangle Δ as any point of Δ which under f_Δ is sent to the central point of the tripod T_Δ. We define internal size of Δ, denoted as $insize(\Delta)$, as the diameter of the set of internal points of Δ,

$$insize(\Delta) = \max d(c_i, c_j)$$

where $c_i (i = 1, 2, 3)$ are the internal points of the geodesic triangle Δ.



Proposition 6.4.2. *Let X be a geodesic metric space. Then we have the following implications:*

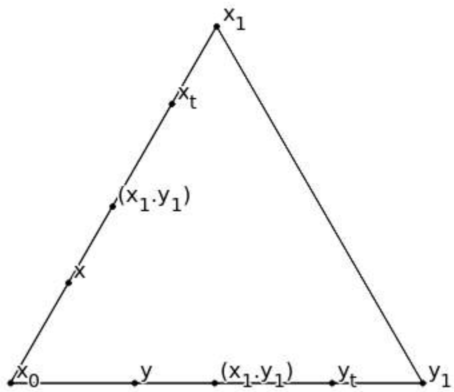
1) If X is δ-hyperbolic, then all the geodesic triangles have an internal size $\leq 4\delta$.

2) If all geodesic triangles in X have an internal size $\leq \delta$, then X is δ-hyperbolic.

Proof.

1) We have seen Proposition 6.3.2 which says that if X is δ-hyperbolic, then all the geodesic triangles of X are 4δ-thin. Now it is clear that a 4δ-thin geodesic triangle has an internal size of 4δ.

2) We hypothesize here that all the geodesic triangles Δ of X satisfy $insize(\Delta) \leq \delta$. We go to show that any geodesic triangle is δ-thin what will finish the proof by the Proposition 6.3.2.



Let $[x_0, x_1, y_1]$ be a geodesic triangle, $x \in [x_0, x_1]$ and $y \in [x_0, y_1]$ such that $f_\Delta(x) = f_\Delta(y)$. Now to show that the triangle is δ-thin, we have to show that $d(x, y) < \delta$.

We have in particular that $d(x, x_0) = d(y, x_0) \leq (x_1.y_1)$ (with x_0 as the base point). Let, for $t \in [0, 1], x_t \in [x_0, x_1]$and $y_t \in [x_0, y_1]$ such that $d(x_t, x_0) = td(x_1, x_0)$ and $d(y_t, x_0) = td(y_1, x_0)$. Now $(x_0.y_0) = 0$ and $(x_1.y_1) \neq 0$. According to the intermediate value theorem, there exists $\alpha \in [0, 1]$ such that $(x_\alpha.y_\alpha) = d(x, x_0) = d(y, x_0)$. The geodesic triangle $\Delta' = [x_0, x_\alpha, y_\alpha]$ satisfies the hypothesis $insize(\Delta) \leq \delta$, and hence we have $d(x, y) \leq \delta$.

□

6.5 Hyperbolicity and Minimum Size of Geodesic Triangles.

Definition 6.5.1. Given a geodesic triangle $\Delta = [x_1, x_2, x_3]$, we define the minimum size of the geodesic triangle Δ, denoted by $minsize(\Delta)$, as the infimum of the diameters of the set of three points situated on each side of the triangle:

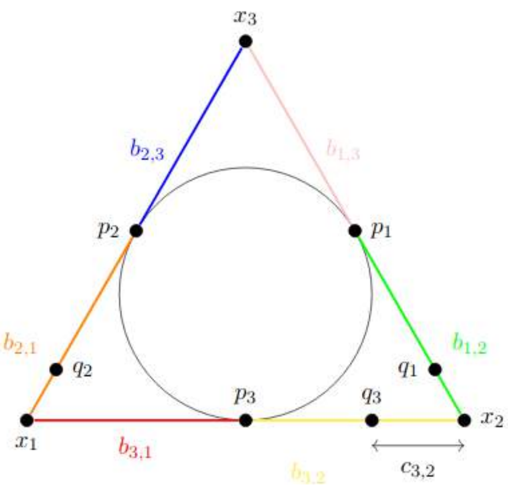
$$minsize(\Delta) = \inf \max_{i,j} d(y_i, y_j)$$

for $y_1 \in [x_2, x_3], y_2 \in [x_3, x_1]$ and $y_3 \in [x_1, x_2]$(This limit is achieved by compactness).

Lemma 6.5.2. *For any geodesic triangle Δ, we have:*

$$minsize(\Delta) \leq insize(\Delta) \leq 4minsize(\Delta).$$

Proof.



The left hand side of the inequality follows trivially. Let $q_1 \in [x_2, x_3], q_2 \in [x_1, x_3]$ and $q_3 \in [x_1, x_2]$ such that $\delta = diam\{q_1, q_2, q_3\}$. Let p_1, p_2 and p_3 be the points of the inscribed circle that meet the geodesic triangle. Let $a_1 = d(x_2, x_3), a_2 = d(x_1, x_3), a_3 = d(x_1, x_2), b_{i,j}$ as drawn on the picture, and $c_{i,j} = d(q_i, x_j)$ for all $i, j \in \{1, 2, 3\}$. Then, by the properties ofthe inscribed circle,

$$b_{i,k} = b_{j,k} = \frac{1}{2}(a_i + a_j - a_k) \text{ and } b_{i,j} + b_{i,k} = c_{i,j} + c_{i,k} = a_i.$$

Since $d(q_i, q_j) \leq \delta$,

$$d(q_i, x_k) - d(q_j, x_k) \leq d(q_i, q_j) \leq \delta \text{ and } d(q_j, x_k) - d(q_i, x_k) \leq \delta.$$

That is, $|c_{i,k} - c_{j,k}| \leq \delta$. As a consequence,

$$\begin{aligned} 2b_{i,k} = a_i + a_j - a_k = c_{i,j} + c_{i,k} + b_{j,i} + b_{j,k} - c_{k,i} - c_{k,j}, \\ c_{i,j} + c_{i,k} + b_{k,i} - b_{i,k} - c_{k,i} - c_{k,j} = 0, \\ |c_{i,k} - b_{i,k} - c_{k,i} + b_{k,i}| = |c_{i,j} - c_{k,j}| \leq \delta. \end{aligned}$$

If we denote $c_{1,2} - b_{1,2}$ by d_1, we have that

$$d_1 = c_{1,2} - b_{1,2} = d(q_1, x_2) - d(p_1, x_2) = d(p_1, x_3) - d(q_1, x_3) = -(c_{1,3} - b_{1,3})$$

In the same way,

$$d_2 = c_{2,3} - b_{2,3} = -(c_{2,1} - b_{2,1}) \text{ and } d_3 = c_{3,1} - c_{3,1} = -(c_{3,2} - b_{3,2}).$$

Hence,

$$|d_1| = d(p_1, q_1), |d_2| = d(p_2, q_2) \text{ and } |d_3| = d(p_3, q_3).$$

In addition, $|d_1 + d_2| = |c_{1,2} - b_{1,2} - c_{2,1} + b_{2,1}| \leq \delta$, and in the same fashion, $|d_i + d_j| \leq \delta$, for $i, j \in \{1, 2, 3\}$.
As a result,

$$|d_i| = \frac{1}{2}|d_i + d_j + d_i + d_k - d_j - d_k| \leq \frac{3}{2}\delta.$$

In conclusion, $d(p_j, p_k) \leq d(p_j, q_j) + d(q_j, q_k) + d(q_k, p_k) \leq \frac{3}{2}\delta + \delta + \frac{3}{2}\delta = 4\delta.$ □

From this lemma and the preceding proposition, we get:

Proposition 6.5.3. *Let X be a geodesic metric space. Then we have the following implications:*

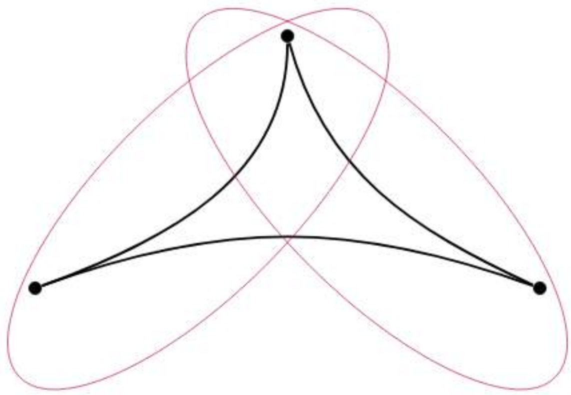
1) If X is δ-hyperbolic, then all the geodesic triangles have a minsize $\leq 4\delta$.

2) If all the geodesic triangles of X have a minsize $\leq \delta$, then X is 4δ-hyperbolic.

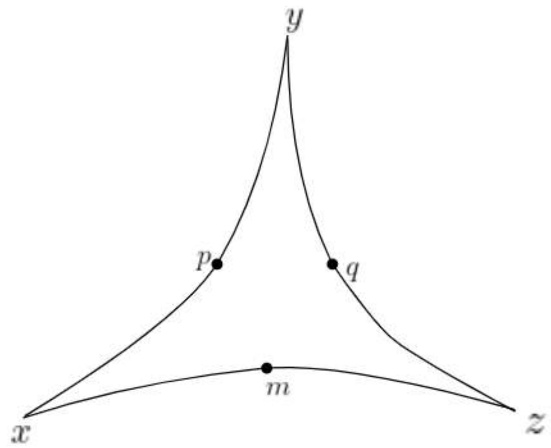
Proof. The proof follows easily from the Lemma 6.5.2 above and the Proposition 6.4.2 above. □

6.6 Hyperbolicity and Slimness of Geodesic Triangles.

Definition 6.6.1. We say that a geodesic triangle is δ-slim if one of its sides is contained in δ-neighbourhood of the meeting of the other two (Notice that when $Y \subset X$, the δ-neighbourhood of Y is the set of points of X which are at a distance $\leq \delta$ of Y).



Lemma 6.6.2. *In a metric space, any δ-slim triangle is of minsize $\leq 2\delta$.*



Proof. Let $[x, y, z]$ be a geodesic triangle such that $[x, z]$ is contained in the $\delta-$ neighbourhood of $[x, y] \cup [y, z]$. The function $f(m) = dist(m, [x, y]) - dist(m, [y, z])$ for all $m \in [x, z]$ satisfies $f(x) \leq 0$ and $f(z) \geq 0$. From the theorem on intermediate values, there exists a point m of $[x, z]$ such that $dist(m, [x, y]) = dist(m, [y, z]) \leq \delta$. Hence the existence of points $p \in [x, y]$ and $q \in [y, z]$ whose distance from m is $\leq \delta$. The diameter of $[m, p, q]$ is $\leq 2\delta$, which shows the lemma. ☐

Proposition 6.6.3. *Let X be a geodesic space. Then we have the following implications:*

1) If X is δ-hyperbolic, then all the geodesic triangles of X are 4δ-slim.

2) If all the geodesic triangles of X are δ-slim, then X is 8δ-hyperbolic.

Proof. For the first implication, it suffices to use the Proposition 6.3.2 and the remark that any side of the δ-thin triangle is contained in the δ-neighbourhood of the meeting of the other two sides of the triangle.

For the second implication, we use the Lemma 6.6.2 and the Proposition 6.5.3. □

Chapter 7

Boundary of Hyperbolic Spaces

7.1 Sequences converging at infinity

Definition 7.1.1. Let X be a metric space with base point x_0. We say that the sequence (a_i) of points of X converges at infinity if

$$\lim(a_i.a_j)_{x_0} = \infty \text{ when } i \text{ and } j \to \infty.$$

Lemma 7.1.2. *This definition does not depend on the choice of the base point x_0.*

Proof. Since for all the pairs of points $x, y \in X$, we have

$$|(x.y)_{x_1} - (x.y)_{x_0}| \leq d(x_0, x_1)$$

Now using the triangle inequality, we see that

$$|(x.y)_{x_1}| \geq |(x.y)_{x_0}| - d(x_0, x_1)$$

Hence the proof follows. □

Note here that if the sequence (a_i) converges at infinity, then $|a_i| \to \infty$ when $i \to \infty$, since $(a_i.a_i) = |a_i|$ (remember that $|a| = d(a, x_0)$.)

We denote by $S_\infty(X)$, the set of all sequences of points in X which, converge at infinity.

7.2 Construction of boundary of hyperbolic spaces

We define a relation $\mathcal{R}$ in $S_\infty(X)$ as

$$(a_i)\mathcal{R}(b_i) \iff \lim(a_i.b_i) = \infty \text{ when } i \to \infty. \tag{7.1}$$

This relation also does not depend on the choice of the base point as before.

Proposition 7.2.1. *The relation $\mathcal{R}$ is an equivalence relation for a hyperbolic space.*

Proof. Let X be a δ-hyperbolic space.

Reflexive: This follows from the fact that the relation is defined on $S_\infty(X)$. Hence for $(a_i) \in S_\infty(X)$, we have

$$\lim(a_i.a_j)_{x_0} = \infty \text{ when } i \text{ and } j \to \infty.$$

In particular,

$$\lim(a_i.a_i)_{x_0} = \infty \text{ when } i \to \infty.$$

Hence $(a_i)\mathcal{R}(a_i)$

Symmetric: We know that the Gromov product is symmetric. In particular,

$$(a_i.b_j)_{x_0} = (b_j.a_i)_{x_0}$$

Hence the relation is transitive as well.

Transitivity: We have $(a_i)\mathcal{R}(b_i)$ and $(b_i)\mathcal{R}(c_i)$. Hence $\lim(a_i.b_i) = \infty$ and $\lim(b_i.c_i) = \infty$ as $i \to \infty$. Now, since X is δ-hyperbolic, we have

$$(a_i.c_i) \geq min((a_i.b_i), (b_i.c_i)) - \delta.$$

In particular, we have

$$\lim(a_i.c_i) = \infty \text{ as } i \to \infty.$$

Hence $(a_i)\mathcal{R}(c_i)$.

□

Lemma 7.2.2. *Note that the above proposition also helps to show that if $(a_i)\mathcal{R}(b_i)$ then $(a_i.b_j) \to \infty$ when i and $j \to \infty$.*

Proof. From the hyperbolicity inequality, we get,

$$(a_i.b_j) \geq min((a_i.b_i),(b_i.b_j)) - \delta \tag{7.2}$$

Since $(b_i) \in S_\infty X$ and $(a_i)\mathcal{R}(b_i)$, we get that $(a_i.b_j) \to \infty$ as $i \to \infty$. Hence proved. □

Definition 7.2.3. If X is a hyperbolic metric space, we call the boundary(hyperbolic) of X and denote by ∂X the quotient of $S_\infty(X)$ by the equivalence relation $\mathcal{R}$.

We say that the sequence (a_i) of the points of X converges to the point $x \in \partial X$ if (a_i) converges to infinity and x is the equivalence class of (a_i).

Example 7.2.4.

1) If X is bounded, we have $S_\infty(X) = \partial X = \phi$.

2) If $X \in \mathbb{R}$, we have $\partial X = \{-\infty, \infty\}$.

3) Let X be any metric space. We have seen (5.2) that the new metric $|\ |' \to log(1 + |\ |)$ is a δ-hyperbolic metric on X, with $\delta = log2$. If X is not bounded, the hyperbolic boundary of X for $|\ |'$ is reduced to a point(since metrics are positive.), and all sequences (a_i) with $a_i \in X$ and $|a_i| \to \infty$ converge to this unique point on the boundary. In effect, we have

$$(x.y) = \frac{1}{2}(\log(1 + |x|) + \log(1 + |y|) - \log(1 + |x - y|)) \tag{7.3}$$

$$= \frac{1}{2}\log(\frac{(1 + |x|)(1 + |y|)}{1 + |x - y|}) \tag{7.4}$$

$$\tag{7.5}$$

Now using the triangle inequality, we get

$$(x.y) \geq \frac{1}{2}\log(\frac{|x||y|}{1 + |x| + |y|})$$

Lemma 7.2.5. *If X and Y are two hyperbolic spaces, and $f : X \to Y$ is an isometry, then the mapping from $S_\infty(X)$ to $S_\infty(Y)$, which associates to $(a_i), (f(a_i))$ goes to the quotient to give an injective mapping $\partial f : \partial X \to \partial Y$.*

Proof. Since f is an isometry, observe that

$$(f(a_i).f(b_i)) = ((a_i).(b_i))$$

Hence to check injectivity of ∂f let

$$\begin{aligned} [(f(a_i))] &= [(f(b_i))] \\ \implies \lim(f(a_i).f(b_i)) &= \infty \\ \implies \lim(a_i.b_i) &= \infty \\ \implies [(a_i)] &= [(b_i)] \end{aligned}$$

Hence ∂f is injective. Hence proved. □

Definition 7.2.6. Let (X_1, d_1) and (X_2, d_2) be metric spaces. A (not necessarily continuous) map $f : X_1 \to X_2$ is called a (λ, ϵ)-quasi-isometric embedding if there exist constants $\lambda \geq 1$ and $\epsilon \geq 0$ such that for all $x, y \in X_1$

$$\frac{1}{\lambda} d_1(x, y) - \epsilon \leq d_2(f(x), f(y)) \leq \lambda d_1(x, y) + \epsilon$$

If, in addition, there exists a constant $C \geq 0$ such that every point of X_2 lies in the C-neighbourhood of the image of f, then f is called a (λ, ϵ)-quasi-isometry. When such a map exists, X_1 and X_2 are said to be quasi-isometric.

