Supersymmetric nonlinear coherent states

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A dissertation submitted for the partial fulfillment of

BS-MS degree in Science

Indian Institute of Science Education and Research Mohali

May 2021

Certificate of Examination

This is to certify that the dissertation titled "Supersymmetric nonlinear coherent states" submitted by Mr. Subhamoy Deb (MS16126) for the partial fulfilment of BS-MS dual degree programme of the Institute, has been examined by the thesis committee duly appointed by the Institute. The committee finds the work done by the candidate satisfactory and recommends that the report be accepted.

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Declaration

The work presented in this dissertation has been carried out by me under the guidance of Dr. Sanjib Dey at the Indian Institute of Science Education and Research Mohali.

This work has not been submitted in part or in full for a degree, a diploma, or a fellowship to any other university or institute. Whenever contributions of others are involved, every effort is made to indicate this clearly, with due acknowledgement of collaborative research and discussions. This thesis is a bonafide record of original work done by me and all sources listed within have been detailed in the bibliography.

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In my capacity as the supervisor of the candidate's project work, I certify that the above statements by the candidate are true to the best of my knowledge.

Dr. Sanjib Dey

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May 30, 2021

Acknowledgement

This is where I pour my heart out to express my deepest gratitude and respect to the people who have guided me and helped me throughout my project. First of all, I extend my gratefulness heartily to my respected advisor Dr. Sanjib Dey for giving me the opportunity to carry out my thesis work under his supervision. I have always found him to be extremely passionate about his science and his unparalleled competence in it is the source of inspiration for me. I am really thankful to him for his patience, attention and guidance he put for the accomplishment of the project.

I convey my gratitude to the most important persons of my life. My maa, baba and bhai for their never-ending love, support and encouragement which has played the greatest role in each and every accomplishment in my life. I am blessed to have dadu, thamma, dima my Aunts and uncles and cousins in my life. Their unconditional compassion has given me the strength throughout the journey.

I take this opportunity to thank my friends here in IISER Mohali as well as my school friends. Gyana and Somnath are friends in my life who have always been there by my side be it in terms of any academic help, in a medical emergency or during any other crisis. Few of my school friends, Arnab, Rahut, bunu are such pillars of my life whom I will remain ever indebted to for their unconditional support and inspiring attitude. I would like to thank my seniors here for the valuable discussions I have had with them regarding my thesis. Their important suggestions and guidance has immensely helped me to shape my ideas.

Above all I extend my soulful regards and gratefulness to the almighty for everything I have got.

Subhamoy Deb

Contents

Abstract

Coherent states are constructed for supersymmetric partner Hamiltonian, starting from an initial Hamiltonian. For this purpose one crucial insight is to analyse the algebra of the supersymmetric partner Hamiltonians. these algebras are not exactly the same as that of the initial Hamiltonian. However, characteristic features of the initial algebra seem to inherit to the algebra associated with the SUSY partner Hamiltonians. Then coherent states are formed from the definition of being the eigenstates of the annihilation operators. Then the initial Hamiltonian is perturbed and it is checked how this perturbation gets inherited by its supersymmetric partners. Finally, the coherent states are constructed for the perturbed SUSY partner Hamiltonians.

Chapter 1

Introduction

One of the pillars of modern physics, namely quantum mechanics, was invented during the first few decades of twentieth century. Almost immediately after its advent as a fundamental theory of nature its applications in explaining the results of different small scale experiments started to be explored. For this purpose physicists tried to solve the Schrodinger equation, which was published in the year 1926, for different simple potentials. Among these the quantum harmonic oscillator is a significant one. People that time tried to solve it analytically and became successful in the endeavour. It is an exactly solvable potential and we get a solution for it involving Hermite polynomial. Later Paul Dirac (as mentioned in many sources) developed the ladder operator method for solving the same. In this method we typically use two operators called raising or lowering operators or creation and annihilation operators respectively. These operators increase or decrease the quantum number describing a state of a system. Eventually we saw that it is way simpler to solve the harmonic oscillator problem with the ladder operator method. This method since then has been extensively used to solve Schrodinger equation for harmonic oscillator potential. What today we known as coherent states are the eigenstates of the annihilation operator. The term coherent state was coined by Roy Jay Glauber. These states are also called Glauber states.

Apart from their algebraic definition of being the eigenstates of annihilation operators coherent states have different defining features too. For example they are the 'most classical' quantum states. It is said so because of the followoing reasons. The fluctuations in the fractional uncertainty for the photon number decrease with increasing in average photon number and if the average photon number increases the state gets localized in phase. The coherent states are moreover not squeezed states. This implies that the family of coherent states distribute the operator uncertainties symmetrically between two non-commuting observables. In quadrature phase space coherent states form a circle which is not squeezed to ellipse. The uncertainties between two quadrature operators are equally distributed.

One limitation of our understanding of coherent states and its application since quite recent days was that it we knew it only in the context of linear systems like quantum harmonic oscillators. For this systems has eqully spaced energy levels and the expression of energy is linearly proportional to the photon number. These systems obey Heisenberg-Weyl algebra which is characterized by the usual commutation relations of the annihilation and creator operators and the Hamiltonian, we see in harmonic oscillator problem. However, it is a pretty recent development that coherent states are far more general than we used to think of it. Coherent states can be constructed for any system, be it linear or non-linear. If we study the way of construction of coherent states for a system closely, we see that it depends on the commutation relations of the system or in general on the algebra ruling the system. This is due to the fact that coherent states are eigenstates of the annihilation operators and annihilation operators are part of this algebra. For harmonic oscillator this algebra is of course the Heisenberg-Weyl algebra. This observation leads to the idea that even when we try to construct coherent state for a non-linear system it must depend on the algebra governing the same. Thus we need to first analyse the non-linear algebra of the system before constructing the coherent states. Once we do so we observe that non-linear algebras can be further classified in intrinsic and natural algebra and the annihilation operator is basically an algebraic deformation to the linear Heisenberg-Weyl annihilation operator. This results in the altered form of its eigenstate or in other words the coherent states.

On the other hand Supersymmetric quantum mechanics (SUSYQM) has proven itself to be very successful in reformulating the framework of standard quantum mechanics and describing it in terms of symmetric bosonic and fermionic parts. This theory makes life a lot simpler. We can find solution to hydrogen atom like problem without explicitly solving a differential equation with power series. Moreover we can find a list of exactly solvable potentials provided we know only one such potential. If the partner potentials satisfy a particular condition called shape invariance, we need not know the energy spectrum of one potential to know that of the other.

Once we have these two apparently disconnected concepts of SUSYQM and coherent states, it is natural to try to combine these in a logical fashion. In this report we bring our attention to this and try to construct coherent states for the nonlinear supersymmetric partner Hamiltonians, given an initian one. Once we are given with an initial Hamiltonian, we can construct its coherent states. Further we can form the entire series of its partner potentials. The natural question here is how the coherent states for the partner potentials can be formed once we are provided with the initial Hamiltonian and its coherent states. For this we must look into the algebra of the system of partner Hamiltonians and find out to what extent the algebra governing the initial Hamiltonian is inherited to that of its successive Supersymmetric partner

Hamiltonians.

We further extend our search to the systems that are not exactly solvable. For most of practical life problems including Hydrogen atom like problems we often require to solve systems which are not exactly solvable. For these we use various approximation techniques. Perturbation theory is one of them. We have tried to show the following things here. Firstly, How a perturbed initial Hamiltonian gives rise to a series of perturbed SUSY partners Hamiltonians. We have used first order perturbation theory for this purpose. Secondly, How coherent states of a perturbed system can be constructed and it is then extended to forming coherent states for Supersymmetric partner Hamiltonians of the given initial one.

Chapter 2

Coherent Sates

2.1 Introduction:

In physics, the correspondence principle states that the behavior of a system described by quantum mechanics, reproduces classical physics in the limit of large quantum numbers. This motivated physicists to conceive of a way by which they could express the behaviour of quantum fields in a comprehensive and classical manner. According to the correspondence principle the most straightforward approach would be to take the limit of extremely large number of photons. The underlying target would be to make the number operator a continuous variable. However this approach leads to a pitfall. The mean field $\langle n|\hat{E}_x|n\rangle = 0$, no matter how large we take the photon numbers. For the field to be classical it must behave sinusoidally at a fixed point in space. Clearly it is not the case with the mean value of the field operator. This indicates we need something else other than just the number states. These new states are called the coherent states which gives rise to classical field in proper limits. Coherent states are thus called the 'most classical' states. This means their dynamics is the closest resemblance of that of a classical harmonic oscillator[Gerry 04]. In this chapter we will explore how these states are constructed, how they behave and what their properties are.

