# Estimation of Pressure of Matter formed in Heavy-Ion Collision

**Saurav Goyal**
**Roll No: MP18007**

*A dissertation submitted for the fulfilment of MS degree in Physics*

Under the guidance of
**Dr. Satyajit Jena**

**May 2021**

**Indian Institute of Science Education and Research Mohali**
**Sector - 81, SAS Nagar, Mohali 140306, Punjab, India**

# Certificate of Examination

This is to certify that the dissertation titled **"Estimation of Pressure of Matter formed in Heavy-ion Collisions"** submitted by **Saurav Goyal** (Reg. No. MP18007) for the fulfillment of MS degree programme of the Institute, has been examined by the thesis committee duly appointed by the Institute. The committee finds the work done by the candidate satisfactory and recommends that the report be accepted.

Dr. Ambresh Shivaji | Dr. Anosh Joseph | Dr. Satyajit Jena (Supervisor)

Dated: 26.5.2021

# Declaration

The work presented in this dissertation has been carried out by me under the guidance of Dr. Satyajit Jena at the Indian Institute of Science Education and Research Mohali.

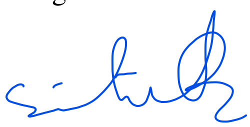
This work has not been submitted in part or in full for a degree, a diploma, or a fellowship to any other university or institute. Whenever contributions of others are involved, every effort is made to indicate this clearly, with due acknowledgement of collaborative research and discussions. This thesis is a bonafide record of original work done by me and all sources listed within have been detailed in the bibliography.


Saurav Goyal

(Candidate)

Dated: May 26, 2021

In my capacity as the supervisor of the candidate's project work, I certify that the above statements by the candidate are true to the best of my knowledge.



Dr. Satyajit Jena

(Supervisor)

# Acknowledgement

First and foremost, I would like to thank my thesis supervisor Dr. Satyajit Jena, without whose help and supervision, this thesis would have never been possible. The discussions that I had with him has enhanced my capabilities as a researcher. By his consistent support helped me feel encouraged and brave to try on new ideas.

I learned most of part of the thesis from having discussions with my lab-mates and seniors in lab. I would like to thank Rohit Gupta, Rutik without their generous help and knowledge at my early stage in the lab, i would not be able to do coding and understand theoretical basics. I am also grateful to Yogesh Verma and Shubhangi Jain who helped me and kept me stayed motivated.

My genuine appreciation extends to IISER Mohali Library for providing support throughout this pandemic by keeping library open. I would also like to thank my friends Bhim, Lakshita, Akash, Rajeshree and especially Nidhi who had always encouraged me to give my best efforts.
Finally, I would like acknowledge with gratitude, the support and love of my family. They all kept me going, no matter how difficult the situation was and this thesis work would not have been possible without them.

Saurav Goyal
MP18007
IISER Mohali

# List of Figures

# Contents

# Abstract

It is believed that after the Big-Bang, in early age of the universe, a hot dense soup of quarks and gluons was formed named QGP(Quark Gluon Plasma), having high energy density and number density. QGP is a fireball consist of quarks and gluons in the deconfined form. Due to high internal Pressure and Temperature it expanded and cooled down, the deconfined-to-confined phase transition occurred and hadrons were formed resulting in the baryonic matter that we observe today. Study about this kind of a phase transition can lead us to understand the early stages of the universe. The promising technique to produce such state of matter in lab is by heavy-ion collisions. Due to complexity of underline theory of these partons i.e. QCD, we rely generally on other effective models like hydrodynamics and Statistical Thermodynamic approaches to study the system.

The Thesis is based on 'Study of Calculation of Pressure' in formed QGP in heavy ion collisions and final hadronic matter formed. Standard statistical models based on Boltzmann-Gibbs distribution(B-G) which is known for its great success on non-interacting classically large systems. Since the number of particle produced in heavy-ion collisions are much less than that of Avogadro number, we need to use non-extensive statistical mechanics to estimate thermal properties of matter formed. As the system undergoes collective expansion, to study the dynamics, hydrodynamics is used, as it provides a simple, intuitive description of dynamical collective behaviour of system under evolution in relativistic heavy-ion collisions. We will use generalised non-extensive statistics known as 'Tsallis-statistics' for the calculations. Tsallis statistics is based on generalization of B-G distribution which in particular limit gives back the standard statistics. For analysis we have used data generated by UrQMD simulator in hydro mode and experimental data extracted from HEPData and carried out the analysis using ROOT.

