L^p Harmonic Analysis on the Heisenberg Group

Gautam Neelakantan M

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Certificate of Examination

This is to certify that the dissertation titled **"L^p Harmonic Analysis on the Heisenberg Group"** submitted by **Mr. Gautam Neelakantan M** (Reg. No. MS16060) for the partial ful- filment of BS-MS dual degree programme of the Institute, has been examined by the thesis committee duly appointed by the Institute. The committee finds the work done by the candidate satisfactory and recommends that the report be accepted.

Dr. Lingaraj Sahu	Dr. Alok Maharana	Dr. Jotsaroop Kaur (Supervisor)

Dated: April, 2021

Declaration

The work presented in this dissertation has been carried out by me under the guidance of Dr. Jotsaroop Kaur at the Indian Institute of Science Education and Research Mohali.

This work has not been submitted in part or in full for a degree, a diploma, or a fellowship to any other university or institute. Whenever contributions of others are involved, every effort is made to indicate this clearly, with due acknowledgement of collaborative research and discussions. This thesis is a bonafide record of original work done by me and all sources listed within have been detailed in the bibliography.

Gautam Neelakantan M

(Candidate)

Dated: April, 2021

In my capacity as the supervisor of the candidate's project work, I certify that the above statements by the candidate are true to the best of my knowledge.

Dr. Jotsaroop Kaur
(Supervisor)

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Notations

$<<$	is much less than
$\approx$	approximately
$\lesssim$	for complex valued functions we write $f \lesssim g$ if there exists a constant positive real number number C such that $f \leq Cg$
$\mathbb{C}$	space of complex numbers
$\mathbb{N}$	space of natural numbers
$\mathbb{R}^n$	Euclidean space of dimension n
$\mathbb{R}_+$	positive real numbers
$\mathcal{S}(H^n)$	Schwartz class functions on H^n
H^n	$2n+1$ dimensional Heisenberg group
a.e	almost everywhere
$\mathrm{C}_c^\infty(H^n)$	space of compactly supported infinitely differentiable functions on H^n
Im	imaginary part

Contents

Introduction

Consider the circle group defined as the quotient group $\mathbb{T} = \mathbb{R}/\mathbb{Z}$, where $\mathbb{R}$ and $\mathbb{Z}$ are the additive groups of real numbers and integers respectively. For any $n \in \mathbb{Z}$, the function given by $\chi_n(t) = e^{2\pi int}$, $t \in [0,1]$ is a unitary representation of $\mathbb{T}$ on the one dimensional unitary group $U(1) = \{z : z \in \mathbb{C}, |Z| = 1\}$. To each function $f \in L^1(\mathbb{T})$, we associate the sequence $\{\widehat{f}(k)\}$ of Fourier coefficients of f, defined by

$$\widehat{f}(n) = \int_0^1 f(t)\chi_n(-t)\ dt.$$

The trignometric series with these coefficients,

$$\sum_{n=-\infty}^{\infty} \widehat{f}(n)\chi_n(t), \tag{1}$$

is called the Fourier series of f. Note that $f \to \widehat{f}(n)\chi_n$ is the spectral projection of the laplacian $\frac{\partial^2}{\partial t^2}$ corresponding to the eigencalue $-4\pi^2 n^2$. One of the fundamental problem is to determine in what sense (1) represents the function f. We have that the partial sums given by $S_N f = \sum_{-N}^{N} \widehat{f}(n)\chi_n \to f$ in $L^p(\mathbb{T})$ norm as $N \to \infty$ for $f \in L^1 \cap L^p(\mathbb{T})$, $1 < p < \infty$. But this is not true for $p = 1, \infty$. Furthermore, one could also ask if $\underset{N\to\infty}{lim} S_N f \to f$ almost everywhere if $f \in L^1 \cap L^p(\mathbb{T})$. This problem turned out to be much more difficult. In 1926, A. Kolmogorov constructed an integrable function that diverges at every point, and hence a.e convergence is not true for $p = 1$. The a.e convergence was established by the well celebrated theorems of L. Carleson (1965, $p = 2$) and R. Hunt(1967, $1 < p < \infty$). Until the result by Carleson, the answer was unknown even for a continuous function. A slightly easier problem is to study the convergence of the Abel sums $A_r f = \sum_{-\infty}^{\infty} r^{|n|}\widehat{f}(n)\chi_n$ as $r \to 1^-$. The Abel sums converge in both L^p norm, $1 \le p < \infty$ and almost everywhere if $f \in L^1 \cap L^p(\mathbb{T})$ (see [**JD**] for further details).

The Heisenberg group has its origins in quantum mechanics: it is a group of unitary operators arising from the "quantization" of "momentum" and "position" operators. It is is extensively studied because it comes up in different facets of mathematics like partial differential equations, harmonic analysis, sub-Riemannian geometry and so on.

The $2n+1$ dimensional Heisenberg group denoted by H^n is $\mathbb{C}^n \times \mathbb{R}$ with the group law given by

$$(z,t).(w,s) = (z - w, t - s + \frac{1}{2}\ Im\ z.\overline{w}), \quad \forall z, w \in \mathbb{C}^n \text{ and } t, s \in \mathbb{R}.$$

The sublaplacian $\mathcal{L}$ (see Defintion 1) on H^n is defined analogously to the laplacian Δ on the Euclidean space. However, the sublaplacian is not ellitpic unlike the laplacian. Nevertheless, by the seminal theorem of Hörmander ([**Ho**]) $\mathcal{L}$ is hypoellitpic, i.e. $f \in C^\infty(H^n)$ implies $\mathcal{L}f \in C^\infty(H^n)$. To read about the construction of fundamental solution for the sublaplacian refer to [**Fo2**].

In [**Str1**], R.S. Strichartz proposes a new notion of "harmonic analysis" on H^n, namely the joint spectral theory for $-\mathcal{L}$ and iT. In the setting of the results in [**Str1**] we intend to study spectral theory for the operator $(-\mathcal{L})(iT)^{-1}$ obtained from the joint functional calculus of $\mathcal{L}$ and $iT = i\frac{\partial}{\partial t}$. Once we have the spectral projections we can study about the convergence of the spectral resolution, which we study in detail in this thesis. One of the reasons why $(-\mathcal{L})(iT)^{-1}$ is an interesting operator to study is because it has a discrete spectrum. Moreover, it turns out that the spectral theory for $(-\mathcal{L})(iT)^{-1}$ is also equivalent to studying the representation theory for the Heisenberg motion group HM_n. (However, we do not discuss Heisenberg motion group in this thesis. Please refer to [**Str1**] for more details.

Plan of the thesis: In Chapter 1, we introduce the preliminaries required to study analysis on the Heisenberg group. In chapter 2, first we provide joint functional calculus of $-\mathcal{L}$ and iT and then compute the spectral projections of the operator $(-\mathcal{L})(iT)^{-1}$ denoted by $P_{k,\epsilon}f$ (see Section 2.1). Corresponding to the Fourier series for the circle group, we wish to obtain a series expansion of $f \in L^p(H^n)$ using the spectral projections $P_{k,\epsilon}$ in the form

$$f = \sum_{\epsilon} \sum_{k=0}^{\infty} \left(P_{k,1} + P_{k,-1}\right) f.$$

Once again, we have to determine in what sense does the above equation hold. This is what we do in the rest of Chapter 2. The following result is proven by R.S. Strichartz in [**Str1**]: for a function $f \in L^2(H^n)$ the partial sums $\sum_{k=0}^{N} \left(P_{k,1} + P_{k,-1}\right) f \to f$ in $L^2(H^n)$ norm as $N \to \infty$ and for $f \in L^p(H^n), 1 < p < \infty$ the Abel sums $\sum_\epsilon \sum_{k=0}^{\infty} r^k \left(P_{k,1} + P_{k,-1}\right) f$ in the $L^p(H^n)$ norm as $r \to 1^-$.

In Chapter 3, we prove that the Abel sums of the spectral projections of $f \in L^p(H^n)$ converges to the function f almost everywhere for $1 < p < \infty$ and hence an extension of the L^p spectral theory proven in [**Str1**]. We prove our result by establishing the L^p boundedness of the Littlewood-Paley g-function for the heat semi-group of $(-\mathcal{L})(iT)^{-1}$.

Future research

Using the L^p boundedness of the Littlewood-Paley g-funtion we have established in Chapter 3, we intend to prove a multiplier theorem for $(-\mathcal{L})(iT)^{-1}$. In [**MRS**], Müller, Ricci and Stein has already proven a version of muliplier theorem for $(-\mathcal{L})(iT)^{-1}$. However, we intend to do it using Littlewood-Paley g-function for the heat semi-group.

Chapter 1

Preliminaries

1.1 Introduction to the Heisenberg Group

The Harmonic analysis on the Euclidean space revolves around the Euclidean Fourier transform, correspondingly, the Heisenberg group is also assigned a notion of “Fourier transform”, which we call as group Fourier transform. Before we formally define the Heisenberg group let us define the Euclidean Fourier transform and certain other unitary operators on $L^2(\mathbb{R}^n)$. Let $f \in L^1 \cap L^2(\mathbb{R}^n)$. Then, we define the Fourier transform of f as

$$\mathcal{F}(f)(\xi) = \int_{\mathbb{R}^n} f(x)\, e^{-2\pi i x.\xi} dx. \tag{1.1}$$

Using the Plancherel theorem, the Fourier transform $\mathcal{F}$ extends as a unitary operator from $\mathcal{F} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$. Next, observe that the operators given by

$$e(x) f(\xi) = e^{-2\pi i x.\xi} f(\xi), \quad \tau(y) f(\xi) = f(\xi + y) \tag{1.2}$$

are also unitary operators in $L^2(\mathbb{R}^n)$ for $x, y \in \mathbb{R}^n$. Moreover, observe that $\{e(x) : x \in \mathbb{R}^n\}$ and $\tau(y) : y \in \mathbb{R}^n$ are groups under composition of unitary operators isomorphic to the group $(\mathbb{R}^n, +)$. The classical Fourier transform given by (1.1) intertwines these two groups, i.e $\mathcal{F}\tau(y)\mathcal{F}^{-1} = e(y)$.

More importantly, the operators $e(x)$ and $\tau(y)$ are the unitary operators motivated by the position and momentum operators in quantum mechanics. Let Q_j, D_j, $j = 1, ..., n$ be the unbounded operators defined on suitable domains by

$$Q_j f(\xi) = \xi_j f(\xi), \quad D_j f(\xi) = -i\frac{\partial}{\partial \xi_j} f(\xi). \tag{1.3}$$

For every $x, y \in \mathbb{R}^n$ we define

$$x.Q = \sum_{j=1}^{n} x_j.Q_j, \quad y.D = \sum_{j=1}^{n} y_j.D_j. \tag{1.4}$$

The operators $ix.Q$ and $iy.D$ are densely defined on $L^2(\mathbb{R}^n)$ and they are skew Hermitian. Also, observe that

$$e\left(x\right)=exp\left(-ix.Q\right),\tau\left(y\right)=exp\left(iy.D\right).$$

Hence by the functional calculus for these operators we have that $e\left(x\right)$ and $\tau\left(y\right)$ are unitary. The operators Q_j and D_j do not commute and $[Q_j, D_j] = iI, j = 1, ..., n$, where I is the identity operator. All other commutators are zero. On the level of the groups $\{e\left(x\right) : x \in \mathbb{R}^n\}$ and $\{\tau\left(y\right) =: y \in \mathbb{R}^n\}$ the commutation relation takes the form

$$e\left(x\right)\tau\left(y\right)=e^{-ix.y}\tau\left(y\right)e\left(x\right). \tag{1.5}$$

Note that,

$$e\left(x\right)\tau\left(y\right)e\left(u\right)\tau\left(v\right)=e^{-iy.u}e\left(x\right)e\left(u\right)\tau\left(y\right)\tau\left(v\right)=e^{-iy.u}e\left(x+u\right)\tau\left(y+v\right).$$

So, the set $\{e\left(x\right)\tau\left(y\right) : x, y \in \mathbb{R}^n\}$ is not closed under multiplication. Hence we introduce another operator $\chi\left(t\right)$ to complete the group structure. Define

$$\chi\left(t\right)f\left(\xi\right)=e^{it}f\left(\xi\right), \quad t\in\mathbb{R}. \tag{1.6}$$

It can be easily seen that $\chi(t) = exp(it\ Id)$, where Id is the identity operator on $L^2(\mathbb{R}^n)$. Moreover, $\chi\left(t\right)$ is unitary and also commutes with $e\left(x\right)$ and $\tau\left(y\right)$. Now, the set

$$H^n_{pol}=\{e\left(x\right)\tau\left(y\right)\chi\left(t\right);x,y\in\mathbb{R}^n,t\in\mathbb{R}\} \tag{1.7}$$

becomes a group under composition as

$$\begin{aligned}e\left(x\right)\tau\left(y\right)\chi\left(t\right)e\left(u\right)\tau\left(v\right)\chi\left(s\right)&=e\left(x\right)\tau\left(y\right)e\left(u\right)\tau\left(v\right)\chi\left(t\right)\chi\left(s\right)\\&=e\left(x\right)e\left(u\right)\tau\left(u\right)\tau\left(v\right)\chi\left(u.y\right)\chi\left(t\right)\chi\left(s\right)\\&=e\left(x+u\right)\tau\left(y+v\right)\chi\left(t+s-u.y\right)\end{aligned}.$$

This group is also called the polarized Heisenberg group. As one can see that this group is not symmetric in all the variables, we introduce a slightly modified group given by

$$\{\rho\left(x,y,t\right):x,y\in\mathbb{R}^n,t\in\mathbb{R}\}, \tag{1.8}$$

where

$$\rho\left(x,y\right)=e^{-\frac{i}{2}x.y}e\left(x\right)\tau\left(y\right), \quad \rho\left(x,y,t\right)=\chi\left(t\right)\rho\left(x,y\right). \tag{1.9}$$

The group law then becomes

$$\rho\left(x,y,t\right)\rho\left(u,v,s\right)=\chi\left(t\right)\rho\left(x,y\right)\chi\left(s\right)\rho\left(u,v\right)=\chi\left(t+s\right)\rho\left(x,y\right)\rho\left(u,v\right)$$
$$=e^{-\frac{i}{2}(x.y+u.v+u.y)}e\left(x+u\right)\chi\left(t+s\right)\tau\left(y+v\right)=\rho\left(x+u,y+v,t+s-\frac{1}{2}\left(u.y-v.x\right)\right).$$

Hence we can make $\mathbb{C}^n \times \mathbb{R}$ into a group by defining

$$(z,t) \circ (z',t') = \left(z+z', t+t' - \frac{1}{2} Im z.\overline{z}'\right),$$

which we call as the $(2n+1)$ dimensional Heisenberg group H^n. Observe that H^n is also a simply connected Lie group. The Haar measure on H^n is the product measure $dz\ dt$, where dz and dt are the Lebesgue measures on $\mathbb{C}^n$ and $\mathbb{R}$ respectively. The vector fields

$$\begin{aligned} X_j &= \frac{\partial}{\partial x_j} - \frac{1}{2} y_j \frac{\partial}{\partial t}, & j &= 1, ..., n \\ Y_j &= \frac{\partial}{\partial y_j} + \frac{1}{2} x_j \frac{\partial}{\partial t}, & j &= 1, ..., n \\ T &= \frac{\partial}{\partial t}, & & \end{aligned} \tag{1.10}$$

forms a basis for the Lie algebra of left-invariant vector fields on H^n. The only non-trivial commutation relation for these vector fields are

$$[X_j, Y_j] = T \quad j = 1, ..., n. \tag{1.11}$$

Recall that, a Lie algebra $\mathfrak{g}$ the lower central series given by

$$\mathfrak{g}_0 \geq \mathfrak{g}_1 \geq \mathfrak{g}_2 \geq ...., \tag{1.12}$$

where we inductively define $\mathfrak{g}_0 = \mathfrak{g}$ and $g_n = [\mathfrak{g}, \mathfrak{g}_{n-1}]$. Now, $\mathfrak{g}$ is said to an n-step nilpotent Lie algebra if n is the smallest natural number such that $\mathfrak{g}_n = 0$. In the case of Heisenberg group, using the commutation relations (1.10) we get that H^n is a 2-step nilpotent Lie group.