2.2 Construction of coherent states:

Coherent states can be defined in several ways. For our purpose the most suitable one is these are the eigenstates of the annihilation operator. As we just knew that the expectation value of the field operator will be identically zero irrespective of what the number states are, it is most logical to try to construct a state which will be a combination of the number states. The reason behind it is the mean value of say, annihilation operator $\hat{a}$ does not vanish. This implies the expectation of field operator will also not vanish identically. Therefore we are expected to be safe to think of the coherent state as the superposition of number states. These states are denoted as $|\alpha\rangle$. By definition

$$\hat{a}\,|\alpha\rangle = \alpha\,|\alpha\rangle \tag{2.1}$$

where α is a complex number, otherwise arbitrary.
Since the number states form a complete set, we can express the state $|\alpha\rangle$ as the following,

$$|\alpha\rangle = \sum_{n=0}^{\infty} C_n |n\rangle \tag{2.2}$$

Operating this expression with $\hat{a}$ we see,

$$\hat{a}|\alpha\rangle = \sum_{n=1}^{\infty} C_n \sqrt{n}|n-1\rangle = \alpha \sum_{n=0}^{\infty} C_n|n\rangle \tag{2.3}$$

equating coefficients of $|n\rangle$ we finally arrive at the recursion relation,

$$\begin{aligned} C_n &= \frac{2}{\sqrt{n}} C_{n-1} = \frac{\alpha^2}{\sqrt{n(n-1)}} C_{n-2} = \dots \\ &= \frac{\alpha^n}{\sqrt{n!}} C_0 \end{aligned} \tag{2.4}$$

This implies,

$$|\alpha\rangle = C_0 \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}}|n\rangle \tag{2.5}$$

Now we are only left with one unknown term C_0. This we derive from the normalization condition as follows,

$$\begin{aligned} \langle \alpha \mid \alpha \rangle = 1 &= |C_0|^2 \sum_n \sum_{n'} \frac{\alpha^{*^n} \alpha^{n'}}{\sqrt{n! n'!}} \langle n \mid n' \rangle \\ &= |C_0|^2 \sum_{n=0}^{\infty} \frac{|\alpha|^{2n}}{n!} = |C_0|^2 \, e^{|\alpha|^2} \end{aligned} \tag{2.6}$$

or,

$$C_0 = \exp\left(-\frac{1}{2}|\alpha|^2\right) \tag{2.7}$$

Hence we have our final expression of the normalised coherent state,

$$|\alpha\rangle = \exp\left(-\frac{1}{2}|\alpha|^2\right) \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}}|n\rangle \tag{2.8}$$

2.3 Expectation value of the field operator and uncertainty:

We now calculate the mean value of the field operator. Considering the number states it was identically zero previously. Now let us see how the expectation behaves. Let us define the field operator as,

$$\hat{E}_x(\mathbf{r}, t) = i \left(\frac{\hbar\omega}{2\varepsilon_0 V} \right)^{\frac{1}{2}} \left[\hat{a} e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)} - \hat{a}^\dagger e^{-i(\mathbf{k}\cdot\mathbf{r} - \omega t)} \right] \tag{2.9}$$

Taking the expectation we arrive at,

$$\left\langle \alpha \left| \hat{E}_x(\mathbf{r}, t) \right| \alpha \right\rangle = i \left(\frac{\hbar\omega}{2\varepsilon_0 V} \right)^{\frac{1}{2}} \left[\alpha e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)} - \alpha^* e^{-i(\mathbf{k}\cdot\mathbf{r} - \omega t)} \right] \tag{2.10}$$

In polar coordinates the mean of the square of the field operator becomes ,

$$\left\langle \alpha \left| \hat{E}_x^2(\mathbf{r}, t) \right| \alpha \right\rangle = \frac{\hbar\omega}{2\varepsilon_0 V} \left[1 + 4|\alpha|^2 \sin^2(\omega t - \mathbf{k}\cdot\mathbf{r} - \theta) \right]. \tag{2.11}$$

Where,

$$\alpha = |\alpha| e^{i\theta}$$

These relations tell that coherent states produces the correct form of the sinusoidal field as the expectation value, which is expected from a field also. Another thing is that if we calculate the uncertainty in the field operator the fluctuation turns out to be,

$$\Delta E_x \equiv \left\langle \left(\Delta \hat{E}_x \right)^2 \right\rangle^{\frac{1}{2}} = \left(\frac{\hbar\omega}{2\varepsilon_0 V} \right)^{\frac{1}{2}} \tag{2.12}$$

Which is exactly equal to the noise of a vacuum state[Dwyer 14]. Thus coherent states minimizes the uncertainty. In other words they are the states of minimum uncertainty which is equal to the vacuum fluctuation. In this sense also this is the closest we can reach to a classical system which is deterministic in nature with no uncertainty.

2.4 Coherent states as displaced vacuum states:

We have seen that coherent state can be defined as the eigenstate of the annihilation operator. Moreover it can be described in another way. It is the state which minimizes the uncertainty. Apart from these two description coherent state has a third one as well. This third approach is the displacement of the vacuum state. This can be shown in the following way. The displacement operator is given as:

$$\hat{D}(\alpha) = \exp\left(\alpha\hat{a}^\dagger - \alpha^*\hat{a}\right) \tag{2.13}$$

Now let us state the operator identity'

$$\begin{aligned} e^{\hat{A}+\hat{B}} &= e^{\hat{A}}e^{\hat{B}}e^{-\frac{1}{2}[\hat{A},\hat{B}]} \\ &= e^{\hat{B}}e^{\hat{A}}e^{\frac{1}{2}[\hat{A},\hat{B}]} \end{aligned} \tag{2.14}$$

Comparing it with the expression of $\hat{D}(\alpha)$ we see that,

$$\hat{D}(\alpha) = e^{\alpha\hat{a}^\dagger - \alpha^*\hat{a}} = e^{-\frac{1}{2}|\alpha|^2\mathbb{1}}e^{\alpha\hat{a}^\dagger}e^{-\alpha^*\hat{a}} \tag{2.15}$$

Where

$$\hat{A} = \alpha\hat{a}^\dagger$$

and

$$\hat{B} = -\alpha^*\hat{a}$$

$$[\hat{A},\hat{B}] = |\alpha|^2\mathbb{1}$$

Expanding $e^{-\alpha^*\hat{a}}$ and operating it on the vacuum state we see that

$$e^{-\alpha^*\hat{a}}|0\rangle = \sum_{l=0}^{\infty}\frac{(-\alpha^*\hat{a})^l}{l!}|0\rangle = |0\rangle \tag{2.16}$$

Now evaluating $e^{\alpha\hat{a}|0\rangle}$ we get,

$$\begin{aligned} e^{\alpha\hat{a}^\dagger}|0\rangle &= \sum_{n=0}^{\infty}\frac{\alpha^n}{n!}\left(\hat{a}^\dagger\right)^n|0\rangle \\ &= \sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle \end{aligned} \tag{2.17}$$

We finally have,

$$\begin{aligned} |\alpha\rangle &= \hat{D}(\alpha)|0\rangle \\ &= e^{-\frac{1}{2}|\alpha|^2}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle \end{aligned} \tag{2.18}$$

Which is exactly the previous form. Hence we see here that the coherent states can be alternatively be defined as the displacement of the vacuum state. Now as we have already seen the definition of coherent states and how these definitions can be used in their construction we now move to explore some important properties of these states.

2.5 Properties of coherent states:

Coherent states have some extremely interesting properties. In this section we will discuss some of them.

2.5.1 Time evolution

We now see how coherent state for a single mode free field evolve in time under the influence of a time evolution operator. The time evolution operator is given by, $\exp\left(\frac{-i\hat{H}t}{\hbar}\right)$. Where the Hamiltonian is given by

$$\hat{H} = \hbar\omega\left(\hat{a}^{\dagger}\hat{a} + \frac{1}{2}\right) \tag{2.19}$$

Now operating the coherent state with this time evolution operator we get the following,

$$\begin{aligned} |\alpha, t\rangle &\equiv \exp\left(-i\hat{H}t/\hbar\right)|\alpha\rangle = e^{-i\omega t/2}e^{-i\omega t\hat{n}}|\alpha\rangle \\ &= e^{-i\omega t/2}\left|\alpha e^{-i\omega t}\right\rangle \end{aligned} \tag{2.20}$$

We see here that a coherent state retains its existance as a coherent state under free field evolution. The only change that occurs under time evolution is in the phase of the state.