Example 7.2.7.

(1) For $v, b \in \mathbb{R}^2$, the map $t \to tv + b$ from $\mathbb{R}$ to $\mathbb{R}^2$ is a quasi-isometric embedding.

(2) The natural inclusion $\mathbb{Z} \hookrightarrow \mathbb{R}$ is a quasi-isometry.

Proposition 7.2.8. *If $f : X \to Y$ is a quasi-isometry,*

$$|(f(x).f(y)) - (x.y)| \leq \frac{3}{2}\epsilon + \frac{1}{2}(\lambda - 1)(x.y)$$

by taking $x_0 \in X$ and $f(x_0) \in Y$ as base points.

Proof.

$$\begin{aligned} |(f(x).f(y)) - (x.y)| &= \frac{1}{2}(|d(f(x), f(x_0)) + d(f(y), f(x_0)) - d(f(x), f(y)) \\ &\quad - d(x, x_0) - d(y, x_0) + d(x, y)|) \qquad (7.6) \\ &\leq \frac{1}{2}(|d(f(x), f(x_0)) - d(x, x_0)| + |d(f(y), f(x_0)) \\ &\quad - d(y, x_0)| + |d(f(x), f(y)) - d(x, y)|) \qquad (7.7) \\ &\leq \frac{3}{2}\epsilon + \frac{1}{2}(\lambda - 1)(x.y) \end{aligned}$$

□

Proposition 7.2.9. *Let X be a hyperbolic space, (a_i) and (b_i) be two sequences of points in X. We suppose that $(a_i) \in S_\infty(X)$ and that $(a_i.b_i) \to \infty$ when $i \to \infty$. Then $(b_i) \in S_\infty(X)$. (Note that, by definition, the two sequences converge to the same point of ∂X)*

Proof. We have, by the hyperbolicity inequality,

$$(a_i.b_j) \geq min((a_i.a_j),(a_j.b_j) - \delta$$

Since $(a_j.b_j) \to \infty$ as $j \to \infty$ and $(a_i) \in S_\infty X, (a_i.b_j) \to \infty$ as well. Now, again by hyperbolicity inequality,

$$(b_i.b_j) \geq min((b_i.a_j),(a_j.b_j)) - \delta$$

which shows that $(b_i.b_j) \to \infty$ when $i, j \to \infty$. □

Proposition 7.2.10. *If X and Y are two hyperbolic spaces, and $f : X \to Y$ is a quasi-isometry, then the mapping from $S_\infty(X)$ to $S_\infty(Y)$, which associates to $(a_i),(f(a_i))$ goes to the quotient to give a bijective mapping $\partial f : \partial X \to \partial Y$.*

Proof. Since f is a quasi-isometry, observe that

$$\begin{aligned} |(f(x).f(y)) - (x.y)| &\leq \frac{3}{2}\epsilon + \frac{1}{2}(\lambda - 1)(x.y) \\ \implies (f(x).f(y)) &\leq \frac{3}{2}\epsilon + \frac{1}{2}(\lambda + 1)(x.y) \end{aligned}$$

Hence to check injectivity of ∂f let

$$\begin{aligned} [(f(a_i))] &= [(f(b_i))] \\ \implies \lim(f(a_i).f(b_i)) &= \infty \\ \implies \lim(a_i.b_i) &= \infty \\ \implies [(a_i)] &= [(b_i)] \end{aligned}$$

Hence ∂f is injective.
Now we show that it is surjective. Let $(y_i) \in Y$ be a sequence of points such that $[(y_i)] \in S_\infty X$. Now by Definition 7.2.6, we can choose for each i, $x_i \in X$ such that

$$d_Y(f(x_i), y_i) < C$$

Now we show that $[(x_i)]$ is the required pre-image i.e.,

$$[(f(x_i))] = [(y_i)]$$

Hence it suffices to show that

$$\lim(f(x_i).y_i) = \infty \text{ as } i \to \infty$$

Now, using the definition of Gromov product,

$$(f(x_i).y_i) = \frac{1}{2}(d(f(x_i), y_0) + d(y_i, y_0) - d(f(x_i), y_i))$$

Now since $|y_i| \to \infty$ as $i \to \infty$ and $d_Y(f(x_i), y_i) < C$, we see that $\lim(f(x_i).y_i) = \infty$. Now using Proposition 7.2.9, we see that $[(f(x_i))] \in \partial Y$ as well. Hence we are done. □

Corollary 7.2.11. Let X be a hyperbolic space, (a_i) and (b_i) be two sequences of points in X such that $d(a_i, b_i) \leq C($ where $C \geq 0$ is a constant). We suppose that the sequence (a_i) belongs to $S_\infty(X)$. Then the sequence (b_i) also belongs to $S_\infty(X)$, and the two sequences (a_i) and (b_i) converges to the same point in ∂X.

Proof. We have

$$\begin{aligned} (a_i.b_i) &= \frac{1}{2}(d(a_i, x_0) + d(b_i, x_0) - d(a_i, b_i)) & (7.8)\\ &= \frac{1}{2}((a_i.a_i) + (b_i.b_i) - d(a_i, b_i)) & (7.9)\\ &\geq \frac{1}{2}((a_i.a_i) + (b_i.b_i) - C). & (7.10) \end{aligned}$$

Now since $(a_i.a_i) \in S_\infty X$, we see that $(a_i.b_i) \to \infty$ as $i \to \infty$. Now it suffices to utilize the preceding proposition. □

7.3 Extension of $(x.y)$

Here we are defining the Gromov product for sequences.

Definition 7.3.1. Let $a = (a_i)$ and $b = (b_i)$ be two sequences of points of X. We pose:

$$(a.b) = \liminf(a_i.b_i), \text{ when } i \to \infty. \quad (7.11)$$

Let x and y be two points of $X \cup \partial X$. We pose:

$$(x.y) = \inf(a.b), \tag{7.12}$$

with $a = (a_i)$ converging to x, and $b = (b_i)$ converging to y.

(Remember that the formulae are correct if the points x and y are in X).

Proposition 7.3.2. *We have then $(x.y) = \infty$ if and only if $x = y \in \partial X$*

Proof.
If $x = y \in \partial X$, then by definition of sequences converging at infinity and sequential boundary, we see that $(x.y) = \infty$.
Now the other way. Let $(x.y) = \inf \lim \inf(a_i.b_i) = \infty$. Hence $\lim(a_i.b_i) = \infty$ as $i \to \infty$. Thus $x = y \in \partial X$. □

We then have always that $(x.y) \geq min((x.z), (y.z)) - \delta$, for all $x, y, z \in X \cup \partial X$.

Chapter 8

The Boundary ∂X as a Set of Rays

8.1 Quasi-Geodesics

Definition 8.1.1. A (λ, ϵ)-quasi-geodesic in a metric space X is a (λ, ϵ)-quasi-isometric embedding $c : I \to X$, where I is an interval of the real line (bounded or unbounded) or else the intersection of $\mathbb{Z}$ with such an interval. More explicitly,

$$\frac{1}{\lambda} d(t, t') - \epsilon \leq d(c(t), c(t')) \leq \lambda d(t, t') + \epsilon$$

for all $t, t' \in I$. If $I = [a, b]$, then $c(a)$ and $c(b)$ are called the endpoints of c. If $I = [0, \infty)$, then c is called a quasi-geodesic ray.

Definition 8.1.2. Two geodesic rays $c, c' : [0, \infty) \to X$ in a metric space X are said to be asymptotic if $\sup_t d(c(t), c'(t))$ is finite; this condition is equivalent to saying that the Hausdorff distance between the images of c and c' is finite.

Lemma 8.1.3. *The relation defined above is an equivalence relation on the set of all geodesic rays.*

Proof.

Reflexive: Since $\forall\ t \in [0, \infty),\ \ d(c(t), c(t)) = 0$, we see that the relation is reflexive.

Symmetric: This follows from the symmetric nature of the metric.

Transitive: Let a, b and b, c be asymptotic geodesics. Hence we have

$$\sup_t d(a(t), b(t)) = k < \infty \text{ and } \sup_t d(b(t), c(t)) = l < \infty$$

Now, we have by triangle inequality, for some $t \in [0, \infty)$,

$$d(a(t), c(t)) \leq d(a(t), b(t)) + d(b(t), c(t)) \leq k + l$$

Hence we have that

$$\sup_t d(a(t), c(t)) \leq k + l < \infty$$

□

Definition 8.1.4. We define quasi-geodesic rays to be asymptotic if the Hausdorff distance between their images is finite.

Lemma 8.1.5. *The relation defined above is an equivalence relation on the set of all quasi-geodesic rays.*

Proof.

Reflexive: For a quasi-geodesic $c, d_H(im(c), im(c)) = 0$. Hence the relation is reflexive.

Symmetric: Let a, b be two quasi-geodesics which are asymptotic. Then we have

$$\begin{aligned} d_H(im(a), im(b)) &= \max\left\{ \sup_{x \in im(a)} \inf_{y \in im(b)} d(x, y), \sup_{y \in im(b)} \inf_{x \in im(a)} d(x, y) \right\} \\ &= d_H(im(b), im(a)) < \infty \end{aligned}$$

Transitive: Let a, b and b, c be asymptotic. Hence we have,

$$d_H(im(a), im(b)) = \inf\{\varepsilon \geq 0\,;\ im(a) \subseteq im(b)_\varepsilon \text{ and } im(b) \subseteq im(a)_\varepsilon\} < \infty$$

$$d_H(im(b), im(c)) = \inf\{\Gamma \geq 0\,;\ im(b) \subseteq im(c)_\Gamma \text{ and } im(c) \subseteq im(b)_\Gamma\} < \infty$$

Now,

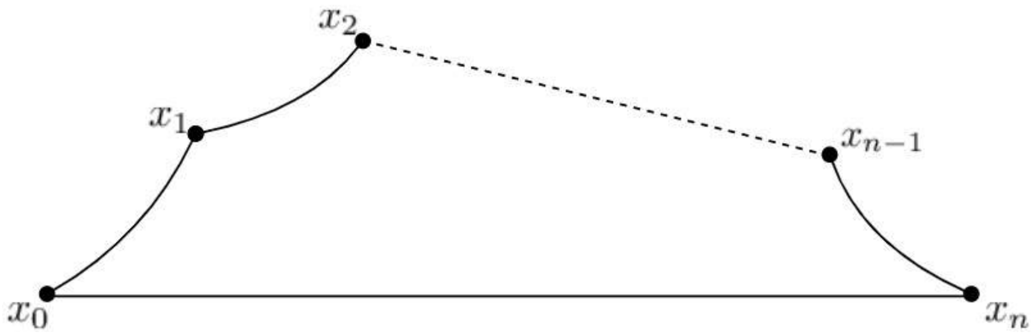
$$\begin{aligned} d_H(im(a), im(c)) &= \inf\{\Delta \geq 0\,;\ im(a) \subseteq im(c)_\Delta \text{ and } im(c) \subseteq im(a)_\Delta\} \\ &\leq d_H(im(a), im(b)) + d_H(im(b), im(c)) \\ &< \infty \end{aligned}$$

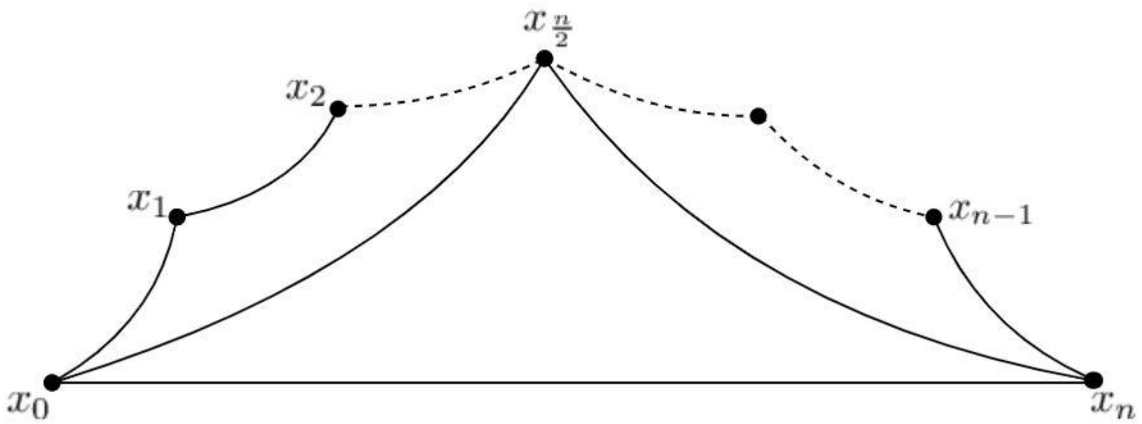
Hence the relation is transitive.

□

8.2 Stability of Quasi-Geodesics

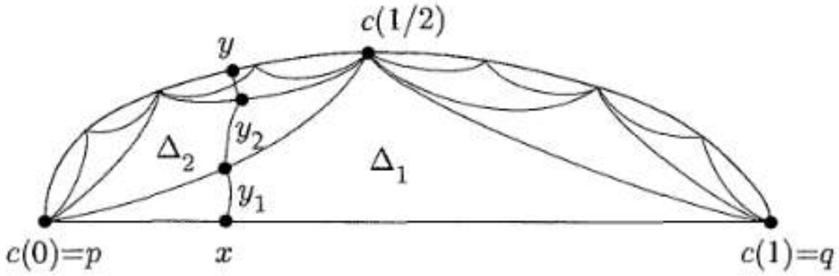
Lemma 8.2.1. *Let X be a $\delta/4$-hyperbolic geodesic metric space, and $Y = [x_0, x_1] \cup [x_1, x_2] \cup ... \cup [x_{n-1}, x_n]$ a chain of n geodesic segments, with $n \leq 2^k$, where k is an integer ≥ 1. Then, for every point x in a geodesic segment $[x_0, x_n]$, we have $dist(x, Y) \leq k\delta$.*





Proof. By subdividing some of the n segments, we come back to the case where $n = 2^k$. We then do induction over k. For $k = 1$, the property follows from the fact that X is δ-thin(If X is δ-hyperbolic, then all geodesic triangles are 4δ-thin). Suppose the lemma is true for $n = 2^k$, and we show it for $n = 2^{k+1}$. We consider two geodesic segment, $[x_0, x_{n/2}]$ and $[x_{n/2}, x_n]$. In the geodesic triangle $\triangle \equiv [x_0, x_n] \cup [x_0, x_{n/2}] \cup [x_{n/2}, x_n]$, the point x in $[x_0, x_n]$ is at a distance $\leq \delta$ from a point m located on $[x_0, x_{n/2}]$ or on $[x_{n/2}, x_n]$. By the hypothesis, m is then situated at a distance $\leq k\delta$ from a point on $[x_0, x_1] \cup ... \cup [x_{n/2-1}, x_{n/2}]$ or on $[x_{n/2}, x_{n/2+1}] \cup ... \cup [x_{n-1}, x_n]$. Hence $dist(x, Y) \leq (k + 1)\delta$, which proves the lemma. □

Proposition 8.2.2. *Let X be a δ-hyperbolic geodesic space. Let c be a continuous rectifiable path in X. If $[p, q]$ is a geodesic segment connecting the endpoints of c,*



then for every $x \in [p, q]$

$$d(x, im(c)) \leq \delta |\log_2 l(c)| + 1.$$

Proof.
If $l(c) \leq 1$, we observe that $d(p, q) \leq 1$. Hence for $x \in [p, q]$, $d(x, im(c)) \leq 1$. Hence $d(x, im(c)) \leq \delta |\log_2 l(c)| + 1$.