# Chapter 1

# Introduction

High Energy Physics(HEP) is an intriguing branch of Physics which studies the nature of particles that constitute matter and radiation. It is focused on understanding the behavior of these particles. The primary goal of HEP is to determine the most fundamental building block of matter and understand the interactions between these elementary particles. In the current scenario,these elementary particles are excitations of 'quantum fields'. 'Standard Model' is currently dominant Quantum Field Theory which almost explains these indivisible fundamental particles and their interaction [2]. It is developed by both theoretical and experimental physicists. According to Standard Model of particle physics[3][4], these elementary particles can be classified into fermion and bosons[1]. Fermions are half-integer-spin particles which are matter particles.They are classified according to how they interact (or equivalently, what charges[2] they carry). There are six quarks and six leptons, which are further divided into pairs exhibiting similar behaviour named by generations. The lightest and most stable particles belongs to first generation , whereas the heavier and less stable make up the second and third generation. Bosons(gauge) are integral-spin particles which are mediators of interaction between these matter particles. Particles of matter transfer discrete amount of energy by exchanging these bosons. These are carrier particles that carries any fundamental interactions of nature.

Fig: 1.1 Summarizes Standard-Model of particle physics.

[1]Also Anyons, these are quasi-particles occurs in 2D systems[5]

[2]charges here include electric charge, color charge.

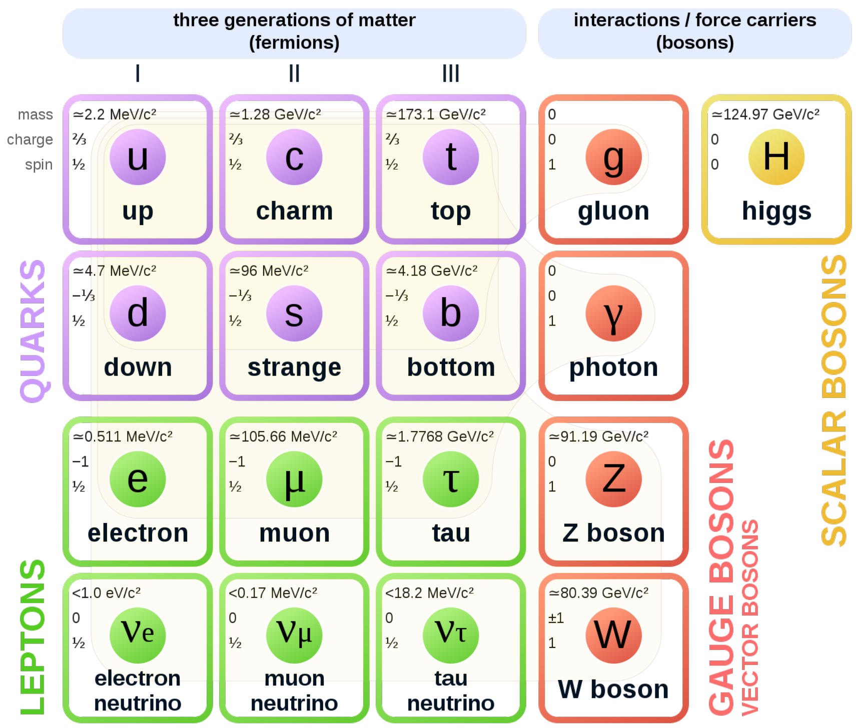



Figure 1.1: Standard Model of Particle Physics [ Wikipedia ]

These handful of fundamental constituent particles interact in known four definite manners:

(i) Electromagnetic interaction | (ii) Strong interaction

(iii) Weak interaction | (iv) Gravitational interaction

***Electromagnetic interaction***: The electromagnetic force is responsible for practically all phenomena one encounters in daily life (with the exception of gravity) that are above the nuclear scale e.g. friction, chemical bonds between atoms etc. It occurs between electric charged particles, it is the reason for keeping negatively charge electron cloud around positively charge nucleus to form atom. It is governed by Lorentz Force which includes both electric and magnetic force, which are both manifestation of same phenomenon. In modern QFT, Quantum Electrodynamics (QED) is the mathematical theory which describes all the phenomenon happens according to electromagnetic interaction.

***Strong interaction***: Before 1970's it was uncertain to explain how nucleus bound together, as it was known that nucleus compose of protons and neutrons, and protons being positively charge particles and neutron being neutral particles, so due to electric repulsion between protons, nucleus should fly apart. However, we do observe nucleus, to explain this, strong interaction was postulated for binding neutron and proton to form atomic nuclei (nuclear-force), and in addition, it was found that protons and neutron are not elementary particles. They are composite system of more fundamental particles, quarks and gluons, which are bounded together by more fundamental force (color-force). Since quarks are fermions and according to Pauli-Exclusion Principle no two or more identical fermionic particles can share same quantum state within a system. The notion of color-charge (a quantum number) was introduced so they can co-exists inside the hadrons without violating the principle. There are three types of color charge labelled red, blue and green along with their complimentary anti-colors. Gluons are also colored, they are the mixture of color and anti-color[3]. The quarks interacts through color exchange and the color carrier is gluon. Quantum Chromo-dynamics (QCD) is the quantum field theory within the Standard Model framework that describes color-force and its residue, the strong nuclear force.