Next, we define the sublaplacian for H^n, which is a formally self-adjoint densely defined operator on H^n. To read more about the sublaplacian the reader is referred to ([**Tv2**], Chapter 2).

Definition 1. *The sublaplacian for H^n is defined as*

$$\mathcal{L} = \sum_{i=1}^{n} \left(X_j^2 + Y_j^2\right). \tag{1.13}$$

1.2 The Schrödinger Representation

The representation theory of Heisenberg group is completely understood with the help of Stone-von Neumann theorem. Here, we will define the Schrödinger representation for the Heisenberg group [**Str1**].

Let us first recall a few definitions in representation theory. Let G be a topological group and $\mathcal{H}$ be a Hilbert space. Denote $\mathcal{U}(\mathcal{H})$ the unitary operators acting on $\mathcal{H}$.

A unitary representation of G into the $\mathcal{U}(\mathcal{H})$ is a continuous group homomorphism $\pi : G \to \mathcal{U}(H)$. The representation π is said to be a strongly continuous unitary representation of G on $\mathcal{H}$, if for every $x \in \mathcal{H}$, the map $g \to \pi(g)x$ is continuous. A subspace M of $\mathcal{H}$ is said to be invariant under π if $\pi(g)x$ belongs to M for all $g \in G$ whenever x belongs to M. A unitary representation π is said to be irreducible if there is no nontrivial closed subspace of M that is invariant under π. Two representations π and ρ are said to be unitarily equivalent if there is $T \in \mathcal{U}(\mathcal{H})$ such that $\rho(g) = T\pi(g)T^*$ for all g.

Definition 2. *The Schrödinger representation is the group homomorphism $\pi_{\epsilon\lambda}$: $H^n \to \mathcal{U}(L^2(\mathbb{R}^n))$ given by*

$$\pi_{\epsilon\lambda}(x,y,t)\,u(\xi) = e^{-i\epsilon\left(\lambda t+\sqrt{\lambda}x.\xi+\lambda x.y/2\right)}u\left(\xi+\sqrt{\lambda}y\right), \tag{1.14}$$

for $\epsilon \in \{-1,1\}$, $\lambda \in \mathbb{R}_+$ and $u \in L^2(\mathbb{R}^n)$.

We will now prove the irreducibility of $\pi_{\epsilon\lambda}$, $\forall\lambda \in \mathbb{R}_+$. We will do this for the case $\epsilon\lambda = 1$; the general case will follow similarly. In order to do this we use the Plancherel theorem for Euclidean Fourier transform. Let us write $\pi_1(x,y,t) = e^{it}\pi(z) = e^{it}\pi(x,y,0)$, where $z = x+iy$ and

$$\pi(z)\,\phi(\xi) = e^{-i\left(x\xi+\frac{1}{2}x.y\right)}\phi(\xi+y).$$

Suppose $M \subset L^2(\mathbb{R}^n)$ is invariant under all $\pi_1(x,y,t)$. If $M \neq \{0\}$ we will show that $M = L^2(\mathbb{R}^n)$ proving the irreducibility of π_1.

If M is a proper subspace of $L^2(\mathbb{R}^n)$ invariant under $\pi_1(x,y,t)$ for all (x,y,t), then if we choose non-trivial functions $f \in M$ and $g \in M^\perp$(the orthogonal complement of M in $L^2(\mathbb{R}^n)$) we have $\pi(z)f$ is perpendicular to g for all z, i.e $\langle\pi(z)f, g\rangle = 0$ for all $z \in \mathbb{C}^n$.

Now, given $\phi, \psi \in L^2(\mathbb{R}^n)$, consider the function

$$V_\phi(\psi,z) = (2\pi)^{-n/2}\langle\pi(z)\,\phi,\psi\rangle. \tag{1.15}$$

This is called the Fourier-Wigner transform of ϕ and ψ. Let us calculate the $L^2(\mathbb{C}^n)$ norm of this function. Explicitly,

$$V_\phi(\psi,z) = (2\pi)^{-n/2}\int_{\mathbb{R}^n} e^{-ix\xi}\phi\left(\psi+\frac{y}{2}\right)\overline{\psi}\left(\xi-\frac{y}{2}\right)\,d\xi\,dy.$$

Applying Plancherel theorem for Fourier transform in x variable, we get

$$\int_{\mathbb{C}^n}|V_\phi(\psi,z)|^2\,dz = \int_{\mathbb{R}^{2n}}\left|\phi\left(\xi+\frac{y}{2}\right)\right|^2\left|\overline{\phi}\left(\xi-\frac{y}{2}\right)\right|^2\,d\xi\,dy.$$

Then by making the change of variable $\xi \to \xi+\frac{y}{2}$ becomes

$$\int_{\mathbb{R}^n}|\phi(\xi)|^2\,d\xi\int_{\mathbb{R}^n}|\overline{\psi}(\xi)|^2\,d\xi$$

Thus we have $||V_\phi(\psi)||_2 = ||\phi||_2||\psi||_2$. Under the assumption that M is non-trivial and proper, we have $V_f(g)(z) = 0, \ \forall z \in \mathbb{C}^n$, which means that $||f||_2||g||_2 = 0$. This is a contradiction as both f and g are non-trivial. Hence M has to be the whole of $L^2(\mathbb{R}^n)$ and therefore the irreducibility of π_1.

Remark 1. *Let $\phi, \psi, f, g \in L^2(\mathbb{R}^n)$. Then by polarization identity we have*

$$\langle V_\phi(\psi), V_f(g)\rangle = \langle \phi, f\rangle\langle \psi, g\rangle. \tag{1.16}$$

Next, we state the Stone-von Neumann theorem below, however we do not prove it here as it is out of the scope of this thesis. For the proof, the reader can the refer to ([**Fo1**], page 35).

Theorem 1. *(The Stone-von Neumann Theorem) Let π be a unitary representation of H^n on a separable Hilbert space $\mathcal{H}$, such that $\pi(0,0,t) = e^{-i\epsilon\lambda t}I$ for some $\lambda \in \mathbb{R}_+$ and $\epsilon \in \{-1,1\}$. Then $\mathcal{H} = \bigoplus \mathcal{H}_\alpha$, where the $\mathcal{H}_\alpha$'s are mutually orthogonal subspaces of $\mathcal{H}$, each invariant under π, such that $\pi|_{\mathcal{H}_\alpha}$ is unitarily equivalent to $\pi_{\epsilon\lambda}$ for each α, i.e. there exsits a Hilbert space isomorphism $\Phi_\pi : \mathcal{H} \to L^2(\mathbb{R}^n)$ such that*

$$\Phi_\pi \circ \pi \circ \Phi_\pi^{-1} = \pi_{\epsilon\lambda},$$

where $\pi_{\epsilon\lambda}$ is given by (1.14). In particular, if π is irreducible then π is unitarily equivalent to $\pi_{\epsilon\lambda}$.

Remark 2. *If the representation π is trivial on the centre $(0,0,t)$ then then the all the unitary irreducible representations on $L^2(\mathbb{R}^{2n})$ are of the form $\pi(x,y,t)f(\xi) = e^{-2\pi i\xi.(x,y)}f(\xi)$ for some $\xi \in \mathbb{R}^{2n}$.*

Next, we give a few definitions

Definition 3. *The non-isotropic dilation δ_r on the Heisenberg group is given by*

$$\delta_r(z,t) = (rz, r^2t). \tag{1.17}$$

Definition 4. *If f, g are functions on H^n, then their convolution is defined by*

$$f * g(z,t) = \int_{H^n} f((z,t).(-w,-s))\, g(w,s)\ dw\ ds \tag{1.18}$$

$$= \int_{H^n} f\left(\left(z-w, t-s+\frac{1}{2}Im\ z.\overline{w}\right)\right) g(w,s)\ dw\ ds \quad . \tag{1.19}$$

1.3 The Fourier and Weyl transforms

It can be easily seen that the Lebesgue measure $dz\ dt$ of $\mathbb{C}^n \times \mathbb{R}$ is the Haar measure of H^n. With this measure, we form the usual function space $L^p(H^n)$.

First we define the Fourier transform for integrable functions f. For each $\epsilon \in \{-1,1\}$ and $\lambda \in \mathbb{R}_+$, $\widehat{f}(\pi_{\epsilon\lambda})$ is the operator acting on $L^2(\mathbb{R}^n)$ given by

$$\widehat{f}(\pi_{\epsilon\lambda})\phi = \int_{H^n} f(z,t)\,\pi^*_{\epsilon\lambda}(z,t)\,\phi\ dz\ dt. \tag{1.20}$$

For the sake of easiness we will denote $\widehat{f}(\epsilon\lambda)$ instead of $\widehat{f}(\pi_{\epsilon\lambda})$ by abuse of notation. The integral defined above is defined as explained for vector valued integrals in the Section in the Appendix. If ψ is another function in $L^2(\mathbb{R}^n)$, then

$$\langle \widehat{f}(\epsilon\lambda)\phi, \psi\rangle = \int_{H^n} f(z,t)\langle \pi^*_{\epsilon\lambda}(z,t)\phi,\psi\rangle dz\ dt.$$

Since $\pi_{\epsilon\lambda}$ are unitary operators, it follows that

$$\langle \pi^*_{\epsilon\lambda}(z,t)\phi,\psi\rangle \leq ||\phi||_2||\psi||_2,$$

and consequently

$$|\langle \widehat{f}(\epsilon\lambda)\phi,\psi\rangle| \leq ||\psi||_2||\psi||_2||f||_1.$$

Hence by Riesz representation theorem, the operator $\widehat{f}(\pi_{\epsilon\lambda})$ is a bounded operator on $L^2(\mathbb{R}^n)$ and the operator norm satisfies $||\widehat{f}(\epsilon\lambda)|| \leq ||f||_1$. We will also show that when f is also in $L^2(H^n)$, $\widehat{f}(\epsilon\lambda)$ is a Hilbert-Schmidt operator (see 4.1 for the definition of a Hilbert-Scmidt operator).

Let us write $\pi_{\epsilon\lambda}(z,t) = e^{-i\epsilon\lambda t}\pi_{\epsilon\lambda}(z)$, where $\pi_{\epsilon\lambda}(z) = \pi_{\epsilon\lambda}(z,0)$ and define

$$f^{\epsilon\lambda}(z) = \int_{-\infty}^{\infty} e^{i\epsilon\lambda t} f(z,t)\ dt$$

to be the inverse Fourier transform of f in t variable (slightly different from our notion of classical Fourier transform). Then it follows that

$$\widehat{f}(\epsilon\lambda)\phi = \int_{\mathbb{C}^n} f^{\epsilon\lambda}(z)\pi^*_{\epsilon\lambda}(z)\phi\ dz.$$

Thus we are led to consider operators of the form

$$W_{\epsilon\lambda}(g) = \int_{\mathbb{C}^n} g(z)\pi^*_{\epsilon\lambda}(z)\ dz \tag{1.21}$$

for functions on $\mathbb{C}^n$. When $\epsilon\lambda = 1$, we call it the Weyl transform and denote it by $W(g)$. We also write $\pi(z)$ in place of $\pi_1(z)$. Thus

$$W(g)\phi(\xi) = \int_{\mathbb{C}^n} g(z)\pi^*(z)\phi(\xi)\ dz.$$

From the explicit description of the representation, it follows that

$$W(g)\phi(\xi) = \int_{\mathbb{R}^{2n}} g(x+iy)\, e^{i\left(x.\xi+\frac{1}{2}x.y\right)}\phi(\xi+y)\ dx\ dy.$$

Thus $W(g)$ is an integral operator with kernel $K_g(\xi,\eta)$ given by

$$K_g(\xi,\eta) = \int_{\mathbb{R}^n} g(x,\eta-\xi)\, e^{\frac{i}{2}x.(\xi+\eta)}\ dx$$

where $g(x,y)$ stands for $g(x+iy)$. Therefore , if $g \in L^1 \cap L^2(\mathbb{C}^n)$, the kernel $K_g(\xi,\eta)$ belongs to $L^2(\mathbb{R}^{2n})$, and hence from the theory of integral operators, it follows that $W(g)$ is a Hilbert-Schmidt operator whose norm is given by

$$||W(g)||_{HS}^2 = \int_{\mathbb{R}^{2n}} |K_g(\xi,\eta)|^2 \, d\xi \, d\eta.$$

Using the explicit formula for the kernel and Plancherel theorem for the Fourier transform, we get

$$||W(g)||_{HS}^2 = (2\pi)^n \int_{\mathbb{R}^{2n}} |g(x,y)|^2 \, dx \, dy. \tag{1.22}$$

This is the Plancherel theorem for Weyl transform.

We can now establish Plancherel theorem for group Fourier transform of the Heisenberg group.

Theorem 2. *For $f \in L^2(H^n)$,*

$$||f||_{L^2(H^n)} = \sum_{\epsilon} \int_0^\infty ||\widehat{f}(\epsilon\lambda)||_{HS}^2 \, d\mu(\lambda). \tag{1.23}$$

Proof. First assume that $f \in L^1 \cap L^2(H^n)$. Using the similar steps as we had used to show (1.22), we can show that for $\lambda > 0$

$$||\widehat{f}(\epsilon\lambda)||^2 = (2\pi)^n \lambda^n \int_{\mathbb{C}^n} |f^{\epsilon\lambda}(z)|^2 dz,$$

by replacing $\pi(z)$ by $\pi_{\epsilon\lambda}(z)$. Now, sum over ϵ and multiply λ^n on both sides of the above equation, and then integrate with respect to $d\lambda$ to get

$$\begin{aligned} \sum_{\epsilon} \int_0^\infty ||\widehat{f}(\epsilon\lambda)||_{HS}^2 \, d\mu(\lambda) &= \sum_{\epsilon} (2\pi)^{-1} \int_0^\infty \int_{\mathbb{C}^n} ||f^{\epsilon\lambda}(z)||^2 dz \, d\lambda \\ &= (2\pi)^{-1} \int_{-\infty}^\infty \int_{\mathbb{C}^n} ||f^{\epsilon\lambda}(z)||^2 dz \, d\lambda. \end{aligned}$$

Then by Euclidean Plancherel theorem we get (1.24). As $L^1 \cap L^2(H^n)$ is dense in $L^2(H^n)$, it is clear from the equality of the norms that we can extend it to all $f \in L^2(H^n)$. □

The next lemma will provide an important formula to complete the inversion formula for the group Fourier transform, which will further give us that the group Fourier transform is an isomorphism.

