Single-Particle Entanglement and Weak Value Amplification

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Certificate of Examination

This is to certify that the dissertation titled "**Single-Particle Entanglement and Weak Value Amplification**" submitted by Mr. Ijaz Ahamed Mohammad (Reg. No. MS16079) for the partial fulfilment of BS-MS dual degree programme of the Institute, has been examined by the thesis committee duly appointed by the Institute. The committee finds the work done by the candidate satisfactory and recommends that the report be accepted.

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Dr. Ramandeep Johal | Dr. Kovita Dorai | Prof. Arvind (Supervisor)

Dated: May 26, 2021

Declaration

The work presented in this dissertation has been carried out by me under the guidance of Prof. Arvind (IISER Mohali) and Prof. Dipankar Home (Bose Institute, Kolkata) at the Indian Institute of Science Education and Research Mohali.

This work has not been submitted in part or in full for a degree, a diploma, or a fellowship to any other university or institute. Whenever contributions of others are involved, every effort is made to indicate this clearly, with due acknowledgement of collaborative research and discussions. This thesis is a bonafide record of original work done by me and all sources listed within have been detailed in the bibliography.

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In my capacity as the supervisor of the candidate's project work, I certify that the above statements by the candidate are true to the best of my knowledge.

26/5/21

Prof. Arvind

(Supervisor)

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Abstract

This thesis is divided into two parts. The first part talks about Single-Particle Entanglement, while the rest talks about Weak Value Amplification. A state in a quantum system with at least two degrees of freedom is said to entangled when it has a particular non-separable form. Usually, entanglement is described in bipartite or multipartite systems. In single particle hybrid entanglement, this quantum correlation is seen in same particle but in different Hilbert spaces of spin, polarization, orbital angular momentum etc. The basics of Quantum Entanglement will be discussed together contrasting Single-Particle Entanglement with the regular entanglement. A major portion deals with the applications of Single-Particle Entanglement highlighting its resource-friendliness. The other part of this thesis is on a novel work which shows that one can achieve exponential enhancement in the post-selection probability for a fixed weak-value in Weak Value Amplification with the selection of certain entanglement generating operator.

Part I

Single-Particle Entanglement

Motivation

Entanglement is one of the key features of quantum systems. An entangled state was first discussed by Einstein in a joint paper with Podolsky and Rosen in 1935 discussing the famous EPR paradox without coining the word entanglement. Entanglement is now considered to be the source of non-locality and is regarded as a crucial ingredient of storage and distribution in the area of quantum information. Entanglement can also be seen where the non-classical correlations between two different parties require different degrees of freedom, which is termed as hybrid entanglement. If the entanglement is realised in different degrees of freedom within the same particle, then this is referred as intraparticle entanglement or single-particle entanglement [Azz+20]. Entanglement was first realised in terms of position-momentum variables and was later realised in other variables spin, photon polarization, orbital angular momentum, nuclear spins etc. Entanglement has been used as a resource for many quantum information processing protocols as teleportation,cryptography, super-dense coding etc. All these different types of entanglement can be easily realised quantum mechanically, but the concept of single-particle entanglement still remains under-appreciated.

Background

The first experimental study on such a locally entangled state was the mesoscopic Schrödinger cat-like state of cold atoms which was realized with trapped Be ions by the group of Wineland, in 1996[Mon+96]. They were able to entangle the local spatial position of the atom wave packet inside the trap with its hyper fine ground state. The violation of the Bell-Clauser, Horne, Shimony and Holt(BCHSH) inequality for measurements performed on different degrees of freedom of a single particle is a demonstration of intraparticle entanglement and of contextual behavior of any realistic hidden variable theory. This idea was introduced by Home and colleagues in 1984 [HS84].The first experiment with single photons was performed by Zukowski and colleagues, and published in 2000[MWŻ00].Then came the articles on applications of single particle entanglement in Teleportation by Pramanik et al. (2010) [Pra+10], Entanglement swapping by Adhikari et al. (2010) [Adh+10] and QKD by Y. Sun et al.(2011)[SWY11]. Finally, a review article is published on single particle entanglement in 2020 by Azzini et al [Azz+20].

Chapter 1

Single-Particle Entanglement

To describe the mathematical formulation of entanglement, let us consider two arbitrary quantum subsystems A and B, belonging to Hilbert spaces H_A and H_B. The Hilbert space of the composite system is given by the tensor product of these two individual Hilbert spaces $H_A \otimes H_B$. Considering the case of pure states, if the state of the subsystem A is $|\psi_A\rangle$ and the state of the subsystem is $|\psi_B\rangle$, then the state of the composite system is represented by $|\psi_A\rangle \otimes |\psi_B\rangle$. Such states of the composite system are known as separable or product states. Fixing the basis $|i_A\rangle$ for H_A and $|i_B\rangle$ for H_B, the most general state of this composite system is given by

$$|\psi_{AB}\rangle = \sum_{ij} c_{ij} |i_A\rangle \otimes |j_B\rangle$$

This general state remains separable if $c_{ij} = c_i^A c_j^B$ yielding $|\psi_A\rangle = \sum_i c_i^A |i_A\rangle$ and $|\psi_B\rangle = \sum_j c_j^B |i_B\rangle$. There is a class of state which do not satisfy this condition and remain inseparable as $c_{ij} \neq c_i^A c_j^B$ for all values of i and j. Such states are known as entangled states. For example, considering $|0_A\rangle$ and $|1_A\rangle$ as basis states for H_A and $|0_B\rangle$ and $|1_B\rangle$ as basis states for H_B, the following is an entangled state:

$$|\psi_{AB}\rangle = 1/\sqrt{2}(|0_A\rangle \otimes |0_B\rangle + |1_A\rangle \otimes |1_B\rangle)$$

This kind of entanglement can be realised in multiple ways. It can be realized in multipartite systems with entanglement within same degree of freedom. It can also be realized in multipartite systems with entanglement between different degrees of freedom, which is known as hybrid entanglement. The entanglement Further, it can also be realized within a single particle with entanglement between same or different degrees of freedom, which is termed as single-particle entanglement or single particle-hybrid entanglement respectively. It is important to note that mathematical formulation of this single particle entanglement stands same as the conventional interparticle entanglement.