Suppose that $l(c) > 1$. Without loss of generality we may assume that $c : [0, 1] \to X$ is a map that parameterizes its image proportional to arc length. Thus $p = c(0)$ and $q = c(1)$. Let N denote the positive integer such that $l(c)/2^{N+1} < 1 \leq l(c)/2^N$ (Equivalently, $2^N \leq l(c) < 2^{N+1}$).

Let $\Delta_1 = \Delta([c(0), c(1/2)], [c(1/2), c(1)], [c(0), c(1)])$ be a geodesic triangle in X containing the given geodesic $[c(0), c(1)]$. Given $x \in [c(0), c(1)]$, we choose $y_1 \in [c(0), c(1/2)] \cup [c(1/2), c(1)]$ with $d(x, y_1) < \delta$(since all triangles in X are δ-slim). If $y_1 \in [c(0), c(1/2)]$ then we consider a geodesic triangle

$$\Delta([c(0), c(1/2)], [c(1/4), c(1/2)], [c(0), c(1/4)]),$$

which has the edge $[c(0), c(1/2)]$ in common with Δ_1 and call this triangle Δ_2. If on the other hand $y_1 \in [c(1/2), c(1)]$, then we consider

$$\Delta([c(1/2), c(3/4)], [c(3/4), c(1)], [c(1/2), c(1)])$$

and call this triangle Δ_2. In either case, we can choose $y \in \Delta_2 - \Delta_1$ such that $d(y_1, y_2) < \delta$.

We proceed inductively: at the $(n+1)$-st stage we consider a geodesic triangle Δ_{n+1} which has in common with Δ_n the side $[c(t_n), c(t'_n)]$ containing y_n and which has as its third vertex $c(t_{n+1})$, where $t_{n+1} = (t_n + t'_n)/2$. We choose $y_{n+1} \in \Delta_{n+1} - [c(t_n), c(t'_n)]$ with $d(y_n, y_{n+1}) \leq \delta$.

At the N-th stage of this construction, we obtain a point y_N which is a distance at most δN from x, and which lies on an interval of length $l(c)/2^N$ with endpoints in the image of c (Observe that $1 \leq l(c)/2^N < 2$). If we define y to be the closest endpoint of this interval, then

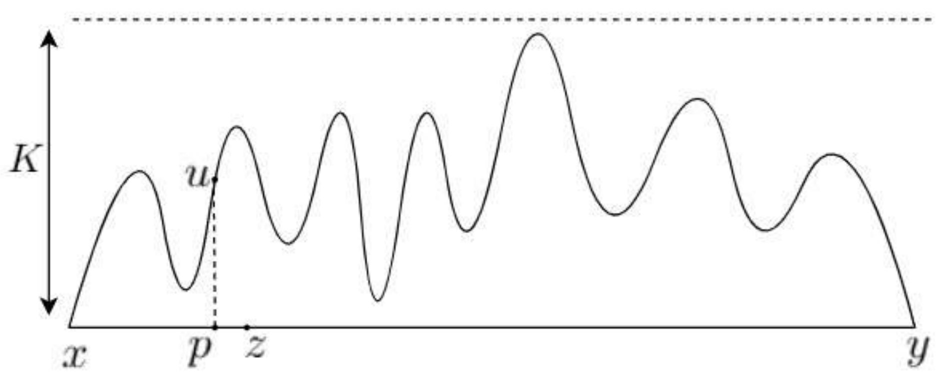
$$d(y, y_N) \leq 1.$$

Now, since $l(c)/2^{N+1} < 1$ and $2^N \leq l(c)$ we have

$$\begin{aligned} d(x,y) &\leq d(x, y_N) + d(y_N, y) \\ &\leq \delta N + 1 \\ &\leq \delta |\log_2 l(c)| + 1 \end{aligned}$$

Hence proved. □

Lemma 8.2.3. *Let, in any metric space $X, \gamma : [a,b] \to X$ be a continuous path and $[x,y]$ be a geodesic segment connecting the extremities $x = \gamma(a)$ and $y = \gamma(b)$ of γ. Let $K \geq 0$ such that γ is contained in the K-neighbourhood of $[x,y]$. Then $[x,y]$ is contained in the $2K$-neighbourhood of γ.*



Proof. Let z be a point in $[x,y]$. From the Intermediate value theorem, we can find a point u in γ such that $|x-u| = |x-z|$. By hypothesis, there exists a point p in $[x,y]$ such that $|u-p| \leq K$. Then we have:

$$|p-z| = ||x-z| - |x-p|| = ||x-u| - |x-p|| \leq |u-p|$$

on applying the triangle inequality, we get

$$|p - z| \leq K$$

where,

$$|u - z| \leq |u - p| + |p - z| \leq 2K$$

which shows the lemma. ☐

Lemma 8.2.4 (Taming Quasi-Geodesics)**.** *Let X be a geodesic space. Given any (λ, ϵ) quasi-geodesic $c : [a, b] \to X$, one can find a continuous (λ, ϵ') quasi-geodesic $c' : [a, b] \to X$ such that:*

(1) $c'(a) = c(a)$ *and* $c'(b) = c(b)$.

(2) $\epsilon' = 2(\lambda + \epsilon)$.

(3) $l(c'|_{[t,t']}) \leq k_1 d(c'(t), c'(t')) + k_2$, *for all* $t, t' \in [a, b]$, *where* $k_1 = \lambda(\lambda + \epsilon)$ *and* $k_2 = (\lambda\epsilon' + 3)(\lambda + \epsilon)$.

(4) The Hausdorff distance between the images of c and c' is less than $(\lambda + 2\epsilon)$.

Proof.

(1) (2) Define c' to agree with c on $\Sigma = \{a, b\} \cup (\mathbb{Z} \cap (a, b))$. Then choose geodesic segments joining the images of successive points in Σ and define c' by concatenating linear reparameterizations of these geodesic segments. Note that the length of each of the geodesic segments is at most $(\lambda + \epsilon)$.

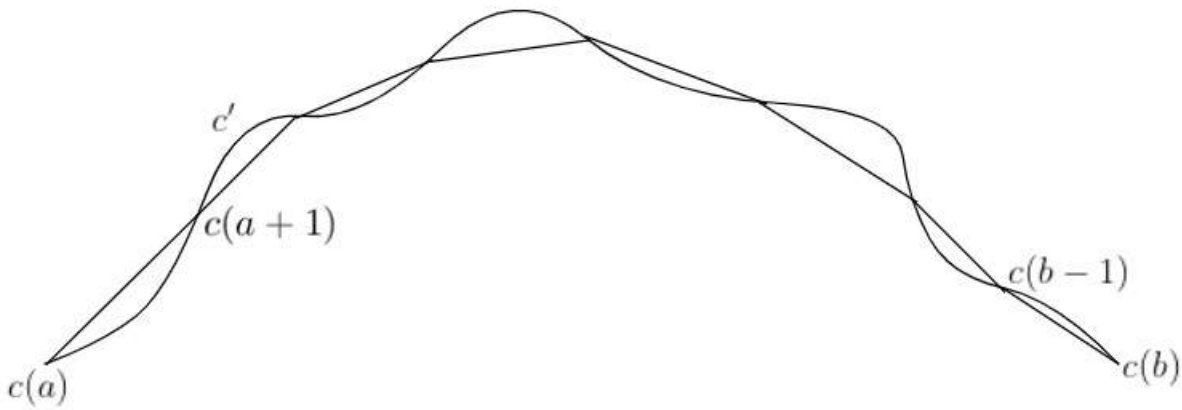
$$d(c(a), c(a + 1)) \leq \lambda d(a, a + 1) + \epsilon = \lambda + \epsilon$$

Every point of $im(c')$ lies in the $(\lambda + \epsilon)/2$ neighbourhood of $c(\Sigma)$ and every point of $im(c)$ lies in the $\lambda/2 + \epsilon$ neighbourhood of $c(\Sigma)$. To see the latter, consider,

$$d(c(a), c(a + 1/2)) \leq \lambda d(a, a + 1/2) + \epsilon = \lambda/2 + \epsilon$$

Let $[t]$ denote the point of Σ closest to $t \in [a, b]$. Now since $im(c')$ is in the $(\lambda + \epsilon)/2$ neighbourhood of $c(\Sigma)$, we have

$$\begin{aligned} d(c'(t), c'(t')) &\leq d(c'(t), c'([t])) + d(c'([t]), c'([t'])) + d(c'(t'), c'([t'])) \\ &\leq d(c'([t]), c'([t'])) + (\lambda + \epsilon) \end{aligned}$$



Because c is a (λ, ϵ) quasi-geodesic and $c([t]) = c'([t])$ for all $t \in [a, b]$, we have

$$\begin{aligned} d(c'(t), c'(t')) &\leq d(c([t]), c([t'])) + (\lambda + \epsilon) \\ &\leq \lambda|[t] - [t']| + \epsilon + (\lambda + \epsilon) \\ &\leq \lambda(|t - t'| + 1) + (\lambda + 2\epsilon) \\ &\leq \lambda(|t - t'|) + 2(\lambda + \epsilon) \end{aligned}$$

and

$$\begin{aligned} \frac{1}{\lambda}|t - t'| - 2(\lambda + \epsilon) &= \frac{1}{\lambda}(|t - t'| - \lambda^2) - (\lambda + 2\epsilon) \\ &\leq \frac{1}{\lambda}(|t - t'| - 1) - (\lambda + 2\epsilon) \\ &\leq \frac{1}{\lambda}(|t - t'| + 1) - (\lambda + 2\epsilon) \\ &\leq \frac{1}{\lambda}|[t] - [t']| - (\lambda + 2\epsilon) \\ &\leq d(c([t]), c([t'])) - (\lambda + \epsilon) \\ &\leq d(c'([t]), c'([t'])) - (\lambda + \epsilon) \\ &\leq d(c'(t), c'(t')). \end{aligned}$$

This proves that c' is a (λ, ϵ) quasi-geodesic with $\epsilon' = 2(\lambda + \epsilon)$.

(3) For all integers $n, m \in [a, b]$,

$$l(c'|_{[n,m]}) = \sum_{i=n}^{m-1} d(c(i), c(i + 1)) \leq (\lambda + \epsilon)|m - n|,$$

and similarly $l(c'|_{[a,m]}) \leq (\lambda + \epsilon)(m - a + 1)$ and $l(c'|_{[n,b]}) \leq (\lambda + \epsilon)(b - n + 1)$.

Thus for all $t, t' \in [a, b]$ we have:

$$l(c'|_{[t,t']}) \leq (\lambda + \epsilon)(|[t] - [t']| + 2),$$

and

$$d(c'(t), c'(t')) \geq \frac{1}{\lambda}|t - t'| - \epsilon' \geq \frac{1}{\lambda}(|[t] - [t']| - 1) - \epsilon'.$$

By combining these inequalities and noting that $\epsilon \leq \epsilon'$ we obtain,

$$\begin{aligned} l(c'|_{[t,t']}) &\leq (\lambda + \epsilon)(|[t] - [t']| + 2) \\ &\leq (\lambda + \epsilon)((\lambda(d(c'(t), c'(t')) + \epsilon') + 1 + 2) \\ &\leq \lambda(\lambda + \epsilon)d(c'(t), c'(t')) + (\lambda + \epsilon)(\lambda\epsilon' + 3) \end{aligned}$$

Hence $l(c'|_{[t,t']}) \leq k_1 d(c'(t), c'(t')) + k_2$, for all $t, t' \in [a, b]$, with $k_1 = \lambda(\lambda + \epsilon)$ and $k_2 = (\lambda\epsilon' + 3)(\lambda + \epsilon)$.

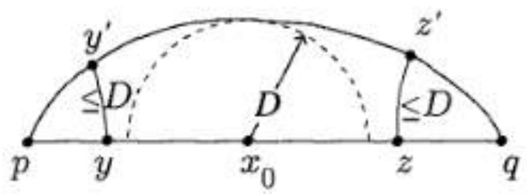
(4) Since every point of $im(c) \cup im(c')$ lies in the $\lambda/2 + \epsilon$ neighbourhood of $c(\Sigma)$, we have that,

$$d(c(t), c'(t)) \leq d(c(t), c([t])) + d(c'(t), c'([t])) \leq \lambda + 2\epsilon.$$

Hence we are done.

□

Theorem 8.2.5 (Stability of Quasi-Geodesics). *For all $\delta > 0, \lambda \geq 1, \epsilon \geq 0$ there exists a constant $R = R(\delta, \lambda, \epsilon)$ with the following property: If X is a δ-hyperbolic geodesic space, c is a (λ, ϵ)-quasi-geodesic in X and $[p, q]$ is a geodesic segment joining the endpoints of c, then the Hausdorff distance between $[p, q]$ and the image of c is less than R.*



Proof. First one tames c. In other words, one replaces it by c' as in the Lemma 8.2.4. We write $im(c')$ for the image of c' and $[p, q]$ for a choice of geodesic segment joining its endpoints. Let $D = \sup\{d(x, im(c'))|x \in [p, q]\}$ and let x_0 be a point on $[p, q]$ at which this supremum is attained. The open ball of radius D with centre x_0 does not meet $im(c')$. We shall use Proposition 8.2.2 to bound D.

Let y be the point of $[p, x_0] \subset [p, q]$ that is a distance $2D$ from x_0 (if $d(x_0, p) < 2D$ then let $y = p$). Choose $z \in [x_0, q]$ similarly. We fix $y', z' \in im(c')$ with $d(y, y') \leq D$ and $d(z, z') \leq D$ and choose geodesic segments $[y, y']$ and $[z', z]$. Consider the path γ from y to z that traverses $[y, y']$ then follows c' from y' to z', then traverses $[z', z]$. This path lies outside $B(x_0, D)$.

Since,

$$d(y', z') \leq d(y', y) + d(y, z) + d(z, z') \leq D + 4D + D \leq 6D$$

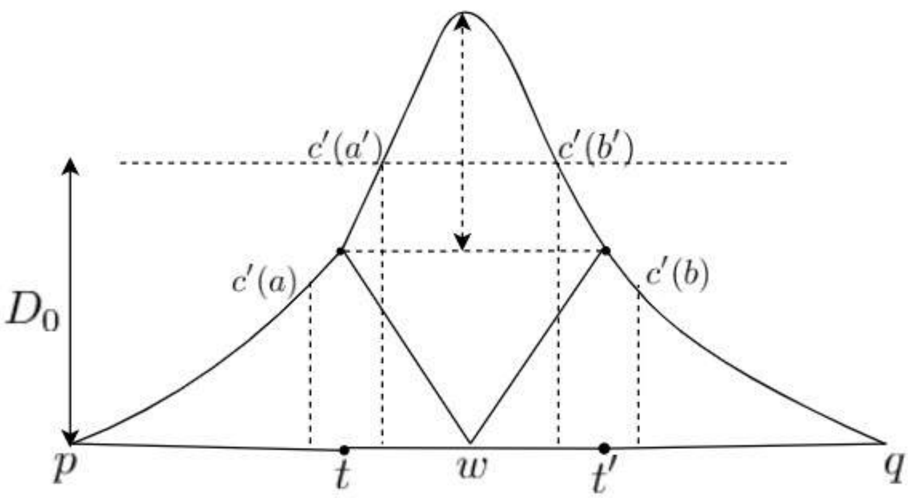
from 8.2.4(3) we have,

$$l(\gamma) = l(\gamma|_{[y', z']}) + d(y, y') + d(z, z') \leq 6Dk_1 + k_2 + 2D$$

And from 8.2.2, as $d(x_0, im(\gamma)) = D$ we have $D - 1 \leq \delta |\log_2 l(\gamma)|$. Thus

$$D - 1 \leq \delta \log_2(6Dk_1 + k_2 + 2D),$$

which gives an upper bound on D depending only on λ, ϵ and δ. Let D_0 be such a bound.