Lemma 1. *If $f \in L^1(H^n)$, then*

$$tr\left(\widehat{f}(\pi_{\epsilon\lambda})\,\pi_{\epsilon\lambda}(z,t)\right) = (2\pi)^n \lambda^{-n} \int_{-\infty}^\infty f(z,t-s)\, e^{i\epsilon\lambda s} ds, \tag{1.24}$$

Proof. If $\phi \in L^2(\mathbb{R}^n)$, and $z = x + iy$ then $\widehat{f}(\pi_{\epsilon\lambda})\phi$ is given by

$$\begin{aligned}\widehat{f}(\pi_{\epsilon\lambda})\phi(\xi) &= \int_{H^n} f(z,t)\,\pi_{\epsilon\lambda}(-z,-t)\,\phi(\xi)\;dz\;dt \\ &= \int_{H^n} f(z,t)\,e^{i\epsilon\left(\lambda t+\sqrt{\lambda}x.\xi-\lambda x.y/2\right)}\phi\left(\xi-\sqrt{\lambda}y\right)\;dx\;dydt \\ &= \lambda^{-n}\int_{H^n} f(x,(\xi-\eta)/\lambda,t)\,e^{i\epsilon(\lambda t+x(\xi+\eta)/2)}\phi(\eta)\;d\eta dx\;dt \qquad .\end{aligned}$$

Thus, $\widehat{f}(\pi_{\epsilon\lambda})$ is an integral operator with kernel

$$K_f^{\epsilon\lambda}(\xi,\eta) = \lambda^{-n}\int_{-\infty}^{\infty}\int_{\mathbb{R}} f(x,(\xi-\eta)/\lambda,t)\,e^{i\epsilon(\lambda t+x(\xi+\eta)/2)}\;dx\;dt$$

Moreover,

$$\begin{aligned}\widehat{f}(\pi_{\epsilon\lambda})\,\pi_{\epsilon\lambda}(x,y,t) &= \int_{H^n} f(x',y',t')\,\pi_{\epsilon\lambda}(-x',-y',-t')\,\pi_{\epsilon\lambda}(x,y,t)\;dx'\;dy'\;dt' \\ &= \int_{H^n} f(x',y',t')\,\pi_{\epsilon\lambda}\left(x-x',y-y',t-t'-\frac{1}{2}(y'x-x'y)\right)\;dx'\;dy'\;dt' \\ &= \widehat{g}(\pi_{\epsilon\lambda})\end{aligned}$$

where

$$g(x',y',t') = f(x-x',y-y',t-t')\,e^{-\frac{i}{2}\epsilon\lambda(y'x-x'y)}.$$

Hence the integral kernel of $\widehat{f}(\pi_{\epsilon\lambda})\,\pi_{\epsilon\lambda}(x,y,t)$ is

$$\begin{aligned}&F(\xi,\eta) \\ &= \lambda^{-n}\int_{-\infty}^{\infty}\int_{\mathbb{R}^n} g(x',(\xi-\eta)/\lambda,t')\,e^{i\epsilon(\lambda t'+x'(\xi+\eta)/2)}dx'\;dt' \\ &= \lambda^{-n}\int_{-\infty}^{\infty}\int_{\mathbb{R}^n} f(x-x',y-(\xi-\eta)/\lambda,t-t')\,e^{-\frac{i}{2}\epsilon\lambda(x(\xi-\eta)/2-x'y)}e^{i\epsilon(\lambda t'+x'(\xi+\eta)/2)}dx'\;dt' \\ &= \lambda^{-n}\int_{-\infty}^{\infty}\int_{\mathbb{R}^n} f(x',y-(\xi-\eta)/2,t')\,e^{-\frac{i}{2}\epsilon\lambda(x(\xi-\eta)/2-(x-x')y)}e^{i\epsilon(\lambda(t-t')+(x-x')(\xi+\eta)/2)}dx'\;dt'.\end{aligned}$$

We will assume that f is a Schwartz class function for now. Then, all the integrals converge nicely. For a general L^1 function, we take a sequence of Schwartz class functions converging to f in $L^1(H^n)$ and then dominated convergence theorem. For $f \in \mathcal{S}(H^n)$, let $\mathcal{F}_1 f(x,y,t)$ denote the n-dimensional Euclidean Fourier transform in the first coordinate of f. Now, we have

$$
\begin{aligned}
& tr\left(\widehat{f}\,(\,\pi_{\epsilon\lambda}\pi_{\epsilon\lambda}\,(x,y,t)\right.\\
&\qquad = \int_{\mathbb{R}^n} F\,(\xi,\xi)\;\;d\xi\\
&\qquad = \lambda^{-n}\int_{-\infty}^{\infty}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f\,(x',y,t')\,e^{\frac{i}{2}\epsilon\lambda(x-x')y}e^{i\epsilon\lambda(t-t')}e^{i\epsilon\lambda\xi(x-x')}\;\;dx'\;\;d\xi\;\;dt'\\
&\qquad = \lambda^{-n}e^{i\epsilon\lambda x.y/2}\int_{-\infty}^{\infty}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f\,(x',y,t')\,e^{-i\epsilon\lambda x'.\left(\frac{y}{2}+\xi\right)}e^{i\epsilon\lambda\xi.x}e^{i\epsilon\lambda(t-t')}\;\;dx'\;\;d\xi\;\;dt'\\
&\qquad = \lambda^{-n}\,(2\pi)^{n/2}\,e^{i\epsilon\lambda x.y/2}\int_{-\infty}^{\infty}\int_{\mathbb{R}^n}\mathcal{F}_1 f\left(\epsilon\lambda\left(\frac{y}{2}+\xi\right),y,t'\right)e^{i\epsilon\lambda\xi.x}e^{i\epsilon\lambda(t-t')}\;\;d\xi dt'\\
&\qquad = \lambda^{-n}\,(2\pi)^{n/2}\int_{-\infty}^{\infty}\int_{\mathbb{R}^n}\mathcal{F}_1 f\,(\xi,y,t')\,e^{i\xi.x}e^{i\epsilon\lambda(t-t')}\;\;d\xi dt'\\
&\qquad = \lambda^{-n}\,(2\pi)^{n}\int_{-\infty}^{\infty} f\,(x,y,t')\,e^{i\epsilon\lambda(t-t')}dt'\\
&\qquad = \lambda^{-n}\,(2\pi)^{n}\int_{-\infty}^{\infty} f\,(x,y,t-s)\,e^{i\epsilon\lambda s}\;\;ds.
\end{aligned}
$$

□

Using the above lemma we have

$$
\begin{aligned}
(2\pi)^{-n-1}\sum_{\epsilon}\int_0^{\infty} tr\left(\widehat{f}\,(\pi_{\epsilon\lambda})\,\pi_{\epsilon\lambda}\,(z,t)\right)\lambda^n d\lambda \qquad (1.25)\\
= \frac{1}{2\pi}\sum_{\epsilon}\int_0^{\infty}\int_{\mathbb{R}} f\,(z,t-s)\,e^{-i\epsilon\lambda(t-s)}e^{+i\epsilon\lambda t}\;\;ds\;\;d\lambda\\
= \frac{1}{2\pi}\int_{\mathbb{R}}\int_{\mathbb{R}} f\,(z,t-s)\,e^{-i\lambda(t-s)}e^{+i\lambda t}ds\;\;d\lambda\\
= f\,(z,t) \qquad (1.26)
\end{aligned}
$$

We obtain the above expression by the Fourier inversion formula in one dimension. For the interchange and well-definedness of all the integrals we take f to be a Schwartz class function in the above calculations.

Chapter 2

L^p Harmonic Analysis of $(-\mathcal{L})(iT)^{-1}$

In [**Str1**], R.S. Strichartz proposes a new notion of what Harmonic Analysis on the Heisenberg group should be; it is the joint spectral theory of the operators $-\mathcal{L}$ and $iT = i\frac{\partial}{\partial t}$. It is not difficult to see that the operator iT commutes with $-\mathcal{L}$. Moreover they are strongly commuting essentially self adjoint operators and hence they have a well defined joint spectrum (see [**Sm**], Chapter 5). One of the interesting operators that arise out of the joint functional calculus of $-\mathcal{L}$ and iT is the operator $(-\mathcal{L})(iT)^{-1}$, having a discrete spectrum. To better understand this operator we will compute its spectral projections and study the $L^p(H^n)$ boundedness of these spectral projections, for $1 < p < \infty$. We will also develop Plancherel theorem and spectral theorem for $L^p(H^n), 1 < p < \infty$.

2.1 L^2 spectrum

Let $u \in L^2(\mathbb{R}^n)$. Recall that the Schrödinger representation $(\pi_{\epsilon\lambda})$ is defined as

$$\pi_{\epsilon\lambda}(x,y,t)\,\phi(\xi) = e^{-i\epsilon\left(\lambda t+\sqrt{\lambda}x.\xi+\lambda x.y/2\right)}u\left(\xi+\sqrt{\lambda}y\right)$$

Let $u, v \in L^2(\mathbb{R}^n)$. Then the entry function for the unitary representation is defined as

$$\begin{aligned} E_{\epsilon\lambda}(u,v)(z,t) &= \langle \pi_{\epsilon\lambda}(z,t)\,u, v\rangle \\ &= e^{-i\epsilon\lambda t}e^{-i\epsilon\lambda x.y/2}\int_{\mathbb{R}^n} e^{-i\epsilon\sqrt{\lambda}x.\xi}u\left(\xi+\sqrt{\lambda}y\right)\overline{v(\xi)}\,d\xi \\ &= e^{-i\epsilon\lambda t}\int_{\mathbb{R}^n} e^{-i\epsilon\sqrt{\lambda}x.\xi}u\left(\xi+\frac{1}{2}\sqrt{\lambda}y\right)v\left(\xi-\frac{1}{2}\sqrt{\lambda}y\right)d\xi \qquad (2.1) \end{aligned}$$

Using $\pi_{\epsilon\lambda}$ we also get a representation of the Lie algebra of H^n given by

$$d\pi_{\epsilon\lambda}(V)\,u = \frac{d}{dt}\pi_{\epsilon\lambda}(exp\ tV)\,u|_{t=0}$$

for any left invariant vector field V and exp is the exponential map for the Lie group. The above representation of V is densely defined unbounded skew adjoint operator acting on $L^2(\mathbb{R}^n)$. Let $u \in C^\infty(\mathbb{R}^n)$, then

$$d\pi_{\epsilon\lambda}(X_j)\,\phi(\xi) = -i\epsilon\sqrt{\lambda}\xi_j\phi(\xi_j)$$
$$d\pi_{\epsilon\lambda}(Y_j)\,\phi(\xi) = \sqrt{\lambda}\frac{\partial}{\partial\xi_j}\phi(\xi)$$

Hence we get that

$$d\pi_{\epsilon\lambda}(\mathcal{L}) = d\pi_\lambda\left(\sum_{j=1}^n \left(X_j^2 + Y_j^2\right)\right) = -\lambda\left(-\Delta + |\xi|^2\right)$$

Next, we will give the spectral decomposition of the Hermite operator. The Hermite functions are defined as (see [**Tv1**])

$$h_\alpha(\xi) = \Pi_{j=1}^n h_{\alpha_j}(\xi_j)\,, \quad \xi \in \mathbb{R}^n$$

where

$$h_k(x) = (-1)^k\left(2^n\sqrt{\pi}k!\right)^{-\frac{1}{2}}\left(\frac{d^k}{dx^k}\{e^{-x^2}\}e^{x^2}\right)e^{-\frac{1}{2}x^2}, \quad x \in \mathbb{R}$$

We know that Hermite functions are the eigenfunctions of the Hermite operator with eigenvalue $2(|\alpha| + n)$. Hence it is easy to see that they are eigenfunctions of the Schrödinger representation of $\mathcal{L}$ with eigenvalues $-(2|\alpha| + n)\,\lambda$. Moreover, recall that the Hermite functions form an orthonormal dense subset of $L^2(\mathbb{R}^n)$. Now observe the following

$$\begin{aligned}
X_j\langle\pi_{\epsilon\lambda}(z,t)\,u, v\rangle &= \langle X_j\pi_{\epsilon\lambda}(z,t)\,u, v\rangle \\
&= \langle -i\epsilon\sqrt{\lambda}\left(\xi_j + \sqrt{\lambda}y_j\right)e^{-i\epsilon\left(\lambda t + \sqrt{\lambda}x.\xi + \lambda x.y/2\right)}u\left(\xi + \sqrt{\lambda}y\right), v\rangle \\
&= \langle\pi_{\epsilon\lambda}(z,t)\,(d\pi_{\epsilon\lambda})\,(X_j)\,u, v\rangle
\end{aligned}$$

Note that one could also have simply applied the fact that X_j is a left invariant to get the above equality. Similar calculation for Y_j also shows that

$$Y_j\langle\pi_{\epsilon\lambda}(z,t)\,u, v\rangle = \langle\pi_{\epsilon\lambda}(z,t)\,(d\pi_{\epsilon\lambda})\,(Y_j)\,u, v\rangle$$

Since every left invariant differential operator is a polynomial of the left invariant vector fields ([**Va**], Chapter 2), we get the above equation for any left invariant differential operator also. Observe that $\langle\pi_{\epsilon\lambda}(z,t)\,u, v\rangle$ is the entry function (matrix coefficient) $E_{\epsilon\lambda}$. Hence if we take $u = h_\alpha$ and $v = h_\beta$ for some multi index α and β, we get that

$$\mathcal{L}E_{\epsilon\lambda}(h_\alpha, h_\beta) = -\lambda\,(2|\alpha| + n)\,E_{\epsilon\lambda}(h_\alpha, h_\beta)$$

From the explicit form of $E_{\epsilon\lambda}$ (2.1) it is easy to see that they are eigenfunctions of the operator iT with eigenvalue $\epsilon\lambda$. Observe that to explicitly compute the joint spectrum we did not really need the abstract theory of strongly commuting operators. Now for the ease of computation we will scale the eigenfunction. The joint spectrum is the closed subset of $\mathbb{R}^2$ which we will refer to as the Heisenberg fan, consisting of union of rays.

$$R_{k,\epsilon} = \{(\lambda, \tau) : \tau = \frac{\epsilon\lambda}{n+2k}, \lambda > 0\} \tag{2.2}$$

for $\epsilon = \pm 1,\ k = 0, 1, 2, ...$ and the limit ray

$$R_\infty = \{(\lambda, \tau) : \tau = 0, \lambda \geq 0\}$$

Now, if $\{h_\alpha\}$ be the Hermite functions, then

$$\begin{aligned} tr\left(\widehat{f}(\pi_{\epsilon\lambda})\,\pi_{\epsilon\lambda}(g)\right) &= \sum_\alpha \int_{H^n} f(g')\,\langle \pi_{\epsilon\lambda}(g)\,h_\alpha, \pi_{\epsilon\lambda}(g')\,h_\alpha\rangle dg' \\ &= \sum_\alpha \sum_\beta \int_{H^n} f(g')\,\langle \pi_{\epsilon\lambda}(g)\,h_\alpha, h_\beta\rangle \overline{\langle \pi_{\epsilon\lambda}(g')\,h_\alpha, h_\beta\rangle} dg' \\ &= \sum_\alpha \sum_\beta \langle f, E_{\epsilon\lambda}(h_\alpha, h_\beta)\rangle E_{\epsilon\lambda}(h_\alpha, h_\beta) \end{aligned}$$

Now we can use the explicit form of the trace function in the Fourier inversion formula for the Heisenberg group Fourier transform of f to get

$$f = (2\pi)^{-n-1} \sum_\epsilon \int_0^\infty \sum_\alpha \sum_\beta \langle f, E_{\epsilon\lambda}(h_\alpha, h_\beta)\rangle E_{\epsilon\lambda}(h_\alpha, h_\beta)\,\lambda^n d\lambda$$

We invariably use $\langle . \rangle$ to represent the inner product on $L^2(H^n)$ and $L^2(\mathbb{R}^n)$ and hence they should be interpreted according to the context. From here on, we write $\epsilon\lambda/(n+2k)$ as $\lambda_{\epsilon,k}$ by abuse of notation. Now we can group together the terms with $|\alpha| = k$ in the above equation and make the change of variable $\lambda \to \lambda/(n+2k)$ to get

$$f = (2\pi)^{-n-1} \sum_{k=0}^\infty \sum_\epsilon \int_0^\infty (n+2k)^{-n-1} \times \sum_\beta \sum_{|\alpha|=k} \langle f, E_{\lambda_{\epsilon,k}}(h_\alpha, h_\beta)\rangle E_{\lambda_{\epsilon,k}}(h_\alpha, h_\beta)\,\lambda^n d\lambda. \tag{2.3}$$

Hence we get the joint spectral decomposition of f written as an integral over the Heisenberg fan of the joint eigenfunctions $E_{\epsilon\lambda}(h_\alpha, h_\beta)$. Next, we intend to explicitly compute the spectral projections denoted by $P_{k,\epsilon}$ such that

$$(-\mathcal{L})(iT)^{-1}(P_{k,\epsilon}f) = (n+2k)\,P_{k,\epsilon}f, \quad \forall f \in \mathcal{S}(H^n). \tag{2.4}$$

Observe that

$$\begin{aligned}
&\sum_{\beta}\langle f, E_{\epsilon\lambda}(h_\alpha, u_\beta)\rangle E_{\lambda_{\epsilon,k}}(h_\alpha, h_\beta)\\
&= \sum_{\beta}\int_{H^n} f(g')\langle \pi_{\lambda_{\epsilon,k}}(.)h_\alpha, h_\beta\rangle \overline{\langle \pi_{\lambda_{\epsilon,k}}(g')h_\alpha, h_\beta\rangle} dg'\\
&= \int_{H^n} f(g')\langle \pi_{\lambda_{\epsilon,k}}(.)h_\alpha, \pi_{\lambda_{\epsilon,k}}(g')h_\alpha\rangle dg'\\
&= \int_{H^n} f(g')\langle \left(\pi_{\lambda_{\epsilon,k}}(g')\right)^* \pi_{\lambda_{\epsilon,k}}(.)h_\alpha, h_\alpha\rangle dg'\\
&= f * E_{\lambda_{\epsilon,k}}(h_\alpha, h_\beta)
\end{aligned}$$