Experimentally, this intraparticle entanglement has been realized in systems as photons, neutrons and atoms. Unlike interparticle entanglement which involves correlations between different particles, one advantage of single particle entanglement is that these entangled states are easier to produce and remain robust under decoherence and dephasing [Pra+10].An interesting point to note is that in the case of single particle entanglement, the system is not comprised of two space-like separated subsystems and the EPR phenomenology does no longer give rise to the non-locality issues [Pra+10]. Since only single particles are needed, the technique utilizing single particle entanglement consumes less resources than those using interparticle entanglement [Pra+10].

1.1 Applications of Single Particle Entanglement

There are many applications of intraparticle entanglement in quantum information processing like Teleportation [Pra+10], Entanglement Swapping [Adh+10], Quantum Key Distribution [SWY11] etc. Three of these application are discussed below.

1.1.1 Teleportation

The first theoretical proposal of implementing teleportation using a intraparticle entangled state was given by Pramanik *et al.* in 2010[]. This protocol is briefly described as follows:
Let us consider a spin-1/2 particle corresponding to an initial spin polarized state along the +z-axis($|0\rangle$). Considering the particle's path variable, the joint spin-path state is given by

$$|S\rangle_{ps}^{1} = |\psi_0\rangle_p \otimes |0\rangle_s \tag{1.1}$$

where the subscripts s and p represent the spin and path variables respectively. Now this state is made to pass through a beam splitter with reflected and transmitted channels designated as $|0\rangle_p$ and $|1\rangle_p$ respectively. This transforms the initial state into

$$|S\rangle_{ps}^{1} \rightarrow (\alpha\,|1\rangle_{p} + i\beta\,|0\rangle_{p}) \otimes |0\rangle_{s} \tag{1.2}$$

where the coefficients α and β represent real numbers corresponding to α^2 and β^2 as reflection and transmissions probabilities.

On passing this above state through a spin-flipper through one of the channels(here considered on path channel 0) results in a single particle hybrid entangled state given by

$$\alpha\,|0\rangle_{s} \otimes |1\rangle_{p} + i\beta\,|1\rangle_{s} \otimes |0\rangle_{p} \tag{1.3}$$

Before the beginning of protocol for teleportation Alice possesses three particles in her lab; the first being the hybrid intraparticle entangled state shown above, an auxiliary particle present in $|0\rangle_s^a$ and the particle which is meant to be teleported $\gamma\,|0\rangle_s^2 + \delta\,|1\rangle_s^2$ whereas Bob possesses only a single particle in state $|0\rangle_s^3$. The whole protocol can be described in five simple steps briefly as follow:

1. Alice makes a CNOT operation with 1^{st} particle's spin state as control and auxiliary particle's spin state as target resulting as

$$\alpha\,|1\rangle_p^1\,|0\rangle_s^1\,|0\rangle_s^a + i\beta\,|1\rangle_p^1\,|1\rangle_s^1\,|1\rangle_s^a \tag{1.4}$$

2. Alice then makes a CNOT operation with 1^{st} particle's spin state as control and 2^{nd} particle's spin state as target. The resulting state after this operation is

$$\begin{aligned}\alpha\gamma\,|1\rangle_p^1\,|0\rangle_s^1\,|\,|0\rangle_s^2\,|0\rangle_s^a + i\beta\gamma\,|0\rangle_p^1\,|1\rangle_s^1\,|0\rangle_s^2\,|0\rangle_s^a \\ + \alpha\delta\,|1\rangle_p^1\,|0\rangle_s^1\,|1\rangle_s^2\,|0\rangle_s^a + i\beta\delta\,|0\rangle_p^1\,|1\rangle_s^1\,|s\rangle_s^2\,|1\rangle_s^a\end{aligned} \tag{1.5}$$

3. Next, Alice sends 1^{st} particle to Bob. After Bob confirms that he has received the particle, Alice measures spin of her both 2^{nd} particle along z-axis and auxiliary particle along x-axis. The above state in the expression (5) can be rewritten as:

$$\begin{aligned}(\alpha\gamma\,|1\rangle_p^1\,|0\rangle_s^1 + i\beta\delta\,|0\rangle_p^1\,|1\rangle_s^1)\,|0\rangle_s^2\,|0_x\rangle_s^a \\ + (\alpha\gamma\,|1\rangle_p^1\,|0\rangle_s^1 - i\beta\delta\,|0\rangle_p^1\,|1\rangle_s^1)\,|0\rangle_s^2\,|1_x\rangle_s^a \\ + (i\beta\gamma\,|0\rangle_p^1\,|1\rangle_s^1 + \alpha\delta\,|1\rangle_p^1\,|0\rangle_s^1)\,|1\rangle_s^2\,|0_x\rangle_s^a \\ + (-i\beta\gamma\,|0\rangle_p^1\,|1\rangle_s^1 + \alpha\delta\,|1\rangle_p^1\,|0\rangle_s^1)\,|1\rangle_s^2\,|1_x\rangle_s^a\end{aligned} \tag{1.6}$$