We claim that $im(c')$ is contained in the R'-neighbourhood of $[p, q]$, where $R' = D_0(1 + k_1) + k_2/2$. Consider a maximal sub-interval $[a', b'] \subset [a, b]$ such that $c'([a', b'])$ lies outside the D_0-neighbourhood $V_{D_0}[p, q]$. Every point of $[p, q]$ lies in $V_{D_0}(im(c'))$, so by connectedness there exist $w \in [p,q], t \in [a,a']$ and $t' \in [b',b]$ such that $d(w,c'(t)) \le D_0$ and $d(w,c'(t')) \le D_0$. In particular

$$d(c'(t),c'(t')) \le d(c'(t),w) + d(w,c'(t')) \le 2D_0$$

so by Lemma 8.2.4(3)

$$l(c'|_{[t,t']}) \le 2k_1D_0 + k_2$$

Now, the maximum height that can be attained by the curve is half of its length i.e., $k_1D_0 + k_2/2$. Hence $im(c')$ is contained in the R' neighbourhood of $[p,q]$ where $R' = D_0(1+k_1) + k_2/2$.
From this and Lemma 8.2.4(4), it follows that $R = R' + \lambda + \epsilon$ satisfies the statement of the theorem(since we have tamed c). □

Definition 8.2.6. A (λ,ϵ)-quasi-geodesic triangle in a metric space X consists of three (λ,ϵ)-quasi-geodesics (its sides) $p_i : [0,T_i] \to X, i = 1,2,3$ with $p_i(T_i) = p_{i+1}(0)$ (indices mod 3). Such a triangle is said to be k-slim (where $k > 0$) if for each $i \in \{1,2,3\}$ every point $x \in im(p_i)$ lies in the k-neighbourhood of $im(p_i) \cup im(p_{i+1})$ (indices mod 3).

Corollary 8.2.7. A geodesic metric space X is hyperbolic if and only if, for every $\lambda \ge 1$ and every $\epsilon \ge 0$, there exists a constant M such that every (λ,ϵ)-quasi-geodesic triangle in X is M-slim. (If X is δ-hyperbolic, then M depends only on δ, λ and ϵ.)

Proof.
Let X be a δ-hyperbolic space. Hence all the geodesic triangles are δ-slim. Now consider a (λ,ϵ)-quasi-geodesic triangle Δ. Now by Theorem 8.2.5, we have that the Hausdorff distance between each side and the corresponding geodesic is at most $R(\delta,\lambda,\epsilon)$. Also, we have the same constant R for all the sides since all the sides are (λ,ϵ)-quasi-geodesic. Hence Δ is M-slim where $M = R + \delta$.

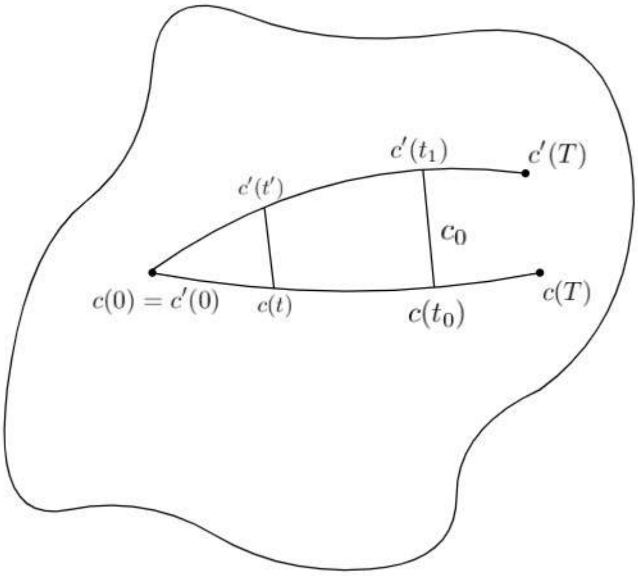
Now in the other direction, if, for every $\lambda \ge 1$ and every $\epsilon \ge 0$, there exists a constant M such that every (λ,ϵ)-quasi-geodesic triangle in X is M-slim, it is obvious that every geodesic triangle is M-slim as well. Hence the space is M-hyperbolic. □

Lemma 8.2.8. *Let X be a geodesic space that is δ-hyperbolic.*

(1) Let $c, c' : [0,T] \to X$ be geodesics with $c(0) = c'(0)$. If $d(c(t_0), im(c')) \le K$, for some $K > 0$ and $t_0 \in [0,T]$, then $d(c(t),c'(t)) \le 2\delta$ for all $t < t_0 - K - \delta$.

(2) *Let* $c_1 : [0, T_1] \to X$ *and* $c_2 : [0, T_2] \to X$ *be geodesics with* $c_1(0) = c_2(0)$. *Let* $T = \max\{T_1, T_2\}$ *and extend the shorter geodesic to* $[0, T]$ *by the constant map. If* $k = d(c_1(T), c_2(T))$, *then* $d(c_1(t), c_2(t)) \leq 2(k + 2\delta)$ *for all* $t \in [0, T]$.

Proof.



(1) To prove the first assertion, we choose a geodesic c_0 joining $c(t_0)$ to a closest point $c'(t_1)$ on the image of c'. By the triangle inequality,

$$\begin{aligned} |t_0 - t_1| &= |d(c(t_0), c(0)) - d(c'(t_1), c'(0))| \\ &\leq d(c(t_0), c'(t_1)) \\ &\leq d(c(t_0), im(c')) \\ &\leq K \end{aligned}$$

Note that $c(t)$ is not δ-close to any point on c_0 if $t < t_0 - K - \delta$. It follows from the δ-slimness of the triangle with sides $c_0, c([0, t_0]), c'([0, t_1])$ that $d(c(t), c'(t')) \leq \delta$ for some t'. By the triangle inequality

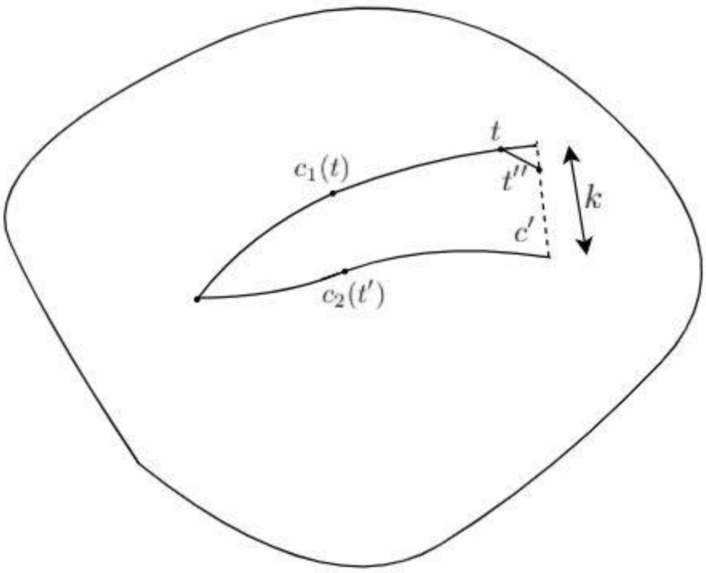
$$\begin{aligned} |t - t'| &= |d(c(t), c(0)) - d(c'(t'), c'(0))| \\ &\leq d(c(t), c'(t')) \\ &\leq \delta \end{aligned}$$

Therefore,

$$\begin{aligned} d(c(t), c'(t)) &\leq d(c(t), c'(t')) + d(c'(t'), c'(t)) \\ &\leq 2\delta \end{aligned}$$

Hence we are done.

(2) To prove the second assertion, we consider a geodesic triangle, two of whose sides are c_1 and c_2. Let the third side be c'. If $c_1(t)$ is δ-close to a point $c_2(t')$ then as above $|t - t'| \leq \delta$ and hence $d(c_1(t), c_2(t)) < 2\delta$. If $c_1(t)$ is δ-close to a point on the third side of the triangle, then it is $(k + \delta)$-close to the endpoint of c_2.



$$\begin{aligned} d(c_1(t), c_2(T)) &\leq d(c_1(t), c'(t'')) + d(c'(t''), c_2(T)) \\ &\leq \delta + k \end{aligned}$$

Now also observe that,

$$\begin{aligned} |t - T| &= |d(c_1(t), c_1(0)) - d(c_2(T), c_2(0))| \\ &\leq d(c_1(t), c_2(T)) \\ &\leq \delta + k \end{aligned}$$

Hence as in the first case, this implies that

$$\begin{aligned} d(c_1(t), c_2(t)) &\le d(c_1(t), c_2(T)) + d(c_2(t), c_2(T)) \\ &\le 2(k + \delta) \end{aligned}$$

Hence we are done.

□

8.3 Boundary ∂X as a set of Rays

We write ∂X to denote the set of equivalence classes of geodesic rays in X, and we write $\partial_q X$ to denote the set of equivalence classes of quasi-geodesic rays. In each case, we write $c(\infty)$ to denote the equivalence class of c.
There is a natural map from ∂X to $\partial_q X$ given by the inclusion of the set of equivalence classes of geodesic rays in the set of equivalence classes of quasi-geodesic rays.

Lemma 8.3.1. *The natural map from ∂X to $\partial_q X$ is well defined.*

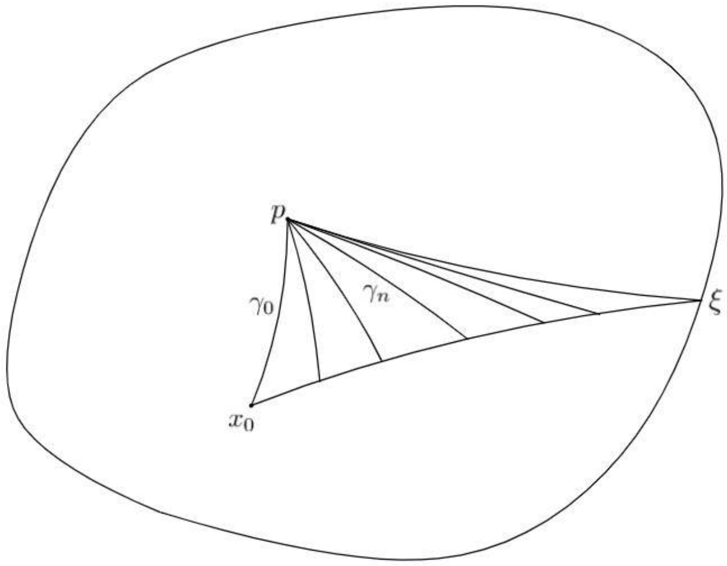
Proof. Let a, b be two geodesic rays belonging to the same equivalence class in ∂X. Hence by the equivalence condition, we have that

$$d_H(a, b) < R$$

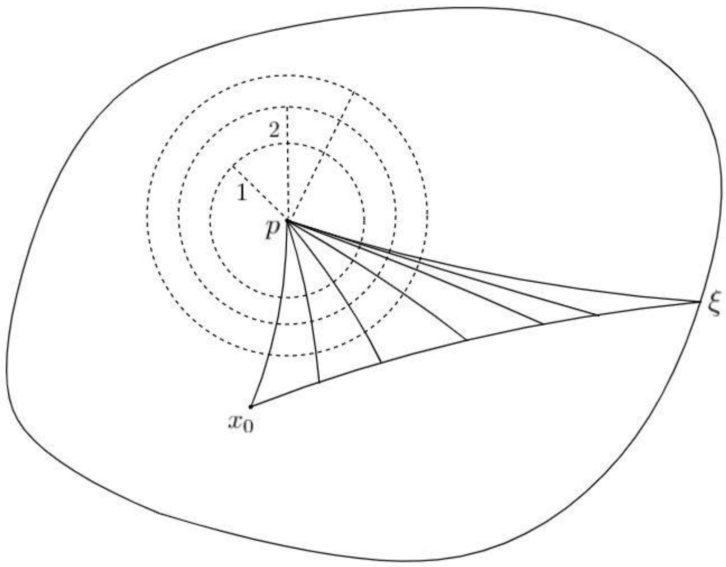
Now since the natural map is an inclusion and the Hausdorff distance between the geodesics is finite, we get that a, b belong to the same equivalence class in $\partial_q X$ the map is well-defined. □

Lemma 8.3.2. *If X is a proper geodesic space that is δ-hyperbolic, then the natural map from ∂X to $\partial_q X$ is a bijection. For each $p \in X$ and $\xi \in \partial X$ there exists a geodesic ray $c : [0, \infty) \to X$ with $c(0) = p$ and $c(\infty) = \xi$.*

Proof. The natural map $\partial X \to \partial_q X$ is injective because the natural map is an inclusion.
Let c be a geodesic ray asymptotic to ξ, with the initial point x_0. Consider a sequence of geodesic segments $\gamma_n : [0, D_n] \to X$, connecting p to $x_n = c(n)$, where $D_n = d(p, c(n))$. The δ-hyperbolicity of X implies that the image of γ_n is at a Hausdorff distance of at most $4\delta + dist(x, x_0)$ from $x_0 x_n$, where $x_0 x_n$ is the initial subsegment of c.

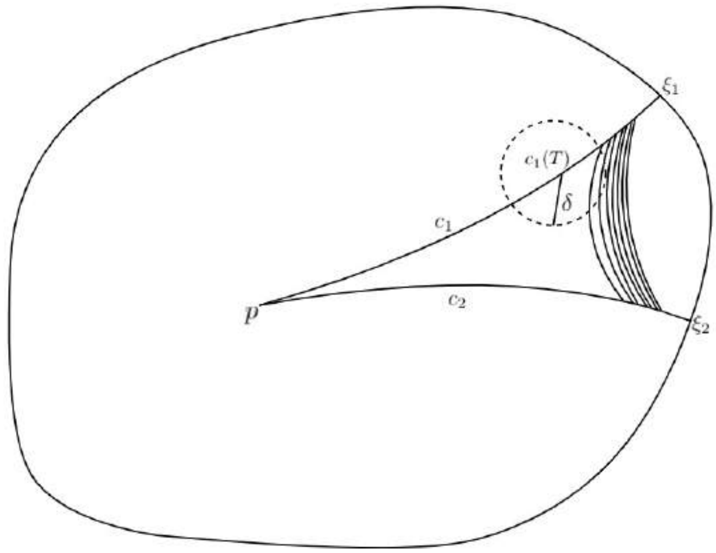


Now consider a closed ball centered at p of radius 1. The sequence of geodesics c_n hits the closed ball of radius 1 at the points p_n. Now since X is proper, we have that the ball is compact, and hence we can apply the Arzela-Ascoli theorem. Then there exists a subsequence c_n^1 of c_n corresponding to $[p, p_n^1]$, which converges. Now we increase the radius of the closed circle, and hence, we get $c_n^2 \subset c_n^1 \subset c_n$. Now proceeding similarly, we get the diagonal subsequence which converges to a geodesic ray $c' : [0, \infty) \to X$.



Clearly, the image of c' is at Hausdorff distance at most $4\delta + dist(x, x_0)$ from the image of c. In particular, c' is asymptotic to c. □

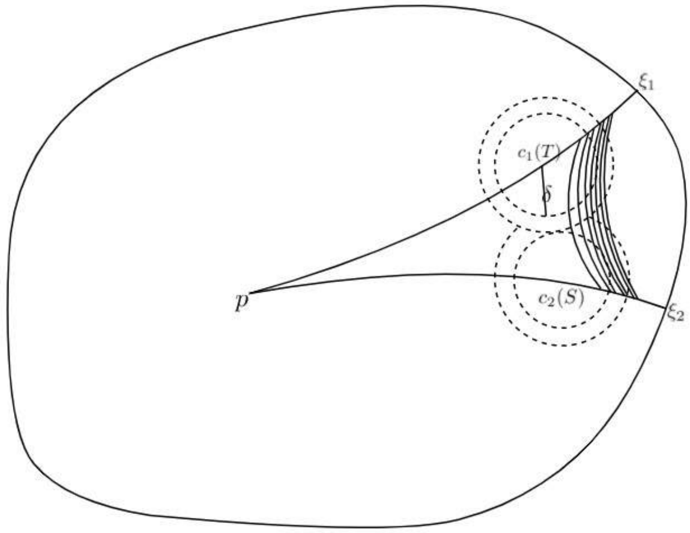
Lemma 8.3.3 (Visibility of ∂X). *If the metric space X is proper, geodesic and δ-hyperbolic, then for each pair of distinct points x and y on the boundary of X, there exists a geodesic $d : \mathbb{R} \to X$ such that $d(-\infty) = x$ and $d(\infty) = y$.*



Proof. Fix $p \in X$ and choose geodesic rays $c_1, c_2 : [0, \infty) \to X$ issuing from p with $c_1(\infty) = \xi_1$ and $c_2(\infty) = \xi_2$. Let T be such that the distance from $c_1(T)$ to the image of c_2 is greater than δ. For each $n > T$ we choose a geodesic segment $[c_1(n), c_2(n)]$ and consider the geodesic triangle with sides $c_1([0, n]), c_2([0, n])$ and $[c_1(n), c_2(n)]$. Since this triangle is δ-slim, $[c_1(n), c_2(n)]$ must intersect the closed (hence compact) ball of radius δ about $c_1(T)$ at a point, say p_n.
By the Arzela-Ascoli theorem on compact sets, as $n \to \infty$ we can find a subsequence of the geodesics c^δ of $[p_n, c_2(n)] \subset [c_1(n), c_2(n)]$, which will converge. Now we increase the radius of the ball around $c_1(T)$ and choose the sequence $c^{\delta+1} \subset c^\delta \subset [c_1(n), c_2(n)]$, which converges. We keep on proceeding in a similar manner and obtain a subsequence c_{n_k} of the sequence $[c_1(n), c_2(n)]$, which converges. The limit of this subsequence is a geodesic line which has an endpoint $c_1(\infty)$.

