Models To Solve The Problems Associated With The Hot Big Bang Model

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Certificate of Examination

This is to certify that the dissertation titled **Models To Solve The Problem With The Hot Big Bang Model** submitted by **Rishi Gangahdar G** (Reg. N for the partial fulfillment of BS- MS Dual Degree programme of the instit examined by the thesis committee duly appointed by the institute. The comm work done by the candidate satisfactory and recommends that the report be a

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Declaration

The work presented in this dissertation has been carried out by me under the guidance of Dr. Kinjalk Lochan at the ndian Institute of Science Education and Research, Mohali and Dr. Jaffino at the Indian Institute of Science Education and Research, Mohali.

This work has not been submitted in part or in full for a degree, a diploma, or a fellowship to any other university or institute. Whenever contributions of others are involved, every effort is made to indicate this clearly, with due acknowledgement of collaborative research and discussions. This thesis is a bonafide record of original work done by me and all sources listed within have been detailed in the bibliography.

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In my capacity as the supervisor of the candidates project work, I certify that the above statements by the candidate are true to the best of my knowledge.

Dr Kinjalk Lochan
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Notation

The signature of the metric used in this paper is $(+,-,-,-)$. We will also set the quantities c (Speed of Light), $\hbar$ (Planck's Constant), and $8\pi G$ (Re-scaled Newton's Gravitational Constant) to be equal to 1. We will use the variable "t" for cosmic time and the variable η for conformal time.

Dots over variables refers to their cosmic time derivative, i.e., $\dot{x} = \frac{dx}{dt}$, and primes over variables refer to their conformal time derivative, i.e., $x' = \frac{dx}{d\eta}$.

The Ricci Tensor that we will use is

$$R_{\mu\nu} = \partial_\nu \Gamma^\lambda_{\mu\lambda} - \partial_\lambda \Gamma^\lambda_{\mu\nu} + \Gamma^\lambda_{\sigma\mu}\Gamma^\sigma_{\lambda\nu} - \Gamma^\lambda_{\mu\nu}\Gamma^\sigma_{\sigma\lambda}$$

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Contents

Abstract

In this thesis we discuss about the problems with the hot big bang model of cosmology and models that attempt to solve them. We will then discuss the idea of inflation – an era of exponential expansion of the universe, and why our universe required an epoch of accelerated expansion. After discussing about the matter fields which could have driven inflation in the beginning of the radiation dominated epoch, we establish the conditions under which inflation can occur. Apart from this, theory of inflation naturally provides a theoretical basis to the inhomogeneities observed in the Cosmic Microwave Background (CMB) radiation that we observe today. After discussing the theory of cosmological perturbations in detail, we show the methods to calculate the power spectrum corresponding to the comoving curvature. Finally, we will discuss about regularized big bang models and why these models come into picture. Afterwards we will discuss about two such models and check whether they solve the problem of singularity in the case of a closed universe.

Chapter 1

Introduction

In this thesis we discuss about some of the problems with the hot big bang model of cosmology, namely the problem of initial conditions and the problem of singularity and discuss some of the models that were proposed as a solution to these problems. A model that was proposed and will be discussed in this thesis as a solution to the problem of initial conditions is called inflation and the model that could offer a solution the problem of singularity is obtained though the introduction of a topological defect.

After a brief introduction to the standard hot big-bang cosmology, we discuss about problems associated with the standard hot big-bang model. Among many problems with the big-bang model, we discuss only the crucial ones: horizon problem and flatness problem. Then we show how the idea of inflation, an epoch of accelerated expansion of the universe, can solve these problems.

Since the cosmological inflationary theory provides a theoretical basis to the origin of inhomogeneities in the CMB, we discuss cosmological perturbation theory in great detail. We further discuss the construction of gauge invariant variables corresponding to the metric perturbations, and derive the necessary dynamical equations that will help us in calculations in the subsequent chapters.

Making use of the gauge invariant variable corresponding to the metric perturbations, we show the methods to calculate the power spectrum corresponding to the comoving curvature.

Finally, we will look at the problem of singularity and compare two proposed models to solve this problem. Since both these models were proposed to solve the problem of singularity in the case of a universe without spatial curvature, they might not work in the case of a closed universe and result in a singularity during the "big crunch". We will then discuss a procedure that will help us to derive such models with a topological defect that solve the problem of singularity for all values of spatial curvature.

Chapter 2

Hot big bang model of Cosmology

Our universe is homogeneous and isotropic at very large scales (scales much larger than the size of a galaxy), which means at a given time the universe appears the same from any given point and direction of observation. The spacetime metric which respects this can be written as (without loss of any generality)

$$ds^2 = dt^2 - S^2(t) h_{ij} dx^i dx^j. \tag{2.1}$$

Here, h_{ij} are functions of the spatial coordinates (x^1, x^2, x^3). We should also note that homogeneity and isotropy implies that the line element $h_{ij} dx^i dx^j$ must correspond to a maximally symmetric 3-space. We can also see this by noticing that homogeneity corresponds to 3 killing vectors and isotropy corresponds to 3 more killing vectors and a maximally symmetric 3-space admits 6 killing vectors. This gives rise to the following line element:

$$ds^2 = dt^2 - S^2(t) \left[\frac{d\bar{r}^2}{1 - \mathscr{K}\bar{r}^2} + \bar{r}^2 (d\theta^2 + sin^2\theta d\phi^2) \right], \tag{2.2}$$

where $\mathscr{K}$ is the spatial curvature. To remove the arbitrariness of $\mathscr{K}$, one can re-scale the radial coordinate as $r = \sqrt{\mathscr{K}}\,\bar{r}$, define $S(t) = \sqrt{\mathscr{K}}\,a(t)$, and $K = \frac{\mathscr{K}}{|\mathscr{K}|}$. This gives rise to the FLRW line element

$$ds^2 = dt^2 - a^2(t) \left[\frac{dr^2}{1 - Kr^2} + r^2 (d\theta^2 + sin^2\theta d\phi^2) \right]. \tag{2.3}$$

Parameter K can only take values 0 and ± 1. Specifically, K is taken to be +1 (-1) if the spatial curvature is positive (negative), and 0 if the spatial part of the metric is flat. Note that the scale factor $a(t)$ describes the relative size of the space-like hypersurfaces at different times.

For the sake of convenience one can rewrite the metric using conformal time, i.e., $d\eta =$

$dt/a(t)$ as

$$ds^2 = a^2(\eta)\left[d\eta^2 - \frac{dr^2}{1-Kr^2} + r^2(d\theta^2 + sin^2\theta d\phi^2)\right]. \quad (2.4)$$

Making use of this metric in the Einstein equation, and assuming the energy content of the universe allows us to understand the dynamics of the universe under different circumstances.

2.1 Einstein Equations

The Einstein field equations is given by

$$R_{\mu\nu} + \frac{1}{2}Rg_{\mu\nu} = -T_{\mu\nu}, \quad (2.5)$$

where $R_{\mu\nu}$ is the Ricci tensor and $R = g^{\mu\nu}R_{\mu\nu}$ is called the Ricci scalar. These two quantities are calculated using the metric and they are purely geometric in nature. Feeding in the Energy-Momentum tensor $T_{\mu\nu}$ in the Einstein equation, one arrives at the form of the scale factor $a(t)$.

2.1.1 Components of Ricci tensor

To compute the Ricci tensor we must first compute the components of the Christoffel symbols, defined as

$$\Gamma^{\alpha}_{\mu\nu} = g^{\alpha\lambda}\left(\partial_\mu g_{\lambda\nu} + \partial_\nu g_{\lambda\mu} - \partial_\lambda g_{\mu\nu}\right). \quad (2.6)$$

The components of the Christoffel symbols corresponding to the FLRW metric in Eq.(2.3) are:

$$\Gamma^0_{11} = \frac{\dot{a}a}{1-Kr^2}, \qquad \Gamma^0_{22} = \dot{a}ar^2, \qquad \Gamma^0_{33} = \dot{a}ar^2\sin^2(\theta) \quad (2.7)$$

$$\Gamma^1_{01} = \frac{\dot{a}}{a}, \qquad \Gamma^1_{11} = \frac{Kr}{1-Kr^2}, \qquad \Gamma^1_{33} = -r(1-Kr^2)\sin^2(\theta), \quad (2.8)$$

$$\Gamma^2_{02} = \frac{\dot{a}}{a}, \qquad \Gamma^2_{12} = \frac{1}{r}, \qquad \Gamma^2_{33} = \sin(\theta)\cos(\theta), \quad (2.9)$$

$$\Gamma^3_{03} = \frac{\dot{a}}{a}, \qquad \Gamma^3_{13} = \frac{1}{r}, \qquad \Gamma^3_{23} = \cot(\theta). \quad (2.10)$$

Making use of the components of Christoffel symbol in the expression for Ricci tensor, which is

$$R_{\mu\nu} = \partial_\nu\Gamma^{\lambda}_{\mu\lambda} - \partial_\lambda\Gamma^{\lambda}_{\mu\nu} + \Gamma^{\lambda}_{\sigma\mu}\Gamma^{\sigma}_{\lambda\nu} - \Gamma^{\lambda}_{\mu\nu}\Gamma^{\sigma}_{\sigma\lambda}, \quad (2.11)$$

one can arrive at

$$R_{00} = 3\frac{\ddot{a}}{a}, \qquad R_{11} = -\frac{\ddot{a}a + 2\dot{a}^2 + 2K}{1 - Kr^2}, \tag{2.12}$$

$$R_{22} = (\ddot{a}a + 2\dot{a}^2 + 2K)r^2, \qquad R_{33} = (\ddot{a}a + 2\dot{a}^2 + 2K)r^2 \sin^2(\theta). \tag{2.13}$$

Employing the components of Ricci tensor, one can calculate Ricci scalar as shown below

$$R = 6\left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{k}{a^2}\right). \tag{2.14}$$

2.1.2 Energy-Momentum Tensor

Having discussed the geometric part of the Einstein equation, let us discuss the energy/matter part of the same in this sub-section. In the forthcoming sections we consider the universe to be dominated by an ideal fluid given by the following energy-momentum Tensor:

$$T^{\mu}_{\nu} = (\rho + p)u^{\mu}u_{\nu} - p\delta^{\mu}_{\nu}, \tag{2.15}$$

where u^{μ} is the velocity vector of the ideal fluid, and we chose to work in a coordinate system where $u^{\mu} = (1,0,0,0)$.

2.2 Friedmann Equations

Employing the expression for Ricci scalar and the components of Ricci tensor in Einstein equation Eq. (2.5), and assuming the universe to be dominated by an ideal fluid we arrive at the following equations, usually called Friedmann equations,

$$\mathscr{H}^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{1}{3}\rho - \frac{K}{a^2} \tag{2.16}$$

$$\dot{\mathscr{H}} + \mathscr{H}^2 = \frac{\ddot{a}}{a} = -\frac{1}{6}(\rho + 3p), \tag{2.17}$$

where $\mathscr{H}$, called the Hubble parameter, defined as $\mathscr{H} = \dot{a}/a$, and the over-dot denotes differentiation with respect to cosmic time t. The Hubble parameter $\mathscr{H}$ denotes the characteristic scale of the universe. Note that Eq. (2.16) and (2.17) correspond to the time-time and space-space component of the Einstein equation respectively.

We also have the following property of the Energy-Momentum Tensor:

$$\partial_{\mu}T^{\mu\nu} = 0 \tag{2.18}$$

This equation gives rise to a relation relating the pressure (p) and the energy density (ρ), and it is given by

$$\frac{d\rho}{dt} + 3\mathscr{H}(\rho + p) = 0. \tag{2.19}$$

The solution of this equation is

$$\rho = \rho_0 a^{-3(1+w)}, \tag{2.20}$$

where, $w = \frac{p}{\rho}$ is called the equation of state parameter and is taken to be a constant.