Assuming that both the spin measurements of Alice resulted in $|0\rangle_s^2\ |0_x\rangle_s^a$ state, the state of the 1^{st} particle collapses to

$$\alpha\gamma\,|1\rangle_p^1\,|0\rangle_s^1 + i\beta\delta\,|0\rangle_p^1\,|1\rangle_s^1 \tag{1.7}$$

After Bob receives the 1st particle, he sends it through a 50-50 beam splitter, which gives rise to two different paths $|a\rangle_p$ and $|b\rangle_p$. The action of Beam splitter is given by $|0\rangle \rightarrow \frac{(|a\rangle_p + i|b_p\rangle)}{\sqrt{2}}$ and $|1\rangle \rightarrow \frac{(|b\rangle_p + i|a_p\rangle)}{\sqrt{2}}$. Then Bob applies a CNOT gate with 1^{st} particle's spin state as control and 3^{rd} particle's spin state as target. Omitting the normalization constants, the resultant state is

$$\alpha\gamma\,|b\rangle_p^1\,|0\rangle_s^1\,|0\rangle_s^3 \;+\; \beta\delta\,|b\rangle_p^1\,|1\rangle_s^1\,|1\rangle_s^3 \;+\; i\alpha\gamma\,|a\rangle_p^1\,|0\rangle_s^1\,|0\rangle_s^3 \;+\; i\beta\delta\,|a\rangle_p^1\,|1\rangle_s^1\,|1\rangle_s^3 \tag{1.8}$$

Bobs finally makes a measurement on 1st particles spin state along x-axis. Assuming the resultant state is in $|a\rangle_p^1\ |0_x\rangle_s^1$, state of the third particle is given by

$$i\alpha\gamma\,|0\rangle_s^3 - i\beta\delta\,|1\rangle_s^3 \tag{1.9}$$

which on application of a unitary operator is transformed to $\alpha\gamma\,|0\rangle_s^3 + \beta\delta\,|1\rangle_s^3$ which is similar to the 2^{nd} particle (2) whose state was meant to be teleported.
The fidelity(F) of this teleportation protocol is given by

$$F = |\,\langle\psi^{in}|\,|\psi^{out}\rangle\,|^2 = \frac{(\alpha\gamma^2 + \beta\delta^2)^2}{\alpha^2\gamma^2 + \beta^2\delta^2} \tag{1.10}$$

Similarly, the cases where the result of Bob's measurement are other than the one mentioned above can be dealt in the related way accordingly.

This protocol is similar to the standard teleportation scheme for a single qubit. The difference here is that the intraparticle entanglement is not be initially shared between the two distant parties(as it is not possible), the particle itself is transferred from Alice to Bob in between the protocol. The particle whose state is teleported remains with Alice, and its initial state is destroyed by Alice's measurement, thus avoiding any conflict with the no-cloning theorem.

It is interesting to note that, even if the particle gets intercepted by Eve in between the protocol, it is not possible for Eve to decode the information encoded in the qubit which is being shared. This can be realized by rewriting Eq. (5) as follows:

$$\alpha\,|1\rangle_p^1\,|0\rangle_s^1\,(\gamma\,|0\rangle_s^2 + \delta\,|1\rangle_s^2)\,|0\rangle_s^a + i\beta\,|0\rangle_p^1\,|1\rangle_s^1\,(\gamma\,|1\rangle_s^2 + \delta\,|0\rangle_s^2)\,|1\rangle_s^a \tag{1.11}$$

It can be clearly seen from the state above, that even if Eve intercepts the 1^{st} particle and measures it, she gains no sort of information of the state that was intended to be teleported.

1.1.2 Entanglement Swapping

Interparticle entanglement is widely used in quantum information processing because of the fact that an entanglement partner can be shared over distances which isn't the case with single particle entanglement. This Entanglement swapping from intraparticle to interparticle shall resolve the issue mentioned above. A protocol for swapping the entanglement from intra to inter was given by Adhikari *et al.* in 2010 [Adh+10]. This protocol is briefly outlined as follows

At the beginning of this protocol, Alice has two particles, one is the spin-path hybrid entangled state and other is an up-spin state. Bob also has a up-spin state to begin with.

1. Alice prepares spin-path hybrid entangled state as described in previous subsection, which is

$$\alpha \left|0\right\rangle_s \otimes \left|1\right\rangle_p + i\beta \left|1\right\rangle_s \otimes \left|0\right\rangle_p \tag{1.12}$$

   The joint state of the particles present with Alice is given by

$$\alpha \left|0\right\rangle_s^1 \left|0\right\rangle_s^2 \otimes \left|1\right\rangle_p^1 + i\beta \left|1\right\rangle_s^1 \left|0\right\rangle_s^2 \otimes \left|0\right\rangle_p^1 \tag{1.13}$$

   Now Alice performs a CNOT gate considering spin of the 1^{st} particle as source and the 2^{nd} particle as the target qubit. The resultant state is given by

$$\alpha \left|0\right\rangle_s^1 \left|0\right\rangle_s^2 \otimes \left|1\right\rangle_p^1 + i\beta \left|1\right\rangle_s^1 \left|1\right\rangle_s^2 \otimes \left|0\right\rangle_p^1 \tag{1.14}$$

2. Alice sends particle 1 to her spatially distant friend Bob who posses another particle '3' in spin up state with him. Bob then performs a CNOT operation on spin part of particle 1 and qubit 3 by considering qubit 3 as the target qubit. Note that, particle '1' and qubit '3' are physically present with Bob whereas particle '2' is sent to a distant party Charlie. After this operation, the joint four qubit state is given by

$$\alpha \left|0\right\rangle_s^1 \left|0\right\rangle_s^2 \left|0\right\rangle_s^3 \otimes \left|1\right\rangle_p^1 + i\beta \left|1\right\rangle_s^1 \left|1\right\rangle_s^2 \left|1\right\rangle_s^3 \otimes \left|0\right\rangle_p^1 \tag{1.15}$$

