Asymptotic Symmetries in Higher Dimensions

Ruchira Mishra

A dissertation submitted for the partial fulfilment of BS-MS dual degree in Science

Indian Institute of Science Education and Research, Mohali

April, 2021

Certificate of Examination

This is to certify that the dissertation titled **"Asymptotic Symmetries in Higher Dimensions"** submitted by Ruchira Mishra (Reg. No. MS16071) for the partial fulfilment of BS-MS dual degree program of the Institute, has been examined by the thesis committee duly appointed by the Institute. The committee finds the work done by the candidate satisfactory and recommends that the report be accepted.

Dr. Anosh Joseph	Dr. KP Yogendran	Dr. Kinjalk Lochan (Supervisor)

Dated: April 28, 2021

Declaration

The work presented in this dissertation has been carried out by me under the guidance of Dr. Suvrat Raju at the International Centre for Theoretical Sciences (ICTS), Bangalore, Dr. Alok Laddha at the Chennai Mathematical Insitute, Chennai and Dr. Kinjalk Lochan at the Indian Institute of Science Education and Research Mohali. This work has not been submitted in part or in full for a degree, a diploma, or a fellowship to any other university or institute. Whenever contributions of others are involved, every effort is made to indicate this clearly, with due acknowledgement of collaborative research and discussions. This thesis is a bonafide record of original work done by me and all sources listed within have been detailed in the bibliography.

Ruchira Mishra
(Candidate)

Dated: April 28, 2021

In my capacity as the supervisor of the candidate's project work, I certify that the above statements by the candidate are true to the best of my knowledge.

Dr. Kinjalk Lochan
(Supervisor)

Acknowledgements

I would like to express my deepest gratitude to Dr. Suvrat Raju for introducing me to this exciting field of research and for giving me the opportunity to work on this project. I am extremely grateful to Dr. Alok Laddha for his constant guidance and unrelenting support without which this project would not have been possible.

I would like to thank Dr. Kinjalk Lochan for providing me guidance at IISER, Mohali as my local guide. I am also grateful to my committee members Dr. Anosh Joseph and Dr. KP Yogendran for providing insights and feedback.

I am indebted to Chandramouli Chowdhury for helping me out with calculations during my project and for always being there when I had doubts. I would like to specially thank Prahar Mitra and Siddharth Prabhu for all the helpful discussions during the project. I am also grateful to Pushkal Shrivastava for his invaluable help during the initial stages of this project. I also want to thank other members of the group, Tuneer Chakraborty, Joydeep Chakravarty, Jewel Ghosh, Olga Papadoulaki and Priyadarshi Paul for all the enlightening discussions during the group meetings. Special thanks to Dr. Abhishek Chaudhuri for his unwavering support and encouragement.

I am grateful to IISER Mohali for providing me with an excellent academic environment to flourish as a budding scientist, especially access to the library where I spent a lot of time working on this project. I would also like to acknowledge the financial support received from DST and the ICTS Long Term Visiting Program.

Last but not the least, I would like to express my sincerest gratitude to my parents, Dr. Mamta Mishra and Dr. Amit Mishra, my sister Medha and my friends Ardra, Kausthub, Manisha, Sasank, Shradha, Soumya and Vedang for their unparalleled love, support and encouragement during this entire project.

List of Figures

Abbreviations

Abbreviation	**Expansion**
LGT	Large Gauge Transformation
ED	Electrodynamics
EOM	Equation of Motion
EL	Euler Lagrange equation

Contents

Abstract

In this thesis, we revisit asymptotic symmetries in electrodynamics and gravity. Our goal is to study the existence of asymptotic symmetries in higher even dimensions. We first analyse the case of electromagnetism and study large gauge transformations. We prove the conservation of large gauge transformation charges by following the procedure in [CE17]. This conservation is related to the soft photon theorem [HMPS14]. Next we study asymptotically flat spacetimes in four and higher even dimensions. We do a full non-linear analysis of the phase space in higher dimensional asymptotically flat spacetimes and study the existence of BMS symmetries.

Chapter 1

Introduction

Lately, there has been renewed interest in studying asymptotic symmetries because of emerging connections between quantum soft theorems in studies of scattering amplitudes, asymptotic symmetries and the memory effect (see [Str18] for a review). Quantum soft theorems describe universal feature of the S matrix in gauge theory or gravity when a finite number of gauge bosons or gravitons become soft, i.e. have energies much less than the characteristic energy of the scattering. The universality of the S matrix in such a kinematic region (where certain number of particles are soft) has an analog in classical scattering. That is, in any classical scattering in which gravitational or electromagnetic radiation is emitted, the soft (low frequency) limit of the radiation displays certain universal features which are independent of the details of the scattering. In the case of gravitational scattering, this universality goes under the name of classical soft graviton theorem [Wei65]. Consider a classical scattering process with n incoming objects with momenta $p_1, \ldots, p_n$ and m outgoing objects with momenta $p_{n+1}, \ldots, p_m$. The scattering states can be composite objects such as black holes or neutron stars and p_i defines the (initial or final) momentum of the center of mass of these objects which may interact via gravitational and other interactions including contact interactions such as collision. In general, the radiative gravitational field $h_{\mu\nu}(t, \bar{x})$ (or it's Fourier transform in the time coordinate $h_{\mu\nu}(\omega, \hat{x})$) depends on the details of the scattering with no closed form expression. But in the low frequency expansion (which corresponds to the gravitational radiation emitted during the scattering at early and late retarded time), we get an interesting result at leading order [Wei65], [SSS20]

$$h_{\mu\nu}(\omega, \hat{x}) = \sum_{a=1}^{m+n} \frac{p_{(a)\mu} p_{(a)\nu}}{p_a \cdot k} \tag{1.1}$$

with $k^\mu = \omega(1, \hat{x})$ and $\hat{x}$ the direction at which the radiation is being measured. This statement is universal in the sense that it does not depend on the details of the scattering process. In fact this (leading) soft behaviour of radiative gravitational field is true in all dimensions $D \geq 4$. Generically in a scattering process, universal constraints on scattering are associated to conservation laws and symmetries. Simplest example is that of conservation of energy during a scattering event which arises from time translation symmetries in the Lagrangian. The soft graviton theorem is a statement of energy conservation at every angle and should have an underlying symmetry associated with it.

In four dimensions, it turns out that such a connection does exist. The conservation of so called super-translation charges (that generate super-translation symmetries) in a gravitational scattering that occurs in asymptotically flat spacetime is associated with the classical soft graviton theorem [HLMS15]. In fact, this is not the whole story. The classical soft graviton theorem of (1.1) also describes the so called memory effect due to point particles. The memory effect is the permanent displacement in the relative positions of inertial observes due to a radiation epoch. It turns out that the shift in the metric components due to the radiation epoch is the same metric fluctuation that one gets from the soft graviton theorem in four dimensions. The soft graviton theorem is precisely the memory effect in four dimensions [SZ14].

Classical soft graviton theorem is thus on one hand conservation of super-translation charges and on the other hand, describes the memory effect. This obviously implies that the gravitational memory effect is nothing but a statement about conservation of super-translation charges.[1] The metric before and after the radiation epoch can actually be related through asymptotic symmetries. We can ask what happens to this trinity of relations in $D > 4$ dimensions. The classical soft graviton theorem in (1.1)) is a dimension independent result. So our naive expectation would be that the relation between asymptotic symmetries and soft theorems should also trivially generalise to higher dimensions. But it turns out that in higher dimensions, the existence of asymptotic symmetries is a rather subtle issue. Our goal in this thesis is to study the existence of asymptotic symmetries in higher even dimensions. The case of odd dimensions is far more subtle as it is not clear if there is a definition

[1] Although for the sake of pedagogy, we restrict ourselves to a statement about gravitational radiation due to point particles, the results can be trivially generalised to emission of gravitational radiation due to fields including gravitational field itself [LS20]

of null infinity for odd-dimensional spacetimes which contain radiation [HW04] [2] As gravitational physics is rather complicated due to the non-linear nature of Einstein equations, we start by analysing the universality of soft Electro-magnetic radiation in four dimensional Minkowski spacetime. A structure analogous to gravity exists here as well. The conservation law associated with asymptotic symmetries is related to the so-called classical soft photon theorem [Wei65]. Since the ED case is more simpler and cleaner, we use it to introduce some key concepts important for the gravity analysis. More specifically, we introduce the notion of asymptotic symmetries or Large Gauge Transformations (LGTs) through this example. We also introduce the powerful covariant phase space formalism which is perhaps the most efficient language to understand the Hamiltonian framework for corresponding charges.

Next, we explicitly prove the conservation of the LGT charges. Equipped with these tools, we then move on to four dimensional gravity. We find the asymptotic symmetry group of asymptotically flat spacetimes, also called the BMS group. We discuss why the four dimensional case is special and the issues associated with extending the analysis to higher dimensions. In the higher dimensional analysis, we first do a linearised analysis of the phase space and asymptotic symmetries. We find from the linearized analysis that a conservation law exists even in higher dimensions. But this presents a new confusion that we try to explain. We finally present a complete non-linear analysis and state our results so far.

[2] There has been interesting recent progress in analysing the case of odd-dimensions [HM19] but the analysis in these works is outside the scope of this thesis.

Chapter 2

Electrodynamics

In this chapter, we will look at Electrodynamics (ED) in 4-dimensional Minkowski spacetime. We will start by reviewing aspects of Minkowski spacetime. Next we will study massless charged matter in this background. After analysing the Equation of Motion (EOM)s, we will move on to the phase space analysis and use it to find charges generating asymptotic symmetries (or Large Gauge Transformations (LGTs) in this context). We will then analyse the EOMs at spatial infinity to prove the conservation of these charges.

2.1 Minkowski spacetime

The metric for a $d+2$ dimensional Minkowski spacetime can be written as

$$ds^2 = -dt^2 + dr^2 + r^2 \gamma_{AB} dx^A dx^B \tag{2.1}$$

where γ_{AB} is the unit round metric on the d sphere. Note that the higher dimensional space-time has additional non-compact dimensions with respect to the four dimensional Minkowski space. That is, we are adding a new dimension in the sense of adding a new spatial direction (for instance going to 5 dimensions from 4 dimensions means adding w to (t,x,y,z) to get (t,x,y,z,w) in the cartesian sense. Then r is defined as $r = \sqrt{x^2 + y^2 + z^2 + w^2}$ and in the corresponding spherical co-ordinates, the celestial sphere is d dimensional. Now in order to represent the spacetime in a finite portion of area, we “compactify” the spacetime and represent it using what is called the Penrose diagram. Distances are not represented faithfully in the Penrose diagram but the causal structure does not get affected by compactification brought about through conformal transformations. We can see this from the fact that light rays still travel at 45 degrees in this diagram. Every pair of points on the left and right side of this diagram is a d sphere labelled by constant $(r>0,t)$ and matched antipodally[1] to each other. Figure 2.1 shows the Penrose diagram. In this diagram the labels represent the following:

- Timelike infinity $i^{\pm}$: This represents the d-sphere that is reached by taking $t \to \pm\infty$ while keeping r constant. Since massive particles cannot move faster than the speed of light, they will always end up being crossed by a light ray which will move to a larger radius r. Hence, they will always end up at i^{+}.

- Spacelike infinity i^{0}: This represents the d-sphere reached by taking $r \to \infty$ while keeping t constant.

- Null infinity $\mathscr{I}^{\pm}$: This represents the codimension-1 hypersurface at $r \to \infty$ formed by the starting and ending points of null geodesics. $\mathscr{I}^{-}$ is where these null geodesics start and $\mathscr{I}^{+}$ is where they end.

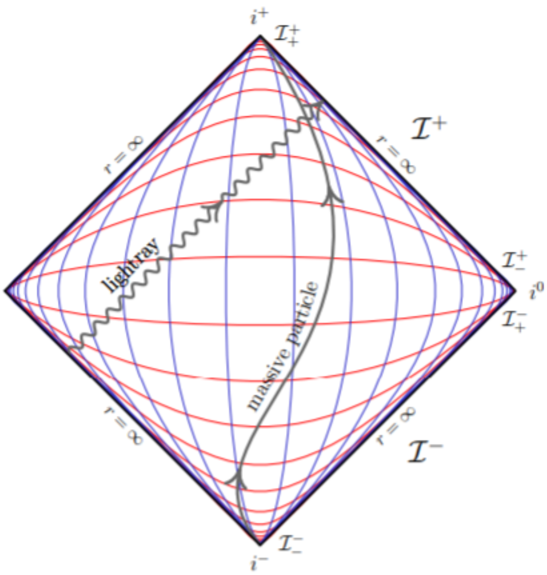

Figure 2.1: *Penrose diagram of Minkowski spacetime. Here, red lines represent constant-t surfaces and blue lines represent constant-r surfaces. The trajectory of a massive particle is shown by the thick gray line and the curly line shows a light ray. Timelike infinities are shown by* $i^{\pm}$ *and spacelike infinity by* i^{0}. $\mathscr{I}^{\pm}$ *label future and past null infinities. Every d-sphere of constant-*$r>0,t$ *is denoted by two antipodally related points in the diagram. [Str18]*

[1] The antipodal map $A: S^d \to S^d$ is defined by $A(x) = -x$ and sends every point on the sphere to its antipodal point

Since t and r both tend to infinity as we go towards $\mathscr{I}^{\pm}$, we introduce advanced and retarded null coordinates u and v defined as

$$u = t - r \qquad v = t + r \tag{2.2}$$

So $\mathscr{I}^{+}$ can be parameterized by $(u, \hat{x})$ and $\mathscr{I}^{-}$ by $(v, \hat{x})$. We can put further labels to denote the $u \to \pm\infty$ limit as $\mathscr{I}^{+}_{\pm}$ and similarly for $v \to \pm\infty$ as $\mathscr{I}^{-}_{\pm}$. These points are all shown in 2.1. The Minkowski metric in these coordinates takes the form

$$ds^2 = -du^2 - 2dudr + r^2\gamma_{AB}dx^A dx^B \tag{2.3}$$

for advanced null coordinates $(u, r, \hat{x})$ and

$$ds^2 = -dv^2 + 2dvdr + r^2\gamma_{AB}dx^A dx^B \tag{2.4}$$

for retarded null coordinates $(v, r, \hat{x})$.

In this chapter, we will study scattering i.e. we will be interested in understanding how the phase space prescribed on the Cauchy slice $i^- \cup \mathscr{I}^-$ is mapped to the phase space defined on $i^+ \cup \mathscr{I}^+$ and try to see if we get any conservation laws. In simpler words, we would like to understand the evolution of particles/wavepackets starting in the past towards the future i.e. how particles starting out in the past at $i^- \cup \mathscr{I}^-$, interact with each other and finally come out in the futute at $i^+ \cup \mathscr{I}^+$. Most of our analysis will deal with massless particles and so we will neglect $i^{\pm}$.

Studying this scattering problem will require us to analyse equations near spatial infinity i^0. In the conformal description given above, the points i^0 and $i^{\pm}$ are singular and values of fields in these regions often depend on the order of limits taken to reach these regions. Or in other words, fields are multivalued in these regions. Resolving these points can help solve this issue. For this purpose, we introduce a second set of coordinates that in some sense "blow up" the region near i^0 and $i^{\pm}$. We will introduce coordinates $(\rho, \tau, \hat{x})$ such that in the $|r| > t$ region,

$$\rho \equiv \sqrt{r^2 - t^2} \qquad \tau \equiv \frac{t}{\sqrt{r^2 - t^2}} \tag{2.5}$$

and the Minkowski metric in these coordinates becomes

$$ds^2 = d\rho^2 + \rho^2 d\sigma^2 \tag{2.6}$$

with the $d\sigma^2$ the $(d+1)$-dimensional de Sitter metric given as

$$d\sigma^2 = -\frac{d\tau^2}{1+\tau^2} + (1+\tau^2)\gamma_{AB}dx^A dx^B \equiv h_{\alpha\beta}dy^\alpha dy^\beta \tag{2.7}$$

here $y^\alpha = (\tau, x^A)$. We can think of the region $|r| > t$ as being foliated by dS_{d+1} slices.

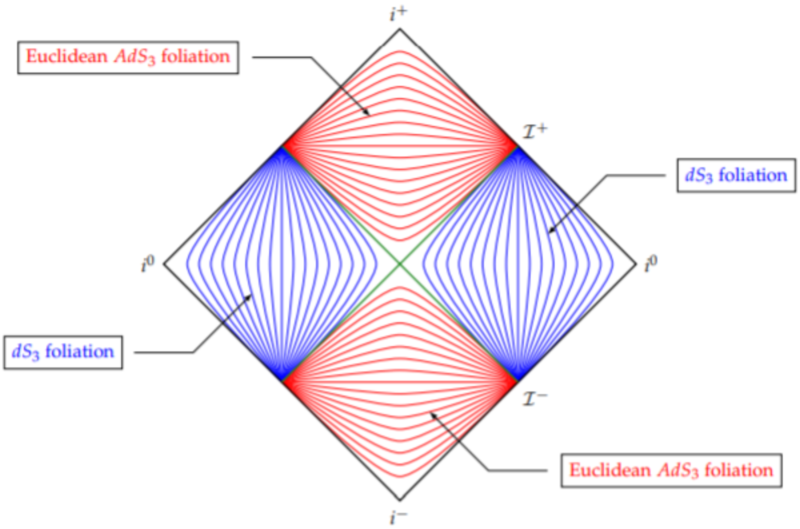

Figure 2.2: *Foliation of 4D Minkowski spacetime with dS_3 slices (drawn in blue) and AdS_3 slices (drawn in red) [CF19]*

We can similarly also foliate other regions of the spacetime. Figure 2.2 shows this for Minkowski in $4D$.

At this point we would also like to introduce some notation that will be used for the rest of the chapter. We will mostly follow [CE17] for notations and analysis of the EOMs at spatial infinity.

- spacetime $\mathbb{R}^{d+2}$: x^a, g_{ab}, ∇_a
- sphere S^d: x^A, γ_{AB}, D_A
- spatial infinity i^0: $y^\alpha = (\tau, x^A)$, $h_{\alpha\beta}$, D_α
- future null infinity $\mathscr{I}^+$: (u, x^A), q_{AB}, (∂_u, D_A)

As noted above also, (ρ, τ) are defined differently depending on the region of analysis.

2.2 Field equations and solutions

We are interested in studying Maxwell fields A_a in four dimensional flat spacetime coupled to massless charged fields ϕ.

