Calibration of Option Pricing Models with Emphasis on Stochastic Calculus

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Certificate of Examination

This is to certify that the dissertation titled "**Calibration of Option Pricing Models with Emphasis on Stochastic Calculus**" submitted by Dinesh Kumar (Reg. No. MS16067) for the partial fulfillment of BS-MS dual degree programme of the institute, has been examined by the thesis committee duly appointed by the institute. The committee finds the work done by the candidate satisfactory and recommends that the report be accepted.

Dr. Soma Maity | Dr. Abhik Ganguli | Dr. Amit Kulshrestha (Supervisor)

Dated: April 21, 2021

Declaration

The work presented in this dissertation has been carried out by me under guidance of Dr. Arun Kumar at Indian Institute of Technology Ropar and Dr. Amit Kulshrestha at the Indian Institute of Science Education and Research Mohali.

This work has not been submitted in part or in full for a degree, a diploma, or a fellowship to any other university or institute. Whenever contributions of others are involved, every effort is made to indicate this clearly, with due acknowledgement of collaborative research and discussions. This thesis is a bonafide record of work done by me and all sources listed within have detailed in the bibliography.

Dinesh Kumar
(Candidate)

Dated: April 21, 2021

In my capacity as the supervisor of the candidate's project work, I certify that the above statements by the candidate are true to the best of my knowledge.

Dr. Amit Kulshrestha
(Supervisor)

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Abstract

The thesis aims to discuss various models for option pricing and their calibration in the Global and Indian market. Construction of Ito integral with respect to Brownian motion has been carried out rigorously. In discrete-time models, single and multiple period binomial Model, CRR model, and multinomial model has been discussed. For continuous-time, Bachelier and BSM model has been discussed. The solution of BSM PDE has been discussed using two methods, first by converting BSM PDE to heat equation and then solved it by Fourier transform technique and second is by changing probability measure. Further implementation and calibration of Apple and Google Stock in Bachelier and geometric Brownian motion model have been carried out. It has been shown that GBM Model is a better fit for the stock path rather than the Bachelier model. It has also been demonstrated that GBM Model also deviates from the real stock path because of the assumption of the log-normal distribution of return, constant mean, and constant volatility. Finally, simulation of Infosys option for CRR and BSM Model has been carried out and it has been shown that the CRR model is a very good estimate for the BSM model for a large number of time steps.

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Chapter 1

Introduction

Modern theory of mathematical finance starting to emerge after Louis Bachelier defended his thesis titled "Theórie de la Spećulation" with his work published in a most influential scientific journal of France [18]. He was the first who recognize the role of Brownian motion in finance and used it to model stock price. In this thesis, we have presented the Bachelier model of stock price and compared it to a more robust Black-Scholes model for Apple and Google stock. Schachermayer, W. and Teichmann, J. carried out a more mathematically rigorous analysis of these two models [19] and argued that Bachelier's model provides good short-time approximations of prices and volatilities for short time.

Brownian motion is also called Wiener process because mathematically rigorus development of Brownian motion was done by Norbert Wiener. He provided construction of Brownian motion by constructing Wiener measure. In this thesis we have omitted the construction and existence of Brownian motion and can be found [1] with great detail. After development of modern probability theory by Andrey Nikolaevich Kolmogorov's in his book "Foundations of the Theory of Probability" [20] in 1933, Kiyoshi Ito came with Ito lemma in his widely cited paper "On stochastic differential equations (1951) [21]" and in his Ito's representation theorem proved that any square integrable martingale with a filtration derived by Brownian motion can be expressed as an Ito integral with respect to Brownian motion.[21] This theorem is essential for derivation of Block Scholes option pricing formula to prove the existence of hedging portfolio.

In 1973 two paper published, one is titled "The Pricing of Options and Corporate Liabilities"[22] by Fischer Black and Myron Scholes and other is titled "On the pricing of corporate debt: the risk structure of interest rates"[23] by Robert Merton. These paper used Ito calculus and geometric Brownian motion to model to price European options.

This thesis is divided in 6 chapters including Introduction. Chapter 2 provides the basic background for probability theory and stochastic process. For more details about topics discussed in this section one can refer to [1], [2], [14] and [15]. Chapter 3 provides the foundation of options and includes single period binomial Model, CRR model and multinomial Model. In chapter 4 we have discussed various property of Brownian motion and constructed the stochastic integral with respect to Brownian motion for process whose Riemann integral exist and are square integrable. Although it is possible to define stochastic Integral with respect to local martingales, but since we require Brownian motion to model asset price, we restricted it's construction to Brownian motion. In chapter 5 we have constructed the Black sholes PDE and it's proof using two methods. First one is by converting BSM PDE to heat equation. Now solution of heat equation can be found using Fourier transfer technique. Second method utilize probabilistic approach. It use Grisanov's theorem to change the drift of the stock price process to convert discounted stock price process to martingale. Finally in first part of chapter 6 we have done simulation,calibration and comparison of Bachelier model to Google and Apple stock price path. In second part of chapter 6 we simulated CRR and BSM model on Infosys stock to price Infosys option. It has been shown that CRR price agrees with BSM price for large value of time steps.

Chapter 2

Preliminaries

2.1 Essential of Probability Theory

We assume that reader is already familiar with basic Probability theory and Notion of random variable. In this section we will define concept of probability theory and stochastic Process which are required to develop our theory. Although we are stating some definition so that reader can refresh his/her memory but we except that reader is familiar with introductory measure theoretic foundation of probability theory. Theory in this section is adapted from [11] and [6].

Definition 2.1.1 (σ-algebra)**.** *Let Ω be a nonempty set.A collection $\mathcal{F}$ of subset of Ω is said to be σ-algebra if following condition are satisfied:*

(i). $\phi \in \mathcal{F}$.

(ii). If $E \in \mathcal{F}$ *then* $E^c \in \mathcal{F}$.

(iii). If sequence of sets $E_1, E_2, ...$ *belongs to* $\mathcal{F}$*, then* $\cup E_i \in \mathcal{F}$.

Proposition 2.1.1. *Any arbitrary intersection of σ-algebra on Ω is also a σ-algebra on Ω.*

Proof. Proof is fairly straightforward. □

Above proposition allows us to define smallest σ-algebra which contains the subset $\mathcal{A}$ of Ω. Precise definition is provided below:

Definition 2.1.2. *Let $\mathcal{C}(\mathcal{A})$ be the collection of all σ-algebra that contains $\mathcal{A}$. The σ-algebra generated by a collection $\mathcal{A} \subset 2^{\Omega}$, denoted by $\sigma(\mathcal{A})$, is defined as following:*

$$\sigma(\mathcal{A}) = \cap\{\mathcal{F} : \mathcal{F} \in \mathcal{C}(\mathcal{A})\}$$

of course $\sigma(\mathcal{A})$ is a σ-algebra by proposition 2.1.1.

Definition 2.1.3. *Consider pair $(\Omega, \mathcal{F})$, where Ω is a set which is non empty and $\mathcal{F}$ is a σ-algebra on Ω. Suppose $\mathbb{P}$ is a function $\mathbb{P} : \mathcal{F} \to [0, 1]$ such that,*

(i). $\mathbb{P}(\Omega) = 1$

(ii). For any countable pairwise disjoint collection $\{E_i\}_{i \geq 1}$:

$$\mathbb{P}(\cup_{i\geq 1} E_i) = \sum_{i\geq 1} \mathbb{P}(E_i)$$

Then $\mathbb{P}$ is called a probability measure. Triplet $(\Omega, \mathcal{F}, \mathbb{P})$ is called a probability space.

Definition 2.1.4. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. A random variable X is a real-valued function $X : \Omega \to \mathbb{R}$ such that inverse image of every Borel set of $\mathbb{R}$ under X lies in σ-algebra $\mathcal{F}$. Alternatively for every borel set B,*

$$\{X \in B\} = \{\omega \in \Omega : X(\omega) \in B\} \in \mathcal{F}$$

In above definition we already have a σ-algebra and we want to behave random variable X in a certain way. One can also define σ-algebra generated by random variable X. Essentially it is the smallest σ-algebra on which we can define X as a random variable. Mathematically σ-algerba $\sigma(X)$ generated by random variable X is defined as collection of all subset of form:

$$\{\omega : X(\omega) \in B` \; \forall B \in \mathcal{B}\}$$

where $\mathcal{B}$ is a set of all borel set in $\mathcal{R}$.

There are various notion of convergence of random variable we will only state that are of our use.

Definition 2.1.5. *A sequence of non negative random variables $X_n, n = 1, 2, ...,$ is said to converge pointwise almost surely to some random variable X and we write $X_n \to X$ (a.s) if,*

$$\mathbb{P}[\omega \in \Omega : \lim_{n\to\infty} X_n(\omega) = X(\omega)] = 1$$

Definition 2.1.6. *A sequence of non negative random variables $X_n, n = 1, 2, ...,$ is said to converge pointwise monotonically almost surely to some random variable X and we write $X_n \uparrow X$ (a.s) if,*

$$\mathbb{P}[\omega \in \Omega : X_1 \leq X_2 ... \leq X] = 1$$

$$\mathbb{P}[\omega \in \Omega : \lim_{n \to \infty} X_n(\omega) = X(\omega)] = 1$$

Given a sequence of converging random variable one can think that their expectation will converge. Unfortunately this is not always true. However we do have some restriction under which their expectation will Always converge. One of which is widely known monotone convergence theorem(MCT). In-fact we stated definition 2.1.6 only to state MCT. Proof of theorem is omitted as it can be found in most of the probability theory text.

Theorem 2.1.1. *Monotone convergence Theorem If $X_n; n = 1, 2, ...$ is a sequence of nonnegative random variables converging pointwise monotonically to X (a.s.) , then $E[X_n] \uparrow E[X]$ almost surely.*

Reader might know that expectation in measure theoretic probability defined by notion of lebesgue integral, but computation under this integral is rather difficult than Riemanian Interal. For this reason one want to know when these integral are equal so that one can compute in Riemanian environment and still have required result. Below theorem precisely state when these two notion are equal.

Theorem 2.1.2. *(Lebesgue versus Riemann Integration)(Theorem 1.3.8, [6]) Let $f : [a, b] \to \mathbb{R}$ be bounded. Then following hold :*

1. *f is Riemann-integrable iff f is continuous almost everywhere.*
2. *If f is Riemann-integrable, then the Lebesgue integral over $[a, b]$ is also defined and the two integrals are the same.*

In determining stock precise one has to include arrived information till a particular time. To capture this idea Notion of condition expectation is widely used.

Let X be a random variable on probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and let $\mathcal{G}$ be a sub-σ-algebra of $\mathcal{F}$.

Definition 2.1.7. *We say that random variable Y is conditional expectation of X given $\mathcal{G}$ and denote it by $Y = E[X|\mathcal{G}]$ if following hold:*

1. *Y is $\mathcal{G}$ measurable random variable, in mathematical notation $\sigma(Y) \subset \mathcal{G}$.*

2. *$E[X\mathbb{I}_B] = E[Y\mathbb{I}_B]$ for every event $B \in \mathcal{G}$*

Example 2.1.1. 1. *If $\mathcal{G} = \mathcal{F}_0 = \{\phi, \Omega\}$, then $E[X|\mathcal{G}] = E[X]$*

2. *for a constant random variable C, $E[C|\mathcal{G}] = C$*

3. *If X is $\mathcal{G}$-measurable then $E[X|\mathcal{G}] = X$.*

Definition of conditional expectation raises to immediate question. One is that does condition expectation always exist? Other is if it does exist will it be unique. Answer to both these questions is Yes and proof for the same requires Radon-Nikodym Theorem.

Definition 2.1.8. *wo probability measures P_1, P_2 defined on the same space $(\Omega, \mathcal{F})$, P_2 is said to be absolutely continuous with respect to P_1 if for every set $A \in \mathcal{F}$, $P_1(A) = 0 \implies P_2(A) = 0$.*

Theorem 2.1.3. *Radon-Nikodym Theorem ((Theorem 1.3.8, [6])) Suppose P_2 is absolutely continuous with respect to P_1. Radon-Nikodym Theorem guarantees the existence of a non-negative and unique random variable upto measure zero, such that $Z : \Omega \to R_+$ and*

$$\mathbb{P}_2[A] = E_{\mathbb{P}_1}[Z\mathbb{I}_A] = \int_A Z d\mathbb{P}_1$$

for every $A \in \mathcal{F}$.

The proof of existence of condition expectation is given below and is adapted from appendix of [6]

Theorem 2.1.4. *The conditional expectation $E[X|\mathcal{G}]$ exists and is unique almost surely.*

Proof. Assume X is non constant such that $X \geq 0$ and $E_{\mathbb{P}}[X] < \infty$. Define a probability measure $\mathbb{P}_2$ on $\mathcal{G}$ as follows:

$$\mathbb{P}_2(B) = \frac{E_{\mathbb{P}}[X\mathbb{I}_B]}{E_{\mathbb{P}}[X]} = \frac{\int_B X d\mathbb{P}}{\int_\Omega X d\mathbb{P}}$$

Where $B \in \mathcal{G}$. Note that $\mathbb{P}_2$ is absolutely continuous with respect to $\mathbb{P}$.By the Radon-Nikodym Theorem there exist a random variable Z which is measurable wrt $\mathcal{G}$ such that for every $B \in \mathcal{G}$

$$\mathbb{P}_2[B] = E_{\mathbb{P}}[Z\mathbb{I}_B]$$

Now take $Y = ZE_{\mathbb{P}}[X]$. We have :

$$E[Y\mathbb{I}_B] = E[(ZE_{\mathbb{P}}[X])\mathbb{I}_B] = E_{\mathbb{P}}[X]E[Z\mathbb{I}_B] = E_{\mathbb{P}}[X\mathbb{I}_B].$$

□

2.2 Essentials of Stochastic Process

Stochastic process is a way to model randomly occurring phenomena. In case of a deterministic function or any map we have a fix value for an input. But in case of stochastic process there is no way to predict what will be the output or possible path. Since in case of stock price path we do not know what will happen in next moment or at a particular time, this gives us clue that stock price can be modeled by some stochastic process. For instance Black and Scholes used geometric Brownian motion to describe stock price path. Theory in this section is adapted [1] and is highly recommended for readers who are more interested in mathematical rigorous theory of stochastic process in particular Brownian motion.

Definition 2.2.1 (Stochastic Process)**.** *A stochastic process is a collection of random variables that are all defined on a common sample space Ω and indexed by $t \in T$:*

$$X = \{X_t(w)\}_{t\in T}, \textit{so that } \forall t \in T \;\; X_t : \Omega \to \mathbb{R}$$

T can be discrete or continuous.

For a fixed outcome ω, the function $t \to X_t(\omega)$ is called the sample path of process X associate with outcome ω. Two stochastic process X and Y defined on same probability space $(\Omega, \mathcal{F}, \mathbb{P})$ are said to be same if $X_t(\omega) = Y_t(\omega)$ $\forall t \in T$and $\omega \in \Omega$.

But by the beauty of probability measure $\mathbb{P}$ we can weaken the above requirement to obtain three different concept of sameness of stochastic process [1]. These are listed as definition below in order of increasing strongness:

Definition 2.2.2. *X and Y are said to be have same distribution if for each borel set B of $\mathbb{R}$, we have:*

$$\mathbb{P}(X_t \in B) = \mathbb{P}(Y_t \in B) \;\forall t \in T$$

Definition 2.2.3. *We say that Y is modification of X if $\forall t \in T$, we have $\mathbb{P}(X_t = Y_t) = 1$.*

Definition 2.2.4. *We say that X and Y are indistinguishable if almost all their sample path agree:*

$$\mathbb{P}(X_t = Y_t) = 1 \ \forall \ t$$

From now on we will assume that $t \in [0, \infty)$

In case of random variable we use σ-algebra for technical reason, but in case of stochastic process there is extremely important non technical reason to include σ-algebra that is we need to keep track of information. In definition of stochastic process we can say that t is a flow of time, so at every t we can talk about past,future or present or in other words one can ask observer of process that what he knows at present compared to past or what he will knew in future. Also note that information can only increase over time.

Above Remark suggest that we need to include a non-decreasing family of σ-algebra, which we will call filtration. Mathematically precise definition of filtration is provided below:

Definition 2.2.5 (filtration)**.** *A filtration on a sample space $(\Omega, \mathcal{F})$ is a non-decreasing family of $\{\mathcal{F}_t\}_{t\geq 0}$ if sub σ-algebra of $\mathcal{F}$ i.e. $\mathcal{F}_s \subseteq \mathcal{F}_t \subseteq \mathcal{F}$ for $0 \leq s \leq t$. Set $\mathcal{F}_\infty = \sigma(\cup_{t\geq 0}\mathcal{F}_t)$.*

Most natural way to obtain a filtration of a stochastic process $\{X_t\}_{t\geq 0}$ is one that is generated by process itself :

$$\mathcal{F}_t^X = \sigma(\{X_s : 0 \leq s \leq t\})$$

Definition 2.2.6. *Let $(\Omega, \mathcal{F})$ be a measurable space. We say that random variable X, which is defined on Ω is $\mathcal{F}$-measurable if we have $\sigma(X) \subset \mathcal{F}$.*

Now we define measurability of stochastic process.

Definition 2.2.7. *A stochastic process is called measurable if for every $B \in \mathcal{B}(\mathbb{R})$ the set $\{(t, \omega) : X_t(\omega) \in B\}$ belongs to the product σ-field $\mathcal{B}([0, \infty)) \otimes \mathcal{F}$*

Introduction of filtration open up a new requirement of measurability of process and in fact this requirement is stronger than above.

Definition 2.2.8. *Let $\mathbb{F} = \{\mathcal{F}_t\}_{t\geq 0}$ be a filtration over Ω. A random process $\{X_t\}_{t\geq 0}$ on Ω is said to be adapted to filtration $\mathbb{F}$ if X_t is $\mathcal{F}_t$-measurable for every t.*

Given a particular time we want to know that our stock price will rise or decline in future. Mathematical term for capture this idea is martingale. Notion of martingale is at core of Modern Theory of Mathematical finance. Intuitively a stochastic process is martingale if on an average it does not show sign of rise or fall. Martingales also used to check fairness of game. A game can be viewed as fair game if its profit or loss process can be modeled as a martingale process.

Definition 2.2.9. *Considered an adapted integrable stochastic process $\mathcal{M}_t$ defined on filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F})$ and an. Let $0 \leq s \leq t < \infty$. It is called*

1. *a martingale iff $E[\mathcal{M}_t|\mathcal{F}_s] = \mathcal{M}_s$ almost surely.*

   A martingale process can be thought as a process which has no tendency to fall or rise.

2. *a sub-martingale iff $E[\mathcal{M}_t|\mathcal{F}_s] \leq \mathcal{M}_s$ almost surely.*

   A sub-martingale process can be thought as a process which has no tendency to fall.

3. *a super-martingale iff $E[\mathcal{M}_t|\mathcal{F}_s] \geq \mathcal{M}_s$ almost surely.*

   A super-martingale process can be thought as a process which has no tendency to rise.

Chapter 3

Introduction to Options and Discrete Time Models

3.1 Properties of Options

Theory in this section is adapted from [3] and [9], but with a more compact approach. Theory about basic properties of option is restricted to our use. Reader who want to know more about option can refer to [9].

A derivative is a type of financial contract whose value depends on other variables (that is why name derivative) called underlying assets. These other variable can be stock price, interest rate, market indices etc.

There are broadly three type of traders who can be interested in trading derivative instruments. These are

Hedgeres : Those who trade derivatives product to minimize their risk in future because of market movement.

speculators : Those who use derivatives to make profit by betting on future market movement.

There are broadly three type of Derivative instruments. These are : Forward contracts, Futures contract and options. Pricing of first two are relatively easy, but pricing of options contract require various tools of probability theory, stochastic calculus and stochastic process. We will discuss various models of pricing option contract.

Option contract (Definition 2.1, [3]) : An option is a contract whose holder not obligated but have right to buy (if it is call option) or sell (if it is a put option) a pre-specified asset (called underlying asset) at a pre-specified price (called the strike price) on or before a pre-specified date (called the expiry date or maturity date).

The options are further divided into two categories based on exercise date. A European option can only be exercise at the expiry date while an American option can also be exercised prior to expiry date.

Now suppose we have an European option whose whose strike price is K, maturity time is T and let S_t denote the price of underlying asset at time t, then payoff δ of option for call and put is given by :

$$\delta_{call} = max(S(T) - K) = (S(T) - K)^+, \ \delta_{put} = max(K - S(T)) = (K - S(T))^+$$

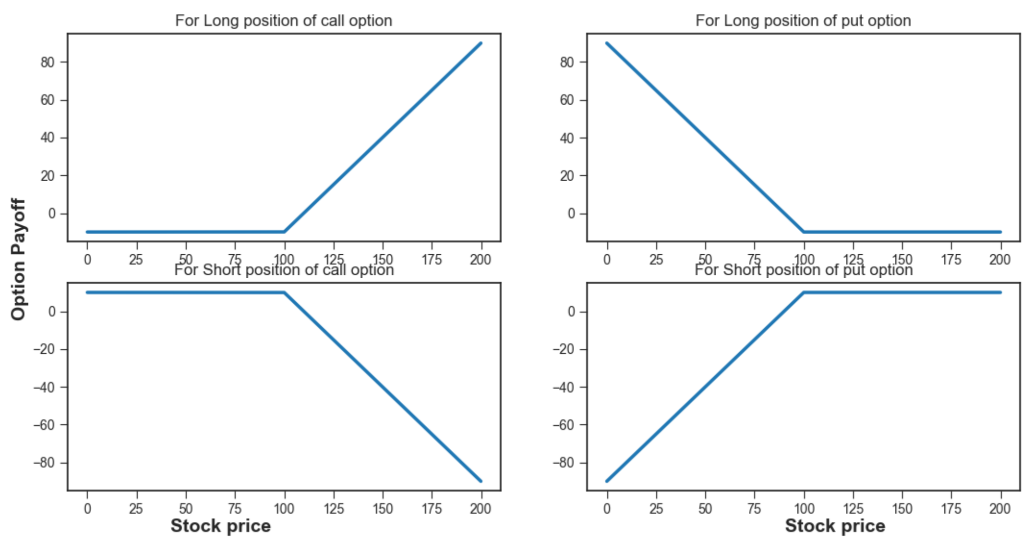


Figure 3.1: Payoff diagram of Europian Option

One essential feature of efficient market is that one should not have non zero chance of getting profit from nothing. Technical term for the same is arbitrage.