***Weak interaction***: It is an interaction between sub-atomic particles which is responsible for radio-active decay of nucleus. It is the only interaction through which quarks can change their flavours. For example, during $\beta^+$ decay(happens inside the nucleus), an up-quark within a proton is changed into a down-quark, thus converting the proton to a neutron and resulting in the emission of an positron and an electron-neutrino. It plays a vital role in the energy generating processes of stars(including our Sun), in fusion of hydrogen into helium.

***Gravitation***: It is an interaction at a distance that is present between any two bodies having mass or energy. At atomic scale, it is the weakest of all the interactions. Before Einstein's General Theory of Relativity (GTR), Gravitation was well explained by Newton's Laws of Gravitation, which considers gravitation as an attractive force between two bodies possessing mass and the strength of which is proportional to the product of masses and inversely proportional to square of distance between them. In GTR, the relativistic version of gravitation which describes gravity as a consequence of curvature of space-time caused by the sources of Energy-Stress Tensor. It describes the macro-scale bulk interaction in matter, quantizing it for micro-scale matter is the current area of research.
In Table:1.1, We have listed the relative strength of these known interaction along with their

[3]This color charge of quarks and gluons is completely unrelated to everyday meaning of color i.e. unrelated to the wavelength of light.

mediators and range.

| Interaction | Relative strength | Theory | Interaction range | Mediated by |
|---|---|---|---|---|
| Strong | $10^{38}$ | QCD | $10^{-15}$ | $Gluons(8)$ |
| Weak | $10^{25}$ | Electro-weak | $10^{-18}$ | $W^+, W^-, Z$ |
| Electromagnetic | $10^{36}$ | QED | infinite | $Photons$ |
| Gravitational | 1 | General Relativity | infinite | $Graviton$ |

Table 1.1: Fundamental Interactions of nature

Even Standard Model being the current best description for understanding sub-atomic world, it doesn't tell us the full story, there are many questions like What is causing universe expansion to faster? What is dark matter? What made big bang to happen? How did the universe come into existence? Why neutrinos have non-zero mass and the observed neutrinos are left-handed? Is there a theory of everything?, which are not answered in the framework of Standard Model and many more unsolved mysteries.

Curious people, Physicists are searching to find answers to above questions and several deep incites in increasing the understanding of the universe. One of very interesting question is how we came into existence. It was explain using Big Bang Theory which explains existence of almost all matter present till now. According to it, in early microseconds universe began with extremely high temperature and energy density of fireball which cooled and expanded into blizzard of ordinary matter. At early time the temperature was high enough that constituents of matter we see today, were in their most elementary form. The fireball, hot and dense formed, consisted of asymptotically free color-charge particles, quarks and gluons(collectively called partons). At such high temperature these "strongly interacting" particles are quasi-free and fairly weakly interacting. This nature of partons is called Asymptotic Freedom.

## 1.1 Asymptotic freedom

Can we get an isolated color-charged particle (quark or gluon)? No, Quarks/Gluons cannot be find in free(isolated) state due to color confinement. When two color-charges(quarks) are tried to separate, in order to increase the separation between two quarks, large amount of energy is required and as the separation is further increased, at some point eventually it becomes energetically favourable for quark-antiquark pair to form from the surrounding virtual sea-quark, turning the pair of quarks into hadrons. This phenomenon is named color-confinement.

It is phenomenologically said to explain the increase of interaction with separation distance, that QCD potential is given by,

$$V(r) = \frac{\alpha_s}{r} + \sigma r$$

comparison to QED,

$$V(r) \approx \frac{-e^2}{r}$$

The strength of interaction between quarks mediated by gluons in QCD is given by coupling constant, as analogous of coupling constant in QED, $\alpha_e = \frac{e^2}{4\pi(\epsilon_0)\hbar c}$, in QCD, $\alpha_s = \frac{g_s^2}{4\pi}$ however, essential difference between both theories is, $\alpha_e$ is a constant whereas $\alpha_s$ decreases with increase in momentum transfer(E) between quarks or momentum carried by gluons.

Due to this asymptotic freedom at high energies, many experiments can be successfully described by perturbative QCD. The relation between coupling constant with exchange momentum between interacting partons(E) up-to a QCD scale($\Lambda \approx 300$ MeV calculated from experimental data),

$$\alpha_s(E^2) = \frac{12\pi}{(11N_c - 2n_f)ln(\frac{E^2}{\Lambda^2})} \tag{1.1}$$

where, $N_c$ is the number of color charge(3) and $n_f$ is the number of quarks flavour(6).