By the properties of Laguerre polynomials (see [**Fo1**], page 64-65) we have

$$E_{\lambda_{\epsilon,k}}(h_\alpha, h_\alpha)(z,t) = exp\left(-\frac{i\epsilon\lambda t}{n+2k}\right) exp\left(-\frac{\lambda|z|^2}{4(n+2k)}\right)\Pi_{j=1}^{n} L_{\alpha_j}^{0}\left(\frac{\lambda|z_j|^2}{2(n+2k)}\right) \tag{2.5}$$

By using the properties of Laguerre polynomials ([**Tv1**], Chapter 1) and summing over $|\alpha| = k$ we get

$$\sum_{|\alpha|=k}\prod_{j=1}^{n} L_{\alpha_j}^{0}\left(\frac{\lambda|z_j|^2}{2(n+2k)}\right) = L_k^{n-1}\left(\frac{\lambda|z|^2}{2(n+2k)}\right)$$

Substituting these values into the spectral resolution equation of f we get that

$$f = \sum_{k}\sum_{\epsilon}\int_0^\infty f * \phi_{\lambda,k,\epsilon} d\lambda, \tag{2.6}$$

where

$$\phi_{\lambda,k,\epsilon}(z,t) = \frac{\lambda^n}{(2\pi)^{n+1}(n+2k)^{n+1}} exp\left(-\frac{i\epsilon\lambda t}{n+2k}\right) exp\left(-\frac{\lambda|z|^2}{4(n+2k)}\right) L_k^{n-1}\left(\frac{\lambda|z|^2}{2(n+2k)}\right) \tag{2.7}$$

Remark 3. *In the above calculations to find the spectral resolution of the sublaplacian we have taken f to be a Schwartz class function and hence all the equations hold point-wise. Once we prove the Plancherel theorem, we can extend to any function in $L^2(H^n)$.*

Plancherel Formula

The Plancherel formula for $(-\mathcal{L})(iT)^{-1}$ is slightly different from the group Placherel formula as it is not an immediate consequence of (2.6). Observe that we have

$$||f||_2^2 = \sum_{k=0}^{\infty}\sum_{\epsilon}\int_0^\infty (2\pi)^{-n-1}(n+2k)^{-n-1} \times \sum_{\beta}\sum_{|\alpha|=k} |\langle f, E_{\lambda_{\epsilon,k}}(h_\alpha, h_\beta)\rangle|^2 \lambda^n d\lambda, \tag{2.8}$$

where we have taken inner product with f on both sides of

$$f = \sum_{k=0}^{\infty}\sum_{\epsilon}\int_0^{\infty}(2\pi)^{-n-1}(n+2k)^{-n-1}\times\sum_{\beta}\sum_{\alpha}\langle f, E_{\lambda_{\epsilon,k}}(h_\alpha,h_\beta)\rangle E_{\lambda_{\epsilon,k}}(h_\alpha,h_\beta)\,\lambda^n d\lambda. \tag{2.9}$$

We also know that

$$f*\phi_{\lambda,k,\epsilon} = (2\pi)^{-n-1}(n+2k)^{-n-1}\times\sum_{\beta}\sum_{|\alpha|=k}\langle f, E_{\lambda_{\epsilon,k}}(h_\alpha,h_\beta)\rangle E_{\lambda_{\epsilon,k}}(h_\alpha,h_\beta). \tag{2.10}$$

We would like to simplify the right side of (2.8) further. A glance at (2.1) shows that $|E_{\lambda_{\epsilon,k}}(h_\alpha,h_\beta)|$ is independent of t and rapidly decreasing $|z|$, so that $\int_{\mathbb{C}^n} f(z,0)\overline{g(z,0)}dz$ defines an inner product on the linear combinations of the matrix coefficients of the Schrödinger representation. Moreover, using the orthonormality of $E_{\epsilon\lambda}(h_\alpha,h_\beta)$ that we get from (1.16) we have

$$\int_{\mathbb{C}^n} E_{\lambda_{\epsilon,k}}(h_\alpha,u)(z,0)\,\overline{E_{\lambda_{\epsilon,k}}(h_{\alpha'},v)(z,0)}\,dz = a(k,\epsilon\lambda)\langle u,v\rangle\delta_{\alpha,\alpha'}\ , \tag{2.11}$$

for $|\alpha| = |\alpha'| = k$ for some constant $a(k,\epsilon\lambda)$. Thus to obtain our Plancherel formula we need to compute the constant $a(k,\epsilon\lambda)$, and use it in (2.10).

Theorem 3. *For $f \in L^2(H^n)$ we have*

$$||f||_2^2 = 2\pi\sum_{k=0}^{\infty}\sum_{\epsilon}(n+2k)\int_0^{\infty}\int_{\mathbb{C}^n}|f*\phi_{\lambda,k,\epsilon}(z,0)|^2\,dz\,d\lambda. \tag{2.12}$$

Proof. To compute the constant $a(k,\epsilon\lambda)$ in (2.11) we choose $\alpha=\alpha'$ and $u=v=h_\alpha$, so that $\langle u,v\rangle = 1$ and by (2.5) we have

$$a(k,\epsilon\lambda) = \int_{\mathbb{C}^n} exp\left(\frac{-\lambda|z|^2}{2(n+2k)}\right)\prod_{j=1}^{n}\left|L_{\alpha_i}^0\left(\frac{\lambda|z_j|^2}{2(n+2k)}\right)\right|^2 dz.$$

The integral breaks up into a product of integrals over $\mathbb{C}$, which we evaluate directly in polar coordinates using orthogonality formula for Laguerre polynomial (see [**Tv1**], page 7)

$$\int_0^{\infty} e^{-s}L_k^0(s)^2\,ds = 1$$

to obtain

$$\int_{\mathbb{C}^n} exp\left(\frac{-\lambda|z|^2}{2(n+2k)}\right)\prod_{j=1}^{n}\left|L_{\alpha_i}^0\left(\frac{\lambda|z_j|^2}{2(n+2k)}\right)\right|^2 dz$$
$$=2\pi\int_0^{\infty} exp\left(\frac{-\lambda r^2}{2(n+2k)}\right)\prod_{j=1}^{n}\left|L_{\alpha_i}^0\left(\frac{\lambda r^2}{2(n+2k)}\right)\right|^2 r\,dr = \frac{2\pi(n+2k)}{\lambda}.$$

hence

$$a(k,\epsilon\lambda) = \left(\frac{2\pi(n+2k)}{\lambda}\right)^n. \tag{2.13}$$

Now applying (2.11) and (2.13) to (2.10) we get

$$\int_{\mathbb{C}^n} |f * \phi_{\lambda,k,\epsilon}(z,0)|^2 \, dz = (2\pi)^{-n-2}(n+2k)^{-n-2}\lambda^n \sum_\beta \sum_{|\alpha|=k} |\langle f, E_{\lambda_\epsilon,k}(h_\alpha,h_\beta)\rangle|^2, \tag{2.14}$$

which then easily gives the required result. □

Spectral Resolution of $(-\mathcal{L})(iT)^{-1}$

Next we compute the spectral projection operator associated with the ray $R_{k,\epsilon}$ of the Heisenberg fan. This is just the operator

$$f \mapsto \int_0^\infty f * \phi_{\lambda,k,\epsilon} \, d\lambda.$$

Using Calderon-Zygmund theory we will later show that we can interchange the convolution with the integrals for L^2 functions in the L^2 sense. Hence it amounts to find the kernel $\int_0^\infty \phi_{\lambda,k,\epsilon}\, d\lambda$. From the generating function for the Laguerre polynomials ([**Tv1**], page 8) we have

$$\sum_{k=0}^\infty r^k L_k^\alpha(x) = (1-r)^{-\alpha-1} e^{-rx/(1-r)}. \tag{2.15}$$

To find the explicit form of the kernel $\int_0^\infty \phi_{\lambda,k,\epsilon}\, d\lambda$ we first find an explicit expression for the Taylor series with the coefficients $\int_0^\infty \phi_{\lambda,k,\epsilon} d\lambda$ and then use (2.15). Using the definition (2.7) of $\phi_{\lambda,k,\epsilon}$. we get

$$\begin{aligned}\sum_{k=0}^\infty r^k \int_0^\infty \phi_{\lambda,k,\epsilon}(z,t)\, d\lambda &= (2\pi)^{-n-1}\int_0^\infty \lambda^n e^{-i\epsilon\lambda t} e^{-\lambda|z|^2/4} \sum_{k=0}^\infty r^k L_k^{n-1}\left(\frac{\lambda|z|^2}{2}\right) d\lambda \\ &= (2\pi)^{-n-1}(1-r)^{-n}\int_0^\infty \lambda^n e^{-\lambda\left(\left(|z|^2\right)/4((1+r)/(1-r))+i\epsilon t\right)} d\lambda \\ &= 2^{n-1}\pi^{-n-1} n!\,(1-r)\left(|z|^2+4i\epsilon t + r\left(|z|^2-4i\epsilon t\right)\right)^{-n-1}.\end{aligned} \tag{2.16}$$

Note that (2.15) allows us to interchange the sum with the integral using Fubini's theorem, for $z \neq 0$. Then by differentiating the above k times and dividing by $k!$ and putting $r=0$ we get

$$\begin{aligned}\int_0^\infty \phi_{\lambda,k,\epsilon}\, d\lambda = 2^{n-1}\pi^{-n-1}(-1)^k &\times \left[\frac{(n+k)!}{k!}\frac{\left(|z|^2-4i\epsilon t\right)^k}{\left(|z|^2+4i\epsilon t\right)^{n+1+k}}\right] \\ &\times \left[1+\left(\frac{k}{n+k}\right)\left(\frac{|z|^2+4i\epsilon t}{|z|^2-4i\epsilon t}\right)\right]\end{aligned} \tag{2.17}$$

which is clearly a homogeneous function of degree $-2n-2$ with respect to the non-isotropic dilations of H^n given by Definition 1.17. Let us denote the kernel in (2.17) by K. Now to establish that the kernel K is a Calderon-Zygmund kernel on the Heisenberg group (refer to Chapter XII of [**St3**] for a discussion on singular integrals on the Heisenberg group) we will show the cancellation property given by

$$\int_{\mathbb{C}^n} \left(K(z,1) + K(z,-1)\right) \, dz = 0. \tag{2.18}$$

Then the fact that K is homogeneous of degree $-2n-2$ will give us that

$$\int_{H^n} K(z,t) \, dz \, dt = 0.$$

Now, observe that to show (2.18) it is sufficient to show that

$$\int_{\mathbb{C}^n} \int_0^\infty \left(\phi_{\lambda,k,\epsilon}(z,1) + \phi_{\lambda,k,\epsilon}(z,-1)\right) d\lambda \, dz = 0.$$

By using the change of variable given by $|z|^2 = w$ we get

$$\begin{aligned}
&\int_{\mathbb{C}^n} \int_0^\infty \left(\phi_{\lambda,k,\epsilon}(z,1) + \phi_{\lambda,k,\epsilon}(z,-1)\right) d\lambda dz \\
&= c_{n,k} \int_{-\infty}^\infty \frac{(w-4i\epsilon)^k}{(w+4i\epsilon)^{n+1+k}} \left(1 + \left(\frac{k}{n+k}\right)\left(\frac{w+4i\epsilon}{w-4i\epsilon}\right)\right) w^{n-1} dw,
\end{aligned}$$

for some constant $c_{n,k}$ depending only on n and k. Observe that if we take w to be a complex number then the integrand has only one pole and that lies in the half space depending upon ϵ and is given by $\epsilon \, Im \, w < 0$ (The reason is that we have $k \geq 1$ and hence the term inside the bracket won't be a problem). Then consider the contour given by the intersection of the circle $|w| = R$ with the appropriate half space. This will give us

$$\int_0^{2\pi} \frac{1}{R^2} \frac{\left(e^{i\theta} - 4i\epsilon\right)^k}{\left(e^{i\theta} + 4i\epsilon/R\right)^{n+1+k}} \left(1 + \left(\frac{k}{n+k}\right)\left(\frac{e^{i\theta} + 4i\epsilon/R}{e^{i\theta} - 4i\epsilon/R}\right)\right) e^{i\theta(n-1)} \, d\theta. \tag{2.19}$$

Now $R \to \infty$ will give us that (2.19) is 0 and hence we have the required result. Let us denote the kernel $\int_0^\infty \phi_{\lambda,k,\epsilon} d\lambda$ by K. The kernel is also locally L^1 in $\mathbb{C}^n \times \mathbb{R}\backslash\{0\}$ (where 0 is the origin in $\mathbb{C}^n \times \mathbb{R}$). Hence we can associate a convolution operator to the kernel via principal value integral such as

$$\lim_{s \to 0^+} \int_{|t'|>s} \int_{\mathbb{C}^n} f\left((z,t) \circ (z',t')^{-1}\right) K(z',t') \, dz' \, dt'. \tag{2.20}$$

We could have also cut off any other compact neighbourhood of the origin. However this wouldn't matter much. Since the kernel is homogeneous of degree $-2n-2$ and hence as a distribution the kernel differs only by a multiple of the delta function. So, the operator differs only by a multiple of the identity operator. But in our case the operators are obtained more naturally by a limiting argument, and we will have to do some work to relate them to principle value integrals of the above form (2.20).

But as we have told before we cannot directly interchange the integral and the convolution and hence we have to introduce a summability factor. Observe the following

$$\int_0^\infty \phi_{\lambda,k,\epsilon}(z,t)\, exp\left(\frac{-s\lambda}{n+2k}\right) d\lambda = 2^{n-1}\pi^{-n-1}(-1)^k \left[\frac{(n+k)!}{k!}\frac{\left(|z|^2-s-4i\epsilon t\right)^k}{\left(|z|^2+s+4i\epsilon t\right)^{n+1+k}}\right.$$
$$\left.\times\left[1+\left(\frac{k}{n+k}\right)\left(\frac{|z|^2+s+4i\epsilon t}{|z|^2-s-4i\epsilon t}\right)\right]\right], \tag{2.21}$$

for $s>0$. The proof of the above equation follows similarly as we calculated $\int_0^\infty \phi_{\lambda,k,\epsilon}(z,t)\,d\lambda$ using the Taylor series. Even though the above kernel is no more homogeneous, it is locally integrable.

Proposition 1. *For $f\in L^2(H^n)$ we have*

$$\int_0^\infty f*\phi_{\lambda,k,\epsilon}(z,t)\, exp\left(\frac{-s\lambda}{n+2k}\right) d\lambda = f*\left(\int_0^\infty \phi_{\lambda,k,\epsilon}(z,t)\, exp\left(\frac{-s\lambda}{n+2k}\right) d\lambda\right), \tag{2.22}$$

as $L^2(H^n)$ functions.