Let S be such that the distance from $c_2(S)$ to the image of c_1 is greater than δ. For each $n_k > S$ we consider the geodesic triangle with sides $c_1([0, n_k]), c_2([0, n_k])$ and $[c_1(n_k), c_2(n_k)]$. Since this triangle is δ-slim, $[c_1(n_k), c_2(n_k)]$ must intersect the closed (hence compact) ball of radius δ about $c_2(S)$ at a point, say q_{n_k}.

By the Arzela-Ascoli theorem on compact sets, as $n_k \to \infty$ we can find a subsequence of the geodesics $c_{n_k}^\delta$ of $[q_n, c_1(n_k)] \subset [c_1(n_k), c_2(n_k)]$, which will converge. Now we increase the radius of the ball around $c_2(S)$ and choose the sequence $c_{n_k}^{\delta+1} \subset$



$c^{\delta}_{n_k} \subset [c_1(n_k), c_2(n_k)]$, which converges. We keep on proceeding in a similar manner and obtain a subsequence c_∞ of the sequence $[c_1(n), c_2(n)]$, which converges. The limit of this subsequence is a geodesic line which has end point $c_1(\infty)$ and $c_2(\infty)$.

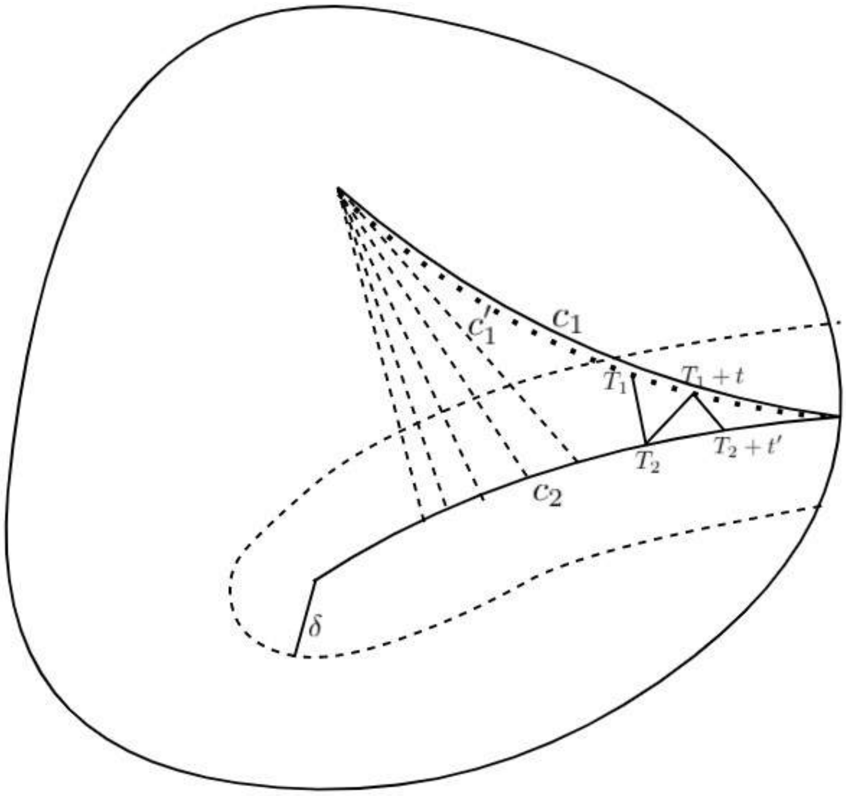
Since each $[c_1(n), c_2(n)]$ is contained in the δ-neighbourhood of the union of the images of c_1 and c_2, the image of c is also contained in this neighbourhood. Thus the endpoints of c are ξ_1 and ξ_2. ☐

Lemma 8.3.4 (Asymptotic Rays are Uniformly Close)**.** *Let X be a proper δ-hyperbolic space and let $c_1, c_2 : [0, \infty) \to X$ be geodesic rays with $c_1(\infty) = c_2(\infty)$.*

1) If $c_1(0) = c_2(0)$ then $d(c_1(t), c_2(t)) < 2\delta$ for all $t > 0$.

2) In general, there exist $T_1, T_2 > 0$ such that $d(c_1(T_1 + t), c_2(T_2 + t)) < 5\delta$ for all $t \geq 0$.

Proof.
(1) follows immediately from 8.2.8. When $c_1(\infty) = c_2(\infty)$, $\sup_t d(c_1(t), c_2(t)) = M < \infty$. Now we apply 8.2.8 (1). $d(c_1(t_0), im(c_2)) \leq K$ for some $K > 0$ and all $t_0 \in [0, \infty)$. Hence $d(c_1(t), c_2(t)) \leq 2\delta$ for all t.
(2) In order to prove this part, we consider a sequence of geodesics $c_n = [c_1(0), c_2(n)]$. Now apply the Arzela-Ascoli theorem to obtain a subsequence of the c_n that converges to a geodesic ray c'_1 with $c'_1(0) = c_1(0)$. Since the triangles $\Delta(c_1(0), c_2(0), c_2(n))$ are δ-slim, all but an initial segment of each c_n is contained in the δ-neighbourhood of the image of c_2, and hence a terminal segment of c'_1 is also contained in this neighbourhood. In other words, there exists $T_1, T_2 > 0$ with $d(c_2(T_2), c'_1(T_1)) \leq \delta$ such



that for all $t \geq 0$ one can find t' with,

$$d(c_2(T_2+t'), c_1'(T_1+t)) \leq \delta$$

(This is because the terminal segments of c_1' and c_2 lies in the δ-neighbourhood). By the triangle inequality,

$$\begin{aligned}
|t-t'| &= |d(c_1'(T_1), c_1'(T_1+t')) - d(c_2(T_2), c_2(T_2+t))| \\
&\leq |d(c_1'(T_1), c_2(T_2)) + d(c_2(T_2), c_1'(T_1+t')) \\
&\quad - (d(c_2(T_2), c_1'(T_1+t')) - d(c_1'(T_1+t'), c_2(T_2+t)))| \\
&\leq |d(c_1'(T_1), c_2(T_2)) + d(c_1'(T_1+t'), c_2(T_2+t))| \\
&\leq 2\delta
\end{aligned}$$

t and t' differ by at most 2δ. Thus for all $t \geq 0$ we have

$$\begin{aligned} d(c_2(T_2+t), c_1'(T_1+t)) &\leq d(c_2(T_2+t), c_1'(T_1+t')) + d(c_1'(T_1+t), c_1'(T_1+t')) \\ &\leq \delta + 2\delta \\ &\leq 3\delta \end{aligned}$$

And from (1), we know that $d(c_1(T+t), c_1'(T+t)) < 2\delta$ for all $t > 0$ (Since c_1' and c_1 have the same starting point). Hence,

$$\begin{aligned} d(c_2(T_2+t), c_1(T_1+t)) &\leq d(c_2(T_2+t), c_1'(T_1+t)) + d(c_1'(T_1+t), c_1(T_1+t)) \\ &\leq 3\delta + 2\delta \\ &\leq 5\delta \end{aligned}$$

Hence we are done. $\square$

Chapter 9

Hyperbolicity and Quasi-Isometries

Definition 9.0.1. Let X_1 and X_2 be two metric spaces, $\lambda \geq 1$ and $k \geq 0$ be any real number, and let f a function between X_1 and X_2. We say that f is a (λ, k)-quasi-isometry in the strong sense if for all pair of points $x, y \in X_1$, we have:

$$\lambda^{-1}|x - y| - k \leq |f(x) - f(y)| \leq \lambda|x - y|.$$

***Remark*.** The above definition is by Thurston. Notice that a quasi-isometry in the strong sense is continuous(because of lipschitzness), which is not necessarily the case for a quasi-isometry in the broad sense.

In this section, we demonstrate the following theorem:

Theorem 9.0.2. *Let X_1 and X_2 be two geodesic metric spaces with X_2 hyperbolic. If $f : X_1 \to X_2$ is a quasi-isometry in the strong sense, then*

1) X_1 is also hyperbolic.

2) The function which associates a sequence (a_i) of X_1 which converges at infinity with $(f(a_i))$, induces a topological embedding.

$$\partial f : \partial X_1 \to \partial X_2.$$

In addition, if f is bounded (that is to say, if the function $dist(., f(X_1))$ is bounded in X_2), then ∂f is a homeomorphism.

We can immediately deduce from Theorem 9.0.2 the following corollary:

Corollary 9.0.3. Let X be a space provided with two metrics $|\ |_1$ and $|\ |_2$, which are geodesic, and which are equivalent (that is to say that there exists $\lambda > 0$ such that $\lambda^{-1}|\ |_1 \leq |\ |_2 \leq \lambda|\ |_1$). If X is hyperbolic for $|\ |_1$, then it is also for $|\ |_2$, and the boundary of X for $|\ |_1$ is a canonical homeomorphism of boundary of X for $|\ |_2$.

Note here that for the equivalent metrics which are not geodesic, the statement is generally true. To prove the Theorem 9.0.2, we apply the following two lemmas:

Lemma 9.0.4. *Let X_1 and X_2 be two metric spaces and $f : X_1 \to X_2$ be a (λ, k)-quasi-isometry in the strong sense. For all geodesic segments $[x, y]$ in X_1, the image of the path $f([x, y])$ is (λ', k')-tamed-quasi-geodesic.*

Proof. For any segment $[x', y']$ of $[x, y]$, we have, in utilizing the fact that f is a (λ, k)-quasi-isometry,

$$\begin{aligned}
l(f|_{[x',y']}) &= \sup_{x'=t_0\leq t_1\leq \ldots \leq t_n=y'} \sum_{i=0}^{n-1} |f(t_i) - f(t_{i+1})| \\
&\leq \sup_{x'=t_0\leq t_1\leq \ldots \leq t_n=y'} \sum_{i=0}^{n-1} \lambda|t_i - t_{i+1}| \\
&\leq \lambda l([x', y']) \\
&\leq \lambda|x' - y'| \\
&\leq \lambda^2(|f(x') - f(y')| + k) \\
&\leq \lambda^2|f(x') - f(y')| + \lambda^2 k
\end{aligned}$$

Hence the image of the path $f([x, y])$ is (λ', k')-tamed-quasi-geodesic with $\lambda'(\lambda' + k') = \lambda^2$ and $(2\lambda'(\lambda' + k') + 3)(\lambda' + k') = \lambda^2 k$. □

Lemma 9.0.5. *Let X_1 and X_2 be two geodesic metric spaces, with X_2 δ-hyperbolic, and let $f : X_1 \to X_2$ a (λ, k)-quasi-isometry in the strong sense. There exists a constant K dependent only on δ, λ and k, such that for every geodesic segment $[f(x), f(y)]$ and for all points z in $f([x, y])$, we have*

$$dist(z, [f(x), f(y)]) \leq K. \tag{9.1}$$

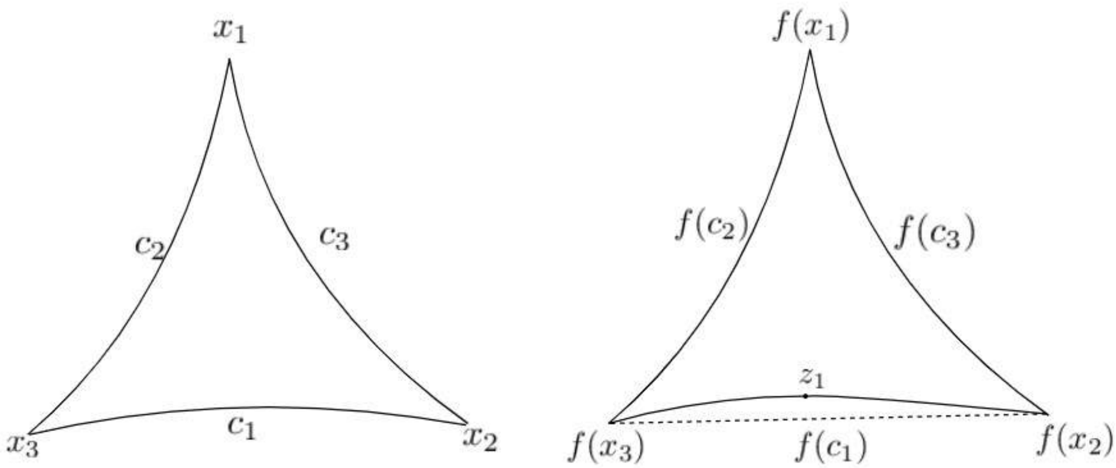
Proof.
The path $f([x, y])$ is from the Lemma 9.0.4 above, (λ', k')-tamed-quasi-geodesic. From the Theorem 8.2.5 above, it is contained in the K-neighbourhood of the segment $[f(x), f(y)]$, where K is a constant which depends only on δ, λ and k. □

Corollary 9.0.6. Under the hypothesis of Lemma 9.0.5, there exists a constant C such that for all geodesic segments $[f(x), f(y)]$ and for all points z in $[f(x), f(y)]$, we have

$$dist(z, f([x, y])) \leq C. \tag{9.2}$$

Proof. It follows from the Lemma 9.0.5 and Lemma 8.2.3, taking $C = 2K$, where K is the constant of Lemma 9.0.5. ☐

Proof of the first part of Theorem 9.0.2 in the case of quasi-isometry in the strong sense.



Suppose that the function f is a (λ, μ)-quasi-isometry in the strong sense, and let $\Delta_1 = [x_1, x_2, x_3]$ be a geodesic triangle of X_1. Consider the geodesic triangle $\Delta_2 = [f(x_1), f(x_2), f(x_3)]$. If X_2 is δ-hyperbolic, then the minimum size of the triangle Δ_2 is $\leq 4\delta$. Consequently, there exists points $z_i (i = 1, 2, 3)$ situated on each side of the triangle Δ_2, such that $|z_i - z_j| \leq 4\delta$. From the lemma 8.2.3 before, there exists a constant C such that z_i is at a distance $\leq C$ from $f(c_i)$, where c_i is the side opposite to x_i in Δ_1. Hence there exists $u_i \in c_i$ such that $|z_i - f(u_i)| \leq C$. From the triangle inequality, we have

$$\begin{aligned} |f(u_i) - f(u_j)| &\leq |f(u_i) - z_i| + |z_i - z_j| + |z_j - f(u_j)| \\ &\leq 2C + 4\delta. \end{aligned}$$

So we have

$$\begin{aligned} |u_i - u_j| &\leq \lambda(|f(u_i) - f(u_j)| + k) \\ &\leq \lambda(2C + 4\delta) + \lambda k = \delta' \end{aligned}$$

The minimum size of the geodesic triangle Δ_1 is $\leq \delta'$. We then deduce that X_1 is $4\delta'$-hyperbolic. □

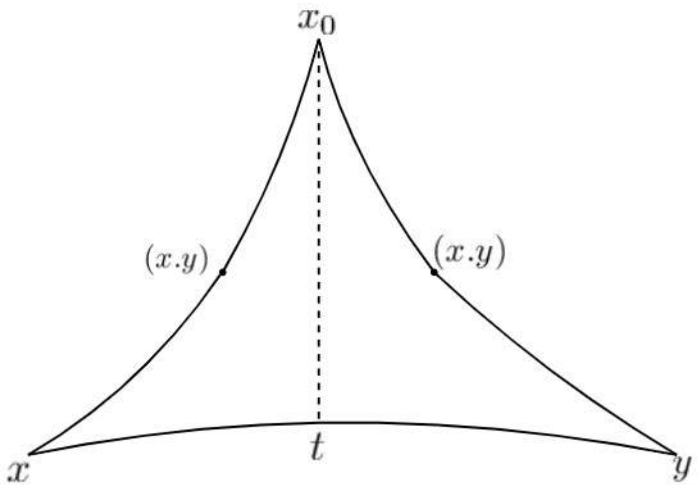
To show the second part of the theorem, we will need the following lemma:

Lemma 9.0.7. *Let X be a metric space, $x_0 \in X$ a base point and $[x, y]$ a geodesic segment joining the points x and y of X. We have then:*

$$(x.y) \leq dist(x_0, [x, y]). \tag{9.3}$$

If, in addition, the space X is geodesic and δ-hyperbolic, we have

$$(x.y) \leq dist(x_0, [x, y]) \leq (x.y) + 4\delta. \tag{9.4}$$



Proof. For the first inequality, we take a point $t \in [x, y]$ such that $|x_0 - t| = dist(x_0, [x, y])$. We then have

$$|x_0 - t| \geq |x - x_0| - |x - t| \text{ and } |x_0 - t| \geq |y - x_0| - |y - t|. \tag{9.5}$$

Hence,

$$\begin{aligned} |x_0 - t| &\geq \frac{1}{2}(|x - x_0| + |y - x_0| - (|x - t| + |y - t|)) \\ &\geq \frac{1}{2}(|x - x_0| + |y - x_0| - |x - y|) \\ &\geq (x.y) \end{aligned}$$

For the second inequality, let c_1, c_2 and c_3 be the internal points of the geodesic triangle $[x_0, x, y]$ with $c_1 \in [x_0, x]$ and $c_3 \in [x_0, y]$. We have from the triangle inequality,

$$|c_3 - x_0| \leq |c_1 - x_0| + |c_1 - c_3|. \tag{9.6}$$

or,

$$|c_1 - x_0| = (x.y) \text{ and } |c_1 - c_3| \leq 4\delta \text{ follows from 6.4.2 and 6.5.3} \tag{9.7}$$

Hence,

$$dist(p, [x, y]) \leq |c_3| \leq (x.y) + 4\delta \tag{9.8}$$

□

Proof of the second part of Theorem 2.2 when f is a quasi isometry in the strong sense.