Definition 3.1.1. *An arbitrage opportunity is a trading opportunity that allows one to have a non zero chance of getting profit and no chance of loss from zero initial wealth.*

In our financial model we denote our set of all possible market scenario as Ω. $\omega \in \Omega$ is any possible market scenario.

Theorem 3.1.1. ***The Law of one Price****(Theorem 2.3), [3] In an arbitrage free market if we have have two portfolio $\prod^x$ and $\prod^y$ with initial price $\prod_0^x$ and $\prod^y{}_0$ respectively. If* *at any future time $T \geq 0$ they have the price of $\prod^x$ and $\prod^x$ are equal in all state of the world i.e $\prod_T^x = \prod_T^y \forall \omega \in \Omega$, then their initial price should be same i.e $\prod_0^x = \prod_0^y$.*

Proof. Outline of proof : Without loss of generality assume $\prod_0^x > \prod_0^y$ and construct a portfolio as a combination of these two portfolio such that resulting portfolio give rise to arbitrage. □

Before jumping into models of pricing options lets look at some of relationship between European put and call options. We start with famous put-call parity.This is an important relationship between call and put European options called put call parity. Assume that price of European call option is denoted by C^E and European put option is denoted by P^E. Assume a continuous compounding interest rate r. Then relation can be derived using replicating portfolio and law of one price:

Relation between the arbitrage free prices of European put and call options

	Portfolio 1	Portfolio 2
	1 European call + cash Ke^{-rt}	1 one European put + one share of stock
At $t = T$	$\max(S(T) - K)$	$\max(S(T) - K)$
At t = 0	$C^E + Ke^{-rT}$	$P^E + S(0)$

By law of one price we have :

$$C^E + Ke^{-rT} = P^E + S(0) \leftarrow \textbf{Put-call parity}$$

Below theorem find some upper and lower bound for options :

Theorem 3.1.2. *Theorem 4.5, [3] Under the assumption of absence of arbitrage followings are true :*

$$(S(0) - Ke^{-rT})^+ \leq C^E < S(0) \tag{3.1}$$

$$(Ke^{-rT} - S(0))^+ \leq P^E < Ke^{-rT} \tag{3.2}$$

Proof. Proof for the call option is outlined. A similar procedure can give the proof of put option.

Suppose $C(E) \geq S(0)$. Consider following portfolio : sell a call option, invest remaining money in risk free market and buy one stock of share. At expiry date sell stock.

	sell call option + buy one share
t=0	$C^E - S(0)$
t=T	$\min\{S(T), K\} + (C^E - S(0))e^{rT} > 0$

This is an arbitrage opportunity, hence $C^E < S(0)$.

Obtain lower bound on call and put using put call parity and non-negativity of option prices:

$$C^E \geq 0 \text{ and } C^E = S(0) - Ke^{-rT} + P^E \geq S(0) - Ke^{-rT}$$
$$\Downarrow$$
$$C^E \geq \max\{0, S(0) - Ke^{-rT}\}$$

□

3.2 Single Period Binomial Model

This is the simplest model for pricing options. Assume that there are only two possible market scenario $\Omega = \{\omega^+, \omega^-\}$. market scenario ω^+ can occur with probability p and market scenario ω^- can occur with probability $1 - p$.These are called real world probability. There two tradable asset in market one is underlying asset S and other is risk free bond B. Since it is a single period model we are assuming that there only two trading dates, namely $t = 0$ and $t = 1$. We wish to find price of a option which gives us payoff $\delta(\omega^+)$ under scenario ω^+ and payoff $\delta(\omega^-)$ under scenario ω^-. S_0 and B_0 are initial price ($at\ t = 0$) of one unit of asset S and bond B respectively. $S_1(\omega^+)$ and $S_1(\omega^-)$ are respectively the price at time $t = 1$ of asset S under market scenario ω^+ and ω^- respectively. risk free one-period interest rate is r, which gives us price of the bond at time $t = 1$ as $B_1 = B_0(1 + r)$. V_0 and V_1 are price of option at time $t = 0$ and $t = 1$ respectively.

One could expect that price of option can be discounted expectation of its payoff. But we will see that real world probability does not impact the price of option. Kind of surprising!

Before proceeding further there are certain assumption that are required to construct our model. These are :

- Fraction quantity of asset and bond is allowed
- any quantity of asset and bond can be sold or bought without disturbing their price
- shorting (negative quantity) of asset and bond is allowed
- there are no transaction cost

We will construct a portfolio (x, y) (where x and y are quantity of asset S and bond B respectively) such that it exactly replicate the payoff of the option, then by law of one price option value at $t = 0$ is equal to initial price of that portfolio, otherwise arbitrage opportunity will arise. Hence we have:

$$V_0 = xS_0 + yB_0 \tag{3.3}$$

To find x and y we need to solve following system of linear equation :

$$xS_1(\omega^+) + yB_1 = \delta(\omega^+) \tag{3.4}$$

$$xS_1(\omega^-) + yB_1 = \delta(\omega^-) \tag{3.5}$$

Above equations gives us :

$$x = \frac{\delta(\omega^+) - \delta(\omega^-)}{S_1(\omega^+) - S_1(\omega^-)} \ ; \ y = \frac{\delta(\omega^-)S_1(\omega^+) - \delta(\omega^+)S_1(\omega^-)}{B_1(S_1(\omega^+) - S_1(\omega^-))} \tag{3.6}$$

In our model ω^+ represent an increase in price of stock S and ω^- represent an decrease in price of stock S. Hence we can write:

$$S_1(\omega^+) = S_0 u \ ; S_1(\omega^-) = S_0 d \tag{3.7}$$

Where $0 < d < 1$ and $u > 1$. substitute $B_1 = B_0(1 + r)$ and 3.7 in 3.6 to get:

$$x = \frac{\delta(\omega^+) - \delta(\omega^-)}{S_0(u - d)} \ ; \ y = \frac{\delta(\omega^-)u - \delta(\omega^+)d}{B_0(1 + r)(u - d)} \tag{3.8}$$

Now substitute 3.8 in 3.3 and rearrange to get the following formula :

$$V_0 = \frac{1}{1 + r}\left(\delta(\omega^+)\frac{1 + r - d}{u - d} + \delta(\omega^-)\frac{u - r - 1}{u - d}\right) \tag{3.9}$$

Write $\frac{1+r-d}{u-d} = \tilde{p}$ to get:

$$V_0 = \frac{1}{1 + r}(\delta(\omega^+)\tilde{p} + \delta(\omega^-)(1 - \tilde{p})) \tag{3.10}$$

$$= \frac{1}{1 + r}\tilde{E}(V_1) \tag{3.11}$$

The number $\tilde{p}$ and $1 - \tilde{p}$ are called risk neutral probability. One should note that under this probability measure option value at $t = 0$ is simply as discounted expectation of payoff of option at $t = 1$. We see that real world probability doesn't appear in pricing formula.

Above model is simple but make sense because our one state represent an increase in stock price and another represent a decrease in stock price. Essentially if we are not concerned about how much increase or decrease then there are only these state of world possible. Secondly this model is particularly important for those who are only concerned about initial and final price. Our trading dates exactly cover this fact.

Now to make our model more realistic, let's introduce more trading dates in our model. Let's suppose our set of maturity date is $\mathcal{D} = \{0, 1, 2, ..., T\}$. Now for each trading dates our stock price will evolve according to rule in single period model. It will rise under market scenario ω^+ and fall under market scenario ω^-. Hence stock price has following probability distribution:

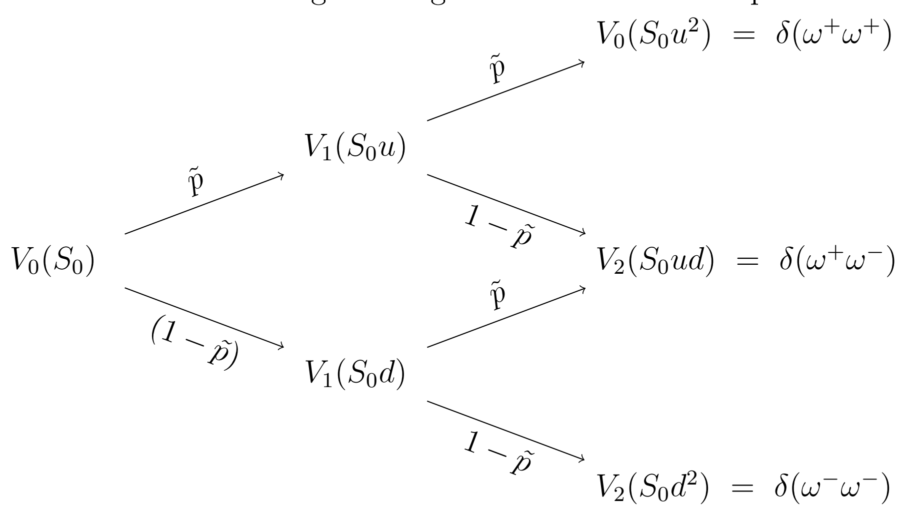
$$\mathbb{P}(S_t = S_0 u^k d^{t-k}) = \binom{t}{k} p^t (1-p)^{t-k} \ \forall \ k \in \mathcal{D} \tag{3.12}$$

We wish to price a European option having payoff function δ, which depends on stock price process. Since price at time t of European option depends on price of stock at that time, we can write $V_t(S_t)$.

In binomial tree model stock price at different time can be described as a recombining binomial tree. This can be thought as a combination of single time binomial tree. Therefore following recurrence relation find option price for different time period:

$$V_t(S_t) = \frac{1}{1+r} \left(\tilde{p} V_{t+1}(S_t u) + (1-\tilde{p}) V_{t+1}(S_t d) \right) \tag{3.13}$$

There are $k+1$ nodes in a T-period binomial model. Therefore there are $\sum_{k=0}^{T} = \frac{(T+1)(T+2)}{2}$ nodes in total. We want to find option price for each node. Following picture reflect this idea for two period binomial tree model. An example for pricing in two period binomial tree model is also given to give a clear idea of the picture.



Theorem 3.2.1. *(Theorem 4.8, [3]) In the binomial tree model, the no-arbitrage initial price of a European option with payoff $\delta(S_T)$ at discrete time $T > 1$, is given by :*

$$\begin{aligned} V_0(S_0) &= \frac{1}{(1+r)^T} \sum_{k=0}^{T} \binom{T}{k} \tilde{p}^k (1-\tilde{p})^{T-k} \delta(S_0 u^k d^{T-k}) \\ &= \frac{1}{(1+r)^T} \tilde{E}[\delta(S_T)] \end{aligned} \tag{3.14}$$

Proof. Followed by induction. □

By using Binomial distribution function one can write European call and Put option price in a more compact form. First recall cumulative distribution function of Binomial Probability distribution:

$$\mathcal{B}(m; n, p) = \sum_{k=0}^{m} \binom{n}{k} p^k (1-p)^{n-k}$$

.

Theorem 3.2.2. *The Cox-Ross-Rubinstein (CRR) Option Price Formula (Theorem 4.9, [3]) In the binomial tree model, the no-arbitrage initial price of standard European call and put options with strike price K and expiry time T are, respectively, given by :*

$$\begin{aligned} C_0^E(S_0) &= S_0 \left(1 - \mathcal{B}(m_T; T, \frac{d}{1+r}\tilde{p})\right) - \frac{K}{(1+r)^T} \left(1 - \mathcal{B}(m_T; T, \frac{d}{1+r}\tilde{p})\right) \\ P_0^E(S_0) &= \frac{K}{(1+r)^T} \mathcal{B}\left(m_T; T, \frac{d}{1+r}\tilde{p}\right) - S_0 \mathcal{B}\left(m_T; T, \frac{d}{1+r}\tilde{p}\right) \end{aligned} \tag{3.15}$$

where $m_T = max\{m : 0 \le m \le T; S_0 u^m d^{T-m} \le K\} = \lfloor \frac{\ln(K/S_0) - T \ln d}{\ln(u/d)} \rfloor$

3.3 Multinomial One Period Model

Till now we are assuming that there are only two tradable assets. Now let's increase number of tradable asset to n and suppose there are m state of world possible. We will represent these assets by $S^1, S^2, ..., S^n$ and state of world by $\Omega = \{\omega^1, \omega^2, ..., \omega^m\}$. We are assuming only two trading dates, these are $t = 0$ and $t = T$. Suppose p^i is probability of occurring of state ω^i. From these probability we can create a probability measure $\mathbb{P} : 2^\Omega \to [0, 1]$ in trivial sense. Now suppose we want to find price of a financial contract. According to our Model it's payoff will be a function χ: $\Omega \to \mathbb{R}$. Since Ω is finite we can represent it as a m dimensional vector $(\chi(\omega^1), \chi(\omega^2), ..., \chi(\omega^m))$. This set of vector can be considered as a vector space. Denote this vector space by V_χ. It's standard basis $\mathcal{E} = \{e_1, e_2, ..., e_m\}$ have a special name called AD securities, named after Arrow and Debreu.

We will represent our initial price of base asset as vector $\mathcal{S}_0 = (S_0^1, S_0^2, ..., S_0^m)^T$. At time T Payoff from these assets can be represented by a matrix Γ such that:

$$\Gamma = \begin{bmatrix} S_T^1(\omega^1) & S_T^1(\omega^2) & \cdots & S_T^1(\omega^m) \\ S_T^2(\omega^1) & S_T^2(\omega^2) & \cdots & S_T^2(\omega^m) \\ \vdots & \vdots & \ddots & \vdots \\ S_T^n(\omega^1) & S_T^n(\omega^2) & \cdots & S_T^n(\omega^m) \end{bmatrix}$$

Number of quantity of each asset is called a portfolio, which we will represent by $\phi = (\phi_1, \phi_2, \cdots, \phi_n)$, where ϕ_i is number of quantity of asset S^i (called position in asset S^i). If number of quantity of an asset is positive than it called a long position in that asset and if it is negative then it called short position in that asset. We will denote portfolio value by $\prod_t^\phi$. At time T there are m possible portfolio value because there are m state of the world. So we can view $\prod_T^\phi$ as an m dimensional vector.

If there exist a nonzero portfolio such that $\prod_T^\phi = 0$. Then there exist an asset S^j such that payoff of this asset at time T can be represented by a linear combination of other asset. This is called a redundant asset. Hence there exist a redundant base asset iff $\text{rank}(\Gamma) < n$. It is clear that a nontrivial solution of system of linear equations $\phi\Gamma = 0$ will guarantees that there exist a redundant asset.

If there exist a portfolio for a payoff $\chi \in V_\chi$, then it is said to be attainable. This portfolio is called a replicating portfolio or hedge of this payoff. Denote set of all attainable payoff by A_χ. Without much difficulty one can prove that A_χ is a vector subspace of V_χ. We can get a replicating portfolio by solving system of linear equations of $\phi\Gamma = \chi$. Following theorem finds criteria for a unique replicating portfolio. Proof is a standard practice in linear algebra.

Theorem 3.3.1. *For every attainable payoff $\chi \in A_\chi$ there is a unique hedge if and only if there are no redundant base asset.*

Below is immediate corollary of above theorem.

Corollary 3.3.1.1. *If there exist a redundant asset then every attainable payoff has infinitely many Hedge.*

An important property of a financial model is weather it is a complete or not, that is can we find a hedge for an arbitrary payoff. In other words if every payoff is attainable i.e. $A_\chi = V_\chi$ then we say that market is complete. Denote set of all payoff such that $\chi(\omega^i) \geq 0 \;\forall\; i$ by V_χ^+.

Theorem 3.3.2. *The following statements are equivalent:*

1. *Every payoff is attainable i.e.* $V_\chi = A_\chi$.
2. $V_\chi^+ \subset A_\chi$.
3. *Every Arrow-Debreu security is attainable.*

Above theorem and preceding discussion provide us useful criterion about the completeness of a market. If rank(Γ)$< m$, then market is incomplete. If rank(Γ)$< n$, then there are redundant assets. If rank(Γ)$< m = n$, then market is complete.

Theorem 3.3.3. *theorem 5.9, [3] There are no arbitrage portfolios iff there exist a strictly positive solution* $\psi = [\psi_1, \psi_2, \ldots, \psi_m]^T \gg 0 \in \mathbb{R}^m$ *of systems of linear equations obtained by*

$$D\psi = S_0$$

Proof. Initial value of portfolio $\phi \in \mathbb{R}^n$ is

$$\Pi_0^\phi = \phi S_0 = \phi(D\psi) = (\phi D)\psi = \sum_{j=1}^{m} \Pi_T^\phi(\omega^j)\psi_j$$

The no arbitrage initial price of a payoff X replicated by ϕ_X is

$$\pi_0(X) = \Pi_0^{\phi_X} = \sum_{j=1}^{m} \Pi_T^{\phi_X}(\omega^j)\psi_j = \sum_{j=1}^{m} X(\omega^j)\psi_j$$

Suppose there are no arbitrage portfolio.

$$\mathbb{R}_+^{m+1} = \{x \in \mathbb{R}^{m+1} : x \geq 0\};\;\; L = \{[-\theta S_0, \theta S_T(\omega_1), ..., \theta S_T(\omega_m)] : \theta \in \mathbb{R}^n\}$$

Suppose $\theta \in \mathbb{R}^n$ be an arbitrage portfolio (either $\theta S_0 < 0$ and $\theta D \geq 0$, or $\theta S_0 \leq 0$ and $\theta D > 0$). then non existence of arbitrage implies $T \cap \mathbb{R}_+^{m+1} = 0$

separating hyperplane theorem There exists a hyperplane $H \subset \mathbb{R}^{m+1}$ (a linear subspace of dimensions m) that separates $\mathbb{R}^{m+1}$ into two half-spaces H^+ and H^- such that

$$\mathbb{R}_+^{m+1} \subseteq H^+,\;\; L \subseteq H^-, H^+ \cap H^- = H.$$

The general equation for a hyperplane in $\mathbb{R}^{m+1}$ passing through the origin is

$$\lambda x^T = 0 \iff \lambda_0 x_0 + \lambda_1 x_1 + \cdots + \lambda_m x_m = 0$$

In our case separating hyperplane theorem implies that either $\lambda x^T > \lambda y^T$ or $\lambda x^T > \lambda y^T$ hold $\forall x \in \mathbb{R}^{m+1} \backslash \{0\}$ and $y \in L$. In particular, the set $\{\lambda y^T : y \in L\}$is bounded from above or below. This is possible iff $\lambda y^T = 0$, which implies L is contained in H. To show this, suppose that there exists $y \in L$ such that $\lambda y^T > 0$. Since L is a vector space, the set $\{a\lambda y^T : a \in \mathbb{R}\} = \mathbb{R}$ is unbounded. We arrive at a contradiction.

Now $\lambda x^T > 0$ $\forall x \in \mathbb{R}_+^{m+1}$. In particular $\lambda e_j^T = \lambda_j > 0$. Since $L \subseteq H$

$$-\lambda_0 \theta S_0 + \sum_{m}^{j=1} \lambda_j \theta S_T(\omega^j) = 0$$

holds for every portfolio $\theta \in \mathbb{R}^n$.

Now set $\theta = e_i \in \mathbb{R}^n$ for each $i = 1, 2, \ldots, n$ to get

$$-\lambda_0 S_0^i + \sum_{m}^{j=1} \lambda_j S_T^i(\omega^j) = 0$$

This implies that

$$-\lambda_0 S_0 + \sum_{m}^{j=1} \lambda_j S_T(\omega^j) = 0$$

substitute $\psi_j = \frac{\lambda_j}{\lambda_0}$ and rearrange to get

$$S_0 = \sum_{j=1}^{m} \psi_j S_T(\omega^j)$$

which in matrix for can be written as $D\psi = S_0$.

□

Following fundamental theorem from [10] of asset pricing ensures that model is arbitrage free and complete.Refer to definition 1.2.1 For definition Equivalent martingale measure (EMM) in descrete time sense.

Theorem 3.3.4. *(Theorem 1.5.2, [10]) The market is arbitrage-free and complete if and only if there exists an unique EMM.*

Chapter 4

Brownian Motion and Construction of Stochastic Integral

4.1 Properties of Brownian Motion

Markove property and martingale property are Two most important property for a stochastic process. Brownian motion satisfies both of them. For almost all stock, if we plot distribution of log return of stock it looks like a bell curve. Brownian motion is a stochastic process which have property that its increment are normally distributed. This property makes Brownian motion a good model for modeling stock path. It is also turns out that Brownian motion is also a martingale. If we look at stock price path they tend to get a positive drift over time. To overcome this deficiency geometric Brownian motion can be used to model stock path. Black Scholes formula is also based on the assumption that stock price path follow geometric Brownian motion. In recent time geometric fraction Brownian motion which in fact has dependent increment has been proposed to model stock price path because of its fatter tales [4],[7]. But it is still not clear under what condition this model is accurate as it admit arbitrage most of the time. There is still a lot of research is going on for finding condition under which it is safe to use geometric fractional Brownian Motion.