Substituting Eq. (2.20) in (2.16), we get

$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{K}{a^2} = \frac{1}{3}\rho_0 a^{-3(1+w)}. \tag{2.21}$$

The solution corresponding to Eq. (2.21) for $K = 0$ is given by

$$a(t) = a_0 t^{\frac{2}{3(1+w)}}, \text{ where } w \neq -1, \tag{2.22}$$

$$a(t) = a_0 e^{\mathscr{H} t}, \text{ where } w = -1. \tag{2.23}$$

For the case of matter dominated universe, $w = 0$, and in the case of radiation dominated universe, $w = \frac{1}{3}$. In both these cases the second time derivative of the scale factor is negative, i.e., $\ddot{a} < 0$. This is pivotal to the problem of initial conditions that necessitates inflation. We will explore this in detail in the section on Cosmological Inflation.

Chapter 3

Cosmological Inflation

3.1 Problem of Initial Conditions

The Hot Big-Bang model is one of the most successful models that describe the evolution of the universe. It is successful in explaining the abundance of the lighter elements, like H and He. However, there are several problems associated with the hot big bang model. Some of them are the problems of initial condition (horizon problem and flatness problem), problem of singularity, etc. The problems that we will be discussing in this chapter is the problems of initial conditions. We will discuss the problem of singularity in chapter 6. The problem of initial condition arises because the CMB photons arriving from even diametrically opposite regions have nearly the same temperature (approximately 3K), even though the region from which the CMB photons are arriving us have never been in causal contact before the time of decoupling (according to the hot big bang model). In this chapter we discuss how inflation solves the problems of initial condition, and give a natural basis for the origin of primordial inhomogeneities.

3.1.1 The Horizon Problem

For the purpose of motivating the horizon problem, it is convenient to work with the conformal time coordinate η, and the FLRW line-element is given by

$$ds^2 = a^2(\eta)(d\eta^2 - d\vec{r}^2). \tag{3.1}$$

Since the line element ds^2 is vanishing for null geodesics, the physical size r of the event horizon can be found from the differential equation

$$\frac{dr}{d\eta} = \pm 1, \tag{3.2}$$

solving which leads to

$$r = \pm\eta + constant. \tag{3.3}$$

Therefore, the horizon size at any instant t is given by

$$r = a(t)\int_{t_1}^{t}\frac{d\tilde{t}}{a(\tilde{t})} = . \tag{3.4}$$

The equation (3.4) can be rewritten in the following form:

$$r = a(t)\int_{0}^{a} dln(\tilde{a})\frac{1}{\tilde{a}\tilde{\mathscr{H}}} \tag{3.5}$$

This form is quite useful while studying inflation.

Since the universe was dominated by radiation from the big-bang ($t = 0$) to the time of decoupling ($t = t_{\rm dec}$), and was dominated by non-relativistic matter after the time of decoupling. The physical size of the universe of the surface of last scattering from which we receive cosmic microwave background (CMB) radiation is

$$l_B(t_0, t_{\rm dec}) = a_{\rm dec}\int_{t_{\rm dec}}^{t_0}\frac{d\tilde{t}}{a(\tilde{t})} = 3(t_{\rm dec}^2 t_0)^{\frac{1}{3}} - 3t_{\rm dec} \approx 3(t_{\rm dec}^2 t_0)^{\frac{1}{3}}, \tag{3.6}$$

where, t_0 is the cosmic time today. The approximation above is the result of the fact that $t_0 \gg t_{\rm dec}$.

Now, the size of the horizon at the time of decoupling is

$$l_F(t_0, t_{\rm dec}) = a_{\rm dec}\int_{0}^{t_{\rm dec}}\frac{d\tilde{t}}{a(\tilde{t})} = 2t_{\rm dec}. \tag{3.7}$$

Thus the ratio (Q) of the backward and forward light cones is

$$Q = \frac{l_B}{l_F} = \frac{3}{2}\left(\frac{t_0}{t_{\rm dec}}\right)^{\frac{1}{3}} \approx 70, \tag{3.8}$$

where we have made use of an observational fact that $t_{\rm dec} \approx 10^5$years and $t_0 \approx 10^{10}$years [Bassett 06]. This shows that the backward light cone is approximately 70 times larger than the forward light cone. This means that the regions in space that have come into causal contact now will never have been in causal contact before decoupling. However, the universe is isotropic and homogeneous at present.

3.1.2 Flatness Problem

Let us take a look at the following Friedmann equation:

$$\mathscr{H}^2 = \frac{1}{3}\rho(a) - \frac{k}{a^2}, \tag{3.9}$$

where k is the scalar curvature of the spatial part of the metric. This can be rewritten as

$$1 - \Omega(a) = -\frac{k}{(a\mathscr{H})^2}, \tag{3.10}$$

where $\Omega(a) = \frac{\rho(a)}{\rho_{crit}(a)}$, and $\rho_{crit} = 3\mathscr{H}^2$.

Since $a(t) = t^{\frac{2}{3(1+w)}}$ $(w \neq -1)$, $(a\mathscr{H})^{-1}$ increases with time for both the radiation and matter dominated cases, and thus the value $\Omega = 1$ is an unstable fixed point. Given the observation on the spatial curvature of the universe, the initial conditions must be very fine-tuned (at the Planck scale the deviation of $\Omega - 1$ from 0 must be less than 10^{-61} units)[Baumann 09].

This problem indicates issue with the predictive power of the model. A theory that would be much more preferable would be the one where these conditions are arrived at dynamically, rather than as an assumption. Inflation turned out to be such a theory. Inflation[Guth 81].

3.2 Theory of Inflation

The most important thing that needs to be understood is that if 2 particles are separated by a distance greater than $(a\mathscr{H})^{-1}$, then they will not be able to communicate with each other now, but, if they would have never been in causal contact only if the particle horizon is greater than their separation. Thus, the value of the particle Horizon can be much greater than $(a\mathscr{H})^{-1}$.

From equation (3.5), we can see that the above condition can be realised if, for a certain period the following condition is met:

$$\frac{d}{dt}(a\mathscr{H})^{-1} < 0 \tag{3.11}$$

This solves the horizon problem if this condition lasts long enough. This also solves the flatness problem, because, in a non-flat universe the solution $\Omega = 1$ is an attractor solution. Thus, inflation drives the universe to flatness, and if inflation lasts long enough, it matches with the results of observation. This scenario where $\frac{d(a\mathscr{H})^{-1}}{dt} < 0$ is known as inflation.

Equation (3.11) can be cast in different ways as

$$\frac{d^2a}{dt^2} > 0 \text{ and } \rho + 3P < 0. \tag{3.12}$$

The second equality comes from the Friedmann equations. Thus, Inflation can be viewed in 3 equivalent ways.

1. The condition (3.11) is equivalent to saying that the Hubble radius $(aH)^{-1}$ must decrease with time for inflation to occur.

2. The first inequality in equation (3.15) implies accelerated expansion. This is the reason why inflation is known as the period of accelerated expansion. We can also write this condition in another form, which will be useful in the next part. Let us define a new quantity ε_H (which is called the slow-roll parameter) as:

$$\varepsilon_H = -\frac{\dot{\mathscr{H}}}{\mathscr{H}^2} \tag{3.13}$$

   From the second Friedmann equation, namely,

$$\ddot{a} = \mathscr{H}^2 + \dot{\mathscr{H}}. \tag{3.14}$$

   This gives the condition in terms of the parameter ε_H, i.e..,

$$\varepsilon_H = -\frac{\dot{\mathscr{H}}}{\mathscr{H}^2} = -\frac{dln(\mathscr{H})}{dN} < 1, \tag{3.15}$$

   where $N = Hdt$ gives the condition as the number of e-folds “N" of expansion. Thus, equation (3.15) says that the fractional change in the Hubble parameter per e-fold is small.

3. The second inequality in equation (3.12) says that $\rho < -3P$, and since the energy-density is always positive, the pressure must be negative! This does not happen in normal hydrodynamic matter, but it is possible in the case of scalar fields, which is exactly what one considers while studying Inflation.

3.2.1 Duration of Inflation

Let us assume for the sake of simplicity that the expansion that occurs during inflation is purely exponential ($\varepsilon_H = 0$). To quantitatively describe the amount of inflation that has occurred, we can find the number of e-folds by which the scale factor has expanded before inflation ends. The size of the backward light-cone is given by

$$l_B = a_{\rm dec}\int_0^{t_{\rm dec}} \frac{d\tilde{t}}{a(\tilde{t})}. \tag{3.16}$$

Since the expansion of the scale factor is exponential in this case, we can neglect the effect of radiation dominated universe while calculating the forward light cone. Also, since the scale factor increases exponentially, $\mathscr{H}$ must be a constant, i.e., $a(t) \propto e^{\mathscr{H}t}$. Therefore, the size of the backward light-cone can be found to be

$$l_B \approx a_{\rm dec}\int_{t_i}^{t_f} \frac{d\tilde{t}}{a(t_i)e^{\mathscr{H}(\tilde{t}-t_i)}}, \tag{3.17}$$

where t_i is the time when inflation starts and t_f is the time when inflation ends. Evaluating the integral, and ignoring the term with scale factor evaluated at the beginning of the inflation we get

$$l_B \approx a_{\rm dec}\frac{a_{dec}}{\mathscr{H}}A, \tag{3.18}$$

where $A \equiv \frac{a(t_f)}{a(t_i)}$. This gives rise to the ratio Q of the forward and backward light cones as

$$Q = \frac{l_B}{l_F} = \frac{A}{10^{26}}, \tag{3.19}$$

where we have taken $\mathscr{H}$ to be 10^{13}GeV[Liddle 03] [Dodelson 03], and the size of the forward light cone l_F is taken from Eq.(3.7). To solve the problem of initial conditions, we require that the ratio Q must be of the order of 1. This implies that A must be at least 10^{26} or e^{60}. Therefore, one can conclude that the minimum number of e-folds that is required to overcome the horizon problem is roughly 60.

3.2.2 Inflation driven by Scalar-fields

Since the radiation ($w = \frac{1}{3}$) or non-relativistic matter($w = 0$) cannot give rise to the negative pressure condition, one needs to find matter fields that can give rise to negative pressure. It can be shown that scalar fields can be a good candidate.
The action corresponding to the scalar field can be written as

$$S = \int d^4x\sqrt{-g}[\frac{1}{2}R + \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)], \tag{3.20}$$

where $V(\phi)$ is the potential of the scalar field. Since the background must be homogeneous and isotropic, we require the scalar field to depend only on time, and not on the spatial coordinates. The non-zero components of the energy-momentum tensor can be computed from the action as

$$T^0_0 = \rho = \frac{\dot{\phi}^2}{2} + V(\phi), \quad T^i_j = -p\delta^i_j = -\left(\frac{\dot{\phi}^2}{2} - V(\phi)\right)\delta^i_j. \tag{3.21}$$

Therefore, the equation of state parameter w can be written as

$$w_\phi = \frac{\dot{\phi}^2/2 - V(\phi)}{\dot{\phi}^2/2 + V(\phi)}. \tag{3.22}$$

Varying the scalar field action in Eq. (3.20) with respect to ϕ, we obtain the equation of motion corresponding to the scalar as

$$\ddot{\phi} + 3\mathscr{H}\dot{\phi} + V_{,\phi} = 0. \tag{3.23}$$

From Eq.(3.21) and the second inequality in Eq. (3.12), one can obtain the condition for inflation to take place

$$\dot{\phi}^2 < V(\phi), \tag{3.24}$$

i.e., the potential term must dominate the kinetic term of the scalar field ϕ. Using the expressions for pressure and energy density that we obtained in Eq. (3.21), one gets the following Friedmann equations:

$$\mathscr{H}^2 = \frac{1}{3}\left(\frac{\dot{\phi}^2}{2} + V(\phi)\right) \tag{3.25}$$

$$\dot{\mathscr{H}} = -\frac{\dot{\phi}^2}{2} \tag{3.26}$$

From these Friedmann equations we can write the potential V and field variable ϕ in terms of the Hubble parameter $\mathscr{H}$ as

$$\phi(t) = \sqrt{2}\int\sqrt{\dot{\mathscr{H}}}dt, \quad V(t) = 3\mathscr{H}^2 + \dot{\mathscr{H}}. \tag{3.27}$$

Thus, given the scale factor, we can derive the form of potential and the field ϕ (However, to arrive at the field ϕ, we require a boundary condition). The reverse, i.e., arriving at the scale factor given a potential is also possible, provided we are given the appropriate boundary conditions (since the above equations are given in terms of the time derivative of the scale factor).