The field equations are

$$\nabla^a F_{ab} = J_b \tag{2.8}$$

$$D_a D^a \phi = 0 \tag{2.9}$$

where $F_{ab} = \partial_a A_b - \partial_b A_a$ is the field strength, ∇_a is the spacetime covariant derivative, $D_a \phi = \partial_a \phi - i A_\mu \phi$ is the gauge-covariant derivative and J_a is the conserved matter current given by

$$J_a = \phi (D_a \phi)^* + c.c. \tag{2.10}$$

Local $U(1)$ gauge transformations, parameterized by scalar Λ act on the fields in the following way leaving the equations unchanged

$$\delta_\Lambda A_a = \partial_a \Lambda \qquad \delta_\Lambda \phi = i \Lambda \phi \tag{2.11}$$

2.3 Electrodynamics (ED) at null infinity

2.3.1 Field expansion near null infinity

In this section, we study fields near $\mathscr{I}^+$ i.e. as $r \to \infty$ with $(u, \hat{x})$ finite. The metric in coordinates $(u, r, \hat{x})$ as noted before in (2.3) is

$$g_{ab} = \begin{pmatrix} -1 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & r^2 \gamma_{AB} \end{pmatrix}, \qquad g^{ab} = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & r^{-2} \gamma^{AB} \end{pmatrix} \tag{2.12}$$

The EOMs in these coordinates are

$$\frac{1}{r^2} \partial_r (r^2 F_{ru}) - \partial_u F_{ru} + \frac{1}{r^2} D^A F_{Au} = J_u \tag{2.13}$$

$$\frac{1}{r^2} \partial_r (r^2 F_{ru}) + \frac{1}{r^2} D^A F_{Ar} = J_r \tag{2.14}$$

$$\partial_r (F_{rA} - F_{uA}) - \partial_u F_{rA} + \frac{1}{r^2} D^A F_{BA} = J_A \tag{2.15}$$

where D_A denotes derivative w.r.t the sphere metric γ_{AB}.

To decide the falloff for the fields near $\mathscr{I}^+$, we set some physical constraints. We would want the total energy, momentum and charge to be finite. Keeping these constraints in mind, we use the following falloffs:

$$F_{AB}(u,r,\hat{x}) = \sum_{n=0}^{\infty} \frac{F_{AB}^{(n)}(u,\hat{x})}{r^n} \tag{2.16}$$

$$F_{ur}(u,r,\hat{x}) = \sum_{n=2}^{\infty} \frac{F_{ur}^{(n)}(u,\hat{x})}{r^n} \tag{2.17}$$

$$F_{uA}(u,r,\hat{x}) = \sum_{n=0}^{\infty} \frac{F_{uA}^{(n)}(u,\hat{x})}{r^n} \tag{2.18}$$

$$F_{rA}(u,r,\hat{x}) = \sum_{n=2}^{\infty} \frac{F_{rA}^{(n)}(u,\hat{x})}{r^n} \tag{2.19}$$

This also gives the falloffs for the massless matter current

$$J_u(u,r,\hat{x}) = \sum_{n=2} \frac{J_u^{(n)}(u,\hat{x})}{r^n} \tag{2.20}$$

$$J_A(u,r,\hat{x}) = \sum_{n=2} \frac{J_A^{(n)}(u,\hat{x})}{r^n} \tag{2.21}$$

$$J_r(u,r,\hat{x}) = \sum_{n=4} \frac{J_r^{(n)}(u,\hat{x})}{r^n} \tag{2.22}$$

With these falloffs, we can solve the field equations near $\mathscr{I}^+$. We will work in the radial gauge i.e.

$$A_r = 0 \tag{2.23}$$

We can write the falloffs in terms of the components of the gauge field as,

$$A_I(u,r,\hat{x}) = \sum_{n=0}^{\infty} \frac{A_I^{(n)}(u,\hat{x})}{r^n} \tag{2.24}$$

$$A_u(u,r,\hat{x}) = \sum_{n=1}^{\infty} \frac{A_u^{(n)}(u,\hat{x})}{r^n} \tag{2.25}$$

On substituting the above falloffs in the EOMs (2.13)-(2.15), we see that $A_I^{(0)}(u,\hat{x})$ is unconstrained free data and all the components can be evaluated in terms of $A_I^{(0)}$ and integration constants at $u = -\infty$. We further assume the following behaviour for $A_I^{(0)}$ as $u \to \pm\infty$.

$$A_I^{(0)} = D_I\Phi(\hat{x}) + \hat{A}_I(u,\hat{x}) \tag{2.26}$$

where $\hat{A}_I(u,\hat{x}) \sim O(|u|^{-\varepsilon})$. The reason we can write the u-independent piece as $D_I\phi$ is because we assume that the magnetic field vanishes as $u \to \infty$. To determine the u-falloffs for a general $A_I^{(n)}$, the following equation (derived from (2.15)) will be useful

$$\partial_u A_I^{(n)} = \frac{1}{2n} D^J F_{IJ}^{(n-1)} \qquad n \geq 1 \tag{2.27}$$

Using the equation above and $A_I^{(0)} \sim O(u^0)$, we get

$$A_I^{(n)} = O(|u|^n) \tag{2.28}$$

Then (2.14) tells us

$$A_u^{(n)} = \frac{1}{n} D \cdot A^{(n-1)} \tag{2.29}$$

And so $A_u^{(n)} \sim O(|u|^{n-1})$. Thus,

$$F_{ur}^{(n)} = A_u^{(n-1)} = O(|u|^{n-2}) \tag{2.30}$$

as $u \to \pm\infty$.

Radial gauge does not completely fix the gauge and there are residual gauge transformations left in the theory. We can also determine the large-r behaviour of the gauge parameter associated to the residual gauge symmetries. It has an expansion of the form

$$\Lambda(u,r,\hat{x}) = \lambda(u,\hat{x}) + O(r^{-1}) \tag{2.31}$$

If $\lambda(u,\hat{x}) = 0$ then we obtain the so-called trivial gauge transformations. These gauge transformations are redundancies of the theory and the Noether charge associated to them is zero. On the other hand, the group of gauge transformations generated by Λ with $\lambda(u,\hat{x}) \neq 0$ are the so-called Large Gauge Transformations (LGTs).

In our case, when we fix the radial gauge, we get on using (2.11),

$$\delta_\Lambda A_r = \partial_r \lambda = 0 \qquad \delta_\lambda A_u^{(0)} = \partial_u \lambda = 0 \tag{2.32}$$

And so

$$\lambda \equiv \lambda(\hat{x}) \qquad \text{in radial gauge} \tag{2.33}$$

To make a distinction between trivial gauge transformations and LGT, we usually need to calculate the charge associated with the transformation (which we will do in the next section). The reason these are called “large” gauge transformations is because the gauge parameter Λ does not die off at infinity. Instead what we get is an angle dependent parameter λ as seen above. The action of the gauge parameter λ on the fields is as follows then

$$\delta_\lambda A_I^{(0)} = D_I \lambda; \qquad \delta_\lambda A_I^{(1)} = 0 \tag{2.34}$$

$$\delta_\lambda \phi = i\lambda\phi \tag{2.35}$$

2.3.2 Phase space at null infinity

We will start with reviewing the covariant phase space formalism in this section and then use it to analyse the phase space at $\mathscr{I}^+$ of massless ED. [HI87] and [HM21] are good references to study this in greater detail.

Covariant phase space formalism is an entirely classical construct. Given the Lagrangian, it tells us how to construct the phase space. Consider a simple example where we have $L \equiv L(\phi, \partial_\mu \phi)$ in a flat background. Then the Euler-Lagrange (EL) is given by

$$\partial_\mu \left(\frac{\partial L}{\partial(\partial_\mu \phi)} \right) = \frac{\partial L}{\partial \phi} \tag{2.36}$$

and we define $\pi = \frac{\partial L}{\partial \dot{\phi}}$. Since it is a second order equation, the phase space or the solution space Γ is parameterized by 2 variables/functions $(\phi(\bar{x}), \pi(\bar{x}))_{t=t_0}$. We can then define the Poisson bracket structure

$$\{\phi(\bar{x}), \pi(\bar{y})\} = \delta(\bar{x} - \bar{y}) \tag{2.37}$$

$$\{\phi(\bar{x}), \phi(\bar{y})\} = 0 \tag{2.38}$$

$$\{\pi(\bar{x}), \pi(\bar{y})\} = 0 \tag{2.39}$$

where all these Poisson brackets are evaluated on a time slice. This procedure can also be understood in a more geometric way. We start with varying the Lagrangian $L \equiv L(\phi, \partial_\mu \phi)$,

$$\delta L = \big[\frac{\partial L}{\partial \phi} - \partial_\mu \frac{\partial L}{\partial(\partial_\mu \phi)}\big]\delta\phi + \partial_\mu \big[\frac{\partial L}{\partial(\partial_\mu \phi)}\delta\phi\big] \tag{2.40}$$

The first term is just the Euler Lagrange equation (EL) and the second term which is a boundary term gives what we call the symplectic potential. We integrate it over a constant-t

surface with normal $n_\mu = (1, \bar{0})$.

$$\begin{aligned}
\Theta(\delta) &= \int d\bar{x} n_\mu \frac{\partial L}{\partial(\partial_\mu \phi(\bar{x}))} \delta\phi(\bar{x}) \\
&= \int d\bar{x} n_\mu \frac{\partial L}{\partial(\dot{\phi}(\bar{x}))} \delta\phi(\bar{x}) \\
&= \int d\bar{x} \pi(\bar{x}) \delta\phi(\bar{x})
\end{aligned} \tag{2.41}$$

From here we can calculate what we call the symplectic form by performing a variation over the field space,

$$\Omega(\delta, \delta') = \int d\bar{x} \big[\delta' \pi(\bar{x}) \wedge \delta\phi(\bar{x})\big] \tag{2.42}$$

The symplectic form is a closed and invertible two-form on the field space and helps us get the Poisson bracket structure. We can also think of it as a matrix

$$\Omega_{ab} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \tag{2.43}$$

where we have $\Omega_{\phi\phi} = \Omega_{\pi,\pi} = 0$, $\Omega_{\phi,\pi} = 1$ and $\Omega_{\pi,\phi} = -1$. We can invert this matrix to find the Poisson bracket structure. In general we have

$$\{A, B\} = \Omega^{ab} \partial_a A \partial_b B \tag{2.44}$$

This gives us for the fields $\phi(\hat{x})$ and $\pi(\hat{y})$ from (2.43),

$$\begin{aligned}
\{\phi(\bar{x}), \pi(\bar{y})\} &= \Omega^{ab} \partial_a \phi(\bar{x}) \partial_b \pi(\bar{y}) \\
&= \delta^3(\bar{x} - \bar{y})
\end{aligned} \tag{2.45}$$

This procedure can be generalised to find the phase space of theories in any arbitrary space-time, with any Lagrangians and on any Cauchy surfaces. In our case, we are interested in analysing the theory at null infinity $\mathscr{I}^+$. The advantage of doing our analysis at null infinity is that all interactions die down at these large distances and so the theory is essentially free. Since we are working with massless fields, we can also neglect $i^\pm$ and just work with defining our initial data on a null slice. For free ED the symplectic form is given by

$$\Omega_\Sigma = \int_\Sigma d\Sigma^a J_a = \int_\Sigma d\Sigma^a \sqrt{-g} \delta F_{ab} \wedge \delta' A^b \tag{2.46}$$

In our case, we are interested in analysing ED with massless matter at $\mathscr{I}^+$. The interaction terms in the Lagrangian do not contribute to the symplectic form at leading order and the symplectic form factorizes into the matter part and pure ED part. We can see this from the fact that the symplectic form at $\mathscr{I}^+$ will have terms of the form

$$\sqrt{-g}\int dud^dx\big(\delta\phi\wedge\delta' D_u\phi\big)=\frac{1}{r^d}\int dud^dx\big(\delta\phi\wedge\delta'\partial_u\phi-i\delta\phi\wedge\delta' A_u\phi\big) \tag{2.47}$$

with $D_\mu=\partial_\mu-iA_\mu$ the gauge covariant derivative. Now for the symplectic form to be finite, we would want $\phi\sim\frac{1}{r^{d/2}}$. We know that $A_u\sim\frac{1}{r^k}$ where $k\geq 1$ in any dimension. So the second term does not contribute at leading order to the symplectic form and the symplectic form factorizes into pure ED part and matter field part.

$$\Omega_{\mathscr{I}^\pm}=\Omega^A_{\mathscr{I}^\pm}+\Omega^{mat}_{\mathscr{I}^\pm} \tag{2.48}$$

where we can calculate $\Omega^A_{\mathscr{I}^+}$ from (2.46). We can reach $\mathscr{I}^+$ by taking a constant-t slice and taking the limit $t\to\infty$ while keeping u fixed.

$$\Omega_{\Sigma_t}=\int_{\Sigma_t}d\Sigma_t J^t \tag{2.49}$$

Now $J^t=J^u+J^r=-J_u$. So then,

$$\begin{aligned}\Omega^A_{\mathscr{I}^+}(\delta,\delta')&=-\int dud^dx\sqrt{-g}\delta F_{ua}\wedge\delta' A^a\\&=-\int_{\mathscr{I}^+}dud^dx\sqrt{-g}(\delta F_{ua}\wedge\delta' A^a)\\&=-\int_{\mathscr{I}^+}dud^dx\sqrt{-g}\big(\delta F_{ur}\wedge\delta' A^r+\delta F_{uI}\wedge\delta' A^I\big)\end{aligned} \tag{2.50}$$

For $d=2$, we get at leading order

$$\Omega^A_{\mathscr{I}^+}(\delta,\delta')=\int_{\mathscr{I}^+}dud^2x\big(\delta F^{(0)}_{Iu}\wedge\delta' A^{I(0)}\big) \tag{2.51}$$

We write $A^{(0)}_I$ as in (2.26) and substitute in the symplectic form

$$\begin{aligned}\Omega^A_{\mathscr{I}^+}(\delta,\delta')&=\int_{\mathscr{I}^+}dud^2x\big(\delta F^{(0)}_{Iu}\wedge\delta' A^{I(0)}\big)\\&=\int_{\mathscr{I}^+}dud^2x\big(\delta\partial_u\hat{A}^{(0)}_I\wedge\delta'\hat{A}^{I(0)}+\delta F^{(0)}_{uI}\wedge\delta' D_I\Phi\big)\\&=\int_{\mathscr{I}^+}dud^2x\big(\delta\partial_u\hat{A}^{(0)}_I\wedge\delta'\hat{A}^{I(0)}-\delta D_I F^{(0)}_{uI}\wedge\delta'\Phi\big)\end{aligned} \tag{2.52}$$

The matter field symplectic form is given as

$$\Omega^{mat}_{\mathscr{I}^+}(\delta,\delta')=\int_{\mathscr{I}^+}dud^2x(\delta\partial_u\phi\wedge\delta'\phi) \tag{2.53}$$

This gives us

$$\Omega_{\mathscr{I}^+}(\delta,\delta') = \int_{\mathscr{I}^+} dud^2x\big(\delta\partial_u\hat{A}_I^{(0)}\wedge\delta'\hat{A}^{I(0)} - \delta D_I F_{uI}^{(0)}\wedge\delta'\Phi\big) + \int_{\mathscr{I}^+} dud^2x\delta\partial_u\phi\wedge\delta'\phi \tag{2.54}$$

Once we have the symplectic structure, we can also define the Poisson bracket structure. Before doing that, we define some more notation that might be useful later. We define

$$D_I N = \int_{-\infty}^{\infty} du F_{uI}^{(0)} \tag{2.55}$$

So then the ED part of the symplectic form can also be written as

$$\begin{aligned}\Omega^A_{\mathscr{I}^+} &= \int_{\mathscr{I}^+} dud^2x\big(\delta\partial_u\hat{A}_I^{(0)}\wedge\delta'\hat{A}^{I(0)} + \delta F_{uI}^{(0)}\wedge\delta' D_I\Phi\big) \\ &= \int_{\mathscr{I}^+} dud^2x\big(\delta\partial_u\hat{A}_I^{(0)}\wedge\delta'\hat{A}^{I(0)}\big) + \int d^2x(D^I N\wedge\delta' D_I\Phi\big) \\ &= \int_{\mathscr{I}^+} dud^2x\big(\delta\partial_u\hat{A}_I^{(0)}\wedge\delta'\hat{A}^{I(0)}\big) - \int d^2x(N\wedge\delta' D^2\Phi\big)\end{aligned} \tag{2.56}$$

where the first equation comes from the second line of (2.52). We then get the following Poisson brackets

$$\{\partial_u A_I^{(0)}(u,\hat{x}), A_I^{(0)}(u',\hat{x}')\} = \delta(u-u')\delta(\hat{x}-\hat{x}') \tag{2.57}$$

$$\{D_I\phi(\hat{x}), D_J N(\hat{x}')\} = \gamma_{IJ}\delta(\hat{x}-\hat{x}') \tag{2.58}$$

$$\{\phi(u,\hat{x}), \partial_u\phi(u',\hat{x}')\} = \delta(u-u')\delta(\hat{x}-\hat{x}') \tag{2.59}$$

2.4 Scattering and conservation laws

Towards the end of 2.1, we discussed about the scattering process. We will now try to work that out explicitly. We are interested to see how solutions at $\mathscr{I}^+_-$ match with the solutions at $\mathscr{I}^-_+$. For this we will analyse the field equations at spatial infinity.

2.4.1 Fields near null infinity

In this section, we will study fields near spatial infinity i^0. As seen before, coordinates $(\rho,\tau,\hat{x})$ as defined in (2.5) are useful in this region. The Minkowski metric in these coordinates can be written as in (2.6)

$$g_{ab} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\rho^2/(1+\tau^2) & 0 \\ 0 & 0 & \rho^2(1+\tau^2)\gamma_{AB} \end{pmatrix}, \qquad g^{ab} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -(1+\tau^2)/\rho^2 & 0 \\ 0 & 0 & 1/\rho^2(1+\tau^2)\gamma^{AB} \end{pmatrix} \tag{2.60}$$

In these coordinates the field equations (2.8) take the following form

$$\frac{1}{\sqrt{-g}}\partial_\tau\big(\sqrt{-g}\frac{(1+\tau^2)}{\rho^2}F_{\tau\rho}\big) + \frac{1}{\rho^2(1+\tau^2)}D^J F_{J\rho}\big] = J_\rho \tag{2.61}$$

$$\frac{\rho^2}{1+\tau^2}\big[\frac{1}{\sqrt{-g}}\partial_\rho(\sqrt{-g}\frac{1+\tau^2}{\rho^2}F_{\rho\tau}) + \frac{1}{\rho^4}D^J F_{J\tau}\big] = J_\tau \tag{2.62}$$

$$\rho^2(1+\tau^2)\frac{1}{\sqrt{-g}}\big[\partial_\rho(\frac{\sqrt{-g}}{\rho^2(1+\tau^2)}F_{\rho I}) + \partial_\tau(\frac{\sqrt{-g}}{\rho^4}F_{\tau I})\big] = J_I \tag{2.63}$$

Here $\sqrt{-g} = \frac{\rho^{d+1}}{(1+\tau^2)^{(d-1)/2}}$ in $D = d+2$ dimensions. If we assume that the matter field has no "soft" component i.e. $\phi \sim \frac{1}{u}$, then we can see that the above equations are essentially free near spatial infinity [CL19]. From coordinate transformations, we can see that $J_\rho, J_A \sim \frac{1}{\rho^2}$ and $J_\tau \sim \frac{1}{\rho^3}$.