Proof. Observe that the right hand side of (2.21) can be written as the sum of an L^1 and L^2 functions by multiplying with a cut off function supported in a neighbourhood of origin. Hence the right hand side of (2.22) makes sense. If $f\in C_c^\infty(H^n)$ then we can apply Fubini's theorem to interchange integrals to obtains (2.22) as $\phi_{\lambda,k,\epsilon}$ is locally integrable. Since, they agree on a dense class it suffices to prove that $f\mapsto\int_0^\infty f*\phi_{\lambda,k,\epsilon}(z,t)\, exp\left(\frac{-s\lambda}{n+2k}\right)d\lambda$ is a bounded operator on $L^2(H^n)$. Then,

$$\begin{aligned} &f*\phi_{\lambda,k,\epsilon}(z,t)\\ &=\frac{\lambda^n e^{-i\lambda_{\epsilon,k}t}}{(n+2k)(2\pi)^{n+1}}\int_{H^n} f(w,s)\, e^{i\lambda_{\epsilon,k}s}e^{-\frac{1}{2}z.\overline{w}}e^{-\frac{\lambda}{4(n+2k)}|z-w|^2}L_k^{n-1}\left(\frac{\lambda}{2(n+2k)}(z-w)\right)dw\ ds\\ &=e^{-i\lambda_{\epsilon,k}t}f*\phi_{\lambda,k,\epsilon}(z,0).\end{aligned}$$

Hence

$$\begin{aligned}\int_{\mathbb{R}}\left|\int_0^\infty f*\phi_{\lambda,k,\epsilon}(z,t)\,d\lambda\right|^2 dt &= \int_{\mathbb{R}}\left|\int_0^\infty e^{-i\lambda_{\epsilon,k}t}f*\phi_{\lambda,k,\epsilon}(z,0)\,d\lambda\right|^2 dt\\ &=\int_0^\infty\left|f*\phi_{\lambda,k,\epsilon}(z,0)\right|^2 d\lambda\end{aligned}$$

by using the Plancherel theorem in the Euclidean space in one variable. Then,

$$\int_{\mathbb{C}^n}\int_{\mathbb{R}}\left|\int_0^\infty f*\phi_{\lambda,k,\epsilon}(z,t)\,d\lambda\right|^2 dt\ dz = \int_{\mathbb{C}^n}\int_0^\infty\left|f*\phi_{\lambda,k,\epsilon}(z,0)\right|^2 d\lambda\ dz.$$

But by the Plancherel theorem for Heisenberg group we know that

$$||f||_2^2 = 2\pi \sum_k \sum_\epsilon (n+2k) \int_0^\infty \int_{\mathbb{C}^n} \left| f * \phi_{\lambda,k,\epsilon}(z,0) \right|^2 dz \; d\lambda.$$

Hence we get that

$$\int_{\mathbb{C}^n} \int_{\mathbb{R}} \left| \int_0^\infty f * \phi_{\lambda,k,\epsilon}(z,t)\, d\lambda \right|^2 dt \; dz \le ||f||_2^2.$$

Now, since $exp\left(-\frac{s\lambda}{n+2k}\right)$ is a bounded function we have

$$\begin{aligned}
&\int_{\mathbb{C}^n} \int_{\mathbb{R}} \left| \int_0^\infty f * \phi_{\lambda,k,\epsilon}(z,t)\, exp\left(\frac{-s\lambda}{n+2k}\right) d\lambda \right|^2 \\
&= \int_{\mathbb{C}^n} \int_0^\infty \left| f * \phi_{\lambda,k,\epsilon}(z,0)\, exp\left(\frac{-s\lambda}{n+2k}\right) \right|^2 d\lambda \; dz \\
&\le \left\| exp\left(\frac{-s\lambda}{n+2k}\right) \right\|_{L^\infty(\mathbb{R})} \int_{\mathbb{C}^n} \left| f * \phi_{\lambda,k,\epsilon}(z,0) \right|^2 dz \; d\lambda \\
&\le C||f||_{L^2(H^n)}.
\end{aligned}$$

Hence $f \mapsto \int_0^\infty f * \phi_{\lambda,k,\epsilon}(z,t)\, exp\left(\frac{-s\lambda}{n+2k}\right) d\lambda$ is a bounded operator on $L^2(H^n)$. □

Lemma 2. *Let $P_{k,\epsilon}$ denote the spectral projection operator associated to the ray $R_{k,\epsilon}$ of the Heisenberg fan. Then*

$$P_{k,\epsilon} f = \lim_{s\to 0^+} f * \left(\int_0^\infty \phi_{\lambda,k,\epsilon} exp\left(-\frac{s\lambda}{n+2k}\right) \, d\lambda \right), \tag{2.23}$$

for $f \in L^2(H^n)$, the limit existing in L^2 norm. We also have

$$P_{k,\epsilon} f = \frac{(-1)^{n+1}(n+k)!\; 2^{n-1}}{n!} f + P.V. f * \left(\int_0^\infty \phi_{\lambda,k,\epsilon} d\lambda \right), \tag{2.24}$$

as $L^2(H^n)$ functions for $f \in L^2(H^n)$, with P.V. convolution given by (2.20).

Proof. We will prove (2.23) using (2.22). We know that $P_{k,\epsilon}$ is given by

$$P_{k,\epsilon} f = \int_0^\infty f * \phi_{\lambda,k,\epsilon} \, d\lambda,$$

for $f \in L^2(H^n)$. By applying dominated convergence theorem we have

$$P_{k,\epsilon} f = \lim_{s\to 0^+} \int_0^\infty f * \phi_{\lambda,k,\epsilon}(z,t)\, exp\left(\frac{-s\lambda}{n+2k}\right) d\lambda,$$

as $L^2(H^n)$ functions. Then, the Proposition 1 easily gives (2.23).

Now to get (2.24) we first restrict f to a test function and compare the two distributions

$$f \mapsto \lim_{s\to 0^+} f * \left(\int_0^\infty \phi_{\lambda,k,\epsilon}\ exp\left(-\frac{s\lambda}{n+2k}\right)\ d\lambda\right)(0),$$

and

$$f \mapsto P.V. f * \left(\int_0^\infty \phi_{\lambda,k,\epsilon} d\lambda\right)(0).$$

Since the difference of the two distributions is supported at origin and their difference is homogeneous (i.e. if the difference of the two distributions is D, then $Df_r(z,t) = Df$, where $f_r(z,t) = f(rz, r^2t)$) we have that the difference is a constant multiple of Dirac delta distribution (see [**Rd**], Chapter 6). Hence it amounts to find that constant. To find that constant we first approximate the indicator function $\mathbb{I}_{[-1,1]}$ of the strip $-1 \le t \le 1$ by restricting z to a ball of radius R and then taking $R \to \infty$. Of course the indicator function is not in L^2 and the approximation is not L^2 approximation, nevertheless it wouldn't be a problem as $\phi_{\lambda,k,\epsilon}$ has a good decay in z variable.

Recall that the kernel $K = \int_0^\infty \phi_{\lambda,k,\epsilon}\ d\lambda$ satisfies

$$\int_S \int_{\mathbb{C}^n} K(z,t)\ dz\ dt = 0, \tag{2.25}$$

for any compact symmetric subset S of $\mathbb{R}$ around the origin. Combining (2.23), (2.25) and the fact that $[-1,1]$ is a symmetric set we get

$$P.V.\ \mathbb{I}_{[-1,1]} * \left(\int_0^\infty \phi_{\lambda,k,\epsilon} d\lambda\right)(0) = 0.$$

Thus we need to compute

$$\begin{aligned}&\lim_{s\to 0^+} \mathbb{I}_{[-1,1]} * \left(\int_0^\infty \phi_{\lambda,k,\epsilon} exp\left(-\frac{s\lambda}{n+2k}\right)\ d\lambda\right)(0)\\ &= \lim_{s\to 0^+} \int_{-1}^1 \int_{\mathbb{C}^n} \int_0^\infty \phi_{\lambda,k,\epsilon} exp\left(-\frac{s\lambda}{n+2k}\right)\ d\lambda\ dz\ dt.\end{aligned}$$

We first compute the integral with respect to z. We can do this because of Fubini's theorem

$$\int_{\mathbb{C}^n} \phi_{\lambda,k,\epsilon}(z,t)\, dz = \frac{exp\left(-i\lambda_{\epsilon,k} t\right)}{(2\pi)^{n+1}(n+2k)} \int_{\mathbb{C}^n} e^{-\lambda|z|^2/4} L_k^{n-1}\left(\lambda|z|^2/2\right) dz.$$

Now we again use the trick of generating function to obtain the above integral

$$\begin{aligned}\sum_k r^k \int_{\mathbb{C}^n} e^{-\lambda|z|^2/4} L_k^{n-1}\left(\lambda|z|^2/2\right) dz &= (1-r)^{-n} \int_{\mathbb{C}^n} exp\left(\frac{|z|^2}{4}\left(\frac{1+r}{1-r}\right) dz\right)\\ &= (4\pi)^n (1+r)^n.\end{aligned}$$

Therefore

$$\int_{\mathbb{C}^n} e^{-\lambda|z|^2/4} L_k^{n-1}\left(\lambda|z|^2/2\right) dz = \frac{(-1)^k (n+k)!\,(4\pi)^n}{n!}.$$

Hence

$$\int_{\mathbb{C}^n} \phi_{\lambda,k,\epsilon}(z,t)\, dz = \frac{(-1)^k (n+k)! 2^{n-1}}{n!\,(n+2k)\,\pi} exp\left(-\frac{i\epsilon\lambda t}{n+2k}\right).$$

Now substituting the above values we get

$$\begin{aligned} &\lim_{s\to 0^+} \frac{(-1)^k (n+k)!\,(4\pi)^n}{n!\,(n+2k)\,\pi} \int_{-1}^{1}\int_0^\infty exp\left(-i\lambda_{\epsilon,k} t\right) exp\left(-s\lambda/(n+2k)\right) d\lambda dt \\ &= \frac{(-1)^k (n+k)!\,(4\pi)^n}{n!\pi} \lim_{s\to 0^+} \int_{-1}^{1} \frac{dt}{s+i\epsilon t} = \frac{(-1)^{k+1}(n+k)! 2^{n-1}}{n!}, \end{aligned}$$

where, for the last equality we have used

$$\begin{aligned} \lim_{s\to 0^+} \int_{-1}^{1} \frac{dt}{s+i\epsilon t} = \lim_{s\to 0^+} \int_{-1}^{1} \frac{s-i\epsilon t}{s^2+t^2} dt &= \lim_{s\to 0^+} \int_{-1}^{1} \frac{s}{s^2+t^2} dt \\ &= \lim_{s\to 0^+} \left(tan^{-1}(1/s) - tan^{-1}(-1/s)\right) = \pi. \end{aligned}$$

□

Corollary 1. *For any complex r, $|r|<1$ we have*

$$\sum_{k=0}^{\infty} r^k P_{k,\epsilon} f = \frac{-2^{n-1}}{(1+r)^n} f + P.V.f * \left(\sum_{k=0}^{\infty} r^k \int_0^\infty \phi_{\lambda,k,\epsilon} d\lambda\right), \tag{2.26}$$

with $\sum_{k=0}^{\infty} r^k \int_0^\infty \phi_{\lambda,k,\epsilon}$ given by (2.16).

Proof. Multiplying (2.24) by r^k and summing over k will give us the required result.

□

2.2 L^p Spectrum of $(-\mathcal{L})(iT)^{-1}$

In this section, we will see to what extent the $L^2(H^n)$ Harmonic analysis corresponding to the spectral projections for $(-\mathcal{L})(iT)^{-1}$ can be applied to functions in $L^p(H^n)$. We have already seen that spectral projection $P_{k,\epsilon}$ is a Calderon-Zygmund operator and hence bounded on $L^p(H^n)$. However, we would like to find an estimate of the $L^p(H^n)$ operator norm.

To obtain L^p estimates for $P_{k,1}+P_{k,-1}$, we will consider $Q_k = P_{k,1} - P_{k,-1}$, which is the Hilbert transform of $P_{k,1}+P_{k,-1}$ in the t variable. The advantage of Q_k over $P_{k,\epsilon}$ is that it is an odd kernel, hence is suited for "method of rotations". The main idea is to use the following lemma by M.Christ [**C1**].

Lemma 3. *Let $K(z,t)$ be an odd function homogeneous of degree $-2n-2$ on H^n. Then the operator norm of $P.V.f*K$ on $L^p(H^n)$ is bounded by $c_p \int_{\mathbb{C}^n} |K(z,1)|\, dz$, $1 < p < \infty$, for some constant c_p depending only on p.*

The proof of the above lemma has been given in a general case of Hilbert space valued kernels in Theorem 7 in the next chapter.

Lemma 4. *The operator norm of $Q_k = P_{k,1} - P_{k,-1}$ is bounded by*

$$c_{p,\epsilon}(1+k)^{2n|1/p-1/2|+\epsilon}, \tag{2.27}$$

for every $\epsilon > 0$, $1 < p < \infty$

Proof. By Lemma 2, Q_k is of the form $P.V.f * K$, where K is given by the imaginary part of (2.17). Now, it is easy to estimate $\int |K(z,1)|$ because $(|z|^2 - 4i\epsilon t)/(|z|^2 + 4i\epsilon t)$ is of absolute value one, so in fact it is only the constant $(n+k)!/k!$ that contributes to the growth of k variable. Then by Stirling's approximation and from Christ's theorem we get that the estimate is $c_p(1+k)^n$. However, for $p = 2$, we know that the operator norm is 1 by Plancherel theorem. Hence by applying Marcinkiewicz interpolation theorem (see [**JD**], Chapter 2)for $p = 2$ and p close to 1 or ∞ we get the intended estimate. □

Observe that, the polynomial growth in 'k' is not sufficient to sum the projection operators absolutely. Nevertheless, it turns out that Abel sums converge in $L^p(H^n)$, i.e.

$$\lim_{r\to 0} \sum_{k=0}^{\infty} r^k (P_{k,1} + P_{k,-1}) f \to f \quad \text{in } L^p(H^n). \tag{2.28}$$

Theorem 4. *For any p, $1 < p < \infty$ the operator norm of $\sum_{k=0}^{\infty} r^k Q_k$ is uniformly bounded for $0 < r < 1$ (or more generally for approaching $r = 1$ in a non-tangential cone).*

Proof. By Corollary 2.1 we have that the $\sum_{k=0}^{\infty} r^k Q_k$ is of the form $P.V.f * K$ with K given by

$$2^{n-1}\pi^{-n-1} n!\,(1-r)\left(\left(|z|^2(1+r) + 4it(1-r)\right)^{-n-1} - \left(|z|^2(1+r) - 4it(1-r)\right)^{-n-1}\right). \tag{2.29}$$

Notice that It is sufficient to get a uniform bound of the integral $(1-r)\int_{\mathbb{C}^n} ||z|^2(1+r) \pm 4i(1-r)|^{-n-1} dz$ in r to apply the Lemma 3. By making the change of variable $z \mapsto z\sqrt{\frac{1-r}{1+r}}$ we get that

$$\begin{aligned}(1-r)\int_{\mathbb{C}^n} ||z|^2(1+r) \pm 4i(1-r)|^{-n-1} dz &= \frac{1}{(1+r)^n}\int_{\mathbb{C}^n} ||z|^2 \pm 4i|^{-n-1} dz \\ &= c_n (1+r)^{-n},\end{aligned}$$

for $0 < r < 1$. For r complex we write $1 - r = se^{i\theta}$, and make the change of variable $z \mapsto s^{1/2}z$ to obtain

$$(1-r)\int_{\mathbb{C}^n} ||z|^2(1+r) \pm 4i(1-r)|^{-n-1}dz = e^{i\theta}\int \mathbb{C}^n|\left(2-se^{i\theta}\right)|z|^2 \pm 4ie^{i\theta}|^{-n-1}dz.$$

It is easy to see that as along as θ is bounded away from $\pi/2$ the integral remains bounded as $s \to 0$ by dominated convergence theorem. □

Before we prove the L^p spectral theorem we will prove the following lemma.

Lemma 5. *For $f \in \mathcal{S}(H^n)$ (the Schwartz class functions on H^n, which is same as the Schwartz class functions on $\mathbb{R}^{2n+1}$), then we have* $\sum_{k=0}^{\infty} r^k (P_{k,1} + P_{k,-1}) f \to f$ *in $L^\infty(H^n)$.*

Proof. Observe that we can write

$$\sum_{k=0}^{\infty} r^k (P_{k,1} + P_{k,-1}f) = f * K_r,$$

where

$$K_r(z,t) = 2^{n-1}\pi^{n-1}n!(1-r)\left(\left(|z|^2\right)(1+r) + 4it(1-r)^{-n-1} + \left(|z|^2\right)(1+r) - 4it(1-r)^{-n-1}\right.$$

It can be easily seen that K is an L^1 function for $r \neq 1$. Hence, we have that $f * K_r \in \mathcal{S}(H^n)$ if $f \in \mathcal{S}(H^n)$ and $r \neq 1$.