Let (a_i) be a sequence of points of X_1 which converges at infinity. We thus have

$$(a_i.a_j) \to \infty \text{ when } i \text{ and } j \to \infty. \tag{9.9}$$

Let x_0 be the basepoint of X_1. Take $f(x_0)$ as the basepoint of X_2 and show that the sequence $f(a_i)$ converges at infinity. By the lemma 9.0.7, we have $dist(x_0, [a_i, a_j]) \to \infty$ when i and $j \to \infty$.

Lemma 9.0.8. *The function f is a quasi-isometry which implies that*

$$(dist(f(x_0), f([a_i, a_j]))) \to \infty \text{ when } i \text{ and } j \to \infty \tag{9.10}$$

Proof. We proceed in a similar manner as in 9.0.7. We see that

$$\begin{aligned}
|f(x_0) - f(t)| &\geq \frac{1}{2}(|f(x) - f(x_0)| + |f(y) - f(x_0)| - (|f(x) - f(t)| + |f(y) - f(t)|)) \\
&\geq \frac{1}{2}(\frac{1}{\lambda}|x - x_0| + \frac{1}{\lambda}|y - x_0| - \frac{1}{\lambda}|x - y|) - \frac{k}{2} \\
&\geq \frac{1}{\lambda}\frac{1}{2}(|x - x_0| + |y - x_0| - |x - y|) - \frac{k}{2} \\
&\geq \frac{(x.y)}{\lambda} - \frac{k}{2}
\end{aligned}$$

Hence we get that $\frac{(x.y)}{\lambda} - \frac{k}{2} \leq dist(f(x_0), f([x, y]))$ Since we have taken (a_i) to be converging at infinity, we see that $(dist(f(x_0), f([a_i, a_j]))) \to \infty$ when i and $j \to \infty$. □

On the other hand we have, from the lemma 9.0.5 and corollary 9.0.6,

$$|dist(f(x_0), [f(a_i), f(a_j)]) - dist(f(x_0), f([a_i, a_j]))| \leq C_1, \tag{9.11}$$

where C_1 depends only on f. Hence,

$$dist(f(x_0), [f(a_i), f(a_j)]) \to \infty \text{ when } i \text{ and } j \to \infty \tag{9.12}$$

which implies by the lemma 9.0.7, that $(f(a_i).f(a_j))$ converges at infinity. The sequence $(f(a_i))$ is convergent.

Lemma 9.0.9.

$$(f(x).f(y)) \leq \lambda(x.y)$$

Proof.

$$\begin{aligned}(f(x).f(y)) &= \frac{1}{2}(|f(x) - f(x_0)| + |f(y) - f(x_0)| - |f(x) - f(y)|) \\ &\leq \frac{1}{2}(\lambda|x - x_0| + \lambda|y - x_0| - \lambda|x - y|) \\ &\leq \lambda(x.y)\end{aligned}$$

□

We then define a function $\partial f : \partial X_1 \to \partial X_2$. The properties of ∂f stated in the theorem are proved analogously.

Since f is a strong quasi-isometry, it is continuous. Hence so is ∂f. Now we show that it is injective.

$$\begin{aligned}&[(f(a_i))] = [(f(b_i))] \\ \implies &\lim(f(a_i).f(b_i)) = \infty \\ \implies &\lim(a_i.b_i) = \infty \\ \implies &[(a_i)] = [(b_i)]\end{aligned}$$

Hence ∂f is a topological embedding. Now we show that it is surjective. Let $(y_i) \in X_2$ be a sequence of points such that $[(y_i)] \in \partial X_2$. Now we can choose for each i, $x_i \in X_1$ such that

$$d_{X_2}(f(x_i), y_i) < \epsilon$$

Now we show that $[(x_i)]$ is the required pre-image i.e.,

$$[(f(x_i))] = [(y_i)]$$

Hence it suffices to show that

$$\lim(f(x_i).y_i) = \infty \text{ as } i \to \infty$$

Now, using the definition of Gromov product,

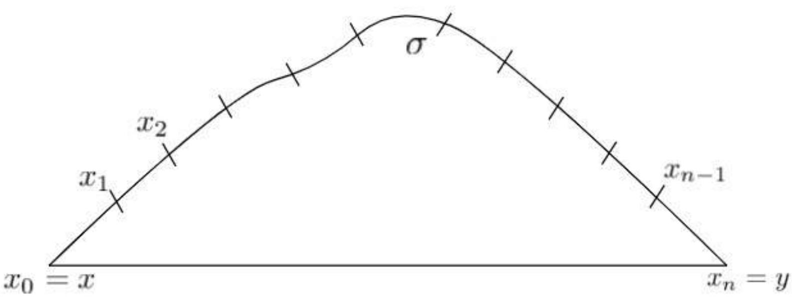
$$(f(x_i).y_i) = \frac{1}{2}(d(f(x_i), y_0) + d(y_i, y_0) - d(f(x_i), y_i))$$

Now since $|y_i| \to \infty$ as $i \to \infty$ and $d_{X_2}(f(x_i), y_i) < \epsilon$, we see that $\lim(f(x_i).y_i) = \infty$. Now using Proposition 7.2.9, we see that $[(f(x_i))] \in \partial Y$ as well. Hence we are done. □

9.1 Application: The 0-hyperbolic geodesic spaces are trees

Theorem 9.1.1. *The 0-hyperbolic geodesic spaces are real trees.*

Proof.
We have already seen that real trees are 0-hyperbolic geodesic spaces. Conversely, let X be a 0-hyperbolic geodesic space. For all $x, y \in X$, let $[x, y]$ be a geodesic segment between the two points. We show that if σ is any topological segment(which is, therefore, injective but is not necessarily geodesic) included between the two points, we have necessarily $\sigma = [x, y]$.



For that let, ϵ be a number > 0. As σ is compact, we can, by uniform continuity, find a sequence $x_0, ..., x_n$ of conseqtive points in σ, with $x_0 = x$ and $x_n = y$ and satisfying $|x_i - x_{i+1}| \leq \epsilon$. (This follows by the definition of uniform continuity.) Consider the broken geodesic $[x_0, x_1] \cup [x_1, x_2] \cup ... \cup [x_{n-1}, x_n]$. From 8.2.1 before, if X is δ-hyperbolic, all points of $[x, y]$ is at a distance $\leq 4k\delta$ of the broken geodesic, where k is an integer satisfying $n \leq 2^k$. As $\delta = 0$, this implies that $[x, y]$ is contained in the broken geodesic. As any point of the broken geodesic is at a distance $\leq \epsilon$

from a point on σ, we can deduce that $[x, y]$ is contained in the ϵ-neighbourhood of σ. As this is true for all $\epsilon = 0$, we obtain, by tending ϵ towards 0, that $[x, y]$ is contained in σ. As $[x, y]$ and σ are both topological paths, we deduce that they are equal. □

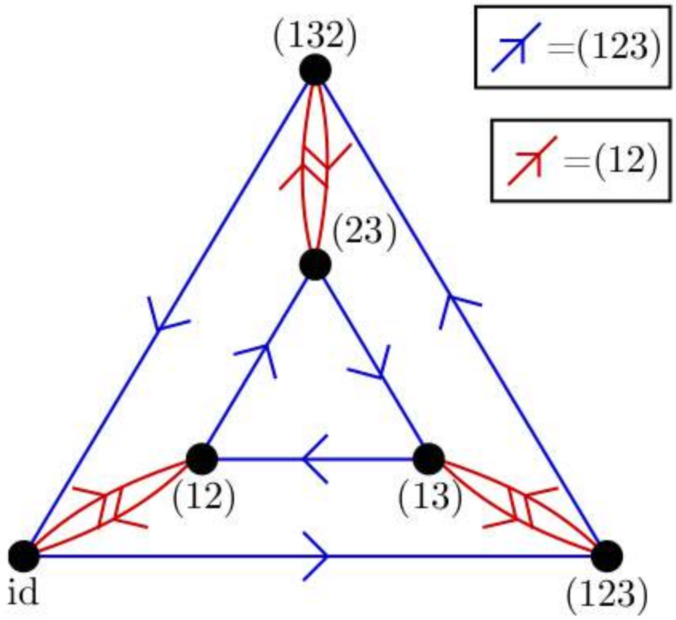
Chapter 10

Hyperbolic Groups

In this chapter, we define the notion of hyperbolic groups, and we see some examples of such groups. This definition involves a system of generators, but we show that the definition does not depend on the choice of the system of generators. As an important example of the hyperbolic groups, we will give that of the group of isometries of the hyperbolic geodesic spaces, acting properly discontinuous, in this space and such that the quotient of this action is compact.
We first recall some definitions in the geometric group theory.

Definition 10.0.1. Let Γ be a finitely generated group and G be a finite system of generators of Γ. We define a metric in this group, named the word metric on Γ(with respect to G), in the following manner:
If s is a word with alphabets $G \cup G^{-1}$(where G^{-1} is the set of inverses of elements of G), that is to say, a finite sequence of letters $g_1, ..., g_n$ with for all $i = 1, ..., n$ $g_i \in G$ where $g_i^{-1} \in G$, we define the length of s as being the number n of elements of the sequence.

For all elements γ of the group Γ, we define then the norm of γ, which we denote as $|\gamma|$, as the infimum of the set of lengths of the word with respect to the alphabets $G \cup G^{-1}$.
The distance between two elements γ_1 and γ_2 of Γ is then defined as $|\gamma_2^{-1}\gamma_1|$. We obtain then a metric on the group Γ, which we call as the word metric associated to the system of generators G. It is invariant upon the action of Γ on itself by the left translations.
We say that the group Γ is δ-hyperbolic relative to the system of generators G if it is δ-hyperbolic with respect to the associated word metric. We will see in section 3 of this chapter that the hyperbolicity of a group does not depend on the choice of the system of generators.



To see that a geodesic metric space is hyperbolic, it is often practical to use the criterions shown in section 3 of chapter 1 concerning the thinness, the slimness, the internal size and the minimal internal size of geodesic triangles in the space. These criteria can also be utilized for metric spaces which are not geodesic. A way of doing this is to embed such a space in a non-geodesic metric space. For a group Γ, provided a system of finite generators G, there exists a concrete object in which we can realize this embedding; it is the Cayley graph K of the group. By definition, it is a simplicial complex of dimension 1 whose vertices are the elements of the group, and such that two vertices γ_1 and γ_2 are lying on a side if and only if we can write $\gamma_2 = \gamma_1.g$ where g is an element of the system of generators G.
We provide the Cayley graph K with a simplicial metric, which on the vertices are the word metric on the group and for which the length of the side is equal to 1. We obtain hence a proper geodesic metric space in which the group Γ is isometric immersion.
We then have the following proposition:

Proposition 10.0.2. *The group Γ is hyperbolic(with respect to the system of generators G) if and only if the Cayley graph associated is hyperbolic metric space.*

Proof. The immersion of the group Γ in the Cayley graph is bounded, and we can apply the Proposition 4.3.1. □

Let Γ be a finitely generated group, and let G_1 and G_2 be two finite systems of generators of Γ. In this section, we demonstrate

Theorem 10.0.3. *The group Γ is hyperbolic with respect to G_1 if and only if it is hyperbolic with respect to G_2.*

The demonstration of this theorem makes use of the following two lemmas.

Lemma 10.0.4. *Let $|\ |_1$ be the metric on Γ associated to G_1, and $|\ |_2$ be the metric associated to G_2. There exists a real number C such that*

$$C^{-1}|\ |_2 \leq |\ |_1 \leq C|\ |_2 \tag{10.1}$$

Proof. We consider an integer C such that each element of G_1 (respectively G_2) can be written as a word of length $\leq C$ in the alphabets of $G_2 \cup G_2^{-1}$ (respectively $G_1 \cup G_1^{-1}$)(We can do this by taking the maximum of norms of each generating set and then taking the maximum of both of them). We then have the inequalities above. □

Let K_1 and K_2 be the Cayley Graphs associated respectively to (Γ, G_1) and (Γ, G_2), each of these graphs being provided with the simplicial metric which, when restricted to the vertices, induces the word metric of the group. We define a function $f : K_1 \to K_2$ such that the restriction of f to Γ is the identity function and that any edge of K_1 is sent linearly to a geodesic segment of K_2 (Note that generally, such a function is not unique).