We can set the field falloffs here using EOMs. Consider the τ component of the EOM (2.62). We can simplify it further to get

$$\lim_{\rho\to\infty} F_{\rho\tau}(\rho,\tau,\hat{x}) = \frac{F^{(k)}_{\rho\tau}}{\rho^k} + O(1/\rho^{k+1}) \tag{2.64}$$

We solve it in the large ρ limit. Say this admits a solution with the leading term for $F_{\rho\tau}$ as,

$$\lim_{\rho\to\infty} F_{\rho\tau}(\rho,\tau,\hat{x}) = \frac{F^{(k)}_{\rho\tau}}{\rho^k} + O(1/\rho^{k+1}). \tag{2.65}$$

And say $F_{I\tau}$ starts as,

$$F_{I\tau} = \frac{F^{(l)}_{I\tau}}{\rho^l} + O(1/\rho^{l+1}) \tag{2.66}$$

Then the leading order equation of motion gives us something like,

$$\left\{\frac{(1+\tau^2)}{\rho^{k-1}}\big[(d-1)-k\big]F^{(k)}_{\rho\tau} + O(1/\rho^{k+1})\right\} + \left\{\frac{1}{\rho^l}\bar{D}^I F_{I\tau} + O(1/\rho^{l+1})\right\} = 0. \tag{2.67}$$

Now say if $l > k-1$ then we must have $k = d-1$. Hence we have the fall off for $F_{\rho\tau}$ in general dimensions for large ρ as,

$$\lim_{\rho\to\infty} F_{\rho\tau}(\rho,\tau,\hat{x}) = \frac{F^{(d-1)}_{\rho\tau}}{\rho^{d-1}} + O(\rho^{d}). \tag{2.68}$$

This for $d = 2$ gives us

$$F_{\rho\tau}(\rho,y) = \sum_{n=1}^{\infty} \frac{F^{(n)}_{\rho\tau}(y)}{\rho^n} \tag{2.69}$$

This matches with [CE17] where they use the following falloffs

$$F_{\alpha\beta}(\rho,y) = \sum_{n=0}^{\infty} \frac{F^{(n)}_{\alpha\beta}(y)}{\rho^n} \tag{2.70}$$

$$F_{\alpha\rho}(\rho,y) = \sum_{n=1}^{\infty} \frac{F^{(n)}_{\alpha\rho}(y)}{\rho^n} \tag{2.71}$$

We can also expand the gauge parameter Λ in the $\rho \to \infty$ limit

$$\Lambda(\rho,y) = \lambda(y) + O(\rho^{-1}) \tag{2.72}$$

After determining these falloffs, we can solve the EOMs near spatial infinity. While working near spatial infinity, equations simplify in the Lorenz gauge and so we will be working in this gauge. The equations (2.8) simplify to

$$\Box A_a = \left(\nabla^2_\rho + \frac{\sigma^{\mu\nu}}{\rho^2}\nabla_a\nabla_b\right)A_a \tag{2.73}$$

where ∇_a denotes the derivative on the whole of spacetime and $\sigma^{\mu\nu}$ is the dS_{d+1} metric as defined in (2.7). We can use these equations to derive the τ falloffs for the fields in the limit $\tau \to \pm\infty$. We get that

$$F^{(n)}_{\rho\tau}(y) = O(|\tau|^{n-4}) \tag{2.74}$$

We show this derivation for a general d in A.

Working in the Lorenz gauge would imply that the gauge parameter is also further constrained and has to satisfy the Lorenz gauge condition i.e.

$$\nabla^a A_a = 0 \quad \Longrightarrow \quad \nabla^a\nabla_a\Lambda = 0 \tag{2.75}$$

This in $\rho \to \infty$ limit will give

$$D^\alpha D_\alpha \lambda = 0 \tag{2.76}$$

We are interested in solving the EOM at spatial infinity and then matching the solutions at $\tau \to \infty$. This will help us match solutions at $\mathscr{I}^+_-$ with solutions at $\mathscr{I}^-_+$. For now we are interested in analysing only the leading order equations at spatial infinity. A similar analysis works at higher orders too [CL19]

$$D^\alpha F^{(-1)}_{\alpha\rho} = 0 \qquad D_{[\alpha} F^{(-1)}_{\beta]\rho} = 0 \tag{2.77}$$

$$D^\beta F^{(0)}_{\alpha\beta} = 0 \qquad D_{[\alpha} F^{(0)}_{\beta\gamma]} = 0 \tag{2.78}$$

where D_α as defined in 2.1 is the covariant derivative on i^0. The Bianchi identity (i.e. the second equation in (2.77)) admits a solution of the form

$$F^{(-1)}_{\alpha\rho} = D_\alpha \psi \tag{2.79}$$

where $\psi \equiv \psi(y)$. The first equation in (2.77) would then give

$$D^\alpha D_\alpha \psi = 0 \tag{2.80}$$

A similar analysis works for (2.78) where we can define a function $\tilde{\psi}$ and write

$$F^{(0)}_{\alpha\beta} = \varepsilon_{\alpha\beta\gamma} D^\gamma \tilde{\psi} \qquad D^\alpha D_\alpha \tilde{\psi} = 0 \tag{2.81}$$

We see that the gauge parameter λ and the function ψ satisfy the same wave equation (2.76). When we solve the wave equation in the limit $\tau \to \pm\infty$ (done in B), we get two types of solutions one corresponding to the large τ behaviour of λ and the other to the large τ behaviour of ψ

$$\lambda(\tau, \hat{x}) = \lambda_\pm(\hat{x}) + O(|\tau|^{-2} \ln|\tau|) \tag{2.82}$$

$$\psi(\tau, \hat{x}) = \frac{1}{\tau^2} \psi_\pm(\hat{x}) + O(\tau^{(-4)}) \tag{2.83}$$

2.4.2 Matching solutions at $\mathscr{I}^+_-$ and $\mathscr{I}^-_+$

We are interested to see how solutions at $\mathscr{I}^+_-$ match with the solutions at $\mathscr{I}^-_+$. For this we will use our knowledge of field equations at spatial infinity. The limits $\tau \to \pm\infty$ at spatial infinity correspond to $\mathscr{I}^+_-$ and $\mathscr{I}^-_+$ respectively. So by matching the solutions in these two limits and doing appropriate coordinate transformation to go to null coordinates, we can see how the solutions evolve.

(ρ,τ) are related to (r,t) as given in (2.5) which gives us

$$r=\rho\sqrt{1+\tau^2} \quad u=\rho(\tau-\sqrt{1+\tau^2}) \tag{2.84}$$

We can relate the field strengths F_{ur} and $F_{\rho\tau}$ at $\mathscr{I}^{\pm}_{\mp}$ by doing just coordinate transformations

$$F_{\alpha\beta}(\rho,\tau,\hat{x})=F_{\mu\nu}\frac{\partial x^{\mu}}{\partial x^{\alpha}}\frac{\partial^{\nu}}{\partial\beta} \tag{2.85}$$

Using this we get that in the limit $\tau\to\infty$

$$F_{ur}=\frac{\tau}{\rho}F_{\rho\tau} \tag{2.86}$$

This combined with (2.30) and (2.74), we get

$$\lim_{\tau\to\infty}\tau^3[\lim_{\rho\to\infty}\rho F_{\rho\tau}(\rho,\tau,\hat{x})]=\lim_{u\to-\infty}[\lim_{r\to\infty}r^2F_{ru}(u,r,\hat{x})] \tag{2.87}$$

$$\implies F_{\rho\tau}^{(1,3)}=F_{ru}^{(2,0)} \tag{2.88}$$

Similarly we also get

$$\lim_{\tau\to-\infty}\tau^3[\lim_{\rho\to\infty}\rho F_{\rho\tau}(\rho,\tau,\hat{x})]=\lim_{v\to\infty}[\lim_{r\to\infty}r^2F_{ru}(u,r,\hat{x})] \tag{2.89}$$

$$\implies F_{\rho\tau}^{(1,3)}=F_{rv}^{(2,0)} \tag{2.90}$$

From (2.83) and the fact that $F_{\alpha\rho}^{(-1)}=D_\alpha\psi$ we can write

$$\psi(\tau,\hat{x})\xrightarrow{\tau\to\pm\infty}k_\pm+\frac{1}{\tau^2}\psi_\pm(\hat{x})+\cdots \tag{2.91}$$

where $k_\pm$ are just constants. Now

$$F_{ru}^{(2)}(u=-\infty,\hat{x})=\lim_{\tau\to\infty}\tau^3F_{\rho\tau}^{(1)} \tag{2.92}$$

$$=\lim_{\tau\to\infty}\tau^3\partial_\tau\psi$$

$$=2\psi_+ \tag{2.93}$$

Similarly we get

$$F_{rv}(v=+\infty,\hat{x})=-2\psi_-(\hat{x}) \tag{2.94}$$

So now our problem reduces to seeing how $\psi_-(\hat{x})$ evolves to $\psi_+(\hat{x})$. It turns out that

$$\psi_+(\hat{x})=-\psi_-(\hat{-x}) \tag{2.95}$$

The calculation is given in B. This implies

$$F_{ur}^{(2)}(u=-\infty,\hat{x}) = F_{vr}^{(2)}(v=\infty,-\hat{x}) \tag{2.96}$$

Thus we have shown that at leading order the fields at $\mathscr{I}^+_-$ and $\mathscr{I}^-_+$ match.
A similar analysis for the gauge parameter $\lambda(\hat{x})$ gives us that

$$\lambda_+(\hat{x}) = \lambda_-(-\hat{x}) \tag{2.97}$$

We can then define the following charges

$$\begin{aligned} Q^+_\lambda &\equiv \int_{S^2} d^2x\lambda_+(\hat{x})F_{ru}^{(2)}(u=-\infty,\hat{x}) \\ &= \int dud^2x\lambda_+(\hat{x})\partial_u F_{ru}^{(2)}(u,\hat{x}) \\ &= \int dud^2x\lambda_+(\hat{x})\left(D^I F_{Iu}^{(0)} - J_u^{(2)}\right) \end{aligned} \tag{2.98}$$

where in the last line we have used the EOM $\nabla_\mu F^{\mu r} = J^r$

$$\begin{aligned} \partial_u F_{ur} &= \frac{1}{r^2}D^I(F_{Ir} - F_{Iu}) + J^r \\ \implies \partial_u F_{ru}^{(2)} &= D^I F_{Iu}^{(0)} - J_u^{(2)} \end{aligned} \tag{2.99}$$

and we have used the current falloffs in (2.20) and (2.22). We can analogously define Q^-_λ. From the matching conditions (2.96) and (2.97), we get

$$Q^+_\lambda = Q^-_\lambda \tag{2.100}$$

Note that these are infinitely many conserved charges each corresponding to a function $\lambda(\hat{x})$ on the 2-sphere.

2.5 Charges associated with Large Gauge Transformations (LGTs)

We expect infinitely many symmetries associated with the infinitely many conserved charges in (2.100). In this section we will show that these conserved charges are the same charges that generate LGT thus showing that LGT charges are conserved. We had found the symplectic form for ED with massless matter in (2.54). We can find the charge that generates

LGT from the symplectic form

$$
\begin{aligned}
Q_\lambda &= \Omega(\delta, \delta_\lambda) \\
&= \int_{\mathscr{I}^+} dud^2x \big(\delta \partial_u \hat{A}_I^{(0)} \wedge \delta'_\lambda \hat{A}^{I(0)} - \delta D_I F_{uI}^{(0)} \wedge \delta'_\lambda \Phi\big) + \int_{\mathscr{I}^+} dud^2x \delta \partial_u \phi \wedge \delta'_\lambda \phi \\
&= \int_{\mathscr{I}^+} dud^2x \lambda(\hat{x}) \big(D^I F_{Iu}^{(0)} - J_u^{(2)}\big)
\end{aligned}
\tag{2.101}
$$

where we have used (2.34) and (2.10). We see that it matches exactly with (2.98).

Now using the symplectic form (2.52) and the form of the charge above, we can get useful commutators

$$\{Q_\lambda^+, \Phi(\hat{x})\} = \lambda(\hat{x}) \tag{2.102}$$

$$\{Q_\lambda^+, \hat{A}(u,\hat{x})\} = 0 \tag{2.103}$$

$$\{Q_\lambda^+, N(\hat{x})\} = 0 \tag{2.104}$$

From these we can calculate

$$\{Q_\lambda, A_I^{(0)}\} = D_I \lambda \tag{2.105}$$

Hence we have seen that these charges indeed generate LGTs and are the same as the conserved charges found from our spatial infinity approach. From the commutators above, we can see that even though A_I gets shifted non-trivially, the associated field F_{uI} remains the same. Since the physical field remains unchanged under these charges, we might think that these symmetries are trivial. But the fact that the vector field A_I changes also has a physical manifestation, in the form of what is also called the "electromagnetic memory effect" [Pas17]. Integrating (2.99) w.r.t u, and taking the variation, we can see that the field F_{ur} gets shifted under a LGT.

A note on conservation and symmetries: Since LGTs keep the action invariant, they are symmetries of the theory. Unlike small gauge transformations which have vanishing Noether charges, every generator of LGT has a non-zero charge associated with it. We however note that these charges are defined on the sphere at $\mathscr{I}^+_-$ or $\mathscr{I}^-_+$. In order to ensure that the charge is conserved in accordance with Noether's theorem we require a mapping of the large gauge parameter from $\mathscr{I}^+_-$ to $\mathscr{I}^-_+$. We note that unlike the charges associated to global symmetries, the charges associated to asymptotic symmetries are not defined on a space-like Cauchy slice at finite time as the large gauge parameters do not have an unambiguous extension in the interior of space-time. The conservation law that we thus aim to prove relates the charges (integrated over 2-sphere at infinity) at $u = -\infty$ and $v = \infty$.

Chapter 3

Asymptotic Symmetries of Gravity in $D = 4$ dimensions

We started the last chapter with discussing flat Minkwoski space and based our entire discussion of ED in this background. We are now interested in studying gravitational theories where the metric asymptotically approaches the flat metric at large distances. We basically want to study spacetimes that approach a notion of $\mathscr{I}^+$. We will work in the Bondi gauge with coordinates $(u, r, \hat{x})$, where the most general metric takes the form

$$ds^2 = e^{2\beta} M du^2 - 2e^{2\beta} du dr + g_{AB}(dx^A - U^A du)(dx^B - U^B du) \tag{3.1}$$

and the Bondi gauge condition is given by

$$g_{rr} = 0; \quad g_{rA} = 0; \quad \partial_r \det\left(\frac{g_{AB}}{r^2}\right) = 0 \tag{3.2}$$

Now to impose asymptotic flatness condition, we need to impose falloffs. These falloffs are set keeping two conditions in mind [HIW17]:

- the falloff conditions should not exclude any physical spacetimes
- total mass and radiated energy flux should be well defined

We set the following falloffs in any even dimension [KLPS17]:

$$g_{uu} = -1 + O(r^{-1}) \quad g_{ur} = -1 + O(r^{-2}) \quad g_{uA} = O(1) \quad g_{AB} = r^2 \gamma_{AB} + O(r) \tag{3.3}$$

where γ_{AB} is the d-sphere metric. The parameters in the metric fall off as

$$M = -1 + \sum_{n=1}^{\infty} \frac{M^{(n)}(u,\hat{x})}{r^n} \tag{3.4}$$

$$\beta = \sum_{n=2}^{\infty} \frac{\beta^{(n)}(u,\hat{x})}{r^n} \tag{3.5}$$

$$U_A = \sum_{n=0}^{\infty} \frac{U_A^{(n)}}{r^n} \tag{3.6}$$

$$g_{AB} = r^2\gamma_{AB} + \sum_{n=-1}^{\infty} \frac{C_{AB}^{(n)}(u,z)}{r^n} \tag{3.7}$$

The falloffs here are set keeping (3.3) in mind. For instance $\beta \sim O(r^{-2})$ because we want $g_{ur} = -1 + O(r^{-2})$. We introduce some notation here for convenience: We will be referring to the $O(r)$ component of g_{AB} as C_{AB} i.e. $C_{AB}^{(-1)} \equiv C_{AB}$. Similarly, $C_{AB}^{(0)} \equiv D_{AB}$.
Once we have the falloffs, we can solve the Einstein's equations near $\mathscr{I}^+$ to get constraints

$$G_{ab} \equiv R_{ab} - \frac{1}{2}Rg_{ab} = T_{ab} \tag{3.8}$$

Assuming there are no matter sources and noting the fact that we are in Bondi gauge where $g_{rr} =$ and $g_{rA} = 0$, we will get $R_{rr} = 0$ and $R_{rA} = 0$. Moreover, it can be shown that R, the Ricci tensor will not contribute to the EOM. Let us look at the contribution to R from our calculation for various terms. From the non-zero values of g^{ab}, we have,

$$R = 2g^{ur}R_{ur} + g^{rr}R_{rr} + 2g^{rA}R_{rA} + g^{AB}R_{AB}. \tag{3.9}$$

From the EOM, $R_{rr} = 0,\ R_{rA} = 0$. Therefore, we have a simplified equation for R,

$$R = 2g^{ur}R_{ur} + g^{AB}R_{AB}. \tag{3.10}$$

Thus the Einstein equations,

$$G_{ur} = 0, \qquad G_{AB} = 0 \tag{3.11}$$

would reduce to,

$$R_{ur} = 0, \qquad R_{AB} = 0. \tag{3.12}$$

This is simple to see. If we consider $G_{ur} = 0$ then we have,

$$G_{ur} = R_{ur} - \frac{1}{2}g_{ur}\left[2g^{ur}R_{ur} + g^{AB}R_{AB}\right] = 0 \implies R_{AB} = 0. \tag{3.13}$$

Therefore, using this result, $R_{AB} = 0$, we can see that,

$$G_{AB} = R_{AB} - \frac{1}{2} g_{AB} \left[g^{AB} R_{AB} + 2 g^{ur} R_{ur} \right] = 0 \implies R_{ur} = 0. \tag{3.14}$$

We also have the Bondi gauge condition which constrains the trace of the tensors in g_{AB}. We get from the condition:

$$C^A_A = 0 \quad D^A_A = \frac{1}{2} C^{CD} C_{CD} \quad C^{(1)A}_A = C^M_A D^A_M - \frac{1}{3} C^M_A C^N_M C^A_N \equiv C \cdot D - \frac{1}{3} C^3 \tag{3.15}$$

On solving the EOM in 4D, we can write the metric at leading order as (note that everything that we will say from here will be in 4D unless stated otherwise)

$$\begin{aligned} ds^2 = & -du^2 - 2dudr + 2r^2 \gamma_{AB} dx^A dx^B \\ & + \frac{2m_B}{r} du^2 + r C_{AB} dx^A dx^B + D^B C_{AB} du dx^A + \frac{1}{16 r^2} C_{AB} C^{AB} du dr + \cdots \end{aligned} \tag{3.16}$$

where D_A is the sphere derivative and all indices are raised and lowered with γ_{AB}. We have written $M^{(1)} = 2m_B$ and $\partial_u C_{AB} = N_{AB}$. We have also used the following EOMs

$$U^{(0)}_A = -\frac{1}{2} D^B C_{AB} \tag{3.17}$$

$$\beta^{(2)} = -\frac{1}{64} C_{AB} C^{AB} \tag{3.18}$$

$$\partial_u m_B = \frac{1}{8} (2 D^A D^B - N^{AB}) N_{AB} \tag{3.19}$$

Here the fields and tensors have the following physical interpretation:

- $m_B(u, \hat{x})$ is called the Bondi mass aspect. Integrating it over the sphere gives what is called the Bondi mass $M(u)$. The Bondi mass aspect measures the angular density of energy from a point on $\mathscr{I}^+$. At $\mathscr{I}^+_-$, the Bondi mass is the same as the ADM energy.