We start with the definition of standard Brownian motion. Norbert Wiener fist described Brownian motion as a stochastic process, that's why Brownian motion is usually denoted by letter W. Theory in this section is adapted from [1],[2], [3],[6] and [10].

Definition 4.1.1. *A continuous adapted stochastic process* $\{W_t\}_{t\geq 0}$ *on a probability space* $(\Omega, \mathcal{F}, \mathbb{P}, \mathcal{F}_t)$ *is said to be a standard Brownian motion if following conditions hold:*

1. $\mathbb{P}[W_0 = 0] = 1$ *(Almost every path starts at origin)*

2. *For , The increment $B_t - B_s$, where $0 \leq s < t$, is normally distributed and independent of $\mathcal{F}_s$ with mean 0 and variance $t - s$.*

3. *For almost all $\omega \in \Omega$, the sample path $W_t(\omega)$ is a continuous function of time t.*

In this project we will not give proof for existence of Brownian motion. But there are various ways by it can be constructed. First is to define a distribution using its property and then construct a probability measure and a process such that it satisfy that distribution. Second method is that is actually used by Wiener is based on Hilbert space theory. There is also an another method which is based on weak limit of random walks. Ioannis Karatzas provides rigorous proof of all these construction. Below are some useful results for Brownian motion.

Theorem 4.1.1. *Assume that $W(t)$ is standard Brownian motion, then followings are true for Brownian motion :*

1. *for $s > 0$, $W(t+s) - W(s); t > 0$ is a standard Brownian motion. This is called differential property.*

2. *for every $c \in \mathbb{R}$, $cW(t)$ is Brownian motion with variance c^2. This is called scaling property.*

3. *for every $c > 0$ $\sqrt{c}W(\frac{t}{c})$ is also a standard Brownian motion.*

4. *Define $W'(0) = 0$ and $W'(t) = tW(\frac{1}{t})$ for $t > 0$, then W' is also a standard Brownian motion.*

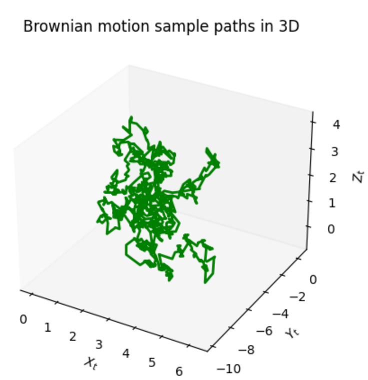


Figure 4.2: Brownian Sample Paths in 3D

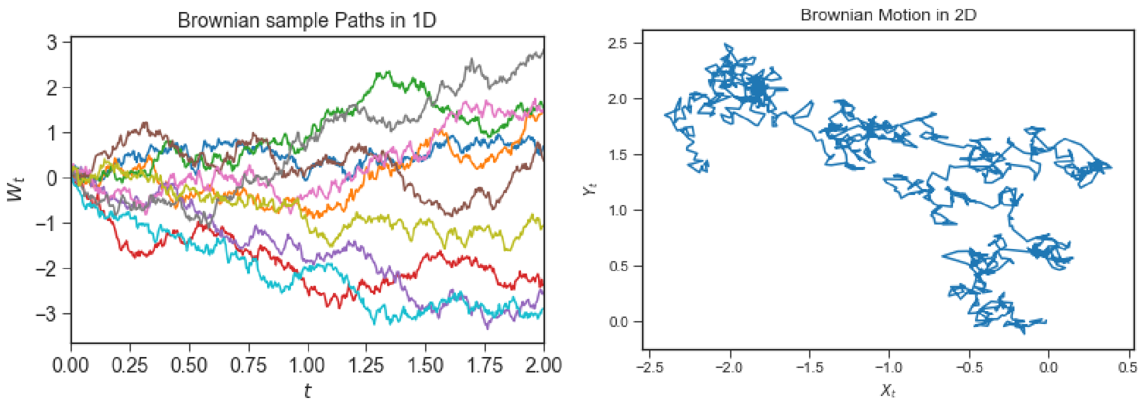


Figure 4.1: Brownian Sample Paths in 1D and 2D

Covariance of standard Brownian motion

take $0 \leq s \leq t$

$$\begin{aligned} Cov(W_s, W_t) &= E[W_s \cdot W_t] - E[W_s] \cdot E[W_t] \\ &= E[W_s(W_t - W_s) + W_s^2] \\ &= E[W_s] \cdot E[W_t - W_s] + Var(W_s) \\ &= 0 + s = s \end{aligned}$$

Brownian motion is Integrable.

$E[|W_t|] \leq \sqrt{E[W_t^2]} = \sqrt{t} < \infty$

Assume that $\{\mathcal{F}_t\}_{t\geq 0}$ be a filtration of Brownian motion.

Theorem 4.1.2. *Brownian motion is martingale w.r.t. filtration* $\{\mathcal{F}_t\}_{t\geq 0}$.

Proof. Since Brownian motion is integrable we only need to prove $E[W_t|\mathcal{F}_s] = W_s$ for $0 \leq s \leq t$. Now W_s is $\mathcal{F}_s$-measurable and $W_t - W_s$ is independent of $\mathcal{F}_s$, hence :

$$\begin{aligned} E[W_t|\mathcal{F}_s] &= E[(W_t - W_s) + W_s|\mathcal{F}_s] \\ &= E[W_t - W_s|\mathcal{F}_s] + E[W_s|\mathcal{F}_s] \\ &= 0 + W_s = W_s \end{aligned} \tag{4.1}$$

□

Theorem 4.1.3. $X_t = \{W_t^2 - t\}_{t\geq 0}$ *is a martingale w.r.t. any filtration* $\{\mathcal{F}_t\}_{t\geq 0}$ *of Brownian motion.*

Proof. $E[|X_t|] \leq E[W_t^2] + t = 2t < \infty$.We only need to show $E[X_t|\mathcal{F}_s] = X_s$ for $s \leq t$. Now:

$$E[W_t^2 - t|\mathcal{F}_s] = E[W_t^2|\mathcal{F}_s] - t$$

Now we need to compute $E[W_t^2|\mathcal{F}_s]$:

$$\begin{aligned}
E_s[W_t^2] &= E_s[(W_t - W_s + W_s)^2] \\
&= E_s[(W_t - W_s)^2 + 2W_s(W_t - W_s) + W_s^2] \\
&= E_s[(W_t - W_s)^2] + 2E_s[W_s(W_t - W_s)] + E_s[W_s^2] \qquad (4.2) \\
&= E[(W_t - W_s)^2] + 2W_s E[W_t - W_s] + W_s^2 \\
&= Var(W_t - W_s) + 2W_s \cdot 0 + W_s^2 = (t - s) + W_s^2
\end{aligned}$$

□

4.2 Sample Path Analysis of Brownian Motion

In this section we will analyse those property of Brownian motion which hold with property one. Assume W_t be a standard Brownian motion. Consider the set of pair of (t, ω) when sample path hit origin:

$$\mathcal{Z} = \{(t, \omega) \in [0, \infty) \times \Omega \mid W_t(\omega) = 0\}$$

and for any fix ω define:

$$\mathcal{Z}_\omega = \{t \in [0, \infty) \mid W_t(\omega) = 0\}$$

Non-differentiability of Brownian Sample Path

Theorem 4.2.1. *For almost all $\omega \in \Omega$, following is true:*

1. *set $\mathcal{Z}_\omega$ has labesgue measure zero.*

2. *set $\mathcal{Z}_\omega$ is an unbounded closed set*

3. *set $\mathcal{Z}_\omega$ is dense in $[0, \infty)$.*

Proof. Proof of 1. based on fubini's theorem, which we will directly use here. Let $\mathcal{M}$ denotes lebesgue measure of a set. Then by fubini's theorem:

$$E[\mathcal{M}(\mathcal{Z}_\omega)] = \int_0^\infty \mathbb{P}[W_t = 0] dt = 0$$

. therefore $(Z_\omega) = 0$ for almost all $\omega \in \Omega$. Note that $\mathcal{Z}_\omega$ is the inverse image of the closed set $\{0\}$ under the almost surely continuous map $t \to W_t(\omega)$. Therefore $\mathcal{Z}_\omega$ is closed set. It is unbounded because SBM return to origin infinitely often with probability one (can be easily proved). $\mathcal{Z}_\omega$ is dense in $[0, \infty)$ can be proved by showing that it has no isolated point in $[0, \infty)$. □

Theorem 4.2.2. *[1] The sample path $W_t(\omega)$ is monotone in no interval for almost all $\omega \in \Omega$.*

Theorem 4.2.3. *[1] The sample path $W_t(\omega)$ of Brownian motion is nowhere differential for almost all $\omega \in \Omega$.*

Non-Differentiability of Brownian sample path can also be shown by probabilistic argument shown below:

$$\tfrac{W_{t+\delta t} - W_t}{\delta t} \overset{d}{=} \tfrac{\sqrt{\delta t}\ Z}{\delta t} = \tfrac{Z}{\sqrt{\delta t}}, Z \sim \mathcal{N}(0, 1)$$

Now observe that for any $c > 0$:

$$\mathbb{P}\left[\left|\tfrac{Z}{\sqrt{\delta t}} > c\right|\right] = \mathbb{P}[|Z| > c\sqrt{\delta t}] \overset{\delta t \to 0}{\longrightarrow} \mathbb{P}[|Z| > 0] = 1$$

Therefore the ratio $\frac{W_{t+\delta t} - W_t}{\delta t}$ is unbounded as $\delta t \to 0$ with probability 1.

Quadratic Variation

Definition 4.2.1. *For $p > 0$, p-variation of a function $f : \mathbb{R} \to \mathbb{R}$ on $[a, b]$ is defined as:*

$$V^p_{[a,b]}(f) = \limsup_{|\Pi|_n \to 0} \sum_{i=1}^{n} |f(x_i) - f(x_{i-1})|^p$$

Where $|\Pi|_n = \max\{(x_i - x_{i-1}) : i \in \{1, 2, ..., n\}\}$.

If $V^p_{[a,b]}(f) < \infty$, then f is said to be a function of bounded p-variation on $[a, b]$.

Proposition 4.2.1. *Bounded monotone functions have bounded first variation.*

Proof. Consider a function f that is non-decreasing on $[a, b]$. For any partition $a = x_0 < x_1 < ... < x_n = b$,

$$\sum_{i=1}^{n} |f(x_i) - f(x_{i-1})| = \sum_{i=1}^{n} (f(x_i) - f(x_{i-1})) = f(b) - f(a)$$

□

Theorem 4.2.4. *Differential functions with a bounded derivative have bounded first variations.*

Proof.

$$\begin{aligned} V^p_{[a,b]}(f) &= \sum_{i=1}^{n} |f(x_i) - f(x_{i-1})| \\ &= \sum_{i=1}^{n} |f'(x'_i)|(x_i - x_{i-1}) \\ &\leq M \sum_{i=1}^{n} (x_i - x_{i-1}) = M(b-a) \end{aligned} \tag{4.3}$$

□

Theorem 4.2.5. *The quadratic variation of a differential function with a bounded derivative is zero.*

Proof.

$$\begin{aligned} \sum_{i=1}^{n} |f(x_i) - f(x_{i-1})|^2 &= \sum_{i=1}^{n} f'(x'_i)^2 (x_i - x_{i-1})^2 \\ &\leq |\Pi|_n M^2 \sum_{i=1}^{n} (x_i - x_{i-1}) = |\Pi|_n M^2 (b-a) \overset{|\Pi|_n \to 0}{\longrightarrow} 0 \end{aligned} \tag{4.4}$$

□

Theorem 4.2.6. *First Variation $V_{[0,t]}(W)^1 = \infty$ and quadratic variation $[W,W](t) = t \; \forall t > 0$ for Brownian motion $\{W_t\}_{t \geq 0}$.*

Proof. Proof for first variation is trivial. For Quadratic variation Consider $V_n = \sum\limits_{i=1}^{n} (W(t_i) - W(t_{i-1}))^2$. Calculate :

$$E[V_n] = \sum_{i=1}^{n} E[(W(t_i) - W(t_{i-1}))^2] = \sum_{i=1}^{n} (t_i - t_{i-1}) = t$$

$$\text{Var}(V_n) = \sum_{i=1}^{n} \text{Var}((W(t_i) - W(t_{i-1}))^2) = \sum_{i=1}^{n} (t_i - t_{i-1})^2 \text{Var}(Z^2) =$$
$$\sum_{i=1}^{n} (t_i - t_{i-1})^2 (E[Z^4] - (E[Z^2])^2) = \sum_{i=1}^{n} (t_i - t_{i-1})^2 (3 - 1^2) = \sum_{i=1}^{n} 2(t_i - t_{i-1})^2 \to 0$$

Therefore $[W,W](t) = \lim\limits_{n \to \infty} = t(a.s.)$. □

Theorem 4.2.7. *(Proposition 11.2, [3]) Assume that f and g do not have discontinuities at the same point in time interval $[0,T]$. If p-variation of f and q variation of g are finite for some $p, q > 0$ and $\frac{1}{p} + \frac{1}{q} > 1$. Then the Riemann-Stieltjes integral $\int_0^T f(t) dg(t)$ exists and finite.*

We have proved that quadratic variation of Brownian motion is finite and first variation is infinite. It is also know that p- variation of Brownian motion is infinite for $p < 2$ and infinite for $p \geq 2$. Now we want to calculate Riemann-Stieltjes Integral $\int_0^t g(s)dW_s$. Now since quadratic variation of Brownian motion is finite, apply theorem 4.2.7 for $q = 2$. Then $\int_0^t g(s)dW_s$ exist only if there exist $p \in (0, 2)$ such that p-variation of g is finite. Now if we want to consider integral like $\int_0^t W(s)dW_s$, then this integral does not exist in Riemann-Stieltjes sense. This lead us to develop stochastic calculus.

Let $\{\mathcal{F}_t\}_{t\geq 0}$ be a filtration.Define σ-field of events strictly prior to $t > 0$:

$$\mathcal{F}_{t^-} = \sigma(\bigcup_{s<t} \mathcal{F}_s)$$

For $t = o$, define :

$$\mathcal{F}_{0^-} = \mathcal{F}_0$$

Define σ-field of events immediately after $t \geq 0$:

$$\mathcal{F}_{t^+} = \bigcap_{\epsilon>0} \mathcal{F}_{t+\epsilon}$$

Definition 4.2.2. *A filtration $\mathcal{F}_t$ is called right continuous (left continuous) if $\mathcal{F}_t = \mathcal{F}_t^+$ ($\mathcal{F}_t = \mathcal{F}_t^-$) for every $t \geq 0$*

Above definition raise one question in our mind. Why the name right continuous (left continuous)? In case of filtration σ-algebra increase by time. In particular for ϵ just greater than 0 we have $\mathcal{F}_t \subset \mathcal{F}_{t+\epsilon}$. By imposing above definition one want to be sure that there is no gap between $\mathcal{F}_t$ and $\mathcal{F}_{t+\epsilon}$ and that is why the name continuous.

Definition 4.2.3. *A random time T is an $\mathcal{F}$-measurable random variable which takes value from* $[0, \infty]$

Intuitively a stopping time is a strategy that tells us when we should stop a particular phenomena based on information available till that moment. Suppose one decide that he will close his position for a certain stock when stock will it at a certain fixed level, is a stopping time.

Definition 4.2.4. *Let X be a stochastic process and T be random time. Define function X_T on event $\{T < \infty\}$ as follows:*

$$X_T(\omega) = X_{T(\omega)}(\omega)$$

Definition 4.2.5. *A random time T is called stopping time of filtration $\{\mathcal{F}_t\}$ if:*

$$\forall t \geq 0, \text{ we have } \{T \leq t\} \in \mathcal{F}_t$$

Definition 4.2.6. *A random time T is called optional time of filtration $\{\mathcal{F}_t\}$ if:*

$$\forall t \geq 0, \text{ we have } \{T < t\} \in \mathcal{F}_t$$

Proposition 4.2.2. *Every stopping time is optional and in case of right continuous filtration both these notions are same.*

Proof. Let T be stopping time.Note that $\{T \leq t - \frac{1}{n}\} \in \mathcal{F}_{t-\frac{1}{n}} \subseteq \mathcal{F}_t$ for $n \geq 1$. Now we have $\{T < t\} = \bigcup_{n=1}^{\infty} \{T \leq t - \frac{1}{n}\} \in \mathcal{F}_t$.

Suppose T is an optional time of right-continuous filtration $\{\mathcal{F}_t\}$. Since $T \leq t\} = \bigcap_{\epsilon>0}\{T < t + \epsilon\}$. we have $\{T \leq t\} \in \mathcal{F}_{t+\epsilon}$ for every $t \geq 0$ and every $\epsilon > 0$. Hence $\{T \leq t\} \in \mathcal{F}_t^+ = \mathcal{F}_t$. □

Chung and Doob (1965) defined another criteria for measurable process given below:

Definition 4.2.7. *A stochastic Process X is said to be progressively measurable with respect to filtration $\{\mathcal{F}_t\}$ if for all $B \in \mathcal{B}(\mathbb{R})$, $\{(s,\omega) \mid 0 \leq s \leq t, \omega \in \Omega, X_s(\omega) \in B\} \in \mathcal{B}([0,t]) \times \mathcal{F}_t$ for each $t \geq 0$.*

Theorem 4.2.8. *A progressively measurable modification exists for a measurable and adapted stochastic process $(X_t, \mathcal{F}_t)$.*

Suppose in a gambling game a gambler is able to make profit by leaving game at a time which is based on some rule which do not depend on future (at stopping time), then this type of game would not be a fair game. Doob's Optional sampling theorem provide condition under which this kind of strategy will not work.

Theorem 4.2.9. *Doob's Optional Sampling Theorem, Theorem 3.22 [1] Suppose X_t is a submartingale and $\mathcal{T}$ be a almost surely bounded stopping time. Then we have:*

$$E[X_{\mathcal{T}}] \geq E[X_0]$$

In case if X_t is supermartingale, then

$$E[X_T] \leq E[X_0]$$

. Equality will hold if X_t is a martingale.

Martingale property also allow us to find convergence of stochastic process under certain weak condition then uniform integrable requirement. Below theorem is stated for supermartingale, a similar resut is also true for submartingale.

Theorem 4.2.10. *Supermartingale convergence theorem, [1] If X_t is a supermartingale such that*

$$sup_n E[|X_n|] < \infty$$

Then $\lim_n X_n$ exist almost surely with finite expectation.

4.3 Construction of Stochastic Integral and Geometric Brownian Motion

Theory in this section is adapted from [1] and [2]

Definition 4.3.1. *An adapted process $\mathcal{A}$ is called increasing if for almost all $\omega \in \Omega$ process starts from origin and function $t \mapsto \mathcal{A}_t(\omega)$ is a non decreasing right continuous function.*

Definition 4.3.2. *Let X be a stochastic process and T be random time. Define function X_T on event $\{T < \infty\}$ as follows:*

$$X_T(\omega) = X_{T(\omega)}(\omega)$$

Let f be a finite number such that $f > 0$. Let $\mathcal{L}_f$ denote the class of all stopping time T such that $\mathbb{P}(T \leq f) = 1$ for all such f.

Definition 4.3.3. *A right continuous process $(\mathcal{X}_t, \mathcal{F}_t)$ is said to be of class $\mathcal{DL}$ if for every f, the family $\{X_T\}_{T \in \mathcal{L}_f}$ is uniformly intergrable.*

Theorem 4.3.1. *Doob-Meyer Decomposition If $(X_t, \mathcal{F}_t)$ is a right continuous submartingale which belongs to class $\mathcal{DL}$. Then it can be decomposed as a summation of right continuous martingale and an increasing process $\mathcal{A}_t$:*

$$X_t = M_t + \mathcal{A}_t \ , 0 \leq t < \infty \tag{4.5}$$

Definition 4.3.4. *A right continuous martingale* $(X_t, \mathcal{F}_t)$ *is said to be square integrable if* $E[X_t^2] < \infty$ *and* $t \geq 0$.

Let $\mathcal{M}_2$ denote the space of all square integrable right continuous martingale. Denote by $\mathcal{M}_2^c$, the subset of $\mathcal{M}_2$ consisting of almost surely continuous processes.

Using Doob-Meyer Decomposition we can provide an alternative definition of quadratic variation and this definition can be proved same as previous one and vice-versa.

Definition 4.3.5. *Consider the Doob-Meyer decomposition of* $X \in \mathcal{M}_2$ *:*

$$X_t = M_t + \mathcal{A}_t$$

Then $\mathcal{A}_t$ *is called the quadratic variation* $\langle X \rangle$ *of* X*. Clearly* $X - \langle X \rangle$ *is a martingale.*

Since quadratic variation of Brownian motion is t, above definition is a direct proof of martingale property of process $\{W_t^2 - t\}_{t\geq 0}$.

Let $\mathcal{L}_2$ be the space of all process such that Riemann integral $\int_0^T X_s ds$ exists almost surely and $E\left[\int_0^T X^2(s)ds\right] < \infty$ for all $T > 0$.