3.2.3 Slow-Roll Conditions

The slow-roll conditions determine the occurrence of inflation (first slow-roll condition), and denote whether inflation lasts for a sufficiently long period of time (the second slow-roll condition). The first slow-roll parameter ε_H is what we defined previously in Eq. (3.15). For inflation to occur, we require this parameter to be less than 1. In the first slow-roll approximation we take the parameter to be approximately 0. In this limit, the scale factor evolves exponentially with time. Using the Friedmann equation we can write ε_H as,

$$\varepsilon_H = \frac{3}{2}(1+w_\phi) = \frac{1}{2}\frac{\dot{\phi}^2}{H^2} \tag{3.28}$$

Comparing equations (3.25) and (3.28) we have

$$\dot{\phi}^2 \ll V(\phi). \tag{3.29}$$

We also want inflation to last in the exponential phase for a long period of time to get rid of the problems of initial condition. This means that we want w_ϕ to be equal to -1 for a sufficiently long period of time, which implies the rate of change of $\dot{\phi}$ needs to be "small". This can be achieved by increasing the magnitude of the dissipative term, i.e., $3\mathscr{H}\dot{\phi}$, in Eq.(3.23). This gives rise to the following condition that

$$\ddot{\phi} \ll 3\mathscr{H}\dot{\phi}. \tag{3.30}$$

This implies the quantity, called the second slow-roll parameter (δ_H), has the following constraint:

$$\delta_H = -\frac{\ddot{\phi}}{\mathscr{H}\dot{\phi}} \approx 0 \tag{3.31}$$

In the slow roll regime, the following conditions hold true:

$$\varepsilon_H \approx \delta_H \approx 0, \tag{3.32}$$

and the Friedmann equations reduce to the following:

$$3\mathscr{H}^2 \approx V(\phi), \qquad 3\dot{\phi} \approx V_{,\phi} \tag{3.33}$$

These slow-roll conditions may also be expressed as the conditions on the shape of the inflationary potential. These parameters, called the **potential slow roll parameters**, determine the shape of the potential, and are

$$\varepsilon_v = \frac{1}{2}\left(\frac{V_{,\phi}}{V}\right)^2, \tag{3.34}$$

$$\delta_v = \frac{V_{,\phi\phi}}{V}. \tag{3.35}$$

In the slow-roll regime, these parameters obey

$$\varepsilon_v \approx \delta_v \approx 0 \tag{3.36}$$

Note that the smallness of the Hubble slow-roll parameters does imply the smallness of the potential slow-roll parameters, but the converse is not true (because for example $\dot{\phi}$ can be as large as we want, even when the potential slow-roll conditions are satisfied as these conditions in no way affect how the kinetic term). Therefore, the smallness of the potential slow-roll parameters is a necessary, but not sufficient condition for inflation.

In the slow-roll regime, these parameters are related to the slow-roll parameters as,

$$\varepsilon_H \approx \varepsilon_v \tag{3.37}$$

$$\delta_H \approx \eta_v - \varepsilon_v \tag{3.38}$$

Inflation ends when the first slow roll parameter equals 1 (by definition). Quantitatively to figure out by how much the scale factor expands during inflation, we can find the number of e-folds before inflation ends. This is given by:

$$N(\phi) = ln\left(\frac{a_{end}}{a}\right) = \int_t^{t_end} \mathscr{H}\,dt = \int_\phi^{\phi_{end}} \frac{\mathscr{H}}{\dot{\phi}} d\phi \approx \int_{\phi_{end}}^{\phi} \frac{V}{V_\phi} d\phi \tag{3.39}$$

The final equality is obtained from the slow-roll condition. This can also be written in terms of the first slow-roll parameter as:

$$\int_{\phi_{end}}^{\phi} \frac{d\phi}{\sqrt{\varepsilon_H}} \approx \int_{\phi_{end}}^{\phi} \frac{d\phi}{\sqrt{\varepsilon_v}} \tag{3.40}$$

As we have discussed before, the minimum number of e-folds necessary is roughly 60.

Chapter 4

Theory of Cosmological Perturbations

Though the idea of inflation was proposed to solve the problems associated with the hot big-bang model, it offers a mechanism on the generation of primordial perturbations. We know from CMB observation that the anisotropy in the CMB is small, i.e., one part in 10^5, which implies that the magnitude of deviation from homogeneity would have been much smaller in the earlier epochs of the universe. This provides us the hint that the generation and evolution of primordial perturbations can be studied using linear perturbation theory.

4.1 The Perturbed Metric

Let us first start with the FLRW metric,whose corresponding line element is given by

$$ds^2 =^{(0)} g_{\mu\nu}dx^\mu dx^\nu = dt^2 - a^2(t)\gamma_{ij}dx^i dx^j = a^2(\eta)(d\eta^2 - \gamma_{ij}dx^i dx^j), \qquad (4.1)$$

where, η is the conformal time, and γ_{ij} is the spatial part of the metric.

Any general form of perturbation in the FLRW line element can be written as (in the conformal time)

$$\delta ds^2 = a^2(\eta)[2\phi d\eta^2 - b_i d\eta dx^i - b_i dx^i d\eta - (-2\psi\gamma_{ij} + 2e_{ij})dx^i dx^j]. \qquad (4.2)$$

Here, we have made use of the fact that the metric is symmetric. The quantity b_i is a vector field, and the quantity e_{ij} is a tensor field. Without proof, we will use the following mathematical theorems[Stewart 90][Bardeen 80] on the decomposition of vector and tensor fields:

- Any vector field (b_i) can be decomposed into a scalar field (B) and a divergence-free vector field (S_i) as

$$b_i = \nabla_i B + S_i. \qquad (4.3)$$

- Any Tensor field (e_{ij}) can be decomposed into a scalar field (E), a divergence-free vector field (F_i), and a divergence-free and trace-free tensor field (h_{ij}) as

$$e_{ij} = \nabla_i \nabla_j E + \nabla_i F_j + \nabla_j F_i + h_{ij}. \tag{4.4}$$

Substituting equations (4.3) and (4.4) into (4.2), and separating into scalar, divergence-free vector, and divergence-free and trace-free tensor parts, we get the following equations:

$$\delta g^{(\text{scalar})} = a^2(\eta) \begin{pmatrix} 2\phi & -\nabla_i B \\ -\nabla_i B & 2(\psi\gamma_{ij} - \nabla_i \nabla_j E) \end{pmatrix} \tag{4.5}$$

$$\delta g^{\text{vector}} = a^2(\eta) \begin{pmatrix} 0 & -S_i \\ -S_i & \nabla_i F_j + \nabla_j F_i) \end{pmatrix} \tag{4.6}$$

$$\delta g^{\text{tensor}} = a^2(\eta) \begin{pmatrix} 0 & 0 \\ 0 & h_{ij} \end{pmatrix} \tag{4.7}$$

Throughout this article, we will only work with scalar perturbations.

4.2 Gauge-Invariant Perturbation Variables

If we take one coordinate chart and move to another coordinate chart, the form of the perturbed metric will change and thus change the values of the perturbation variables. For example, the form of the metric (perturbed) will be different in Cartesian and spherical polar coordinates, and thus the values of the perturbation variables will be very different. To get around this problem of non-covariance, one needs to introduce gauge invariant variables. This can be achieved by first considering only smooth chart transformations which relates coordinate systems that are infinitesimally close to each other. More formally, such a transformation is given by the Lie derivative of the metric along the vector field. Mathematically, this can be written as

$$\delta g^{\xi}_{\alpha\beta} = \mathscr{L}_\xi g_{\alpha\beta}, \tag{4.8}$$

where, the left hand side is the change in the metric effected by the vector field (infinitesimal coordinate transformation) $\xi = (\xi^0, \xi^{|i})$ (the fact that this is the generator of scalar transformations is a mathematical theorem and the proof of which is beyond the scope of this thesis), and $\mathscr{L}_\xi$ is the lie derivative along the vector field ξ. We want to see how the perturbation variables change due to such a coordinate transformation. The right-hand-side of (4.8) is given by

$$\mathscr{L}_\xi g_{\alpha\beta} = \xi^\gamma g_{\alpha\beta,\gamma} + \xi^\gamma_{,\alpha} g_{\gamma\beta} + \xi^\gamma_{,\beta} g_{\gamma\alpha}. \tag{4.9}$$

$$\therefore \mathscr{L}_\xi g_{\alpha\beta} = \xi^0 g_{\alpha\beta,0} + \xi^{|i} g_{\alpha\beta,|i} + \xi^0_{,\alpha} g_{0\beta} + \xi^{|i}_{,\alpha} g_{i\beta} + \xi^0_{,\beta} g_{0\alpha} + \xi^{|i}_{,\beta} g_{i\alpha}. \tag{4.10}$$

We shall now derive how each of component of the metric tensor transforms under such a transformation, and then construct variables, with the help of these transformation laws, that are invariant under such transformations. Here, we shall use "overdots" for "physical time" derivatives and "primes" for "conformal time" derivatives. We will also use "[.];" for covariant derivatives and "[.]," for partial derivatives.

$$\mathscr{L}_\xi g_{00} = 2a'a\xi^0 + 0 + \xi^{0\prime} a^2 + \xi^{|i}\prime \times 0 + \xi^{0\prime} a^2 + \xi^{|i}_{,\beta} \times 0 \tag{4.11}$$

$$\implies \mathscr{L}_\xi g_{00} = 2a'a\xi^0 + 2a^2\xi^{0\prime} \tag{4.12}$$

Similarly,

$$\mathscr{L}_\xi g_{0j} = -a^2 \xi'_{|j} + a^2 \xi^0_{|j} \tag{4.13}$$

For the spatial part, the calculation is slightly more tedious, and only the important steps in its derivation will be shown below.

$$\mathscr{L}_\xi g_{jk} = \xi^0 (a^2\gamma_{jk})' - \xi^{|i} (a^2\gamma_{jk})_{|i} + 0 - \xi^{|i}_{,j} (a^2 \gamma_{ik}) + 0 - \xi^{|i}_{,k} (\gamma_{ij} a^2) \tag{4.14}$$

$$\therefore \mathscr{L}_\xi g_{jk} = 2aa'\xi^0 \gamma_{jk} - 2a^2 [\xi_{(,i,k)} - \frac{1}{2} \xi^{|i} [\gamma_{ik,j} + \gamma_{ij,k} - \gamma_{jk,i}]] \tag{4.15}$$

Here, the circular bracket in the subscript denotes symmetrization. The term inside the square brackets in the RHS of the above equation is none other than the covariant derivative (symmetrized covariant derivative). Thus, we get:

$$\mathscr{L}_\xi g_{jk} = 2aa'\xi^0 \gamma_{jk} - 2a^2 \xi_{(|j|k)} \tag{4.16}$$

Now, we can see how each of the perturbation variables change with such a coordinate transformation.

$$\mathscr{L}_\xi g_{00} = 2a^2 \Delta_\xi \phi \tag{4.17}$$

$$\mathscr{L}_\xi g_{0j} = -2a^2 \Delta_\xi B_j \tag{4.18}$$

$$\mathscr{L}_\xi g_{jk} = 2a^2 \Delta_\xi (\psi \gamma_{jk} - \nabla_j \nabla_k E) \tag{4.19}$$

Thus, equating the above equations (4.12 with 4.17, etc..), we get the following results for the change in the perturbation variables (for the space-space part, use the fact that the tensor $\nabla_j \nabla_k E$ is trace-free):

$$\Delta_\xi \phi = \frac{a'}{a} \xi^0 + \xi^{0\prime}, \qquad \Delta_\xi B = \xi' - \xi^0, \tag{4.20}$$

$$\Delta_\xi \psi = -\xi^0 \frac{a'}{a}, \qquad \Delta_\xi E = \xi. \tag{4.21}$$

It can now easily be shown that the following combinations of the perturbation variables are Gauge invariant:

$$\Phi = \phi + \frac{1}{a}[(B-E')a]', \qquad \Psi = \psi - \frac{a'}{a}(B-E') \tag{4.22}$$

We can now move on to Gauge fixing, where we use a certain Gauge (this is like working in a particular coordinate system, like the Cartesian coordinates for example) to simplify our calculations.