2.5.2 Orthonormality

Coherent states themselves are not orthonormal. However the number states are orthonormal. We can check this non-orthonormal behaviour of the coherent states in the following way,

$$
\begin{aligned}
\langle \beta \mid \alpha \rangle =& e^{-\frac{1}{2}|\alpha|^2-\frac{1}{2}|\beta|^2} \\
&\times \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{\beta^{*^n}\alpha^m}{\sqrt{n!m!}}\langle n \mid m\rangle \\
=& e^{-\frac{1}{2}|\alpha|^2-\frac{1}{2}|\beta|^2}\sum_{n=0}^{\infty}\frac{(\beta^*\alpha)^n}{n!} \\
=& e^{-\frac{1}{2}|\alpha|^2-\frac{1}{2}|\beta|^2+\beta^*\alpha} \\
=& \exp\left[\frac{1}{2}\left(\beta^*\alpha-\beta\alpha^*\right)\right]\exp\left[-\frac{1}{2}|\beta-\alpha|^2\right]
\end{aligned}
\tag{2.21}
$$

which is clearly nonzero in general.

Hence the coherent states are not orthonormal themselves. However if $|\beta - \alpha|$ is large enough, coherent states are approximately orthonormal.

2.5.3 Completeness

The completeness relation of the coherent state is given as the complex integral,

$$
\int |\alpha\rangle\langle\alpha|\frac{d^2\alpha}{\pi} = 1 \tag{2.22}
$$

This relation can be proved in a straightforward manner. We just need to express α and $d^2\alpha$ in the polar coordinate. This will give us,

$$
\int |\alpha\rangle\langle\alpha| d^2\alpha = \sum_n\sum_m \frac{|n\rangle\langle m|}{\sqrt{n!m!}}\int_0^{\infty} dre^{-r^2}r^{n+m+1}\int_0^{2\pi} d\theta e^{i(n-m)\theta} \tag{2.23}
$$

Here we use the relation,

$$
\int_0^{2\pi} d\theta e^{i(n-m)\theta} = 2\pi\delta_{nm} \tag{2.24}
$$

We again change the variable $r^2 = y$, hence $2rdr = dy$, and this gives us,

$$
\int |\alpha\rangle\langle\alpha| d^2\alpha = \pi\sum_{n=0}^{\infty}\frac{|n\rangle\langle n|}{n!}\int_0^{\infty} dye^{-y}y^n \tag{2.25}
$$

and since,

$$
\int_0^{\infty} dye^{-y}y^n = n! \tag{2.26}
$$

we finally arrive at,

$$\int |\alpha\rangle \langle\alpha| \, d^2\alpha = \pi \sum_{n=0}^{\infty} |n\rangle \langle n| = \pi \tag{2.27}$$

Any state vector in a Hilbert space can be expressed in terms of the coherent states. There are in fact more than enough number of such states which can be used to express any state vector in that Hilbert space. This is why coherent states are called to be 'overcomplete'. Coherent states are not linearly independent. This directly follows from the fact that they are 'overcomplete'. We always find more number of coherent states than we need to express any state in terms of coherent states.

2.6 Conclusion:

In this chapter we saw what coherent states are. They are extremely important in the field of quantum optics. Although here coherent states are discussed in the context of quantum harmonic oscillator where the energy levels are integer spaced, the idea can be generalized to other quantum systems which are not essentially linear. The construction of such coherent states are nontrivial. These aspect will be explored in the later chapters. We have behold some of the interesting properties of these states. These preliminary concepts will eventually enable us to appreciate the following sections.

Chapter 3

Basics of SUSYQM

3.1 Introduction:

Supersymmetric quantum mechanics is a comparatively newer branch in theoretical physics. Concepts of Supersymmetry was historically first applied to quantum field theory. Thus it was chiefly a field theoretic approach to understand the behaviour of matter and force at a more fundamental level. However this method turned out not to be so evidently acceptable because we could never detect any partner particle of the same mass as proposed in the theory. This lead physicists to apply the format of SUSY in a simpler setting than quantum field theory. They applied the concepts to our good old quantum mechanics, rather than field theory. The idea was to introduce the SUSY tehniques to the language (quantum mechanics) first and then to any theory written within the framework of quantum mechanics. This is how we got Supersymmetric quantum mechanics or SUSY-QM. In this chapter we will see how the principles of susy makes life simpler in handling the typical mathematically rigorous problems of standard quantum mechanics. We will then explore how we can know about other exactly solvable potentials once we are provided with one solvable potential [Andrianov 93; Sato 02; Andrianov 03; Leiva 03] We will later see how we can form Superpartner Hamiltonian of an initial Hamiltonian.

3.2 Basic principles of supersymmetric quantum mechanics:

Supersymmetry gives an elegant and alternative description of the mathematical foundation of quantum mechanics. In this section we will see how it can be an extremely useful tool for us to deal with quantum systems with enormous ease unlike the conventional ways of tackling such problems. Here we will start with the quantum harmonic oscillator which is exactly solvable. Same techniques can be applied for other potentials also. For example for 'particle in box' problem. Let us delve into the problem of quantum harmonic oscillator now. The Hamiltonian of

a quantum harmonic oscillator which is a bosonic oscillator can be written as the following:

$$H_B = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{1}{2}m\omega_B^2 x^2 \tag{3.1}$$

Where ω_B is the natural frequency of the oscillator and we adopt such unit for convenience where $\hbar = \mathrm{m} = 1$. Here we can define a pair of creation and annihilation operators corresponding to the hamiltonian given. We name them say, b^+ and b respectively.

These are given as,

$$\begin{aligned} b &= \frac{i}{\sqrt{2\omega_B}}\left(p - i\omega_B x\right) \\ b^+ &= -\frac{i}{\sqrt{2\omega_B}}\left(p + i\omega_B x\right) \end{aligned} \tag{3.2}$$

Where $p = -i\frac{d}{dx}$. Now the Hamiltonian can be expressed in terms of the creation and annihilation operators as

$$H_B = \frac{1}{2}\omega_B \left\{b^+, b\right\} \tag{3.3}$$

Where $\{b^+, b\}$ is the anticommutator of b^+ and b. The standard action of these operators on the number states are given as follows,

$$\begin{aligned} b \mid n > &= \sqrt{n}\,|n-1\rangle \\ b^+ \mid n > &= \sqrt{n+1}\,|n+1\rangle \end{aligned} \tag{3.4}$$

Here $[b, b^+] = \mathbb{I}$ and

$$\begin{aligned} [b, b] &= 0 \\ \left[b^+, b^+\right] &= 0 \\ [b, H_B] &= \omega_B b, \\ \left[b^+, H_B\right] &= -\omega_B b^+ \end{aligned} \tag{3.5}$$

Now we can express H_B as,

$$H_B = \omega_B\left(b^+ b + \frac{1}{2}\right) = \omega_B\left(N_B + \frac{1}{2}\right) \tag{3.6}$$

With the energy spectrum,

$$E_B = \omega_B(n_B + \frac{1}{2}) \tag{3.7}$$

Up to this it was standard undergrad quantum mechanics. Here we introduce a comparatively new thing. We replace the bosonic operators b and b^+ by its corresponding fermionic operatros which are annihilation and creation operators of a fermionic quantum harmonic oscillator. Here we introduce two such operators a and a^+. Following the same treatment just done we can write down a fermionic Hamiltonian with the help of these operators as we just did in case of the bosonic one. we thus have,

$$H_F = \frac{1}{2}\omega_F[a^+, a]$$

(3.8)

These operators satisfy the relations,

$$\begin{aligned} \{a, a^+\} &= \mathbb{I} \\ \{a, a\} &= 0, \{a^+, a^+\} = 0 \end{aligned} \tag{3.9}$$

We may also define the fermionic number operators analogous to the bosonic one as N_F. Here

$$N_F = a^+a \tag{3.10}$$

Now we can express H_F as

$$H_F = \omega_F(N_F - \frac{1}{2}) \tag{3.11}$$

with the spectrum

$$E_F = \omega_F(n_F - \frac{1}{2}) \tag{3.12}$$

Now if we look at the complete system which is essentially a superposition of bosonic and fermionic systems we get,

$$E = \omega_B(n_B + \frac{1}{2}) + \omega_F(n_F - \frac{1}{2}) \tag{3.13}$$

At this point we need to stop by a little. This last equation deserves our attention.