Now, we will show that for $g_m \in \mathcal{S}(H^n)$ and $Dg_m \to 0$ in $L^2(H^n)$ for any polynomial D in $\mathcal{L}$ and T implies $||g_m||_\infty \to 0$. By the Fourier inversion formula for group Fourier transform we have

$$g_m(z,t) = (2\pi)^{-n-1}\sum_{\epsilon}\int_0^\infty tr\left(\widehat{g_m}(\pi_{\epsilon\lambda})\pi_{\epsilon\lambda}(z,t)\right)\, d\mu(\lambda),$$

where $d\mu(\lambda) = (2\pi)^{-n-1}\lambda^n d\lambda$ is the Plancherel measure. Then,

$$\begin{aligned}
|g_m(z,t)| &\lesssim \sum_{\epsilon}\int_0^\infty \left|tr\left(\widehat{g_m}(\pi_{\epsilon\lambda})\pi_{\epsilon\lambda}(z,t)\right)\right|\, d\mu(\lambda) \\
&\approx \sum_{\epsilon}\int_1^\infty \frac{\lambda^{n+2}}{\lambda^{n+2}}\left|tr\left(\widehat{g_m}(\pi_{\epsilon\lambda})\pi_{\epsilon\lambda}(z,t)\right)\right|\, d\mu(\lambda) \\
&+ \sum_{\epsilon}\int_0^1 \frac{\lambda^{n}}{\lambda^{n}}\left|tr\left(\widehat{g_m}(\pi_{\epsilon\lambda})\pi_{\epsilon\lambda}(z,t)\right)\right|\, d\mu(\lambda) \\
&\lesssim \sum_{e}\left(\int_1^\infty \left(\lambda^{n+2}\ tr\left(\widehat{g_m}(\pi_{\epsilon\lambda}(z,t))\pi_{\epsilon\lambda}\right)\right)^2\, d\mu(\lambda)\right)^{1/2} \\
&+ \sum_{e}\left(\int_0^1 \left(\lambda^{n}\ tr\left(\widehat{g_m}(\pi_{\epsilon\lambda}(z,t))\pi_{\epsilon\lambda}\right)\right)^2\, d\mu(\lambda)\right)^{1/2}.
\end{aligned}$$

Now, for $\lambda > 1$ observe that

$$
\begin{aligned}
&\left|(1+\lambda)^{(n+2)}\ tr\left(\widehat{g_m}\left(\pi_{\epsilon\lambda}\right)\pi_{\epsilon\lambda}(z,t)\right)\right| \\
&\lesssim \sum_{\alpha}\left|\left((n+2|\alpha|)\right)^{-(n+2)}(n+2|\alpha|)^{(n+2)}\lambda^{n+2}\langle\widehat{g_m}\left(\pi_{\epsilon\lambda}\right)\pi_{\epsilon\lambda}(z,t)\ h_\alpha, h_\alpha\rangle\right| \\
&= \sum_{\alpha}\left|(n+2|\alpha|)^{-(n+2)}\langle\widehat{g_m}\left(\pi_{\epsilon\lambda}\right)\pi_{\epsilon\lambda}(z,t)\ h_\alpha, d\pi_{\epsilon\lambda}(\mathcal{L})^{(n+2)}h_\alpha\rangle\right| \\
&= \sum_{\alpha}\left|(n+2|\alpha|)^{-(n+2)}\langle\pi_{\epsilon\lambda}(z,t)\ h_\alpha, \left(\widehat{g_m}\left(\pi_{\epsilon\lambda}\right)\right)^{*}(z,t)\pi_{\epsilon\lambda}(\mathcal{L})^{(n+2)}h_\alpha\rangle\right| \\
&= \sum_{\alpha}\left|(n+2|\alpha|)^{-(n+2)}\langle\widehat{\mathcal{L}^{(n+2)}g_m}\left(\pi_{\epsilon\lambda}\right)\pi_{\epsilon\lambda}(z,t)\ h_\alpha, h_\alpha\rangle\right|.
\end{aligned}
$$

Now, Cauchy-Schwarz inequality will give us

$$
\begin{aligned}
&\left|\lambda^{n+2}\ tr\left(\widehat{g_m}\left(\pi_{\epsilon\lambda}\right)\pi_{\epsilon\lambda}(z,t)\right)\right|^2 \\
&\lesssim \left(\sum_{\alpha}(n+2|\alpha|)^{-2(n+2)}\right)\left(\sum_{\alpha}\left|\langle\widehat{\mathcal{L}^{(n+2)}g_m}\left(\pi_{\epsilon\lambda}\right)\pi_{\epsilon\lambda}(z,t)\ h_\alpha, h_\alpha\rangle\right|^2\right) \\
&\lesssim \left(\sum_{\alpha}(n+2|\alpha|)^{-2(n+2)}\right)\left(\sum_{\alpha}\left|\widehat{\mathcal{L}^{(n+2)}g_m}\left(\pi_{\epsilon\lambda}\right)h_\alpha\right|^2\left|\pi_{\epsilon\lambda}(z,t)h_\alpha\right|^2\right) \\
&\lesssim \left(\sum_{\alpha}(n+2|\alpha|)^{-2(n+2)}\right)\left|\left|\widehat{\mathcal{L}^{(n+2)}g_m}(\pi_{\epsilon\lambda})\right|\right|_{HS}.
\end{aligned}
$$

Similarly for $0 < \lambda < 1$ we get that

$$
\left|\lambda^{n}\ tr\left(\widehat{g_m}\left(\pi_{\epsilon\lambda}\right)\pi_{\epsilon\lambda}(z,t)\right)\right|^2 \lesssim \left(\sum_{\alpha}(n+2|\alpha|)^{-2n}\right)\left|\left|\widehat{\mathcal{L}^{n}g_m}(\pi_{\epsilon\lambda})\right|\right|_{HS}.
$$

For any integer k, the number of ordered partitioning of k into n parts is $\binom{n+k-1}{k}$. By Sterling's formula we have

$$
\binom{n+k-1}{k} \lesssim \frac{(n+k-1)^{n+k-1+1/2}}{k^{k+1/2}} \lesssim k^{n-1}.
$$

We will show that $\sum_\alpha (n+2|\alpha|)^{-2n} < \infty$ for $n \geq 1$, which also implies that $\sum_\alpha (n+2|\alpha|)^{-2(n+2)} < \infty$ for $n \geq 1$

$$
\sum_{\alpha}(n+2|\alpha|)^{-2n} = \sum_{k=0}^{\infty}\sum_{|\alpha|=k}(n+2k)^{-2n} \lesssim \sum_{k=0}^{\infty}k^{n-1}(n+2k)^{-2n} \lesssim \sum_{k=0}^{\infty}k^{-n-1} < \infty.
$$

Therefore,

$$\underset{(z,t)}{sup}|g_m(z,t)| \lesssim \sum_{\epsilon}\left(\int_1^{\infty} ||\widehat{\mathcal{L}^{(n+2)}g_m}(\pi_{\epsilon\lambda})||_{HS}^2 \, d\mu(\lambda)\right)^{1/2} \tag{2.30}$$

$$+\sum_{\epsilon}\left(\int_0^{1} ||\widehat{\mathcal{L}^{n}g_m}(\pi_{\epsilon\lambda})||_{HS}^2 \, d\mu(\lambda)\right)^{1/2}$$

$$\lesssim \sum_{\epsilon}\left(\int_0^{\infty} ||\widehat{\mathcal{L}^{(n+2)}g_m}(\pi_{\epsilon\lambda})||_{HS}^2 \, d\mu(\lambda)\right)^{1/2}$$

$$+\sum_{\epsilon}\left(\int_0^{\infty} ||\widehat{\mathcal{L}^{n}g_m}(\pi_{\epsilon\lambda})||_{HS}^2 \, d\mu(\lambda)\right)^{1/2}$$

$$\lesssim \left|\left|\mathcal{L}^{(n+2)}g_m\right|\right|_2 + ||\mathcal{L}^n g_m||_2 . \tag{2.31}$$

Hence it is sufficient to show that R.H.S goes to 0 as $g_m \to 0$ in $L^2(H^n)$. The operators $P_{k,\epsilon}$ commutes with $\mathcal{L}$ and hence with any power $\mathcal{L}^j$, $j \geq 0$ (in fact this holds for any left invariant differential operator). Therefore, $\underset{r\to 1}{lim}\ \mathcal{L}^j \sum_{k=0}^{\infty}(P_{k,1}+P_{k,-1}) f = \mathcal{L}^j f$ in L^2 norm if $\mathcal{L}^j f \in L^2(H^n)$. Here, we anyway have $\mathcal{L}^{(n+2)}f, \mathcal{L}^n f \in L^2(H^n)$ as $f \in \mathcal{S}(H^n)$. Now, we take a sequence $\{r_m\}$ such that $r_m \to 1$ as $m \to \infty$ and let $g_m := \sum_{k=0}^{\infty} r_m^k (P_{k,1}+P_{k,-1}) f - f$. Hence, from the above observation for g_m we have the intended result. ☐

Corollary 2. *For any $f \in L^p(H^n)$, $1 < p < \infty$,*

$$\underset{r\to 1}{lim}\sum_{k=0}^{\infty} r^k (P_{k,1}+P_{k,-1}) f = f, \tag{2.32}$$

the limit existing in L^p norm. Similarly we have

$$\underset{r\to 1}{lim}\sum_{k=0}^{(1-r)^2} r^k (P_{k,1}+P_{k,-1}) f = f, \tag{2.33}$$

and

$$\underset{r\to 1}{lim}\sum_{k=0}^{(1-r)^2} r^k \int_0^{(n+2k)/(1-r)} (f*\phi_{\lambda,k,1}+f*\phi_{\lambda,k,-1}) \, d\lambda = f \tag{2.34}$$

In particular, if $\int_0^s f*\phi_{\lambda,k,\epsilon}\, d\lambda =$ for all k,ϵ and s, then $f \equiv 0$.

Proof. We know that the operators $\sum_{k=0}^{\infty} r^k (P_{k,1}+P_{k,-1})$ are uniformly bounded and hence to show (2.32) it is sufficient to prove that the same holds for a dense subspace. Recall that we have (2.32) for $p=2$ as we have that the partial sums converge to f in L^2 and therefore the Abel sums also converge in L^2 norm to f. By using the previous lemma we have the convergence of the Abel sums in $L^{\infty}(H^n)$ and then by interpolating L^2 and L^{∞} convergence we have L^p convergence for any $2 \leq p < \infty$ for

Schwartz class functions.

Next, we will show the equivalence of (2.32) and (2.33) for any p, $1 < p < \infty$. From Lemma 4 the operator norm of the tail of the series

$$\sum_{k=(1-r)^{-2}}^{\infty} r^k (P_{k,1} + P_{k,-1})$$

is bounded by a multiple of

$$\sum_{k=(1-r)^{-2}} r^k k^\alpha$$

where $\alpha = 2n|1/p - 1/2| + \epsilon$ for any $\epsilon > 0$. Since $0 < r < 1$, we can write $r = e^{-s}$ for some $s \in \mathbb{R}_+$. Then, it is easy to observe that

$$\begin{aligned}\sum_{k=(1-r)^{-2}} r^k k^\alpha \leq \int_{s^{-2}}^{\infty} e^{-sx} x^\alpha dx &= \frac{1}{s^{\alpha+1}} \int_{s^{-1}}^{\infty} e^{-x} x^\alpha dx \\ &\leq \frac{\Gamma(\alpha+1)}{s^{\alpha+1}}\end{aligned}$$

where Γ is standard Gamma function. Therefore, R.H.S goes to 0 as $s \to \infty$.

For $1 < p \leq 2$ we will establish (2.33) using duality argument. Similar to the trignometric functions in Euclidean Harmonic analysis, we consider the subspace S of functions whose spectral projections are supported in a finite number of rays. i.e,

$$S = \{f : f \in L^p(H^n) \text{ and } f = \sum_{k=0}^{N} (P_{k,1} + P_{k,-1}) f \text{ for some } N \in \mathbb{N}\}$$

Observe that (2.33) holds on this subspace S. To prove the density of the subspace we need to show that for any $g \in L^{p'}(H^n)$(the dual of $L^p(H^n)$) , $\langle g, f\rangle$ for all f in the subspace S implies that $g = 0$. Now, we know that the operators $P_{k,\epsilon}$ are projections on $L^2(H^n)$, hence on L^p for all $1 \leq p < \infty$. This is not difficult to see as $P_{k,\epsilon}^2 = P_{k,\epsilon}$ extends from the dense subspace $L^2 \cap L^p(H^n)$, since $P_{k,\epsilon}$ is bounded on $L^p(H^n)$. Therefore, for any $f \in L^p(H^n)$ we know $P_{k,\epsilon} f$ is in the subspace S. Hence, $\langle P_{k,\epsilon} g, f\rangle = \langle g, P_{k,\epsilon} f\rangle = 0$ by hypothesis. Thus

$$\left\langle \sum_{k=0}^{(1-r)^{-2}} r^k (P_{k,1} + P_{k,-1}) g, f \right\rangle = 0.$$

Then we can use (2.33) for $L^{p'}(H^n)$ as $2 \leq p < \infty$. By using (2.32) and taking $r \to 1$ we get that $\langle g, f\rangle = 0$. Since this is true for all $f \in L^p(H^n)$ it follows that $g = 0$.

To show (2.34), a similar technique as we did to establish (2.32) works. i.e. we have the convergence in L^2 norm by Plancherel theorem. Hence if we can show the boundedness of the operators

$$\sum_{k=0}^{(1-r)^{-2}} r^k \int_0^{(n+2k)/(1-r)} \left(f * \phi_{\lambda,k,1} + f * \phi_{\lambda,k,-1}\right) \, d\lambda,$$

we are done as we have the convergence in $L^2 \cap L^p(H^n)$, which is a dense subspace. The boundedness of these operators are quite easy to prove. They are obtained by composing the operators

$$\sum_{k=0}^{(1-r)^{-2}} r^k \left(P_{k,1} + P_{k,-1}\right) f \tag{2.35}$$

with the Fourier multipliers in t variable corresponding to the characteristic function of the intervals $|\tau| \leq 1/(1-r)$. Moreover, both these operators are bounded on L^p and hence (2.35) too is bounded on $L^p(H^n)$. □

Chapter 3

Littlewood-Paley Theory for $(-\mathcal{L})(iT)^{-1}$

3.1 Introduction

In this chapter, the main aim is to develop the Littlewood-Paley theory for the operator $(-\mathcal{L})(iT)^{-1}$. One of the applications of Littlewood-Paley theory is to study the L^p boundedness of multiplier operators.

We will first introduce multiplier operators in the Euclidean setting before we move onto the case of Heisenberg group. Fourier multipliers (or simply multipliers) form a very important class of linear operators on $L^p(\mathbb{R}^n)$. Given a function $m : \mathbb{R}^n \to \mathbb{C}$, the multiplier operator given by m on $L^p(\mathbb{R}^n)$ is defined as

$$f \mapsto \int_{\mathbb{R}^n} m(\xi)\,\hat{f}(\xi)\,e^{2\pi i\langle \xi,\cdot\rangle}d\xi, \tag{3.1}$$

with sufficient conditions on m and f such that all the integrals are well defined. For $f \in L^2(\mathbb{R}^n)$, it follows from the Fourier inversion formula that the identity function $m \equiv 1$ denotes the identity operator on $L^2(\mathbb{R}^n)$. One of the basic questions is to study the boundedness of the multiplier operators under different conditons imposed on m. For $L^2(\mathbb{R}^n)$, we have a complete characterization of the bounded multipliers using Plancherel theorem; an operator given by (3.1) is bounded on $L^2(\mathbb{R}^n)$ if and only if $m \in L^\infty(\mathbb{R}^n)$.

The boundedness on L^1 and L^∞ are slightly more complicated, nevertheless it has been completely resolved; a function $m(\xi)$ is bounded $L^1(\mathbb{R}^n)$ or $L^\infty(\mathbb{R}^n)$ multiplier if and only if there exists a finite Borel measure μ such that $m = \widehat{\mu}$. For $1 < p < \infty$, the question has not been completely settled. However, results are available with sufficient conditions on m using techniques like Littlewood-Paley theory. For example, we have the following theorem (see [**St2**], page 96).

