Geometrising Gauge Theory Feynman Diagrams Using AdS/CFT

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A dissertation submitted for the partial fulfilment of BS-MS dual degree in Science

Indian Institute of Science Education and Research, Mohali

May 19, 2021

Certificate of Examination

This is to certify that the dissertation titled "Geometrising Gauge Theory Feynman Diagrams Using AdS/CFT" submitted by R Bharathkumar (Reg. No. MS16097) for the partial fulfilment of BS-MS dual degree program of the Institute, has been examined by the thesis committee duly appointed by the Institute. The committee finds the work done by the candidate satisfactory and recommends that the report be accepted.

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Declaration

The work presented in this dissertation has been carried out by me under the guidance of Prof. Rajesh Gopakumar at the International Centre for Theoretical Sciences Bangalore and Dr. Anosh Joseph at the Indian Institute of Science Education and Research Mohali. This work has not been submitted in part or in full for a degree, a diploma, or a fellowship to any other university or institute. Whenever contributions of others are involved, every effort is made to indicate this clearly, with due acknowledgement of collaborative research and discussions. This thesis is a bonafide record of original work done by me and all sources listed within have been detailed in the bibliography.

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In my capacity as the supervisor of the candidate's project work, I certify that the above statements by the candidate are true to the best of my knowledge.

Dr. Anosh Joseph
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Acknowledgements

> No one who achieves success does so without acknowledging the help of others. The wise and confident acknowledge this help with gratitude.
>
> - Alfred North Whitehead

With the above note, I would like to extend my deepest gratitude to my Master's thesis external supervisor, Prof. Rajesh Gopakumar, without whose constant support this thesis would not have been possible. His continuous guidance in discussions, exchanging ideas, and moral support whenever required helped me complete this thesis on time and learn about some remarkable recent developments in the program of trying to prove the AdS/CFT correspondence. I would also like to thank my local supervisor Dr. Anosh Joseph for his support.

I would like to thank my Master's thesis committee members, Dr. Kinjalk Lochan and Dr. Ambresh Shivaji, to agree to be a part of my committee and provide me with important feedback.

I would also like to thank Faizan Bhat and Pronobesh Maity, who collaborated and constantly provided me with feedback. This thesis would not have been possible without the various informative discussions we had over time.

And as always, the constant support of peers is of utmost importance. I choose to not mention their names out of fear of missing to mention someone. But all your love and support throughout these five years is what pulled me through.

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Chapter 1

Introduction

Dualities have had a long-standing role in collectively improving our understanding of physical systems. Only one duality, however, has come far enough to revolutionise our understanding of quantum gravity, and high energy physics in general. The AdS/CFT correspondence [Mal98, Wit98] posits that a quantum gravitational system living on an AdS background can be completely represented in terms of a conformal field theory living on the boundary of the AdS spacetime. To be more explicit, the correlators on the two sides should match identically.

This identification is what is called as a strong-weak duality. When one considers a weakly coupled CFT, the dual AdS bulk lives in a highly curved background while when one considers a strongly coupled CFT, the dual AdS bulk lives in a weakly curved, "supergravity" background.

In this thesis we consider a specific problem that lets us probe the inner workings of this correspondence. Before we begin, we set the stage up for the motivation behind considering the question.

1.1 Setup and Motivation

Imagine a bunch of strings scattering off in flat space. To simplify the problem, we specifically consider four strings. What are the geometric properties of the worldsheet hence formed? Gross and Mende [GM87] showed that for strings moving on a flat spacetime at high energies and fixed scattering angle, the momentum space expression for the amplitude localises to a saddle point and the resulting expression has an interpretation of the worldsheet localising to a minimal area surface.

The analogous scenario in the AdS/CFT case, specifically for AdS_5/CFT_4, is as follows. Consider four closed strings shooting off the AdS boundary into the bulk. This, through the gauge-gravity duality dictionary, is equivalent to considering the following four-point function of $\frac{1}{2}$−BPS operators

$$\langle \mathcal{O}_{p_1}(x_1,y_1)\mathcal{O}_{p_2}(x_2,y_2)\mathcal{O}_{p_3}(x_3,y_3)\mathcal{O}_{p_4}(x_4,y_4)\rangle. \tag{1.1.1}$$

Here, the operator $\mathcal{O}_p(x,y)$ is defined as

$$\mathcal{O}_p(x,y) = y_{i_1}\dots y_{i_p}\mathrm{Tr}\left(\Phi^{i_1}\dots\Phi^{i_p}\right), \tag{1.1.2}$$

with Φ^i being the six bosonic degress of freedom and Tr denoting trace over the gauge group $SU(N)$ of the boundary $\mathcal{N}=4$ SYM, and y_i being any six-dimensional vector satisfying $y_i \cdot y_i = 0$. In the large coupling limit (defined as taking $\lambda = g^2_{\text{SYM}}N \to \infty$), the dual string theory lives in the so called supergravity limit [DF02] where the AdS spacetime only has a small but non-zero curvature. In this limit, the connected piece of the four-point function can be rewritten in terms of its Mellin transform[1]. Having done that using bootstrap/integrability techniques [AV20], one arrives at similar results as those of flat space scattering.

Having established the progress in considering a Gross-Mende like behaviour in the supergravity limit, the natural question is to ask if it can be generalised to the case where we consider the free-field, or the so-called tensionless, limit. Although the question might seem a little haphazard, there are multiple reasons for why this limit is interesting to explore.

The first and most important of all, our understanding of the tensionless limit in AdS_5 is grossly limited. Studying this region of the parameter space allows us to understand the "stronger" version of the correspondence where in contrast with the widely explored "weakest" form, we maintain the condition on large N while relaxing the condition on large λ. Secondly, recent progress in AdS_3/CFT_2 [GG18, EGG19, EGG20] has shown that to arrive at a physicist's notion of a derivation of the correspondence, the free-field limit is easier to handle. Finally, it lets us make headway into the grand dream of realising gauge theory correlators in the large N limit as closed string amplitudes (sum over Riemann surfaces) [tH74].

1.2 Outline

In chapter 2, we set the stage up for dealing with $\mathcal{N}=4$ SYM in the limit where the t'Hooft coupling is taken to zero. The connected tree-level correlator is given in terms of simple Wick contractions between the operators. Using this technique, we arrive at the expressions for the position space result of four-point functions.

In chapter 3, we introduce Strebel differentials and argue for their utility in

[1]The Mellin amplitude is the AdS equivalent of the momentum space for flat space scattering. See [Pen11, FKP+11] for details.

understanding the behaviour of the worldsheet in the tensionless limit. Further, we connect Strebel differentials to recent developments in the AdS_3 land vis-a-vis covering maps.

We argue for the possible generalisation of these techniques in AdS_5 using twistorial realisation of free field solutions, inspired by similar results in AdS_3, in chapter 4. We conclude with a summary of the progress made and possible future research in chapter 5.

In the appendices, as a proof of concept, we calculate the four-point function of twist operators in orbifold CFTs using Strebel differentials and match with existing results in appendix A. We also provide a small calculation in appendix B that would further help the reader understand the map from the worldsheet to the boundary.

Chapter 2

$\mathcal{N} = 4$ SYM in the Weak Coupling Limit

In this chapter we provide a brief introduction to the field and symmetry contents of $\mathcal{N} = 4$ super Yang Mills, review the calculation of two and three point functions, and then generalise it to four-point functions. The discussions in the review closely follow the set of lectures in [Kom17]. For more general analysis of correlators of $\mathcal{N} = 4$ super Yang Mills, we refer the reader to [CDHS16, APSS03, DO01, DO02].

2.1 Properties of $\mathcal{N} = 4$ SYM

As the name of the theory suggests, it is a supersymmetric theory with four sets of supercharges $\{Q_\alpha^A, \bar{Q}_{\dot{\alpha}A},\}$ where α and $\dot{\alpha}$ are spinorial indices taking values from $(1, 2)$ and the index A runs from 1 to 4. Thus in total, the theory has 16 supercharges. The theory additionally has a $SU(4) \cong SO(6)$ internal symmetry acting on these four sets of supercharges.

The theory is a conformal field theory, with a $SO(4, 2)$ conformal symmetry. Of the generators of the conformal symmetry group, the commutator of the generator for special conformal transformation, K_μ, with the supercharges leads to an additional set of generators, denoted by S. Together, the whole symmetry algebra forms the $\mathcal{N} = 4$ superconformal algebra, which is isomorphic to the $PSU(2, 2|4)$ supergroup.

Since the group is sufficiently large, the fields in the theory can be accomodated in a single superconformal multiplet. The multiplet consists of six scalars, four Weyl fermions, and one gauge field. This leads to all the fields falling in the adjoint representation of $SU(N_c)$. We are primarily concerned with the scalars, denoted as Φ_i, and operators constructed out of these scalars. In particular, we would be considering so-called $\frac{1}{2}$-BPS operators constructed out of single trace operators.