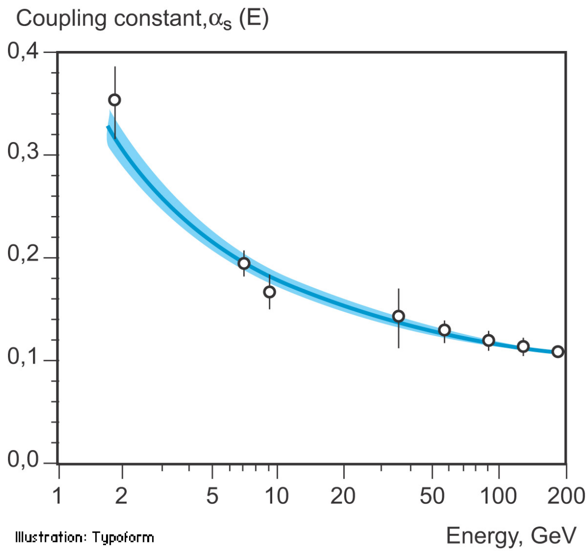



Figure 1.2: Running Coupling Constant[ Nobelprize.org]

This asymptotic freedom was discovered by David Gross and Frank Wilczek and independently by David Politzer , back in 1973 which rewarded them Nobel prize in Physics in 2004. To explore this behaviour and study the system formed in early universe, one promising technique to form QGP is by having high energy density and number density in very small volume, which can be created in relativistic heavy-ion collisions.

## 1.2 Heavy-ion Collision

The main aim of Ultra-relativistic heavy-ion collision is to study the basic building blocks of matter and to probe and dive further in-search of new physics or understanding the structure of these building blocks. When bunches of nuclei collides at ultra-relativistic energy, that results in high temperature and high energy density creating a hot dense state. This state created is thought to consist of quasi-free quarks and gluons, which are the basic building blocks of hadronic matter. This state of matter is called 'Quark Gluon Plasma' (QGP)[6]. The collision energies that are available at the Large Hadron Collider(LHC)[7] at CERN and Relativistic Heavy-Ion Collide(RHIC)[8] at BNL have illuminated new challenges in understanding the possible formation of droplets of this deconfined matter of partonic degrees of freedom in hadronic collisions. When a large fraction of energy is deposited in very small volume of space in very short period of time it produces high energy density and hence Temperature. The theoretically understanding of this new state of matter is given by Lattice QCD. Before QGP was detected in LHC(2000), Lattice-QCD predicted the existence of this new state at sufficient high temperature.

## 1.3 QGP

It was proposed that after the Big-Bang, in its early stages, the universe was filled with an extremely hot and dense soup of quarks and gluons. At extremely high temperature and energy density, even the strongly interacting particles, quarks and gluons, would interact very weakly due to asymptotic freedom. Such high energy density can be mimicked in laboratories by colliding heavy ions at Ultra-relativistic energies (doing little bangs).

Since proton and neutron, constituents of atomic nuclei which are binded by strong(nuclear) interaction, are themselves confined of more elementary particles, quarks which interact with each other via color interaction. Our best theoretical understanding of this strong interaction which is responsible for confinement of quarks into hadrons is 'QCD'[2]. It is non-abelian SU(3) guage theory with color charge as the generator of the theory. QCD deals with interaction between quarks mediated by spin-1 particles named gluons. One of the most interesting property of QCD is that its fundamental degrees of freedom, quarks and gluons combined named partons, carries 'colour' charge, such charged objects have never been observed directly, partons are always found in composite colour-neutral objects called hadrons. Quarks are spin-1/2 particles, which carry electric charge as well as color charge. There are six flavours of quarks: up, down, charm, strange, top, and bottom, see Fig: 1.1. Gluons are spin-1, mass-less particles which are also colored charge. There are 8 independent types of gluons, based on color combination. Quarks change their color state by exchanging gluon.

QGP is a (locally) thermally equilibrated state of matter in which quarks and gluons are deconfined from hadrons, so that they propagate over nuclear, rather than merely nucleonic volume. Composite hadrons appears to lose its identity at sufficiently high energy density ($\approx 1\, GeV fm^{-3}$)[9]. At low densities, the particular quark inside its parent hadron knows its sibling quarks. At high densities, when the hadrons starts to interpenetrate into each other i.e. the quarks no longer confined inside the hadron boundary, particular quark will not be able to identify the partner quark which was there at lower densities. Nucleus–nucleus collisions are used to study nuclear matter. With increasing collision energy, nuclear matter is probed at finer and finer resolution and several facts of nuclear matter are revealed.

### 1.3.1 Space-Time Evolution of QGP

As mentioned we do not detect quarks or gluons directly due to colour confinement, what we detect are the final state hadrons, photons and other kind of particles that survived to reach the detectors. Most of these particles are formed during the expansion of hot and dense fireball, QGP.