Definition 4.3.6. *We will say that a process* $X \in \mathcal{L}_2$ *is simple if there exist countable partition* $\{t_n\}_{n\in\mathbb{N}\cup\{0\}}$ *such that* $X_t(\omega) = X_{t_k}(\omega)\ \forall\ t \in [t_k, t_{k+1})$*, where* $k \in \mathbb{N}\cup\{0\}$.

for such type of process we will define its Ito integral by:

$$I_t(X) = \int_0^t X_s dW_s = \sum_{k=0}^{n-1} X_{t_k}(\omega)(W_{t_{k+1}}(\omega) - W_{t_k}(\omega)) + X_{t_n}(\omega)(W_t(\omega) - W_{t_n}(\omega)) \quad (4.6)$$

where $n = \max\{k\ ;\ t_k \leq t\}$.

To define Ito integral for any arbitrary process from $\mathcal{L}_2$, it has been shown that any process $X \in \mathcal{L}_2$ can be approximated by sequence of simple process and this sequence will converge in the sense of following two theorems:

Theorem 4.3.2. *For any process* $X \in \mathcal{L}_2$ *there exist a sequence of simple process* X_n *such that*

$$\lim_{n\to\infty} E\left[\int_0^t (X_n(s) - X(s))^2 ds\right] = 0 \quad (4.7)$$

Theorem 4.3.3. *Assume we have got a sequence of simple process* X^n *satisfying above. Then there exist a process* $\Gamma_t \in \mathcal{M}_2^c$ *such that:*

$$\lim_{n\to\infty} E\left[(\Gamma_s - I_s(X^n))^2\right] = 0\ \forall\ 0 \leq s \leq t. \quad (4.8)$$

This process Γ_t *is unique upto indistinguishable class.*

Theorem 4.3.4. *For a stochastic Process $X_t \in \mathcal{L}_2$, It's Ito Inegral with respect to Brownian motion is the unique process $\Gamma_t \in \mathcal{M}_2^c$ defined in theorem 4.3.3.*

Now to define Ito integral for general process we can just choose a sequence of simple process by theorem 4.3.2 and this sequence converge by theorem 4.3.3

Theorem 4.3.5. *Ito integral satisfies following:*

1. $I_t(\alpha X + \beta Y) = \alpha I_t(X) + \beta I_t(Y)$.
2. ***Ito Isometry*** *:*$E\left[I_t^2(X)\right] = E\left[\int_0^t X_s^2 ds\right]$.
3. $I_t(X) \in \mathcal{M}_2^c$.

Definition 4.3.7. *If a stochastic process X_t on $(\Omega, \mathcal{F}, \mathbb{P}, \mathcal{F}_t)$ can be written as following:*

$$X_t = X_0 + \int_0^t \alpha_s dW_s + \int_0^t \beta_s ds \tag{4.9}$$

where $\alpha, \beta \in \mathcal{L}_2$, then it is called Ito Process.

As we will see later for making computation handy we will write above as:

$$dX_t = \alpha_t dW_t + \beta_t dt \tag{4.10}$$

We will denote $\alpha_t^2 dt$ by $(dX_t)^2$. This is also convenient since it can be obtain by simple computation rule. Because

$$(dX_t)^2 = dX_t \cdot dX_t = (\alpha_t dW_t + \beta_t dt) \cdot (\alpha_t dW_t + \beta_t dt) =$$

Now using $dt \cdot dt = dW_t \cdot dt = dt \cdot dW_t = 0$ and $dW_t \cdot dW_t = dt$, we will obtain $\alpha_t^2 dt$.

Theorem 4.3.6. ***Ito-Doeblin Formula for Ito Process*** *Let X_t be an Ito process. Let $f(x,t)$ be a function whose partial derivatives $f_x(x,t), f_t(x,t)$ and $f_{xx}(x,t)$ exists and continuous. Then we have:*

$$f(X_t, t) = f(X_0, 0) + \int_0^t f_x(X_s, s) dX_s + \int_0^t f_s(X_s, s) ds + \frac{1}{2}\int_0^t f_{xx}(X_s, s)(dX_t)^2 \tag{4.11}$$

Above can be written as below to make computation handy:

$$df(X_t, t) = f_x(X_t, t)\alpha_t dW_t + f_x(X_t, t)\beta_t dt + f_t(X_t, t)dt + \frac{1}{2} f_{xx}(X_t, t)\alpha_t^2 dt \tag{4.12}$$

There are various process which are derived from Brownian motion and have applications in various area of natural science. We start with drifted Brownian motion.

Let $\mu, \sigma \in \mathbb{R}$, $\sigma > 0$ be constant. Consider the following process:

$$W_t^{(\mu,\sigma)} = \mu t + \sigma W_t \tag{4.13}$$

This is called Brownian motion with drift μ and scale parameter (volatility) σ.

Some Properties of Brownian motion with drift

1. $W_t^{(\mu,\sigma)} \sim \mathcal{N}(\mu t, \sigma^2 t)$ and $Cov(W_s^{(\mu,\sigma)}, W_t^{(\mu,\sigma)}) = \sigma^2 \min(s, t)$.

2. Brownian motion with drift is not a martingale

3. Brownian motion with positive drift is a submartingale

4. Brownian motion with negative drift is a supermartingale

Geometric Brownian motion is defined as the exponential of drifted Brownian motion. In development of Black-scholes model it is required that asset will follow a geometric Brownian motion. Hence we will denote this by S_t :

$$S(t) = e^{c + W_t^{(\mu,\sigma)}} = S_0 e^{\mu t + \sigma W_t} \tag{4.14}$$

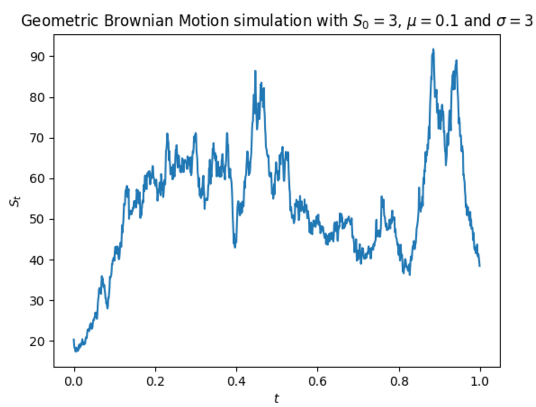


Figure 4.3: Sample Path of Geometric Brownian motion

Chapter 5

Black Scholes Option Pricing Model

So far we have developed the machinery which needed for option price Model. The core idea behind this derivation is based on forming a replicating portfolio which produce same payoff as option. Thus we will form a portfolio in risk-free bond, underlying stock and option. Black and Scholes just after their ground braking paper in 1973, provided a method by which option pricing problem can be transformed to solving a PDE, in particular parabolic PDE with a terminal condition. There are few assumption that are required to develop our model. These are :

1. We are assuming that option is European
2. Market that we are dealing with is free of arbitrage.
3. Log returns of stock is normally distributed.
4. We are not considering transaction cost and dividend.
5. risk free rate and volatility are constant and known.

Now in Black Scholes Model there is an assumption that stock price will follow geometric Brownian motion. Which means stock has constant rate of return and volatility. But in general rate of return and volatility can be time varying. So in general stock priced as exponential of an Ito process X_t (given below) multiplied by it's Initial Value.

$$X_t = \int_0^t \sigma(s)dW_s + \int_0^t (\mu(s) - \frac{1}{2}\sigma^2(s))ds \tag{5.1}$$

As mention Above asset price process S_t will be given by:

$$S_t = S_0 e^{X_t} = S_0 e^{\left(\int_0^t \sigma(s)dW_s + \int_0^t (\mu(s) - \frac{1}{2}\sigma^2(s))ds\right)} \tag{5.2}$$

Now choose $f(x,t) = S_0 e^x$, then we have $S_t = f(X_t,t)$. Now since X_t is an Ito process, by using Ito Doeblin formula for Ito process we get:

$$dS_t = df(X_t,t) \tag{5.3}$$
$$= f_x(X_t,t)\sigma(t)dW_t + f_x(X_t,t)\left(\mu(t) - \frac{1}{2}\sigma^2(t)\right)dt + f_t(X_t,t)dt + \frac{1}{2}f_{xx}(X_t,t)\sigma^2(t)dt \tag{5.4}$$
$$= S_0 e^{X_t}\sigma(t)dW_t + S_0 e^{X_t}\left(\mu(t) - \frac{1}{2}\sigma^2(t)\right)dt + \frac{1}{2}S_0 e^{X_t}\sigma^2(t)dt \tag{5.5}$$
$$= S_t\sigma(t)dW_t + S_t\mu(t)dt \tag{5.6}$$
$$= \mu(t)S_t dt + \sigma(t)S_t dW_t \tag{5.7}$$

Now in Black-Sholes model we are assuming volatility and rate of return are not time varying and random,hence we have:

$$dS_t = \mu S_t dt + \sigma S_t dW_t \tag{5.8}$$

Now in case of constant rate of return and volatility equation 5.2 becomes:

$$S_t = S_0 e^{(\sigma W_t + \left(\mu - \frac{1}{2}\sigma^2\right)t)} \tag{5.9}$$

This is the mathematical expression of geometric Brownian motion.

Now this Geometric Brownian motion is considered as a solution of following stochastic differential equation:

$$dS_t = \mu S_t\ dt + \sigma S_t\ dW_t \tag{5.10}$$

Solution can also obtained by Using Ito Doeblin formula on function $f(x,t) = \ln x$. Note that Above equation is just a shorthand Notation of Ito process:

$$S_t = S_0 + \int_0^t S_s dW_s + \int_0^t \mu S_s ds \tag{5.11}$$

Now using Ito-Doeblin formula we get:

$$\begin{aligned} d\ln(S_t) &= \frac{1}{S_t}\left(\mu S_t\ dt + \sigma S_t\ dW_t\right) - \frac{1}{2} \times \frac{1}{S_t^2}\sigma^2 S_t^2\ dt \\ &= \mu\ dt + \sigma\ dW_t - \frac{1}{2}\sigma^2\ dt \\ &= \left(\mu - \frac{1}{2}\sigma^2\right)dt + \sigma\ dW_t \end{aligned}$$

Above will give us:

$$S_t = S_0 e^{(\sigma W_t + \left(\mu - \frac{1}{2}\sigma^2\right)t)} \tag{5.12}$$

So Now it is clear that In our model stock price is driven by Geometric Brownian motion which can represented by SDE 5.10.

Let $V = V(S,t)$ denote the value of an option (or a contingent claim) that is sufficiently smooth, namely, its second-order derivatives with respect to S and first-order derivative with respect to t are continuous.

Apply Ito formula for Ito process S_t to get:

$$dV = \left(\mu S \frac{\partial V}{\partial S} + \frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}\right) dt + \sigma S \frac{\partial V}{\partial S} dW.$$

Since both stochastic processes S and V are driven by the same Wiener process (Brownian motion) W, the stochastic term, $\sigma S \frac{\partial V}{\partial S} dW$, can be eliminated by constructing a portfolio that consists of the option and the underlying asset. Let P be the wealth of the portfolio that consists of one short position in option with value V and Δ units of the underlying asset with the price S. Now at time t, value of above portfolio will be

$$P = -V + \Delta S,$$

suppose P_0 is the initial capital of the portfolio.

Now the infinitesimal change in the portfolio becomes

$$\begin{aligned} dP &= -dV + \Delta dS \\ &= -\left[\left(\mu S \frac{\partial V}{\partial S} + \frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}\right) dt + \sigma S \frac{\partial V}{\partial S} dW\right] + \Delta\left(\mu S dt + \sigma S dW\right) \\ &= -\left(\mu S \left(\Delta - \frac{\partial V}{\partial S}\right) + \frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}\right) dt + \left(-\frac{\partial V}{\partial S} + \Delta\right) \sigma S dW. \end{aligned}$$

If we choose $\Delta = \frac{\partial V}{\partial S}$, we can hedge the portfolio by eliminating the stochastic term. Now the infinitesimal change dP of the portfolio within the time interval dt is:

$$dP = -\left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}\right) dt \tag{5.13}$$

Further, in infinitesimal time dt the portfolio capital will grow by $rPdt$;

$$dP = rPdt = r(-V + \Delta S)dt \tag{5.14}$$

$$= -rV + rS\frac{\partial V}{\partial S} \tag{5.15}$$

Now under no arbitrage principle, the two investment should give the same infinitesimal change dP. Thus we have:

$$-rV + rS\frac{\partial V}{\partial S} = -\left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}\right) \tag{5.16}$$

Hence,

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0. \tag{5.17}$$

This is the celebrated Black-Scholes-Merton equation for European options.

5.1 Solution of the Black-Scholes Equation

We can obtain solution of Black-Scholes equation by two ways, first is by converting it to heat equation and using Fourier transform technique and other is by changing probability measure. We first discuss solution by Fourier transform technique.

5.1.1 Solution of BSM PDE by Converting it to Heat Equation

Definition 5.1.1 (Fourier transform)**.** *For a function f, the Fourier transform is given by*

$$\hat{f}(\omega) = \int_{-\infty}^{\infty} e^{-i\omega x} f(x)dx, \quad -\infty < \omega < \infty.$$

Definition 5.1.2 (Inverse Fourier transform)**.** *The inverse Fourier transform is given by*

$$\hat{f}(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega x} \hat{f}(\omega)d\omega.$$

Example 5.1.1. *Suppose $f(x) = e^{-ax^2/2}$, $a > 0$. Then*

$$f'(x) = -axf(x),$$

taking Fourier transform in both side yields

$$\begin{aligned} i\omega\hat{f}(\omega) &= -ai\frac{d}{d\omega}\hat{f}(\omega) \\ &\implies \frac{d}{d\omega}\hat{f}(\omega) = -\frac{\omega}{a}\hat{f}(\omega) \\ &\implies f'(\omega) = Ae^{-\frac{\omega^2}{2a}}. \end{aligned}$$

To calculate A, put $\omega = 0$, which implies

$$A = \hat{f}(0) = \int_{-\infty}^{\infty} e^{-\frac{ax^2}{2}} dx = \sqrt{\frac{2\pi}{a}}.$$

Thus

$$\mathcal{F}\left(e^{-\frac{ax^2}{2}}\right) = \sqrt{\frac{2\pi}{a}} e^{-\frac{\omega^2}{2a}}.$$

Solution of the Heat Equation

The standard heat equation is

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \; ; -\infty < x < \infty, \; t > 0,$$
$$u(x, 0) = f(x).$$

Taking Fourier transform in both hand side implies

$$\frac{d}{dt}\hat{u}(\omega, t) = c^2 (i\omega)^2 \hat{u}(\omega, t)$$
$$\implies \hat{u}(\omega, t) = A(\omega) e^{-c^2\omega^2 t},$$

where $A(\omega)$ is a constant depending on ω.

We have $u(x, 0) = f(x)$, which implies $\hat{u}(\omega, 0) = \hat{f}(\omega)$, leading to $A(\omega) = \hat{f}(\omega)$. Hence

$$\hat{u}(\omega, t) = \hat{f}(\omega) e^{-c^2\omega^2 t}.$$

Using convolution property of Fourier transform, we have

$$\mathcal{F}(f * g) = \mathcal{F}(f)\mathcal{F}(g) \implies \mathcal{F}^{-1}(\hat{f}\hat{g}) = f * g.$$

Thus

$$u(x, t) = \mathcal{F}^{-1}(u(\omega, t)) = \frac{1}{2c\sqrt{\pi t}} \int_{-\infty}^{\infty} f(s) e^{-\frac{(x-s)^2}{4c^2 t}} ds.$$

Here, we have used that

$$\mathcal{F}^{-1}(e^{-c^2\omega^2 t}) = \frac{1}{2c\sqrt{\pi t}} e^{-\frac{x^2}{4c^2 t}}.$$

The Black-Scholes equation is given by,

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0, \tag{5.18}$$

where $V(0, t) = 0$ and $V(S, T) = \max\{S - K, 0\}$. PDE of this type are called Parabolic PDE by below classification.

Classification of Second Order PDE

Suppose the linear second order PDE has the form

$$A\frac{\partial^2 u}{\partial x^2} + 2B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + D = 0,$$

then this PDE is called

- Parabolic if $B^2 - AC = 0$
- Elliptic if $B^2 - AC < 0$
- Hyperbolic if $B^2 - AC > 0$.

Now make the following change of variables in Black-Scholes PDE

$t = T - \frac{\tau}{\sigma^2/2}$, which gives $\tau = \frac{\sigma^2}{2}(T-t)$, $S = Ke^x$, which gives $x = \log(\frac{S}{K})$, $V(S,t) = Kv(x,\tau)$. Thus we have $V(S,T) = Kv(x,0)$, since $T = T - \frac{\tau}{\sigma^2/2} \implies \tau = 0$.

Now we have $\frac{\partial V}{\partial t} = \frac{\partial V}{\partial \tau}\frac{d\tau}{dt} = K\frac{\partial v}{\partial \tau}(-\frac{\sigma^2}{2})$, $\frac{\partial V}{\partial S} = \frac{\partial V}{\partial x}\frac{dx}{dS} = K\frac{\partial v}{\partial x}(\frac{1}{S})$

Also,

$$\begin{aligned}\frac{\partial^2 V}{\partial S^2} &= \frac{\partial}{\partial S}\left(\frac{\partial V}{\partial S}\right) = \frac{\partial}{\partial S}\left(K\frac{\partial v}{\partial x}\frac{1}{S}\right) = K\frac{\partial v}{\partial x}\left(-\frac{1}{S^2}\right) + K\frac{\partial}{\partial S}\left(\frac{\partial v}{\partial x}\right)\frac{1}{S}\\ &= K\frac{\partial v}{\partial x}\left(-\frac{1}{S^2}\right) + K\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial x}\right)\frac{dx}{dS}\frac{1}{S} = K\frac{\partial v}{\partial x}\left(-\frac{1}{S^2}\right) + K\frac{\partial^2 v}{\partial x^2}\left(\frac{1}{S^2}\right).\end{aligned}$$

Further, $V(S,T) = \max\{S-K, 0\} = K\max\{e^x - 1, 0\}$. But $V(S,T) = Kv(x,0)$ which implies $v(x,0) = \max\{e^x - 1, 0\}$.

Substituting these values in BS equation, yields

$$\frac{\partial v}{\partial \tau} = \frac{\partial^2 v}{\partial x^2} + (p-1)\frac{\partial v}{\partial x} - pv = 0, \tag{5.19}$$

where $p = \frac{r}{\sigma^2/2}$ and $v(x,0) = \max\{e^x - 1, 0\}$.

Now above equation has only one parameter p. Now since we have $x = \log(S/K)$, where $S, K > 0$. Above equation is defined for $x \in (-\infty, \infty)$.

Now substitute

$$v(x,\tau) = e^{\alpha x + \beta \tau}u(x,\tau) \tag{5.20}$$

where α and β will be chosen appropriately.

Substituting equation 5.20 in equation 5.19 gives:

$$u_\tau = u_{xx} + (2\alpha + p - 1)u_x + (\alpha^2 + (p-1)\alpha - p - \beta)u.$$

Choose $2\alpha + p - 1 = 0$ and $\alpha^2 + (p-1)\alpha - p - \beta = 0$. This gives $\alpha = \frac{1-p}{2}$ and $\beta = -\frac{1}{4}(p+1)^2$.Now the above PDE reduces to

$$u_\tau = u_{xx},$$

which is the standard heat equation.

Now, the initial condition will change to

$$\begin{aligned} u(x,0) &= e^{-\alpha x} v(x,0) = e^{\frac{(p-1)}{2}x} \max\{e^x - 1, 0\} \\ &= \max\{e^{\frac{(p+1)}{2}x} - e^{\frac{(p-1)}{2}x}, 0\}. \end{aligned}$$

Note that $u(x,0) > 0$ when $x > 0$ and $u(x,0) = 0$ for $x \leq 0$.

Now the solution of the heat equation is

$$u(x,\tau) = \frac{1}{2\sqrt{\pi\tau}} \int_{-\infty}^{\infty} u_0(s) e^{-\frac{(x-s)^2}{4t}} ds.$$

Take $z = \frac{(s-x)}{\sqrt{2\tau}}$, which gives $dz = -\frac{1}{\sqrt{2\tau}} dx$, to get the following;

$$u(x,\tau) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u_0(z\sqrt{2\tau} + x) e^{-\frac{z^2}{2}} dz.$$

Now note that $u_0 > 0$ for $z > -\frac{x}{\sqrt{2\tau}}$. Hence above integrataion can be written as;

$$\begin{aligned} u(x,\tau) = &\frac{1}{\sqrt{2\pi}} \int_{-x/\sqrt{2\tau}}^{\infty} e^{\frac{k+1}{2}(x+z\sqrt{2\tau})} e^{-\frac{z^2}{2}} dz \\ &- \frac{1}{\sqrt{2\pi}} \int_{-x/\sqrt{2\tau}}^{\infty} e^{\frac{k-1}{2}(x+z\sqrt{2\tau})} e^{-\frac{z^2}{2}} dz \\ = &I_1 - I_2. \end{aligned}$$

For I_1;

$$\begin{aligned} I_1 &= \frac{1}{\sqrt{2\pi}} \int_{-x/\sqrt{2\tau}}^{\infty} e^{\frac{k+1}{2}(x+z\sqrt{2\tau})} e^{-\frac{z^2}{2}} dz \\ &= \frac{1}{\sqrt{2\pi}} e^{\frac{(k+1)}{2}x + \tau\frac{(k+1)^2}{4}}) \int_{-x/\sqrt{2\tau}}^{\infty} e^{-\frac{1}{2}(z - \sqrt{\frac{\tau}{2}}(k+1))^2} dz. \end{aligned}$$

Substitute $z = z + \sqrt{\frac{\tau}{2}}(k+1))^2$, to get;

$$\begin{aligned} I_1 &= \frac{1}{\sqrt{2\pi}} e^{\frac{(k+1)}{2}x + \tau\frac{(k+1)^2}{4}} \int_{-x/\sqrt{2\tau} - \sqrt{\frac{\tau}{2}}(k+1)}^{\infty} e^{-\frac{y^2}{2}} dy \\ &= e^{\frac{(k+1)}{2}x + \tau\frac{(k+1)^2}{4}} \Phi(d_1), \end{aligned}$$

where $d_1 = x/\sqrt{2\tau} + \sqrt{\frac{\tau}{2}}(k+1)$ and Φ is the standard normal CDF.