We should also note that from the Eq.(4.20) and Eq.(4.21), the variables in the transformed Gauges is given by the following:

$$\tilde{\phi} = \phi - \frac{a'}{a}\xi^0 - \xi^{0\prime}, \tag{4.23}$$

$$\tilde{\psi} = \psi + \frac{a'}{a}\xi^0, \tag{4.24}$$

$$\tilde{B} = B + \xi^0 - \xi', \tag{4.25}$$

$$\tilde{E} = E - \xi. \tag{4.26}$$

4.2.1 Synchronous Gauge

In this gauge, the following perturbation variables take the following values:

$$\tilde{\phi} = 0, \qquad \tilde{B} = 0 \tag{4.27}$$

Here, the variables with a tilde, refer to the variables in the synchronous Gauge. To go to this gauge from any general Gauge, we use the following transformation relation:

$$\eta \to \eta = \eta \tilde{+} \xi^0(\eta, x), \qquad x^i \to \tilde{x^i} = x^i + \gamma^{ij}\xi_{|j}(\eta, x) \tag{4.28}$$

Substituting (4.27) in (4.23) and using product rule we have,

$$a\phi = (a\xi^0)' \tag{4.29}$$

$$\therefore \xi^0 = a^{-1}\int a\phi d\eta \tag{4.30}$$

Using (4.30) and (4.25), we have,

$$B = \xi' - \xi^0 \tag{4.31}$$

$$\therefore \xi = \int B d\eta + \int a^{-1} d\eta \int a\phi d\eta \tag{4.32}$$

Thus, the synchronous time and space components of vector field is given by:

$$\eta \to \eta_s = \eta + a^{-1} \int a\phi d\eta \tag{4.33}$$

$$x^i \to x^i_s = x^i + \gamma^{ij} \left[\int B d\eta + \int a^{-1} d\eta \int a\phi d\eta \right]_{|j} \tag{4.34}$$

There is a problem though. Since all of the transformations in the Synchronous Gauge are defined with integrals, there is a lot of Gauge freedom left in this Gauge and therefore, this transformation is not unique.

4.2.2 Longitudinal / Newtonian Gauge

In this Gauge, the following perturbation variables take the following values:

$$\tilde{B} = \tilde{E} = 0 \tag{4.35}$$

$$\therefore \xi = E \tag{4.36}$$

Here, tilde refers to the variables in the Synchronous Gauge. Also,

$$\xi^0 = E' - B \tag{4.37}$$

The transformation vector field is therefore given by:

$$\eta \to \eta_L = \eta - (B - E'), \qquad x^i \to x^i_L = x^i + \gamma^{ij} E_{|j} \tag{4.38}$$

From these equations, we can see that this transformation is unique and thus this is the preferred gauge to work with. Moreover, the Gauge invariant variables in this Gauge are given as:

$$\Phi = \phi, \qquad \Psi = \psi \tag{4.39}$$

4.3 Perturbation Equations

Let us first rewrite the metric along with the perturbation.

$$ds^2 = a^2(\eta)[(1+2\phi)d\eta^2 - 2B_{|i} dx^i d\eta - [(1-2\psi)\gamma_{ij} + 2E_{|i|j}]dx^i dx^j] \tag{4.40}$$

Upto first order, the Perturbed Einstein Tensor (the total Einstein tensor − the Einstein tensor for the unperturbed case) can be computed to be[Mukhanov 92]:

$$\delta G^0_0 = 2a^{-2}[-3H(H\phi + \psi') + \nabla^2[\psi - H(B-E')] + 3\mathscr{K}h\psi] = 8\pi G\delta T^0_0 \tag{4.41}$$

$$\delta G^0_i = 2a^{-2}[H\phi + \psi' - \mathscr{K}(B-E')]_{|i} = 8\pi G\delta T^0_i \tag{4.42}$$

$$\delta G^i_j = -2a^{-2}[[2H' + H^2]\phi + H\phi' + \psi'' + 2H\psi' - \mathscr{K}\psi + \frac{1}{2}\nabla^2 D]\delta^i_j + 2a^{-2}D_{|i|j} = 8\pi G\delta T^i_j \tag{4.43}$$

Here,

$$D = (\phi - \psi) + 2H(B-E') + (B-E')' \tag{4.44}$$

We have defined this new quantity D for a reason that will be shown later in this article.

These equations are not invariant under the particular class of Gauge transformation that we are interested in. To find the invariant equations, we shall first find out how this tensor transforms under the transformation effected by the vector field ξ^μ. Here we will use the notation that any tensor M^α_β consists of the background part as well as the perturbed part of that tensor and ${}^{(0)}M^\alpha_\beta$ refers to the background part.

$$\Delta_\xi G^\mu_\nu = \mathscr{L}_\xi G^\mu_\nu = \xi^\alpha \partial_\alpha G^\mu_\nu - G^\alpha_\nu \partial_\alpha \xi^\mu + G^\mu_\alpha \partial_\nu \xi^\alpha \tag{4.45}$$

The components of this can be simplified to:

$$\Delta_\xi G^0_0 = \xi^0({}^{(0)}G^0_0)' \tag{4.46}$$

For the time-space component, we will make use of the fact that the space-space part of the Einstein Tensor is diagonal. Therefore, $G^j_i = \frac{1}{3}G^k_k\delta^j_i$. Using this, we get,

$$\Delta_\xi G^0_i = ({}^{(0)}G^0_0 - \frac{1}{3}{}^{(0)}G^k_k)\xi^0_{|i} \tag{4.47}$$

Again, making use of the fact that we used to derive (3.3.8), we will get $G^k_j\partial_k\xi^{|i} = G^i_k\partial_j\xi^{|k}$. Therefore,

$$\Delta_\xi G^0_0 = 2a^{-2} = \xi^0({}^{(0)}G^i_j)' \tag{4.48}$$

The RHS of these equations only contain the background part of the Einstein tensor because we are only interested up to the first order in perturbations. Before writing the perturbed Einstein Tensor in its Gauge invariant form, we will first convert the perturbation variables ϕ and ψ to their Gauge invariant counterparts Φ and Ψ. To do so, we can make use of the equations (**??**) and (4.22). After doing so, we get,

$$\Delta_\xi G^0_0 = 2a^{-2}[-3H(H\Phi+\Psi')+\nabla^2\Psi+3\mathscr{K}\Psi+3H(H^2-H'+\mathscr{K})(B-E')] \tag{4.49}$$

$$\Delta_\xi G^0_i = 2a^{-2}[H\Phi+\Psi'-(H^2-H'+\mathscr{K})(B-E')]_{|i} \tag{4.50}$$

$$\begin{aligned}\Delta_\xi G^i_j &= -2a^{-2}[(2H'+H^2)\Phi+H\Phi'+\Psi''-\mathscr{K}\Psi \\ &+ \frac{1}{2}\nabla^2 D+(H''-HH'-H^3-H\mathscr{K})(B-E')]\delta^i_j+2a^{-2}\gamma^{ik}D_{|k|j}\end{aligned} \tag{4.51}$$

Here,

$$D = \Phi - \Psi \tag{4.52}$$

The equations (4.46)-(4.48) are also valid for the Energy-Momentum Tensor. From there, the perturbed, and Gauge invariant Einstein and Energy-Momentum Tensors respectively are given by,

$$\delta^{(0)}G^{0(gi)}_0 = \delta G^0_0 + ({}^{(0)}G^0_0)'(B-E') \tag{4.53}$$

$$\delta^{(0)}G^{0(gi)}_i = \delta G^0_i + (\frac{1}{3}{}^{(0)}G^k_k)(B-E')_{|i} \tag{4.54}$$

$$\delta^{(0)}G^{i(gi)}_j = \delta G^i_j + ({}^{(0)}G^i_j)'(B-E') \tag{4.55}$$

$$\delta^{(0)}T^{0(gi)}_0 = \delta T^0_0 + ({}^{(0)}T^0_0)'(B-E') \tag{4.56}$$

$$\delta^{(0)}T^{0(gi)}_i = \delta T^0_i + (\frac{1}{3}{}^{(0)}T^k_k)(B-E')_{|i} \tag{4.57}$$

$$\delta^{(0)}T^{i(gi)}_j = \delta T^i_j + ({}^{(0)}T^i_j)'(B-E') \tag{4.58}$$

From these equations, we get the resulting Einstein's equations as:

$$\delta G^{\mu(gi)}_\nu = 8\pi G\delta T^{\mu(gi)}_\nu \tag{4.59}$$

Thus, the Gauge-invariant perturbed Einstein tensor (component wise) is:

$$\delta G^{0(gi)}_0 = 2a^{-2}[-3H(H\Phi+\Psi')+\nabla^2\Psi+3\mathscr{K}\Psi] \tag{4.60}$$

$$\delta G^{0(gi)}_i = 2a^-2[H\Phi+\Psi']_{|i} \tag{4.61}$$

$$\delta G_i^{0(gi)} = -2a^-2[[(2H'+H^2)\Phi + H\Phi' + \Psi'' + 2H\Psi' - \mathscr{K}\Psi + \frac{1}{2}\nabla^2 D]\delta_j^i - \frac{1}{2}\gamma^{ik} D_{|k|j}] \tag{4.62}$$

Thus the Gauge-invariant Einstein's equations for perturbation are:

$$-3H(H\Phi + \Psi') + \nabla^2\Psi + 3\mathscr{K}\Psi = 4\pi Ga^2 \delta T_0^{0(gi)} \tag{4.63}$$

$$[H\Phi + \Psi']_{|i} = 4\pi Ga^2 \delta T_i^{0(gi)} \tag{4.64}$$

$$[(2H'+H^2)\Phi + H\Phi' + \Psi'' + 2H\Psi' - \mathscr{K}\Psi + \frac{1}{2}\nabla^2 D]\delta_j^i - \frac{1}{2}\gamma^{ik} D_{|k|j} = -4\pi Ga^2 \delta T_j^{i(gi)} \tag{4.65}$$

Now, let us analyse equation (4.65) for the case where the perturbed Energy-Momentum tensor has no non-diagonal elements. In that case, when $i \neq j$, we get that the mixed spatial derivatives of $\Phi - \Psi$ is 0. If we now consider $D(x_1, x_2, x_3)$, we have that $D_{|i}$ is only a function of x_i. Therefore, we get,

$$D(x_1, x_2, x_3) = D_1(x_1) + D_2(x_2) + D_3(x_3) = \sum_1^3 D_i(x_i) \tag{4.66}$$

Now, let us perform the following coordinate transformation:

$$x_1 \to x_1 + \alpha x_2 \tag{4.67}$$

Here, α is a very small number. Therefore, we have,

$$D(x_1 + \alpha x_2, x_2, x_3) = D(x_1, x_2, x_3) + \alpha x_2 \frac{\partial D}{\partial x_1} = D_1(x_1) + D_2(x_2) + D_3(x_3) + \alpha x_2 \frac{\partial D}{\partial x_1} \tag{4.68}$$