Theorem 5. *(Mihlin-Hörmander Theorem) Suppose* $m \in C^k(\mathbb{R}^n\backslash\{0\})$*, where* k *is an integer* $> n/2$*. Assume also that for every differential monomial* $\left(\frac{\partial}{\partial x}\right)^\alpha, \alpha =$

$(\alpha_1, ..., \alpha_n)$, *with* $|\alpha| = \alpha_1 + ... + \alpha_n$, *we have*

$$\left|\left(\frac{\partial}{\partial x}\right)^{\alpha} m(x)\right| \leq B|x|^{-|\alpha|} \text{ whenever } |\alpha| \leq k. \tag{3.2}$$

Then there exists a constant A_p *such that* $||Tf||_p \leq ||f||_p$, $1 < p < \infty$.

To read about Littlewood-Paley theory for Euclidean Laplacian see ([**St2**], Chapter IV)

Now, we define multiplier operator for $(-\mathcal{L})(iT)^{-1}$. Given a function $f \in \mathcal{S}(H^n)$ and $m \in L^\infty(\mathbb{R}_+)$, the action of the multiplier given by m on f is

$$f \to \sum_{k=0}^{\infty} m(n+2k)\left(P_{k,1} + P_{k,-1}\right) f, \tag{3.3}$$

For the case of $(-\mathcal{L})(iT)^{-1}$ too Plancherel theorem will immediately give that m defines bounded multiplier on $L^2(H^n)$ if and only if m is a bounded function.

3.2 Littelwood-Paley g-function

In this section, first we will define Littlewood-Paley g function, which is the fundamental object in Littlewood-Paley theory. Here, we will define Littlewood-Paley g-function using the heat semi-group associated to the operator $(-\mathcal{L})(iT)^{-1}$, using which we will try give a useful characterization of L^p multipliers of H^n.

The heat semi-group associated to $(-\mathcal{L})(iT)^{-1}$ is given by

$$\begin{aligned} h_y(z,t) &= \sum_{k=0}^{\infty} e^{-(n+2k)y}\left(P_{k,1} + P_{k,-1}\right)(z,t) \\ &= e^{-ny}2^{n-1}\pi^{-n-1}n!\left(1 - e^{-2y}\right)\left[\left(|z|^2 + 4it + e^{-2y}\left(|z|^2 - 4it\right)\right)^{-n-1}\right. \\ &\quad \left. + \left(|z|^2 - 4it + e^{-2y}\left(|z|^2 + 4it\right)\right)^{-n-1}\right] \end{aligned} \tag{3.4}$$

Then we define $H_y(f)(z,t) = h_y * f(z,t)$ and we define the g_l-function for an integer $l \geq 0$ as

$$g_l(f)(z,t) = \left(\int_0^\infty \left|\frac{\partial^l}{\partial y^l} H_y f(z,t)\right|^2 y^{2l-1}\, dy\right)^{1/2} \tag{3.5}$$

Observe that for a fixed y, $K_{l,y} := \frac{\partial^l}{\partial y^l} h_y(z,t)$ takes value in the Hilbert space $\mathcal{H} = L^2\left(\mathbb{R}_+, y^{2l-1}dy\right)$.

Theorem 6. *For each integer* $l \geq 1$ *and* $f \in L^2(H^n)$ *there exists* $c_l > 0$ *such that*

$$||g_l(f)||_2 = c_l ||f||_2$$

Proof. Observe that

$$\begin{aligned}||g_l(f)||_2^2 &= \int_{H^n}\int_0^\infty \left|\frac{\partial^l}{\partial y^l} H_y f(z,t)\right|^2 y^{2l-1}\, dydz\, dt \\ &= \int_{H^n}\int_0^\infty \left|\frac{\partial^l}{\partial y^l} \sum_k e^{-(n+2k)y}(P_{k,1}+P_{k,-1})(f)(z,t)\right|^2 y^{2l-1}\, dy\, dz\, dt \\ &= \int_{H^n}\int_0^\infty \left|\sum_k (-n-2k)^l e^{-(n+2k)y}(P_{k,1}+P_{k,-1})(f)(z,t)\right|^2 y^{2l-1}\, dy\, dz\, dt\end{aligned}$$

By the orthogonality of $P_{k,\epsilon}(z,t)$ and Fubini's theorem we can write the above as

$$\begin{aligned}&\int_{H^n}\int_0^\infty \left|\sum_k (n+2k)^l e^{-(n+2k)y}(P_{k,1}+P_{k,-1})(f)(z,t)\right|^2 y^{2l-1}\, dy\, dz\, dt \\ &= \int_{H^n}\int_0^\infty \sum_k \left|(n+2k)^l e^{-(n+2k)y}(P_{k,1}+P_{k,-1})(f)(z,t)\right|^2 y^{2l-1}\, dy\, dz\, dt \\ &= \int_{H^n}\sum_k \int_0^\infty |(n+2k)^l e^{-(n+2k)y}(P_{k,1}+P_{k,-1})(f)(z,t)|^2 y^{2l-1}\, dy\, dz\, dt \\ &= \left(\frac{1}{2^{2l}}\int_0^\infty e^{-y}y^{2l-1}\, dy\right)\left(\sum_k \int_{H^n}\int_0^\infty |(P_{k,1}+P_{k,-1})(f)(z,t)|^2\, dz\, dt\right)\end{aligned}$$

We obtained the last equality by a simple change of variable. Now, by applying Plancherel's theorem we get that

$$||g_l(f)||_2 = c_l||f||_2$$

where $c_l = \left(\frac{1}{2^{2l}}\int_0^\infty e^{-y}y^{2l-1}\, dy\right)^{1/2}$. □

Next, we will prove that $f \to g_l f$ is a bounded on $L^p(H^n)$ for every $l > 0$. First, we will prove a vector valued analogue of Lemma 3.1 of [**Str1**].

Theorem 7. *Let $\mathcal{H}$ be a Hilbert space and let $K : H^n \backslash \{0,0\} \to \mathcal{H}$ be homogeneous function of degree $-2n-2$ and an odd function in the 't' variable. Moreover, if $K(z,1) \in L^1(\mathbb{C}^n, \mathcal{H})$ (the space of all integrable functions from $\mathbb{C}^n \to \mathcal{H}$, see Section 4.2), then the L^p operator norm of $P.V. f \to f * K$ is bounded by $c_p \int_{\mathbb{C}^n} ||K(z,1)||_{\mathcal{H}}\, dz$ for some constant c_p depending on p, $1 < p < \infty$. i.e*

$$||P.V. f * K||_{L^p(H^n,\mathcal{H})} \le \left(c_p \int_{\mathbb{C}^n} ||K(z,1)||_{\mathcal{H}} dz\right) ||f||_{L^p(H^n,\mathbb{C})}$$

Proof. As in [**Str1**], the idea is to reduce the problem of L^p boundedness of K to the Euclidean convolution estimate that $P.V. \int_{-\infty}^{\infty} f(x-s, y-s^2)(ds/s)$ is bounded on

$L^p(\mathbb{R}^2)$, $1 < p < \infty$ (see to [SW] for the proof of L^p boundedness of Hilbert transform along a parabola). In fact, the exact same proof as in [**Str1**] follows through, except that the integrals in our case are vector valued.

Now, for the integral on $(-\infty, 0)$, we first make a change of variable $s \to -s$, then use the fact that K is an odd kernel in $'s'$ variable. Then, we make the change of variable $s \to s^2$ to get

$$\begin{aligned} P.V.f * K(z,t) = P.V. &\int_{\mathbb{C}^n} \int_{-\infty}^{\infty} f\left(z-w, t-s^2+\frac{1}{2}Im\ z.\overline{w}\right) sK(w,s^2)\ ds\ dw \\ &+ P.V. \int_{\mathbb{C}^n} \int_{-\infty}^{\infty} f\left(z-w, t+s^2+\frac{1}{2}Im\ z.\overline{w}\right) sK(w,s^2)\ ds\ dw, \end{aligned}$$

By using homegeneity and making a change of variable $w \to w/s$ we get that the above integral is,

$$\begin{aligned} &= \int_{\mathbb{C}^n} P.V\left(\int_{-\infty}^{\infty} f\left(z-sw, t-s^2+\frac{1}{2}s\ Im\ z.\overline{w}\right)\frac{ds}{s}\right) K(w,1)\ dw, \\ &+ \int_{\mathbb{C}^n} P.V\left(\int_{-\infty}^{\infty} f\left(z-sw, t+s^2+\frac{1}{2}s\ Im\ z.\overline{w}\right)\frac{ds}{s}\right) K(w,1)\ dw, \end{aligned} \tag{3.6}$$

By applying triangle inequality, we will find L^p operator bound of the first and second integral in (3.6) separately. For the first integral in (3.6), by applying Minkowski's integral inequality we get that it is sufficient to show that the L^p operator bound of

$$P.V. \int_{-\infty}^{\infty} f\left(z-sw, t-s^2+\frac{1}{2}s\ Im\ z.\overline{w}\right)\frac{ds}{s}$$

is independent of w. Clearly we can rotate w to be of the form $(\rho, 0, ...0)$, ρ is real without changing the L^p norm estimate as $\int_{\mathbb{C}^n} ||K(w,1)||\ dw$ is invariant under rotation of w, and hence the issue is the operator bound of

$$P.V. \int_{-\infty}^{\infty} f\left(x-s\rho, y, t-s^2+\frac{1}{2}s\rho y\right)\frac{ds}{s}$$

on $L^p(\mathbb{R}^3)$, or equivalently the operator bound of

$$P.V. \int_{-\infty}^{\infty} f\left(x-s\rho, t-s^2+\frac{1}{2}s\rho y\right)\frac{ds}{s} \tag{3.7}$$

on $L^p(\mathbb{R}^2)$ for all values of ρ and y. Since the L^p operator norm is invariant under rotations, we will only consider (3.7). Now, we invoke the conjugation transform by two parameter family of dilations $\delta(\lambda_1, \lambda_2) f(x,t) = f(\lambda_1 x, \lambda_2 t)$ which does not alter the $L^p(\mathbb{R}^2)$ operator bound.

Case 1 $(y = 0)$:

Take $\lambda_1 = \rho$ and $\lambda_2 = 1$ Then

$$\delta\left(\rho^{-1},1\right) \circ P.V. \int_{-\infty}^{\infty} f\left(\rho x - s\rho, t - s^2\right) \frac{ds}{s} = P.V. \int_{-\infty}^{\infty} f\left(\frac{1}{\rho}\left(\rho x - s\rho\right), t - s^2\right) \frac{ds}{s}$$
$$= P.V. \int_{-\infty}^{\infty} f\left(x - s, t - s^2\right) \frac{ds}{s} \tag{3.8}$$

Case 2 ($y \neq 0$):
First we make a change of variable $s \to \frac{\rho y}{2} s$. Then

$$P.V. \int_{-\infty}^{\infty} f\left(x - s\rho, t - s^2 + \frac{1}{2} s\rho y\right) \frac{ds}{s}$$
$$= P.V. \int_{-\infty}^{\infty} f\left(x - \frac{1}{2}\rho^2 y s, t - \frac{1}{4}\rho^2 y^2 s^2 + \frac{1}{4}\rho^2 y^2 s\right) \frac{ds}{s}$$

Now, we take $\lambda_1 = \frac{1}{2}\rho^2 y$ and $\lambda_2 = \frac{1}{2}\rho^2 y^2$ and calculate similarly as in case 1 to get

$$\delta\left(2\rho^{-2}y^{-1}, 4\rho^{-2}y^{-2}\right) \circ P.V. \int_{-\infty}^{\infty} f\left(x - \frac{1}{2}\rho^2 y s, t - \frac{1}{4}\rho^2 y^2 s^2 + \frac{1}{4}\rho^2 y^2 s\right) \frac{ds}{s}$$
$$= P.V. \int_{-\infty}^{\infty} f\left(x - s, t - s^2 + s\right) \frac{ds}{s}$$

The form in the second case can again be converted to the one in the first case again by conjugation transform by $(x,t) \mapsto (x, t + x)$ just as we did before. Now, for the second integral in (3.6), using the similar calculations we get that it is sufficient to find the $L^p(\mathbb{R}^2)$ operator bound of

$$P.V. \int_{-\infty}^{\infty} f\left(x - s, t + s^2\right) \frac{ds}{s}$$

Once again we can apply a conjugation by the dilation given by $\lambda_1 = 1$ and $\lambda_2 = -1$ to get (3.8). □

Observe that we require K to be an odd function in $'t'$ variable to apply the earlier theorem. Hence we will take the Hilbert transform of $h_y(z,t)$ in $'t'$ variable to get the kernel $\sum_k e^{-(n+2k)y}\left(P_{k,1} - P_{k,-1}\right)(z,t)$, which is an odd kernel in t as we require. Now, by the L^p boundedness of Hilbert transform for $1 < p < \infty$ it is sufficient to show the boundedness of the new convolution operator. It is easy to see that $\sum_k e^{-(n+2k)y}\left(P_{k,1} - P_{k,-1}\right)(z,t)$ is homogeneous of degree $-2n-2$. Therefore, to show that g_l is L^p bounded it is sufficient to show that the Hilbert transform of the corresponding kernel is in $L^1\left(H^n, L^2\left((0,\infty), y^{2l-1}dy\right)\right)$.

Lemma 6. *For every integer $l \geq 0$ and $n > 0$, the function*

$$K_y(z) = e^{-ny}\left(1 - e^{-2y}\right)\left(|z|^2 + 4i + e^{-2y}\left(|z|^2 - 4i\right)\right)^{-n-1}$$

is in $L^1\left(\mathbb{C}^n, L^2\left((0,\infty), y^{2l-1}dy\right)\right)$.

Proof. Observe that the integral that we have to estimate is

$$\int_{\mathbb{C}^n}\left(\int_0^\infty e^{-2ny}\left(1-e^{-2y}\right)^2\left(|z|^4\left(1+e^{-2y}\right)^2+16\left(1-e^{-2y}\right)^2\right)^{-n-1}\, y^{2l-1}dy\right)^{1/2}dz \tag{3.9}$$

We will first show the result for $l \geq 1$ as there is a slight variation required for $l = 0$. Now, we fix an $\epsilon << 1$ and we split the integral on $\mathbb{C}^n$ into two parts, i.e $|z| < \epsilon$ and it's complement. Observe that in the case of $|z| \geq \epsilon$ we have

$$\left(\int_0^\infty e^{-2ny}\left(1-e^{-2y}\right)^2\left(|z|^4\left(1+e^{-2y}\right)^2+16\left(1-e^{-2y}\right)^2\right)^{-n-1}\, y^{2l-1}dy\right)^{1/2}$$
$$\lesssim \frac{1}{|z|^{2n+2}}\left(\int_0^\infty e^{-2ny}\left(1-e^{-2y}\right)^2\, y^{2l-1}dy\right)^{1/2} \lesssim \frac{1}{|z|^{2n+2}},$$

which is integrable on $|z| \geq \epsilon$. To deal with the case of $|z| < \epsilon$ we will have to split the inner integral too. i.e

$$\int_0^\infty e^{-2ny}\left(1-e^{-2y}\right)^2\left(|z|^4\left(1+e^{-2y}\right)^2+16\left(1-e^{-2y}\right)^2\right)^{-n-1}\, y^{2l-1}dy$$
$$\lesssim \int_0^{|z|^{3/(l+1)}} y^{2l+1}\left(|z|^4+y^2\right)^{-n-1}dy$$
$$+\int_{|z|^{3/(l+1)}}^\infty e^{-2ny}\left(1-e^{-2y}\right)^2\left(|z|^4+\left(1-e^{-2y}\right)^2\right)^{-n-1}\, y^{2l-1}dy$$

The reason for choosing $|z|^{3/(l+1)}$ will be evident as the calculation proceeds. Let us denote the first and second integral in the right hand side as I_1 and I_2 respectively. For I_1, we make the change of variable $y \to |z|^2 y$ to get

$$I_1 \lesssim \frac{|z|^{4l+4}}{|z|^{4n+4}}\int_0^{|z|^{3/(l+1)-2}} y^{2l+1}dy \lesssim \frac{|z|^{4l+4}}{z|^{4n+4}}|z|^{6-(4l+4)} \lesssim \frac{|z|^6}{|z|^{4n+4}}.$$

Now,

$$\int_{|z|<\epsilon}(I_1)^{1/2} \lesssim \int_{|z|<\epsilon}\frac{|z|^6}{|z|^{2n+2}}dz \lesssim \int_0^\epsilon \frac{x^3}{x^{2n+2}}x^{2n-1}dx < \infty.$$

Next,

$$I_2 \lesssim \frac{1}{|z|^{4n+4}}\int_{|z|^{3/(l+1)}}^\infty e^{-2ny}y^{2l+1}dy = \frac{|z|^6}{|z|^{4n+4}}\int_1^\infty e^{-2ny|z|^{3/(l+1)}}y^{2l+1}dy \lesssim \frac{|z|^6}{|z|^{4n+4}}\frac{1}{|z|^{3/(l+1)}}.$$

Hence

$$\int_{|z|<\epsilon}(I_2)^{1/2} \lesssim \int_{|z|<\epsilon}\frac{|z|^3}{|z|^{2n+2}}\frac{1}{|z|^{3/(2l+2)}}dz \lesssim \int_0^\epsilon \frac{1}{x^{3/(2l+2)}}dx < \infty$$

The last integral is finite as $3 < 2l + 2$ for $l \geq 1$.