A phenomenological scheme of various stages in ultra-relativistic collision of heavy-ions is shown in Fig: 1.3, Two lorentz contracted nuclei approach each other with speed nearly equal to the speed of light in t-z plane, and collide at the origin (t=0,z=0), there is an enormous amount of energy density generated which results in diffusion of hadronic boundary (deconfinement) of protons and neutrons[10]. Initially the system is out-of-equilibrium, its constituents collide repeatedly to establish local equilibrium, as system starts expanding due to inertia and internal pressure created,as the result energy density decreases and hence temperature,as the energy density(temperature) becomes smaller then the critical energy density(temperature) there is deconfinement-to-confinement phase transition occurs, and hadrons are formed which after freeze-out fly towards detectors. As one cannot measure the QGP directly,however one can measure these final-state particle spectra to extract thermal and other characteristics of the system formed and give a judgment on properties of it.

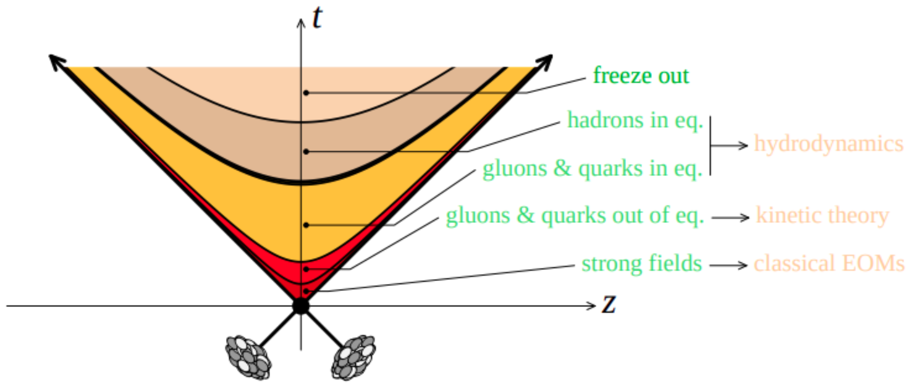



Figure 1.3: Space-Time Evolution[1]

One can Classify Stages of the collision in the following way [9]:

***Pre-equilibrium stage***: The lorentz contracted nucleus having pancake-like structure collides at ultra-relativistic energies, at nucleonic level as they collide, due to high energy density and number density there is high compression due to which nucleon boundary becomes irrelevant, and we get a fireball of quarks and gluons which is in highly non-equilibrium stage. The quarks and gluons formed looses memory of there parent hadrons. By multi-scattering between its constituents, system achieves local momentum-isotropy, hence local thermal-equilibrium.

***Expansion of QGP***: The constituents of the fireball(QGP) collide frequently to establish a local thermal equilibrium state named ‘thermalized state’. The system has high internal thermal-pressure and temperature due to which it expands and the system evolves. The fireball undergoes collective expansion. This expansion can be well described by hydrodynamics, where the expansion and cooling of system is governed on bases of conservation laws of energy and momentum along with conserved current density like charge density, baryon density. As the system expands, the energy density and temperature decreases below critical energy density($\approx 1GeV/fm^3$) or critical Temperature(≈200 MeV), the deconfinement-to-confinement phase-transition occurs and partons get confined in hadrons. This stage is named hadronizaiton.

***Freeze-out***: Even after the hadronization, the matter continues expansion/evolution. After this stage, hadrons can collide in-elastically or can decay and change their identities. Local equilibrium is still maintain and system expands and cools, when this inelastic collision become too small, the hadron abundances will become fixed we named this stage 'chemical freeze-out'. Hadrons then collide elastically which can change the final momentum distribution of the particles. Hadrons then collide elastically which can change the final momentum distribution of the particles. But with further expansion, a stage will arrive where average distance between hadrons become large, even the elastic collisions will become very infrequent and a 'local' equilibrium could no longer be maintained. The hadrons decouple and stop interecting with each other. This stage where no more collision and interaction prevails is called 'kinetic freeze-out'. Hadrons from this kinetic freeze-out surface then subsequently free-stream to the detectors.
These stages of evolution can be nicely summarized in Fig: 1.4.

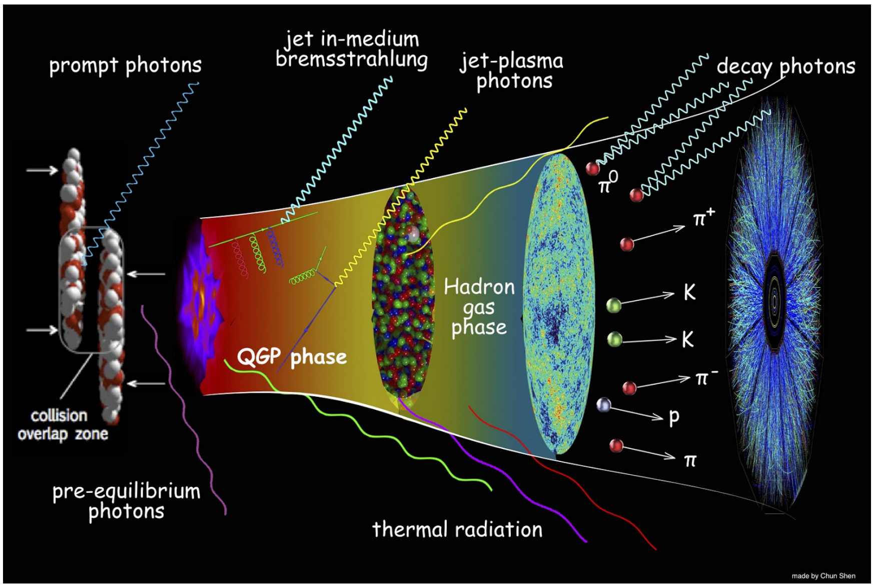



Figure 1.4: Evolution[IEBE-VISHNU]