This equation is simply the spectrum of the entire system which arise from the superposition of the bosonic and fermionic systems. The speciality of this equation lies with the fact that it remains unchanged if one boson gets annihilated and one fermion gets created or if one fermion is annihilated and one boson is created provided the natural frequencies of the bosonic part of the system and the fermionic part of the system are set equal. This symmetry is what we call 'supersymmetry'. The equation (3.13) can be rewritten as

$$E = \omega(n_B + n_F) \tag{3.14}$$

So we see that the symmetry comes from a simultaneous destruction of one boson and creation of one fermion or vice versa. This implies one crucial thing. The generators of such symmetry will be like ba^+ or ab^+ because these operators when act upon the number state can do the simultaneous creation and annihilation of these particles. This motivates to define quantities like

$$\begin{aligned} Q &= \sqrt{\omega} b \otimes a^+ \\ Q^+ &= \sqrt{\omega} b^+ \otimes a \end{aligned} \tag{3.15}$$

The supersymmetric Hamiltonian can be defined as

$$\begin{aligned} H_s &= \omega \left(b^+ b + a^+ a\right) \\ &= \left\{Q, Q^+\right\} \end{aligned} \tag{3.16}$$

The commutation relations are the following,

$$\begin{aligned} [Q, H_s] &= 0 \\ [Q^+, H_s] &= 0 \\ \{Q, Q\} &= 0 \\ \{Q^+, Q^+\} &= 0 \end{aligned} \tag{3.17}$$

The entire Hamiltonian can be written using the Pauli spin matrix as

$$H_s = \frac{1}{2}\left(p^2 + \omega^2 x^2\right)\mathbb{I} + \frac{1}{2}\omega\sigma_3 \tag{3.18}$$

Where I is a 2×2 identity matrix

The entire Hamiltonian can then be split into its two components as,

$$\begin{aligned} H_+ &= -\tfrac{1}{2}\tfrac{d^2}{dx^2} + \tfrac{1}{2}\left(\omega^2 x^2 - \omega\right) \equiv \omega b^+ b \\ H_- &= -\tfrac{1}{2}\tfrac{d^2}{dx^2} + \tfrac{1}{2}\left(\omega^2 x^2 + \omega\right) \equiv \omega b b^+ \end{aligned} \tag{3.19}$$

These two Hamiltonians describe the same harmonic oscillator and are different from each other only by a constant shift in the spectrum. These are called supersymmetric partner Hamiltonians to each other. One is bosonic part and the other is fermionic one. This idea can be generalized further to construct a series of such partner Hamiltonians, which we will see soon.

3.3 Superpotential:

In this section we will discuss about superpotentials. The kinetic part of the partner Hamiltonians are obviously the same. The only difference they have in the potential part. Hence it is justified to call them as partner potentials as they are the ones to give rise to the partner Hamiltonians. We call these partner potentials as V_+ and V_-. These can be expressed as,

$$V_\pm(x) = \frac{1}{2}\left[W^2(x) \mp W'(x)\right] \tag{3.20}$$

Where

$$W(x) = \omega x$$

for harmonic oscillator.

This function $W(x)$ is what we call superpotential.

$$H_s = \frac{1}{2}\left(p^2 + W^2\right)\mathbb{I} + \frac{1}{2}\sigma_3 W' \tag{3.21}$$

Accordingly the corresponding supercharges Q and Q^+ can be expressed as,

$$\begin{aligned} Q &= \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & W + ip \\ 0 & 0 \end{pmatrix} \\ Q^+ &= \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 0 \\ W - ip & 0 \end{pmatrix} \end{aligned} \tag{3.22}$$

and

$$H_s = \{Q, Q^+\} \tag{3.23}$$

The commutation relations of these supercharges with the supersymmetric Hamiltonian are given by,

$$[Q, H_s] = 0 \tag{3.24}$$

$$[Q^+, H_s] = 0 \tag{3.25}$$

The bosonic operators b and b^+ reveals themselves in more general form.

$$\sqrt{2\omega}b \to A = W(x) + \frac{d}{dx} \tag{3.26}$$

$$\sqrt{2\omega}b^+ \to A^+ = W(x) - \frac{d}{dx} \tag{3.27}$$

We can express the Hamiltonian H_s in terms of A and A^+ as,

$$\begin{aligned} H_s &\equiv \text{diag}\left(H_-, H_+\right) \\ &= \frac{1}{2}\,\text{diag}\left(AA^+, A^+A\right) \end{aligned} \tag{3.28}$$

3.4 Properties of the partner Hamiltonian:

Here we discuss some of the properties of the supersymmetric partner Hamiltonians. The first interesting property is that the partner potentials are nearly isospectral. If we consider,

$$H_+\psi_n^+ = E_n^+\psi_n^+ \tag{3.29}$$

we can show that,

$$\begin{aligned} H_-\left(A\psi_n^+\right) &= \frac{1}{2}AA^+\left(A\psi_n^+\right) \\ &= A\left(\frac{1}{2}A^+A\psi_n^+\right) \\ &= E_n^+\left(A\psi_n^+\right) \end{aligned} \tag{3.30}$$

which proves that E_n^+ is the energy spectrum of H_- as well. We see that the energy spectrum of H_- and H_+ are almost identical except for the ground state. The ground state is nondegenerate.

Apart from that we see the standard eigenvalue relations,

$$H_+\psi_{n+1}^{(+)} = E_{n+1}^{(+)}\psi_{n+1}^{(+)} \tag{3.31}$$

$$H_-\psi_n^{(-)} = E_n^{(-)}\psi_n^{(-)} \tag{3.32}$$

We now turn toward how the spectrum and wave functions of H_- and H_+. For this purpose we perform the following steps,

$$\begin{aligned} H_+\left(A^+\psi_n^-\right) &= \frac{1}{2}A^+A\left(A^+\psi_n^-\right) = A^+H_-\psi_n^- = E_n^-\left(A^+\psi_n^-\right) \\ H_-\left(A\psi_n^+\right) &= \frac{1}{2}AA^+\left(A\psi_n^+\right) = AH_+\psi_n^+ = E_n^+\left(A\psi_n^+\right) \end{aligned} \tag{3.33}$$

Hence we can conclude that the spectra and wave functions of H_- and H_+ are related by,

$$E_n^- = E_{n+1}^+, n = 0, 1, 2, \ldots; E_0^+ = 0 \tag{3.34}$$

$$\psi_n^- = \left(2E_{n+1}^+\right)^{-\frac{1}{2}} A\psi_{n+1}^+ \tag{3.35}$$

$$\psi_{n+1}^+ = \left(2E_n^-\right)^{-\frac{1}{2}} A^+\psi_n^- \tag{3.36}$$

3.5 Shape invariance and its application:

Let us discuss about shape invariance properties of the potentials. Shape invariance of partner potentials is a crucial idea which says if partner potentials have shape invariance we need not know the energy spectrum of one potential to know that of the other[Bagchi 01].

3.5.1 Condition of shape invariance

The condition for the bosonic and the fermionic potential to be shape invariant is the following:

$$V^{(2)}\left(x, a_1\right) = V^{(1)}\left(x, a_2\right) + R\left(a_1\right) \tag{3.37}$$

Here the partner potentials have been denoted by $V^{(1)}$ and $V^{(2)}$.

If the superpotential falls in this class of shape invariant potentials we can construct the entire series of the Hamiltonians using the property as follows:

$$H^{(1)} = \frac{-\hbar^2}{2m}\frac{d^2}{dx^2} + V^{(1)}\left(x, a_1\right) \tag{3.38}$$

$$H^{(2)} = \frac{-\hbar^2}{2m}\frac{d^2}{dx^2} + V^{(1)}\left(x, a_2\right) + R\left(a_1\right) \tag{3.39}$$

$$H^{(3)} = \frac{-\hbar^2}{2m}\frac{d^2}{dx^2} + V^{(1)}\left(x, a_3\right) + R\left(a_2\right) + R\left(a_1\right) \tag{3.40}$$

Using the k^{th} Hamiltonian of the ground state eigenfunction we get the ground state energy of the k^{th} partner potential as,

$$E_0^{(k)} = \sum_{i=1}^{k-1} R\left(a_i\right) \tag{3.41}$$

Since the ground state energy of the k^{th} Hamiltonian corresponds to the k-1 energy level for the first Hamiltonian, it is possible to express the entire spectrum for the first Hamiltonian by

$$\begin{gathered} E_n^{(1)}\left(a_1\right) = \sum_{i=1}^{n} R\left(a_i\right) \\ E_0^{(1)} = 0 \end{gathered} \tag{3.42}$$

If the initial problem does not have E_0^1 the spectrum needs to be shifted back to the initial problem after using the SIP method in order to obtain the real spectrum.

3.6 Conclusion:

In this chapter we saw a completely different approach in formulating standard quantum mechanics. Supersymmetry splits the Hamiltonian into its bosonic and fermionic parts. These makes things enormously easy to handle. Apart from this the concept of shape invariance is also extremely useful. Once we find the potential to have this specific property of shape invariance we can instantly have the entire series of superpartners of the initial Hamiltonian.