Now, for the case of $l = 0$ we consider $|z|^{3/2}$ instead of $|z|^{3/(l+1)}$. Thus,

$$I_1 \approx \int_0^{|z|^{3/2}} y\left(|z|^4 + y^2\right)^{-n-1} dy \approx \int_0^{|z|^4+|z|^3} y^{-n-1} dy \approx \frac{1}{|z|^{3n}}$$

So,

$$\int_{|z|<\epsilon} (I_1)^{1/2} \lesssim \int_{|z|<\epsilon} \frac{1}{|z|^{3n/2}} |z|^{2n-1} dz \lesssim \int_0^{\epsilon} x^{n/2-1} dx < \infty \quad (\text{since } n \geq 1)$$

For I_2, the same calculation as above works, except that

$$I_2 \lesssim \frac{|z|^6}{|z|^{4n+4}} \frac{1}{|z|^{3/2}}.$$

Hence,

$$\int_{|z|<\epsilon} (I_2)^{1/2} \lesssim \int_{|z|<\epsilon} \frac{|z|^3}{|z|^{2n+2}} \frac{1}{|z|^{3/4}} \lesssim \int_0^{\epsilon} \frac{1}{x^{3/4}} dx < \infty.$$

Hence, we have the required result in all the cases. □

Remark 4. *For $l \geq n+1$ one can alternatively use the theory of Calderon-Zygmund integrals to prove the L^p boundedness of g_l. The verification is left to the reader.*

Corollary 3. *For $l \geq 0$, $f \to g_l(f)$ is a bounded semilinear operator from $L^p(H^n) \to L^p(H^n)$, for all $1 < p < \infty$.*

Corollary 4. *$H_y(f) \to f$ as $y \to 0^+$ a.e, $\forall f \in L^p(\mathbb{R}^n)$, $1 < p < \infty$.*

Proof. First we will show that if the operator

$$\mathcal{C}(f) = \sup_{y>0} |H_y f| \tag{3.10}$$

is L^p bounded then,

$$L_f = \lim_{y\to 0+} \sup |f(z,t) - H_y(f)(z,t)| = 0 \quad a.e.$$

So, assume that $\mathcal{C}$ is bounded on $L^p(H^n)$. Observe that it is sufficient to show that that for all $\epsilon > 0$, $|\{L_f > \epsilon\}| \lesssim \epsilon$. Now, we take g to be a compactly supported smooth function so that $||f-g||_p \leq \epsilon^{(p+1)/p}$. For g we know that $H_y g \to g$ a.e (as we had shown in Lemma 5). Hence $L_f \leq \mathcal{C}(f-g) + |f-g|$. Since we have assumed that $\mathcal{C}$ is L^p bounded, we also have the weak L^p boundedness. Therefore

$$|\{\mathcal{C}(f-g) > \epsilon\}| \leq \epsilon^{-p} ||f-g||_p^p \lesssim \epsilon.$$

So, all we have to show is the L^p boundedness of the operator $\mathcal{C}$. We will show the L^p boundedness on a dense class, say compactly supported smooth functions. Suppose that $f \in C_c^\infty(H^n)$. Then, by fundamental theorem of calculus in one dimension we get that

$$(H_y f(z,t))^2 = \int_0^y \frac{\partial}{\partial y}(H_y f(z,t))^2 dy = 2\int_0^y (H_y f(z,t)) \frac{\partial}{\partial y}(H_y f(z,t))\ dy$$

Then,

$$\begin{aligned} |H_y f(z,t)|^2 &\lesssim \int_0^\infty |(H_y f(z,t)) \frac{\partial}{\partial y}(H_y f(z,t))|dy \\ &\lesssim \int_0^\infty \left(\frac{1}{\sqrt{y}}|(H_y f(z,t))|\right)\left(\sqrt{y}\left|\frac{\partial}{\partial y}(H_y f(z,t))\right|\right) dy. \end{aligned}$$

By Cauchy-Schwarz inequality

$$\begin{aligned} &|H_y f(z,t)|^2 \lesssim g_1(f)(z,t).g_0(f)(z,t)\ \ \forall\ y > 0 \\ &\Rightarrow \sup_{y>0}|H_y f(z,t)| \lesssim (g_1(f)(z,t))^{1/2}(g_0(f)(z,t))^{1/2} \end{aligned}$$

Again by Cauchy-Schwarz inequality,

$$\begin{aligned} &\left(\int_{H^n}\left(\sup_{y>0}|H_y f(z,t)|\right)^p dz\ dt\right)^{1/p} \\ &\lesssim \left(\int_{H^n}(g_1(f)(z,t))^{p/2}(g_0(f)(z,t))^{p/2} dz\ dt\right)^{1/p} \\ &\lesssim \left(\int_{H^n}(g_1(f)(z,t))^p dz\ dt\right)^{1/2p}\left(\int_{H^n}(g_0(f)(z,t))^p dz\ dt\right)^{1/2p} \\ &= ||g_1(f)||_p^{1/2}||g_0(f)||_p^{1/2} \lesssim ||f||_p^{1/2}||f||_p^{1/2} \lesssim ||f||_p \end{aligned}$$

Therefore, we have shown the L^p boundedness of $\mathcal{C}$ and hence the a.e convergence of $H_y(f)$. □

Corollary 5. *$\sum_k r^k(P_{k,1}+P_{k,-1})f \to f$ a.e as $r \to 1^-$.*

Proof. This is an easy consequence of the earlier corollary that $H_y f \to f$ a.e as $y \to 1^-$. Every r such that $0 < r < 1$ can be expressed as e^{-y} for some $0 < y < \infty$. Hence the result follows immediately. □

Theorem 8. *Suppose $f \in L^2(H^n)$ and $g_l(f) \in L^p(H^n)$, $1 < p < \infty$ for any integer $l \geq 1$. Then $f \in L^p(H^n)$ and there exists a constant A_p' such that*

$$A_p'||f||_p \leq ||g_l(f)||_p$$

Proof. We have

$$||g_l(f)||_2 = c_l||f||_2$$

Hence by polarization identity, for $f_1, f_2 \in L^2(H^n)$ we have

$$\int_{H^n} f_1(z,t)\,\overline{f}_2(z,t)\ dz\ dt = c_l^{-2}\int_{H^n} g_l(f_1)(z,t)\,g_l(f_2)(z,t)\ dz\ dt.$$

Suppose now in addition we also have that $f_1 \in L^p(H^n)$ and $f_2 \in L^q(H^n)$ with $||f_2||_q \leq 1$, and $1/p + 1/q = 1$. Now, using Hölder's inequality and the fact that $||g(f)||_q \leq A_q||f||_q$ we get that

$$\left|\int_{\mathbb{R}^n} f_1(z,t)\,\overline{f_2}(x)\ dx\right| \leq c_l^{-2}||g_l(f_1)||_p||g_l(f_2)||_q \leq c_l^{-2}A_q||g_l(f_1)||p \qquad (3.11)$$

Now take the supremum in the above inequality as f_2 ranges over all functions in $L^2 \cap L^p(H^n)$, with $||f_2||_q \leq 1$. We thus obtain the required inequality with with $A'_p = c_l^2/(A_q)$ for $f \in L^2 \cap L^p(H^n)$. To obtain the inequality for a general functions in $L^p(H^n)$ we will use limiting arguments. Let f_m be a sequence of functions in $L^2 \cap L^p(H^n)$, which converge to a general function $f \in L^p(H^n)$ in the $L^p(H^n)$ norm. Now,

$$\begin{aligned}
&|g_l(f_{m_1})(x) - g_l(f_{m_2})(x)| \\
&= \left(\int_0^\infty \left|\frac{\partial^l}{\partial y^l}H_y f_{m_1}(z,t)\right|^2 y^{2l-1}dy\right)^{1/2} - \left(\int_0^\infty \left|\frac{\partial^l}{\partial y^l}H_y f_{m_2}(z,t)\right|^2 y^{2l-1}dy\right)^{1/2} \\
&\leq \left(\int_0^\infty \left|\frac{\partial^l}{\partial y^l}H_y(f_{m_1} - f_{m_2})(z,t)\right|^2 y^{2l-1}dy\right)^{1/2} = g_l(f_m - f_n).
\end{aligned}$$

Hence, by the L^p boundedness of g_l we get that $g_l(f_m)$ converge to $g_l(f)$ in $L^p(H^n)$ for $1 < p < \infty$. □

Chapter 4

Appendix

4.1 Hilbert-Schmidt Operator

Definition 5. *Let $\{u_\alpha\}_{\alpha\in A}$ (where A is a countable set) be a complete orthonormal set in a separable Hilbert space $\mathcal{H}$. A bounded linear operator T is said to be a Hilbert-Schmidt operator in case the quantity $||T||_{HS}$ defined by the equation*

$$||T||_{HS} = \left(\sum_{\alpha\in A} ||Tu_\alpha||^2\right)^{1/2}$$

is finite. The number $||T||$ is called the Hilbert-Schmidt norm of T. The class of all Hilbert-Schmidt operators on $\mathcal{H}$ will be denoted by HS.

In the definition of the class HS a particular orthonormal sequence was used. However, the following lemma will show that HS depends only on the Hilbert space and not on the basis.

Lemma 7. *The Hilbert-Schmidt norm is independent of the orthonormal basis used in its definition. If T is in HS and U is a unitary operator in $\mathcal{H}$, then $U^{-1}TU$ is in HS and and $||T||_{HS} = ||U^{-1}TU||_{HS}$. In addition, $||T||_{\mathcal{H}} \leq ||T||_{HS}$ and $||T^*||_{HS} = ||T||_{HS}$*

See [**DS**], Chapter XI for the proof of above lemma. Now, one can verify that, given any orthonormal basis $\{u_\alpha\}$ of $\mathcal{H}$

$$\langle S, T\rangle_{HS} = \sum_\alpha \langle T^*Su_\alpha, u_\alpha\rangle, \qquad S, T \in HS \tag{4.1}$$

defines an inner product on $\mathcal{H}$ and $\langle T, T^*\rangle_{HS} = \langle T^*, T\rangle_{HS} = ||T||^2_{HS}$. Moreover, the inner product is independent of the basis chosen by polarization identity. Now, we define the trace of a Hilbert-Schmidt operator as

$$tr\,(T) = \langle T, Id\rangle_{HS}, \tag{4.2}$$

where Id is the identity operator on $\mathcal{H}$. If the Hilbert space $\mathcal{H}$ is represented as $L^2(S, \Sigma, \mu)$ for a positive measure μ, then the Hilbert-Schmidt operators are those operators K having a representation in the form

$$(Kf)(s) = \int_S k(s,t)f(t)\ d\mu(t)$$

where

$$\int_S \int_S |k(s,t)|^2\ d\mu(s)\ d\mu(t) < \infty.$$

They are also called Hilbert-Schmidt integral operators, and $||K||_{HS}$, the Hilbert-Schmidt norm, is equal to $\int_S \int_S |k(s,t)|^2\ d\mu(s)\ d\mu(t)$.

4.2 Vector valued integrals

Let $\mathcal{H}$ be a separable Hilbert space. Then, we say that the function $f(x)$ from H^n to $\mathcal{H}$ is measurable if $(x), \phi\rangle$ is measurable, wehre $\langle ., . \rangle$ denotes the inner product on $\mathcal{H}$ and the ϕ is an arbitrary element of $\mathcal{H}$. Since the norm $||.||$ function is continuous we have that $||f||$ is also measurable as a real valued function if f is measurable. Now, we define $L^p(H^n, \mathcal{H})$ as the equivalence class of measurable functions f from H^n to $\mathcal{H}$ such that the norm $||f||_p = \left(\int_{H^n} |f(x)|^p\ dx\right)^{1/p}$ is finite, for $p = \infty$ we have similar definition with $||f||_p = ess\ sup||f(x)||$.

Next, let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two separable Hilbert spaces and let $B(\mathcal{H}_1, \mathcal{H}_2)$ denote the Banach space of bounded linear operators from $\mathcal{H}_1$ to $\mathcal{H}_2$ with the usual operator norm. We say that the function f from H^n to $B(\mathcal{H}_1, \mathcal{H}_2)$ is measurable if $f\phi$ is an $\mathcal{H}_2$ valued measurable function for every $\phi \in \mathcal{H}_2$. In this case also $||f||_{B(\mathcal{H}_1,\mathcal{H}_2)}$ is measurable and we can define $L^p(H^n, B(\mathcal{H}_1, \mathcal{H}_2))$ as before. It easy see that the fundamental results in integration theory like Dominated convergence theorem, Fubini's theorem, Young's integral inequality, Minkowski's integral inequality for complex valued functions also hold in this case where we replace the $|.|$ function on $\mathbb{C}$ with the operator norm $||.||_{B(\mathcal{H}_1,\mathcal{H}_\in)}$. Observe $\mathcal{H}_2$ valued functions are a special case of $B(\mathcal{H}_1, \mathcal{H}_2)$ valued functions if we take $\mathcal{H}_1 = \mathbb{C}$.

Next, we will see what exactly does it mean to integrate Hilbert space valued functions. Let $f \in L^p(H^n, \mathbb{C})$ and let $b \in \mathcal{H}$. Now, we can define the function $f.h : H^n \to \mathcal{H}$ as $(f.h)(x) = f(x)h$. Observe that $f.h \in L^p(H^n, \mathcal{H})$ as $||f.h||_{L^p(H^n,\mathcal{H})} \leq ||f||_{L^p(H^n,\mathbb{C})}||h||_{\mathcal{H}}$. Let the subspace of $L^p(H^n, \mathcal{H})$ consisting of finite linear combinations of functions of this type be denoted as $L^p \otimes \mathcal{H}$. Note that the functions in $L^p \otimes \mathcal{H}$ with f being a characteristic function replaces the role of simple functions in scalar valued integration theory. Hence, a proof similar to the one in scalar valued integration theory will show that $L^p \otimes \mathcal{H}$ is dense in $L^p(H^n, \mathcal{H})$ if $1 \leq p < \infty$.

Given $F = \sum_j f_j.h_j \in L^1 \otimes \mathcal{H}$, we define its integral to be the element of $\mathcal{H}$ given by

$$\int_{H^n} F(x)\ d\mu(x) = \sum_j \left(\int_{H^n} f_j(x)\ d\mu(x) \right) h_j.$$

The map $F \to \int F(x)\ d\mu(x)$ extends to $L^1(H^n, \mathcal{H})$ by continuity. For a function in $L^1(H^n, \mathcal{H})$, by Riesz representation theorem, this integral is a unique element of $\mathcal{H}$ given by

$$\left\langle h, \int_{H^n} F(x)\ d\mu(x) \right\rangle = \int_{H^n} \langle h, F(x) \rangle\ d\mu(x), \qquad \forall\ h \in \mathcal{H}.$$

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