- C_{AB} is the shear mode. It is traceless as we saw above and symmetric. It contains information about gravitational radiation near $\mathscr{I}^+$. C_{AB} is analogous to $A^{(0)}_I$ from ED.

- N_{AB} is the news tensor. In ED we had $F^{(0)}_{uI}$ and $D_I N = \int d^2 x F^{(0)}_{uI}$. The news tensor is analogous to $F^{(0)}_{uI}$ and its square is proportional to the energy flux across $\mathscr{I}^+$.

If we assume that the news tensor $N_{AB}(u, \hat{x}) \sim \frac{1}{u^{1+\delta}}$ near $\mathscr{I}^+_\pm$, then we can determine $C_{AB}(u, \hat{x})$ upto an integration constant from the news tensor by just integrating it. Infact this falloff for N_{AB} was proven by Christodoulou and Klainerman to hold in a finite neighbourhood of flat space [CK93]. Next if we assume that there are no "magnetic charges" i.e. the magnetic part of Weyl tensor is zero, then we get the condition [Str14]

$$[D_A U_B - D_B U_A]_{\mathscr{I}^+_\pm} = 0 \tag{3.20}$$

$$\implies [D_A D^C C_{CB} - D_B D^C C_{AC}]_{\mathscr{I}^+_\pm} = 0 \tag{3.21}$$

This gives us

$$C_{AB}|_{\mathscr{I}^+_-} = -2D_A D_B \psi|_{\mathscr{I}^+_-} \tag{3.22}$$

So we can take $\psi|_{\mathscr{I}^+_-}$ to be the integration constant while trying to get C_{AB} from N_{AB}. In other regions on $\mathscr{I}^+$, $C_{AB} = -2D_A D_B \psi(u,\hat{x}) + \gamma_{AB} D^2 \psi(u,\hat{x})$. This condition is the same like in ED, where by demanding no magnetic charges we got (2.26). Once we have N_{AB} and C_{AB}, we can integrate (3.19) to get m_B upto integration constant $m_B(\hat{x})|_{\mathscr{I}^+_-}$. Hence we see that all data can be specified in terms of the intial data

$$\{N_{AB}(u,\hat{x}), \psi(\hat{x})|_{\mathscr{I}^+_-}, m_B(\hat{x})|_{\mathscr{I}^+_-}\} \tag{3.23}$$

which in other words is the Cauchy data. An analogous construction holds at $\mathscr{I}^-$ too and we define the Cauchy data there as

$$\{N_{AB}(v,\hat{x}), \psi(\hat{x})|_{\mathscr{I}^-_+}, m_B(\hat{x})|_{\mathscr{I}^-_+}\} \tag{3.24}$$

3.1 BMS group

In the ED case we had found LGTs by trying to preserve the gauge conditions and falloffs. Here also we are interested to find diffeomorphisms that preserve the Bondi gauge conditions and the falloffs. We would expect that since the spacetime is almost flat here, we should recover the Poincare group. But it turns out that we get an infinite dimensional group call the BMS group.

Let ξ be the generator of these diffeomorphisms. To preserve the Bondi gauge, we need to impose the following conditions:

$$\mathscr{L}_\xi g_{rr} = 0, \qquad \mathscr{L}_\xi g_{rA} = 0, \qquad \mathscr{L}_\xi \partial_r \det\left(\frac{g_{AB}}{r^2}\right) = 0 \tag{3.25}$$

and to preserve the falloffs we need,

$$\mathscr{L}_\xi g_{uu} = O(r^{-2}), \quad \mathscr{L}_\xi g_{ur} = O(r^{-2}), \quad \mathscr{L}_\xi g_{uA} = O(r^0), \quad \mathscr{L}_\xi g_{AB} = O(r) \tag{3.26}$$

From the above conditions we get that

$$\xi = f\partial_u + D^2 f\partial_r - \frac{1}{r}D^A f D_A + \cdots \tag{3.27}$$

where $f \equiv f(x)$ is a function on the 2-sphere and the $\cdots$ represent higher order terms. These transformations generated by ξ are parameterized by f and are called supertranslations. They can be understood as generalizations of the four translations in Minkowski spacetime given by $f = constant, Y_{1,\pm 1}, Y_{1,0}$. Here $f = constant$ generates u-translations and $f = Y_{l,m}$ for $l = 1$ generates spatial translations.

We can also calculate the action of supertranslation on other components of the metric by calculating the lie derivative of the appropriate component of the metric and then picking the right order. For instance to calculate $L_\xi C_{AB}$, we will need to extract the $O(r)$ component in g_{AB}. Doing this gives us

$$L_\xi C_{AB} = f\partial_u C_{AB} - 2D_A D_B f + \gamma_{AB} D^2 f \tag{3.28}$$

$$L_\xi m_B = f\partial_u m_B + \frac{1}{4}\left[N^{AB} D_A D_B f + 2D_A N^{AB} D_B f\right] \tag{3.29}$$

$$L_\xi N_{AB} = f\partial_u N_{AB} \tag{3.30}$$

From here we can see that if we were to start with Minkowski spacetime i.e. where $m_B = N_{AB} = C_{AB} = 0$ and supertranslate it, the resultant spacetime would still have zero Bondi mass and news tensor. This agrees with our understanding that a diffeomorphism cannot create mass or gravitational radiation. The shear mode C_{AB} as we can see will shift by $L_\xi C_{AB} = -2D_A D_B f + \gamma_{AB} D^2 f$. Since in a non-radiative configuration $N_{AB} = 0$, it means C_{AB} is u-independent and we have

$$C_{AB} = -2D_A D_B \psi(\hat{x}) + \gamma_{AB} D^2 \psi(\hat{x}) \tag{3.31}$$

This tells us that the field ψ is shifted as follows under a supertranslation

$$\delta_{\xi_f} \psi(\hat{x}) = f(\hat{x}) \tag{3.32}$$

So, $\psi(\hat{x})$ labels the degenerate vacuum states upto the first 4 spherical harmonics which label translations. By vacuum we mean non-radiative configurations.

As we mentioned above, time translations here are generated by $f = constant$. What we mean by time translation here is translation in u along $\mathscr{I}^+$. This certainly commutes with other supertranslations and so as a result the associated charges (generating these transformations) should also commute. But note that this is not a statement of the conservation of charge. When we say time translation here, we are really just moving along $\mathscr{I}^+$ and changing u. In order to establish charge conservation, we would have to show that charges on different Cauchy slices are equal. Or in other words, the generator that takes us from one Cauchy slice to the other commutes with supertranslation charges. But showing that the commutator is zero is difficult because LGT charges are unambiguously defined in the bulk. We could add a small gauge transformation to the generator in the bulk and end up getting the same large gauge transformation at the boundary. On the other hand, showing an explicit matching of the charges like in the ED case turns out to be easier. Analogous to the ED construction, one analyses the Weyl tensor components at spatial infinity to get the conservation laws [Pra19].

3.2 Phase space analysis and supertranslation charge

We now move on to finding the symplectic form. The symplectic current is given by

$$J^\alpha = \delta\Gamma^\alpha_{\mu\nu}\delta g^{\mu\nu} - \delta\Gamma^\nu_{\mu\nu}\delta g^{\alpha\mu} \tag{3.33}$$

Like before, we will calculate the symplectic form on a constant-t slice and take $t \to \infty$ while keeping u fixed. We have $J^t = J^r + J^u$. We get the symplectic form at leading order as

$$\begin{aligned}\Omega_{\mathscr{I}^+} &= \int dud^2x\sqrt{\gamma}\big[\delta C_{AB} \wedge \delta' N^{AB}\big] \\ &= \int dud^2x\sqrt{\gamma}\big[-2\delta D_A D_B \psi(\hat{x}) \wedge \delta' N^{AB} + \delta\hat{C}_{AB}(u,\hat{x}) \wedge \delta' N^{AB}\big]\end{aligned} \tag{3.34}$$

where $\sqrt{\gamma}$ is the determinant of the sphere metric. In the last line we have decomposed the shear mode into a "soft" mode (i.e. u independent) and hard mode that falls off as $1/u^\varepsilon$. Here C_{AB} is like the $A_I^{(0)}$ mode in ED

$$C_{AB}(u,\hat{x}) = -2D_A D_B \psi(\hat{x}) + \hat{C}_{AB}(u,\hat{x}) \tag{3.35}$$

We can also define a mode analogous to N in (2.55)

$$D_A D_B N = \int_{-\infty}^{\infty} du N_{AB} \tag{3.36}$$

giving us

$$\Omega_{\mathscr{I}^+}(\delta,\delta') = \int dud^2x\sqrt{\gamma}\delta\hat{C}_{AB}(u,\hat{x}) \wedge \delta' N^{AB} - 2\int dud^2x\sqrt{\gamma}\delta D_A D_B \psi(\hat{x}) \wedge \delta' D^A D^B N \tag{3.37}$$

We can find the supertranslation charge from the symplectic form. We use (3.28) and the fact that C_{AB} is traceless to get

$$\begin{aligned} Q_\xi &= \Omega_{\mathscr{I}^+}(\delta_\xi,\delta) && (3.38)\\ &= \int dud^2x\sqrt{\gamma}f\left(N_{AB}N^{AB} - 2D_A D_B N^{AB}\right) \\ &= k\int d^2x\sqrt{\gamma}fm_B && (3.39)\end{aligned}$$

where k is some constant that can be fixed by (3.19). Note that in all our analysis we have used $8\pi G = 1$. It was shown in [Pra19] that these charges are conserved. The analysis is similar to the one in ED where we study the EOMs at spatial infinity to prove the conservation law. In [HLMS15], it was shown that this conservation law is related to the soft graviton theorem. Not only that, supertranslations in four dimensions are also related to the memory effect [SZ14]. The memory effect can be understood as follows: say we have two inertial observers near $\mathscr{I}^+$. They are localized in a region where there is no gravitational radiation initially. Then say some radiation is turned on at $u = u_i$ and stopped at $u = u_f$. It can then be shown that these inertial observers are permanently shifted from each other because of the null radiation. It turns out that the shifted metric after the radiation epoch can be related with the metric before it through a supertranslation.

Chapter 4

BMS in higher dimensions

We would like to now extend our analysis of asymptotically flat spacetimes to higher dimensions. One would think that this is just a direct extension of our results in 4 dimensions. But it turns out that the extension is in fact not so straightforward both due to physical and technical reasons that we will list out in this chapter. To start with, one is not even sure if supertranslations or the BMS group exists in higher dimensions. In all our discussions here, we will work with even dimensions. In the case of odd dimensions, there are fractional r-falloffs near $\mathscr{I}^{\pm}$ and we leave that analysis for later.

4.1 BMS-to exist or not to exist?

If we start with the metric in Bondi gauge (3.1) and the falloffs (3.4)-(3.7), we see that even in higher even dimensions, the mode affected non-trivially by supertranslations is C_{AB} i.e. supertranslations cause a change at $O(r^{-1})$ of the metric in all dimensions. On the other hand, the radiative data, in general $d+2$ dimensions is stored at $O(r^{-d/2})$ [LS20]. For d=2, these two match. So it is the radiative mode that is shifted non-trivially by supertranslations in 4 dimensions. This means that if we eliminate supertranslations by fixing C_{AB}, we also end up removing radiative solutions. So in 4 dimensions, supertranslations or the BMS group is inevitable. But this is not the case in higher dimensions. We can set $C_{AB} = 0$ to eliminate supertranslations without affecting any radiative solutions.

In [HIW17], it was argued unlike in four dimensions, existence of supertranslations is not mandated by gravitational radiation in higher dimensions. More in detail, it was shown in [HIW17] that there is no memory effect in higher dimensions at leading order in $\frac{1}{r}$. Since radiative data is stored at $O(r^{-d/2})$, during the radiation epoch, the radiative metric components appear at this order. And as shown in [HIW17], the difference in the metric components before and after the radiation is at $O(r^{-d+1})$. For $d=2$, these two are equal. But in higher dimensions, $d/2 < d-1$. So the change in the metric components due to the radiation epoch is not detected at leading order by observers at $\mathscr{I}^+$. Hence there is no memory effect at leading order in higher dimensions.

For $d=2$, it can be shown that the change in the metric components is related to supertranslations such that the metric before and after the radiation epoch are related through a supertranslation [SZ14]. So the presence of supertranslations is necessary if we want to hold the metrics before and after the radiation epoch at the same footing. But in higher dimensions, since there is no memory effect, existence of supertranslations is a necessary. We can just do away with supertranslations by setting appropriate boundary conditions and setting $C_{AB}=0$. We can choose to thus only work with the Poincare group in higher dimensions. Furthermore, even if we were to have supertranslations, the authors of [HIW17] claimed that the associated charge generating them would diverge.

But, if we look at it from the perspective of classical soft theorem, we expect supertranslations to exist. Soft theorems are dimension independent results. So if they are related to a conservation law of asymptotic symmetries in four dimensions, as shown in [HLMS15], it is natural to ask if a similar relation holds in higher dimensions too. Or in other words we want to ask if there are asymptotic symmetries in higher dimensions whose conservation laws are equivalent to the soft theorem. Moreover, there has been no explicit calculation done that completely rules out the possibility of having supertranslations with carefully chosen counter terms at the boundary to regularize the divergences. In our work, we attempted to do this analysis in full non-linear General Relativity and tried to analyse the role of relaxed boundary conditions that allow for supertranslations.

We start with a linearized analysis of the phase space of asymptotically flat spacetimes in higher even dimensions. What we mean by that is that we will only keep terms that contribute to terms linear in metric fluctuations in the final computation of the supertranslation charge. After stating major results here, we move on to a complete non-linear analysis.

4.2 Linearized Gravity

We will mostly follow the analysis done in [Agg19] and state major results. The metric (3.1) gets simplified in the linearized limit and we have

$$ds^2 = Mdu^2 - 2dudr + g_{AB}dx^B dx^B - 2U_A dx^A du \tag{4.1}$$

and its inverse

$$g^{ab} = \begin{pmatrix} 0 & -1 & 0 \\ -1 & -M & -U^B \\ 0 & -U^A & g^{AB} \end{pmatrix} \tag{4.2}$$

We assume the same falloffs stated in (3.3). M, U_A and g_{AB} have the falloffs stated in (3.4)-(3.7).

The Bondi gauge condition in the linearized limit gives the constraint that all $C^{(n)}_{AB}$ are traceless. We will state results for six dimensions here but they can be generalized to arbitrary even dimensions as done in [Agg19]. In six dimensions, radiative data is carried by D_{AB}. As we will see from the EOMs, all metric components can be entirely specfied in terms of D_{AB} and C_{AB}. In the linearized limit, the EOMs give the following constraints

$$R_{ur} = 0 \quad \Longrightarrow \quad M^{(1)} = 0; \quad M^{(2)} = -\frac{1}{2} D \cdot U^{(1)} \tag{4.3}$$

$$R_{rA} = 0 \quad \Longrightarrow \quad U^{(0)}_A = -\frac{1}{6} D^B C_{AB}; \quad U^{(1)}_A = -\frac{1}{3} D^B D_{AB} \tag{4.4}$$

$$R_{AB} = 0 \quad \Longrightarrow \quad \partial_u C_{AB} = 0 \tag{4.5}$$

4.2.1 Phase space analysis

As usual, we will calculate the symplectic form on a constant-t slice and then take $t \to \infty$ while keeping u fixed. The symplectic current $J^t = J^u + J^r$ can be calculated using (3.33). We can split this into a finite and divergent piece

$$J^t = J^t_{div} + J^t_{fin} \tag{4.6}$$

After using (3.33) and the EOM (4.5), we get

$$\left(\frac{2}{\sqrt{g}}\right) J^t_{div} = \frac{1}{r^3}\left[\delta C_{AB} \wedge \delta' \left(D_{(A} U^{(0)}_{B)} + \frac{1}{2}\partial_u D_{AB}\right)\right] \tag{4.7}$$

This can be further simplified by using (4.4) and noting that the first term becomes a total derivative. And so the only term that contributes is

$$\begin{aligned}\left(\frac{2}{\sqrt{g}}\right) J^t_{div} &= \frac{r}{4}\delta\partial_u D_{AB}\wedge\delta' C^{AB} \\ &= \frac{(t-u)\sqrt{\gamma}}{4}\delta\partial_u D_{AB}\wedge\delta' C^{AB} \\ &= \frac{\sqrt{\gamma}}{4}\delta D_{AB}\wedge\delta' C^{AB} + \partial_u\big(\frac{(t-u)\sqrt{\gamma}}{4}\delta D_{AB}\wedge\delta' C^{AB}\big)\end{aligned} \tag{4.8}$$

Similarly,

$$\begin{aligned}\left(\frac{2}{\sqrt{g}}\right) J^t_{fin} = \frac{1}{r^4}\Big[&\delta U^{(1)A}\wedge\delta' U^{(0)}_A + \delta C_{AB}\wedge\delta' D_A U^{(1)}_B - \delta D_A U^{(0)}_B\wedge\delta' D^{AB} \\ &+\frac{1}{2}(\delta\partial_u C^{(1)}_{AB}\wedge\delta' C^{AB} + \delta\partial_u D^{AB}\wedge\delta' D_{AB})\Big]\end{aligned} \tag{4.9}$$

4.2.2 Supertranslation Charge

We can find the vector field that generates supertranslations like in the four dimensional case. We get a form similar to (3.27)

$$\xi = f\partial_u + \frac{1}{4}D^2 f\partial_r - \frac{1}{r}\gamma^{AB}D_A f D_B + \cdots \tag{4.10}$$

We can see that even in higher dimensions, supertranslations are entirely parameterized by $f(\hat{x})$. We derive this in D for the full non-linear metric. For our purpose, we need to calculate $\delta_\xi C_{AB}$ and $\delta_\xi D_{AB}$ to compute the supertranslation charge from the symplectic form. We have

$$\delta_f C_{AB} = \frac{1}{2}\gamma_{AB}D^2 f - (D_A D_B + D_B D_A)f \tag{4.11}$$

The expression for $\delta_f D_{AB}$ is slightly more complicated. But the good thing is that it does not contribute to the charge in the linearized limit. After substituting the variation above in the symplectic form we get that

$$\Omega_{div}(\delta,\delta_\xi) = \lim_{t\to\infty}\left[-\int_{\mathscr{I}^+} dud^4x\frac{\sqrt{\gamma}}{2}\delta D^{AB}D_A D_B f - \int_{\mathscr{I}^+} dud^4x\partial_u\big(\frac{(t-u)\sqrt{\gamma}}{4}\delta D_{AB}\wedge\delta' C^{AB}\big)\right] \tag{4.12}$$

This diverges as $t\to\infty$. But if we assume that

$$D_A D_B D^{AB}(u=-\infty,\hat{x}) = O(|u|^{-1-\varepsilon}) \tag{4.13}$$

we can avoid the divergence. Note that satisfying this condition under supertranslations requires a redefinition of D_{AB}. We will discuss this in more detail during the non-linear analysis. On imposing the above condition, we get a finite contribution to the charge from the divergent piece of the symplectic form

$$Q_\xi^D = -\int du d^4x f D_A D_B D^{AB} \tag{4.14}$$

where the superscript D denotes that this is the contribution from the divergent piece. Similarly the finite piece gives on substituting the EOM (4.4)

$$Q_\xi^F = \int_{\mathscr{I}^+} \frac{\sqrt{\gamma}}{3} f\left(\frac{D^2}{4}+1\right) D^A D^B D_{AB} \tag{4.15}$$

Hence the total soft charge (i.e. without matter) is

$$Q_\xi = \int_{\mathscr{I}^+} \frac{\sqrt{\gamma}}{12} f(D^2-2) D^A D^B D_{AB} \tag{4.16}$$

As is evident, at least in the linear theory, there are no divergences and one can define a finite supertranslation charge. This charge matches with what was derived in [KLPS17] from writing the soft theorem as a Ward identity and identifying the conserved charge. This presents a confusion. The derivation of the supertranslation charge as seen above, requires $C_{AB} \neq 0$. But the derivation of the soft theorem does not require C_{AB} at all. So it is confusing what role C_{AB} plays in higher dimensions. Since non-linearities contribute even at higher order terms, we probably shouldn't ignore them. And it is possible that a complete non-linear analysis will provide insight into the role C_{AB} plays.