2.2 Half-BPS Operators

The most basic gauge-invariant operator that can be constructed out of the fields available in the superconformal multiplet are the single trace operators involving the scalars of the theory,

$$\text{Tr}((y \cdot \Phi)^p), \tag{2.2.1}$$

where y is a six-dimensional vector that mixes the scalars together. From the $SO(6)$ internal symmetry of the fields, it is evident that the above single trace operator is $SO(6)$ invariant, where the group acts on the y. The set of these operators that satisfy $y^2 = 0$ are annihilated by half of the supercharges Q and hence the name $\frac{1}{2}$-BPS. The exact set of operators that annihilate the operator depends on the choice of y. We denote these operators as $\mathcal{O}^p$, thus

$$\mathcal{O}^p(x, y) = \text{Tr}((y.\Phi(x))^p). \tag{2.2.2}$$

At the tree level, correlation functions involving these scalars are purely given by Wick contractions. We start with the two-point function of two scalars. In this case the Wick contraction reads

$$\langle \Phi_i(x_1)\Phi_j(x_2)\rangle = \frac{\delta_{ij}}{|x_1 - x_2|^2}. \tag{2.2.3}$$

Thus upon mixing them up using six-dimensional vectors y_1^i and y_2^j, we obtain

$$\langle (y_1 \cdot \Phi(x_1))(y_2 \cdot \Phi_j(x_2)\rangle = \frac{y_1 \cdot y_2}{|x_1 - x_2|^2}. \tag{2.2.4}$$

Here we caution the reader that the $SU(N_c)$ indices have been suppressed for brevity. Including the gauge indices leads to an addiditional $\delta_{ac}\delta_{bd}$ piece in Eq. (2.2.3). Thus, upon exponentiating these operators with an integer p and taking a trace, we arrive at the two-point function of $\frac{1}{2}$-BPS operators as

$$\langle \text{Tr}((y_1 \cdot \Phi(x_1))^p)\text{Tr}((y_2 \cdot \Phi(x_2))^p)\rangle = pN_c^p \frac{(y_1 \cdot y_2)^p}{|x_1 - x_2|^{2p}}. \tag{2.2.5}$$

The p in the above equation is taken to be identical since these $\frac{1}{2}$-BPS operators are conformal primaries and two-point functions of primaries are non-zero only when the operators entering the correlator are of the same conformal dimensions. In order to make our lives easier, the single trace operators in Eq. (2.2.1) are normalised by a factor of $1/\sqrt{pN_c^p}$.

In the double-line notation of t'Hooft, the two-point function can be interpreted as follows. In a single operator $\mathcal{O}^p$, there are p threads, each having a pair of gauge indices. In the presence of two of these operators, the $\delta_{ac}\delta_{bd}$ piece acts as a term that glues the threads of the two operators together. Each thread that was glued together contributes a factor of $\frac{1}{|x_1-x_2|^2}$, thus giving us the expressions in Eq. (2.2.5). Developing this pictorial interpretation helps in calculating higher point functions as follows.

While the next possibility is the three-point function, we jump directly to four-point functions since the methods are the same. Given a set of four operator insertions as

$$G = \langle \mathcal{O}_{p_1}(x_1, y_1)\mathcal{O}_{p_2}(x_2, y_2)\mathcal{O}_{p_3}(x_3, y_3)\mathcal{O}_{p_4}(x_4, y_4)\rangle, \tag{2.2.6}$$

we start arriving at the position space expression for this correlator as follows. As outlined in the previous paragraph, we need to start counting the number of threads coming out of each operator, and the number of times each of these threads are glued to each other between a pair of operators. We also obviously need to make sure that we are dealing with planar diagrams in the large N_c limit, but we arrive at that at a later stage.

Using the expression for Wick contractions, the four-point function can be written as follows. For the operators in Eq. (2.2.6), the number of threads are p_i. Denoting the number of "connections" between two operators as n_{ij}, we have

$$G = \sum_{n_{ij}} \prod_{1\leq i<j\leq 2} C_{n_{ij}} \left(\frac{y_{ij}^2}{x_{ij}^2}\right)^{n_{ij}} \tag{2.2.7}$$

Since the operators only have p_i threads, we have the condition

$$\sum_{j,j\neq i} n_{ij} = p_j, \qquad \forall j. \tag{2.2.8}$$

These conditions leave two independent parameters. We choose them to be n_{12} and n_{14}. For simplicity, we additionally take all the p_i to be equal. The case of unequal p_i follows the same procedure. Piecing all the information together, we get the expression for G as

$$G = \frac{1}{N_c^2}\left(\frac{y_{13}^2 y_{24}^2}{x_{13}^2 x_{24}^2}\right)^p \sum_{n_{12}=0}^{p} \sum_{n_{14}=0}^{p-n_{12}} C_{n_{ij}} \left(\frac{\sigma}{u}\right)^{n_{12}} \left(\frac{\tau}{v}\right)^{n_{14}}. \tag{2.2.9}$$

Here, we have rewritten the propagators in terms of the conformal cross ratios

$$u = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}, \qquad v = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2} \tag{2.2.10}$$

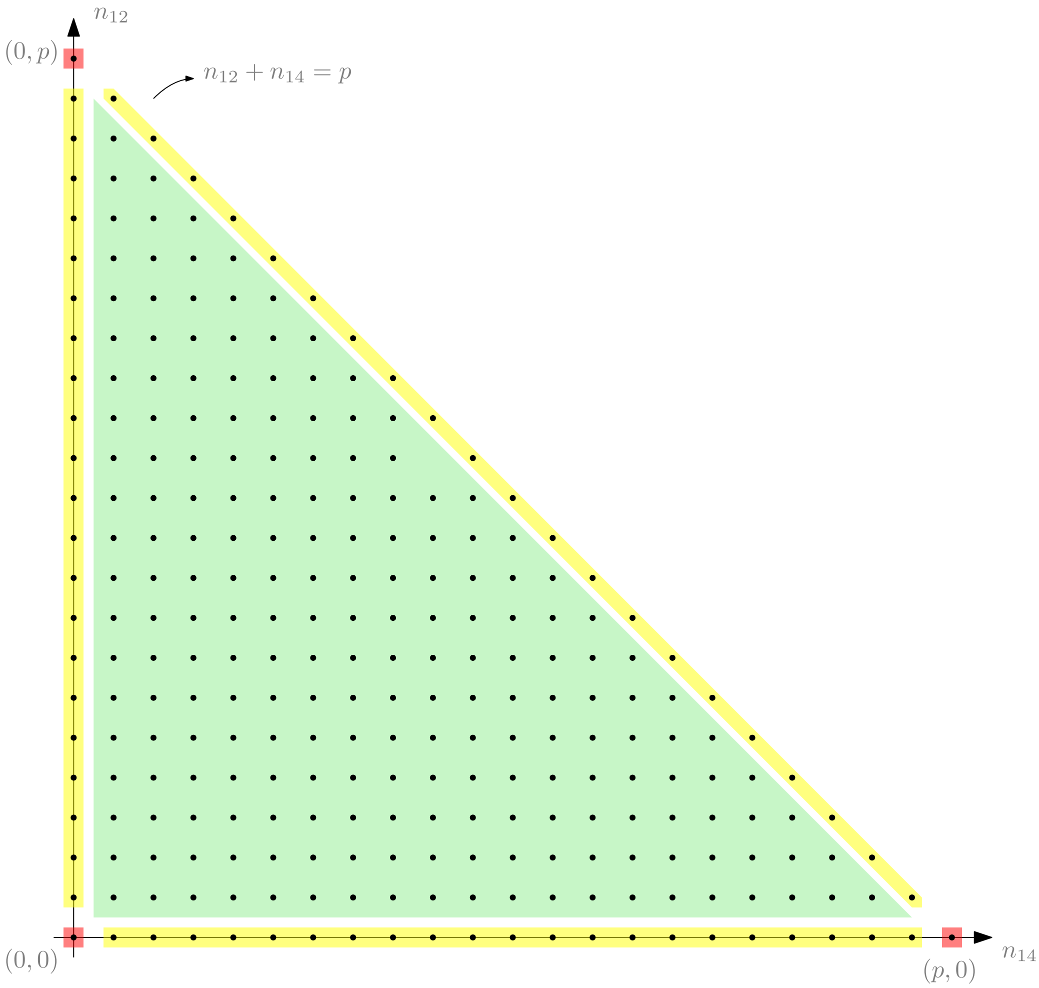

Figure 2.2.1: *A schematic depiction of the combinatorial factors appearing in the correlator. Green regions correspond to $C = 2$, yellow regions correspond to $C = 1$, and red regions correspond to $C = 0$.*

and an equivalent set of "$SO(6)$ cross ratios"

$$\sigma = \frac{y_{12}^2 y_{34}^2}{y_{13}^2 y_{24}^2}, \qquad \tau = \frac{y_{14}^2 y_{23}^2}{y_{13}^2 y_{24}^2}. \tag{2.2.11}$$

There is one piece remaining that needs to be determined - the combinatorial factor $C_{n_{12},n_{14}}$. This is given as

$$C_{n_{12},n_{14}} = \begin{cases} 0 & (n_{12}, n_{14}) = (0,0), (0,p), (p,0), \\ 1 & \text{only one of } n_{ij} = 0 \\ 2 & \text{otherwise} \end{cases} \tag{2.2.12}$$

We refer the reader to [Cor19] for a detailed analysis of the combinatorial factor. These conditions can be visualised in terms of a triangle depicting n_{12} and n_{14} as demonstrated in Fig. 2.2.1.