3. Bob then uses a 50-50 beam splitter to recombine the paths 1 and 0. The action of beam splitter is in the similar way as described in the previous section; $|0\rangle \rightarrow \frac{(|a\rangle_p + i|b_p\rangle)}{\sqrt{2}}$ and $|1\rangle \rightarrow \frac{(|b\rangle_p + i|a_p\rangle)}{\sqrt{2}}$. Using this transformation, the above state is rearranged as

$$\frac{i}{\sqrt{2}}[(\alpha\,|0\rangle_s^1\,|0\rangle_s^2\,|0\rangle_s^3 + \beta\,|1\rangle_s^1\,|1\rangle_s^2\,|1\rangle_s^3) \otimes |a\rangle_p^1] \\ + \frac{1}{\sqrt{2}}[(|0\rangle_s^1\,|0\rangle_s^2\,|0\rangle_s^3 - \beta\,|1\rangle_s^1\,|1\rangle_s^2\,|1\rangle_s^3) \otimes |b\rangle_p^1] \quad (1.16)$$

4. Bob then measures the path of the particle 1 and also measures spin qubit of particle 1 along x-axis. Let us assume the particle travels along $|a\rangle_p$, then after its interaction of Stern-Gerlach apparatus, the joint state is given by

$$\frac{1}{\sqrt{2}}[|0_x\rangle_s^1 \otimes (\alpha\,|0\rangle_s^2\,|0\rangle_s^3 + \beta\,|1\rangle_s^2\,|1\rangle_s^3) + |1_x\rangle_s^1 \otimes (\alpha\,|0\rangle_s^2\,|0\rangle_s^3 - \beta\,|1\rangle_s^2\,|1\rangle_s^3)] \quad (1.17)$$

5. If the outcome of the Bob's spin measurement was $|0_x\rangle_1$, then the resultant joint state of qubit 2 and 3 is

$$\alpha\,|0\rangle_s^2\,|0\rangle_s^3 + \beta\,|1\rangle_s^2\,|1\rangle_s^3 \quad (1.18)$$

This state is similar to the spin-path hybrid entangled state we began with, a simple unitary operation $I \otimes S$ (S is a phase flip operator) can result the above state same as the hybrid entangled state. Similarly, in other cases on application of other unitary operators we get the desired entangled state shared between Charlie and Bob. It should be noted that the particles '2' and '3' are kept separated and never interact with each other in this protocol.

1.2 Future Outlook

- One can study along this line of single particle hybrid entanglement by involving three different degrees of freedom within the same particle as was shown between spin, trajectory and energy in [She+20] . By continuing in this one can achieve the benefits of multipartite entanglement by mimicking it in a single particle.

- Other potential research along this line of study is to generate multipartite inter-particle entanglement beginning with a single particle entangled state (having entanglement over multiple degrees of freedom like spin, path, orbital angular momentum etc.) whose entanglement can be transformed to multipartite entanglement between different particles using the above entanglement swapping protocol mentioned above [Adh+10; Kum+19].

Part II

Weak Value Amplification

Background

In 1988, Aharonov, Albert, and Vaidman(AAV) [AAV88] claimed in an article titled ""How the result of a measurement of a component of the spin of a spin-1/2 particle can turn out to be 100" that certain procedure consisting of preparation of states and post selection, can lead to a result of an measurement observable whose value can lie outside the range of eigenvalues of the observables. Most of the people around the world thought then that it was an impossible task. This thinking was strengthened as there were multiple errors in the article. After careful study Duck, Stevenson and Sudharshan had shown in their article [DSS89] the validity of the main point of AAV's article still remains and clarified it to put out the clear picture.

Weak measurement is widely accepted as one of the most potential research tools in quantum physics. Weak value amplification is one of the important applications of Weak values which can be used to amplify signals in experimental studies.

Chapter 2

Weak Value Amplification

2.1 Weak Measurement

There are many inspirations behind generalizing the usual quantum projective measurements into weak measurements. Some of them are to determine the state of the system between two strong measurements, disclosing the abnormal weak values etc [TC13]. In this process of weak measurement both the measurement device and the system under consideration are quantum in nature. This process is of two steps. At first, one weakly couples the quantum system to the measurement device and then finally one strongly measures the measurement device. The weakly couple means, the standard deviation of the outcome of measurement should be more than the difference between eigenvalues of the quantum system. Tasks believed to be self contradictory by nature such as 'determining a particle's state between two measurements' prove to be perfectly possible with the aid of this technique. Weak measurements can reveal some information about the amplitudes of a quantum state without collapsing the state into eigenvectors. Weak measurements generalize ordinary quantum (projective) measurements: following the weak coupling the state vector is not collapsed but biased by a small angle, and the measurement device does not show a clear eigenvalue, but a superposition of several values.

Weak measurement has found many ways into application is used in real life now-a-days. One of the first realisations of the importance of weak measurements was given in the article titled "The sense in which a 'weak measurement' of a spin1/2 particle's spin component yields a value 100"[DSS89]. This can be briefly described as follows.

- Consider the standard von Neumann model of measurement in which a quantum system is coupled with measuring apparatus. The coupling Hamilton is given by

$$H = -g(t)\hat{q} \otimes \hat{A} \tag{2.1}$$

where q is the canonical variable, g(t) is a relevant coupling constant whose integral over time is unity and A is the operator that is to be measured on the quantum system.