4.3 Non-linear analysis

We explicitly work out everything in the non-linear theory. We will work in six dimensions but all the results can be generalized to higher dimensions too. We will work with the full non-linear metric (3.1),

$$g_{\mu\nu} = \begin{pmatrix} Me^{2\beta}+U^2 & -e^{2\beta} & -U_A \\ -e^{2\beta} & 0 & 0 \\ -U_B & 0 & g_{AB} \end{pmatrix}, \qquad g^{\mu\nu} = \begin{pmatrix} 0 & -e^{-2\beta} & 0 \\ -e^{-2\beta} & -e^{-2\beta}M & -e^{-2\beta}U^A \\ 0 & -e^{-2\beta}U^B & g^{AB} \end{pmatrix} \tag{4.17}$$

The inverse g^{AB} can be calculated by using $g^{AC}g_{BC} = \delta^A_B$. This gives us upto $O(r^{-5})$

$$g^{AB} = \frac{\gamma^{AB}}{r^2} - \frac{C^{AB}}{r^3} + \frac{C^A_C C^{BC} - D^{AB}}{r^4} + \frac{C^{AC}D^B_C + D^{AC}C^B_C - C^A_M C^{MN} C^B_N - C^{(1)AB}}{r^5}. \quad (4.18)$$

Here U_A is raised with g^{AB} but $U_A^{(n)}$ is raised with γ^{AB}. In the non-linear theory, the Bondi gauge condition implies that

$$C^A_A = 0\,, \quad D^A_A = \frac{1}{2}C^{AB}C_{AB}, \quad C^{(1)A}_A = C\cdot D - \frac{1}{3}C^3. \quad (4.19)$$

4.3.1 Equations of Motion

The EOM in the non-linear analysis give the following constraints. We list out all the Christoffel symbols used in the calculation in C.

$\mathbf{R_{rr}}$

We can use the definition of R_{rr}

$$R_{rr} = -\partial_r \Gamma^A_{rA} + \Gamma^A_{rA}\Gamma^r_{rr} - \Gamma^A_{Br}\Gamma^B_{Ar}. \quad (4.20)$$

to compute it to R_{rr} to subleading order.

$$\beta^{(2)} = -\frac{1}{64}C^{AB}C_{AB} \quad (4.21)$$

And,

$$\beta^{(3)} = \frac{1}{48}(C^3 - 2C\cdot D) \quad (4.22)$$

$\mathbf{R_{rA}}$

We start with

$$\begin{aligned} R_{rA} = {} & \frac{1}{2}\partial_r\left(g^{ru}\partial_r g_{uA} + g^{rB}\partial_r g_{AB} - g^{ur}\partial_A g_{ur}\right) \\ & + \frac{1}{2}\left[\partial_B(g^{BC}\partial_r g_{AC}) - \partial_r(g^{BC}\partial_A g_{BC})\right] + \frac{1}{2}g^{CD}\partial_r g_{CD}\Gamma^r_{rA} \\ & + \frac{1}{2}\left(g^{CD}\partial_B g_{CD} - g^{ur}\partial_r g_{uB} - g^{rC}\partial_r g_{BC} + g^{ur}\partial_B g_{ur}\right)\Gamma^B_{rA} - \Gamma^B_{ur}\Gamma^u_{AB} - \Gamma^C_{Br}\Gamma^B_{AC} \end{aligned} \quad (4.23)$$

From here we get,

$$U_A^{(0)} = -\frac{1}{6}D_B C^B_A \quad (4.24)$$

And at $O(1/r^3)$,

$$3U_A^{(1)} = -D_B D_A^B + C_{AB} U^{(0)B} + \frac{1}{2} D_B (C^{BM} C_{AM}) + 6 D_A \beta^{(2)} + \frac{1}{8} D_A (C^{BC} C_{BC}) \tag{4.25}$$

$\mathbf{R_{ur}}$

This is given as,

$$R_{ur} = \left(\partial_r \Gamma^r_{ur} + \partial_A \Gamma^A_{ur}\right) - \left(\partial_u \Gamma^r_{rr} + \partial_u \Gamma^A_{Ar}\right) + \left(\Gamma^A_{rA} \Gamma^r_{ur} + \Gamma^B_{AB} \Gamma^A_{ur}\right) - \left(\Gamma^A_{uB} \Gamma^B_{rA}\right). \tag{4.26}$$

From here we get,

$$M^{(1)} = 0. \tag{4.27}$$

And,

$$-M^{(2)} = \frac{1}{2} D^A U_A^{(1)} + D^2 \beta^{(2)} - 2\beta^{(2)} + U^{(0)2} \, . \tag{4.28}$$

$\mathbf{R_{AB}}$

The expansion for this is given as,

$$R_{AB} = \partial_a \Gamma^a_{AB} - \partial_A \Gamma^a_{aB} + \Gamma^a_{ab} \Gamma^b_{AB} - \Gamma^a_{Ab} \Gamma^b_{aB}. \tag{4.29}$$

From here we get

$$\partial_u C_{AB} = 0 \tag{4.30}$$

And,

$$\begin{aligned} \partial_u C_{AB}^{(1)} = & -M^{(2)} + 2\beta^{(2)} - 2 D_A D_B \beta^{(2)} - U^{(0)C} D_B C_{AC} \\ & + D_B U_A^{(1)} + \frac{1}{2} C_{AB} D_C U^{(0)C} + D_C \Big[\gamma_{AB} (U^{(1)C} - C^{CD} U_D^{(0)}) \\ & - \frac{1}{2} C^{CD} (2 D_B C_{AD} - D_D C_{BA}) + \frac{1}{2} (2 D_B D_A^C - D^C D_{AB}) \Big] \\ & - 5 U_A^{(0)} U_B^{(0)} - D_{(A} C_{C)}^D D_{(B} C_{D)}^C + D^D C_{AC} D_{(B} C_{D)}^C - \frac{1}{4} D^D C_{AC} D^C C_{BD} \\ & - \frac{1}{2} C_B^C D_A U_C^{(0)} + \frac{3}{2} C_B^C D_C U_A^{(0)} + C_B^M \partial_u D_{AM} + 2 D_{AB} - C_{AM} C_B^M \end{aligned} \tag{4.31}$$

Demanding that the magnetic part of the Weyl tensor be zero gives us the same constraint like in four dimensions (3.21). Since $\partial_u C_{AB} = 0$, we can define C_{AB} as

$$C_{AB} = -2 D_A D_B \psi(\hat{x}) + \frac{1}{2} \gamma_{AB} D^2 \psi \tag{4.32}$$

We can see from the EOMs above that all quantities can be defined in terms of D_{AB} and C_{AB}. Since C_{AB} is also described in terms of $\psi(\hat{x})$, we see that D_{AB} and $\psi(\hat{x})$ determine all the quantities upto integration constants.

4.3.2 Supertranslations

Transformations preserving the Bondi gauge and falloffs are generated by

$$\xi = f\partial_u + \big[\frac{1}{4}D^2 f - \frac{1}{8r}(\frac{4}{3}D_A C^{AB} D_B f - C^{AB} D_A D_B f)\big]\partial_r \tag{4.33}$$

$$+ \big(-\frac{1}{r}D^A f + \frac{1}{2r^2}C^{AB} D_B f\big) D_A \tag{4.34}$$

This is derived in D. The action of supertranslations on the metric components is also calculated there. At $O(r)$, we get

$$\delta_f C_{AB} = -2\left(D_A D_B f - \frac{1}{4}\gamma_{AB} D^2 f\right) \tag{4.35}$$

From (4.32), we then see that

$$\delta_f \psi = f \tag{4.36}$$

At $O(r^0)$, we get

$$\begin{aligned}\delta_f D_{AB} = f\partial_u D_{AB} + \frac{1}{4}\gamma_{AB}\left[-\frac{4}{3}D_C C^{CD} D_D f - C^{CD} D_C D_D f\right] \\ + \frac{1}{4}C_{AB} D^2 f - D_C C_{AB} D^C f - \frac{1}{2}\big(C_{BC} D_A D^C f + C_{AC} D_B D^C f\big) \\ + \frac{1}{2}\big[D_A C_{BC} D^C f + D_B C_{AC} D^C f\big] \\ + \frac{1}{6}\big[D^C C_{BC} D_A f + D^C C_{AC} D_B f\big]\end{aligned} \tag{4.37}$$

During the linear analysis in six dimensions, we had imposed the condition that the radiative data D_{AB} should have a certain falloff in u near $\mathscr{I}^+_\pm$ for the symplectic form to be finite (4.13). But from the form of $\delta_f D_{AB}$ above, it is clear that such a falloff is violated not just for supertranslations but even simple translations. The C_{AB} terms in $\delta_f D_{AB}$ go as $O(u^0)$ thus violating the $\frac{1}{|u|}$ falloff of D_{AB}. This requires us to redefine D_{AB}. We define $\hat{D}_{AB}$ such that $\delta_f \hat{D}_{AB} = f\partial_u \hat{D}_{AB}$

$$\hat{D}_{AB} = D_{AB} - \frac{1}{4}C_{AC} C^C_B \tag{4.38}$$

Since the trace of this renormalized tensor $\hat{D}_{AB}$ can be expressed in terms of C_{AB},

$$\text{Tr}\big(\hat{D}_{AB}\big) = \frac{1}{4}C^2 \tag{4.39}$$

it is the trace free part which is the free data,

$$\begin{aligned} \hat{D}^{tf}_{AB} &= \hat{D}_{AB} - \frac{1}{16}\gamma_{AB}C^2 \\ &= D_{AB} - \frac{1}{4}C_{AC}C^C_B - \frac{1}{16}\gamma_{AB}C^2 \end{aligned} \tag{4.40}$$

And we demand that $\hat{D}^{tf}_{AB} \sim \frac{1}{|u|}$.

4.3.3 Phase Space analysis

Next we find the symplectic form using the full non-linear metric (3.1). All the relevant Christoffel symbols used are given in C and the EOMs in 4.3.1. It was claimed in [HIW17] that given the falloffs in higher dimensions, the supertranslation charge would diverge. In the linear analysis we saw that even though the symplectic form was diverging, the supertranslation charge found from it was finite. We want to see if this still holds true in the full non-linear analysis. We will calculate like always $J^t = J^u + J^r$.

$$\begin{aligned} J^u &= \frac{\sqrt{g}}{2}\left\{\delta g^{ru}\delta\left[g^{ur}\partial_r g_{ru}\right] - \frac{1}{2}\delta g^{AB}\delta\left[g^{ur}\partial_r g_{AB}\right] - \delta g^{ur}\delta\left[g^{ur}\partial_r g_{ur}\right] - \frac{1}{2}\delta g^{ur}\delta\left[g^{AB}\partial_r g_{AB}\right]\right\} \\ &= -\frac{\sqrt{g}}{4}\left\{\delta g^{AB}\delta\left[g^{ur}\partial_r g_{AB}\right] + \delta g^{ur}\delta\left[g^{AB}\partial_r g_{AB}\right]\right\} \end{aligned} \tag{4.41}$$

This comes out as,

$$\begin{aligned} \left(\frac{2}{\sqrt{g}}\right)J^u &= -\frac{1}{2r^4}\delta D^{AB}\wedge\delta' C_{AB} \\ &= -\frac{1}{2r^4}\left(\delta\hat{D}^{tf}_{AB}\wedge\delta' C_{AB} + \frac{1}{4}\delta C^{AC}C^B_C\wedge\delta' C_{AB}\right) \end{aligned} \tag{4.42}$$

And,

$$J^r = \frac{\sqrt{g}}{2}\Bigg\{\delta g^{ur}\wedge\delta\Big[-e^{-2\beta}\big(\partial_r(Me^{2\beta})+U^A\partial_r U_A+\partial_u e^{2\beta}\big)-2U^A\partial_A\beta-\frac{1}{2}g^{AB}\partial_u g_{AB}\Big] \tag{4.43}$$

$$-\frac{1}{2}\delta g^{rr}\wedge\delta(g^{AB}\partial_r g_{AB})$$

$$+\delta g^{rA}\wedge\delta\Big[e^{-2\beta}\partial_r U_A - e^{-2\beta}U^B\partial_r g_{AB}-\frac{1}{2}g^{BC}\partial_A g_{BC}\Big]$$

$$+\frac{1}{2}\delta g^{AB}\wedge\delta\Big[g^{ru}\big(\partial_A g_{uB}+\partial_B g_{uA}-\partial_u g_{AB}\big)-g^{rr}(\partial_r g_{AB})+g^{rC}\big(\partial_A g_{BC}+\partial_B g_{AC}-\partial_C g_{AB}\big)\Big]\Bigg\}$$

This has a finite piece and a divergent piece,

$$\begin{aligned}\left(\frac{2}{\sqrt{g}}\right)J^r_{div} &= -\frac{1}{2r^3}\delta C^{AB}\wedge\delta'(\partial_u D_{AB}) \\ &= -\frac{1}{2r^3}\delta C^{AB}\wedge\delta'(\partial_u \hat{D}^{tf}_{AB})\end{aligned} \tag{4.44}$$

This is the same divergent term we had in the linear case as well. On working with renormalised D_{AB} i.e. $\hat{D}^{tf}_{AB}$, this term vanishes when integrated over whole $\mathscr{I}^+$ due to fall-off conditions satisfied by $\hat{D}^{tf}_{AB}$. The finite part of J^r is given by,

$$\begin{aligned}\left(\frac{2}{\sqrt{g}}\right)J^r_{finite} &= \frac{1}{r^4}\delta U^{(0)A}\wedge\delta'\big[U^{(1)}_A + C^B_A U^{(0)}_B\big] \\ &\quad -\frac{1}{2r^4}\Bigg\{\big[\delta D^{AB}-\delta(C^A_C C^{BC})\big]\wedge\delta'\big[2D_A U^{(0)}_B-\partial_u D_{AB}-C_{AB}\big] \\ &\qquad +\delta C^{AB}\wedge\delta'\Big[2D_A U^{(1)}_B-\partial_u C^{(1)}_{AB}-U^{(0)C}\big(D_A C_{BC}+D_B C_{AC}-D_C C_{AB}\big)\Big]\Bigg\} \\ &= \frac{1}{r^4}\delta U^{(0)A}\wedge\delta'\big[U^{(1)}_A + C^B_A U^{(0)}_B\big] \\ &\quad -\frac{1}{2r^4}\Big\{\delta\big[\hat{D}^{ABtf}-\frac{3}{4}(C^A_C C^{BC})+\frac{1}{16}\gamma^{AB}C^2\big]\wedge\delta'\Big[2D_A U^{(0)}_B-\partial_u\hat{D}^{tf}_{AB}-C_{AB}\Big] \\ &\qquad +\delta C^{AB}\wedge\delta'\Big[2D_A U^{(1)}_B-\partial_u C^{(1)}_{AB}-U^{(0)C}\left(D_A C_{BC}+D_B C_{AC}-D_C C_{AB}\right)\Big]\Big\}\end{aligned} \tag{4.45}$$

where in the second line, we have rexpressed it in terms of $\hat{D}^{tf}_{AB}$. Hence, we get

$$\begin{aligned}
J^t = J^u + J^r = \frac{1}{r^4}\Big[&\delta U^{(0)A}\wedge\delta' U^{(1)}_A - \delta C^{AB}\wedge\delta\hat{D}_A U^{(1)}_B - \delta\hat{D}^{ABtf}\wedge\delta'\hat{D}_A U^{(0)}_B \\
&+\frac{1}{2}\delta C^{AB}\wedge\delta'\partial_u C^{(1)}_{AB} + \delta\hat{D}^{ABtf}\wedge\delta'\partial_u\hat{D}_{ABtf}\Big] \\
&+\frac{1}{r^4}\Big[\delta U^{(0)A}\wedge\delta' C^B_A U^{(0)}_B - \frac{1}{2}\delta(C^A_C C^{BC})\wedge\delta' C_{AB} \\
&-\delta C^{AB}\wedge\delta' U^{(0)C}(D_A C_{BC} + D_B C_{AC} - D_C C_{AB}) \\
&-\delta\big(\frac{3}{4}(C^A_C C^{BC}) + \frac{1}{16}\gamma^{AB}C^2\big)\wedge\delta' D_A U^{(0)}_B \\
&+\frac{1}{2}\delta\big(\frac{3}{4}(C^A_C C^{BC}) - \frac{1}{16}\gamma^{AB}C^2\big)\wedge\delta'\partial_u\hat{D}^{tf}_{AB}\Big]
\end{aligned} \tag{4.46}$$

where the last line does not contribute to the symplectic structure because $\hat{D}^{tf}_{AB}\sim\frac{1}{u}$. We can see that the finite (in r) part of symplectic structure has a set of terms which are independent of D_{AB} and hence diverge when integrated over u. As these terms are sphere integrals of local functionals, we assume that there is a counter-term that can be added to the action such that the effect on $\Omega(\delta,\delta')$ is to precisely cancel the divergent terms. Although investigation of such a boundary term is outside the scope of this thesis, we conjecture that there is a regularisation of the radiative symplectic structure which is finite. This symplectic structure then has the same form as in linearised gravity with the crucial difference being that D_{AB} is renormalised to $\hat{D}^{tf}_{AB}$. Thus in terms of such a renormalised D_{AB}, the super-translation charge takes the same form as in the linearised theory and the resulting conservation law is equivalent to classical soft graviton theorem.

Chapter 5

Conclusion and Future Prospects

In this project, we were interested in studying asymptotic symmetries in higher even dimensions. We drew our motivation from the fact that the universal soft graviton (and photon) theorem is related to the conservation of these symmetries in four dimensions. While looking for such connections in higher dimensions, we realized that even proving the existence of these symmetries in higher dimensions was a non-trivial task. We started with studying these symmetries in four dimensions. On the way, we also introduced several new tools such as the covariant phase space formalism to study the phase space of a given theory. Next, we moved on to studying these symmetries in higher dimensions. While we were able to show that such symmetries can exist in higher dimensions by arguing that the divergent pieces in the charge can be cancelled by adding appropriate terms to the action, there are still some issues that remain unresolved.