Once we have the expression for the four-point function, the question we want to address with respect to the problem at hand is can the expression for the four-point function be arrived at as an integral of an action of a worldsheet that lets us give an interpretation of a minimal area surface. To arrive at an answer to this question, we need a machinery to deal with strings in this regime of a highly curved background. We discuss a suitable way to handle strings on such backgrounds – the Strebel differential – in the next chapter.

Chapter 3

A Short Primer on Strebel Differentials in AdS/CFT

In this chapter we review existing work on a framework that allows us to go from free field correlators to the so-called Strebel differentials over the moduli space of Riemann surfaces, and lay the groundwork for calculating free field correlators as worldsheet correlators.

3.1 Physical Motivation

Imagine there exists some worldsheet that describes a free correlator of a field theory. What are the properties that are expected to be satisfied?

First, if we are working in the large N expansion of a gauge theory correlator M as

$$M = \sum_{g=0}^{\infty} f_g(\lambda) N^{2-2g}, \tag{3.1.1}$$

where g is the genus of the Riemann surface and λ is the t'Hooft coupling $\lambda = g^2 N$, we require that we focus specifically on a particular genus to recover the term $f_g(\lambda)$. We denote the space of such Riemann surfaces as $\mathcal{M}_g$ – the "moduli" space of Riemann surfaces with genus g. The Riemann surface is required to have punctures for the operator insertions. Including this, we have a moduli space that we denote using $\mathcal{M}_{g,s}$ where s is the number of punctures. Once we fix this, the next step is to fix the energies of the operators that are inserted on the worldsheet. These are added in by hand as residues around the marked points, leading to the "decorated moduli space" of Riemann surfaces, $\mathcal{M}_{g,s} \times \mathbb{R}_s^+$. It is expected that every Riemann surface $\Sigma_{g,s}$ in the moduli space contributes to $f_g(\lambda)$.

In order to map our correlators to this decorated moduli space, we utilise the isomorphism [Str84] between metric graphs and a specific kind of differentials on Riemann surfaces as follows [Gop04a, Gop04b, Gop05]. For a free theory, we achieve this by first gluing the homotopically equivalent edges of the Feynman graphs, intuitively leading to what are called as the skeleton graphs. These skeleton graphs are devoid of information beyond the fact that "two operators have propagators between them". The next step is to make a one-to-one association between these graphs and points in the decorated moduli space of worldsheets. This performed through the mathematical tool of Strebel differentials.

3.2 Strebel Differentials

A quadratic differential $\phi_S(z)dz^2$ is a differential on a Riemann surface $\Sigma_{g,s}$ in the moduli space $\mathcal{M}_{g,s}$. These differentials have the property that under a holomorphic redefinition of the worldsheet coordinate z as $z \to z'(z)$, they transform as

$$\phi_S(z)dz^2 = \phi'_S(z')dz'^2. \tag{3.2.1}$$

Using a quadratic differential, one can define a line element on the worldsheet as

$$dl = \sqrt{\phi_S}dz. \tag{3.2.2}$$

This line element in general can take complex values. Given a curve $\gamma(t)$ on this Riemann surface, we can define a horizontal curve as a curve satisfying

$$\phi(\gamma)\left(\frac{d\gamma}{dt}\right)^2 > 0. \tag{3.2.3}$$

These curves can be either closed or open, with the open ends necessarily ending on either a pole or a zero. When the quadratic differential has only double poles, these open curves only end on zeros and divide the full worldsheet into ring domains, with each ring domain consisting at most one double pole. These curves are called critical curves. Since these open curves bound the double poles, fixing the residues fixes the length of these bounding horizontal curves.

Thus we have for a specific Riemann surface with s marked points and genus g, a set of quadratic differentials such that they have $(4g+2s-4)$ zeros with $(6g+3s-6)$ curves ending on these zeros. Fixing the residue at the double poles leads to a unique quadratic differential on a specific Riemann surface, called the Strebel differential. In order to recover the number of Wick contractions between the operators, we associate the length of every critical curve of the Strebel differential with the number of Wick contractions of the two operators that lie on the sides of the critical curve. For more details, see Appendix A.

Once we have this information, how do we go from a sum over Feynman graphs to worldsheet correlators? We demonstrate this in the following section using the recent machinery of covering maps from the worldsheet to the boundary in AdS_3/CFT_2 [GGKM20].

3.3 Orbifold Correlators

To understand how Strebel differentials help in going from free correlators to a worldsheet description, we discuss the case of AdS_3/CFT_2. This lower dimensional version of the gauge-gravity duality has recently been explored widely [GGKM20, EGG19, EGG20, GG18] and gives us a firm ground to try and build our understanding of AdS_5.

The exact bulk theory is the $AdS_3 \times S^3 \times \mathbb{T}^4$, which is dual to the free symmetric orbifold CFT $\text{Sym}^N(\mathbb{T}^4)$ in the large N limit, analogous to the five dimensional version. There, the following four-point function of the twisted sector correlator is considered

$$G_{\text{orbifold}\, CFT_2} = \langle \sigma_{[w_1]}(x_1)\, \sigma_{[w_2]}(x_2)\, \sigma_{[w_3]}(x_3)\, \sigma_{[w_4]}(x_4) \rangle. \tag{3.3.1}$$

These w are similar to the operator energy p in $\mathcal{N} = 4$ SYM. These correlators are constructed using the Lunin and Mathur prescription where one exploits the conformal symmetry to lift these correlators via a covering map $\Gamma : \Sigma \to S^2$ to single valued functions on the covering surface Σ. In general, this covering surface can have any genus g, but we will choose to focus on the case where $g = 0$. The condition that these functions be single-valued translates to the requirement that near $\Gamma^{-1}(x_i) = z_i$, one has

$$\Gamma(z) \sim x_i + a_i^\Gamma (z - z_i)^{w_i}, \qquad z \sim z_i, \ \ \forall i. \tag{3.3.2}$$

Since σ are twisted sector ground states, after moving to the covering surface, they vanish and are invisible to the chiral fields. Therefore, we have translated the problem of calculating the correlator to the problem of finding the partition function of the CFT living on the covering surface, which is equal to the (exponential of the) conformal factor/anomaly associated with the covering map. This is given by the Liouville action

$$S_L(\Phi) = \frac{c}{96\pi} \int d^2 z \sqrt{-g}\, \left(g^{\mu\nu} \partial_\mu \Phi \partial_\nu \Phi + 2R\, \Phi \right), \tag{3.3.3}$$

with c being the central charge of the theory on a single $\mathbb{T}^4$. Therefore, we have the required correlator as

$$G_{\text{orbifold }CFT_2} = \sum_{\Gamma} \tilde{W}_\Gamma \exp[S_L(\Phi_\Gamma)] \tag{3.3.4}$$

with

$$\Phi_\Gamma = \log \partial_z \Gamma(z) + \log \partial_{\bar{z}} \bar{\Gamma}(\bar{z}). \tag{3.3.5}$$

The W_Γ are constants expected to be independent of Γ [DE20]. The number of covering maps that satisfy the conditions outlined above (and hence the number of different configurations of the worldsheet) is in general very large and scales with w. Thus in the large twist limit, we obtain a very large number of covering maps, and the sum over all these covering maps become an integral over the moduli space of the worldsheets. Further, by choosing the worldsheet metric in the Strebel gauge (see Eq. (3.2.2), one arrives at the conclusion that the integration measure over the moduli space is the Nambu-Goto action. For more details about the exact logic through which this correspondence emerges, we refer the reader to [GGKM20].