- Consider the initial state of the measuring device as $|\Phi_{in}\rangle$ whose p-representation of wave function $\tilde{\phi_{in}}$(p) is a Gaussian centred at p=0 with a spread of Δp. In von Neumann Measurement, the Hamiltonian has a coupling between the quantum system whose observable is supposed to be measured and the measuring device itself. The ideal measuring device has well-defined initial and final values of $\hat{p}$ where the difference between these values is the pointer reading which registers the value of $\hat{A}$. The p and q representations of wave forms are Fourier transforms of each other, and as we know that the Fourier transform of a Gaussian is also a Gaussian.

$$|\Phi_{in}\rangle = \begin{cases} \int dq\phi_{in}(q)(q - representation) \\ \int dp\tilde{\phi_{in}}(p)(p - representation) \end{cases} \tag{2.2}$$

where

$$\phi_{in}(q) \equiv \langle q|\Phi_{in}\rangle = exp[\frac{-q^2}{4\Delta^2}],$$
$$\tilde{\phi_{in}}(p) = \langle p|\Phi_{in}\rangle = exp[-\Delta^2 p^2],\ \Delta q \equiv \Delta$$
$$\text{and } \Delta p = 1/2\Delta \text{ with } \hbar = 1$$

- Consider the quantum system to be in a definite state $|\Psi_{in}\rangle$ which can be a superposition of eigenstates of $\hat{A}$.

$$|\Psi_{in}\rangle = \sum_n \alpha_n |A = a_n\rangle \tag{2.3}$$

By choosing the superposition of the eigenstates of variable $\hat{A}$ as the quantum state and acting coupled Hamiltonian on it, the whole system evolves unitarily.

The effects of "weak measurement" is only seen when the Δp is much greater than the spread of $a_n^{'}$s and the effects of ideal measurement are seen when this Δ p tends to zero with measurement always producing one of the eigenvalues instead of the superposition of the states.

- When the coupling Hamiltonian acts for short duration of measurement, during which it is assumed to dominate over other terms of full Hamiltonian, the whole system is evolved into

$$exp[-i\int \hat{H}dt]\left|\Psi_{in}\right\rangle\left|\Phi_{in}\right\rangle = \sum_n \alpha_n e^{iqa_n} exp[\frac{-q^2}{4\Delta^2}]\left|A = a_n\left|q\right\rangle\right\rangle$$
$$= \sum_n \alpha_n \int dp exp[-\Delta^2(p-a_n)^2]\left|A = a_n\right\rangle\left|p\right\rangle \quad (2.4)$$
$$\text{using } \hat{I} = \int dp\left|p\right\rangle\left\langle p\right|$$

- When Δp is small when compared to the spacing between the eigenvalues a_n's, then the above equation shows that the whole state in a superposition of widely separated peaks centred at each eigenvalues. One should note that when Δ p $\rightarrow$ 0 this process tends to the ideal measurement process. When Δ p is much larger when compared to spread of a_n's the case referred by AAV as the "weak measurement", the equation above approximates to a single broad Gaussian peaked at mean value of $\hat{A}$, which is $\langle A\rangle = \sum_n |\alpha_n^2| a_n$.

- Interesting results appear on post selection of the state of the quantum system. Just after the "weak measurement" of $\hat{A}$, make a strong measurement of some other observable $\hat{B}$ and select one outcome 'b', which puts the quantum state in $\left|\Psi_f\right\rangle = \left|B = b\right\rangle$, which can be written as a state which is composed of eigenvectors of $\hat{A}$; $\sum_n \alpha_n^{'}\left|A = a_n\right\rangle$. This post selection puts the final state of measuring device in

$$\left|\Phi_f\right\rangle = \left\langle\Psi_f\right|exp[-i\int \hat{H}dt]\left|\Psi_{in}\right\rangle\left|\Phi_{in}\right\rangle$$
$$= \sum_n \alpha_n\alpha_n^{'*}\int dq e^{iqa_n} exp[\frac{-q^2}{4\Delta^2}]\left|q\right\rangle \quad (2.5)$$
$$= \sum_n \alpha_n\alpha_n^{'*}\int dp exp[-\Delta^2(p-a_n)^2]\left|p\right\rangle$$

- AAV introduced a quantity A_w, which is called the weak value of $\hat{A}$

$$A_w \equiv \frac{\langle\Psi_f|A|\Psi_{in}\rangle}{\langle\Psi_f|\Psi_{in}\rangle} \quad (2.6)$$

and showed that the final state of the measuring device can be approximately given by

$$|\Phi_f\rangle \approx \langle\Psi_f|\Psi_{in}\rangle \int dpexp[-\Delta^2(p - A_w)^2]\,|p\rangle \tag{2.7}$$

This above wave function is a single Gaussian peaked at A_w. What makes it interesting is by considering Ψ_f and Ψ_{in} almost orthogonal and a non-zero matrix element $\langle\Psi_f|\hat{A}|\Psi_{in}\rangle$ makes A_w lie far outside the range of eigenvalues a_n.

How can a superposition of Gaussians peaked at eigenvalues be approximate to a single Gaussian peaked at A_w, when A_w can itself lie far away from any value of a_n? This paradox can be explained by the complex nature of the coefficients involved in the state.

2.1.1 Example

- Consider a beam of spin-1/2 particles moving along y-axis, having their spins aligned at an angle α with the x-axis in the xz-plane and the spatial wave function of the particles is Gaussian with a width of Δp in z direction.