To start with, it is not clear what counter terms have to be added to the action to exactly cancel the divergences in the symplectic form and therefore the supertranslation charge. It is also not understood what physical significance the $O(r^{-1})$ mode in the metric, C_{AB} has. It is intriguing that irrespective of the dimension we work in, existence of super translations implies existence of C_{AB} as Super-translation acts on this mode in homogeneously. On the other hand, the radiative gravitational field in $D > 4$ dimensions does not require existence of such a mode. Thus if we are to understand the classical soft graviton theorem as a consequence of asymptotic conservation laws, we need to expand the radiative data to include C_{AB}, even though this mode is not explicitly present in soft radiation in a classical scattering.

Similarly, the issue of memory and its relation with asymptotic symmetries at $\mathscr{I}$ in higher dimensions is not completel understood. It is indeed possible that $C_{AB} \neq 0$ allows for memory (gravitational field at $\frac{1}{r^{D-3}}$ in higher dimensions. But to the best of understanding, this has not been shown before. A detailed analysis of this issue will help us in establishing the classical version of the so-called IR triangle in higher dimensions which equates (classical) soft graviton theorem, super translation conservation law and the memory effect on equal footing [PRS18].

We hope to address these issues in the near future.

Bibliography

[Agg19] Ankit Aggarwal. Supertranslations in higher dimensions revisited. *Physical Review D*, 99(2), Jan 2019.

[CE17] Miguel Campiglia and Rodrigo Eyheralde. Asymptotic u(1) charges at spatial infinity. *Journal of High Energy Physics*, 2017(11), Nov 2017.

[CF19] Geoffrey Compère and Adrien Fiorucci. Advanced lectures on general relativity, 2019.

[CK93] Demetrios Christodoulou and Sergiu Klainerman. *The Global Nonlinear Stability of the Minkowski Space (PMS-41)*. Princeton University Press, 1993.

[CL19] Miguel Campiglia and Alok Laddha. Asymptotic charges in massless qed revisited: a view from spatial infinity. *Journal of High Energy Physics*, 2019(5), May 2019.

[HI87] S. W. Hawking and W. Israel, editors. *THREE HUNDRED YEARS OF GRAVITATION*. 1987.

[HIW17] Stefan Hollands, Akihiro Ishibashi, and Robert M Wald. Bms supertranslations and memory in four and higher dimensions. *Classical and Quantum Gravity*, 34(15):155005, Jul 2017.

[HLMS15] Temple He, Vyacheslav Lysov, Prahar Mitra, and Andrew Strominger. Bms supertranslations and weinberg's soft graviton theorem. *Journal of High Energy Physics*, 2015(5), May 2015.

[HM19] Temple He and Prahar Mitra. Asymptotic symmetries and weinberg's soft photon theorem in minkd+2. *Journal of High Energy Physics*, 2019(10), Oct 2019.

[HM21] Temple He and Prahar Mitra. Covariant phase space and soft factorization in non-abelian gauge theories. *Journal of High Energy Physics*, 2021(3), Mar 2021.

[HMPS14] Temple He, Prahar Mitra, Achilleas P Porfyriadis, and Andrew Strominger. New symmetries of massless qed. *Journal of High Energy Physics*, 2014(10):112, 2014.

[HW04] Stefan Hollands and Robert M Wald. Conformal null infinity does not exist for radiating solutions in odd spacetime dimensions. *Classical and Quantum Gravity*, 21(22):5139–5145, Oct 2004.

[KLPS17] Daniel Kapec, Vyacheslav Lysov, Sabrina Pasterski, and Andrew Strominger. Higher-dimensional supertranslations and weinberg's soft graviton theorem. *Annals of Mathematical Sciences and Applications*, 2(1):69–94, 2017.

[LS20] Alok Laddha and Ashoke Sen. Classical proof of the classical soft graviton theorem in d¿4. *Physical Review D*, 101(8), Apr 2020.

[Pas17] Sabrina Pasterski. Asymptotic symmetries and electromagnetic memory. *Journal of High Energy Physics*, 2017(9), Sep 2017.

[Pra19] Kartik Prabhu. Conservation of asymptotic charges from past to future null infinity: supermomentum in general relativity. *Journal of High Energy Physics*, 2019(3), Mar 2019.

[PRS18] Monica Pate, Ana-Maria Raclariu, and Andrew Strominger. Gravitational memory in higher dimensions. *Journal of High Energy Physics*, 2018(6), Jun 2018.

[SSS20] Arnab Priya Saha, Biswajit Sahoo, and Ashoke Sen. Proof of the classical soft graviton theorem in $D = 4$. *JHEP*, 06:153, 2020.

[SSV01] Marcus Spradlin, Andrew Strominger, and Anastasia Volovich. Les Houches lectures on de Sitter space. In *Les Houches Summer School: Session 76: Euro Summer School on Unity of Fundamental Physics: Gravity, Gauge Theory and Strings*, 10 2001.

[Str14] Andrew Strominger. On bms invariance of gravitational scattering. *Journal of High Energy Physics*, 2014(7), Jul 2014.

[Str18] Andrew Strominger. Lectures on the infrared structure of gravity and gauge theory, 2018.

[SZ14] Andrew Strominger and Alexander Zhiboedov. Gravitational memory, bms supertranslations and soft theorems, 2014.

[Wei65] Steven Weinberg. Infrared photons and gravitons. *Phys. Rev.*, 140:B516–B524, Oct 1965.

Appendix A

Deriving (2.74)

We want to solve $\Box A_\rho$ in the limit $\tau \to \pm\infty$ to get the τ falloffs for A_ρ and consecutively other fields in this limit. The Christoffel symbols that will be useful here can be written in a compact form

$$\Gamma^{\rho}_{\mu\nu} = -\rho\,\sigma_{\mu\nu} \quad \Gamma^{\mu}_{\nu\rho} = \frac{1}{\rho}\delta^{\mu}_{\nu} \quad \Gamma^{\mu}_{\alpha\beta} \tag{A.1}$$

Note that as defined in the notations in 2.1, Latin indices run over $(\rho, \tau, \hat{x})$ and Greek indices run over $(\tau, \hat{x})$. We are interested in solving

$$\Box A_\rho = \nabla^2_\rho A_\rho + \frac{\sigma^{\alpha\beta}}{\rho^2}\nabla_\alpha\nabla_\beta A_\rho = \partial^2_\rho A_\rho + \frac{\sigma^{\alpha\beta}}{\rho^2}\nabla_\alpha\nabla_\beta A_\rho \tag{A.2}$$

Now, let us look at the second term in this, namely,

$$\sigma^{\alpha\beta}\nabla_\alpha\nabla_\beta A_\rho = \sigma^{\alpha\beta}\partial_\alpha(\nabla_\beta A_\rho) - \sigma^{\alpha\beta}\Gamma^{\gamma}_{\alpha\beta}\nabla_\gamma A_\rho - \sigma^{\alpha\beta}\Gamma^{\gamma}_{\alpha\rho}\nabla_\beta A_\gamma. \tag{A.3}$$

Consider the three terms in this individually.

1. $\sigma^{\alpha\beta}\partial_\alpha(\nabla_\beta A_\rho)$ This can be written as,

$$\sigma^{\alpha\beta}\partial_\alpha(\nabla_\beta A_\rho) = \sigma^{\alpha\beta}\partial_\alpha\partial_\beta A_\rho - \frac{1}{\rho}\sigma^{\alpha\beta}\partial_\alpha A_\beta. \tag{A.4}$$

2. $\sigma^{\alpha\beta}\Gamma^{\gamma}_{\alpha\beta}\nabla_\gamma A_\rho$

$$\sigma^{\alpha\beta}\Gamma^{\gamma}_{\alpha\beta}\nabla_\gamma A_\rho = -\rho\ (d+1)\partial_\rho A_\rho + \sigma^{\alpha\beta}\Gamma^{\mu}_{\alpha\beta}\partial_\mu A_\rho - \frac{1}{\rho}\sigma^{\alpha\beta}\Gamma^{\mu}_{\alpha\beta}A_\mu \tag{A.5}$$

3. $\sigma^{\alpha\beta}\Gamma^{a}_{\alpha\rho}\nabla_\beta A_a$

$$\sigma^{\alpha\beta}\Gamma^{a}_{\alpha\rho}\nabla_\beta A_a = (d+1)A_\rho + \frac{1}{\rho}\sigma^{\alpha\beta}D_\beta A_\alpha \tag{A.6}$$

Upon combining these, you get,

$$\sigma^{\alpha\beta}\nabla_\alpha\nabla_\beta A_\rho = \sigma^{\alpha\beta}\partial_\alpha\partial_\beta A_\rho - \frac{2}{\rho}\sigma^{\alpha\beta}D_\alpha A_\beta - \sigma^{\alpha\beta}\Gamma^\mu_{\alpha\beta}\partial_\mu A_\rho + (d+1)\left[\rho\partial_\rho A_\rho - A_\rho\right]. \tag{A.7}$$

Hence we have,

$$\Box A_\rho = \partial^2_\rho A_\rho + \frac{(d+1)}{\rho^2}\left[\rho\partial_\rho A_\rho - A_\rho\right] + \frac{1}{\rho^2}\left[D^2 A_\rho - \frac{2}{\rho}\sigma^{\alpha\beta}D_\alpha A_\beta\right] \tag{A.8}$$

We also need the Lorenz gauge condition, through which we will be able to simplify the final term in that.

$$\nabla^a A_a = 0 \implies D_\alpha A^\alpha = -\rho^2\partial_\rho A_\rho - \rho(d+1)A_\rho. \tag{A.9}$$

Using the condition above, the equation can be simplified to give,

$$D_\alpha A^\alpha = \sigma^{\alpha\beta}D_\alpha A_\beta = -\rho^2\partial_\rho A_\rho - \rho(d+1)A_\rho. \tag{A.10}$$

Here A^α is lowered with $\sigma_{\alpha\beta}$. This gives us $\Box A_\rho = 0$ for any ρ as,

$$D^2 A^{(n)}_\rho + (n-1)(n-d-1)A^{(n)}_\rho = 0. \tag{A.11}$$

We also need to know the value of (here $\sqrt{-\sigma} = (1+\tau^2)^{\frac{d-1}{2}}$),

$$\begin{aligned} D^2 A^{(n)}_\rho &= \frac{1}{\sqrt{-\sigma}}\partial_\alpha(\sqrt{-\sigma}\sigma^{\alpha\beta}\partial_\beta A^{(n)}_\rho) \\ &= -\frac{1}{(1+\tau^2)^{\frac{d-1}{2}}}\partial_\tau\Big((1+\tau^2)^{\frac{d+1}{2}}\partial_\tau A^{(n)}_\rho\Big) + \text{lower order in } \tau \\ &= -(1+d)\tau\partial_\tau A^{(n)}_\rho(\tau) - (1+\tau^2)\partial^2_\tau A^{(n)}_\rho(\tau) + \cdots \end{aligned} \tag{A.12}$$

For large τ we have,

$$\lim_{\tau\to\infty} D^2 A^{(n)}_\rho = -(1+d)\tau\partial_\tau A^{(n)}_\rho - \tau^2\partial^2_\tau A^{(n)}_\rho \tag{A.13}$$

This gives us two solutions,

$$A^{(n)}_\rho = O(\tau^{1-n}), \quad A^{(n)}_\rho = O(\tau^{n-d-1}). \tag{A.14}$$

The first one leads to divergent charges, and hence the one we choose is the second one. Thus, $A^{(n)}_\rho = O(\tau^{n-d-1})$. This leads to $F^{(n)}_{\rho\tau} = O(\tau^{n-d-2})$. This for $d=2$, gives us $F^{(n)}_{\rho\tau} =$

$O(\tau^{n-4})$ which matches with (2.74).

Appendix B

Laplacian on dS_{d+1}

We know that λ and ψ satisfy the wave equation

$$\begin{aligned} & D^{\alpha}D_{\alpha}\phi = 0 \\ \implies & -\partial_\tau[(1+\tau^2)^{\frac{d+1}{2}}\partial_t\phi] + (1+\tau^2)^{\frac{d-3}{2}}D_AD^A\phi = 0 \end{aligned} \tag{B.1}$$

Solving this in the large τ limit and assuming $\phi \sim O(\tau^{-k})$, we get in the limit $\tau \to \infty$

$$dk - k^2 = 0 \quad \implies \quad k = 0, d \tag{B.2}$$

The $k = 0$ solution corresponds to the large τ behaviour of λ while $k = d$ corresponds to the large τ limit of ψ. So we have (2.83) and (2.82)

$$\lambda(\tau,\hat{x}) = \lambda_-(\hat{x}) + \cdots \tag{B.3}$$

$$\psi(\tau,\hat{x}) = \frac{1}{\tau^d}\psi_-(\hat{x}) + \cdots \tag{B.4}$$

where $\cdots$ represent higher order terms in τ. We can use Green's functions to determine the solution in terms of λ_- and ψ_-

$$\lambda(y) = \int d^2V' G^{(0)}(y,\hat{x}')\lambda_-(\hat{x}') \tag{B.5}$$

$$\psi(y) = \int d^2V' G^{(2)}(y,\hat{x}')\psi_-(\hat{x}') \tag{B.6}$$

We will now analyse the Green's functions in terms of following variables

$$Y^\mu \equiv (\tau, \sqrt{1+\tau^2}\hat{x}) \tag{B.7}$$

$$\sigma \equiv \tau + \sqrt{1+\tau^2}\hat{x}\cdot\hat{x}' \tag{B.8}$$

$$P = Y^\mu Y'_\mu \tag{B.9}$$

From the assumption that the dS_{d+1} vacuum is invariant under $SO(d+1,1)$ de Sitter group, $G(y,y')$ is also expected to be de Sitter invariant. This then tells us that $G(y,y') \equiv G(P)$ and it satisfies the following equation [SSV01].

$$(P^2-1)\partial_P^2 G + (d+1)P\partial_P G = 0 \tag{B.10}$$

This gives a hypergeometric solution

$$G^{\mathbb{S}^d}(P) = c_d P (P^2-1)^{\frac{1-d}{2}} F[1, 1-\frac{d}{2}; \frac{3}{2}; P^2]. \tag{B.11}$$

Here c_d is a constant, which depends on d. For $d=2$, this gives us,

$$G^{\mathbb{S}^2}(P) = \frac{c_2 P}{\sqrt{P^2-1}}. \tag{B.12}$$

Taking the discontinuities across the branch cuts we get

$$G_R^{\mathbb{S}^2}(P) = \frac{c_2 P}{\sqrt{P^2-1}} \Theta(\tau-\tau')\Theta(P-1). \tag{B.13}$$

This matches with [CE17] where they have $c = 1/2\pi$. Now in the limit when $\tau \to -\infty$ we get,

$$\lim_{\tau' \to -\infty} P = -|\tau'|(1 + x\cdot\hat{x}\sqrt{1+\tau^2}) = -|\tau'|\sigma, \tag{B.14}$$

So $G_R^{\mathbb{S}^2}(P)$ for $d=2$ in this limit becomes

$$G_R^{\mathbb{S}^2}(P) = \frac{1}{2\pi}\Theta(\sigma) \tag{B.15}$$

Now we will use Kirchoff's integral representation for $\psi(y)$

$$\psi(y) = \lim_{\tau' \to -\infty} |\tau'|^3 \int_{\tau'=\text{const}} d^d V' \Big[G_R(y,y')\partial_{\tau'}\psi(y') - \partial_{\tau'} G_R(y,y')\psi(y') \Big]. \tag{B.16}$$

We look at the terms in the Green function separately. First consider the term $G_R^{\mathbb{S}^2}(y,y')\partial_{\tau'}\psi(y')$. This in the limit $\tau \to -\infty$ becomes,

$$\lim_{\tau' \to -\infty} |\tau'|^3 G_R^{\mathbb{S}^2}(y,y')\partial_{\tau'}\psi(y') = |\tau'|^3 \times \frac{1}{2\pi}\Theta(\sigma)\left(\frac{2}{|\tau'|^3}\psi_-(\hat{x}')\right) = \frac{1}{\pi}\Theta(\sigma)\psi_-(\hat{x}'). \tag{B.17}$$

The other term, which is $\partial_{\tau'} G_R^{\mathbb{S}^2}(y,y')\psi(y')$ becomes 0. This is because of the derivative acting on the Green function here,

$$\partial_{\tau'} G_R^{\mathbb{S}^2}(y,y') = \frac{1}{2\pi}\partial_{\tau'}\Theta(\sigma) = \frac{1}{2\pi}\delta(\sigma)\frac{\partial\sigma}{\partial\tau'} \sim (1-\hat{x}\cdot\hat{x}')\delta(\tau'(1-\hat{x}\cdot\hat{x}')) = 0. \quad \text{(B.18)}$$

In concluding that the equation above does not result in a contribution to the final result of $\psi(y)$, there is an assumption that $\psi_-(\hat{x}')$ is a smooth function. Thus we have the equation,

$$\psi(\hat{x},\tau) = \frac{1}{\pi}\int d^2V'\Theta(\sigma)\psi_-(\hat{x}'). \quad \text{(B.19)}$$

Now, in order to do this integral we need to study the $\Theta(\sigma)$ function, and where it is non-vanishing. This can be easily done by noticing that this is non-zero only when $\sigma > 0$. And from the definition of σ in B.14 and $\sigma > 0$ we have,

$$\hat{x}\cdot\hat{x}' > -\frac{\tau}{\sqrt{1+\tau^2}}. \quad \text{(B.20)}$$

We can study this inequality in various limits of τ. For the case when $\tau \to -\infty$ we have,

$$\lim_{\tau\to-\infty} -\frac{\tau}{\sqrt{1+\tau^2}} \approx 1 - \frac{1}{2\tau^2} + O(1/\tau^3). \quad \text{(B.21)}$$

This implies that we have,

$$\hat{x}\cdot\hat{x}' > 1 - \frac{1}{2\tau^2} \implies 2\hat{x}\cdot\hat{x}' > 2 - \frac{1}{\tau^2} \implies \frac{1}{\tau^2} > 2 - 2\hat{x}\cdot\hat{x}' = |\hat{x}-\hat{x}'|^2. \quad \text{(B.22)}$$

Here we have $|\hat{x}|^2 = |\hat{x}'|^2 = 1$. Note that the integral is over the variable $\hat{x}'$. Diagrammatically we have this region as,

This integral will clearly click only when $\hat{x}'$ is centered around $\hat{x}$, and that should give us a delta function over that. Along with it, we will get the volume of the 2d sphere (disk), which is $\pi\tau^{-2}$. Therefore, in the case when $\tau \to -\infty$ B.19 becomes,

$$\psi(\hat{x}) = \tau^{-2}\psi_-(\hat{x}') \quad \text{(B.23)}$$

as expected when $k_- = 0$.