The key takeaway message of the above discussion that helps in providing the direction of approach in the case of AdS_5 is the following. Modulo the regularisation of the correlator, the classical Liouville action to leading order under large w limit becomes

$$S_L[\Gamma] = \frac{cK^2}{48\pi} \int d^2z |\phi_S(z)| \tag{3.3.6}$$

where $\phi_S(z)$ is the Strebel differential defined in terms of the covering map as

$$Ki\sqrt{\phi_S(z)} = \partial(\log \partial \Gamma(z)). \tag{3.3.7}$$

K is a parameter defined for genus zero as

$$K = 1 + \frac{1}{2}\sum_i (w_i - 1). \tag{3.3.8}$$

This leads to a direct association of the covering map to the Strebel differential. A continuation of the discussion in this chapter can be found in Appendix A where we calculate the orbifold correlator as Strebel areas to drive home the usefulness of Strebel differentials in arriving at a geometric understanding of free correlators.

This approach has all the nice properties (such as the appearance of a Nambu-Goto action) that one would seek for showing a Gross-Mende like behaviour in the tensionless limit for AdS_5. However, there is one major missing piece -

What does it mean to have a covering map from the worldsheet to the boundary of AdS_5?

From the discussion on Strebel differentials, it is evident that one of the most important aspects of the map from the worldsheet to the boundary (or vice versa) is the existence of the notion of holomorphicity. Unfortunately, the boundary of AdS_5 (i.e. S^4) cannot be endowed with a complex structure [NN97]. This prevents us from discussing the holomorphic nature of the map since the boundary is not a complex manifold. Therefore, is there no hope in trying to realise a generalisation of the method to AdS_5? The answer is an emphatic no!

While one cannot impose a complex structure on S^4, one does have the tool of twistors which are complex realisations of real manifolds. In the case of AdS_3, the boundary is itself a complex manifold (perhaps the simplest possible complex manifold), thus making it easier to deal with. But the lesson here is that the more natural way of approaching AdS/CFT in higher dimensions is perhaps through an auxiliary space of twistors. Using this, one must be able to arrive at a beautiful symphony of mathematical structures that allow us to reach the goal of showing a Gross-Mende like behaviour.

In the following chapter, we develop the mechanism that lets us translate everything to the language of twistors, frequently drawing inspiration from the three dimensional case to keep us grounded.

Chapter 4

Twistors

In the previous chapter, we stressed the importance of Strebel differentials in the context of the tensionless limit of AdS/CFT and discussed the appearance of twistors for the correspondence in higher dimensions. In this chapter, we provide a review of the twistor techniques for general spacetimes following [Ada18], refine it for the case of AdS [ASW16], and provide a translation of the results in three dimensions in the language of twistors.

4.1 General Twistor Theory

The primary tool that is used to construct the twistor space of any manifold is spinors. But before we move to spinors, we start by considering the easiest case of Minkowski spacetime to help us constructively arrive at spinors.

Minkowski space in Lorentzian signature is given by

$$ds^2 = \eta_{ab} dx^a dx^b = (dx^0)^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2. \tag{4.1.1}$$

We *complexify* the Minkowski space as $\mathbb{M}_{\mathbb{C}}$ with the same metric but where the coordinates are now allowed to take complex values. Here, there is no notion of a signature since one can take any slice and land at a desired signature. The convention of the metric here is, as the name suggests, just a convention.

Now, the spin group of this complexified Minkowski space is SO$(4, \mathbb{C})$, which is locally isomorphic to a double copy of SL$(2, \mathbb{C})$. One can write a vector in $\mathbb{M}_{\mathbb{C}}$ as a $(\frac{\mathbf{1}}{\mathbf{2}}, \frac{\mathbf{1}}{\mathbf{2}})$ representation of SL$(2, \mathbb{C})\times$SL$(2, \mathbb{C})$, or a pair of indices denoting a $(\mathbf{0}, \frac{\mathbf{1}}{\mathbf{2}})$ and $(\frac{\mathbf{1}}{\mathbf{2}}, \mathbf{0})$ representation respectively. This is the usual representation of the flat space coordinates in terms of the Pauli matrices

$$x^{\alpha\dot{\alpha}} = \frac{\sigma_a^{\alpha\dot{\alpha}}}{\sqrt{2}} x^a \tag{4.1.2}$$

The undotted indices live in the $(\frac{1}{2}, \mathbf{0})$ representation and are referred to as the negative chirality spinor indices. The dotted indices live in the $(\mathbf{0}, \frac{1}{2})$ representation and are referred to as the positive chirality spinor indices.

A nice consequence of writing spinors in this form is that the norm of a vector can now be written as

$$\eta_{ab}v^a v^b = 2\text{det}(v^{\alpha\dot{\alpha}}) \tag{4.1.3}$$

which means that a vector is null if and only if the rank of the $v^{\alpha\dot{\alpha}}$ matrix less than two, leading to

$$v^{\alpha\dot{\alpha}}_{\text{null}} = a^\alpha \tilde{a}^{\dot{\alpha}} \tag{4.1.4}$$

for some spinors a and $\tilde{a}$. This is perhaps a good time to try to connect with the twistor space of AdS.

4.2 Twistors for AdS

Now, consider the five-dimensional complex projective space $\mathbb{CP}^5$. By a counting of degrees of freedom, it is easy to convince that this space can be spanned by an four dimensional antisymmetric matrix X with the requirement that $X \sim \lambda X$. We can now define a metric on such a space as

$$ds^2 = -\frac{dX^2}{X^2} + \left(\frac{X.dX}{X^2}\right)^2 \tag{4.2.1}$$

where the products are defined in terms of a contraction of ^{AB} with the totally antisymmetric tensor ϵ_{ABCD}. This metric is not defined globally since on the hypersurface $X^2 = 0$, the metric becomes singular. Thus the metric is defined globally on $\mathbb{CP}^5 \backslash M$ where M is defined as

$$M = \{X \in \mathbb{CP}^5 | X^2 = 0\}. \tag{4.2.2}$$

It is this $\mathbb{CP}^5 \backslash M$ that describes AdS_5, with the subspace M as its four dimensional conformal boundary.

It is in fact interesting and convenient to note that this twistor space of AdS_5 is isomorphic to the *ambitwistor* space of AdS_5 defined as

$$\{(Z^A, Y_A) \in \mathbb{CP}^3 \times \mathbb{CP}^3 | Y \cdot Z = 0\} \tag{4.2.3}$$

where Z and Y are homogeneous coordinates of the two copies of $\mathbb{CP}^3$ with each having its own scaling rule. The two spaces are related to each other by the incidence relation

$$Z^A = X^{AB} Y_B. \tag{4.2.4}$$

This amounts to the translation of the adjoint representation of the AdS fields to a bifundamental representation.

The relevance of these ambitwistor spaces is as follows. Since the coordinates of the ambitwistor spaces can be represented as two vectors, we can work with these coordinates (in the case of $\mathbb{CP}^3$) as a pair of Weyl spinors of opposite chirality as

$$Z^A = (\mu^{\dot{\alpha}}, \lambda_\alpha), \tag{4.2.5}$$

and an analogous pair of conjugated spinors for Y. This rewriting helps in quantisation, since the currents of the theory can now be written as bilinears formed using these spinors. In the next section, we discuss this process for AdS_3, which lets us relate these ambitwistor variables to the covering map, giving a much need glimpse into what the AdS_5 situation would look like.

4.3 Twistors ⇔ Covering Maps

The key to relating the covering map to twistors is as follows. The currents of the WZW model that describes the AdS_3 side of the duality admits a free-field realisation of the model given by the symplectic bosons ξ^a, η^b satisfying the commutation relation

$$[\xi^a, \eta^b] = \epsilon^{ab} \tag{4.3.1}$$

where $a, b \in -, +$. These can be interpreted as the analogues of the ambitwistor variables for the three dimensional case[1]. The $SL(2, \mathbb{R})$ currents of this theory (which forms the bosonic part of the full $\mathfrak{psu}(1,1|2)$ algebra) are given as [DGGK21]

$$J^3 = -\eta^+\xi^- \tag{4.3.2}$$

$$J^\pm = \eta^\pm\xi^\pm, \tag{4.3.3}$$

with the condition that $\eta^+\xi^- = \eta^-\xi^+$. Defining our ambitwistors as

$$Z^A = (\xi^+, \xi^-), \quad Y_A = (-\eta^-, \eta^+), \tag{4.3.4}$$

we immediately see that the currents can be represented as the bilinears formed with Z and Y, and the condition that $\eta^+\xi^- = \eta^-\xi^+$ is equivalent to a tracelessness condition on the bilinears as

$$Y_A Z^A = -\eta^+\xi^- + \eta^-\xi^+ = 0. \tag{4.3.5}$$

[1]The exact set of fields also contains a fermionic part that we choose to ignore since we are interested in the bosonic sector.