# Chapter 2

# Tools to study QGP

As the formation and expansion of QGP is in very very small scale (in femto-secs), probing and analysing this state of matter is near to impossible by todays technology. Like after explosion, we can scan back to initial condition by observing the after-effects of the explosion, similarly we study the spectrum of final state particles to extract the properties and characteristics of Quark Gluon Plasma. Some of the tools to study the fingerprint of QGP are: '$p_T$' (Transverse momentum) spectra, '$y$' Rapidity spectra,'$\eta$' Pseudo-rapidity spectra etc. Our focus will be on '$p_T$ spectra', which provides information about the equilibrium dynamics as well as the anisotropy of the system produced in heavy-ion collision and fitting it with thermodynamical distribution functions provides insights to thermodynamical quantities.

Transverse momentum ($p_T$) spectra is an important observable, it is the transverse component of total momentum to the beam direction. As being energy dependent,it provides information of thermal properties of system formed in high energy collisions. Also transverse momentum is Lorentz invariant for boosts along the beam direction, experimentally once kinetic freeze-out is reached, the kinetic freeze-out properties of the system freezes and so does $p_T$ spectra.

## 2.1 Kinematics Variables

There are some kinematic variable when studying high-energy collisions, these variables are used frequently in experiments also [11]. We will discuss some of those variables which have significant role in the thesis.

### 2.1.1 Centrality

In ultra-relativistic nucleus collision performed in LHC or RHIC, stream of bunches of nucleus are collided from opposite direction. When a nuclei scatters with nuclei of other beam this is named as an ‘event’. Scattering between nuclei-nuclei depends on the impact parameter(b) between them, so we classify the event-by-event collisions by ‘b’. Most central collisions have b≈0, and peripheral collisions have b around 2R (R,mean nuclei radii). As impact parameter of an event cannot be measured experimentally, so we define centrality class. Centrality is an important concept which characterizes the amount of overlap or size of the fireball in the collision region in heavy-ion collisions. It can also be defined by the number of nucleons $N_{Part}$ (participants nucleons) in the overlap region of the colliding nucleus[12].Centrality and impact parameter have one-to-one correspondence. For most central collisions, (0-5)$\%$ centrality for Pb-Pb collision at 2.76 TeV have impact-parameter range from 0 to 3.50 fm. Using the $\%$ centrality in collisions, the initial geometric configuration can be estimated using Glauber model.

### 2.1.2 Rapidity

Choose some direction (usually the beam direction) for the z-axis named longitudnal direction, the velocity of nuclei can be written in terms of dimensionless quatity, rapidity as $\tanh y = \frac{v_z}{c} = \beta_L$, and the lorentz factor($\gamma$) can also be written as $\cosh y = \gamma$. One can see as the relativistic addition of velocity is non-linear, but rapidity which is the function of velocity itself, is an additive quantity.The advantage of introducing this rapidity variable is that as its additive, the shape of its distribution remains invariant under longitudinal Lorentz boost. Also rapidity in natural units can be expressed as,

$$y = \tanh^{-1}\beta_L = \tanh^{-1}\frac{p_z}{E} = \frac{1}{2}\ln\frac{E+p_z}{E-p_z} \tag{2.1}$$

Rapidity can be expressed in terms of polar angle($\theta$), angle made by particle emitted with respect to beam axis,

$$y = \frac{1}{2}\ln\frac{E+p_zc}{E-p_zc} = \frac{1}{2}\ln\frac{\sqrt{m^2+p^2}+pcos\theta}{\sqrt{m^2+p^2}-pcos\theta} \tag{2.2}$$

When emitting particle have high $p_z$ value one can neglect mass term, so, $E \approx p$ and this approximated rapidity is called pseudorapidity ($\eta$).

$$\eta = \frac{1}{2}\ln\frac{p+pcos\theta}{p-pcos\theta} = -\ln\tan\frac{\theta}{2} \tag{2.3}$$

It is a convenient parameter, to study when particle emitted have high longitudinal momentum and we know the polar angle of it even without knowing its mass. More properties of rapidity and pseudo-rapidity can be studied from [13].