And with this in place, we should now explore the region $\tau \to \infty$. In this case, we have,

$$\lim_{\tau\to\infty} -\frac{\tau}{\sqrt{1+\tau^2}} \approx -1 + \frac{1}{2\tau^2} + O(1/\tau^3). \quad \text{(B.24)}$$

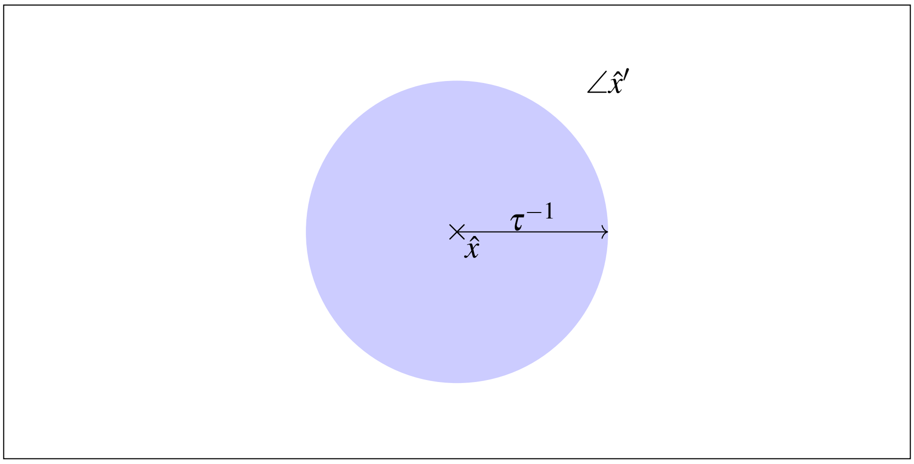

Figure B.1: *The blue region, inside the circle of radius τ^{-1} and centered about $\hat{x}$, denotes the domain of integration for the variable $\hat{x}'$.*

Which implies that $\sigma > 0$ gives us,

$$\hat{x}\cdot\hat{x}' > -1+\frac{1}{2\tau^2} \implies 2\hat{x}\cdot\hat{x}' > -2+\frac{1}{\tau^2} \implies |\hat{x}'+\hat{x}'|^2 = \frac{1}{\tau^2}. \tag{B.25}$$

This region is shown in the diagram below,

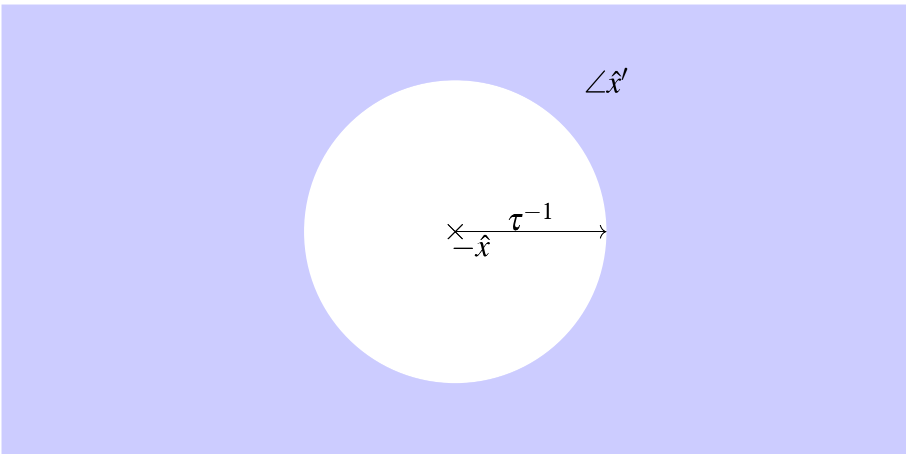

Figure B.2: *The blue region, outside the circle of radius τ^{-1} and centered about $-\hat{x}$, denotes the domain of integration for the variable $\hat{x}'$.*

Thus, the integral in the region shown above can be decomposed into an integral over the whole region $\mathbb{S}^2$ subtracted by an integral inside the circle. Therefore, we have,

$$\lim_{\tau\to\infty} \psi(\hat{x},\tau) = \frac{1}{\pi}\int d^2V'\Theta(\sigma)\psi_-(\hat{x}') = \frac{1}{\pi}\int_{\mathbb{S}^2} d^2V'\psi_-(\hat{x}') - \frac{1}{\pi}\int d^2V'\Theta(-\sigma)\psi_-(\hat{x}')$$
$$= \frac{1}{\pi}\left(\int_{\mathbb{S}^2} d^2V'\psi_-(\hat{x}') - \pi\tau^{-2}\psi_-(-\hat{x}')\right) \equiv k_+ + \tau^{-2}\psi_+(\hat{x}). \tag{B.26}$$

Therefore, from the equation above, we have,

$$k_+ = \frac{1}{\pi}\int_{\mathbb{S}^2} d^2V'\psi_-(\hat{x}'), \tag{B.27}$$

$$\psi_+(\hat{x}) = -\psi_-(-\hat{x}). \tag{B.28}$$

Appendix C

Tools for gravity calculations

C.1 Christoffel Symbols

We list here all the relevant christoffel symbols used in our computation of the EOMs and symplectic form. All the non-zero Christoffel symbols are

$$\Gamma^u_{uu} = \frac{1}{2} g^{ur} \left(2\partial_u g_{ur} - \partial_r g_{uu}\right) \tag{C.1}$$

$$\Gamma^u_{uA} = \frac{1}{2} g^{ur} \left(\partial_A g_{ur} - \partial_r g_{uA}\right) \tag{C.2}$$

$$\Gamma^u_{AB} = -\frac{1}{2} g^{ur} \partial_r g_{AB} \tag{C.3}$$

$$\Gamma^r_{rr} = g^{ur} \partial_r g_{ur} \tag{C.4}$$

$$\Gamma^r_{ur} = \frac{1}{2} \left[g^{ur} \partial_r g_{uu} + g^{rA} \left(\partial_r g_{uA} - \partial_A g_{ur}\right)\right] \tag{C.5}$$

$$\Gamma^r_{uA} = \frac{1}{2} \left[g^{ur} \partial_A g_{uu} + g^{rr} \left(\partial_A g_{ur} - \partial_r g_{uA}\right) + g^{rB} \left(\partial_A g_{uB} - \partial_B g_{uA} + \partial_u g_{AB}\right)\right] \tag{C.6}$$

$$\Gamma^r_{rA} = \frac{1}{2} \left[g^{ru} \left(\partial_r g_{uA} + \partial_A g_{ur}\right) + g^{rB} \partial_r g_{AB}\right] \tag{C.7}$$

$$\Gamma^r_{uu} = \frac{1}{2} \left[g^{ru} \partial_u g_{uu} + g^{rr} \left(2\partial_u g_{ur} - \partial_r g_{uu}\right) + g^{rA} \left(2\partial_u g_{uA} - \partial_A g_{uu}\right)\right] \tag{C.8}$$

$$\Gamma^r_{AB} = \frac{1}{2} \left[g^{ur} \left(\partial_B g_{uA} + \partial_A g_{uB} - \partial_u g_{AB}\right) - g^{rr} \partial_r g_{AB} + g^{rc} \left(\partial_B g_{CA} + \partial_A g_{CB} - \partial_C g_{AB}\right)\right] \tag{C.9}$$

$$\Gamma^A_{uu} = \frac{1}{2} \left[g^{Ar} \left(2\partial_u g_{ur} - \partial_r g_{uu}\right) + g^{AB} \left(2\partial_u g_{uB} - \partial_B g_{uu}\right)\right] \tag{C.10}$$

$$\Gamma^A_{ur} = \frac{1}{2} g^{AB} \left(\partial_r g_{uB} - \partial_B g_{ur}\right) \tag{C.11}$$

$$\Gamma^A_{uB} = \frac{1}{2} \left[g^{Ar} \left(\partial_B g_{ur} - \partial_r g_{uB}\right) + g^{AC} \left(\partial_u g_{BC} + \partial_B g_{uC} - \partial_C g_{uB}\right)\right] \tag{C.12}$$

$$\Gamma^A_{rB} = \frac{1}{2} g^{AC} \partial_r g_{CB} \tag{C.13}$$

$$\Gamma^A_{BC} = \frac{1}{2} \left[-g^{Ar} \partial_r g_{BC} + g^{AD} \left(\partial_B g_{DC} + \partial_C g_{BD} - \partial_D g_{BC}\right)\right] \tag{C.14}$$

We next substitute the metric components and evaluate the Christoffel symbols upto the order needed for our analysis.

List of Christoffel symbols computed below are,

$$\Gamma^A_{Br},\ \Gamma^A_{Ar},\ \Gamma^r_{rr},\ \Gamma^r_{ur},\ \Gamma^A_{ur},\ \Gamma^A_{uB},\ \Gamma^u_{AB},\ \Gamma^r_{AB},\ \Gamma^A_{BC},\ \Gamma^a_{aB},\ \partial_A\Gamma^a_{aB},\ \Gamma^a_{ar},\ \Gamma^a_{au} \tag{C.15}$$

C.1.1 Γ^A_{Br}

$$\Gamma^A_{Br} = \frac{1}{2}g^{AC}\partial_r g_{BC}. \tag{C.16}$$

Here we have,

$$g^{AC} = \frac{\gamma^{AC}}{r^2} - \frac{C^{AC}}{r^3} + \frac{C^A_D C^{CD} - D^{AC}}{r^4} + \frac{g^{(2)AC}}{r^5}. \tag{C.17}$$

where,

$$g^{(2)AC} = C^{AM}D^C_M + C^{MC}D^A_M - C^A_M C^{MN} C^C_N - C^{(1)AC} \tag{C.18}$$

$$\Longrightarrow\ g^{(2)A}_B = C^{AM}D_{BM} + C_{BM}D^{AM} - C^A_M C^{MN} C_{NB} - C^{(1)A}_B. \tag{C.19}$$

And,

$$\partial_r g_{BC} = 2r\gamma_{BC} + C_{BC} - \frac{C^{(1)}_{BC}}{r^2}. \tag{C.20}$$

Therefore,

$$\Gamma^A_{Br} = \frac{1}{2}\left(\frac{\gamma^{AC}}{r^2} - \frac{C^{AC}}{r^3} + \frac{C^A_D C^{CD} - D^{AC}}{r^4} + \frac{g^{(2)AC}}{r^5}\right)\left(2r\gamma_{BC} + C_{BC} - \frac{C^{(1)}_{BC}}{r^2}\right) \tag{C.21}$$

$$\begin{aligned}
&= \frac{\delta^A_B}{r} - \frac{C^A_B}{2r^2} + \frac{C^{AC}C_{BC} - 2D^A_B}{2r^3} + \frac{1}{2r^4}\left(2g^{(2)A}_B + C_{BC}(C^A_D C^{CD} - D^{AC}) - C^{(1)A}_B\right) \\
&= \frac{\delta^A_B}{r} - \frac{C^A_B}{2r^2} + \frac{C^{AC}C_{BC} - 2D^A_B}{2r^3} \\
&+ \frac{1}{2r^4}\left(2C^{AM}D_{BM} + 2C_{BM}D^{AM} - 2C^A_M C^{MN}C_{NB} - 2C^{(1)A}_B + C_{BN}(C^A_M C^{NM} - D^{AN}) - C^{(1)A}_B\right) \\
&= \frac{\delta^A_B}{r} - \frac{C^A_B}{2r^2} + \frac{C^{AC}C_{BC} - 2D^A_B}{2r^3} + \frac{2C^{AM}D_{BM} + C_{BM}D^{AM} - C^A_M C^{MN}C_{NB} - 3C^{(1)A}_B}{2r^4}.
\end{aligned} \tag{C.22}$$

Final result,

$$\boxed{\Gamma^A_{Br} = \frac{\delta^A_B}{r} - \frac{C^A_B}{2r^2} + \frac{C^{AC}C_{BC} - 2D^A_B}{2r^3} + \frac{2C^{AM}D_{BM} + C_{BM}D^{AM} - C^A_M C^{MN}C_{NB} - 3C^{(1)A}_B}{2r^4}.} \tag{C.23}$$

C.1.2 Γ^A_{Ar}

This is easily computed using the previous one,

$$\Gamma^A_{Ar} = \frac{4}{r} + \frac{3C\cdot D - C^3 - 3C^{(1)}}{2r^4}. \tag{C.24}$$

Using $3C\cdot D - 3C^{(1)} = C^3$ we get,

$$\boxed{\Gamma^A_{Ar} = \frac{4}{r} + O(r^{-5}).} \tag{C.25}$$

C.1.3 Γ^r_{rr}

This is simple to compute and is,

$$\boxed{\Gamma^r_{rr} = -\frac{4\beta^{(2)}}{r^3} - \frac{6\beta^{(3)}}{r^4}.} \tag{C.26}$$

C.1.4 Γ^r_{ur}

$$\Gamma^r_{ur} = \frac{1}{2}\Big[g^{ur}\partial_r g_{uu} + g^{rA}(\partial_r g_{uA} - \partial_A g_{ur})\Big] \tag{C.27}$$

In this we first have,

$$\begin{aligned}
g^{ur}\partial_r g_{uu} &= -e^{-2\beta}\partial_r(Me^{2\beta} + U^2) \qquad \text{(C.28)}\\
&= -\left(1 - \frac{2\beta^{(2)}}{r^2} - \frac{2\beta^{(3)}}{r^3}\right)\partial_r\left(-\frac{2\beta^{(2)}}{r^2} + \frac{M^{(2)}}{r^2}\left(1 + \frac{2\beta^{(2)}}{r^2} + \frac{2\beta^{(3)}}{r^3}\right) + \frac{U^{(0)2}}{r^2}\right)\\
&= -\left(1 - \frac{2\beta^{(2)}}{r^2} - \frac{2\beta^{(3)}}{r^3}\right)\left[\frac{4\beta^{(2)}}{r^3} - \frac{2M^{(2)}}{r^3} - \frac{2U^{(0)2}}{r^3}\right]\\
&= -\frac{4\beta^{(2)}}{r^3} + \frac{2M^{(2)}}{r^3} + \frac{2U^{(0)2}}{r^3}\ .
\end{aligned}$$

Next we have,

$$g^{rA}\partial_r g_{uA} = U^A e^{-2\beta}\partial_r U_A = O(r^{-4}). \tag{C.29}$$

The last one,

$$g^{rA}\partial_A g_{ur} = U^A e^{-2\beta}\partial_A e^{2\beta} = O(r^{-4}). \tag{C.30}$$

And therefore we finally have,

$$\boxed{\Gamma^{r}_{ur} = \frac{1}{r^3}\Big(M^{(2)} + U^{(0)2} - 2\beta^{(2)}\Big)\ .} \tag{C.31}$$

C.1.5 Γ^{A}_{ur}

$$\Gamma^{A}_{ur} = \frac{1}{2} g^{AB}(\partial_r g_{uB} - \partial_B g_{ur}) \tag{C.32}$$

Here we have,

$$\begin{aligned}\Gamma^{A}_{ur} &= \frac{1}{2}\frac{\gamma^{AB}}{r^2}\Big(\frac{U^{(1)}_B}{r^2} + \frac{2}{r^2}\partial_B\beta^{(2)}\Big) \\ &= \frac{1}{2r^4}\Big(\frac{U^{A(1)}}{r^2} + \frac{2}{r^2}D^A\beta^{(2)}\Big).\end{aligned} \tag{C.33}$$

Thus we have,

$$\boxed{\Gamma^{A}_{ur} = \frac{1}{2r^4}\big[U^{A(1)} + 2D^A\beta^{(2)}\big].} \tag{C.34}$$

C.1.6 Γ^{A}_{uB}

This is,

$$\boxed{\begin{aligned}\Gamma^{A}_{uB} &= \frac{\gamma^{AC}}{2r^2}\Big(-D_B U^{(0)}_C + D_C U^{(0)}_B + \partial_u D_{BC}\Big) \\ &+ \frac{1}{2r^3}\Big[-C^{AC}(-D_B U^{(0)}_C + D_C U^{(0)}_B + \partial_u D_{BC}) + (-D_B U^{(1)A} + D^A U^{(1)B} + \partial_u C^{(1)A}_B)\Big]\end{aligned}} \tag{C.35}$$

C.1.7 Γ^{u}_{uA}

This is given as,

$$\begin{aligned}\Gamma^{u}_{Au} &= \frac{1}{2} g^{ur}\big(\partial_A g_{ur} - \partial_r g_{uA}\big) \\ &= \frac{1}{2}e^{-2\beta}\partial_A e^{2\beta} - \frac{1}{2}e^{-2\beta}\partial_r U_A \\ &= \frac{1}{r^2}\partial_A\beta^{(2)} - \frac{1}{r^3}\partial_A\beta^{(3)} - \frac{1}{2}\Big(1 - \frac{2\beta^{(2)}}{r^2} - \frac{2\beta^{(3)}}{r^3}\Big)\Big(-\frac{U^{(1)}_A}{r^2}\Big) \\ &= \frac{U^{(1)}_A + 2\partial_A\beta^{(2)}}{2r^2} + O(r^{-3})\end{aligned} \tag{C.36}$$

Thus,

$$\boxed{\Gamma^u_{Au} = \frac{U_A^{(1)} + 2\partial_A\beta^{(2)}}{2r^2} + O(r^{-3}).} \tag{C.37}$$

C.1.8 Γ^r_{Ar}

This is simple and is given as,

$$\Gamma^r_{Ar} = \frac{1}{2}\Big[g^{ru}(\partial_r g_{uA} + \partial_A g_{ur}) + g^{rB}\partial_r g_{AB}\Big] \tag{C.38}$$

The terms in this are given as follows. Consider the first,

$$g^{ru}\partial_r g_{uA} = e^{-2\beta}\partial_r U_A = -\frac{U_A^{(1)}}{r^2} \tag{C.39}$$

We consider the other terms,

$$g^{ru}\partial_A g_{ur} = e^{-2\beta}\partial_A e^{2\beta} = \frac{2}{r^2}\partial_A\beta^{(2)} + O(r^{-3}), \tag{C.40}$$

and

$$\begin{aligned} g^{rB}\partial_r g_{AB} &= -U^B e^{-2\beta}(2r\gamma_{AB} + C_{AB}) &\text{(C.41)}\\ &= -\left[\frac{U^{(0)B}}{r^2} + \frac{1}{r^3}(U^{(1)B} - C^{CB}U_C^{(0)})\right]\left(1 - \frac{2\beta^{(2)}}{r^2}\right)(2r\gamma_{AB} + C_{AB}) \\ &= -\left[\frac{U^{(0)B}}{r^2} + \frac{1}{r^3}(U^{(1)B} - C^{CB}U_C^{(0)})\right](2r\gamma_{AB} + C_{AB}) \\ &= -\frac{2U_B^{(0)}}{r} - \frac{1}{r^2}\Big(U^{(0)B}C_{AB} + 2(U_A^{(1)} - C_{AC}U^{(0)C})\Big) \\ &= -\frac{2U_B^{(0)}}{r} - \frac{1}{r^2}\Big(2U_A^{(1)} - C_{AC}U^{(0)C}\Big). \end{aligned}$$

Therefore,

$$\boxed{\Gamma^r_{Ar} = \frac{1}{2}\left[-\frac{2U_B^{(0)}}{r} - \frac{1}{r^2}\Big(3U_A^{(1)} - C_{AC}U^{(0)C} - \partial_A\beta^{(2)}\Big)\right].} \tag{C.42}$$