This gives us a method to represent the currents of the theory in terms of the ambitwistor variables. These currents can also be represented in terms of the covering map, using the Wakimoto representation of the currents as follows [Wak86, EGG20]

$$J^+ = \beta, \tag{4.3.6}$$

$$J^3 = -\partial\Phi + \beta\gamma, \tag{4.3.7}$$

$$J^- = -2\partial\Phi\gamma + \beta\gamma\gamma - 3\partial\gamma, \tag{4.3.8}$$

where $(\beta(z), \Phi(z), \gamma(z))$ are holomorphic functions. In (cite), these functions were related to the covering map as

$$\gamma(z) = \Gamma(z), \quad \partial\Phi(z) = -\frac{\partial^2\Gamma(z)}{2\partial\Gamma(z)}, \tag{4.3.9}$$

and the function $\beta(z)$ is defined as

$$\beta(z) = -\frac{(\partial\Phi(z))^2}{\Gamma(z)}. \tag{4.3.10}$$

By plugging these expressions into the current and comparing with the current in terms of the twistor variables, we obtain the conditions that

$$\eta^+ = \frac{\partial\Phi - \beta\gamma}{\xi^-}, \tag{4.3.11}$$

$$\eta^- = \frac{-2\gamma\partial\Phi + \beta\gamma^2 - \partial\gamma}{\xi^-}, \tag{4.3.12}$$

$$\xi^+ = \frac{\beta}{\eta^+}, \tag{4.3.13}$$

$$\xi^- = \frac{\beta}{\partial\Phi - \beta\gamma}. \tag{4.3.14}$$

These expressions let us relate the ambitwistor variables to the covering map, up to a choice in ξ^-. Interestingly, the Schwarzian of the covering map can be written in terms of these variables as

$$Y \cdot \partial Z, \tag{4.3.15}$$

which is related directly to the Strebel differential. By choosing the covering map to be a rational function

$$\Gamma(z) = \frac{P(z)}{Q(z)}, \tag{4.3.16}$$

we make a judicious choice of ξ^- such that we obtain fields of equal weights.

Performing this, we obtain

$$\eta^+ = -\frac{1}{2W^{3/2}}(W'Q - 2Q'W), \tag{4.3.17}$$

$$\eta^- = \frac{1}{2W^{3/2}}(W'P - 2P'W), \tag{4.3.18}$$

$$\xi^+ = \frac{1}{2W^{3/2}}(W'Q - 2Q'W), \tag{4.3.19}$$

$$\xi^- = -\frac{1}{2W^{3/2}}(W'P - 2P'W). \tag{4.3.20}$$

where, W is the Wronskian of the two polynomials

$$W = P'Q - Q'P. \tag{4.3.21}$$

This provides us a complete picture of how the covering map relates to the ambitwistor variables, and hence the twistor representation of the boundary degrees of freedom. Emulating this procedure for the five dimensional case would amount to finding a similar set of symplectic bosons that lead to a representation of the currents of the theory in terms of the bilinears formed from these bosons. Then by comparing this current with the current of the theory in terms of the Poincaré coordinates, we expect the five dimensional analogue of Eq. (4.3.11) to emerge.

As discussed in section 4.2, the ambitwistor variables for the five dimensional case can be represented using two symplectic bosons - packaged as spinors - as

$$Y_I = (\mu_\alpha, \lambda_{\dot\alpha}), \quad Z^I = (\xi^\alpha, \eta^{\dot\alpha}), \tag{4.3.22}$$

with $\alpha, \dot\alpha \in \{+,-\}$. The generators of the bosonic part of the $\mathfrak{psu}(2,2|4)$ can be written as bilinears of the form

$$j_\alpha^{\dot\alpha} = \begin{pmatrix} \mu_\alpha \xi^\alpha & \lambda_{\dot\alpha}\xi^\alpha \\ \mu_\alpha \eta^{\dot\alpha} & \lambda_{\dot\alpha}\eta^{\dot\alpha} \end{pmatrix} \tag{4.3.23}$$

or in a slightly more illuminating form [GG21]

$$\mathcal{L}^\alpha_\beta = \mu_\beta \xi^\alpha - \frac{1}{2}\delta^\alpha_\beta \mu_\gamma \xi^\gamma, \tag{4.3.24}$$

$$\dot{\mathcal{L}}^{\dot\alpha}_{\dot\beta} = \lambda_{\dot\beta}\eta^{\dot\alpha} - \frac{1}{2}\delta^{\dot\alpha}_{\dot\beta}\lambda_{\dot\gamma}\eta^{\dot\gamma}, \tag{4.3.25}$$

$$\mathcal{P}^{\dot\alpha}_\beta = \eta^{\dot\alpha}\mu_\beta, \tag{4.3.26}$$

$$\mathcal{K}^\alpha_{\dot\beta} = \xi^\alpha \lambda_{\dot\beta}. \tag{4.3.27}$$

The ambitwistor condition corresponds to setting the overall $\mathfrak{u}(1)$ generator, $\frac{1}{2}Y_I Z^I$ to zero, which leads to the condition

$$\xi^-\mu^+ + \lambda^-\eta^+ = \xi^+\mu^- + \lambda^+\eta^-. \tag{4.3.28}$$

These generators, togther with the ambitwistor condition, thus simmer down to four generators $\mathcal{P}$ (equivalent to translations), four generators $\mathcal{K}$ (equivalent to the generators for special conformal transformations), and seven generators $\mathcal{L}$ and $\dot{\mathcal{L}}$, which are linear combinations of the Lorenz generators and the generator for dilatations. The remaining non-trivial task however, is finding a suitable expression for currents in terms of the Poincaré coordinates that would allow us to relate these ambitwistors to a notion of holomorphic maps. We discuss our approach, together with the aspects that make the task difficult, in chapter 5. In appendix B, we provide a short back-of-the-envelope calculation that would help in understand this business in five dimensions using a handmade holomorphic map.

Chapter 5

Conclusion and Future Directions

After a long and mathematically arduous ride, we ground the reader and give a recap of the goal and where we stand in the process of achieving the goal.

Our primary question is whether a Gross-Mende like behaviour can be observed in the case of tensionless strings (free field theory). More specifically, can the worldsheet dual of the four-point function

$$\mathcal{O}_p(x,y) = y_{i_1}\dots y_{i_p}\mathrm{Tr}\left(\Phi^{i_1}\dots\Phi^{i_p}\right), \tag{5.0.1}$$

be shown to localise to a minimal area worldsheet?

After providing a brief overview of the expressions for a four-point function of $\frac{1}{2}$–BPS operators on the field theory side in position space in Chapter 2, we changed gears and looked at the usefulness of Strebel differentials in dealing with this particular region of the parameter space of the gauge-gravity duality in Chapter 3. In particular, we related the Strebel differential on a worldsheet to the covering map that takes us from the worldsheet to the boundary, giving a firm physical definition of the Strebel differential. We further noted that the sum over covering maps translates to an integral over the moduli space of worldsheets with an integration measure that looks like the Nambu-Goto action. A short calculation that further explains this relation between sum over covering maps and integration over the moduli space is provided in Appendix A.

The key ingredient in the discussion on Strebel differentials in the case of AdS3/CFT$_2$ is the existence of the holomorphic covering map. Since the conformal boundary of the five-dimensional AdS spacetime cannot be endowed with a complex structure, we resorted to, and argued for, using twistor space to define holomorphic maps. We demonstrated the translation of existing results in AdS3/CFT$_2$ in terms of the ambitwistor space variables, which provides us a firm ground to base our generalisation to higher dimensions.

We showed the form of the currents for the higher dimensional case in terms of the ambitwistor variables. The remaining nontrivial task is to find a suitable expression for the currents in terms of the Poincaré coordinates. This would allow us to compare the current in terms of the ambitwistor variables, leading to a better understanding of what these holomorphic maps look like.

The nontriviality of the task stems from the fact that in the AdS_3 case, the bulk is itself a group manifold, leading to an easy derivation of the currents. AdS_5 however, is not a group manifold but a coset manifold. Thus, the currents depend on the coset representative, which can be thought of as the multiple possible coordinates on AdS_5. Thus finding the current boils down to finding a suitable coset representative, which is more of a lucky draw. We still hope however, that such a suitable candidate can be found by making the process less and less random by exploiting the structure of the coset itself and comparing with the slices in twistor space.

Looking back, we have made tremendous progress in arriving at a framework that allows us to answer the main thesis. In the process, we have discovered many gems that we believe would lead to rapid progress in this yet unexplored region of the parameter space. We must note that although we have reached closer to the goal than before, there is a lot of progress yet to be made. This applies to both AdS_3 and, obviously, AdS_5.