We can notice that the last term in the above equation has mixed terms containing x_1 (because D_{x_1} is only a function of x_1) and x_2, and thus, there isn't any way of writing this in the form of equation (4.66). This means that $\frac{\partial f}{\partial x_i}$ is a constant number, and therefore, $f_i(x_i)$ is a linear in x_i, i.e..,

$$f_i(x_i) = a_i x_i + b_i \tag{4.69}$$

In the cases that we are interested in (Scalar Field Perturbations and Hydrodynamic Perturbations), the spatial average of the perturbations must vanish. Without loss of generality we can work with only one of the functions (say D_1).

$$\therefore \lim_{\alpha \to +\infty} \frac{1}{2\alpha} \int_{-\alpha}^{\alpha} D_1(x_1) dx_1 = 0 \tag{4.70}$$

$$\implies \lim_{\alpha \to +\infty} \frac{1}{2\alpha} \int_{-\alpha}^{\alpha} (a_1 x_1 + x b_1) dx_1 = 0 \tag{4.71}$$

$$\implies \frac{1}{2\alpha}2\underline{1}\alpha = 0 \tag{4.72}$$

$$\implies b_1 = 0 \tag{4.73}$$

This is true for all b_i.

$$\therefore b_i = 0 \forall i \in (1,2,3) \tag{4.74}$$

$$\therefore D_i(x_i) = a_i x_i \tag{4.75}$$

$$\therefore D = \sum_{i=1}^{3} a_i x_i \tag{4.76}$$

If we make another coordinate transformation, namely, $x_1 \to x_1 + k$, where k is a very small parameter, we get,

$$D \to D_1 + D_2 + D_3 + ka_1 \tag{4.77}$$

This is clearly not of the form (4.76), unless $ka_i = 0$, $\forall i \in (1,2,3)$. Since $k \neq 0$, we have $a_i = 0$, $\forall i \in (1,2,3)$.

$$\therefore D_i = 0 \implies D = 0 \tag{4.78}$$

$$\therefore \Phi = \Psi \tag{4.79}$$

This result will be very useful in simplifying a lot of calculations.

Chapter 5

Calculation of the Power Spectrum

An important statistical measure of scalar fluctuations is the power spectrum corresponding to the comoving curvature perturbation ($\mathscr{R}$). We will discuss about the comoving curvature perturbation in great detail in the forthcoming sections. The power spectrum $P_R(k)$ is defined using an ensemble average as

$$\langle R(k)R(k') = (2\pi)^3\delta(k+k')P_R(k). \tag{5.1}$$

One can also construct a dimensionless quantity from the power spectrum as

$$\Delta_R^2 = \frac{k^3}{2\pi^2}P_R(k). \tag{5.2}$$

In this chapter we discuss about the methods to calculate the power spectrum corresponding to the comoving curvature perturbation.

5.1 Comoving Curvature Perturbation

The comoving curvature perturbation (R) is defined as, as[Mukhanov 92][Durrer 08]:

$$R = \Phi + \left(\frac{2\rho}{3H}\right)\left(\frac{\Phi' + H\Phi}{\rho + p}\right) \tag{5.3}$$

Let us rewrite this in terms of the equation of state parameter ($\omega = \frac{p}{\rho}$). We get:

$$R = \Phi + \left(\frac{2\rho}{3(1+\omega)}\right)\left(H^{-1}\Phi' + \Phi\right) \tag{5.4}$$

Differentiating this and rearranging terms we have,

$$R' = \frac{2H^{-1}}{3(1+\omega)}\Phi'' - \left(\frac{2\omega' H^{-1}}{3(1+\omega)^2} + \frac{2H'}{3(1+\omega)H^2} - \frac{2}{3(1+\omega)}\right)\Phi' - \frac{2\omega'}{3(1+\omega)^2} \tag{5.5}$$

Now,

$$\rho' = -3H(\rho + p) \tag{5.6}$$

The above is the continuity equation (2.19). Thus,

$$\omega' = \frac{p'}{\rho} - \frac{p}{\rho^2}\rho' \tag{5.7}$$

Substituting (5.6) in (5.7) and simplifying we have,

$$\frac{\omega'}{1+\omega} = 3H(\omega - c_s^2) \tag{5.8}$$

Here, $c_s^2 = \frac{p'}{\rho'}$. This quantity (c_s) is the speed of sound in the medium of interest. Substituting (5.8) in (5.5), we have,

$$\frac{3}{2}(1+\omega)R'H = \Phi'' + 3(1+c_s^2)H\Phi' + 3H^2(c_s^2 - \omega)\Phi \tag{5.9}$$

The perturbed part of the Einstein's equations (using the fact that $\Phi = \Psi$ and that the space is flat) are:

$$\nabla^2\Phi - 3H\Phi' - 3H^2\Phi = \frac{a^2}{2}\delta\rho \tag{5.10}$$

$$\Phi'' + 3H\Phi' + (2H' + H^2)\Phi = \frac{a^2}{2}\delta p \tag{5.11}$$

We can also split the perturbation in pressure into an adiabatic part and a non-adiabatic part[Sriramkumar 09].

$$\delta p = \frac{\partial p}{\partial \rho}\delta\rho + \delta p^{NA} = c_s^2\delta\rho + \delta p^{NA} \tag{5.12}$$

Using (5.12) in (5.10) and (5.11), we get:

$$\Phi'' + 3H(1+c_s^2)\Phi' - c_s^2\nabla^2\Phi + [2H' + (1+3c_s^2)H^2]\Phi = \frac{a^2}{2}\delta p^{NA} \tag{5.13}$$

Now, combining equations (5.13) and (5.9) and using the Friedmann equations and writing in the Fourier space, we have,

$$R_k' = \left(\frac{H}{H^2 - H'}\right)\left[\frac{a^2}{2}\delta p^{NA} - k^2 c_s^2\Phi\right] \tag{5.14}$$

From the above equation we can see that for the case of adiabatic perturbations ($\delta p^{NA} = 0$) and in the super-horizon scales ($|k| \ll H$), the right hand side of (5.14) vanishes. Thus, the comoving curvature perturbation remains constant for adiabatic perturbations in the super-horizon scales.

Let us now take a look at the components of the perturbed part of the Energy-Momentum Tensor for a scalar field.

$$\delta T^0_0 = a^{-2}[-\phi'^2\Phi + \phi'\delta\phi + V_\phi a^2\delta\phi] = \delta\rho \tag{5.15}$$

$$\delta T^0_i = a^{-2}(\phi'\delta\phi)_{,i} = (\delta\sigma)_{,i} \tag{5.16}$$

$$\delta T^i_j = -a^{-2}[-\phi'^2\Phi + \phi'\delta\phi - V_\phi a^2\delta\phi]\delta^i_j = -\delta p \tag{5.17}$$

Using the above 3 equations, it can be shown that the non-adiabatic pressure perturbation (δp^{NA}) for the case of a scalar field is given by:

$$\delta p^{NA} = 2\left(\frac{1-c_s^2}{a^2}\right)\nabla^2\Phi \tag{5.18}$$

Substituting (5.18) in (5.14) we have,

$$R'_k = -\left(\frac{H}{H^2 - H'}\right)(k^2\Phi_k) \tag{5.19}$$

Differentiating the above and using the equation (5.13) and (5.18) we get the following important equation [Durrer 08] [Bassett 06]:

$$R''_k + 2\left(\frac{z'}{z}\right)R'_k + k^2 R_k = 0 \tag{5.20}$$

Here,

$$z = \frac{a\phi'}{H} \tag{5.21}$$

Now, let us make a change of variables as follows:

$$v_k = R_k z \tag{5.22}$$

Here, v_k is called the Mukhanov variable[Mukhanov 87][Sasaki 86]. Therefore,

$$R'_k = \frac{v'}{z} - \frac{vz'}{z^2} \tag{5.23}$$

$$R_k'' = \frac{v''}{z} - 2\frac{v'z'}{z^2} + 2\frac{vz'^2}{z^3} - \frac{vz''}{z^2} \tag{5.24}$$

Substituting (5.22), (5.23) and (5.24) in (5.20) we get,

$$v_k'' + \left[k^2 - \left(\frac{z''}{z}\right)\right] v_k = 0 \tag{5.25}$$

The above equation is known as the Mukhanov-Sasaki equation. It governs the evolution of the field variable v_k.

5.2 Quantization

Now, we would like to quantize the field variable v_k, i.e., we want to write the operator as linear combinations of a creation and annihilation operators. We can write that as follows[Baumann 09]:

$$v \to \hat{v} = \int \frac{d^3\vec{k}}{(2\pi)^3} [v_{\vec{k}} \hat{a}_{\vec{k}} e^{i\vec{k}.\vec{x}} + v_{\vec{k}}^* \hat{a}_{\vec{k}}^\dagger e^{-i\vec{k}.\vec{x}}] \tag{5.26}$$

Let us also note that the equation (5.25) comes from the following action[Baumann 09]:

$$S = \frac{1}{2} \int d\eta d^3x \left[(v')^2 + (\partial_i v)^2 + \frac{z''}{z} v^2\right] \tag{5.27}$$

The momentum conjugate to the variable v is given by:

$$p_v = \frac{\partial \mathscr{L}}{\partial v'} = v' \tag{5.28}$$

We impose the following canonical commutation relation:

$$v, p_v] = [v, c'] = i \tag{5.29}$$

Substituting (5.26) in (5.29) we get the following commutation relation between the creation and annihilation operators:

$$\langle v, v \rangle [\hat{a}, \hat{a}^\dagger] = 1 \tag{5.30}$$

Here,

$$\langle v, w \rangle = i(v^*(\partial_t w) - (\partial_t v^*) w) \tag{5.31}$$

We will now impose the following condition:

$$\langle v, v \rangle = 1 \tag{5.32}$$

This gives,

$$[\hat{a}, \hat{a}^{\dagger}] = 1 \tag{5.33}$$

Equation (5.32) is one of the boundary conditions that will be used to solve equation (5.25). The other boundary condition comes from the choice of vacuum that we discuss below.

5.3 Choice of Vacuum

The choice of vacuum [Liddle 00] also corresponds to choosing the second boundary condition for solving (5.25). The choice that we will use in this article is the Minkowski vacuum of a comoving observer in the far past (when all the comoving scales were within the Hubble radius) , i.e., $\eta \to -\infty$ or $|k\eta| \gg 1$ or $k \gg aH$. In the limit equation (5.25) reduces to:

$$v_k'' + k^2 v_k = 0 \tag{5.34}$$

The solution to the equation is:

$$\lim_{\eta \to -\infty} v_k = \frac{e^{-ik\eta}}{\sqrt{2k}} \tag{5.35}$$

5.4 Solving the Mukhanov-Sasaki Equation

First let us derive an expression for $\frac{z''}{z}$ in terms of the quantities ε_H and δ_H. We have,

$$z = \sqrt{2} a \sqrt{\varepsilon_H} \tag{5.36}$$

$$\varepsilon_H = 1 - \frac{H'}{H^2} \implies H' = (1 - \varepsilon_H) H^2 \tag{5.37}$$

$$\delta_H = \varepsilon_H - \left(\frac{\varepsilon_H'}{2H\varepsilon_H} \right) \implies \varepsilon_H' = (\varepsilon_H - \delta_H) 2H\varepsilon_H \tag{5.38}$$

Differentiating equation (5.36), we have,

$$z' = \sqrt{2} a \sqrt{\varepsilon_H} H + \frac{1}{\sqrt{2}} \frac{a}{\sqrt{\varepsilon_H}} \varepsilon_H' \tag{5.39}$$

Substituting equation (5.38) in (5.39) we have,

$$z' = zH(1 + \varepsilon_H - \delta_H) \tag{5.40}$$

Differentiating equation (5.40) and simplifying we get,

$$z'' = zH^2 \left[2 - \delta_H + (\varepsilon_H - \delta_H)(2 - \delta_H) + \frac{\varepsilon_H' - \delta_H'}{H} \right] \tag{5.41}$$

Therefore,

$$\frac{z''}{z} = H^2\left[2-\delta_H+(\varepsilon_H-\delta_H)(2-\delta_H)+\frac{\varepsilon'_H-\delta'_H}{H}\right] \tag{5.42}$$