Lemma 10.0.5. *The function f is a quasi-isometry in the strong sense.*

Proof. Notice first that f satisfies the following property(where C is a constant of 10.0.4 above)

$$\forall x_1, x_2 \in K_1, |f(x_1) - f(x_2)| \leq C|x_1 - x_2|. \tag{10.2}$$

To this effect, consider a geodesic segment $[x_1, x_2]$ joining these points. We have

$$|f(x_1) - f(x_2)| \leq l(f|_{[x_1,x_2]}) \leq C|x_1 - x_2| \tag{10.3}$$

since each side and each piece of the side dilates by the same factor $\leq C$.
Let γ_1 and γ_2 be two vertices of K_1 such that for $i = 1, 2$, we have

$$|x_i - \gamma_i| \leq 1/2$$

We have

$$\begin{aligned}|\gamma_1-\gamma_2| &\le C|f(\gamma_1)-f(\gamma_2)|\\ &\le C(|f(\gamma_1)-f(x_1)|+|f(x_1)-f(x_2)|+|f(x_2)-f(\gamma_2)|)\\ &\le C(|f(x_1)-f(x_2)|+2C)\end{aligned}$$

Hence,

$$|x_1-x_2| \le |x_1-\gamma_1|+|\gamma_1-\gamma_2|+|\gamma_2-x_2| \tag{10.4}$$
$$\le |\gamma_1-\gamma_2|+1 \tag{10.5}$$
$$\le C|f(x_1)-f(x_2)|+2C^2+1 \tag{10.6}$$

The inequalities 10.2 and 10.4 gives

$$C^{-1}|x_1-x_2|-2C-1/C \le |f(x_1)-f(x_2)| \le C|x_1-x_2|,$$

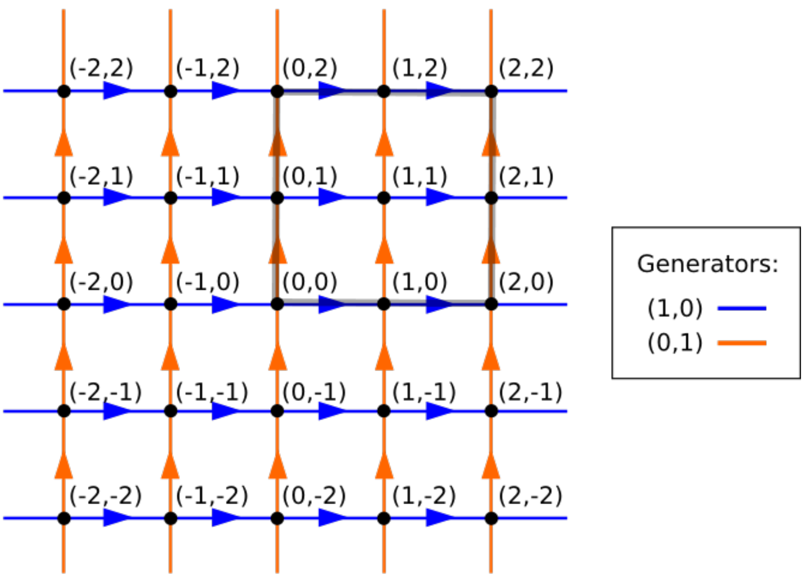
which implies that f is a $(C, 2C+1/C)$-quasi isometry in the strong sense. Thus we are done. □

Proof of Theorem 10.0.3. From the Lemma 10.0.5, we obtain the proof of Theorem 10.0.3 thanks to the Proposition 10.0.2 above and to Theorem 9.0.2.
Let us also point out that the Theorem 9.0.2 shows that if Γ is a hyperbolic group, then there is a boundary $\partial\Gamma$ canonically attached to Γ. □

Chapter 11

Examples of Hyperbolic Groups

All finite groups are hyperbolic. A free group is hyperbolic. In effect, its Cayley graph, with respect to a system of finite generators, is a tree, which is 0-hyperbolic 5.1.5.
Consider the abelian free group $\mathbb{Z}^n$, with its standard system of generators. We show that for all $n \geq 2$, this group is not hyperbolic. For that, construct in its Cayley graph a sequence of geodesic triangles whose internal size is not bounded. Let us first take the case when $n = 2$, and let us look at the Cayley graph of $\mathbb{Z}^n$ like the meeting, in the euclidean plane $\mathbb{R}^2$, of horizontal and vertical lines passing through the integer points.



Let $(0, 0)$ be a basepoint for the space. For any positive integer n, we consider the geodesic triangle Δ_n defined in the following manner. The vertices of this triangle are at $(n, n), (0, n)$ and $(n, 0)$. The vertices (n, n) and $(n, 0)$ are joined by a vertical line segment, the vertices (n, n) and $(0, n)$ are joined by a horizontal segment, and the vertices $(0, n)$ and $(n, 0)$ are joined by a vertical segment followed by a horizontal segment (In the above drawing, the triangle is, therefore, a square). Now we see that for all n

$$((n, n).(0, n)) = \frac{1}{2}(d((n, n), (0, 0)) + d((0, n), (0, 0)) - d((n, n), (0, n))) = n$$
$$((n, n).(n, 0)) = \frac{1}{2}(d((n, n), (0, 0)) + d((n, 0), (0, 0)) - d((n, 0), (0, n))) = n/2$$
$$((n, 0).(0, n)) = \frac{1}{2}(d((n, 0), (0, 0)) + d((0, n), (0, 0)) - d((n, 0), (0, n))) = 0$$

Hence for the space to be hyperbolic, we should have that

$$0 \geq min(n, n/2) - \delta \ \forall \ n$$

i.e.,

$$\delta \geq n/2 \ \forall \ n,$$

which does not hold. Hence $\mathbb{Z}^2$ is not hyperbolic.
For all $n \geq 0$, we consider a subspace $\mathbb{Z}^2$ of $\mathbb{Z}^n$. The Cayley graph of $\mathbb{Z}^2$ is contained like a totally geodesic subspace of the Cayley graph of $\mathbb{Z}^n$, and the geodesic triangles of $\mathbb{Z}^2$ are geodesic triangles of $\mathbb{Z}^n$. By consequence, $\mathbb{Z}^n$ is not hyperbolic.
Important examples of hyperbolic groups follows from the following theorem:

Theorem 11.0.1. *Let X be a proper geodesic space, and Γ the group of isometries of X acting properly discontinuous on this space, and such that the quotient of the action is compact. Then Γ is hyperbolic if and only if X is. In addition, if Γ (and X) is hyperbolic, we have a canonical homeomorphism $\partial\Gamma \to \partial X$.*

We recall that a group Γ acts properly discontinuous on a space if for all compact K, the set $\{\gamma \in \Gamma | \gamma(K) \cap K \neq \emptyset\}$ is finite.
In all that follows, Γ is assumed to satisfy the hypothesis of Theorem 11.0.1. For showing this theorem, we utilize the techniques mainly due to Milnor.
Fix a points m of X. We define then a function

$$\phi : \Gamma \to X$$

which to all elements γ of Γ associates the point $\gamma(m)$ of X. We define the metric on X/Γ as follows

$$d(\Gamma.x, \Gamma.y) = \min\{d(p,q) | p \in \Gamma.x \text{ and } q \in \Gamma.y\}$$
$$= \min\{d(x, \gamma y) | \gamma \in \Gamma\}$$

If D is the diameter of X/Γ, we denote by Δ the closed ball of radius $2D$ centred at m, and we define G as $\{\gamma \in \Gamma | \gamma(\Delta) \cap \Delta \neq \emptyset\}$. We then have

Lemma 11.0.2. *The set G and the set Δ possesses the following properties:*

1) $G^{-1} = G$

2) *For any point $x \in X$, there exists a point $x' \in \Delta$ and an element $\gamma \in \Gamma$ such that $x = \gamma(x')$.*

3) *G is finite*

4) *G generates Γ*

Proof.

1) If $\gamma \in G \implies \gamma(\Delta) \cap \Delta \neq \emptyset$. Now, $\exists\, \gamma^{-1} \in \Gamma$, so that $\gamma^{-1}(\gamma(\Delta) \cap \Delta) = \gamma^{-1}(\Delta) \cap \Delta \neq \emptyset \implies \gamma^{-1} \in G \implies \gamma \in G^{-1}$. Hence $G \subseteq G^{-1}$.

   Similarly, if $\gamma \in G^{-1} \implies \gamma^{-1}(\Delta) \cap \Delta \neq \emptyset$. Now $\exists\, \gamma \in \Gamma$, so that $\gamma(\gamma^{-1}(\Delta) \cap \Delta) = \gamma(\Delta) \cap \Delta \neq \emptyset \implies \gamma \in G$. Hence $G^{-1} \subseteq G$.

2) Let $x \in X$. If $x \in \Delta$, then $x' = x$ and $\gamma = id$.

   If $x \notin \Delta$, since the diameter of X/Γ is equal to D, we have

$$d(\Gamma.x, \Gamma.m) = \min\{d(x, \gamma m) | \gamma \in \Gamma\} \leq D.$$

   Hence $\exists\, \gamma$ such that $d(x, \gamma m) \leq D$. Hence $x \in \gamma(\Delta)$. Now, since Γ acts by isometries on X, $\exists\, x' \in \Delta$ such that

$$d(x, \gamma m) = d(x', m).$$

   Hence $x = \gamma(x')$.

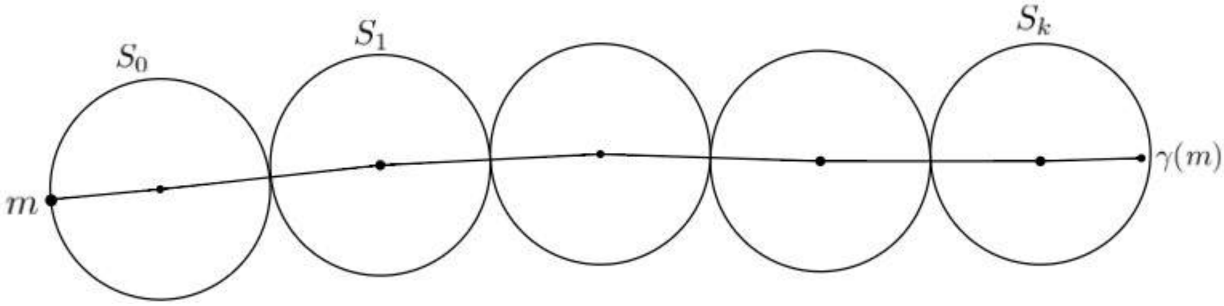
3) This results due to the fact that the action of Γ is properly discontinuous that Δ is compact because the space X is proper.

4) Let us show now that G generates Γ.

Given an element $\gamma \in \Gamma$, we consider a geodesic segment $[m, \gamma(m)]$ joining the points m and $\gamma(m)$, and we cover $[m, \gamma m]$ by k closed discs $S_0, ..., S_k$, each of radius D and whose centres x_i are on this geodesic segment (we suppose that the centres are ordered according to the order indicated by the enumeration of the discs), with

$$k \leq [\frac{|m - \gamma(m)|}{2D}] + 1 \text{ (where [ ] designates the integer part)}$$

As the diameter of X/Γ is equal to D, we can find in each of the discs S_i a



point of the form $g_i(m)$, where g_i is an element of Γ(we will take $g_0 = id$ and $g_k = \gamma$). This follows from part(2).

$$d(\Gamma.x, \Gamma.m) = \min\{d(x, gm) | g \in \Gamma\} \leq D.$$

Hence for every disc S_i with centre x_i, we can find a $g \in \Gamma$ such that $d(x_i, g(m)) \leq d$. This implies that $g(m) \in S_i$.

For any i, $g_i(\Delta)$ is the disc of radius $2D$ and the center $g_i(m)$. Two successive points $g_i(m)$ and $g_{i+1}(m)$, are situated at a distance $\leq 4D$ from one another.

$$\begin{aligned} d(g_i(m), g_{i+1}(m)) &\leq d(g_i(m), x_i) + d(x_i, x_{i+1}) + d(x_{i+1}, g_{i+1}(m)) \\ &\leq D + 2D + D = 4D. \end{aligned}$$

We have, $\forall i = 0, ..., k-1$, $g_{i+1}(\Delta) \cap g_i(\Delta) \neq \emptyset$, which implies $g_i^{-1} g_{i+1}(\Delta) \cap \Delta \neq \emptyset$, and hence $g_i^{-1} g_{i+1} \in G$. Hence each element g_i is written as the product of elements of G. Now,

$$\gamma = g_0(g_0^{-1} g_1)(g_1^{-1} g_2)....(g_{k-1}^{-1} g_k) \tag{11.1}$$

This demonstrates the last part of the lemma.

□

We deduce from the above proof the following lemma, which will be useful to us later. Consider the Cayley graph, K, of the group Γ equipped with the system of generators G. We recall that Γ is naturally immersed in K as the set of vertices of this graph and that this immersion is isometric.
We extend the function $\phi : \Gamma \to X$ to $\Phi : K \to X$, by sending linearly each edge joining the vertices γ_1 and γ_2 to a geodesic segment of X which joins the points $\phi(\gamma_1)$ and $\phi(\gamma_2)$.

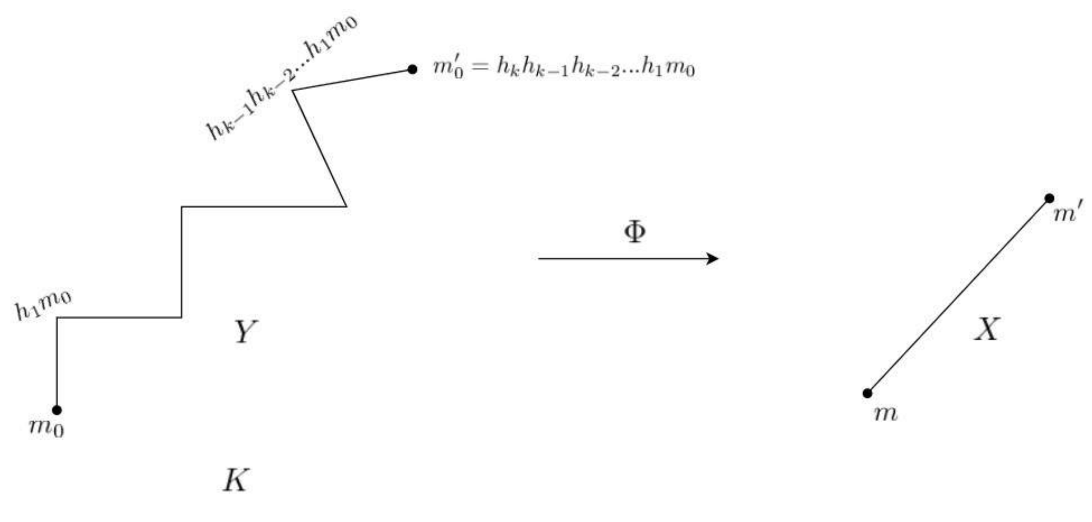
Lemma 11.0.3. *Given two points $m = \phi(m_0)$ and $m' = \phi(m'_0)$ of the image of $\phi(\Gamma)$ of the group Γ, and a geodesic segment $[m, m']$ in X joining these points, we can find a sequence of k elements $h_1, h_2, ..., h_k$ of the system of generators G verify the following properties:*

(4.3.1) $k \leq [\frac{|m-m'|}{2D}] + 1$, *and m and m' are joined by a path of the form $\Phi(Y)$ where Y is a simplicial path in K defined as*

$$Y = [m_0, h_1m_0] \cup [h_1m_0, h_2h_1m_0] \cup ... \cup [h_{k-1}...h_1m_0, h_kh_{k-1}...h_1m_0],$$

(4.3.2) $\Phi(Y)$ *is contained in the d'-neighbourhood of the geodesic $[m, m']$, where d' is a constant which does not depend on the choice of the two points m and m'.*

(4.3.3) Y *is a (q, q')-quasi-geodesic in K, with q and q' independent of m and m'.*



Proof. We utilise the notations which were used in the proof of the Proposition 11.0.2. The first two statements are clear (we can take $d' = D + s$, with $s = \sup |(\phi(g_i) - \phi(Id))|$, where g_i describes the elements of the system of generators G). To prove 4.3.3, consider an element γ of Γ such that $m'_0 = \gamma m_0$, and suppose that γ is written as $\gamma = g'_1 g'_2 ... g'_l$, where for all $i = 1, ..., L$, $g'_i \in G$. We have:

$$\begin{aligned}|m - g'_1(m)| &\leq 4D, \\ |m - g'_1 g'_2(m)| &\leq |m - g'_1(m)| + |g'_1(m) - g'_1 g'_2(m)| \\ . &\leq |m - g'_1(m)| + |m - g'_2(m)| \\ &\leq 8D.\end{aligned}$$

Hence we have,

$$|m - g'_1 g'_2 ... g'_L(m)| \leq 4DL$$

We then have $\frac{|m-\gamma(m)|}{4D} \leq L$.
So if $|\gamma|$ is the length of γ for the word metric associated to the system of generators, we have

$$\frac{|m - \gamma(m)|}{4D} \leq |\gamma| \tag{11.2}$$

If $l(Y)$ is the length of the path Y, we have from (4.3.1) and 11.2:

$$l(Y) \leq |m - m'| + 1 \leq 4D|\gamma| + 1.$$

Note also that $|\gamma|$ designates similarly the distance between the extremities of the path Y. We then deduce easily that Y is a (q, q')-quasi-geodesic in K, with two constants, q and q', independent of the points m and m'.

□

Consider the system G of generators of Γ. We denote by $|\ |_\Gamma$ the associated word metric and $|\ |_X$ the distance in X. We then have the following proposition, due to Milnor.

Proposition 11.0.4. *There exists a constant $C > 0$ such that for γ_1 and γ_2 in Γ, we have*

$$C^{-1}|\gamma_1 - \gamma_2|_\Gamma \leq |\phi(\gamma_1) - \phi(\gamma_2)|_X \leq C|\gamma_1 - \gamma_2|_\Gamma$$

Proof. As Γ acts isometrically on X, it suffices to show that for any element $\gamma \in \Gamma$, we have

$$C^{-1}|\gamma| \leq |m - \gamma(m)|_X \leq C|\gamma|.$$

Let γ be an element of Γ of length n. We can the write $\gamma = g_1.g_2...g_n$ with $\forall\ i = 1,...,n \ \ g_i \in G$. As $diam(\Delta) \leq 4D$, we have, $\forall\ g \in G, |m - g(m)| \leq 8D$. We can deduce that $|m - \gamma(m)| \leq 8nD$, and that

$$\forall\ \gamma \in \Gamma, |m - \gamma(m)| \leq 8D|\gamma|. \tag{11.3}$$

We show next the left-hand side of the inequality.
Let $v = min\{dist(\Delta, \gamma\Delta) | \gamma \in \Gamma \text{ and } \Delta \cap \gamma\Delta = \emptyset\}$. As Δ is compact, and as Γ act properly discontinuously on X, we have $v > 0$.
Let k be the smallest integer such that $|m - \gamma(m)| < kv$, and show that $|\gamma| \leq k$.