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Appendix A

Orbifold Correlators from Strebel Area

A.1 Correlators

As discussed in Chapter 3, we are interested in the four-point function of twist fields

$$G_{\text{orbifold } CFT_2} = \langle \sigma_{[w_1]}(x_1)\, \sigma_{[w_2]}(x_2)\, \sigma_{[w_3]}(x_3)\, \sigma_{[w_4]}(x_4) \rangle$$

in the free symmetric product orbifold CFT. The correlator is given by a sum over contributions from all possible branch covering maps $\Gamma : \mathbb{CP}^1[z] \to \mathbb{CP}^1[x]$ with the specified branching behaviour at the positions of the twist fields-

$$\Gamma(z) \sim x_i + a_i^\Gamma\,(z - z_i)^{w_i} \quad z \sim z_i$$

as

$$G_{\text{orbifold } CFT_2} = \sum_\Gamma \tilde{W}_\Gamma \exp[S_L(\Phi_\Gamma)] \prod_{i=1}^{4} |a_i^\Gamma|^{-2\Delta h_i}, \tag{A.1.1}$$

where the Lioville action $S_L(\Phi_\Gamma)$ is given by

$$S_L(\Phi) = \frac{c}{96\pi} \int d^2 z \sqrt{-g}\,(g^{\mu\nu}\partial_\mu \Phi \partial_\nu \Phi + 2R\,\Phi)\,, \tag{A.1.2}$$

with,

$$\Phi_\Gamma = \log \partial_z \Gamma(z) + \log \partial_{\bar{z}} \bar{\Gamma}(\bar{z}).$$

Here, $\Delta h_i = (h_i - h_i^0)$ is the the conformal dimension from extra dressings over the bare twist-field of dimension $h_i^0 = \frac{c}{24}(w - \frac{1}{w})$. For our calculations, we take it to be zero (that is, the ground state). It has been speculated that W_Γs are independent of Γ [DE20]. In the AdS_3/CFT_2 set-up [GGKM20], the Strebel differential was

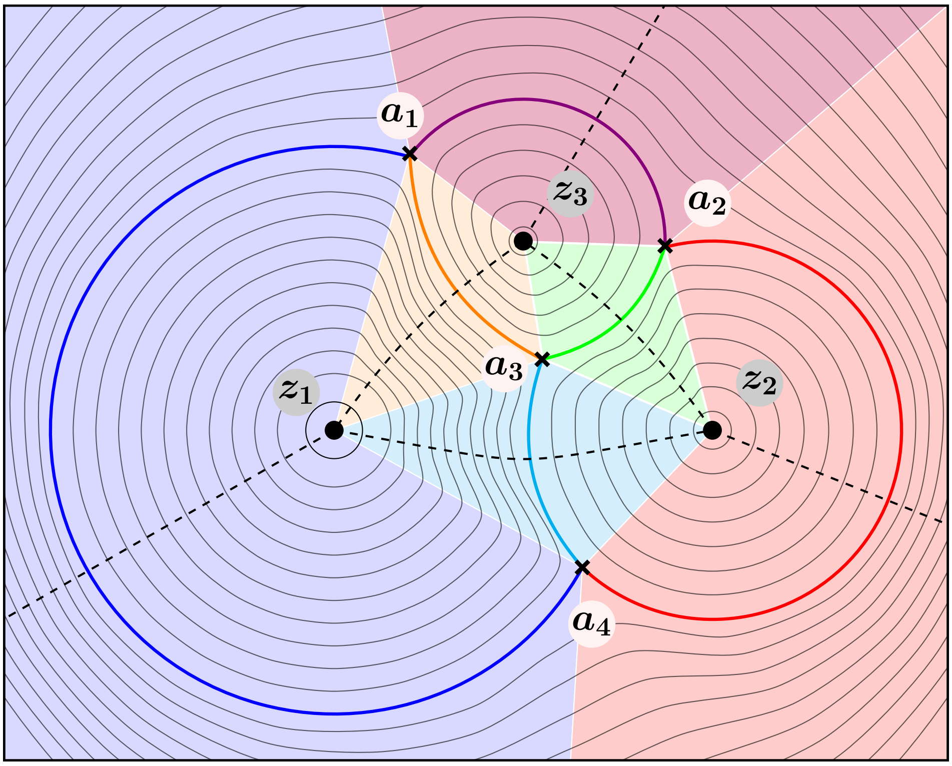

Figure A.2.1: *A depiction of the ring domains and the strip geometries of the worldsheet*

identified with the covering map as

$$\partial_z (\log \partial_z \Gamma(z)) (= \partial_z \Phi_\Gamma) = K\, i \sqrt{\phi_S(z)} \tag{A.1.3}$$

where, $\phi(z) dz^2$ is the Strebel differential on the worldsheet. It was further realised that Γ is specified by the A and B-cycle periods l_A and l_B. Also we note that, $a_i^\Gamma = \partial^{w_i}\Gamma(z_i)$. Putting all these together, the correlator becomes,

$$\begin{aligned} G_{\text{orbifold } CFT_2} = &\sum_{l_A + l_B \leq 1} \tilde{W}_\Gamma \exp\left(-N^2 \frac{c}{48\pi} 2 \int d^2 z g^{z\bar{z}} \sqrt{\phi(z)\bar{\phi}(\bar{z})} \right. \\ &\left. + \frac{c}{48\pi} \int d^2 z \sqrt{-g} R\, \Phi_\Gamma - \sum_{i=1}^{4} \Delta h_i \log |\partial^{w_i} \Gamma(z_i)|^2 \right) . \end{aligned} \tag{A.1.4}$$

A.2 Strebel Area

Area of the quadralateral between the poles z_1 and z_2,

$$\begin{aligned} A_{12} &= A_{12}^{z_1} + A_{12}^{z_2} \\ &= \int_{a_4}^{a_3} d\lambda\, \rho^{z_1}(\lambda)\, \Pi_{\lambda z_1} + \int_{a_3}^{a_4} d\lambda\, \rho^{z_2}(\lambda)\, \Pi_{\lambda z_2}, \end{aligned} \tag{A.2.1}$$

where

$$\Pi_{\lambda z_i} = \int_{|\lambda-\delta_\lambda|}^{|z_i-\delta_{z_i}|} \sqrt{\phi_S(z)}dz. \tag{A.2.2}$$

Here λ denotes the positions of the poles of the covering map, which form the branch cuts of $\sqrt{\phi_S(z)}$ and z_i's are the insertion points of twist operators, which are double poles $\phi_S(z)$. $\rho^{z_1}(\lambda)$ and $\rho^{z_2}(\lambda)$ denote the density of poles along the cut $a_4 - a_3$ in the two half-strips $z_1 a_4 a_3$ and $z_2 a_3 a_4$ respectively.

In the large K limit, from the AdS_3/CFT_2 dictionary (A.1.3),

$$\sqrt{\phi_S(z)} = -\frac{i}{K}\partial(\log \partial\Gamma(z)).$$

Using this we get,

$$\Pi_{z_i\lambda} = -\frac{i}{K}\log\left[\frac{\partial\Gamma(|z_i-\delta_{z_i}|)}{\partial\Gamma(|\lambda-\delta_\lambda|)}\right]. \tag{A.2.3}$$

From the poles of the Strebel differential:

We have cut off discs of same radius ϵ at the insertion points of the twist fields on the x-sphere. The corresponding discs on the z-sphere will have different radii δ_{z_i} at z_i. For $z_i \neq \infty$, we have

$$\Gamma(|z_i-\delta_{z_i}|) \sim x_i + a_i^\Gamma\, \delta_{z_i}^{w_i} \Rightarrow \epsilon = a_i^\Gamma\, \delta_{z_i}^{w_i}. \tag{A.2.4}$$

Thus we now have the result,

$$\partial\Gamma(|z_i-\delta_{z_i}|) = w_i\, a_i^\Gamma\, \delta_{z_i}^{w_i-1} = w_i\,(a_i^\Gamma)^{1/w_i}\,\epsilon^{\frac{w_i-1}{w_i}}. \tag{A.2.5}$$

This gives (for $i \neq 4$)

$$\log\partial\Gamma(|z_i-\delta_{z_i}|) = \log[w_i(a_i^\Gamma)^{1/w_i}] + \epsilon\text{-dependent part}. \tag{A.2.6}$$

Now we consider the case $z_4 = \infty$. We have the following branching behaviour,

$$\begin{aligned} &\frac{1}{\Gamma(z)} - \frac{1}{x_4} = \frac{1}{\widetilde{a}_4^\Gamma}\left(1-\frac{1}{x_4}\right)\left(\frac{1}{z}-0\right)^{w_4} \qquad z\sim\infty \\ &\Rightarrow \Gamma(z) \sim x_4 + a_4^\Gamma z^{-w_4} \quad \text{with} \quad a_4^\Gamma = \frac{x_4(1-x_4)}{\widetilde{a}_4^\Gamma}. \end{aligned} \tag{A.2.7}$$