Similarly get

$$I_2 = \frac{1}{\sqrt{2\pi}} e^{\frac{(k-1)}{2}x + \tau\frac{(k-1)^2}{4}} \int_{-x/\sqrt{2\tau}-\sqrt{\frac{\tau}{2}}(k-1)}^{\infty} e^{-\frac{y^2}{2}} dy$$
$$= e^{\frac{(k-1)}{2}x + \tau\frac{(k-1)^2}{4}} \Phi(d_2),$$

where $d_2 = x/\sqrt{2\tau} + \sqrt{\frac{\tau}{2}}(k-1)$ and Φ is the standard normal CDF.

Thus the transformed heat equation has solution

$$u(x,\tau) = e^{\frac{(k+1)}{2}x + \tau\frac{(k+1)^2}{4}} \Phi(d_1) - e^{\frac{(k-1)}{2}x + \tau\frac{(k-1)^2}{4}} \Phi(d_2).$$

Now substitute the following in above equation;
$v(x,\tau) = e^{-\frac{(k-1)}{2}x - \frac{(k+1)^2}{4}\tau} u(x,\tau)$, $x = \log(S/K)$, $\tau = \frac{\sigma^2}{2}(T-t)$, $V(S,T) = Kv(x,\tau)$.
This gives us:

$$V(S,K,r,\sigma,T,t) = S\Phi(d_1) - Ke^{-r(T-t)}\Phi(d_2),$$

This is the price of Europian call option. where Φ is standard normal CDF and d_1 and d_2 are defined as

$$d_1 = \frac{\log(S/K) + (r + \sigma^2/2)(T-t)}{\sigma\sqrt{(T-t)}}$$

and

$$d_2 = \frac{\log(S/K) + (r - \sigma^2/2)(T-t)}{\sigma\sqrt{(T-t)}}.$$

Where S is the price of underlying asset, r is risk free rate, σ is stock volatility, T is exercise or maturity time and K is strike price.

We can get the price of Put option using put-call parity.

5.1.2 Solution of BSM PDE by Changing Probability Measure

Girsonov theorem is a unique element of stochastic calculus which makes it different from ordinary calculus. If we have two probability measure we can relate them by theorem 2.1.3, this gives us notion of equivalent measure.

Definition 5.1.3. *Suppose $\mathbb{P}$ and $\mathbb{Q}$ are two probability measures which are defined on same σ-field $\mathcal{F}$. If for every $A \in \mathcal{F}$ we have:*

$$\mathbb{Q}(A) = \int_A h(\omega) d\mathbb{P}(\omega)$$

then $\mathbb{Q}$ is said to be absolutely continuous with respect to $\mathbb{P}$. we shall call function h to be density of Q w.r.t. $\mathbb{P}$.

Definition 5.1.4 (Equivalent (martingale) measure)**.** *In above definition if $\mathbb{P}$ is also absolutely continuous with respect to $\mathbb{Q}$, then they are called equivalent measure. If there is a stochastic process X_t which is martingale under $\mathbb{Q}$, then probability measure $\mathbb{P}$ and $\mathbb{Q}$ are called equivalent martingale measure.*

Theorem 5.1.1 (Girsanov's Theorem)**.** *Consider the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let $W_t,\ 0 \le t \le T$ be the standard Brownian motion and let $\mathcal{F}_t,\ 0 \le t \le T$ is associated filtration with W_t. Suppose $\Theta(t),\ 0 \le t \le T$ be an adapted process. Define*

$$Z(t) = e^{-\int_0^t \Theta(u) dW_u - \frac{1}{2}\int_0^t \Theta^2(u) du}.$$

$$\tilde{W}_t = W_t + \int_0^t \Theta(u) du.$$

Let $Z = Z(T)$, we define

$$\mathbb{Q}(A) = \int_A Z d\mathbb{P}(\omega), \quad A \in \mathcal{F}.$$

Then $\mathbb{E}_{\mathbb{P}}[Z] = 1$ and $\tilde{W}_t$ is a standard Brownian motion under the measure $\mathbb{Q}$.

For our purpose, We are interested in process of the form

$$\tilde{W}_t = W_t + ct.$$

the process $\tilde{W}_t$ is not a standard Brownian motion unless $c = 0$. Girsonov theorem gives us tool to make it standard Brownian motion. change the probability measure $\mathbb{P}$ with probability measure $\mathbb{Q}$ according to Girsanov theorem such that process $\tilde{W}_t$ becomes a standard Brownian motion. By Girsanov's theorem consider

$$Z := Z(T) = e^{-cW_T - \frac{1}{2}c^2 T}$$

and

$$\mathbb{Q}(A) = \int_A Z(\omega) d\mathbb{P}(\omega), \quad A \in \mathcal{F}.$$

Then $\tilde{W}_t$ is standard Brownian motion under $\mathbb{Q}$.

We want to have a probability measure $\mathbb{Q}$ such that the discounted price process is a martingale. Thus we want

$$\begin{aligned}\tilde{S}(t) &= e^{-rt}S(t) = e^{-rt}S_0 e^{(\mu-\frac{1}{2}\sigma^2)t+\sigma W_t} \\ &= S_0 e^{(\mu-r-\frac{1}{2}\sigma^2)t+\sigma W_t}\end{aligned}$$

to be a martingale. We know that if W_t is a standard Brownian motion then $e^{\sigma W_t - \frac{1}{2}\sigma^2 t}$ is a martingale. Thus the discounted price process is a martingale if $\mu = r$. Thus under the risk neutral measure $\mathbb{Q}$, the equity price process can be represented as

$$\begin{aligned}S(t) &= S_0 e^{(r-\frac{1}{2}\sigma^2)t+\sigma W_t} \\ &= S_0 e^{(r-\frac{1}{2}\sigma^2)t+\sigma\sqrt{t}Z}.\end{aligned}$$

Note that

$$\begin{aligned}\mathbb{P}(S(t) > K) &= \mathbb{P}\left(S_0 e^{(r-\frac{1}{2}\sigma^2)t+\sigma\sqrt{t}Z} > K\right) \\ &= \mathbb{P}\left(Z > \frac{\log(K/S_0) - (r-\sigma^2/2)t}{\sigma\sqrt{t}}\right) \\ &= \mathbb{P}\left(Z > \sigma\sqrt{t} - \omega\right),\end{aligned}$$

where

$$\omega = \frac{rt + \sigma^2 t/2 - \log(K/S_0)}{\sigma\sqrt{t}}.$$

Let I be the indicator random variable for the event that the option finishes in the money. That is

$$I = \begin{cases} 1 & \text{if } S(t) > K \\ 0 & \text{if } S(t) \le K. \end{cases}$$

Then, we have

$$\begin{aligned}\mathbb{E}(I) &= \mathbb{P}(S(t) > K) \\ &= \mathbb{P}\left(Z > \sigma\sqrt{t} - \omega\right) \\ &= \mathbb{P}(Z \le \omega - \sigma\sqrt{t}) \\ &= \Phi(\omega - \sigma\sqrt{t}).\end{aligned}$$

Let $z_1 = \sigma\sqrt{t} - \omega$. The option price is given by

$$\begin{aligned}
c &= e^{-rT}\mathbb{E}_Q\left[(S(T)-K)^+\right] \\
&= e^{-rT}\int_{z_1}^{\infty}\left(S_0 e^{(r-\frac{1}{2}\sigma^2)T+\sigma\sqrt{T}y} - K\right)\frac{1}{\sqrt{2\pi}}e^{-y^2/2}dy \\
&= S_0 e^{-\frac{1}{2}\sigma^2 T}\frac{1}{\sqrt{2\pi}}\int_{z_1}^{\infty} e^{\sigma\sqrt{T}y - y^2/2}dy - Ke^{-rT}\int_{z_1}^{\infty}\frac{1}{\sqrt{2\pi}}e^{-y^2/2}dy \\
&= S_0\frac{1}{\sqrt{2\pi}}\int_{z_1}^{\infty} e^{-(y-\sigma\sqrt{t})^2/2}dy - Ke^{-rT}\Phi(-z_1) \\
&= S_0\frac{1}{\sqrt{2\pi}}\int_{-\omega}^{\infty} e^{-u^2/2}du - Ke^{-rT}\Phi(-z_1) \;\; \text{(by putting } u = y - \sigma\sqrt{t}\text{)} \\
&= S_0\mathbb{P}(Z > -\omega) - Ke^{-rT}\Phi(-z_1) \\
&= S_0\Phi(\omega) - Ke^{-rT}\Phi(\omega - \sigma\sqrt{T}),
\end{aligned}$$

where $\omega = \frac{rT+\sigma^2 T/2 - \log(K/S_0)}{\sigma\sqrt{T}}$.

In general if t is the time to maturity of the European call option then

$$C(s,t,K,\sigma,r) = s\Phi(\omega) - Ke^{-rt}\Phi(\omega - \sigma\sqrt{t}),$$

where

$$\omega = \frac{rt + \sigma^2 t/2 - \log(K/s)}{\sigma\sqrt{t}}$$

and s is the initial price of the equity.

Chapter 6

Simulation of Option Pricing Models on Global and Indian equity

6.1 Prediction of Stock Price Path for Apple and Google Under Bachelier Model Using Historical Volatility[1]

We predicted the stock price of Apple and Google using Bachelier Model and geometric Brownian motion of stock. To estimate volatility of stocks 11 year of stock price data (from 01-02-2010) have been used.

Under Bachlier model stock price is given by:

$$S_t = S_0 + \sigma W_t \tag{6.1}$$

Where σ is volatility. Now

$$S_t - S_{t-1} = \sigma(W_t - W_{t-1}).$$

where σ is volatility term.

which implies that $S_t - S_{t-1} \sim N(0, \sigma^2)$. Furthermore, since $S_t - S_{t-1}, t = 1, 2, \ldots$ are independent, we will estimate σ by $\hat{\sigma} = sd(S_t - S_{t-1})$.

Google Stock

Stock price of data of Google is obtained from "quantmod" library of R. fig 6.1 shows the closing price data of Google. Daily price change of Google stock is shown in figure

[1]All the data that has been used in this section for simulation purpose is obtained from "Yahoo finance"

size	mean	std dv	skewness	kurtosis	skew.2SE	kurt.2SE
2811	0.6660389	15.18709	0.1022575	15.84188	1.107264	85.79995

Table 6.1: Some statistics of daily price change of Google stock

6.2 . We can see high volatility after march 2020 because of COVID19 Pandemic. Some statistics of daily price change of Google stock is also shown in table 6.1

First thing to note is skew.2SE and kurt.2SE are greater than one, suggesting strong deviation from normality of daily price change. High positive value of kurtosis (also known as leptokurtic curve) suggests that tail of data will die out slowly and low value of skewness indicate that data is may be evenly distributed.

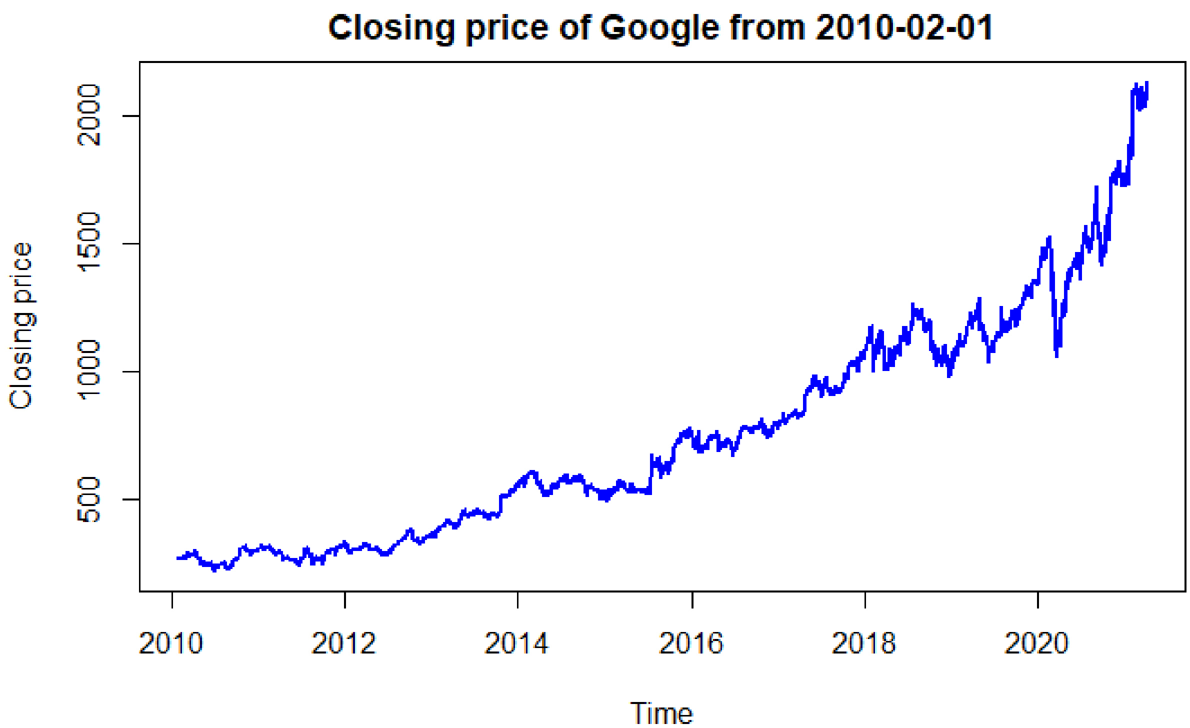


Figure 6.1: Google closing price Data of Google from 2010-01-02

Test	Test Value	p-value
Jarque–Bera test	X-squared = 29449	2.2e-16
Kolmogorov–Smirnov test	D = 0.99964	2.2e-16
Shapiro–Wilk test	W = 0.7879712	2.453127e-51

Table 6.2: Result of test for normality of daily price change of Google

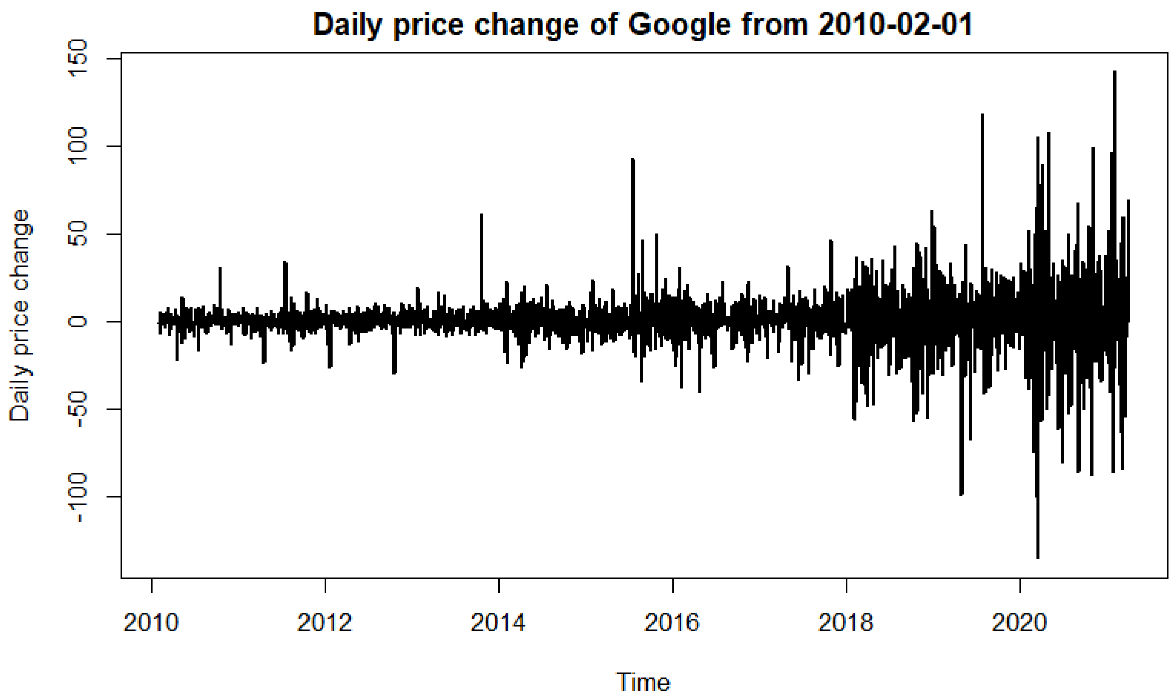


Figure 6.2: Daily price change of Google stock

Jarque–Bera test, Kolmogorov–Smirnov test and Shapiro–Wilk test performed to check normality of price change of stock. Table 6.5 represent the result obtained from these test :

Since value of X-squared is very high, value of D and W is very close to 1 and p-value is significantly low; All these three test strongly reject the null hypothesis or the normality of price change data.

We compared daily price change distribution of google stock from real Data and the distribution which is derived by Bachelier model. For finding distribution under Bachelier model we have used 11 years of data from 2010. For obtaining distribution of Empirical data (again same data of 11 years) we have used kernel density estimation

(KDE) technique. figure 6.3 shows comparison between daily price change distribution of Empirical data and according to Bachelier model for Google. Bandwidth is selected by 'density' function of R which utilise silverman's rule of thumb. It says bandwidth h is given by:

$$h = 0.9 \times \min\left(\hat{\sigma}; \frac{q_3 - q_1}{1.349}\right)^{n^{-\frac{1}{5}}} \tag{6.2}$$

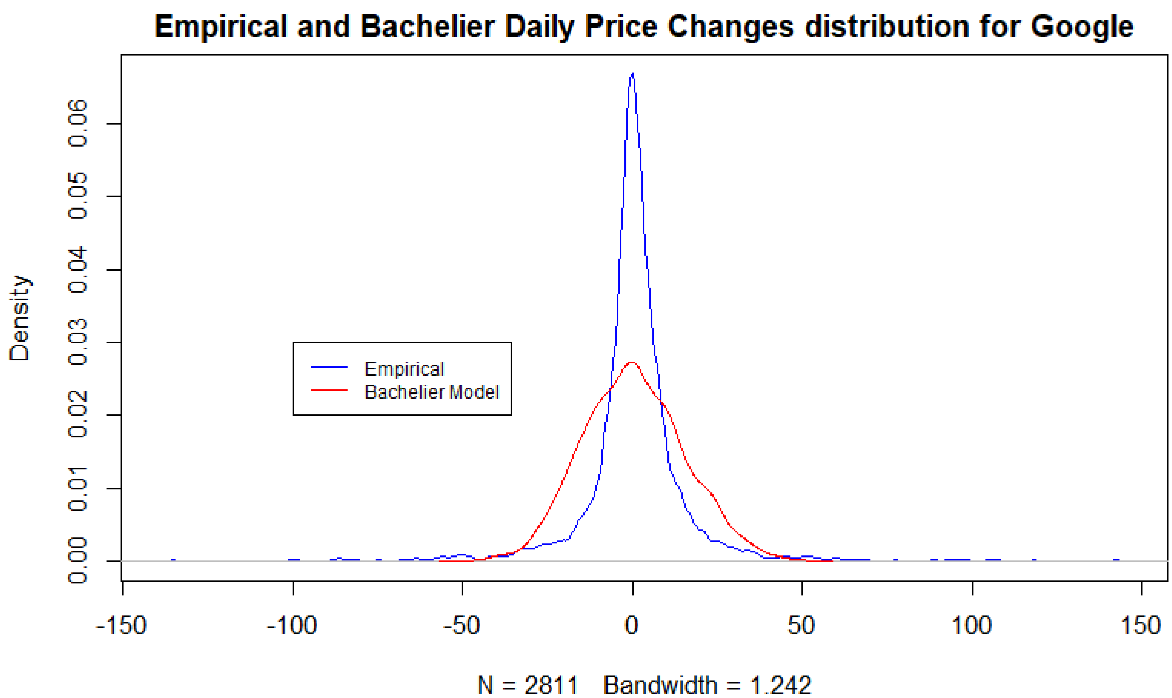


Figure 6.3: Empirical and Bachelier model Daily price change distribution comparison using kernel density estimation of Google stock

Fig 6.3 shows that Bachelier model of stock price is a very bad fit for a leptokurtic curve of daily price change of Google stock. To test the fitness of Bachelier model to Empirical data we used "Two-sample Kolmogorov-Smirnov test". Result of the test is shown in table 6.6 and test strongly reject the null hypothesis that they came from same population.

Test	Test Value	p-value
Two-sample Kolmogorov-Smirnov test	D = 0.16578	2.2e-16

Table 6.3: Goodness of fit test for Bachelier Model of Google stock

We predicted Google stock price based on 11 years of volatility and compared it to observed price. Figure 6.4 shows possible paths for Google stock price under Bachelier model. In Figure 6.4 we generated very large number of possible Bachelier model path to get a good estimate of mean Bachelier path. Clearly mean Bachelier model path deviates strongly from observed stock price for Google.