Upto leading order (i.e., $\varepsilon_H^2=\delta_H^2=\varepsilon_H\delta_H=\varepsilon'_H=\delta'_H=0$) the equation (5.42) simplifies to,

$$\frac{z''}{z} = H^2(2+2\varepsilon_H-3\delta_H) \tag{5.43}$$

We now want to express H upto first order in η. To do so, let us start with equation (5.37).

$$\frac{dH}{d\eta} = (1-\varepsilon_H)H^2 \tag{5.44}$$

Therefore,

$$d\eta = \frac{dH}{H^2(1-\varepsilon_H)} \tag{5.45}$$

Integrating the above equation we get,

$$\eta = \int\frac{dH}{H^2(1-\varepsilon_H)} = -\int\frac{1}{1-\varepsilon_H}d\left(\frac{1}{H}\right) \tag{5.46}$$

Using integration by parts we obtain,

$$\eta = -\left[\frac{1}{H(1-\varepsilon_H)}\right]+\int\frac{1}{H}\frac{\varepsilon'_H}{(1-\varepsilon_H)^2}d\eta \tag{5.47}$$

Substituting equations (5.45) and (5.38) in (5.47) we obtain,

$$\eta = -\left[\frac{1}{H(1-\varepsilon_H)}\right]+\int\left[\frac{2\varepsilon_H(\varepsilon_H-\delta_H)}{(1-\varepsilon_H)^3}\right]d\left(\frac{1}{H}\right) \tag{5.48}$$

The integrand is of the order of $\varepsilon_H\delta_H$ and therefore the integral can be ignored. Therefore we get,

$$H \approx -\frac{1}{\eta(1-\varepsilon_H)} \tag{5.49}$$

Substituting equation (5.49) in (5.43) we have,

$$\frac{z''}{z} \approx \frac{1}{(1-\varepsilon_H)^2\eta^2}[2+2\varepsilon_H-3\delta_H] \tag{5.50}$$

Now, using the first order approximation we get,

$$\frac{z''}{z} \approx \frac{1}{(1-2\varepsilon_H)\eta^2}[2+2\varepsilon_H-3\delta_H] \tag{5.51}$$

We can now get rid of the denominator using the geometric progression approximation ($\sum_{n=1}^{\infty} ar^n = \frac{a}{1-r}$, where $r < 1$) and then using the first order approximation we get,

$$\frac{z''}{z} \approx \frac{2 + 6\varepsilon_H - 3\delta_H}{\eta^2} \tag{5.52}$$

In the leading order, we have,

$$\frac{z''}{z} = \frac{2}{\eta^2} \tag{5.53}$$

In this limit, the Mukhanov-Sasaki equation reduces to,

$$v''_{\vec{K}} + \left(k^2 - \frac{2}{\eta^2}\right) \tag{5.54}$$

To solve the above differential equation let us use the following ansatz:

$$v_k = \frac{e^{\pm ik\eta}}{\sqrt{2k}} \tag{5.55}$$

The above ansatz is motivated by the two boundary conditions. Upon doing this substitution we get the following differential equation in f:

$$f'' \pm 2ikf^{-} \frac{2}{\eta^2} f = 0 \tag{5.56}$$

We shall solve the differential equation for the negative case using the ansatz:

$$f = \frac{c}{\eta} + d \tag{5.57}$$

Upon doing this substitution into the differential equation and using the first boundary condition the solution we get for the negative case is:

$$f = 1 + \frac{1}{ik\eta} \tag{5.58}$$

Similarly the other solution that we obtain is:

$$f = 1 - \frac{1}{ik\eta} \tag{5.59}$$

Thus the solution of the differential equation must be a linear combination of the two solutions.

$$v_{\vec{k}} = \alpha \frac{e^{-ik\eta}}{\sqrt{2k}} \left(1 - \frac{1}{ik\eta}\right) + \beta \frac{e^{ik\eta}}{\sqrt{2k}} \left(1 + \frac{1}{ik\eta}\right) \tag{5.60}$$

Upon using the boundary condition (5.35) we get,

$$v_{\vec{k}} = \frac{e^{-ik\eta}}{\sqrt{2k}}\left(1 - \frac{1}{ik\eta}\right) \tag{5.61}$$

5.5 Power Spectrum

One of the quantities that can be observed is the power spectrum. Let us rewrite its definition here

$$\langle R_k R_{k'}\rangle = (2\pi)^3\delta(k+k')P_R(k) \tag{5.62}$$

$$\Delta_R^2 = \frac{k^3}{2\pi^2}P_R(k) \tag{5.63}$$

Here, $¶_R^2$ is the power spectrum, $\langle..\rangle$ is the Ensemble average of fluctuations, and Δ_R^2 is a dimensionless quantity that is constructed using the power spectrum. Rewriting this in terms of the variable $v_{\vec{k}}$ we have,

$$\langle R_k R_{k'}\rangle = \frac{1}{z^2}\langle v_k v_{k'}\rangle \tag{5.64}$$

Substituting (5.61) in equation (5.64) we get,

$$\langle R_k R_{k'}\rangle = \frac{H^2}{a^2\phi'^2}(2\pi)^3\delta(k+k')\frac{1}{2k}\left(1 + \frac{1}{k^2\eta^2}\right) \tag{5.65}$$

Note that the delta function arises as a result of integration ($\int e^{i(k+k')x}dx = (2\pi)^3\delta(k+k')$). Now, using equation (5.49) at leading order ($\varepsilon_H \approx 0$) in equation (5.65) we get,

$$\langle R_k R_{k'}\rangle = (2\pi)^3\delta(k+k')\frac{H^2}{2k^3}\frac{H^2}{a^2\phi'^2}(1+k^2\eta^2) \tag{5.66}$$

On super-horizon scales, i.e., when $|k\eta| \ll 1$ we get,

$$\langle R_k R_{k'}\rangle = (2\pi)^3\delta(k+k')\frac{H^2}{2k^3}\frac{H^2}{a^2\phi'^2} \tag{5.67}$$

We can evaluate this quantity at horizon crossing ($k = a_* H_*$) since R stays constant in super-horizon scales. Thus evaluating (5.67) at horizon crossing, the power spectrum we obtain is (using equations (5.62) and (5.63))

$$\Delta_R^2(k) = \frac{H_*^2}{(2\pi)^2}\frac{H_*^2}{a^2\phi_*'^2}. \tag{5.68}$$

Here $*$ in the subscript denotes the quantity being evaluated at horizon crossing. We can also write the power spectrum in terms of the potential as opposed to the field ϕ itself [Bassett 06]. This is done by taking the leading order limit of the scalar field Friedmann equations. They are:

$$H^2 \approx a^2 \frac{V}{3} \tag{5.69}$$

$$3H\phi' \approx -a^2 V_{,\phi} \tag{5.70}$$

$$\phi'^2 = \frac{a^2 V_{,\phi}^2}{3V} \tag{5.71}$$

Thus, we get,

$$\Delta_R^2(k) = \frac{1}{12\pi^2} \left(\frac{V^3}{V_{,\phi}^2} \right)_{k=aH} \tag{5.72}$$

As an example, for the case of $\frac{1}{2}m^2\phi^2$ potential, we have,

$$\Delta_R^2(k) = \frac{1}{96\pi^2} \left(m^2 \phi_*^4 \right) \tag{5.73}$$

Chapter 6

Regularized big bang models of Cosmology

6.1 Another Problem With the Hot Big Bang Model

Another problem (aside from the problem of initial conditions) that the hot big bang model faces is the problem of singularity. This problem arises as a result of the divergences encountered by some quantities like the energy density, temperature, etc. Such a divergence is undesirable. In this chapter we propose two models that attempt to solve the problem of singularity. We shall compare these models and see whether both of them completely solve the problem of singularity. We will then give a prescription that will enable us to construct such models.

6.1.1 Problem of Singularity

To explain the problem of singularity, let us consider the case where the universe is flat (K = 0), and is radiation dominated. The time dependence of the scale factor is therefore (2.22),

$$a(t) = \sqrt{\frac{t}{t_0}}. \tag{6.1}$$

From the equations (6.1) and (2.20), we get the following dependence of the energy density on time:

$$\rho(t) \propto \frac{1}{a^4(t)} \propto \frac{1}{t^2}. \tag{6.2}$$

Finally, using Stefan Boltzmann law we have,

$$T(t) \propto \left(\frac{1}{a^4(t)}\right)^{\frac{1}{4}} \propto \frac{1}{\sqrt{t}} \tag{6.3}$$

From the above expressions for time dependence of temperature (6.3) and energy density (6.2) we can see that both these quantities diverge during the big bang (according to the hot big bang model).

To cure this problem, we propose two models where we modify the metric by introducing a topological defect.

6.2 Regularized Big-Bang model - First Ansatz

In this section, we consider the following line element [Klinkhamer 19]:

$$ds^2 = \frac{t^2}{b^2+t^2}dt^2 - a^2(t)d\vec{r}^2 \tag{6.4}$$

$$d\vec{r}^2 = \frac{dr^2}{1-kr^2} + d\Omega^2 \tag{6.5}$$

The modified Friedmann equations will now be derived, and then the derived equation will be thoroughly analyzed to check for further singularities.

Before stating the modified Friedmann equations for this metric we shall consider a more general metric and derive the Friedmann equations for that metric, so that we can use that result for analysis for other metrics. The general metric is as follows:

$$ds^2 = f(t)dt^2 - a^2(t)d\vec{r}^2 \tag{6.6}$$

The Friedmann equations that follow this metric are:

$$\frac{\dot{a}^2}{a^2 f} + \frac{k}{a^2} = \frac{8\pi G}{3}\rho \tag{6.7}$$

$$\frac{2\ddot{a}}{af} - \frac{\dot{a}}{a}\frac{\dot{f}}{f^2} + \frac{\dot{a}^2}{a^2 f} + \frac{k}{a^2} = -8\pi G\rho \tag{6.8}$$

The continuity equation is identical to that of FLRW universe (we have assumed the case of Hydrodynamic matter, and therefore, the Energy-Momentum Tensor is the same).

$$\frac{d}{da}[a^3\rho(a)] + 3a^2P(a) = 0 \tag{6.9}$$

$$w_M = \frac{P}{\rho} \tag{6.10}$$

Substituting $\frac{t^2}{b^2+t^2}$ in place of f in equation (6.7) and using (6.9) and (6.10) we have,

$$\left(1+\frac{b^2}{t^2}\right)\left(\frac{\dot{a}}{a}\right)^2+\frac{k}{a^2}=\frac{1}{3}r_0 a^{-3(1+w_M)} \tag{6.11}$$

6.2.1 Analytic results

This problem can be mathematically analyzed by finding the points of extremum in the time evolution of the scale factor a. So, the values of a for which $\dot{a}=0$ must be found. First, take the case where the equation of state parameter $w_M=0$. Thus, the RHS of the equation (6.11) is reduced to $\frac{1}{3}\frac{r_0}{a^3}$. When $\dot{a}$ (and $a\neq 0$) is 0, we get the following result:

$$a=\frac{r_0}{3k} \tag{6.12}$$

This is just the maximum that is observed in any closed universe model. Now, the important part here is to look for other "bounces", because that is exactly what this article is concerned with.

The focus will only be on cases where $a>0$, because $a<0$ is absurd and $a=0$ is exactly what needs to be avoided. Note that at $t=0$ the value of $\dot{a}$ is zero, because that is the initial condition. Moreover, since the equations are symmetric about $t=0$, the focus will be narrowed down to values for which $t>0$. Since r_0 and k are constants, if $\dot{a}=0$ is substituted, the only value of a for which this is true, is the maxima that had already been derived in this subsection. This implies that there is no other bounce except for the one at $t=0$.