Chapter 4

Coherent states of Nonlinear Supersymmetric Partner Hamiltonians

Here we are equipped with sufficient background knowledge to proceed deeper. In this section we will see how these ideas can be of our use in building coherent states of nonlinear systems. Coherent states are beautiful in the underlying physical concept. This beautiful idea however was restricted to harmonic oscillator which is governed by the well-known Heisenberg-Weyl algebra. Now while we try to construct the coherent state of any system, we primarily use its definitions and properties in doing so[Ghosh 12]. For example coherent states are the eigenstates of the annihilation operator. This definition is extensively used to form coherent state algebraically. Therefore it is of immense importance in order to construct coherent state of any other quantum system other than the Harmonic oscillator to properly identify the the algebra that governs the system. We can use various definitions or properties of coherent states for this purpose [Glauber 06; Klauder 85; Perelomov 86] . The underlying algebraic structures associated with nonlinear systems are surprisingly a very recent development. This development in turn immediately leads to the question of how the coherent states of these systems should look like.

On the other hand supersymmetric quantum mechanics has proven itself extremely useful in finding exactly solvable potentials once we are provided with an initial one. Now as we know the algebraic structure of different systems and are capable of forming coherent states for them as well, it is very natural to ask how much of this governing algebra of the system is inherited by its Supersymmetric partner Hamiltonians and how the corresponding coherent states will be like[Fernández 07]. In this chapter we mainly focus our attention in finding the answers to these questions.

4.1 Algebraic structure of $\mathbf{H}_0$:

The standard initial Schrodinger Hamiltonian is,

$$H_0 = -\frac{1}{2}\frac{d^2}{dx^2} + V_0(x) \tag{4.1}$$

Its eigenstates and eigenvalues are related by,

$$H_0 \left|\psi_n\right\rangle = E_n \left|\psi_n\right\rangle, \quad E_0 < E_1 < E_2 < \dots \tag{4.2}$$

We know that the number operator N_0 of the system will give us,

$$N_0 \left|\psi_n\right\rangle = n \left|\psi_n\right\rangle \tag{4.3}$$

We now turn to explore the intrinsic algebra associated with the system. This is described by the following expressions:

$$a_0^- \left|\psi_n\right\rangle = r_{\mathcal{I}}(n) \left|\psi_{n-1}\right\rangle, \quad a_0^+ \left|\psi_n\right\rangle = \bar{r}_{\mathcal{I}}(n+1) \left|\psi_{n+1}\right\rangle, \tag{4.4}$$

$$r_{\mathcal{I}}(n) = e^{i\alpha(E_n - E_{n-1})}\sqrt{E_n - E_0}, \quad \alpha \in \mathbb{R} \tag{4.5}$$

In terms of the operators,

$$a_0^+ a_0^- = E(N_0) - E_0, \quad a_0^- a_0^+ = E(N_0+1) - E_0 \tag{4.6}$$

Therefore we arrive at the following commutation relations that characterizes the algebra,

$$\left[N_0, a_0^\pm\right] = \pm a_0^\pm, \quad \left[a_0^-, a_0^+\right] = E(N_0+1) - E(N_0) \equiv f(N_0), \tag{4.7}$$

$$\left[H_0, a_0^\pm\right] = \pm a_0^\pm f\left(N_0 - \frac{1}{2} \pm \frac{1}{2}\right) = \pm f\left(N_0 - \frac{1}{2} \mp \frac{1}{2}\right) a_0^\pm \tag{4.8}$$

Here we see that the algebra is not linear. However the intrinsic nonlinear algebra admits a linearization procedure. This is as follows,

$$a_{0_{\mathcal{L}}}^- = b(N_0)\, a_0^-, \quad a_{0_{\mathcal{L}}}^+ = a_0^+ b(N_0) \tag{4.9}$$

with

$$b(n) = \frac{r_{\mathcal{L}}(n+1)}{r_{\mathcal{I}}(n+1)} = \sqrt{\frac{n+1}{E(n+1) - E_0}}, \quad r_{\mathcal{L}}(n) = e^{i\alpha f(n-1)}\sqrt{n} \tag{4.10}$$

The action of the linearized annihilation and creation operators on the state $|\psi_n\rangle$ is,

$$a_{0_{\mathcal{L}}}^{-}\,|\psi_n\rangle = r_{\mathcal{L}}(n)\,|\psi_{n-1}\rangle\,, \quad a_{0_{\mathcal{L}}}^{+}\,|\psi_n\rangle = \bar{r}_{\mathcal{L}}(n+1)\,|\psi_{n+1}\rangle \tag{4.11}$$

These linear annihilation and creation operators which are derived as a deformation of the intrinsic ones satisfy the standard Heisenberg-Weyl algebra.

4.2 Coherent states of the initial Hamiltonian $\mathbf{H}_0$:

Here we try to construct the coherent state of H_0. The definition of coherent state used here is given mathematically,

$$a_0^{-}|z,\alpha\rangle_0 = z|z,\alpha\rangle_0, \quad z \in \mathbb{C} \tag{4.12}$$

The formal procedure that is discussed in the earlier chapter on coherent states leads to the coherent states as,

$$\begin{aligned} |z,\alpha\rangle_0 &= \left(\sum_{m=0}^{\infty}\frac{|z|^{2m}}{\rho_m}\right)^{-\frac{1}{2}}\sum_{m=0}^{\infty} e^{-i\alpha(E_m-E_0)\frac{z^m}{\sqrt{\rho m}}|\psi_m\rangle}, \\ \rho_m &= \begin{cases} 1 & \text{if } m=0 \\ (E_m-E_0)\dots(E_1-E_0) & \text{if } m>0 \end{cases} \end{aligned} \tag{4.13}$$

If we consider the coherent states of H_0 associated with the linear algebra of H_0 as the eigenstates of the linear annihilation operator, Following the same standard procedure it comes out to be,

$$|z,\alpha\rangle_{0_{\mathcal{L}}} = e^{-\frac{|z|^2}{2}}\sum_{m=0}^{\infty} e^{-i\alpha(E_m-E_0)}\frac{z^m}{\sqrt{m!}}\,|\psi_m\rangle \tag{4.14}$$

These cohrent states formed from different algebras of the initial Hamiltonian have all the characteristic feature of a coherent state. They evolve in time and remain a coherent state. Their completeness relation still holds. Their interpretation as the displacement of the vacuum state holds for coherent states in particular.

4.3 Supersymmetric partner Hamiltonians of $\mathbf{H}_0$:

Here we introduce two intertwining operators. These are the operators which connect the SUSY partner Hamiltonians to each other. We name these as B_k and

B_k^+

$$H_k B_k^+ = B_k^+ H_0, \quad H_0 B_k = B_k H_k \tag{4.15}$$

Where H_k is

$$H_k = -\frac{1}{2}\frac{d^2}{dx^2} + V_k(x) \tag{4.16}$$

The SUSY partner potential is related to the initial one by,

$$V_k(x) = V_0(x) - \sum_{i=1}^{k} \alpha_i'(x, \epsilon_i) \tag{4.17}$$

Where, $\alpha_i'(x, \epsilon_i)$ satisfy the Riccati equation

$$\alpha_1'(x, \epsilon_i) + \alpha_1^2(x, \epsilon_i) = 2[V_0(x) - \epsilon_i], \qquad i = 1, \ldots, k \tag{4.18}$$

This is equivalent to the stationary Schrodinger equation for the factorization energies ϵ_i ,if we make the substitution

$$\alpha_1(x, \epsilon_i) = u_i'(x)/u_i(x)$$

With this the equation (4.18) turns out to be

$$-\frac{1}{2}u_i'' + V_0(x)u_i = \epsilon_i u_i$$

This is a Schrodinger equation in one dimension.

Let us now suppose that, as a result of the kth order intertwining technique, the states annihilated by B_k are as well physical eigenstates of H_k associated with the eigenvalues ϵ_i.
By convenience, they will be specially denoted by $|\theta_{\epsilon_i}\rangle$

$$B_k |\theta_{\epsilon_i}\rangle = 0, \; H_k |\theta_{\epsilon_i}\rangle = \epsilon_i |\theta_{\epsilon_i}\rangle \; i = 1, ..., k \tag{4.19}$$

However, we assume that the procedure creates just additional levels with respect to $Sp(H_0)$. but without deleting any of the original levels of H_0, i.e.,

$$Sp(H_k) = \{\epsilon_1, \ldots, \epsilon_q, E_0, E_1, \ldots\}, \tag{4.20}$$

Here the levels $\{E_0, E_1, E_2, ...\}$ come from the eigenstate θ_n of H_k. Summarizing all information we see that the eigenstates obey,

$$\begin{aligned} &H_k \left|\theta_n\right\rangle = E_n \left|\theta_n\right\rangle, \quad H_k \left|\theta_{\epsilon_i}\right\rangle = \epsilon_i \left|\theta_{\epsilon_i}\right\rangle \\ &\left\langle\theta_{\epsilon_i} \mid \theta_n\right\rangle = 0, \quad \left\langle\theta_m \mid \theta_n\right\rangle = \delta_{mn}, \quad \left\langle\theta_{\epsilon_i} \mid \theta_{\epsilon_j}\right\rangle = \delta_{ij} \end{aligned} \tag{4.21}$$

Here n varies from 0 to m_p

This means that $p = s - q$ factorization energies ϵ_{q+j} coincide with p energy levels E_{m_j} of H_0 ,i.e. $\epsilon_{q+j} = E_{m_j}$, $j = 1, ..., p$, $m_j < m_{j+1}$ and thus $B_k^+ \left|\psi_{m_j}\right\rangle = 0$

4.4 Algebraic construction of $\mathbf{H}_k$:

Here we see the similar algebraic construction of H_k as we saw of H_0 . Starting from the intrinsic operators of the initial Hamiltonian and using the intertwining operator we arrive at a different algebra for the SUSY partner Hamiltonian, namely the natural algebra.