Also note that, for $x_4 = \infty$, we get back usual branching rule: $\Gamma(z) \sim \widetilde{a}_4^\Gamma\, z^{w_4}$. Now on the worldsheet, we have cut off a disc at $z = \infty$ as $|z| \geq \frac{1}{\delta_\infty}$. So

$$\epsilon = \Gamma(z) - x_4 = a_4^\Gamma\left(\frac{1}{z}\right)^{w_4} = a_4^\Gamma\,\delta_\infty^{w_4}, \tag{A.2.8}$$

and taking derivative with respect to $1/z$ (which is the local co-ordinate near $z_4 =$

∞),

$$\partial_{1/z}\Gamma(|\infty - 1/\delta_\infty|) = w_4 a_4^\Gamma \delta_\infty^{w_4-1} = w_4\,(a_4^\Gamma)^{1/w_4}\,\epsilon^{\frac{w_4-1}{w_4}}. \tag{A.2.9}$$

This gives,

$$\log\partial_{1/z}\Gamma(|\infty - 1/\delta_\infty|) = \log[w_4\,(a_4^\Gamma)^{1/w_4}] + \epsilon\text{-dependent part.} \tag{A.2.10}$$

From the branch cuts of the Strebel differential:

We have cut off the x-plane at a large radius $|x| = \frac{1}{\eta}$ and there will be corresponding disc of infinitesimal radius δ_λ at the branch points of $\sqrt{\phi_S(z)}$. We also recall,

$$\begin{aligned}&\Gamma(z) \sim \frac{C_a}{z-\lambda_a} \qquad z\sim\lambda_a\ \forall\, a = 1,...,K,\\ &\Rightarrow \eta = \frac{1}{C_a}\delta_a.\end{aligned} \tag{A.2.11}$$

Thus,

$$\partial\Gamma(|\lambda - \delta_a|) = -\frac{C_a}{\delta_a^2} = -C_a^{-1}\eta^{-2}. \tag{A.2.12}$$

Taking logarithm, this gives

$$\log\partial\Gamma(|\lambda - \delta_\lambda|) = -\log C_\lambda + \eta\text{ dependent part.} \tag{A.2.13}$$

Net result:

Combining all these, the perpendicular length in (A.2.3) is given by,

$$\Pi_{\lambda z_i} = -\frac{i}{K}\left[\log[w_i(a_i^\Gamma)^{1/w_i}] + \log C_\lambda\right] + \epsilon,\,\eta\text{ dependent part.} \tag{A.2.14}$$

Hence A_{12} is gievn by,

$$\begin{aligned}A_{12}^{z_1} &= \int_{a_4}^{a_3} d\lambda\,\rho^{z_1}(\lambda)\,\Pi_{\lambda z_1} = -i\left[l^{z_1}_{a_4a_3}\log[w_1(a_1^\Gamma)^{1/w_1}] + \frac{1}{K}\log\prod_{a=1}^{n_{12}} C_a\right],\\ A_{12}^{z_2} &= \int_{a_3}^{a_4} d\lambda\,\rho^{z_2}(\lambda)\,\Pi_{\lambda z_2} = -i\left[l^{z_2}_{a_3a_4}\log[w_2(a_2^\Gamma)^{1/w_2}] + \frac{1}{K}\log\prod_{a=1}^{n_{12}} C_a\right],\end{aligned} \tag{A.2.15}$$

where we have used,

$$\frac{1}{N}\int_{a_k}^{a_l} d\lambda\,\rho^{z_1,z_2}(\lambda) = l^{z_1,z_2}_{a_ka_l} \quad\text{and}\quad \int_{a_k}^{a_l} d\lambda\,\rho^{z_1,z_2}(\lambda)\log C_\lambda = \log\prod_{a=1}^{n_{a_ka_l}} C_a. \tag{A.2.16}$$

The Strebel area which is the sum of areas of the quadrilaterals when they are glued together (from the figure),

$$
\begin{aligned}
A &= A^{z_1}_{12} + A^{z_2}_{12} + A^{z_1}_{13} + A^{z_3}_{13} + A^{z_1}_{14} + A^{z_4}_{14} + A^{z_2}_{23} + A^{z_3}_{23} + A^{z_2}_{24} + A^{z_4}_{24} + A^{z_3}_{34} + A^{z_4}_{34} \\
&= -\frac{i}{K} \log \left[(a^{\Gamma}_1)^{(w_1-1)/w_1} (a^{\Gamma}_2)^{(w_2-1)/w_2} (a^{\Gamma}_3)^{(w_3-1)/w_3} (a^{\Gamma}_4)^{(w_4-1)/w_4} \prod_{a=1}^{N} C_a^2 \right],
\end{aligned}
\tag{A.2.17}
$$

where we used,

$$
\begin{aligned}
l^{z_1}_{a_4 a_3} + l^{z_1}_{a_3 a_1} + l^{z_1}_{a_1 a_4} &= \frac{w_1 - 1}{K}, \\
l^{z_2}_{a_3 a_4} + l^{z_2}_{a_4 a_2} + l^{z_2}_{a_2 a_3} &= \frac{w_2 - 1}{K}, \\
l^{z_3}_{a_3 a_2} + l^{z_3}_{a_2 a_1} + l^{z_3}_{a_1 a_3} &= \frac{w_3 - 1}{K}, \\
l^{z_4}_{a_4 a_1} + l^{z_4}_{a_1 a_2} + l^{z_4}_{a_2 a_4} &= \frac{w_4 - 1}{K}.
\end{aligned}
\tag{A.2.18}
$$

Note the last equality, where even though the contour looks clock-wise from the figure, when viewed from infinity (or the north-pole on the Riemann sphere in the stereographic projection), it is clock-wise, so the sign on the r.h.s is +ve.

Hence our final correlator of the orbifold CFT has the form,

$$
\begin{aligned}
G_{\text{orbifold CFT}} &= \sum_{\Gamma} \exp\left[+i\, N \frac{c}{24} \text{Area} \right] \\
&= \sum_{\Gamma} C_* \, (a^{\Gamma}_1)^{-\frac{c(w_1-1)}{24 w_1}} (a^{\Gamma}_2)^{-\frac{c(w_2-1)}{24 w_2}} (a^{\Gamma}_3)^{-\frac{c(w_3-1)}{24 w_3}} (a^{\Gamma}_4)^{-\frac{c(w_4-1)}{24 w_4}} \prod_{a=1}^{N} C_a^{-c/12}.
\end{aligned}
\tag{A.2.19}
$$

This matches precisely with the results found in [LM01, DE20].

Appendix B

Holomorphic Map Example

B.1 Covering Map Picture

In order to compute the four-point correlation function,

$$\langle \mathcal{O}_{p_1}(x_1, y_1)\mathcal{O}_{p_2}(x_2, y_2)\mathcal{O}_{p_3}(x_3, y_3)\mathcal{O}_{p_4}(x_4, y_4)\rangle, \tag{B.1.1}$$

we propose a covering map from $\mathbb{S}^2 \to \mathbb{R}^4$ with the explicit form:

$$X^i(w) = \frac{x_a^i + x_b^i}{2} + \frac{x_a^i - x_b^i}{2i}\,\tan(\pi w), \quad w \in [0, n_{ab}], \tag{B.1.2}$$

where $i = 1, \cdots, 4$. This is a hand constructed covering map based on the expected properties of the diamond-like strip region formed by the two poles and two zeros. Denoting

$$X^i(w_a) = \widetilde{x}_a^i, \tag{B.1.3}$$

and using the following asymptotic behaviour of $\tan z$

$$\begin{aligned} i\tan z_1 &= -1 + 2\,e^{2iz_1} \quad z_1 \to i\infty, \\ i\tan z_2 &= 1 - 2\,e^{-2iz_2} \quad z_2 \to -i\infty, \end{aligned} \tag{B.1.4}$$

we get

$$\begin{aligned} \widetilde{x}_a^i - x_a^i &\sim -(x_a^i - x_b^i)\,\exp(2\pi i w_a), \quad w_a \sim i\infty \\ \widetilde{x}_b^i - x_b^i &\sim (x_a^i - x_b^i)\,\exp(-2\pi i w_b), \quad w_b \sim -i\infty. \end{aligned} \tag{B.1.5}$$

In the euclidean signature,

$$\begin{aligned} &\sum_{i=1}^{4}(\widetilde{x}_a^i - x_a^i)(\widetilde{x}_b^i - x_b^i) \sim -\left[\sum_{i=1}^{4} x_{ab}^i x_{ab}^i\right]\exp[2\pi i(w_a - w_b)] \\ &\Rightarrow i(w_a - w_b) = -\frac{1}{2\pi}\log[x_{ab}^2] + \log\epsilon. \end{aligned} \tag{B.1.6}$$