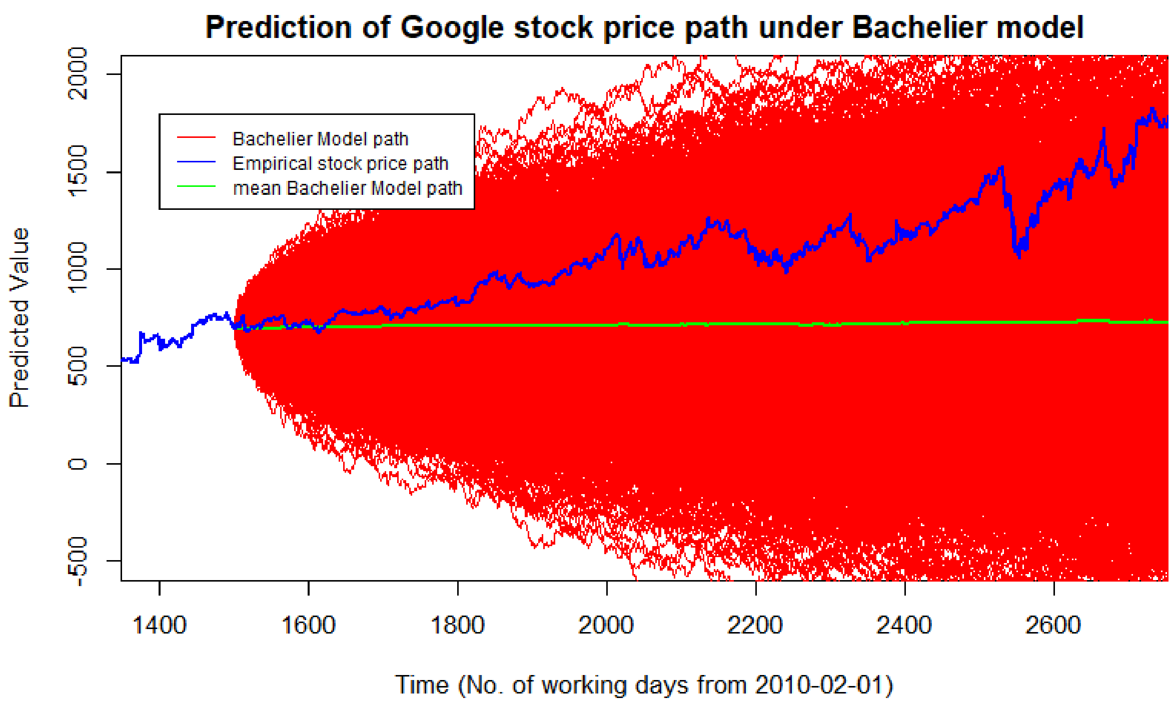


Figure 6.4: Comparison of Bachelier model and observed closing price for Google stock

Apple Stock

Stock price data of Apple is also obtained from "quantmod" library of R. fig 6.5 shows the closing price data of Apple. Daily price change of Apple stock is also shown in figure 6.6 . Like Google, in Apple also, We can see high volatility in 2020 because of COVID19 pandemic. Some statistics of daily price change of Apple stock is also shown in table 6.4

Note that magnitude of skew.2SE and kurt.2SE are greater than one, suggesting strong deviation from normality of daily price change. High positive value of kurtosis suggests that tail of data will die out very slowly and low value of skewness indicate that data is somewhat evenly distributed.

While Google daily price change data has positive skewness, Apple daily price change shows negative skewness. Hence Google dpc (daily price change) data has slightly high density on left side and Apple dpc has slightly high density on the right side of the bell curve.

size	mean	std dv	skewness	kurtosis	skew.2SE	kurt.2SE
2811	0.04162559	0.9717078	-0.145019	24.97615	-1.570298	135.2714

Table 6.4: Some statistics of daily price change of Apple stock

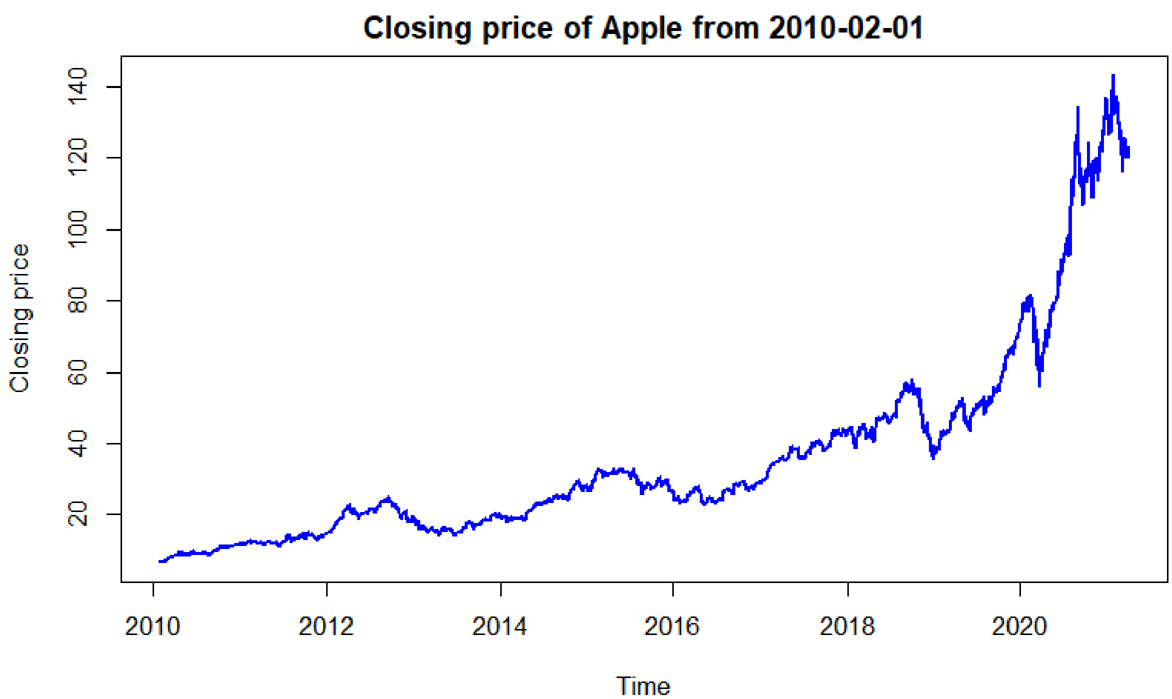


Figure 6.5: Closing price path of Apple from 2010-01-02

Test	Test Value	p-value
Jarque–Bera test	X-squared = 73190	2.2e-16
Kolmogorov–Smirnov test	D = 0.99643	2.2e-16
Shapiro–Wilk test	W = 0.6746677	7.972452e-59

Table 6.5: Result of test for normality of daily price change of Apple stock

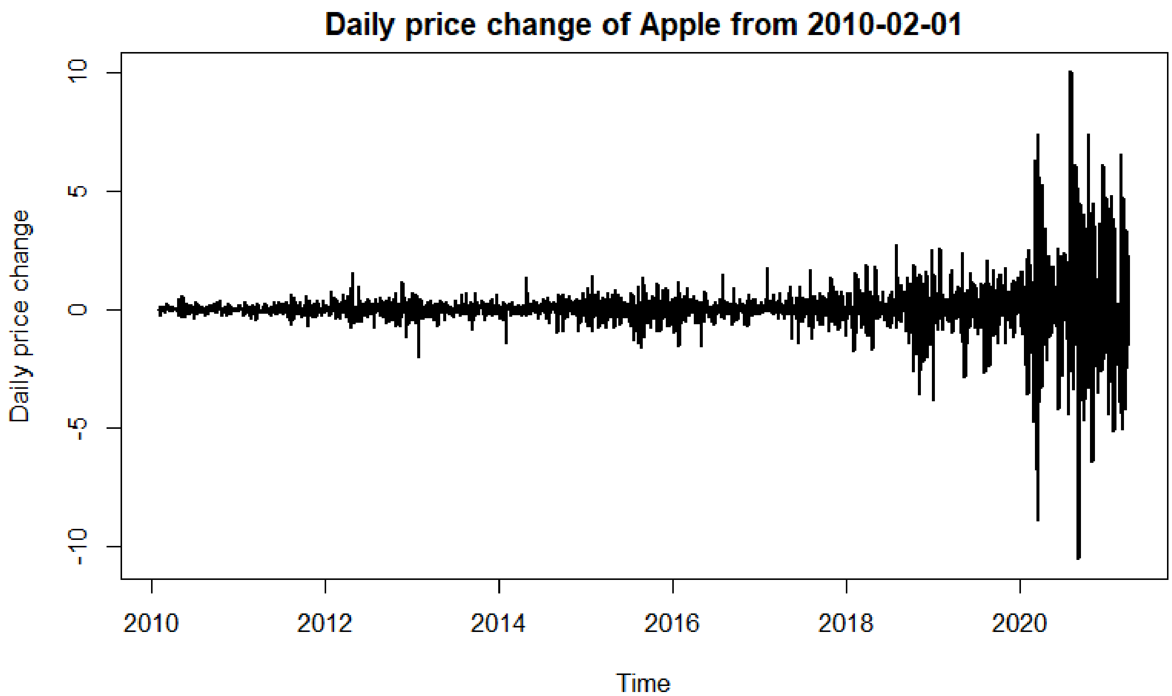


Figure 6.6: Daily price change of Apple stock

Similar to Google; Jarque–Bera test, Kolmogorov–Smirnov test and Shapiro–Wilk test performed to check normality of price change of Apple stock. Table 6.5 represent the result obtained from these test :

Since value of X-squared is very high, value of D and W is very close to 1 and p-value is significantly low; All these three test strongly reject the null hypothesis or the normality of daily price change data of Apple.

Similar to Google, We compared daily price change distribution of Apple stock from real Data and the distribution which is derived by Bachelier Model. For finding distribution under Bachelier model we have used 11 years of data from 2010. For obtaining distribution of Empirical data (again same data of 11 years) we have used kernel density estimation (KDE) technique. figure 6.7 shows comparison between daily price change distribution of Empirical data and according to Bachelier model for Apple.

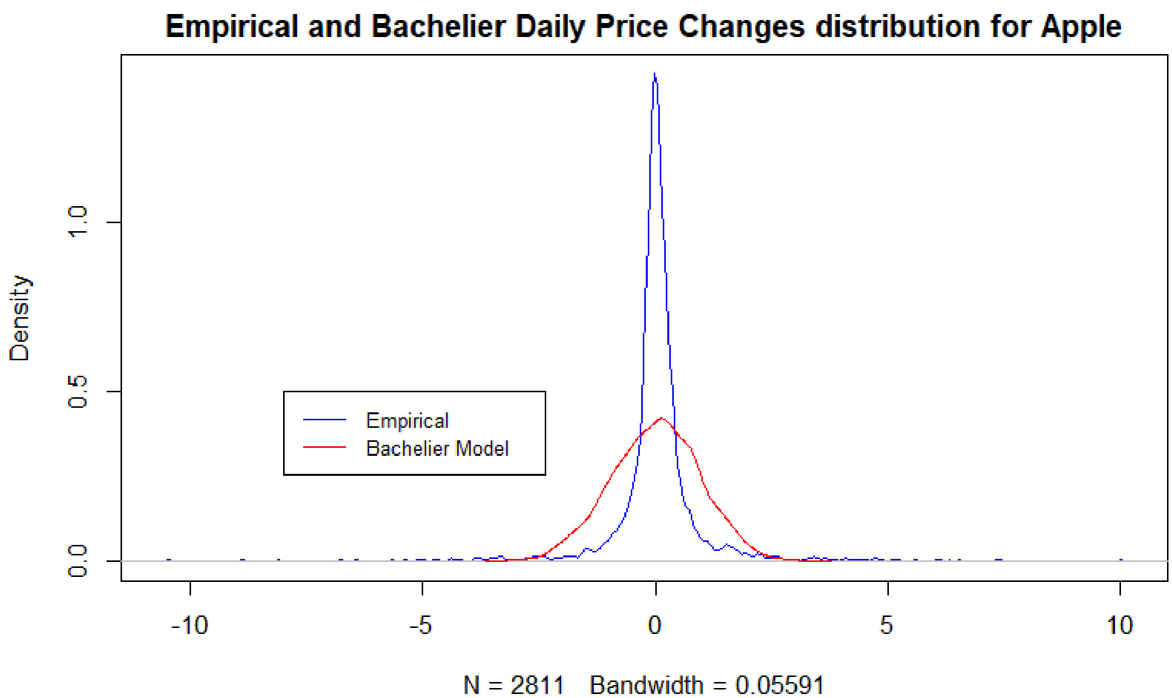


Figure 6.7: Empirical and Bachelier model Daily price change distribution comparison using kernel density estimation of Apple stock

Figure 6.7 shows that Bachelier model of stock price is a very bad fit for a leptokurtic curve of daily price change of Apple stock as we got in Google stock. To test the fitness of Bachelier model to Empirical data we used "Two-sample Kolmogorov-Smirnov test". Result of the test is shown in table 6.6 and test strongly reject the null hypothesis that they came from same population.

Test	Test Value	p-value
Two-sample Kolmogorov-Smirnov test	D = 0.38278	2.2e-16

Table 6.6: Goodness of fit test for Bachelier Model of Google stock

Similar to Google, We predicted Apple stock price based on 11 years of volatility and compared it to observed price. Figure 6.8 shows possible paths for Apple stock price under Bachelier model. In figure 6.8 we generated very large number of possible Bachelier model path to get a good estimate of mean Bachelier path. Clearly mean Bachelier model path deviates strongly from observed stock price for Google.

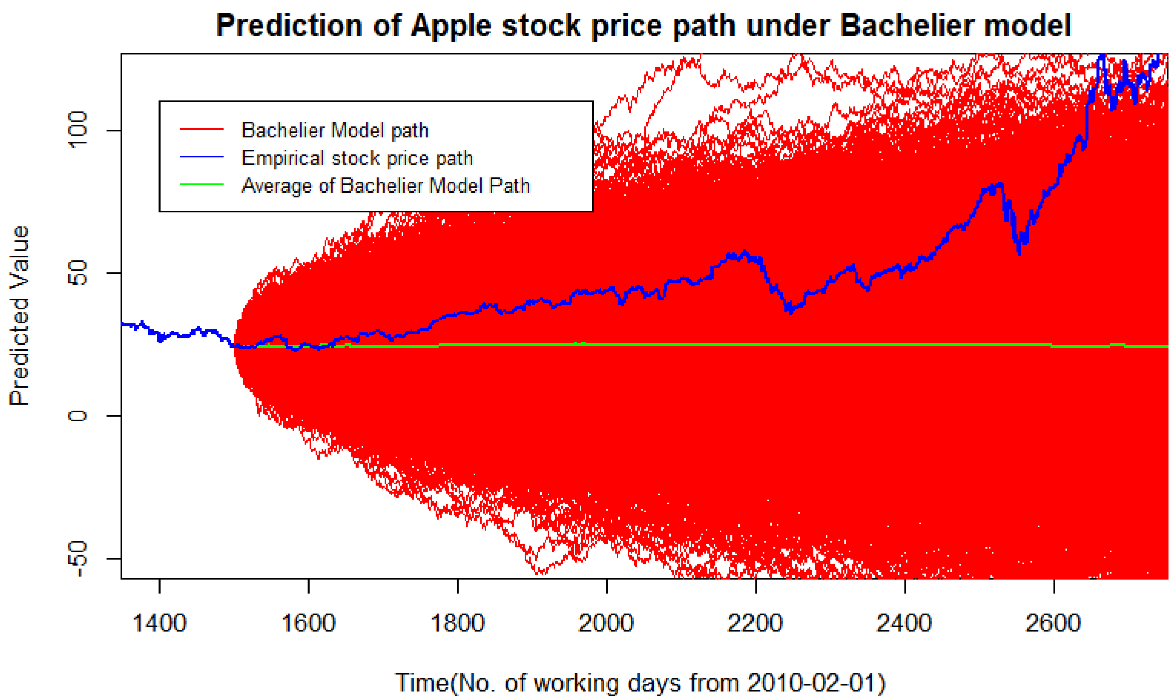


Figure 6.8: Comparison of Bachelier model and observed closing price for Apple stock

6.2 Prediction of Stock Price Path for Apple and Google Under BSM Model Using Historical Volatility[2]

Recall that under Black-scholes model stock price process can be described by below SDE:

$$dS_t = \mu S_t \, dt + \sigma S_t \, dW_t \tag{6.3}$$

We obtained below geometric Brownian motion as the solution of above SDE:

$$S_t = S_0 e^{\left(\sigma W_t + \left(\mu - \frac{1}{2}\sigma^2\right)t\right)} \tag{6.4}$$

Which gives us:

$$\begin{aligned}
\frac{S_{t+1}}{S_t} &= \frac{e^{\left(\sigma W_{t+1} + \left(\mu - \frac{1}{2}\sigma^2\right)(t+1)\right)}}{e^{\left(\sigma W_t + \left(\mu - \frac{1}{2}\sigma^2\right)t\right)}} \\
\frac{S_{t+1}}{S_t} &= e^{\sigma(W_{t+1} - W_t) + (\mu - \frac{1}{2}\sigma^2)} \\
\ln\left(\frac{S_{t+1}}{S_t}\right) &= \sigma(W_{t+1} - W_t) + (\mu - \frac{1}{2}\sigma^2)
\end{aligned}$$

Hence we have

$$\ln\left(\frac{S_{t+1}}{S_t}\right) = (\mu - \frac{1}{2}\sigma^2) + \epsilon \tag{6.5}$$

where $\epsilon \sim \mathcal{N}(0, \sigma^2)$.

For estimating mean and volatility of log return we have used 11 years of log return data of Google and apple. Let $\hat{\mu}$ and $\hat{\sigma}$ be our estimated mean and volatility respectively of log return then μ in equation 6.4 is given by

$$\mu = \hat{\mu} + \frac{1}{2}\hat{\sigma}^2 \tag{6.6}$$

Some statistics of log return of Google and Apple stock is give in table 6.7 and table 6.8 respectively.

size	mean	std dv	skewness	kurtosis	skew.2SE	kurt.2SE
2811	7.420273e-04	1.639007e-02	0.3164235	9.822617	3.426298	5.319951

Table 6.7: Some statistics of log return of Google stock

[2] All the data that has been used in this section for simulation purpose is obtained from "Yahoo finance"

size	mean	std dv	skewness	kurtosis	skew.2SE	kurt.2SE
2811	1.075067e-03	1.787012e-02	-0.306114	6.267043	-3.314668	33.94244

Table 6.8: Some statistics of log return of Apple stock

Daily log return of Google and Apple is shown in figure 6.9 and figure 6.10 respectively. Similar to daily change price the magnitude of skew.2SE and kurt.2SE in case of log return are also greater than one, suggesting deviation from normality of daily log return distribution. High positive value of kurtosis suggests that tail of data will die out very slowly and low value of skewness indicate that data is somewhat evenly distributed.

Also similar to daily change price, while Google daily price change data has positive skewness, Apple daily price change shows negative skewness. Hence Google log return data has slightly high density on left side whereas Apple log return data has slightly high density on the right side of the bell curve.