- Consider the coupling Hamiltonian as $\hat{H} = -\lambda g(t)\hat{\sigma_z}$ where λ is proportional to the particle's magnetic moment. Following the previous notations, $|\psi\rangle$ corresponds to particle's spin state and $|\phi\rangle$ corresponds to its spatial wavefunction, $\hat{A} = \lambda\hat{\sigma_z}$ and $q = z$.

$$|\psi_{in}\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} cos(\alpha/2) + sin(\alpha/2) \\ cos(\alpha/2) - sin(\alpha/2) \end{pmatrix} \tag{2.8}$$

$$|\psi_f\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{2.9}$$

$$\implies \langle\psi_f|\psi_{in}\rangle = cos(\alpha/2)$$

$$\implies \langle\psi_f|\hat{\sigma_z}|\psi_{in}\rangle = sin(\alpha/2)\ \&$$

$$\implies A_w = \lambda tan(\alpha/2)$$

- The initial spatial wave function in z direction is

$$\phi_{in}(q) = exp[-z^2/4\Delta^2] \tag{2.10}$$

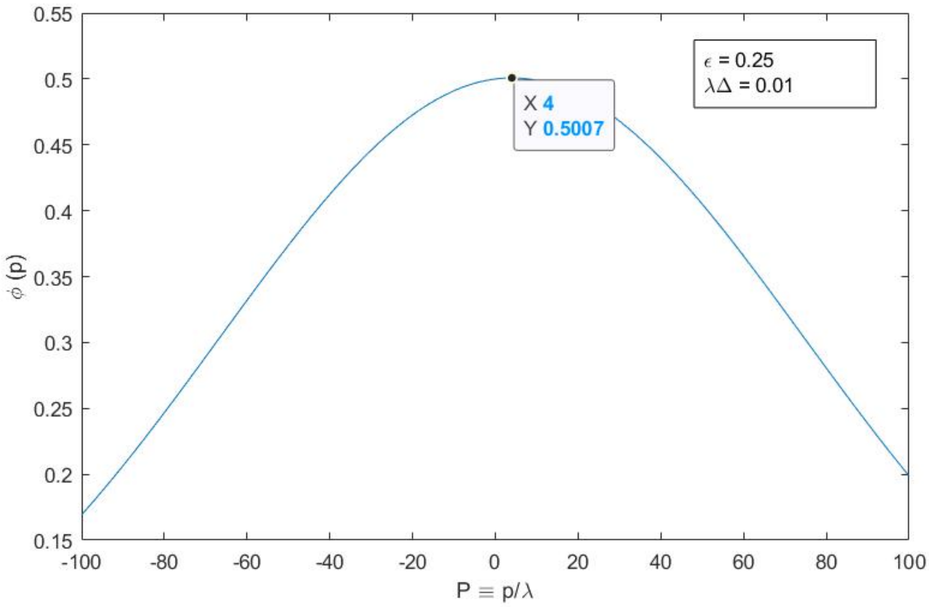

Figure 2.1: The above plot shows how the summation of two Guassians results in a Guassian with it peak at $1/\epsilon$.

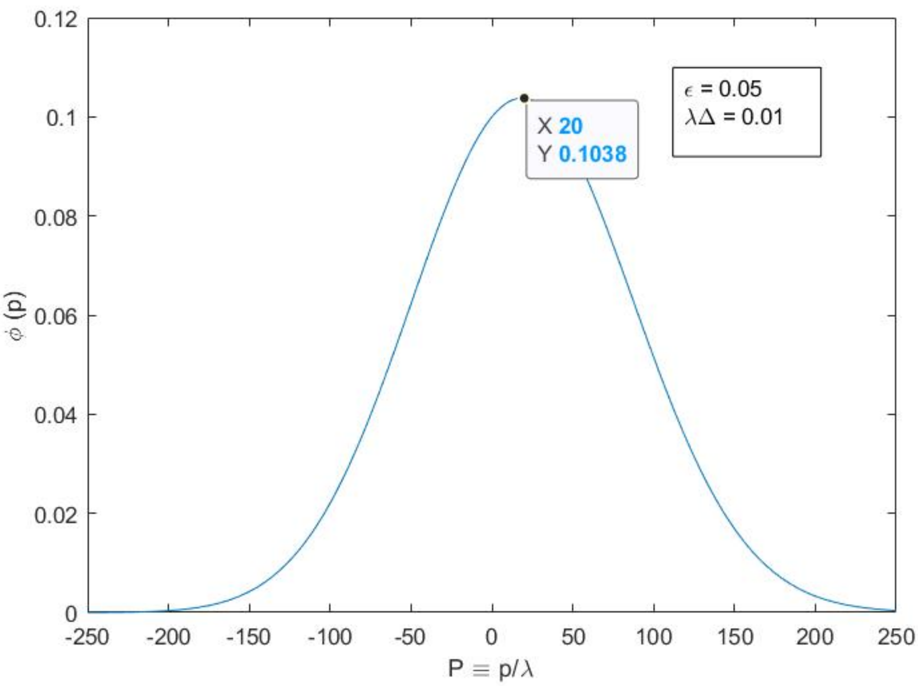

Figure 2.2: The shifthing of the peak increases with smaller values of ϵ only util AAV's approximations hold true.

- The final wave function is

$$\tilde{\phi_f}(p) \approx (cos(\alpha/2))exp[-\Delta^2(p_z - \lambda tan\alpha/2)^2] \; (AAV form) \tag{2.11}$$

  On substituting $\alpha = \pi - 2\epsilon$ where $\epsilon << 1$ and $\lambda\Delta << 1$, the above equation is approximated to

$$\tilde{\phi_f}(p) = \epsilon exp[-\Delta^2(p_z - \lambda/\epsilon)^2] \tag{2.12}$$

- The final wave function according to Eq. 2.5 is given by

$$\phi \equiv 1/2\left((1+\epsilon)exp[-\Delta^2(p-\lambda^2)] - (1-\epsilon)exp[-\Delta^2(p+\lambda^2)]\right) \tag{2.13}$$

  ignoring the scaling constants, the above equation can be scaled as

$$\equiv (1+\epsilon)exp[-\Delta^2\lambda^2(P^2-1)] - (1-\epsilon)exp[-\Delta^2\lambda^2(P^2+1)] \tag{2.14}$$

  where p/λ is considered as P. These plots are shown in Figs. 2.1 & 2.2.