4.4.1 Natural algebra of $\mathbf{H}_k$

The natural annihilation and creation operators are given by

$$a_{k_{\mathcal{N}}}^{\pm} = B_k^+ a_0^{\pm} B_k \tag{4.22}$$

The action of the natural creation and annihilation operators on the eigen states of H_k are as follows:

$$a_{k_{\mathcal{N}}} \left|\theta_n\right\rangle = 0 \quad i = 1, ..., q \tag{4.23}$$

The action of natural annihilation and creation operators on the eigenstates of H_k is given as following:

$$a_{k_{\mathcal{N}}}^{\pm} \left|\theta_{\epsilon_i}\right\rangle = 0, i = 1, \ldots, q \tag{4.24}$$

$$a_{k_{\mathcal{N}}}^{-} \left|\theta_n\right\rangle = r_{\mathcal{N}}(n) \left|\theta_{n-1}\right\rangle, a_{k_{\mathcal{N}}}^{+} \left|\theta_n\right\rangle = \bar{r}_{\mathcal{N}}(n+1) \left|\theta_{n+1}\right\rangle \tag{4.25}$$

Where,

$$r_{\mathcal{N}}(n) = \left\{\prod_{i=1}^{k} \left[E(n) - \epsilon_i\right] \left[E(n-1) - \epsilon_i\right]\right\}^{\frac{1}{2}} r_{\mathcal{I}}(n) \tag{4.26}$$

As we saw in the previous section that the intrinsic and linear algebra are related, likewise natural algebra and intrinsic algebra of a system are also related to each other. They are so by the following equations. Basically they can be viewed as deformities[Román-Ancheyta 15].

$$a_{k_{\mathcal{N}}}^{-} = \frac{r_{\mathcal{N}}(N_k+1)}{r_{\mathcal{I}}(N_k+1)} a_k^{-}, \quad a_{k_{\mathcal{N}}}^{+} = \frac{r_{\mathcal{N}}(N_k)}{r_{\mathcal{I}}(N_k)} a_k^{+}, \quad a_{k_{\mathcal{N}}}^{+} a_{k_{\mathcal{N}}}^{-} = [E(N_k) - E_0] \left[\frac{r_{\mathcal{N}}(N_k)}{r_{\mathcal{I}}(N_k)}\right]^2 \tag{4.27}$$

4.4.2 Intrinsic algebra of $\mathbf{H}_k$

Let us now analyse the intrinsic algebra of H_k. This is generated by the below mentioned annihilation and creation operators.

$$a_k^{-} = r_{\mathcal{I}}(N_k+1) \sum_{m=0}^{\infty} |\theta_m\rangle \langle\theta_{m+1}| \qquad a_k^{+} = \bar{r}_{\mathcal{I}}(N_k) \sum_{m=0}^{\infty} |\theta_{m+1}\rangle \langle\theta_m| \tag{4.28}$$

The action of these operators on the eigenstates of H_k are the following,

$$a_k^{\pm} |\theta_{\epsilon_i}\rangle = 0 \qquad i = 1, ..., q \tag{4.29}$$

$$a_k^{-} |\theta_n\rangle = r_{\mathcal{I}}(n) |\theta_{n-1}\rangle \qquad a_k^{+} |\theta_n\rangle = \bar{r}_{\mathcal{I}}(n+1) |\theta_{n+1}\rangle \tag{4.30}$$

4.4.3 Linear algebra of $\mathbf{H}_k$

Finally, we analyse the linear algebra of H_k The linear annihilation and creation operators are given by

$$a_{k_{\mathcal{L}}}^{-} = r_{\mathcal{L}}(N_k+1) \sum_{m=0}^{\infty} |\theta_m\rangle \langle\theta_{m+1}| \qquad a_{k_{\mathcal{L}}}^{+} = \bar{r}_{\mathcal{L}}(N_k) \sum_{m=0}^{\infty} |\theta_{m+1}\rangle \langle\theta_m| \tag{4.31}$$

Again, the action of these operators on the eigenstates of H_k is given by

$$a_{k_{\mathcal{L}}}^{\pm} |\theta_{\epsilon_i}\rangle = 0, \qquad i = 1, \ldots, q \tag{4.32}$$

$$a_{k_{\mathcal{L}}}^{-} |\theta_n\rangle = r_{\mathcal{L}}(n) |\theta_{n-1}\rangle \qquad a_{k_{\mathcal{L}}}^{+} |\theta_n\rangle = \bar{r}_{\mathcal{L}}(n+1) |\theta_{n+1}\rangle \tag{4.33}$$

The relation between the intrinsic algebra and the linear algebra of H_k is given by,

$$a_{k_{\mathcal{L}}}^{-} = \frac{r_{\mathcal{L}}(N_k+1)}{r_{\mathcal{I}}(N_k+1)} a_k^{-}, \quad a_{k_{\mathcal{L}}}^{+} = \frac{r_{\mathcal{L}}(N_k)}{r_{\mathcal{I}}(N_k)} a_k^{+}, \quad a_{k_{\mathcal{L}}}^{+} a_{k_{\mathcal{L}}}^{-} = N_k \tag{4.34}$$

4.5 Construction of coherent states of $\mathbf{H}_K$:

In this section we will see how the coherent states of these chain of partner Hamiltonians look like. Once we know the algebra governing the system and how the annihilation and creation operators behave we can perform our good old straight forward calculation for coherent states as the eigenstates of the annihilation operator.

4.5.1 Natural coherent state of $\mathbf{H}_k$

By performing that standard procedure we derive the expression for coherent state as,

$$|z,\alpha\rangle_{k_{\mathcal{N}}} = \left[\sum_{m=0}^{\infty} \frac{|z|^{2m}}{\widetilde{\rho}_m}\right]^{-\frac{1}{2}} \sum_{m=0}^{\infty} e^{-i\alpha\left(E_{m+m_p+1}-E_{m_p+1}\right)} \frac{z^m}{\sqrt{\widetilde{\rho}_m}} \left|\theta_{m+m_p+1}\right\rangle \tag{4.35}$$

Where, $\widetilde{\rho}_0 = 1$

and, for m > 0

$$\widetilde{\rho}_m = \frac{\rho_{m+m_p+1}}{\rho_{m_p+1}} \prod_{i=1}^{k} \left(E_{m+m_p+1}-\epsilon_i\right)\left(E_{m+m_p}-\epsilon_i\right)^2 \ldots \left(E_{m_p+2}-\epsilon_i\right)^2 \left(E_{m_p+1}-\epsilon_i\right) \tag{4.36}$$

The intrinsic and linear coherent states which are the eigenstates of the a_k^- and $a_{k_{\mathcal{L}}}^-$ respectively, are the same as that of the initial Hamiltonian. This is due to the fact that the linear and intrinsic algebras of H_0 characterize as well the H_k on the subspace associated to the initial levels.