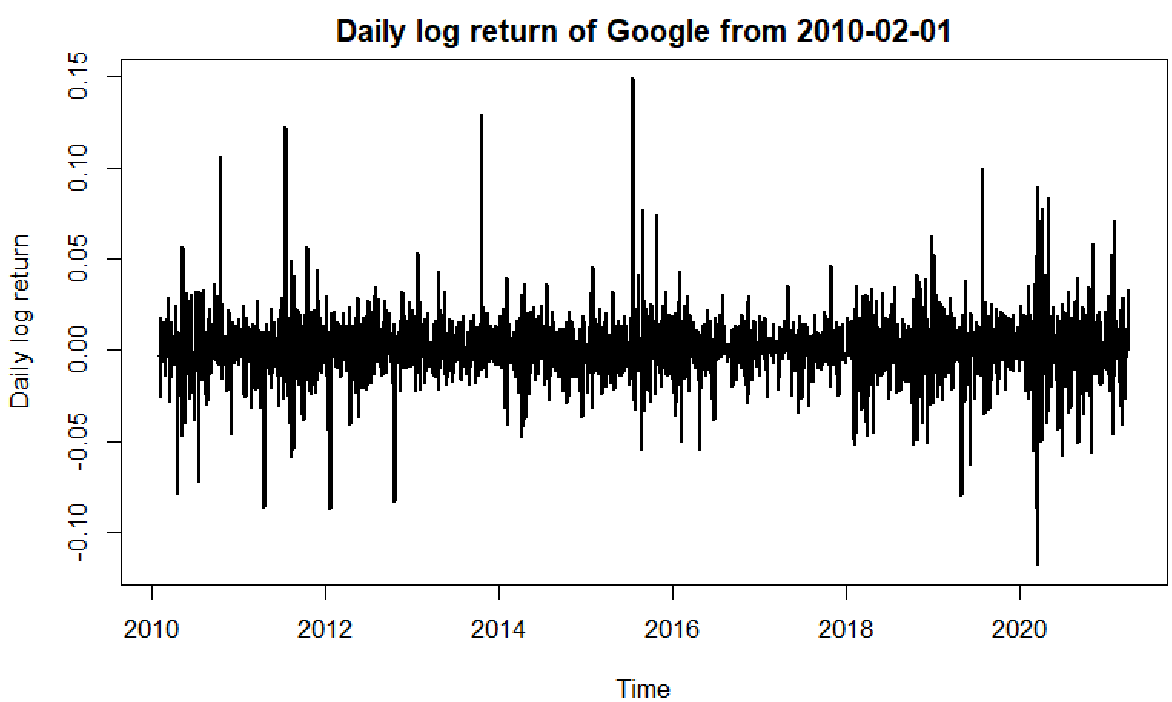


Figure 6.9: Daily log return of Google stock

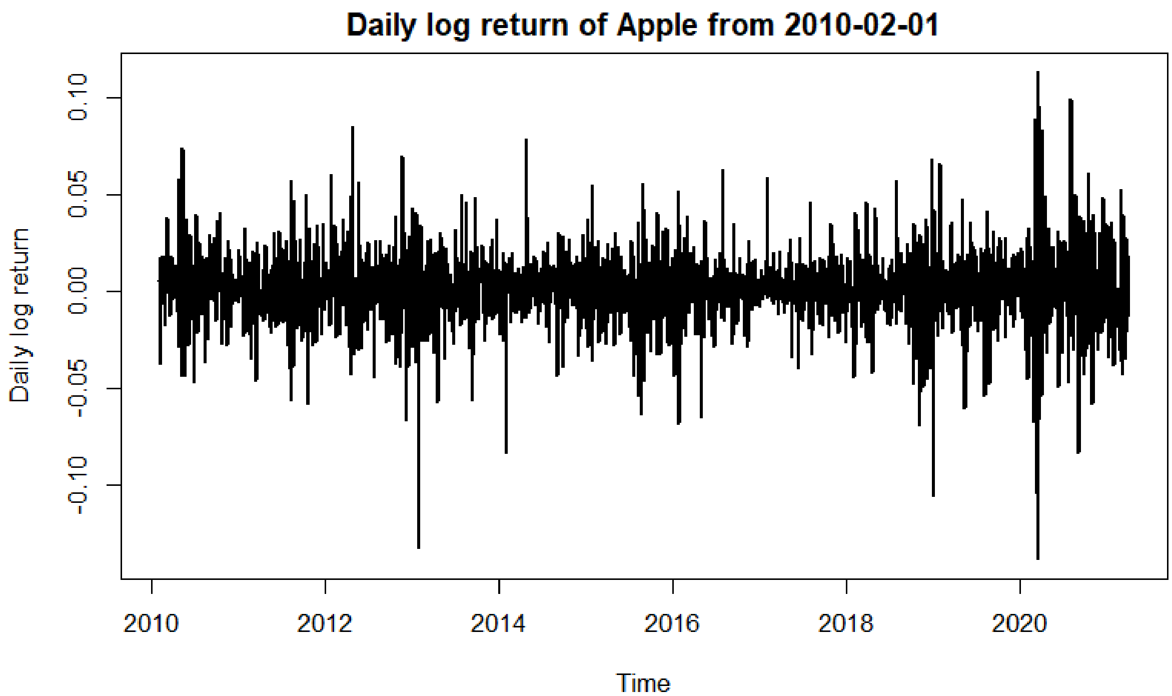


Figure 6.10: Daily log return of Apple stock

Jarque–Bera test, Kolmogorov–Smirnov test and Shapiro–Wilk test performed to check normality of daily log return of Google and Apple stock. Table 6.9 and table 6.10 represent the result obtained from these test for Google and Apple stock respectively.

Test	Test Value	p-value
Jarque–Bera test	X-squared = 11369	2.2e-16
Kolmogorov–Smirnov test	D = 0.50988	2.2e-16
Shapiro–Wilk test	W = 0.9026485	4.553790e-39

Table 6.9: Result of test for normality of Daily log return of le Google stock

Test	Test Value	p-value
Jarque–Bera test	X-squared = 4653.8	2.2e-16
Kolmogorov–Smirnov test	D = 0.51057	2.2e-16
Shapiro–Wilk test	W = 0.9347182	1.918410e-33

Table 6.10: Result of test for normality of Daily log return of Apple stock

Since value of X-squared is high and W is close to 1 and p-value is significantly low for Jarque-Bera test and Shapiro-wilk test hypothesis or the normality of log returns is rejected by these test for both stocks

For Kolmorogov-Smirnov p-value is significantly low, test D value is close to 0.5 which is also less than it's critical value for 95% confidence interval which is 0.077552 for both stocks. Hence it is also safe to reject normality of Daily log return of these stocks for this test.

We compared daily log return distribution of both stock from real Data and the distribution which is derived under GBM Model using kernel density estimation (KDE) technique. Figure 6.11 6.12 shows comparison between daily log return distribution of Empirical data and according to GBM model for Google and Apple respectively.

KDE plot shows that GBM model of stock price is although not a perfect fit to Empirical data, but still is very good fit compared to Bachelier Model. The imperfection came because we are assuming constant drift and volatility which is in general not true. To test the fitness of GBM model to Empirical data we used "Two-sample Kolmogorov-Smirnov test".Table 6.11 and table 6.12 respectively shows the result of the test for Google and Apple. The test strongly reject the null hypothesis that they came from same population.

Test	Test Value	p-value
Two-sample Kolmogorov-Smirnov test	D = 0.079331	4.149e-08

Table 6.11: Goodness of fit test for GBM Model of Google stock

Test	Test Value	p-value
Two-sample Kolmogorov-Smirnov test	D = 0.077552	9.092e-08

Table 6.12: Goodness of fit test for GBM Model of Google stock

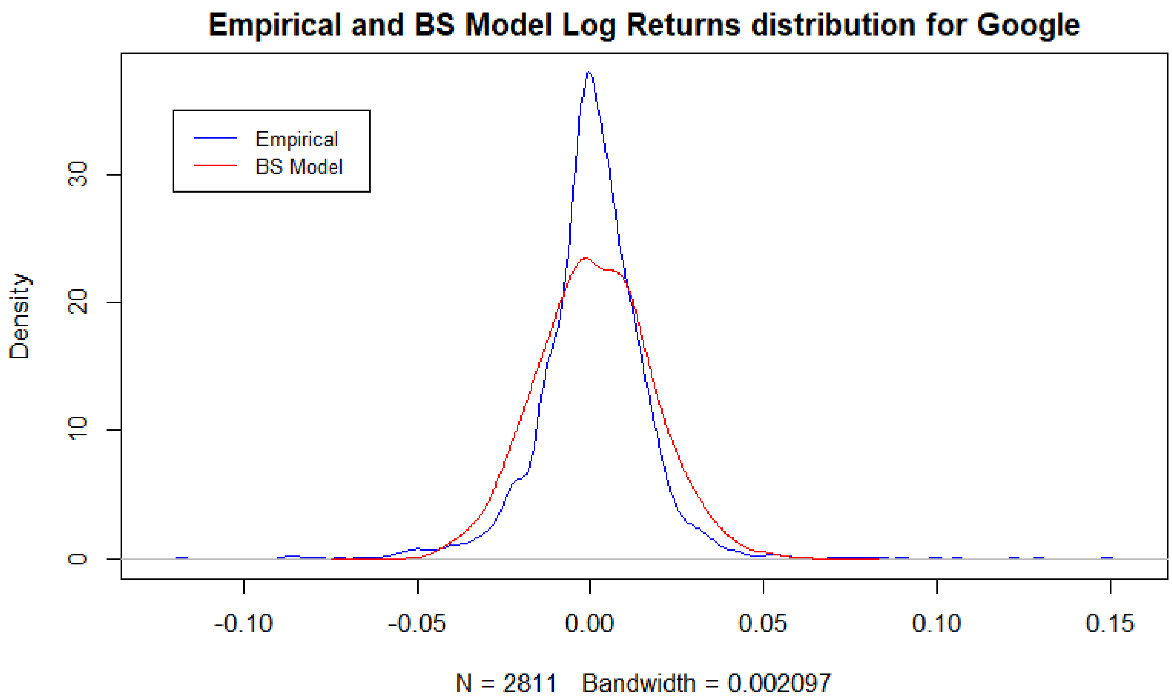


Figure 6.11: Empirical and GBM model Daily log return distribution comparison using kernel density estimation of Google stock

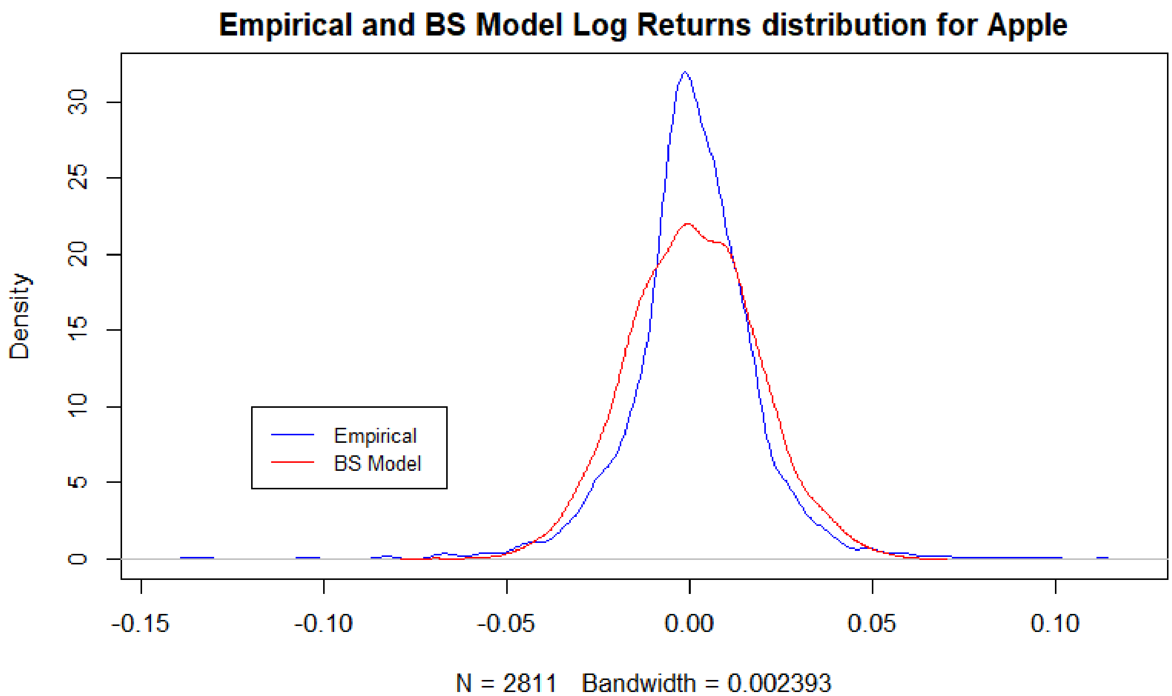


Figure 6.12: Empirical and GBM model Daily log return distribution comparison using kernel density estimation of Apple stock

We predicted the both stock prices according to GBM model based on 11 years of volatility using Monte Carlo simulation and compared it to observed price. Figure 6.13 6.14 shows possible paths for Google and Apple stock price under GBM model respectively. we generated 3000 samples get a good estimate of mean GBM path.

There is deviation from the GBM path from 2020 because of COVID19 Pandemic. In all other time period both stock GBM price path is a good estimate of observed price path. Again some imperfection because we are using constant drift and volatility. Also note that compared to Google, Apple stock price path fit better in GBM model.

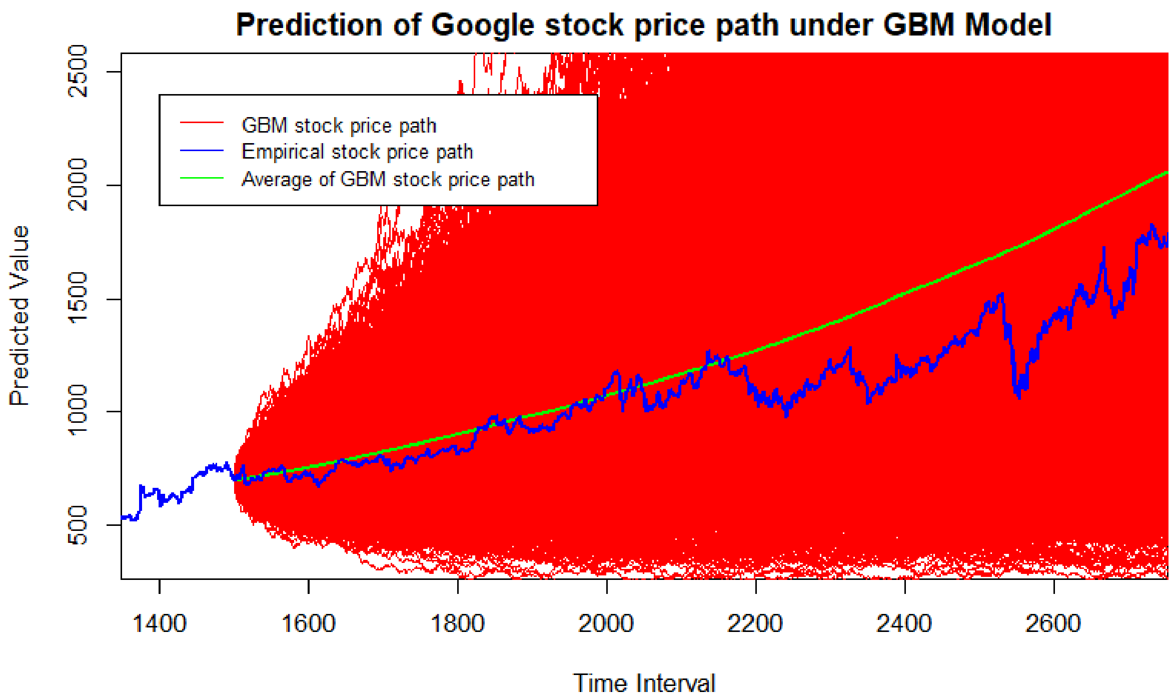


Figure 6.13: Comparison of GBM model and observed closing price for Google stock

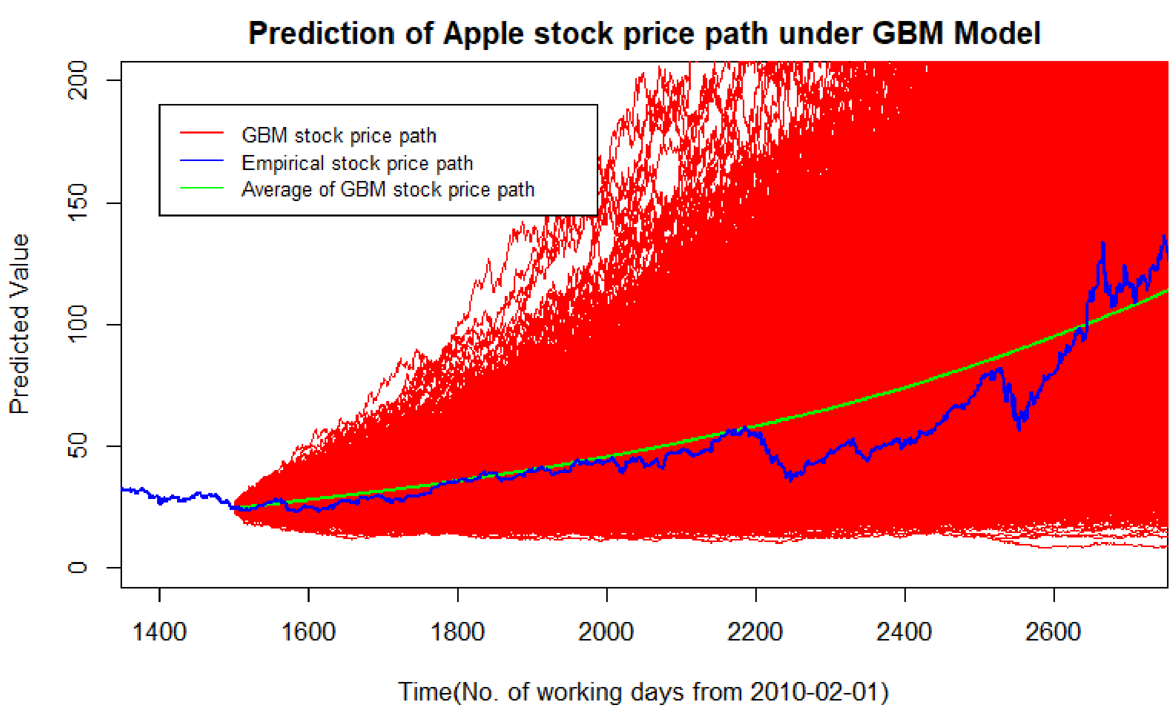


Figure 6.14: Comparison of GBM model and observed closing price for Apple stock

6.3 Prediction of Option Price of Infosys Stock Under CRR and BSM Model[3]

In this section Cox, Ross and Rubinstein (CRR) approach of pricing option is presented and we implemented same model on Infosys option.

CRR Model is a Binomial tree based model such when time period Δ between two consecutive tradeble period approaches zero, model approaches to Black-scholes Model. To accomplish this first discretize time such that $t_0 = 0, t_1 = \Delta, t_2 = 2\Delta, ..., t_n = n\Delta = T$, where $\Delta = \frac{T}{n}$.

Let σ is the volatility of stock. If we restrict u and d such that $u \cdot d = 1$, then by choosing below value for parameter u and $\tilde{p}$, we get a good approximation of Black-scholes model:

$$u = e^{\sigma\sqrt{\Delta}}, \quad \tilde{p} = \frac{(e^{r\Delta} - d)}{(u - d)} \approx \frac{1}{2} + \frac{1}{2}(r - \frac{1}{2}\sigma^2)\frac{\sqrt{\Delta}}{\sigma},$$

Example 6.3.1. *Find the Initial price of European call option with no dividend whose features are given in table 6.13*

Initial stock price S_0	150
Strike price K	135
risk free rate r	0.045
Expiration Time T	0.4
volatility σ	0.20
time steps n	5

Table 6.13: European call with no dividend

We get $\Delta = T/n = 0.08$, $u = e^{\sigma\sqrt{\Delta}} = e^{0.2\sqrt{0.08}} = 1.0823$, *and* $d = 1/u = 0.9234$.. *Figure 6.15 is Recombining tree generated by a R program shows option price evolution at different node.*

In this example if we consider continuous time i.e $n \to \infty$ *then option price found by Black scholes model is 18.96391. Which is close to CRR price. In CRR model if we keep increasing* n *then we keep getting a better estimate of option price. Figure 6.16 shows*

[3]All the data that has been used in this section for simulation purpose is obtained from "Yahoo finance"

trajectory of option price for increasing n. We found option price equal to 18.94796 for $n = 50$

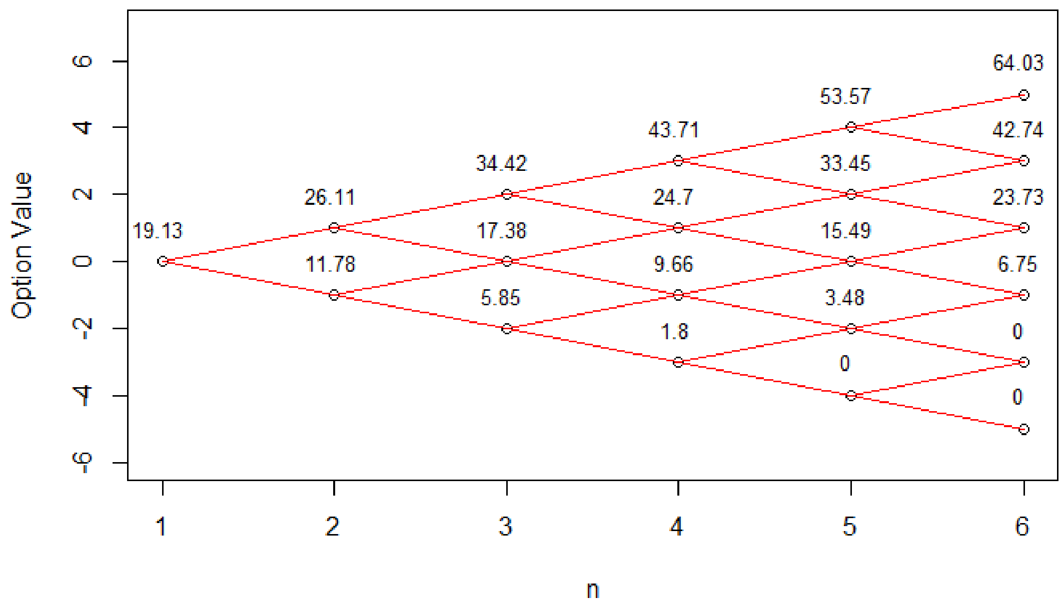


Figure 6.15: Option price Evolution as a recombining Binomial tree

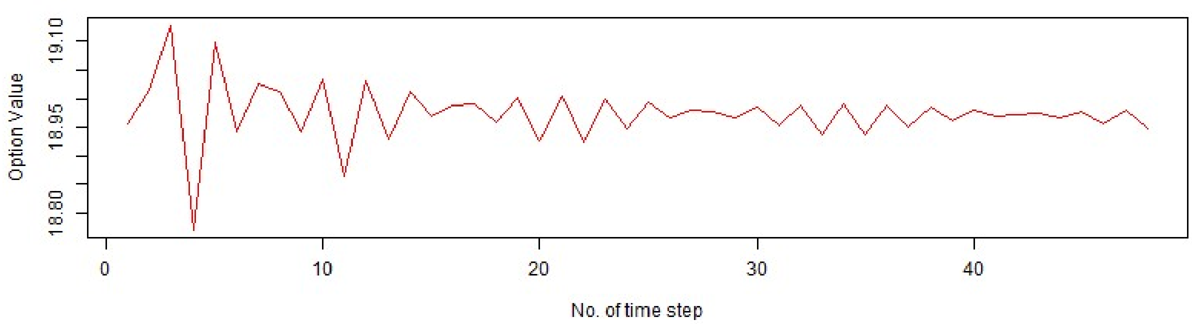


Figure 6.16: CRR option price with increasing n

Pricing Infosys Option under CRR model Using Recent Historical Volatility (5 years)

We priced Infosys Limited (INFY.NS) call option and put option using historical volatility of last 5 years. We obtained historical data from yahoo finance server using quantmod library of R. We priced Infosys option which will expire on 1 June 2021 i.e. 54 days from 6 April 2021. Table 6.14 represent features of Infosys stock on 6 April 2021:

Initial stock price S_0	1409.900024
Strike price K	From 1300 to 1500
risk free rate r	0.031
Expiration Time T	$\frac{54}{365}$
volatility σ	0.2908
time steps n	9

Table 6.14: Infosys call option with no dividend

Table 6.15 Provides Price of Infosys option with given strike price for CRR Model of step size 9 and Black Scholes Model.