2.2 Weak Value Amplification

Weak Value Amplification is one of the important applications of Weak Measurement. As mentioned in the earlier section, choosing Ψ_f almost orthogonal to Ψ_{in}, $|A_w|$ can be amplified to the desired extent.

Following an article published by Sangshi Pang *et al.*[PDB14], which describes about quadratic enhancement of post-selection probability for a fixed weak value in weak value amplification, we plan to show that it is possible to achieve exponential enhancement in P_s by using a suitable entanglement generating operator.

Consider an interaction Hamiltonian of the form

$$H_{int} = g\hat{A} \otimes \hat{F}\delta(t - t_0) \tag{2.15}$$

where A is the ancilla observable, F is the meter observable and g is the coupling parameter that one would estimate. The time factor $\delta(t-t_0)$ indicates that the interaction between the two observables is impulsive. Considering a pure meter state $|\phi\rangle$, a pure ancilla state $|\psi_i\rangle$, then weakly coupling them using the interaction Hamiltonian and post selecting the ancilla state in $|\psi_f\rangle$. This procedures changes the meter state to

$$|\phi^{'}\rangle = \hat{M}\,|\phi\rangle\,/||\hat{M}\,|\phi\rangle\,|| \tag{2.16}$$

where $\hat{M} = \langle\psi_f|\,|exp(-ig\hat{A}\otimes\hat{F}|\,|\psi_i\rangle)$

For a new meter observable $\hat{R}$ and for small g, the average of the observable in the updated meter state is given by

$$\langle\hat{R}\rangle_{|\phi'\rangle} = \frac{2gIm(\alpha A_w) + g^2\beta|A_w|^2}{1 + g^2\sigma^2|A_2|^2} \tag{2.17}$$

where $\alpha = \langle\hat{R}\hat{F}\rangle_{|\phi\rangle}$, $\beta = \langle\hat{F}\hat{R}\hat{F}\rangle_{|\phi\rangle}$ and $\sigma^2 = \langle\hat{F^2}\rangle_{|\phi\rangle}$

Ignoring the second order terms, the above equation can be rewritten as

$$\langle\hat{R}\rangle_{|\phi'\rangle} = 2g[ReA_wIm\alpha + ImA_wRe\alpha] \tag{2.18}$$

This shows how larger weak value can amplify the sensitivity of the meter for small changes in g.

As mentioned above, post selection probability and weak value do not go hand in hand, i.e., they are inversely proportional, where $|\,\langle\Psi_f|\Psi_i\rangle\,|^2 \sim$ post-selection probability (P_s). Note that $|\psi_f\rangle$ is orthogonal to $(\hat{A} - A_w)\,|\psi_i\rangle$.

According to the article published by Shangshi Pang *et al.*, maximization of P_s is done over subspace where $|\psi_f\rangle$ is orthogonal to $(\hat{A} - A_w)\,|\psi_i\rangle$ results with

$$maxP_s = \frac{Var(\hat{A})_{|\psi_i\rangle}}{\langle\psi_i|\hat{A}^2|\psi_i\rangle - 2\langle\psi_i|\hat{A}|\psi_i\rangle ReA_w + |A_w|^2} \tag{2.19}$$

To enhance this post-selection probability the authors consider a joint ancilla observable such as

$$\hat{A} = \hat{A}_1 + \hat{A}_2 + \hat{A}_3 + \hat{A}_4 ...... + \hat{A}_n \tag{2.20}$$

where $\hat{A}_k = I\otimes ......\hat{a}.....\otimes I$ is shorthand for the observable $\hat{a}$ of the k^{th} ancilla.

It was shown that on considering an above observable, the maximum variance of $\hat{A}$, one could attain has a quadratic scaling in n.

$$maxVar(\hat{A}_{|\psi_i\rangle}) = \frac{n^2}{4}(\lambda_{max} - \lambda_{min})^2 \tag{2.21}$$

where λ_{max} and λ_{min} are the maximum and minimum eigenvalues of observable $\hat{a}$.

The states which show such kind of quadratic scaling of variance are entangled.

$$|\psi_i\rangle = \frac{1}{\sqrt{2}}(|\lambda_{max}^{\otimes n}\rangle + e^{i\theta}\,|\lambda_{min}^{\otimes n}\rangle) \tag{2.22}$$

$$|\psi_f\rangle \propto (n\lambda_{min} - A_w^*)\,|\lambda_{max}\rangle^{\otimes n} + e^{i\theta}(n\lambda_{max} - A_w^*)\,|\lambda_{min}\rangle^{\otimes n} \tag{2.23}$$

2.2.1 Exponential Enhancement of Post-selection probability for a fixed weak value

Following an article by S. M. Roy *et al.* [RB08], we consider our ancilla observable to be of certain form whose variance has an exponential dependence on n.

$$\hat{X} = (\sigma_x + i\sigma_y)^{\otimes n} = \hat{H}\prime + i\hat{A}\prime \tag{2.24}$$

The observables $\hat{H}\prime$ and $\hat{A}\prime$ have an average of zero and variance with an exponential dependence on n.