### 2.1.3 Transverse Momentum

The component of momentum transverse (perpendicular) to the beam direction is called Transverse momentum. As in collisions the beams are along longitudinal direction, so the nuclei colliding have only longitudinal momentum. However, after the collision the final-state particles coming out of the collision region will have non-zero transverse momentum. Thus a study of the transverse momentum distribution of outgoing particles can give a significant insight into the physics involved in the collision. Thus the transverse momentum distributions of the out-coming particles is the first observations done in the high-energy experiments. Also $p_T$ is Lorentz invariant along longitudinal direction. We can write energy and total momentum in the form of $p_T$,

$$\begin{aligned} p_T &= \sqrt{p_x^2 + p_y^2}, \\ E = \sqrt{p^2 + m^2} &= \sqrt{p_T^2 + p_z^2 + m^2} = \sqrt{p_z^2 + m_T^2} \\ \text{where,} \qquad m_T &= \sqrt{p_T^2 + m^2} \end{aligned} \tag{2.4}$$

here, $m_T$ is transverse mass, we can write eq: 2.4 in terms of rapidity,

$$E = m_T \cosh y, \qquad p_z = p_T \sinh y \tag{2.5}$$

## 2.2 Invariant Yield

Thermodynamical behaviour of the system formed can be extrapolate from $p_T$ spectra as it carries the kinetic freeze-out properties of the system. The invariant yield when plotted as a function of $p_T$ is called $p_T$ spectrum. Experimentally at kinetic freeze-out the $p_T$ spectra is frozen, as particle number becomes constant.

The Differential-Yield,

$$E\frac{d^3N}{dp^3} = E\frac{d^3N}{(dp_x dp_y)dp_z} = \frac{d^3N}{(|p_T|dp_T d\phi)dy} = \frac{d^2N}{2\pi|p_T|dp_T dy} \tag{2.6}$$

is lorentz invariant, here we have used $dp_z = Edy$ and $dp_x dp_y = |p_T|dp_T d\phi$, $\phi$ is the azimuthal angle and considering azimuthal symmetry. To show eq: 2.6 is lorentz invariant, firstly we have to show $\frac{d^3p}{E}$ is lorentz invariant, The change in longitudinal momentum($p_z$) due to boost in longitudinal direction(beam direction) is given by,

$$\begin{aligned} dp_z^{'} =& \gamma(dp_z - \beta dE) = \gamma(dp_z - \beta\frac{p_z dp_z}{E}), \\ =& \frac{dp_z}{E}\gamma(E - \beta p_z) = \frac{dp_z}{E}E^{'} \end{aligned} \tag{2.7}$$

where, we have used $E^2 = p_T^2 + p_z^2 + m^2$, and we know $p_T$ is lorentz invariant to longitudinal boost, so $EdE = p_z dp_z$. $\frac{d^3p}{E}$ is lorentz invariant, hence the yield is invariant. To measure the yield eq: 2.6 is used experimentally. We can derive the form of yield from standard statistical mechanics[14], Number of particles(N) in a grand-canonical ensemble is given by,

$$N = \frac{gV}{(2\pi)^3}\int \frac{d^3p}{exp(\frac{E(p)-\mu}{k_BT}) + \eta} \tag{2.8}$$

where,

$$\eta = \begin{cases} 0 & \text{for Boltzmann-Gibbs statistics,} \\ +1 & \text{Fermion,} \\ -1 & \text{Boson.} \end{cases} \tag{2.9}$$

Let's denote f(E,T) as the integrand, So,

$$N = \frac{gV}{(2\pi)^3}\int d^3p f(E,T), \tag{2.10}$$

or, in differential form,

$$\frac{d^3N}{dp^3} = \frac{gV}{(2\pi)^3} f(E,T) \tag{2.11}$$

here, g is the degeneracy factor, T is the temperature of the system, V is the volume of the system of particles. So Yield have the form,

$$\frac{d^2N}{2\pi|p_T|dp_T dy} = \frac{EgV}{(2\pi)^3} f(E,T) \tag{2.12}$$

In our analysis, we will work in mid-rapidity regime, y $\in$ (-0.5,0.5), so, average value of $\cosh y \approx 1$, and expressing yield in form of $p_T$ using eq: 2.5,

$$\begin{aligned} \frac{d^2N}{2\pi|p_T|dp_T dy} &= \frac{gV(m_T\cosh y)}{(2\pi)^3}\frac{1}{exp(\frac{(m_T\cosh y)-\mu}{k_BT}) + \eta}, \\ \frac{d^2N}{2\pi|p_T|dp_T dy}\Big|_{y=0} &= \frac{gV(m_T)}{(2\pi)^3}\frac{1}{exp(\frac{m_T-\mu}{k_BT}) + \eta} \end{aligned} \tag{2.13}$$

We will generalize this Yield form to Tsallis statistics (4.2) for our analysis.