C.1.9 Γ^u_{AB}

This is given as,

$$\boxed{\Gamma^u_{AB} = r\gamma_{AB} + \frac{C_{AB}}{2} + \frac{2\beta^{(2)}\gamma_{AB}}{r} - \frac{1}{r^2}\Big(\frac{1}{2}C_{AB}^{(1)} + C_{AB}\beta^{(2)} + 2\beta^{(3)}\gamma_{AB}\Big).} \tag{C.43}$$

$$\Gamma^u_{AB} = r\gamma_{AB} + \frac{C_{AB}}{2} - \frac{2\beta^{(2)}\gamma_{AB}}{r} - \frac{1}{r^2}\Big(\frac{1}{2}C^{(1)}_{AB} + C_{AB}\beta^{(2)} + 2\beta^{(3)}\gamma_{AB}\Big). \tag{C.44}$$

C.1.10 Γ^r_{AB}

This is slightly long and is given as,

$$\boxed{\begin{aligned}\Gamma^r_{AB} = r\gamma_{AB} + \frac{1}{2}\Big[D_B U^{(0)}_A + D_A U^{(0)}_B + \partial_u D_{AB} + C_{AB}\Big] \\ + \frac{1}{2r}\Big[2\gamma_{AB}(+M^{(2)} - 2\beta^{(2)}) - U^{(0)C}\big(D_B C_{AC} + D_A C_{BC} - D_C C_{AB}\big) \\ + D_B U^{(1)}_A + D_A U^{(1)}_B + \partial_u C^{(1)}_{AB}\Big].\end{aligned}} \tag{C.45}$$

C.1.11 Γ^A_{BC}

This is given as,

$$\boxed{\begin{aligned}\Gamma^A_{BC} = \frac{1}{2r}\big[2U^{(0)A}\gamma_{BC} + (D_B C^A_C + D_C C^A_B - D^A C_{BC})\big] \\ + \frac{1}{2r^2}\Big[U^{(0)A}C_{BC} + 2\gamma_{BC}(U^{(1)A} - C^{AD}U^{(0)}_D) - C^{AD}(D_B C_{CD} + D_C C_{BD} - D_D C_{BC}) \\ + (D_B D^A_C + D_C D^A_B - D^A D_{BC})\Big].\end{aligned}} \tag{C.46}$$

C.1.12 Γ^A_{AB}

This follows in a simple way from the expression above if we use $D^A_A = 1/2C^2$. Thus we have,

$$\boxed{\Gamma^A_{AB} = \frac{U_{(0)B}}{r} + \frac{1}{2r^2}\Big[2U^{(1)}_B - C_{BD}U^{(0)D}\Big].} \tag{C.47}$$

C.1.13 Γ^a_{aB}

This can be written as,

$$\Gamma^a_{aB} = \frac{1}{2}(2g^{ur}\partial_B g_{ur} + g^{CD}\partial_B g_{CD}). \tag{C.48}$$

These terms can be written as follows,

$$g^{ur}\partial_B g_{ur} = e^{-2\beta}\partial_B e^{2\beta} = \frac{2}{r^2}D_B\beta^{(2)} + \frac{2}{r^3}D_B\beta^{(3)}. \tag{C.49}$$

The other term is,

$$\begin{aligned}
g^{CD}\partial_B g_{CD} &= \Big[\frac{\gamma^{CD}}{r^2} - \frac{C^{CD}}{r^3} + \frac{C^{CM}C^{D}_{M} - D^{CD}}{r^4}\Big]\Big[rD_BC_{CD} + D_BD_{CD} + \frac{D_BC^{(1)}_{CD}}{r}\Big] \\
&= \frac{1}{r^3}\Big[D_BC^{(1)A}_{A} - C^{MN}D_BD_{MN} + (C^{CM}C^{D}_{M} - D^{CD})D_BC_{CD}\Big] \\
&= \frac{1}{r^3}\Big[D_B(C^{MN}D_{MN} - \frac{1}{3}C^{MN}C_{NA}C^{A}_{M}) - C^{MN}D_BD_{MN} + (C^{CM}C^{D}_{M} - D^{CD})D_BC_{CD}\Big] \\
&= \frac{1}{r^3}\Big[-\frac{1}{3}D_B(C^{MN}C_{NA}C^{A}_{M}) + C^{NM}C^{A}_{M}D_BC_{AN}\Big] = 0.
\end{aligned} \tag{C.50}$$

Hence we have,

$$\boxed{\Gamma^{a}_{aB} = \frac{2}{r^2}D_B\beta^{(2)} + \frac{2}{r^3}D_B\beta^{(3)}.} \tag{C.51}$$

C.1.14 $D_A\Gamma^{a}_{aB}$

From the equation derived above, we have,

$$\boxed{D_A\Gamma^{a}_{aB} = \frac{2}{r^2}D_AD_B\beta^{(2)} + \frac{2}{r^3}D_AD_B\beta^{(3)}.} \tag{C.52}$$

C.1.15 Γ^{a}_{ar}

This is given as,

$$\Gamma^{a}_{ar} = \frac{1}{2}(2g^{ur}\partial_r g_{ur} + g^{CD}\partial_r g_{CD}) \tag{C.53}$$

Let us compute the terms one by one. The first one is simple to compute,

$$g^{ur}\partial_r g_{ur} = e^{-2\beta}\partial_r e^{2\beta} = -\frac{4\beta^{(2)}}{r^3} - \frac{6\beta^{(3)}}{r^4}. \tag{C.54}$$

The other term is given by,

$$g^{CD}\partial_r g_{CD} = \Big(\frac{\gamma^{CD}}{r^2} - \frac{C^{CD}}{r^3} + \frac{C^C_M C^{MD} - D^{CD}}{r^4} + \frac{g^{(2)CD}}{r^5}\Big)\Big(2r\gamma_{CD} + C_{CD} - \frac{C^{(1)}{}_{CD}}{r^2}\Big) \quad \text{(C.55)}$$
$$= \frac{8}{r} + \frac{1}{r^4}\Big[-C_A^{(1)A} + C^3 - C\cdot D + 2g_A^{(2)A}\Big]$$
$$= \frac{8}{r} + \frac{1}{r^4}\Big[-C^{(1)} + C^3 - C\cdot D + 2(2C\cdot D - C^3 - C^{(1)})\Big]$$
$$= \frac{8}{r} + \frac{1}{r^4}\Big[-C_A^{(1)A} + C^3 - C\cdot D + 4C\cdot D - 2C^3 - 2C^{(1)}\Big]$$
$$= \frac{8}{r} + \frac{1}{r^4}\Big[3C\cdot D - C^3 - 3C^{(1)}\Big]$$
$$= \frac{8}{r} + \frac{1}{r^4}\Big[3C\cdot D - C^3 - (3C\cdot D - C^3)\Big] = \frac{8}{r}$$

Thus we have,

$$\boxed{\Gamma^a_{ar} = \frac{4}{r} - \frac{4\beta^{(2)}}{r^3}.} \quad \text{(C.56)}$$

C.1.16 Γ^a_{au}

This can be written as,

$$\Gamma^a_{au} = \frac{1}{2}(2g^{ur}\partial_u g_{ur} + g^{CD}\partial_u g_{CD}) \quad \text{(C.57)}$$

Consider the first term in this.

$$g^{ur}\partial_u g_{ur} = e^{-2\beta}\partial_u e^{2\beta} = \left(1 - \frac{2\beta^{(2)}}{r^2} - \frac{2\beta^{(3)}}{r^4}\right)\left(\frac{2}{r^3}\partial_u\beta^{(3)}\right) = \frac{2}{r^3}\partial_u\beta^{(3)}. \quad \text{(C.58)}$$

The other term is given as,

$$g^{CD}\partial_u g_{CD} = \Big(\frac{\gamma^{CD}}{r^2} - \frac{C^{CD}}{r^3} + \frac{C^C_M C^{MD} - D^{CD}}{r^4}\Big)\Big(\partial_u D_{CD} + \frac{\partial_u C^{(1)CD}}{r}\Big) \quad \text{(C.59)}$$
$$= \frac{1}{r^3}\Big[\partial_u C^{(1)} - C^{CD}\partial_u D_{CD}\Big]$$
$$= \frac{1}{r^3}\Big[\partial_u(C\cdot D - 1/3C^3) - C^{CD}\partial_u D_{CD}\Big]$$
$$= O(1/r^4).$$

Hence,

$$\boxed{\Gamma^a_{au} = \frac{2}{r^3}\partial_u\beta^{(3)} + O(r^{-4}).} \quad \text{(C.60)}$$

Appendix D

Deriving BMS generator and its action on metric components

We will first derive the vector that generates supertranslations in six dimensions and then look at its action on metric components. The procedure we follow here is quite general and holds in all dimensions.

D.1 Deriving ξ

Supertranslations preserve gauge choice and falloffs. Let ξ be the generator of supertranslations. To preserve the Bondi gauge, we need to impose the following conditions

$$\mathscr{L}_\xi g_{rr} = 0, \qquad \mathscr{L}_\xi g_{rA} = 0, \qquad \mathscr{L}_\xi \partial_r \det\left(\frac{g_{AB}}{r^2}\right) = 0 \tag{D.1}$$

and to preserve the falloffs we need,

$$\mathscr{L}_\xi g_{uu} = O(r^{-2}), \quad \mathscr{L}_\xi g_{ur} = O(r^{-2}), \quad \mathscr{L}_\xi g_{uA} = O(r^0), \quad \mathscr{L}_\xi g_{AB} = O(r) \tag{D.2}$$

Starting with the Bondi gauge condition $g_{rr} = 0$, we get

1. $\mathscr{L}_\xi g_{rr}$

$$\begin{aligned}\mathscr{L}_\xi g_{rr} &= \xi^\alpha \partial_\alpha g_{rr} + g_{r\alpha}\partial_r \xi^\alpha + g_{\alpha r}\partial_r \xi^\alpha = 0 \\ &\implies g_{ru}\partial_r \xi^u = 0 \\ &\implies \partial_r \xi^u = 0.\end{aligned} \tag{D.3}$$

Hence ξ^u is independent of r.

$$\boxed{\partial_r \xi^u = 0.} \tag{D.4}$$

Which means that we only have the following power in the r-expansion,

$$\xi^u = \xi^{(0)u}. \tag{D.5}$$

2. $\mathscr{L}_\xi g_{ur}$ The fall off on this is $\mathscr{L}_\xi g_{ur} = O(r^{-2})$.

$$\begin{aligned} \mathscr{L}_\xi g_{ur} &= \xi^\alpha \partial_\alpha g_{ur} + g_{u\alpha}\partial_r \xi^\alpha + g_{r\alpha}\partial_u \xi^\alpha \\ &= \xi^\alpha \partial_\alpha g_{ur} + g_{u\alpha}\partial_r \xi^\alpha + g_{ru}\partial_u \xi^u \\ &= \xi^\alpha \partial_\alpha g_{ur} + g_{u\alpha}\partial_r \xi^\alpha + g_{ru}\partial_u \xi^u. \end{aligned} \tag{D.6}$$

Then at $O(r^0)$ we have,

$$g_{ru}\partial_u \xi^{(0)u} = 0 \implies \partial_u \xi^{(0)u} = 0. \tag{D.7}$$

Therefore let us choose,

$$\boxed{\xi^{(0)u} = f(x^A)} \tag{D.8}$$

At $O(r^{-1})$, the equation above is trivially satisfied.

3. $\mathscr{L}_\xi g_{rA}$. This should be zero at all orders in r.

$$\begin{aligned} \mathscr{L}_\xi g_{rA} &= \xi^\alpha \partial_\alpha g_{rA} + g_{r\alpha}\partial_A \xi^\alpha + g_{\alpha A}\partial_r \xi^\alpha = 0 \\ &\implies g_{r\alpha}\partial_A \xi^\alpha + g_{\alpha A}\partial_r \xi^\alpha = 0 \\ &\implies g_{ru}\partial_A \xi^u + g_{uA}\partial_r \xi^u + g_{AB}\partial_r \xi^B = 0 \\ &\implies g_{ru}\partial_A \xi^u + g_{AB}\partial_r \xi^B = 0. \end{aligned} \tag{D.9}$$

Now at $O(r^0)$ we have,

$$\boxed{-\partial_A f - \gamma_{AB}\xi^{(1)B} = 0 \implies \xi^{(1)A} = -D^A f.} \tag{D.10}$$

At the next order, i.e $O(r^{-1})$, we have,

$$(r^2\gamma_{AB} + rC_{AB})\left[-\frac{\xi^{(1)B}}{r^2} - \frac{2\xi^{(2)B}}{r^3}\right] = 0 \implies -2\gamma_{AB}\xi^{(2)B} - C_{AB}\xi^{(1)B} = 0. \tag{D.11}$$

Thus we have,

$$\boxed{\xi^{(2)A} = \frac{1}{2}C^{AB}D_B f.} \tag{D.12}$$

4. $\mathscr{L}_\xi g_{uu}$. The fall off on this is $\mathscr{L}_\xi g_{uu} = O(r^{-2})$.

$$\begin{aligned}\mathscr{L}_\xi g_{uu} &= \xi^\alpha \partial_\alpha g_{uu} + g_{u\alpha}\partial_u \xi^\alpha + g_{\alpha u}\partial_u \xi^\alpha \\ &= 2 g_{u\alpha}\partial_u \xi^\alpha + O(r^{-2}) \\ &= 2\Big[g_{uu}\partial_u \xi^u + g_{ur}\partial_u \xi^r + g_{uA}\partial_u \xi^A\Big] + O(r^{-2}).\end{aligned} \tag{D.13}$$

At $O(r^0)$ we have,

$$\begin{aligned}& g_{uu}\partial_u \xi^{(0)u} + g_{ur}\partial_u \xi^{(0)r} + g^{(0)}_{uA}\partial_u \xi^{(0)A} = 0 \\ \implies & -\partial_u \xi^{(0)r} - U^{(0)}_A \partial_u \xi^{(0)A} = 0 \\ \implies & -\partial_u \xi^{(0)r} - U^{(0)}_A \partial_u \xi^{(0)A} = 0.\end{aligned} \tag{D.14}$$

If $\xi^{(0)A} = 0$ (since we want to eliminate boosts and rotations), then we have,

$$\boxed{\partial_u \xi^{(0)r} = 0.} \tag{D.15}$$

At $O(r^{-1})$ we have,

$$\begin{aligned}& -\partial_u \xi^{(1)u} - \partial_u \xi^{(1)r} + g^{(0)}_{uA}\partial_u \xi^{(1)A} = 0 \\ \implies & -\partial_u \xi^{(1)r} + g^{(0)}_{uA}\partial_u \xi^{(1)A} = 0 \\ \implies & -\partial_u \xi^{(1)r} - g^{(0)}_{uA}\partial_u D_A f = 0.\end{aligned} \tag{D.16}$$

But this just tells us that,

$$\boxed{\partial_u \xi^{(1)r} = 0.} \tag{D.17}$$

5. $\mathscr{L}_\xi g_{AB}$. This allows us to fix $\xi^{r(0)}$ and $\xi^{r(1)}$. These are fixed by using the trace conditions on $\mathscr{L}_\xi g^A_A$. From the computation of $\delta_f C_{AB}$ we get,

$$\delta_f C_{AB} = 2\xi^{(0)r}\gamma_{AB} + 2 g^{(-2)}_{C(A} D_{B)} \xi^{C(1)}. \tag{D.18}$$

And from the trace condition we get,

$$8\xi^{r(0)} + 2 g^{(-2)}_{CA} D^A \xi^{C(1)} = 0 \implies \xi^{r(0)} = \frac{1}{4}\gamma_{AC} D^A D^C f = \frac{1}{4} D^2 f. \tag{D.19}$$

For getting $\xi^{r(1)}$, we need the value of $\delta_f D_{AB}$, which is given as,

$$\delta_f D_{AB} = f\partial_u D_{AB} + \Big[2\gamma_{AB}\xi^{r(1)} + \xi^{C(1)\alpha} D_C C_{AB} + \xi^{r(0)} C_{AB}\Big] + 2\Big[g^{(-2)}_{C(A} D_{B)}\xi^{C(2)} + g^{(-1)}_{C(A} D_{B)}\xi^{C(1)} + g^{(0)}_{u(A} D_{B)}\xi^{u(0)}\Big]. \tag{D.20}$$

And the trace condition now gives,

$$\delta_f D^A_A = C^{AB}\delta_f C_{AB} = \Big[8\xi^{r(1)}\Big] + 2\Big[g^{(-2)}_{C(A} D^{A)}\xi^{C(2)} + g^{(-1)}_{C(A} D^{A)}\xi^{C(1)} + g^{(0)}_{u(A} D^{A)}\xi^{u(0)}\Big]. \tag{D.21}$$

Thus,

$$\begin{aligned}
-8\xi^{r(1)} &= -C^{AB}\delta_f C_{AB} + 2\Big[D_A\xi^{A(2)} + C_{AC}D^A\xi^{C(1)} - U^{(0)}_A D^A\xi^{u(0)}\Big] \\
&= 2C^{AB}D_A D_B f + 2\Big[\frac{1}{2}D_A(C^{AB}D_B f) - C_{AB}D^A D^B f + \frac{1}{6}D^B C_{AB} D^A f\Big] \\
&= 2\Big[\frac{1}{2}(D_A C^{AB} D_B f + C^{AB} D_A D_B f) + \frac{1}{6}D^B C_{AB} D^A f\Big] \\
&= \frac{4}{3}D_A C^{AB} D_B f + C^{AB} D_A D_B f.
\end{aligned} \tag{D.22}$$

Thus,

$$\xi^{r(1)} = \frac{1}{8}\Big[-\frac{4}{3}D_A C^{AB} D_B f - C^{AB} D_A D_B f\Big] \tag{D.23}$$

D.2 Calculating $\delta_f D_{AB}$ and $\delta_f C_{AB}$

We can calculate this by picking components at the right order from $L_\xi g_{AB}$.

$$\mathscr{L}_\xi g_{AB} = \delta_f g_{AB} = r\delta_f C_{AB} + \delta_f D_{AB} + O(r^{-1}) \tag{D.24}$$

We have

$$\mathscr{L}_x i g_{AB} = \xi^u \partial_u g_{AB} + \xi^r \partial_r g_{AB} + \xi^C D_C g_{AB} + g_{u(A} D_{B)}\xi^u \tag{D.25}$$

At $O(r)$ then we get

$$\begin{aligned}
\delta_f C_{AB} &= -2D_A D_B f + 2\gamma_{AB}\xi^{r(0)} \\
&= -2\left(D_A D_B f - \frac{1}{4}\gamma_{AB}D^2 f\right)
\end{aligned} \tag{D.26}$$

At $O(r^0)$, we get

$$\begin{aligned}
\delta_f D_{AB} = f\partial_u D_{AB} + \frac{1}{4}\gamma_{AB}\left[-\frac{4}{3}D_C C^{CD} D_D f - C^{CD} D_C D_D f\right] \\
+ \frac{1}{4}C_{AB}D^2 f - D_C C_{AB} D^C f - \frac{1}{2}\left(C_{BC}D_A D^C f + C_{AC}D_B D^C f\right) \\
+ \frac{1}{2}\left[D_A C_{BC} D^C f + D_B C_{AC} D^C f\right] \\
+ \frac{1}{6}\left[D^C C_{BC} D_A f + D^C C_{AC} D_B f\right]
\end{aligned} \tag{D.27}$$