Unlike the above method, as we know that the $maxP_s$ depends on the variance of the operator, it was shown by S.M Roy *et al.* [RB08] that the maximum variance is achieved for the state which is not entangled i.e., separable in nature.

n=2

Considering the $|\psi_{in}\rangle = |00\rangle$

$$(\sigma_x + i\sigma_y)^1 \otimes (\sigma_x + i\sigma_y)^2$$

$$= (\sigma_x^1\sigma_x^2 - \sigma_y^1\sigma_y^2) + i(\sigma_x^1\sigma_y^2 + \sigma_y^1\sigma_x^2)$$

$$\implies H' = \sigma_x^1\sigma_x^2 - \sigma_y^1\sigma_y^2 \& A' = \sigma_x^1\sigma_y^2 + \sigma_y^1\sigma_x^2$$

Considering A = A' or H'

$$\langle\psi_f|A|\psi_{in}\rangle = 0$$

$$\langle\psi_f|A^2|\psi_{in}\rangle = 4$$

$$\implies maxP_s = \frac{4}{4 + |A_w|^2}$$

n=3

$|\psi_{in}\rangle = |000\rangle$

$$(\sigma_x + i\sigma_y)^1 \otimes (\sigma_x + i\sigma_y)^2 \otimes (\sigma_x + i\sigma_y)^3$$

$$= [(\sigma_x^1\sigma_x^2 - \sigma_y^1\sigma_y^2) + i(\sigma_x^1\sigma_y^2 + \sigma_y^1\sigma_x^2)] \otimes (\sigma_x + i\sigma_y)^3$$

$$\implies H' = \sigma_x^1\sigma_x^2\sigma_x^3 - \sigma_y^1\sigma_y^2\sigma_x^3 - \sigma_x^1\sigma_y^2\sigma_y^3 - \sigma_y^1\sigma_x^2\sigma_y^3$$

$$A' = \sigma_x^1\sigma_y^2\sigma_x^3 + \sigma_y^1\sigma_x^2\sigma_x^3 + \sigma_x^1\sigma_x^2\sigma_y^3 - \sigma_y^1\sigma_y^2\sigma_y^3$$

Considering A = A' or H'

$$\langle\psi_f|A|\psi_{in}\rangle = 0$$

$$\langle\psi_f|A^2|\psi_{in}\rangle = 16$$

$$\implies maxP_s = \frac{16}{16 + |A_w|^2}$$

For general n,

$$maxP_s = \frac{2^{2n-2}}{2^{2n-2} + |A_w|^2} \tag{2.25}$$

The result above shows this approach is better than the one given by authors of [PDB14], considering $\hat{a} = \sigma_z$, gives the maximum value of post-selection probability for a fixed weak value as

$$maxP_s = \frac{n^2}{n^2 + |A_w|^2} \tag{2.26}$$

Observations and Outlook

- It is important to note that exponential dependence on variance is obtained for a initial state which is separable unlike the previous method where it uses a entangled state, showing that this method is more resource friendly.
- Due to the nature of exponential enhancement of this method, the maximum post selection probability quickly saturates to 1 with respect to n when compared to the previous method as shown in the Figs. 2.3 & 2.4.
- This work is still in progress and soon there shall be a pre-print of this work online.

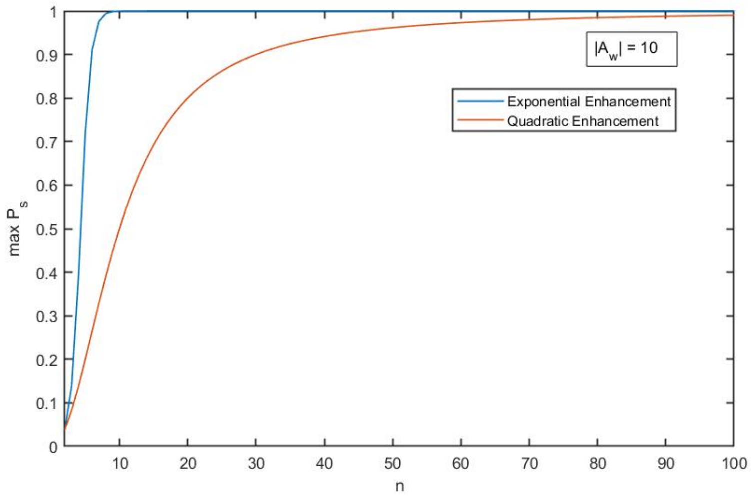

Figure 2.3: It can be clearly seen that for a fixed value of $|A_w| = 10$, method of exponential enhancement saturates the post-selection probability close to n = 11, whereas the quadratic enhancement method doesn't even get saturated after 100 qubits.

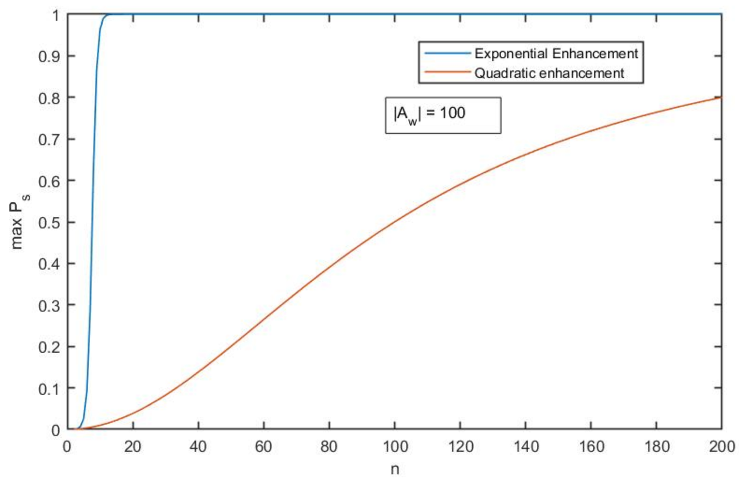

Figure 2.4: Similar to the previous graph, it can be clearly seen that for a fixed value of $|A_w| = 100$, method of exponential enhancement saturates the post-selection probability close to n = 20, whereas the quadratic enhancement method doesn't even get saturated after 200 qubits.

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