Here the completeness relation is given as,

$$\sum_{i=1}^{q} |\theta_{\epsilon_i}\rangle \langle\theta_{\epsilon_i}| + \sum_{m=0}^{m_p} |\theta_m\rangle\langle\theta_m| + \int |z,\alpha\rangle_{k_{\mathcal{N}}k_{\mathcal{N}}} \langle z,\alpha|\, d\widetilde{\mu}(z) = 1$$

where the measure reads:

$$d\widetilde{\mu}(z) = \frac{1}{\pi}\left(\sum_{m=0}^{\infty} \frac{|z|^{2m}}{\widetilde{\rho}_m}\right) \tilde{\rho}\left(|z|^2\right) d^2 z.$$

4.5.2 Intrinsic coherent state of $\mathbf{H}_k$

The intrinsic nonlinear coherent state follows from the intrinsic algebra of H_k. We again follow the standard procedure to calculate cohrent state as the eigenstate of the intrinsic annihilation operator. We finally arrive at the expression,

$$|z, \alpha\rangle_k = \left(\sum_{m=0}^{\infty} \frac{|z|^{2m}}{\rho_m} \right)^{-\frac{1}{2}} \sum_{m=0}^{\infty} e^{-i\alpha(E_m - E_0)} \frac{z^m}{\sqrt{\rho_m}} |\theta_m\rangle \tag{4.37}$$

The completeness relation is given by,

$$\sum_{i=1}^{q} |\theta_{\epsilon_i}\rangle \langle \theta_{\epsilon_i}| + \int |z, \alpha\rangle_{kk} \langle z, \alpha| d\mu(z) = 1$$

4.5.3 Linear coherent state of $\mathbf{H}_k$

In a similar way we can construct the linear coherent state of H_k from the linear annihilation operator. This is given by,

$$|z, \alpha\rangle_{k_{\mathcal{L}}} = e^{-\frac{|z|^2}{2}} \sum_{m=0}^{\infty} e^{-i\alpha(E_m - E_0)} \frac{z^m}{\sqrt{m!}} |\theta_m\rangle \tag{4.38}$$

The completeness relation in this case is given by,

$$\sum_{i=1}^{q} |\theta_{\epsilon_i}\rangle \langle \theta_{\epsilon_i}| + \frac{1}{\pi} \int |z, \alpha\rangle_{k_{\mathcal{L}} k_{\mathcal{L}}} \langle z, \alpha| d^2(z) = 1$$

4.6 Conclusion:

In this chapter we have seen how the algebra of the initial Hamiltonian is inherited by its supersymmetric partner Hamiltonians. We have also seen how coherent states can be formed once we know the complete description of the algebra ruling the system. Nonlinear systems admit a specific linearization scheme and can be exploited to construct the coherent states. In this way the idea of coherent states as well as annihilation and creation operators are further generalized out of the well known harmonic oscillator problem.

Chapter 5

Consequences of perturbation in construction of coherent states

We have so far seen how coherent states can be formed from the initial given Hamiltonian of a linear system like harmonic oscillator. Then we saw how this idea of coherent states can be generalized to nonlinear systems as well. We discussed how an initial Hamiltonian gives rise to an infinite series of its supersymmetric partner Hamiltonians and how they are interconnected. We again tried to elaborate the way an underlying algebra governing a system is transferred or inherited to the SUSY partner Hamiltonians of the same and how the coherent states of SUSY partners are.

However, everything that we discussed so far is about exactly solvable systems. We have taken initially harmonic oscillator which is one of that kind. Later we delved into SUSYQM, whose one of the most important purposes is to find more exactly solvable potentials given an initial exactly solvable one. But what if the system is not exactly solvable and there is a slight perturbation which is deflecting the system from being an exact one. In a recent work, without any consideration of SUSY this question is addressed in a simpler setting of a perturbative harmonic oscillator[Naila 15]. We have not discussed anything about such scenarios so far. Here in this chapter we try to present a comprehensible view of the problem to 1st order approximation.

5.1 Introducing the perturbation:

Let us perturb the initial Hamiltonian H_0 and it becomes Now we perturb the initial Hamiltonian $H_0$and it becomes $H_0 + \lambda H_1$

It can be shown that this perturbation in the initial Hamiltonian inherits to the SUSY partner Hamiltonians.However, the form of the perturbation does not in general remain the same. By which it is meant that if the initial Hamiltonian is perturbed by an amount it is certain that the Partner Hamiltonians will also be affected by the perturbation but not by the same amount as it was for the initial one. We recall that,

$$-\frac{1}{2}u_i'' + V_0(x)u_i = \epsilon_i u_i$$

Which is the Schrodinger equation for ϵ_i eigenvalues and u_i eigenfunctions.

Now with the perturbation the u_i up to the first order approximation becomes

$$|\widetilde{u_i}\rangle = |u_i\rangle + \sum_{i \neq k} \frac{\langle u_k|\lambda H_1|u_i\rangle}{\epsilon_i - \epsilon_k}|u_k\rangle$$

Therefore $\alpha_1 (x, \epsilon_i)$ becomes

$$\widetilde{\alpha_1 (x, \epsilon_i)} = \frac{\widetilde{u_i}'}{\widetilde{u_i}} \tag{5.1}$$

Here this equation tells us how the factor α_1 changes under the influence of the perturbation. This implies that H_k will also be altered since H_k depends on α_1. We therefore look into how H_k changes. Hence H_k becomes

$$\widetilde{H_k} = H_0 + \lambda H_1 - \sum_{i=1}^{k} \widetilde{\alpha_1' (x, \epsilon_i)}$$

Where,

$$\widetilde{\alpha_1 (x, \epsilon_i)} = \frac{\widetilde{u_i}'}{\widetilde{u_i}} \tag{5.3}$$

This equation is important. It tells us the way a perturbation of any general form to the initial Hamiltonian is inherited to its SUSY partner Hamiltonians. From this we can see that the form of the perturbation changes in general. Once we get this result the rest of the task is straight forward. We need to do 1st order perturbation theory for the partner Hamiltonians and need to find how the eigenvalues changes. Finally we need the spectrum in order to calculate for the coherent states.

The perturbation to H_k is

$$\lambda H_{k1} = \lambda H_1 + \lambda[\frac{1}{\lambda}(\sum_{i=1}^{k} \alpha_1' (x, \epsilon_i) - \sum_{i=1}^{k} \widetilde{\alpha_1' (x, \epsilon_i)})]$$

Therefore, the eigenvalues of H_k will be shifted up to first order correction to

$$\epsilon_i \to \epsilon_i + \langle \theta_{\epsilon_i} | \lambda H_{k1} | \theta_{\epsilon_i} \rangle = \widetilde{\epsilon_i}$$

Here we see that although previously H_0 and H_k had same spectrum except for the extra eigen values added in the spectrum of H_k, it is no longer isospectral. Previously we saw that,

$$\epsilon_{q+j} = E_{m_j} \tag{5.5}$$

But now

$$\widetilde{\epsilon_{q+j}} \neq \widetilde{E_{m_j}}$$

The eigen energies of initial Hamiltonian becomes,

$$E_n \to E_n + \langle \psi_0 | \lambda H_1 | \psi_0 \rangle = \widetilde{E_n}$$

Now if we look for how the eigenstate of the perturbed Hamiltonians changes because of the perturbation, we see using the standard technique,

$$\left| \widetilde{\theta_n}^{\,1} \right\rangle = \sum_{l \neq n} \frac{\langle \theta_n^0 | \lambda H_{k1} | \theta_l^0 \rangle}{(E_n^0 - E_l^0)} \left| \theta_n^0 \right\rangle \tag{5.7}$$

Here we have all the tools needed for constructing the coherent state of the perturbed system. We need basically the eigenvalues and the eigenstate of the perturbed Hamiltonians.

5.2 Natural coherent state of the SUSY partner Hamiltonians of the initial perturbed Hamiltonian:

In this part we try to see how the coherent state of a nonlinear system which is governed by its natural algebra gets affected by the initial perturbation. This is true in general for the intrinsic and the linear algebra also. However, those are calculable in pretty similar way as we do here with the natural one.

We recall,

$$|z,\alpha\rangle_{k_{\mathcal{N}}} = \left[\sum_{m=0}^{\infty}\frac{|z|^{2m}}{\widetilde{\rho}_m}\right]^{-\frac{1}{2}}\sum_{m=0}^{\infty}e^{-i\alpha\left(E_{m+m_p+1}-E_{m_p+1}\right)}\frac{z^m}{\sqrt{\widetilde{\rho}_m}}\left|\theta_{m+m_p+1}\right\rangle \tag{5.8}$$

Where,

$$\widetilde{\rho}_0 = 1 \tag{5.9}$$

and, for m>0

$$\widetilde{\rho}_m = \frac{\rho_{m+m_p+1}}{\rho_{m_p+1}}\prod_{i=1}^{k}\left(E_{m+m_p+1}-\epsilon_i\right)\left(E_{m+m_p}-\epsilon_i\right)^2\ldots\left(E_{m_p+2}-\epsilon_i\right)^2\left(E_{m_p+1}-\epsilon_i\right) \tag{5.10}$$

Here $\widetilde{\rho}_m$ becomes, to first order approximation,

$$\widehat{\rho} = \frac{\widehat{\rho}_{m+m_p+1}}{\widehat{\rho}_{m_p+1}}\prod_{i=1}^{k}\left(\widetilde{E}_{m+m_p+1}-\widetilde{\epsilon}_i\right)\left(\widetilde{E}_{m+m_p}-\widetilde{\epsilon}_i\right)^2\ldots\left(\widetilde{E}_{m_p+2}-\widetilde{\epsilon}_i\right)^2\left(\widetilde{E}_{m_p+1}-\widetilde{\epsilon}_i\right)$$

Hence the natural coherent state changes to,

$$\widetilde{|z,\alpha\rangle}_{k_{\mathcal{N}}} = \left[\sum_{m=0}^{\infty}\frac{|z|^{2m}}{\widehat{\rho}_m}\right]^{-\frac{1}{2}}\sum_{m=0}^{\infty}e^{-i\alpha\left(\widetilde{E}_{m+m_p+1}-\widetilde{E}_{m_p+1}\right)}\frac{z^m}{\sqrt{\widehat{\rho}_m}}\left|\widetilde{\theta}_{m+m_p+1}\right\rangle$$

Here $\left|\widetilde{\theta_m}\right\rangle$ is the eigenstate of the perturbed SUSY partner potential. However, $\left|\widetilde{\theta}_{m+m_p+1}\right\rangle$ is not the eigenstate of the H_k, neither is $\widetilde{E}_{m+m_p+1}$ or $\widetilde{E}_{m_p+1}$ the eigen energies of the same.