One more interesting result that can be inferred from this is that, since the big-bang is the time when the scale factor a has its minimum value, it is also necessary that the maximum value of a (equation (6.12)) should to be greater than a at the big-bang (let us call this a_B). Therefore,

$$\frac{r_0}{3k}>a_B \tag{6.13}$$

This implies that,

$$k<\frac{r_0}{3a_B} \tag{6.14}$$

This is a constraint on the maximum value that the spatial curvature can take.

6.2.2 Numerical Results

For this computation, the constant parameters must have some already chosen value, and for this article, they are $r_0=9$, and $b=1$. The matter will also be taken to be dust-like (i.e.

$w_M = 0$). Thus, the equation (6.11) reduces to,

$$\left(1+\frac{1}{t^2}\right)\left(\frac{\dot{a}}{a}\right)^2+\frac{k}{a^2}=\frac{1}{a^3} \tag{6.15}$$

After simplification, the following result is arrived at:

$$\dot{a}=\pm\sqrt{\left(\frac{3-ka}{a}\right)\left(\frac{t^2}{1+t^2}\right)}$$

To graphically visualize the time-evolution of a, Euler's method will be used. The iterative result is:

$$a_{i+1}=a_i\pm\Delta t\sqrt{\left(\frac{3-ka_i}{a_i}\right)\left(\frac{t_i^2}{1+t_i^2}\right)}$$

The positive root (root with the positive sign in place of $\pm$) will be used up to the maxima, after which the negative root (root with the negative sign in place of $\pm$) will be used. The resulting plot is:

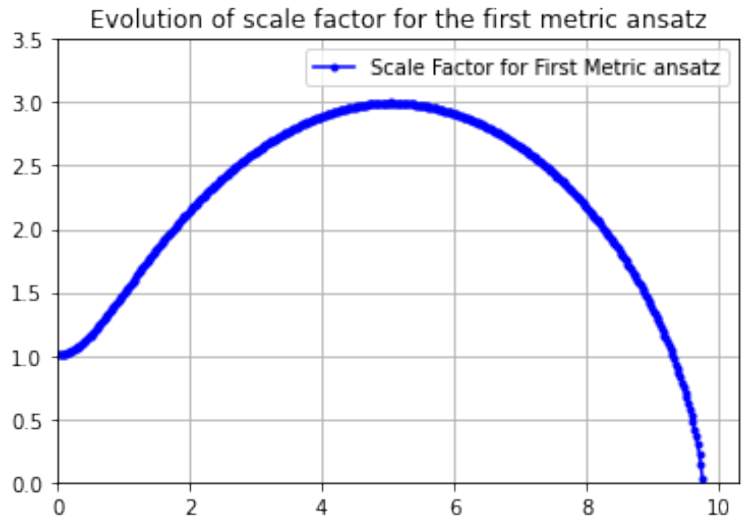

Figure 6.1: Scale Factor evolution for the First Regularized Big-Bang model with k=1, and b=1

Thus in this section, we have shown that this ansatz does not solve the problem of Singularity completely.

6.3 Regularized Big-Bang model - Second Ansatz

In this section, we will introduce a metric ansatz where f depends on time implicitly through the scale factor. It is given by [Klinkhamer 20]:

$$ds^2 = \frac{[a(t) - a_B]^2}{[a(t) - a_B]^2 + b^2[\frac{a'(t)}{2}]^2} dt^2 - a^2(t) d\vec{r}^2 \tag{6.16}$$

where, $(b^2, a_B) > 0, a(t) \in \mathscr{R}, t \in (-\infty, \infty), x^k \in (-\infty, \infty)$.

Using equation (6.7), (6.9), (6.10), we get,

$$(\frac{\dot{a}}{a})^2 + \frac{1}{4}\frac{a^2}{(a-1)^2}(\frac{\dot{a}}{a})^4 + \frac{k}{a^2} = \frac{1}{3} r_0 a^{-3(1+w_M)} \tag{6.17}$$

This is the equation that we will analyze now.

6.3.1 Analytic Results

Again, the equation of state parameter w_M will be taken to be 0. Thus, the equation becomes,

$$(\frac{\dot{a}}{a})^2 + \frac{1}{4}\frac{a^2}{(a-1)^2}(\frac{\dot{a}}{a})^4 + \frac{k}{a^2} = \frac{1}{3}\frac{r_0}{a^3} \tag{6.18}$$

The biggest difference between equations (6.18) and (6.11) is that there is no explicit time dependence anywhere in this equation. From the initial condition ($\dot{a} = 0$ when $a = a_B = 1$),it is clear that whenever $a = 1$, $\dot{a} = 0$ (Otherwise there will be a divergent term in this equation). Since the interest is on values for which $a > 0$ and $a \neq 1$, the calculation can be performed by substituting $\dot{a} = 0$ in equation (6.18). In this case, there is only one value of a for which this is true; namely,

$$a = \frac{r_0}{3k} \tag{6.19}$$

Since this point has to be the maximum, and $a = a_B = 1$ the minimum (because that is the value the scale factor takes at the big-bang), and also since these are the only 2 points of extremum, it is clear see that a cannot go below a_B. Also, there is no explicit time dependence in this equation. Therefore, the above equations must be invariant under time translations of the period of the age of the universe (time at big-crunch - time at big-bang) (A more thorough argument can be made using the second modified Friedmann equation). Thus, this bounce behaviour will be observed again and again for this model. Numerical methods will now be used to determine some quantities like the age of the universe.

6.3.2 Numerical Results

Here, the parameter r_0 will be taken to be equal to 9. Thus equation (6.18) is thus reduced to:

$$(\frac{\dot{a}}{a})^2 + \frac{1}{4}\frac{a^2}{(a-1)^2}(\frac{\dot{a}}{a})^4 + \frac{k}{a^2} = \frac{1}{a^3} \tag{6.20}$$

This equation will now be analysed numerically so that it can be visualized graphically. Euler's method will be the technique to do so.

As can be seen, the equation (6.20) is a quadratic equation in $\frac{\dot{a}}{a}$. Using the quadratic formula, the following result is obtained:

$$\dot{a}^2 = -2(a-1)^2 \pm 2(a-1)^2\sqrt{1+b^2\frac{3-ka}{a(a-1)^2}} \tag{6.21}$$

The negative root can be rejected because $\dot{a}$ has to be real. Therefore,

$$\dot{a}^2 = -2(a-1)^2 + 2(a-1)^2\sqrt{1+b^2\frac{3-ka}{a(a-1)^2}} \tag{6.22}$$

Now, 2 roots for $\dot{a}$ will be obtained. To figure out which root must be used and when, note that the sign of the roots should be changed when an extremum is reached (because of the argument made in the previous section that the bounce is periodic and because $\ddot{a}$ is non-zero at these extremum points). Therefore, the positive root will be taken in the right of the bounce up to the point at which $\dot{a}$ becomes 0. Then, the negative root will be used until the "big-crunch".

The two roots are,

$$\dot{a} = \pm\sqrt{2}(a-1)\sqrt{\sqrt{1+b^2\frac{3-ka}{a(a-1)^2}}-1} \tag{6.23}$$

The initial condition is $a = 1$ at $t = 0$, and then, iteratively time-evolve “a" using equation (6.23).

If a_i is the value of “a" at time t, then a_{i+1} is the value of “a” at time $t+\Delta t$. After simplification, equation (6.23) becomes:

$$a_{i+1} = a_i \pm \Delta t\left(\sqrt{2}(a_i-1)\sqrt{\sqrt{1+b^2\frac{3-ka_i}{a_i(a_i-1)^2}}-1}\right) \tag{6.24}$$

The positive root will be used up to the point at which $ka_i = 1$, for some i, and then the negative root will be used. The age of the universe computed (using the parameters that have been taken in this article) numerically is about 9.1 units. We will take $b = 1$ for the following plot.

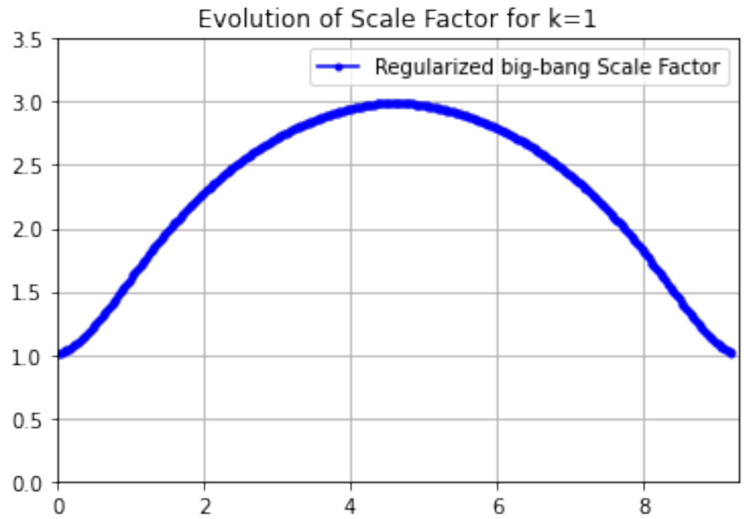

Figure 6.2: Evolution of Scale factor for the second metric ansatz

We will now compare the scale factor corresponding to this regularized big-bang metric ansatz with the scale factor corresponding to the FLRW metric.

Figure 6.3 shows the plots of the scale factors corresponding to both the FLRW model and the second ansatz of the Regularized Big-Bang models for b=1, and Figure 6.4 shows the same, but for $b = \frac{1}{2}$

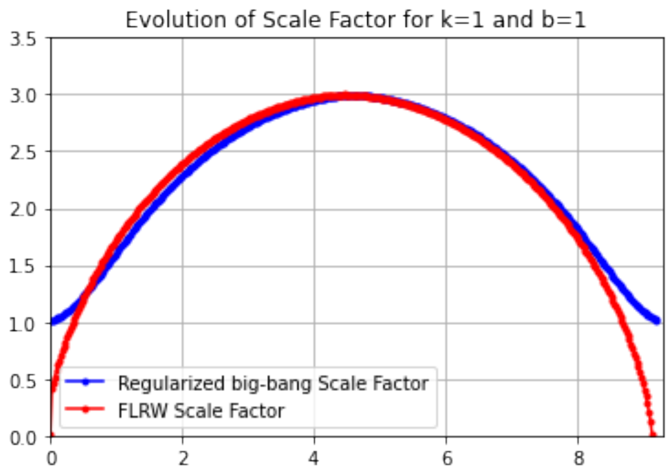

Figure 6.3: Comparison of the scale factor between the FLRW model (red) and the second Regularised Big-Bang model (bluw) when b=1

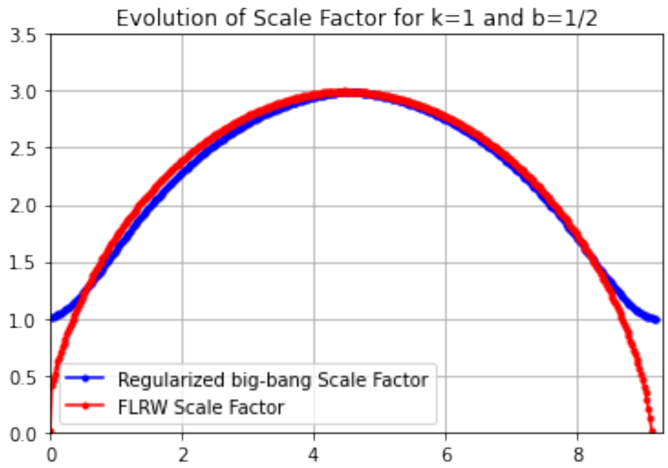

Figure 6.4: Comparison of the scale factor between the FLRW model (red) and the second Regularised Big-Bang model (bluw) when $b = \frac{1}{2}$

6.4 A general metric with the "bounce" behaviour

In this section, we shall work with the following line element:

$$ds^2 = f(t)dt^2 - a^2(t)d\vec{r}^2; \vec{r}^2 = \frac{1}{1-kr^2}dr^2 + r^2 d\Omega^2 \tag{6.25}$$