Thus, the Strebel area of the strip between X_a and X_b gives the space-time propagator,

$$\exp[A_{ab}] = \exp[2\pi i\, n_{ab}(w_a - w_b)] = \left(\frac{1}{x_{ab}^2}\right)^{n_{ab}}. \tag{B.1.7}$$

As shown above,

$$X^i(w) \sim x_a^i - (x_a^i - x_b^i)e^{2\pi i w} \quad w \sim w_a \tag{B.1.8}$$

Near the pole of the strebel differential at $w \sim w_a$, the standard parametrisation is related with the canonical one as,

$$z(w) \sim C_a \exp\left[\frac{2\pi i w}{p_a}\right] \quad w \sim w_a \tag{B.1.9}$$

where

$$C_a = \exp\left[2\pi i \frac{L_1 + L_2 + \cdots + L_{n-1}}{p_a}\right] \tag{B.1.10}$$

with $\{L_j\}_{j=1,\cdots,n-1} \equiv \{n_{cd}\}_{(c,d)\neq(a,b)}$ the set of wick contractions with $\mathcal{O}_{p_a}(X_a)$. Note that $z(w_a) = 0$. Hence we have the following branching behaviour at the operator insertion at X_a,

$$X^i(z) \sim x_a^i - C_a^{-p_a} x_{ab}^i\, z^{p_a}, \quad z \sim 0. \tag{B.1.11}$$

Similarly,

$$X^i(w) \sim x_b^i + (x_a^i - x_b^i)e^{-2\pi i w} \quad w \sim w_b \tag{B.1.12}$$

and $z(w) \sim C_b \exp\left[\frac{2\pi i w}{p_b}\right]$ near $w \sim w_b$. Thus,

$$X^i(z) \sim x_b^i + C_b^{p_b} x_{ab}^i\, z^{-p_b}, \quad z \sim \infty. \tag{B.1.13}$$

B.1.1 Two-Point Function

For two-point functions,

$$p_a = p_b (= p)$$

and the global covering map with two operator insertions at $x_a = 0$ and $x_b = \infty$ is

$$\widetilde{X}^i(z) = z^p \tag{B.1.14}$$

where $z = C \exp\left[\frac{2\pi i w}{p}\right]$. Now we can rewrite our covering map (B.1.2) as,

$$\begin{aligned} X^i(w) &= \frac{x_b\, e^{2\pi i w} + x_a^i}{e^{2\pi i w} + 1} \\ &= \frac{x_b^i\, z^p + x_a^i\, c^p}{z^p + c^p} \end{aligned} \tag{B.1.15}$$

Thus our covering map is the same with the global covering map after the mobius transformation,

$$X^i(w) = \frac{x_b^i \widetilde{X}^i(z) + x_a^i c^p}{\widetilde{X}^i(z) + c^p} \tag{B.1.16}$$

B.2 Twistors

As discussed in Chapter 4, we consider the complex projective space $\mathbb{CP}^5$ with homogeneous co-ordinates X^{IJ} which is a skew-symmetric 4×4 matrix with the identification $X \sim \lambda X$ with $\lambda \in \mathbb{C}^*$. The simplest holomorphic well-defined metric in $\mathbb{CP}^5$ which is invariant under $X \to \lambda X$

$$ds^2 = -\frac{dX^2}{X^2} + \left(\frac{X \cdot d\,X}{X^2}\right)^2 \tag{B.2.1}$$

where the contraction of indices is performed using ϵ_{IJKL}. This metric is not global, with a singular behaviour on the manifold

$$M = \{X^2 = 0 \,|\, X \in \mathbb{CP}^5\} \tag{B.2.2}$$

$\mathbb{CP}^5 \backslash M$ with the above metric (B.2.1) is equivalent to the complexified AdS_5 whereas the quadric M corresponds to the four-dimensional conformal boundary. We can write the points in M in the following fashion,

$$X^{IJ} = \begin{pmatrix} \epsilon_{\alpha\,\beta} & -i\,x^{\dot\beta}{}_{\alpha} \\ i\,x^{\dot\alpha}{}_{\beta} & x^2 \epsilon^{\dot\alpha\dot\beta} \end{pmatrix} \tag{B.2.3}$$

with

$$\epsilon_{\alpha\beta} = -\epsilon^{\alpha\beta} = \epsilon_{\dot\alpha\dot\beta} = -\epsilon^{\dot\alpha\dot\beta} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{B.2.4}$$

This form ensures $X^2 = 0$ i.e $\det(X) = 0$. Writing explicitly, we have

$$X^{IJ} = \begin{pmatrix} 0 & -1 & -i\chi & iy^- \\ 1 & 0 & -iy^+ & i\bar\chi \\ i\chi & iy^+ & 0 & x^2 \\ -iy^- & -i\bar\chi & -x^2 & 0 \end{pmatrix}, \tag{B.2.5}$$

where $\chi = x^1 + ix^2$, $y^+ - x^0 - x^3$ and $y^- = x^0 + x^3$.

Our covering map

$$x^i = \frac{x_a^i c^p + x_b^i z^p}{z^p + c^p}, \quad i = 0, \cdots, 3, \tag{B.2.6}$$

where

$$z = c\exp\left[\frac{2\pi i w}{p}\right], \tag{B.2.7}$$

and w is the canonical co-ordinate. We list the following immediate outcomes,

$$\begin{aligned}\chi &= \frac{\chi_a c^p + \chi_b z^p}{z^p + c^p},\\ y^{\pm} &= \frac{y_a^{\pm} c^p + y_b^{\pm} z^p}{z^p + c^p},\\ x^2 &= \frac{x_a^2 c^{2p} + x_b^2 z^{2p} + 2x\cdot y z^p c^p}{z^p + c^p}.\end{aligned} \tag{B.2.8}$$

Corresponding to each space-time point, there is a line in twistor space given by the incidence relation

$$X_{IJ} Z^J = 0. \tag{B.2.9}$$

That is, the homogeneous co-ordinates of the twistor variable Z^I consitute the kernel of X_{IJ}. We can explicitly find them from the above expression to be

$$\begin{pmatrix} i\chi \\ iy^+ \\ 0 \\ x^2 \end{pmatrix}, \begin{pmatrix} iy^- \\ i\bar{\chi} \\ x^2 \\ 0 \end{pmatrix} \tag{B.2.10}$$

We normalize the twistors in the following manner

$$Z_1 = \begin{pmatrix} i(\chi_a c^p + \chi_b z^p)(z^p + c^p) \\ i(y_a^+ c^p + y_b^+ z^p)(z^p + c^p) \\ 0 \\ x_a^2 c^{2p} + x_b^2 z^{2p} + 2x\cdot y z^p c^p \end{pmatrix}, Z_2 = \begin{pmatrix} i(y_a^- c^p + y_b^- z^p)(z^p + c^p) \\ i(\bar{\chi}_a c^p + \bar{\chi}_b z^p)(z^p + c^p) \\ x_a^2 c^{2p} + x_b^2 z^{2p} + 2x\cdot y z^p c^p \\ 0 \end{pmatrix}. \tag{B.2.11}$$

Note that the twistor fields are polynomial of z (i.e the ”worldsheet” co-ordinate) in this normalization. We can reconstruct the original matrix X^{IJ} from this null space,

$$X^{IJ} = -\frac{1}{x^2(z^p + c^p)^2}(Z_1^I Z_2^J - Z_2^I Z_1^J). \tag{B.2.12}$$

The twistor space is charted by the lines

$$Z^I = v^1[z] Z_1^I + v^2[z] Z_2^I, \tag{B.2.13}$$

where $v^a = (v^1, v^2)[z]$ are the local co-ordinates on the line. These lines define a surface of complex dimension two in the twistor space with boundaries at $z \to 0$ and $z \to \infty$ which is the anologue of the covering surface in twistor spcae $\mathbb{CP}^3$. The twistors corresponding to the points x_a and x_b are given by,

$$
\begin{aligned}
Z_{1a} &= c^{2p} \begin{pmatrix} i\chi_a \\ iy_a^+ \\ 0 \\ x_a^2 \end{pmatrix}, \; Z_{2a} = c^{2p} \begin{pmatrix} iy_a^- \\ i\bar{\chi}_a \\ x_a^2 \\ 0 \end{pmatrix}, \quad z \sim 0, \\
Z_{1b} &= z^{2p} \begin{pmatrix} i\chi_b \\ iy_b^+ \\ 0 \\ x_b^2 \end{pmatrix}, \; Z_{2b} = z^{2p} \begin{pmatrix} iy_b^- \\ i\bar{\chi}_b \\ x_b^2 \\ 0 \end{pmatrix}, \quad z \sim \infty.
\end{aligned}
\tag{B.2.14}
$$