5.3 Intrinsic coherent state of the SUSY partner Hamiltonians of the initial perturbed Hamiltonian:

The intrinsci coherent state is given by,

$$|z,\alpha\rangle_k = \left(\sum_{m=0}^{\infty}\frac{|z|^{2m}}{\rho_m}\right)^{-\frac{1}{2}}\sum_{m=0}^{\infty}e^{-i\alpha(E_m-E_0)}\frac{z^m}{\sqrt{\rho_m}}\left|\theta_m\right\rangle \tag{5.12}$$

Where,

$$\rho_m = \begin{cases} 1 & \text{if } m = 0 \\ (E_m - E_0)\dots(E_1 - E_0) & \text{if } m > 0 \end{cases} \tag{5.13}$$

We have seen that The eigen energies of initial Hamiltonian becomes,

$$E_n \to E_n + \langle\psi_0|\lambda H_1|\psi_0\rangle = \widetilde{E_n}$$

Therefore we can simply substitute the E_ms by $\widetilde{E_m}$s.
Hence ρ_m becomes

$$\hat{\rho_m} = \begin{cases} 1 & \text{if } m = 0 \\ \left(\widetilde{E}_m - \widetilde{E}_0\right)\dots\left(\widetilde{E}_1 - \widetilde{E}_0\right) & \text{if } m > 0 \end{cases} \tag{5.14}$$

Now if we replace ρ_m by $\hat{\rho}_m$ this gives,

$$|z,\alpha\rangle_k = \left(\sum_{m=0}^{\infty} \frac{|z|^{2m}}{\hat{\rho}_m}\right)^{-\frac{1}{2}} \sum_{m=0}^{\infty} e^{-i\alpha(E_m - E_0)} \frac{z^m}{\sqrt{\rho_m}} \left|\widetilde{\theta_m}\right\rangle \tag{5.15}$$

Where $\left|\widetilde{\theta_m}\right\rangle$ is the eigenstate of the perturbed Hamiltonian.

5.4 Conclusion:

In this section we saw that a little perturbation to the initial Hamiltonian is propagated along the series of the Partner Hamiltonians. The perturbations to the partner Hamiltonians are however not quantitatively equal in general. These propagation of perturbation eventually results in the shift in the spectrum of H_k. and in the eigenstates as well. The fact that the coherent states constructed out from different algebras depend on the spectrum and eigenstates leads to the modification in the coherent states too. We have, at a very naive level, checked with the help of generic forms of expressions how these modification in different coherent states can be obtained.

Chapter 6

Conclusion and Future Prospects

In this work we have seen that the the idea of coherent states is far more general than it was previously conceived. We can construct coherent states for nonlinear systems just like we had done it for linear quantum system like quantum Harmonic oscillator. This generalization is comparatively a recent development and needs further investigation. Moreover, we have argued that the supersymmetric quantum mechanics which is an excellent tool for studying different quantum systems with unparalleled ease and it can be used to find exactly solvable potentials once we are provided with one of them. The emergence and role of partner Hamiltonians of an initial one is discussed. Construction of coherent states from its definition as the eigenstate of the annihilation operator is an algebraic process.For long Annihilation and creation operators were known to be relevant only to the context of Heisenberg-Weyl algebra. But the systems which do not obey that can also have annihilation and creation operators corresponding their underlying algebraic structure. Therefore it is of enormous importance to study the algebra of a sytem which do not fit in Heisenberg-Weyl algebra or for that matter a non-linear system. Once we explore their algebra we can immediatey identify that the annihilation and creation operators associated with those are merely a distortion of the ones associated with the Heisenberg-Weyl algebra.

Once we recognize the different algebras of a nonlinear systems, one natural question raises. How much of these algebraic features of a system are inherited to its SUSY partner Hamiltonians. We try to find the answer to this and see that it is indeed inherited along the series of the partner Hamiltonians.Further, as we know that the construction of coherent states depend on the underlying algebra it is evident that the cohrenet states of the SUSY partners of Hamiltonian of the nonlinear system will also be modified. In this work we present a comprehensible study on the same.

Another aspect of the work is to study the affect of perturbation to the coherent states. In practical world most of the systems are not exactly solvable and we must take help of approximation methods. Here we have discussed if the initial Hamiltonians is perturbed slightly how this perturbation is propagated along the chain of partner Hamiltonians. After that we tried to construct the coherent states for those perturbed systems.

This project leaves many areas for future investigation. We have seen only an algebraic construction of coherent states. It may be looked for in future how the process of linearization of the nonlinear algebras can influence at a differential level. In the part of perturbation the basic idea and calculations could be refined further and the generic description could be applied to a real problem. Coherent states are extremely important in the filed of quantum optics. This study can, with required alterations, be implemented in studying quantum optical phenomena.

Bibliography

[Andrianov 93] A.A. Andrianov, M.V. Ioffe & V.P. Spiridonov. *Higher-derivative supersymmetry and the Witten index*. Physics Letters A, vol. 174, no. 4, page 273–279, Mar 1993.

[Andrianov 03] A.A. Andrianov & A.V. Sokolov. *Nonlinear supersymmetry in quantum mechanics: algebraic properties and differential representation*. Nuclear Physics B, vol. 660, no. 1-2, page 25–50, Jun 2003.

[Bagchi 01] Bijan Kumar Bagchi. Supersymmetry in quantum and classical mechanics / bijan kumar bagchi. Chapman Hall/CRC monographs and surveys in pure and applied mathematics ; 116. Chapman Hall/CRC, Boca Raton, Fla, 2001.

[Dwyer 14] Sheila Dwyer. *Squeezing quantum noise*. Physics Today, vol. 67, no. 11, pages 72–73, 2014.

[Fernández 07] David J Fernández, Véronique Hussin & Oscar Rosas-Ortiz. *Coherent states for Hamiltonians generated by supersymmetry*. Journal of Physics A: Mathematical and Theoretical, vol. 40, no. 24, page 6491–6511, May 2007.

[Gerry 04] Christopher Gerry & Peter Knight. Coherent states, page 43–73. Cambridge University Press, 2004.

[Ghosh 12] Subir Ghosh. *Coherent states for the nonlinear harmonic oscillator*. Journal of Mathematical Physics, vol. 53, no. 6, page 062104, Jun 2012.

[Glauber 06] Roy J. Glauber. *One Hundred Years of Light Quanta (Nobel Lecture)*. ChemPhysChem, vol. 7, no. 8, pages 1618–1639, 2006.

[Klauder 85] J.R. Klauder & Skagerstam. Coherent states: Applications in physics and mathematical physics. 1985.

[Leiva 03] Carlos Leiva & Mikhail S Plyushchay. *Superconformal mechanics and nonlinear supersymmetry.* Journal of High Energy Physics, vol. 2003, no. 10, page 069–069, Oct 2003.

[Naila 15] Amir Naila & Iqbal Shahid. *Coherent states for nonlinear harmonic oscillator and some of its properties.* Journal of Mathematical Physics, vol. 56, no. 6, page 062108, 2015.

[Perelomov 86] Askold Perelomov. Generalized coherent states and their applications. 1986.

[Román-Ancheyta 15] Ricardo Román-Ancheyta & José Récamier. *Approximate Coherent States for Nonlinear Systems.* Concepts of Mathematical Physics in Chemistry: A Tribute to Frank E. Harris - Part A, page 299–322, 2015.

[Sato 02] Masatoshi Sato & Toshiaki Tanaka. *N-fold supersymmetry in quantum mechanics—analyses of particular models.* Journal of Mathematical Physics, vol. 43, no. 7, page 3484–3510, Jul 2002.