Here, k is the scalar curvature of the spatial part of the metric. The function f is a function of time, be it explicit or implicit. The Einstein's equations that arise from this metric are the following:

$$\frac{\dot{a}^2}{a^2 f} + \frac{k}{a^2} = \frac{8\pi G}{3}\rho \tag{6.26}$$

$$\frac{2\ddot{a}}{af} - \frac{\dot{a}}{a}\frac{\dot{f}}{f^2} + \frac{\dot{a}^2}{a^2 f} + \frac{k}{a^2} = -8\pi G\rho \tag{6.27}$$

Here, ρ is the Energy density of matter and P is the pressure. Instead of using equation (6.27), we can couple equation (6.26) with the continuity equation, given by the following:

$$\frac{d}{da}[a^3\rho(a)] + 3a^2 P(a) = 0 \tag{6.28}$$

We shall also assume a constant equation of state parameter (w_M):

$$\frac{P}{\rho} = w_M \tag{6.29}$$

We want this scale factor to experience a "bounce" behaviour at $t = 0$, and in the case of a positively curved space, also experience this "bounce" behaviour periodically. The final equation is:

$$\frac{\dot{a}^2}{a^2 f} + \frac{k}{a^2} = \frac{8\pi G}{3}\rho_0 a^{-3(1+w_M)} \tag{6.30}$$

Let us consider the case of dust like matter where $w_M = 0$, and that the curvature of the universe is positive. In this case, the evolution equation reduces to the following:

$$\frac{\dot{a}^2}{a^2 f} + \frac{k}{a^2} = \frac{8\pi G}{3}\rho_0 a^{-3} \tag{6.31}$$

First let us find the condition for the appearance of a "bounce". In this case, we have, $\dot{a} = 0$. There are two possibilities:

$$ka = \frac{8\pi G}{3}\rho_0 \tag{6.32}$$

$$f = 0 \tag{6.33}$$

The first possibility (6.32) is the condition for the maxima (in the case of a positively curved universe) which we had seen in the previous section. The second possibility is the only way the "bounce" could appear. Thus the metric has to be degenerate for a bounce behaviour to occur. Therefore, if the bounce must occur periodically with a period "T" (let's say), then the function "f" must be periodic with the same period "T". Thus, we can expand the functions "a" and "f" as [Berg 88]:

$$a(t) = a_0 + \sum_{n=1}^{\infty} a_n cos\left(\frac{2n\pi t}{T}\right) + \sum_{n=1}^{\infty} b_n sin\left(\frac{2n\pi t}{T}\right) \tag{6.34}$$

$$f(t) = c_0 + \sum_{n=1}^{\infty} c_n cos\left(\frac{2n\pi t}{T}\right) + \sum_{n=1}^{\infty} d_n sin\left(\frac{2n\pi t}{T}\right) \tag{6.35}$$

Now, it will be useful to analyse $\dot{a}$ (its behaviour is well known). From (6.34) we have,

$$\dot{a} = -\sum_{n=1}^{\infty} \frac{2n\pi}{T} a_n sin\left(\frac{2n\pi t}{T}\right) + \sum_{n=1}^{\infty} \frac{2n\pi}{T} b_n cos\left(\frac{2n\pi t}{T}\right) \tag{6.36}$$

Now, "a" attains a minima a $t = 0$ and we will take it to be symmetric about $t = 0$ (because this is what we want) and thus is an even function about that point. Thus, $b_n = 0\ \forall n$. Also, the function "f" must always be greater than or equal to 0 (otherwise, the signature of the metric will change and will thus lead to absurd results), and thus we can use the same argument that we used for "a" and thus, $d_n = 0$. Thus the form of the equations reduce to:

$$a(t) = a_0 + \sum_{n=1}^{\infty} a_n cos\left(\frac{2n\pi t}{T}\right) \tag{6.37}$$

$$f(t) = c_0 + \sum_{n=1}^{\infty} c_n cos\left(\frac{2n\pi t}{T}\right) \tag{6.38}$$

$$\dot{a} = -\sum_{n=1}^{\infty} \frac{2n\pi}{T} a_n sin\left(\frac{2n\pi t}{T}\right) \tag{6.39}$$

Now, we shall look at the result of the initial conditions, $a(0) = a_B$ (this condition is automatically satisfied), $\dot{a}(0) = 0$, $f(0) = 0$. We therefore get:

$$\sum_{n=0}^{\infty} a_n = 0 \tag{6.40}$$

$$\sum_{n=0}^{\infty} c_n = 0 \tag{6.41}$$

Now two possibilities arise:

1. f(t) is an explicit function of time and its time dependence is given.
2. f(t) is an implicit function of time through a variable that is purely time dependent.

In the first case, the only set of variables are the coefficients of the cosines and the constant term in "a", i.e., "a_n". We can substitute these in equation (6.31) and find these coefficients "a_n". We must terminate the sequence somewhere to be able to get results. Let us consider a simple example,

$$a(t) = a_B - a_1 + a_1 cos\left(\frac{2\pi t}{T}\right) \tag{6.42}$$

$$f(t) = -c_1 + c_1 cos\left(\frac{2\pi t}{T}\right) \tag{6.43}$$

Substituting these in equation (6.31) we have,

$$\dot{a}^2 a = \frac{2\pi^2}{T^2} a_1^2 (a_B - a_1) \frac{\pi^2}{T^2} a_1^3 cos\left(\frac{2\pi t}{T}\right) - \frac{2\pi^2}{T^2} cos\left(\frac{4\pi t}{T}\right) - \frac{\pi^2}{T^2} a_1^3 cos\left(\frac{6\pi t}{T}\right)$$

$$kaf = -kc_1(a_B - \frac{1}{2}a_1) + kc_1(a_B - 2a_1) cos\left(\frac{2\pi t}{T}\right) + kc_1 a_1 cos\left(\frac{4\pi t}{T}\right)$$

$$8\pi G\rho_0 f = -8\pi G\rho_0 c_1 + 8\pi G\rho_0 c_1 cos\left(\frac{2\pi t}{T}\right)$$

We have only 1 variable in this equation, namely a_1, and thus we shall compare the constant terms, and we get,

$$-\frac{2\pi^2}{T^2} a_1^3 + \frac{2\pi^2}{T^2} a_1^2 a_B - \frac{1}{2} kc_1 a_1 + kc_1 a_B + 8\pi G\rho_0 c_1 = 0$$

This is a cubic equation in "a_1" and can be solved using the cubic formula. We should choose the solution for which a_1 is greater than a_B (otherwise, we will get an "a" that is lesser than a_B).

Note: By analytic continuation, for the case of negative spatial curvature, we can replace all the trigonometric functions with their hyperbolic counterparts (for example replace $sin(x)$ with $sinh(x)$).

The problem however with this possibility is that the time-period of "a" is independent of the curvature of space (because "f" is an explicit function of "t" and its form is given), and the equation of state parameter w_M (because the time-period is not a variable of equation (6.31)). This is absurd. Thus, we can reject the first possibility.

The second possibility leaves the function undetermined because the function "f" itself depends on another time-dependent function. Since the only other function involved is the scale factor itself, we shall say that the function that "f" depends upon is the scale factor "a". Therefore, in this case, the time-period can be arrived at from equation (6.31), and thus it depends on the curvature "k". It also changes when the equation of state parameter changes because the form of the equation itself depends on the equation of state parameter. The function "f" must go to 0 whenever $a = a_B$.

Now, to be able to arrive at the functional dependence of f on a, we shall consider a universe which is spatially flat. In the limit when $a = a_B$, f becomes 0. In the large time limit (also the large "a" limit because the space is flat), this metric should become the same as the FLRW metric, i.e., $f = 1$ in that limit. We shall take the function to be increasing monotonically as a case study. Thus we can write f as:

$$f = \frac{g(a)}{h(a)} \tag{6.44}$$

Here, the two functions have the following properties:

1. Both g(a) and h(a) must always be non-negative, with g(a) being strictly positive (the signature of the metric must not change).

2. g(a) must be 0 when $a = a_B$

3. h(a) must always be greater than g(a) because f(a) starts at 0 and approaches 1 asymptotically.

Regardless of the value of "a", "f" must always be non-negative (because the signature of the metric must not change). In the simplest case, we shall take the following to be g(a):

$$g(a) = (a - a_B)^2 \tag{6.45}$$

Since h(a) must be positive and also approach g(a) in the large "a" limit, we shall have,

$$h(a) = (a - a_B)^2 + \frac{1}{2}b^2\dot{a}^2 \tag{6.46}$$

The value "b" is what gives rise to the $f = 0$ limit when $a = a_B$. The factor $\frac{1}{2}$ is by convention (because in the small time limit where we can take "a" to be equal to "$a_B + c_1 t^2$", this function "f" boils down to that taken in the first metric ansatz), and we take "$\dot{a}(t)$" because it grows slower than "$a(t)$" in the large "a" (or large time limit for flat space) limit (FLRW limit). This is also valid due to dimensional reasons ("b" has the dimensions of time) and also due to that fact the "f" is an even function. Thus the final form of "f" is,

$$f = \frac{(a - a_B)^2}{(a - a_B)^2 + \frac{1}{2}b^2\dot{a}^2} \tag{6.47}$$

This is the value of the function "f" that is considered in [Klinkhamer 20].

Thus, using this approach we can arrive at other metric ansatz (with different function forms for both $g(a)$ and $h(a)$) that solve the problem of singularity for all values of spatial curvature. We should just keep in mind the FLRW limit and the large time (or large "a" limit for non-positive spatial curvature) limit.

From this section, we can see that the only case where the problem of singularity is solved indefinitely is when "f" is an implicit function of "t" through "$a(t)$".

Chapter 7

Conclusion

In this thesis we first looked at the hot big-bang model in detail. Using the FLRW metric, we arrived at the Einstein equations, assuming that the universe is dominated by an ideal fluid. The Friedmann equations, derived from Einstein equation, determine the evolution of the universe at large scales.

After demonstrating some of the demerits of the hot big-bang model, especially the problems of initial conditions (horizon problem and flatness problem), we showed how the idea of inflation can solve these issues. We also discussed that the conventional matter/fields, say, an ideal fluid, cannot drive inflation, since inflation requires the matter field to possess negative pressure. We also showed that during inflation the scale factor undergoes exponential expansion. Such an exponential expansion can prevail for a sufficiently longer duration of time when the slow-roll conditions are satisfied, and we studied about these slow-roll conditions in detail.

Though the idea of inflation was proposed to solve the puzzles due to hot big-bang model, it was realized later that inflation naturally provides a theoretical base to the occurrence of inhomogeneities in CMB. Therefore, we discussed the theory of cosmological perturbations, where we introduced perturbations in the metric. We also assumed these perturbations to be small. One of the problems that we encounter when we perturb the metric is the problem of non-covariance. One can fix this problem by introducing gauge invariant variables. Then, using the Einstein equations we studied the evolution of these perturbations.

Finally, we dealt with an important statistical quantity called the power spectrum of comoving curvature perturbations. We discussed the prescription to compute the dimensionless power spectrum for the case of slow-roll inflation. The expressions corresponding the power spectrum in Eq. (5.68) and (5.72) enable us to compute the same, given the Hubble slow-roll parameter and the potential slow-roll parameter respectively. This means that we can compute the power spectrum if we are either given the inflationary potential ($V, V_{,\phi}$) or how the inflationary field itself changes ($\dot{\phi}$).

We then demonstrated yet another problem with the hot big bang model, the problem of singularity. To solve this problem we modified the metric in such a way that it is no longer degenerate. This modification led to a "bounce" behaviour, wherein the scale factor exhibited a positive minimum. This also allows us to extrapolate to a pre big bang phase. We dealt with two such models and in the first model, we showed that it does not solve the problem of singularity in the case where the spatial curvature of the universe is positive. However, we showed that the second model doesn't exhibit such problems. Finally, we attempted to find a general class of such models that exhibit such a "bounce" behaviour.

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