Consider a geodesic segment of X joining the points m and $\gamma(m)$. On this segment, we can place $k+1$ points $y_0, y_1, ..., y_k$ with $y_0 = m, y_k = \gamma(m)$, and such that

$$\forall\ i = 0, ..., k-2, \text{ we have } |y_i - y_{i+1}| = v, \text{ and } |y_{k-1} - y_k| \leq v.$$

By the Proposition 11.0.2, we can find, for all $i = 0, ..., k$ a point $y'_i \in \Delta$ and an element $\gamma_i \in \Gamma$ such that $y_i = \gamma_i.y'_i$. (We assume $y'_0 = y'_k = m, \ \ \gamma_0 = id$ and $\gamma_k = \gamma$.)
As $|y_i - y_{i+1}| \leq v$, we have $|\gamma_i(y'_i) - \gamma_{i+1}(y'_{i+1})| \leq v$, and hence

$$\gamma_i^{-1}\gamma_{i+1}(\Delta) \cap \Delta \neq \emptyset, \text{ and } \gamma_i^{-1}\gamma_{i+1} \in G.$$

We can write $\gamma = (\gamma_0^{-1}.\gamma_1).(\gamma_1^{-1}.\gamma_2)....(\gamma_{k-1}^{-1}.\gamma_k)$. Hence $|\gamma|_\Gamma \leq k$.
By defintion of the integer k, we have on the other hand:

$$\begin{aligned} &(k-1)v \leq |m - \gamma(m)| \\ &\Longrightarrow\ |\gamma| \leq v^{-1}.|m - \gamma(m)| + 1 \\ &\Longrightarrow\ |\gamma| \leq v^{-1}.|m - \gamma(m)| + (|m - \gamma(m)|)^{-1}|m - \gamma(m)| \end{aligned}$$

On taking $\mu = min\{|m - \gamma(m)|, \gamma \in \Gamma \text{ and } \gamma(m) \neq m\}$, we obtain

$$|\gamma| \leq (v^{-1} + \mu^{-1}).|m - \gamma(m)| \tag{11.4}$$

The inequalities 11.3 and 11.4 shows the Proposition 11.0.4 □

Proof of Theorem 11.0.1 .

Suppose that the space X is hyperbolic.
From the Proposition 11.0.4, and by the same proof as that of Lemma 10.0.5, we see that the function Φ is a quasi isometry in the strong sense.
If X is hyperbolic, the Cayley graph K is also hyperbolic(by Theorem 9.0.2), which implies(by Proposition 10.0.2 before) that the group Γ is then hyperbolic.
Proof of the inverse implication(i.e., K hyperbolic $\implies$ X hyperbolic).
For that, we take a geodesic triangle $[x_1, x_2, x_3]$ in X, and we show that the minimal size is uniformly bounded. For each of the points x_i, we can find a point $x'_i \in \Gamma.m$ in $\Phi(\Gamma)$ with $|x_i - x'_i| \leq D$. By utilizing the Lemma 11.0.3, we can, for all i and j, join the points x'_i and x'_j by a path Y'_{ij} in $\Phi(K)$, with $Y'_{ij} = \Phi(Y_{ij})$, where Y_{ij} is a (k, k')-quasi-geodesic in K, with k and k' independent of the points x_i.
By Theorem 8.2.5, Y_{ij} is contained in the d_2-neighbourhood of a geodesic segment in K joining its extremities, where d_2 is a uniform constant.
By utilizing the Proposition 11.0.4, we can deduce from the fact that the minimal size of geodesic triangles $Y_{12} \cup Y_{23} \cup Y_{13}$ is uniformly bounded that minimal size of $[x_1, x_2, x_3]$ is also uniformly bounded (the proof of Theorem 10.0.3). This proves the theorem. □

Corollary 11.0.5. A group Γ is hyperbolic if and only if there exists a hyperbolic proper geodesic space X on which Γ acts properly discontinuously with X/Γ compact.

Chapter 12

Hyperbolic Spaces

There are three models for hyperbolic spaces which are equivalent to one another. We introduce them in the following proposition.

Proposition 12.0.1. *For any fixed $R > 0$, the following Riemannian manifolds are all mutually isometric.*

(a) (HYPERBOLOID MODEL) $\mathbb{H}^n(R)$ is the "upper sheet" $\{\tau > 0\}$ of the two sheeted hyperboloid in $\mathbb{R}^{n+1}$ defined in coordinates $(\xi_1, ..., \xi_n, \tau)$ by the equation $\tau^2 - |\xi|^2 = R^2$, with the metric

$$g_1 = i^* m,$$

where $i : \mathbb{H}^n(R) \to \mathbb{R}^{n+1}$ is inclusion, and m is the Minkowski metric on $\mathbb{R}^{n+1}$

(b) (POINCARÉ BALL MODEL) $\mathbb{B}^n(R)$ is the ball of radius R in $\mathbb{R}^n$, with the metric given in coordinates $(u_1, ..., u_n)$ by

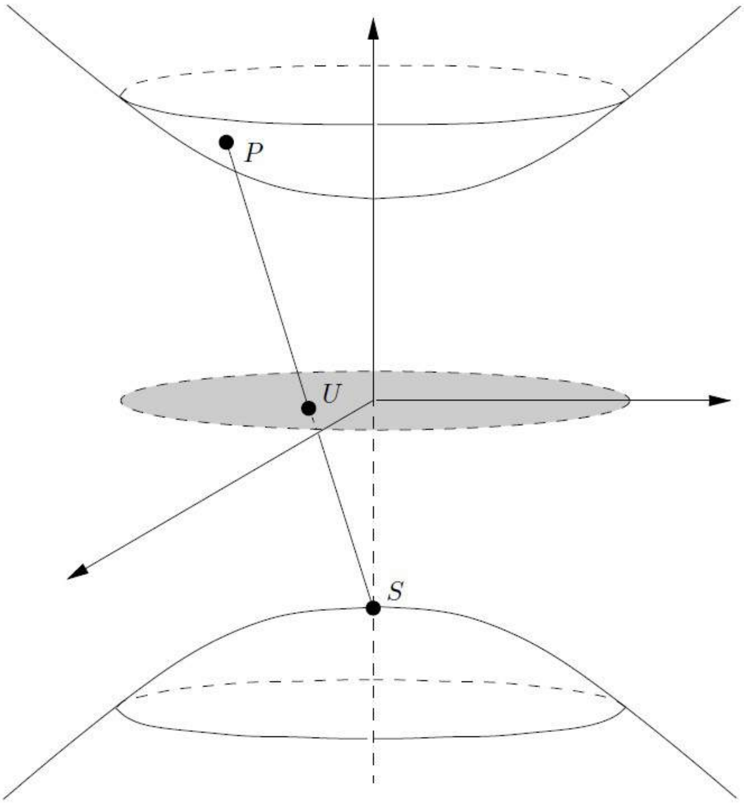
$$g_2 = \frac{4R^4 \sum_{i=1}^{n} (du_i)^2}{(R^2 - |u|^2)^2}$$

(c) (POINCARÉ HALF-SPACE MODEL) $\mathbb{U}^n(R)$ is the upper half space in $\mathbb{R}^n$ defined in coordinates $(x_1, .., x_{n-1}, y)$ by $\{y > 0\}$, with the metric

$$g_3 = \frac{R^2(\sum_{i=1}^{n-1} (dx_i)^2 + dy^2)}{y^2}$$

Proof.
First consider the Hyperboloid model $\mathbb{H}^n(R)$ and the Poincaré model $\mathbb{B}^n(R)$. We construct a diffeomorphism $\phi_1 : \mathbb{H}^n(R) \to \mathbb{B}^n(R)$ between these spaces called the

hyperbolic stereographic projection. This would also serve as an isometry between these spaces.



Let $S = (0, 0, ..., 0, -R) \in \mathbb{R}^{n+1}$ and $P = (\xi_1, ..., \xi_n, \tau) \in \mathbb{H}^n(R)$. Now, let $U = \phi_1(P) = (u_1, ..., u_n, 0)$ be the point where the line joining P and S intersects the hyperplane $\tau = 0$. Take λ such that

$$(\phi_1(P) - S) = \lambda(P - S)$$
$$(u_1, ..., u_n, R) = \lambda(\xi_1, ..., \xi_n, \tau + R)$$

Then, comparing the coordinates,

$$u_i = \lambda \xi_i$$
$$R = \lambda(\tau + R)$$

Hence,

$$\lambda = \frac{R}{\tau + R}$$

which implies

$$u_i = \frac{R\xi_i}{\tau + R}.$$

Therefore,

$$\phi_1(\xi_1, ..., \xi_n, \tau) = (\frac{R\xi_1}{\tau + R}, ..., \frac{R\xi_n}{\tau + R}, 0).$$

Now observe that,

$$\begin{aligned}
|\phi_1(P)| &= (\frac{R^2|\xi|^2}{(\tau + R)^2})^{1/2} \\
&= (\frac{R^2(\tau^2 - R^2)}{(\tau + R)^2})^{1/2} \\
&= (\frac{R^2(\tau - R)}{\tau + R})^{1/2} < R \implies \phi_1(P) \in \mathbb{B}^n(R)
\end{aligned}$$

Now for the inverse. Let $\phi_1^{-1}(u_1, ..., u_n, 0) = (\xi_1, ..., \xi_n, \tau)$. Since P has to lie in $\mathbb{H}^n(R)$,

$$(\xi_1^2 + ... + \xi_n^2) - \tau^2 = -R^2. \tag{12.1}$$

Now consider the line joining U and S. Any point in this line can be represented as

$$\lambda' U + (1 - \lambda')S = (\lambda' u_1, ..., \lambda' u_n, -(1 - \lambda')R).$$

Since it has to satisfy 12.1 Therefore,

$$\begin{aligned}
\lambda'^2 u_1^2 + \lambda'^2 u_2^2 + ... + \lambda'^2 u_n^2 - R^2(\lambda' - 1)^2 = -R^2 \\
\implies \lambda'^2(|u|^2 - R^2) + 2R^2\lambda' = 0
\end{aligned}$$

Hence,

$$\lambda' = \frac{-2R^2}{(|u|^2 - R^2)} = \frac{2R^2}{R^2 - |u|^2}.$$

Therefore,

$$\begin{aligned}
\xi_i &= \frac{2R^2 u_i}{R^2 - |u|^2} \\
\tau &= (\frac{2R^2}{R^2 - |u|^2} - 1)R \\
&= R\frac{R^2 + |u|^2}{R^2 - |u|^2}
\end{aligned}$$

Therefore,

$$\phi_1^{-1}(u_1, .., u_n, 0) = (\frac{2R^2 u_1}{R^2 - |u|^2}, ..., \frac{2R^2 u_n}{R^2 - |u|^2}, R\frac{R^2 + |u|^2}{R^2 - |u|^2})$$

To check isometry, we also need to show that,

$$(\phi_1^{-1})^* g_1 = g_2$$

$$g_1 = (d\xi_1)^2 + ... + (d\xi_n)^2 - (d\tau)^2$$

Therefore,

$$\begin{aligned}
(\phi_1^{-1})^* g_1 &= \sum_{i=1}^{n} (d(\frac{2R^2 u_i}{R^2 - |u|^2}))^2 - (d(R\frac{R^2 + |u|^2}{R^2 - |u|^2}))^2 \\
&= \sum_{i=1}^{n} (\frac{2R^2 du_i}{R^2 - |u|^2} + \frac{4R^2 u_i \sum u_j du_j}{(R^2 - |u|^2)^2})^2 - (d(R\frac{R^2 + |u|^2}{R^2 - |u|^2}))^2 \\
&= \sum_{i=1}^{n} (\frac{4R^4 (du_i)^2}{(R^2 - |u|^2)^2} + \frac{16R^4 u_i du_i \sum u_j du_j}{(R^2 - |u|^2)^3} + \frac{16R^4 (u_i)^2 (\sum u_j du_j)^2}{(R^2 - |u|^2)^4}) - (\frac{4R^3 \sum u_j du_j}{(R^2 - |u|^2)^2})^2 \\
&= \frac{4R^4 \sum (du_i)^2}{(R^2 - |u|^2)^2} + \frac{16R^4}{(R^2 - |u|^2)^3} (\sum u_i du_i)^2 (1 + \frac{|u|^2}{R^2 - |u|^2}) - \frac{16R^6 (\sum u_j du_j)^2}{(R^2 - |u|^2)^4} \\
&= \frac{4R^4 \sum (du_i)^2}{(R^2 - |u|^2)^2} + \frac{16R^4 R^2}{(R^2 - |u|^2)^4} (\sum u_i du_i)^2 - \frac{16R^6 (\sum u_j du_j)^2}{(R^2 - |u|^2)^4} \\
&= \frac{4R^4 \sum (du_i)^2}{(R^2 - |u|^2)^2} \\
&= g_2
\end{aligned}$$

Now we construct an isometry $\phi_3 : \mathbb{U}^n(R) \to \mathbb{B}^n(R)$. Let $z = (x_1, ..., x_{n-1}, y)$. Considering the 2-dimensional space. Take $z = x + iy$. We know the Cayley transform sends upper half plane to the disk of radius R via

$$z \to iR\frac{z - iR}{z + iR}$$

or,

$$x + iy \to \frac{2R^2 x}{|x|^2 + (y + R)^2} + iR\frac{|x|^2 + |y|^2 - R^2}{|x|^2 + (y + R)^2}$$

Similarly we take the map

$$\begin{aligned}
\phi_3(x_1,...,x_{n-1},y) &= (\frac{2R^2x_1}{|x|^2+(y+R)^2},...,\frac{2R^2x_{n-1}}{|x|^2+(y+R)^2},R\frac{|x|^2+|y|^2-R^2}{|x|^2+(y+R)^2}) \\
|\phi_3(x_1,...,x_{n-1},y)|^2 &= \frac{R^2(4R^2|x|^2+(|x|^2+|y|^2-R^2)^2)}{(|x|^2+(y+R)^2)^2} \\
&< \frac{R^2(4R^2(|x|^2+y^2)+(|x|^2+|y|^2-R^2)^2)}{(|x|^2+(y+R)^2)^2} \\
&< R^2
\end{aligned}$$

Its inverse can be constructed using the inverse Cayley transform. This takes the disc of radius R to the upper half plane .

$$w \to -iR\frac{w+iR}{w-iR}$$

or,

$$u_1+iu_2 \to \frac{2R^2u_1}{|u_1|^2+(u_2-R)^2}+iR\frac{R^2-|u_1|^2-|u_2|^2}{|u_1|^2+(u_2-R)^2}$$

$$\phi_3^{-1}(u_1,...,u_{n-1},u_n) = (\frac{2R^2u_1}{|u'|^2+(u_n-R)^2},...,\frac{2R^2u_{n-1}}{|u'|^2+(u_n-R)^2},R\frac{R^2-|u'|^2-u_n^2}{|u'|^2+(u_n-R)^2})$$

Therefore, ϕ_3 is a diffeomorphism.

$(\phi_3^{-1})^*g_3 = g_2$ through direct calculation.

$$\begin{aligned}
(\phi_3^{-1})^*g_3 &= \frac{R^2(\sum_{i=1}^{n-1}(d(\frac{2R^2u_i}{|u'|^2+(u_n-R)^2}))^2+(d(R\frac{R^2-|u'|^2-u_n^2}{|u'|^2+(u_n-R)^2}))^2)}{(R\frac{R^2-|u'|^2-u_n^2}{|u'|^2+(u_n-R)^2})^2} \\
&= \frac{4R^4\sum(du_i)^2}{(R^2-|u|^2)^2} \\
&= g_2
\end{aligned}$$

Hence $\mathbb{H}^n(R) \xrightarrow{\phi_1} \mathbb{B}^n(R) \xleftarrow{\phi_3} \mathbb{U}^n(R)$. Hence the three models are equivalent. □

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