Strike Price K	CRR Model	BSM Model
1300	136.15439	135.09051
1339	107.38090	107.22709
1379	82.96576	82.42351
1419	63.30096	61.64829
1459	43.63616	44.84517
1499	32.63268	31.72608

Table 6.15: Price of Infosys option under CRR and BSM Model for different strike price

Figure 6.17 shows the comparison of Infosys option price with feature mentioned in table 6.14 for CRR and BSM Model with strike price between 1300-1500.

When we increased the step size to 50, we got the figure 6.18. From figure 6.18 it is clear that CRR Model price is almost equal to BSM Model price for Infosys option.

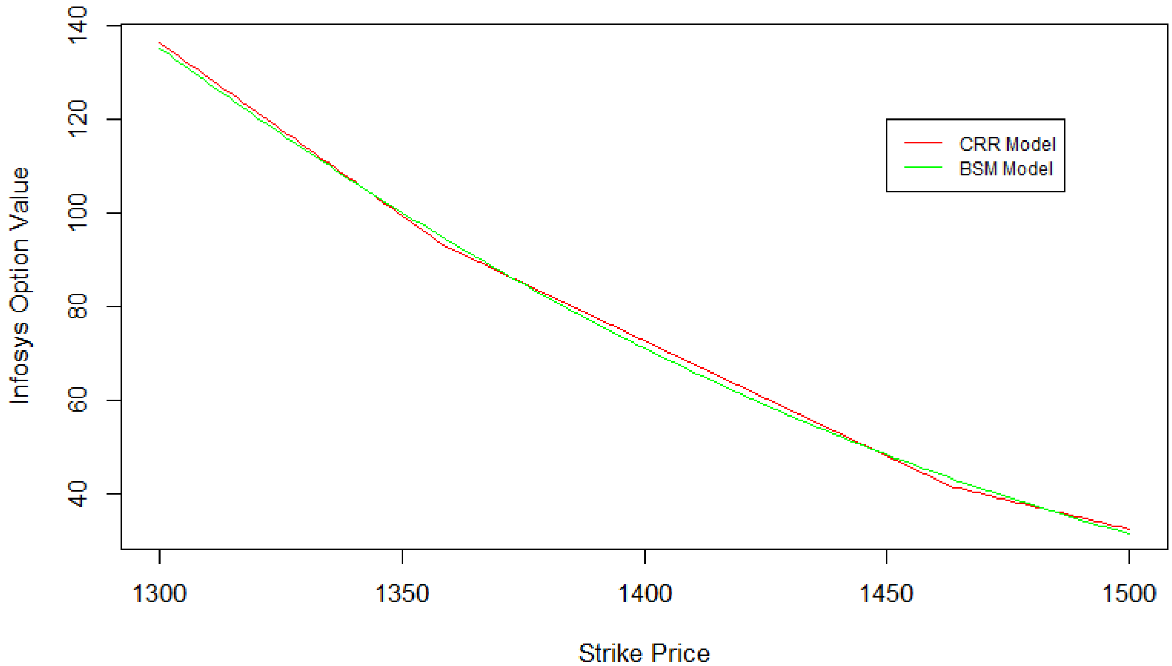


Figure 6.17: Infosys Option price comparison for CRR (step size = 9) and BSM Model

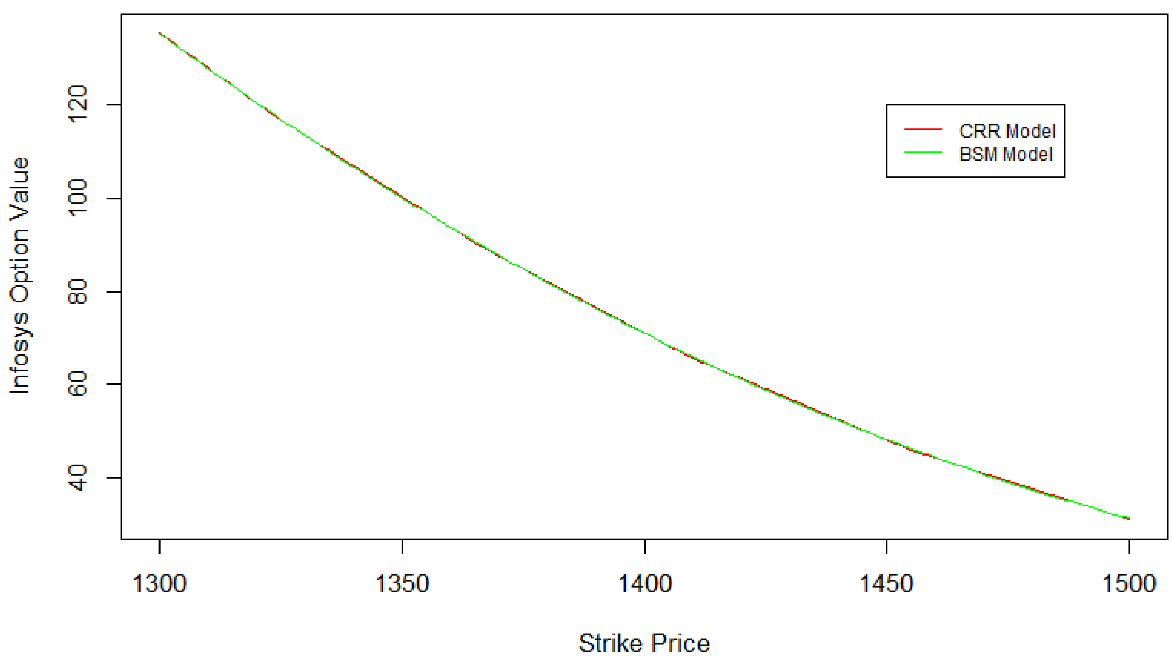


Figure 6.18: Infosys Option price comparison for CRR (step size = 50) and BSM Model

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Appendix A

Source Code Used for Simulations

A.1 R Code Used to Simulate Google Price Path and All Other Plot of Google for GBM Model

```
library(pastecs)
library(quantmod)
library(stats)
library(tseries)
library(quantmod)
goog  = getSymbols('GOOG',src='yahoo', from = "2010-02-01")
data = GOOG[,6]
goog.lr = diff(log(data))[-1]
goog.sd = sd(goog.lr)
N = as.numeric(nrow(GOOG))
mu = mean(goog.lr) + 0.5*(goog.sd*goog.sd)
S0 = as.numeric(GOOG[1501,4])

Wt = cumsum(rnorm(N-1500,0,1));
t = (1500:N);
p1 = (mu - 0.5*(goog.sd*goog.sd))*(t-1501);
p2 = goog.sd*Wt
St = S0* exp(p1 + p2);

op_days = length(goog.lr)
```

```
stat.desc(goog.lr$GOOG.Adjusted, basic=TRUE,

desc=TRUE, norm=TRUE)


ks.test(goog.lr$GOOG.Adjusted, 'pnorm')

jarque.bera.test(goog.lr$GOOG.Adjusted)

plot(index(data[2:N]),goog.lr,type = "l", main =
"Daily log return of Google from 2010-02-01",

xlab = "Time", ylab= "Daily log return", col="black", lwd=2)
dates = index(data)
axis(1, at = 1:N, padj = 1 ,labels=dates)


plot(t, St,type= "l", main = "Prediction of

Google stock price path under GBM Model", xlab =
"Time Interval", ylab= "Predicted Value", ylim =
c(350,2500),xlim=c(1400,2700), col="red")
dates = index(data)
#axis(1, at = 1:N, labels=dates)
dates = index(data)
j=30
TSt = 0
for (i in 1:j){
  Wt = cumsum(rnorm(N-1500,0,1));
  t = (1500:N);
  p1 = (mu - 0.5*(goog.sd*goog.sd))*(t-1500);
  p2 = goog.sd*Wt;
  St = S0* exp(p1 + p2);
  TSt = TSt + St;
```

```
  lines(t,St,col="red")

}
lines(t,TSt/j,col="green",lwd=2)
lines(1:N,GOOG[,4],col="blue", lwd=2)
legend(1400,2400,c("GBM stock price

path","Empirical stock price path","Average of

GBM stock price path"),col=c("red",

"blue","green"), lty=1:1, cex=0.8)

#axis(1, at = 1:N, labels=dates)

```
```


library(quantmod)
goog = getSymbols('GOOG',src='yahoo', from = "2010-02-01")
data = GOOG[,6]
goog.lr = diff(log(data))[-1]
goog.sd = sd(goog.lr)
N = as.numeric(nrow(GOOG))
mu = mean(goog.lr) + 0.5*goog.sd*goog.sd

p1 = (mu - 0.5*(goog.sd*goog.sd));
p2 = goog.sd*goog.sd

theor = rnorm(2811,p1,goog.sd)

emp_d<-density(goog.lr)
theor_d <- density(theor)
plot(emp_d, col = 'blue', type= "l", main =
```

```
"Empirical␣and␣BS␣Model␣Log␣Returns␣distribution
for␣Google")

lines(theor_d, col = 'red')
legend(-0.12,35,c("Empirical","BS

Model"),col=c("blue", "red"), lty=1:1, cex=0.8)


goog.lr_core = coredata(goog.lr)
ks.test(goog.lr_core,theor)
chisq.test(table(goog.lr_core,theor),correct = FALSE)

#ks.test(goog.r,theor)
'''
```

A.2 R Code Used to Simulate Apple Price Path and All Other Plot of Apple for GBM Model

```
library(pastecs)
library(quantmod)
library(stats)
library(tseries)
library(quantmod)
appl = getSymbols('AAPL',src='yahoo', from =

"2010-02-01")
data = AAPL[,6]
class(data)
appl.lr = diff(log(data))[-1]
class(appl.lr)
appl.sd = sd(appl.lr)
```

```
N = as.numeric(nrow(AAPL))
cl_price = AAPL[,4]
mu = mean(appl.lr) + 0.5*(appl.sd*appl.sd)
S0 = as.numeric(AAPL[1501,4])

Wt = cumsum(rnorm(N-1500,0,1));
t = (1500:N);
p1 = (mu - 0.5*(appl.sd*appl.sd))*(t-1501);
p2 = appl.sd*Wt
St = S0* exp(p1 + p2);

op_days = length(appl.lr)
stat.desc(appl.lr$AAPL.Adjusted, basic=TRUE,

desc=TRUE, norm=TRUE)

ks.test(appl.lr$AAPL.Adjusted, 'pnorm')

jarque.bera.test(appl.lr$AAPL.Adjusted)

plot(index(data[2:N]),appl.lr,type = "l", main =
"Daily log return of Apple from 2010-02-01",

xlab = "Time", ylab= "Daily log return",

col="black", lwd=2)
dates = index(data)
axis(1, at = 1:N, padj = 1 ,labels=dates)




#apcl_cr = coredata(AAPL[,4])
```

```
#plot(t,apcl_cr[1501:N]-St,type = "l", main =

"Daily price change of Google from 2010-02-01",

xlab = "Time", ylab= "Daily price change",

col="black", lwd=2)
#dates = index(data)
#axis(1, at = 1:N, padj = 1 ,labels=dates)



plot(t, St,type= "l", main = "Prediction of

Apple stock price path under GBM Model", xlab =

"Time(No. of working days from 2010-02-01)",

ylab= "Predicted Value", ylim =

c(0,200),xlim=c(1400,2700), col="red")
dates = index(data)
#axis(1, at = 1:N, labels=dates)
dates = index(data)
j=30
TSt = 0
for (i in 1:j){
  Wt = cumsum(rnorm(N-1500,0,1));
  t = (1500:N);
  p1 = (mu - 0.5*(goog.sd*goog.sd))*(t-1500);
  p2 = goog.sd*Wt;
  St = S0* exp(p1 + p2);
  TSt = TSt + St;
  lines(t,St,col="red")
```

```
}
lines(t,TSt/j,col="green", lwd=2)
lines(1:N,AAPL[,4],col="blue", lwd=2)
legend(1400,190,c("GBM stock price

path","Empirical stock price path", "Average of

GBM stock price path"),col=c("red", "blue",

"green"), lty=1:1, cex=0.8)
```

A.3 R Code Used to Simulate Google Price Path and All Other Plot of Google for Bachelier Model

```
library(pastecs)
library(quantmod)
library(stats)
library(tseries)
getSymbols('GOOG',src='yahoo', from = "2010-02-01")
data = GOOG[,6]
goog.diff = diff(data)[-1]
sd = sd(goog.diff)
S0 = as.numeric(GOOG[1501,4])
lastPrice = GOOG[,4]
N = as.numeric(nrow(lastPrice))
St = cumsum(rnorm(N-1500,0, sd))
St= St+S0
t = (1501:N);

op_days = length(goog.diff)
stat.desc(goog.diff$GOOG.Adjusted, basic=TRUE,
```

```
desc=TRUE, norm=TRUE)


ks.test(goog.diff$GOOG.Adjusted, 'pnorm')

jarque.bera.test(goog.diff$GOOG.Adjusted)



dates = index(data)
axis(1, at = 1:N, padj = 1 ,labels=dates)



plot(index(data),lastPrice,type = "l", main =

"Closing price of Google from 2010-02-01", xlab

= "Time", ylab= "Closing price", col="blue", lwd=2)
dates = index(data)
axis(1, at = 1:N, padj = 1 ,labels=dates)


j=30
TSt=0
for (i in 1:j){
  t = (1501:N);
  St = S0 + cumsum(rnorm(N-1500,0,sd))
  TSt = TSt + St
  lines(t,St,col="red")
}
lines(t,TSt/j,col="green",lwd=2)
lines(1:N,lastPrice,col="blue", lwd=2)
```

```
# Comparison of KDE
theor = rnorm(op_days, 0, sd)

emp_d<-density(goog.diff)
theor_d <- density(theor)

lines(theor_d, col = 'red')


goog.diff_core = coredata(goog.diff)
ks.test(goog.diff_core,theor)
chisq.test(table(goog.diff_core,theor),correct = FALSE)
```

A.4 R Code Used to Simulate Apple Price Path and All Other Plot of Apple for Bachelier Model

```
library(pastecs)
library(quantmod)
library(stats)
library(tseries)
library(quantmod)
appl = getSymbols('AAPL',src='yahoo', from = "2010-02-01")
data = AAPL[,6]
class(data)
appl.lr = diff(log(data))[-1]
class(appl.lr)
appl.sd = sd(appl.lr)
N = as.numeric(nrow(AAPL))
```

```
cl_price = AAPL[,4]
mu = mean(appl.lr) + 0.5*(appl.sd*appl.sd)
S0 = as.numeric(AAPL[1501,4])

Wt = cumsum(rnorm(N-1500,0,1));
t = (1500:N);
p1 = (mu - 0.5*(appl.sd*appl.sd))*(t-1501);
p2 = appl.sd*Wt
St = S0* exp(p1 + p2);

op_days = length(appl.lr)
stat.desc(appl.lr$AAPL.Adjusted, basic=TRUE,

desc=TRUE, norm=TRUE)

ks.test(appl.lr$AAPL.Adjusted, 'pnorm')

jarque.bera.test(appl.lr$AAPL.Adjusted)

plot(index(data[2:N]),appl.lr,type = "l", main =
"Daily log return of Apple from 2010-02-01",

xlab = "Time", ylab= "Daily log return",

col="black", lwd=2)
dates = index(data)
axis(1, at = 1:N, padj = 1 ,labels=dates)




#apcl_cr = coredata(AAPL[,4])

#plot(t,apcl_cr[1501:N]-St,type = "l", main =
```

```
"Daily price change of Google from 2010-02-01",

xlab = "Time", ylab= "Daily price change",

col="black", lwd=2)
#dates = index(data)
#axis(1, at = 1:N, padj = 1 ,labels=dates)

dates = index(data)
#axis(1, at = 1:N, labels=dates)
dates = index(data)
j=30
TSt = 0
for (i in 1:j){
  Wt = cumsum(rnorm(N-1500,0,1));
  t = (1500:N);
  p1 = (mu - 0.5*(goog.sd*goog.sd))*(t-1500);
  p2 = goog.sd*Wt;
  St = S0* exp(p1 + p2);
  TSt = TSt + St;
  lines(t,St,col="red")

}
lines(t,TSt/j,col="green", lwd=2)
lines(1:N,AAPL[,4],col="blue", lwd=2)
```

A.5 R Code Used to Simulate Infosys Option Price and All Other Plot in Section 6.3

```
m = 50
```

```
CRRV = rep(NA, times = n)

for (n in 3:m) \{

CRRV[n] = CRRBinomialTreeOption(TypeFlag = "ce", S = 150,

X = 135, Time =0.4, r = 0.045, b = 0.045, sigma

= 0.20, n = n)@price
\}

CRRV

plot(CRRV[3:m], type = "l", col = "red", xlab =

"No. of step", ylab = " Value of option")


CRRV[m]

GBSOption(TypeFlag = "c", S = 150,

X = 135, Time =0.4, r = 0.04335, b = 0.03335, sigma = 0.20)

library(quantmod)

library("fOptions")

getSymbols('INFY.NS',src='yahoo', from = "2016-11-01")


INFY  = INFY.NS[,6]

INFY.rets = diff(log(INFY))[-1]
```

```
# Annualized Volatility by Daily Volatility

INFY.sd = sd(INFY.rets, na.rm=TRUE)*sqrt(252)
print(c("Value of annual volatiity is", INFY.sd ))


S = as.numeric(INFY.NS[nrow(INFY.NS), 4])

print(c("Value of S is", S))

k = 54

T = k/365;

r = .031;

b = r;

sigma = INFY.sd;

n=50;

GBSV = rep(NA, times = 201)
CRRV = rep(NA, times = 201)
for (i in 1300:1500){

CRRV[i-1299] = CRRBinomialTreeOption(TypeFlag =

"ce", S = S, X = i,Time = T, r = r, b = r, sigma
= sigma, n = n)@price

GBSV[i-1299] = GBSOption(TypeFlag = "c", S = S,

X = i,Time = T, r = r, b = r, sigma =
```

```
sigma)@price

}

t = c(1,40,80,120,160,200)

print(c(t+1299,CRRV[t]))

print(c(t+1299,GBSV[t]))

lines(1300:1500,GBSV[1:201],col="green")

library("fOptions")

#CRRBinomialTreeOption(TypeFlag = "ce", S = 150,
X = 135,Time = 0.4, r = 0.04335, b = 0.03335,

sigma = 0.20, n = 4);
#CRRBinomialTreeOption(TypeFlag = "ce", S = 100, X = 100,
#Time = 1, r = 0.1, b = 0.1, sigma = 0.25, n = 50)
#GBSOption(TypeFlag = "c", S = 100, X = 100,

#Time = 1, r = 0.1, b = 0.1, sigma = 0.25)

steps = 50
CRRV = rep(NA, times = steps)
for (n in 3:steps) {
CRRV[n] = CRRBinomialTreeOption(TypeFlag = "ce",
S = 150,
```

```
X = 135, Time =0.4, r = 0.045, b = 0.045, sigma

= 0.20, n = n)@price
}
CRRV
plot(CRRVe[3:steps], type = "l", col =

"red", xlab = "No. of time step", ylab = "Option
Value")


CRRV[steps]

GBSOption(TypeFlag = "c", S = 150,

X = 135, Time =0.4, r = 0.04335, b = 0.03335,

sigma = 0.20)

library(quantmod)
library("fOptions")
getSymbols('INFY.NS',src='yahoo', from = "2016-11-01")
# Read Data from Computer
#INFY.NS <- read.csv("INFY_NS6M.csv")
tail(INFY.NS, 5)
# Select the Adjusted Price
INFY  = INFY.NS[,6]
# Calculate Daily Log Returns
INFY.rets = diff(log(INFY))[-1]
# Annualized Volatility by Daily Volatility
INFY.sd = sd(INFY.rets, na.rm=TRUE)*sqrt(252)

print(c("Value of annual volatiity is", INFY.sd ))
```

```
S = as.numeric(INFY.NS[nrow(INFY.NS), 4])
print(c("Value of S is", S))
k = 54
T = k/365;
r = .031;
b = r;
sigma = INFY.sd;
n=50;
GBSV = rep(NA, times = 201)
CRRV = rep(NA, times = 201)
for (i in 1300:1500){

CRRV[i-1299] = CRRBinomialTreeOption(TypeFlag =

"ce", S = S, X = i,Time = T, r = r, b = r, sigma
= sigma, n = n)@price

GBSV[i-1299] = GBSOption(TypeFlag = "c", S = S,

X = i,Time = T, r = r, b = r, sigma =

sigma)@price



}
t = c(1,40,80,120,160,200)
print(c(t+1299,CRRV[t]))
print(c(t+1299,GBSV[t]))
plot(1300:1500,CRRV[1:201], type =

"l", col = "red", xlab = "Strike Price", ylab =
```

```
"Infosys Option Value")
lines(1300:1500,GBSV[1:201], col="green")
```