Till now we have studied the physics behind QGP and its expansion. And also the experimental tools for its detection. Event generators are used widely to mimic the real experiment happening in colliders. These Monte-Carlo generators based on underline physics are the powerful tools to gain detailed and realistic theory behind the real experiment. Different models are made to simulate different stages of heavy-ion collision. “The theoretical tools that are the usual workhorses of quantum field theories, such as perturbative methods and lattice calculations, can only describe very specific features of relativistic heavy-ion collisions. As such, a first principles description of the complex-dynamics of heavy-ion collisions is not yet possible. On the other hand, very successful multi-stage models of heavy-ion collisions have been built by combining lattice and perturbative calculations with effective models such as hydrodynamics”[15].

In the thesis we have used UrQMD (Ultra-Relativistic Quantum Molecular Dynamics) model. It is microscopic transport theory based on propagation of all hadrons on classical trajectories in combination of stochastic binary scattering, color string formation, resonance decay[16]. We have used data generated by UrQMD in hydro mode, in which the target and projectile are treated as ideal fluids but the collision interaction is calculated using kinetic theory[17]. We have calculated ‘Pressure’ with time in scope of it, and also pressure of final state particles like charged pions.

# Chapter 3

# Hydrodynamics

Ideally, one may want to describe the experimental data or dynamical knowledge of QGP from its underline theory i.e. QCD. The QCD Lagrangian density is given by:

$$L = \bar{\psi}_i(\mathrm{i}\gamma_\mu D^\mu_{ij} - m\delta_{ij})\psi_j - \frac{1}{4}F_{\mu\nu\alpha}F^{\mu\nu\alpha} \tag{3.1}$$

where $\psi_i$ is a quark field, $i$(1,2,3) is a color index for quarks, $D^\mu$ is a contra-variant derivative, $m$ is a quark mass, $F^{\mu\nu}_\alpha$ is a field strength of gluons, and $\alpha$ (= 1, 2 . . , 8) is a color index for gluons. This simple-looking lagrangian is mathematically complex, it is difficult to make prediction directly due to its internal complexity. One of the promising techniques used now to connect the first principle with phenomena is to introduce hydrodynamical models to describe the complex-dynamics of heavy-ion collision [18].

## 3.1 Introduction

Thermodynamics deals with the system in global thermal equilibrium with Pressure, Temperature, Volume as globally static variables. As in statistical thermodynamics ensemble average is equivalent to the time average for a system, this is ergodicity theorem. Hydrodynamics deals with study of flow of fluid (continuous medium) having implicit assumption of local thermal equilibrium of fluid. It is a dynamical study of macroscopically average value of densities, which give information of collective-flow of the system using conservation laws with taking statistical inputs [19]. Relativistic hydrodynamics is an important theoretical tool in astrophysics, cosmology, heavy ion-collision etc. The use of relativistic hydrodynamics in high-energy physics started back by Landau in 1953[20], long before

QCD was formulated, as the system is at high temperature and partons are asymptotically free. It is applicable when the mean-free-path of the partons is much smaller than typical scale of the fireball on which its properties are changing. If the macroscopic properties of the fluid e.g. local energy density, pressure, the fluid velocity etc. are known at an initial time, we can use hydrodynamic equations to obtain the space–time evolution of the hot dense fireball formed in heavy ion collision until the freeze-out.The concept of local equilibrium may still apply provided that the expansion rate is much slower than the microscopic interaction rate.

By, ADF/CFT calculations it was found that $\frac{\eta}{s} = \frac{1}{4\pi}$ (specific-viscosity is very low) for QGP system ,which leads to paradigm that the fireball behave as *nearly 'Perfect Fluid'* which makes the system formed, to study using hydrodynamical framework[21]. In this chapter we'll first discuss in-viscid(ideal) hydrodynamcis.

## 3.2 Ideal Hydrodynamics

Ideal hydrodynamic models are largely successful, in explaining a variety of experimental data e.g. transverse momentum spectra and elliptic flow of final state particles in high energy collisons [9]. Like Classical Electromagnetism, most of the information about dynamics of field can be predicted using Maxwell relations, similarly in hydrodynamics we have conservation equations. Energy and momentum in a relativistic theory are encoded in the energy-momentum tensor( $T_o^{\mu\nu}$),

$$T_o^{\mu\nu} = \begin{bmatrix} e & 0 & 0 & 0 \\ 0 & P & 0 & 0 \\ 0 & 0 & P & 0 \\ 0 & 0 & 0 & P \end{bmatrix} \tag{3.2}$$

$T_o^{00}$ tells the energy density

$T_o^{0i}$ is the density of the $i^{th}$ component of momentum with i= 1,2,3

$T_o^{i0}$ is the energy flux along the axis i

$T_o^{ij}$ is the flux along axis $i$ of the $j^{th}$ component of momentum

Stress-energy tensor in a symmetric second-rank-tensor form can be easily derivable.

$$T_o^{\mu\nu} = c_1 a u^\mu u^\nu + c_2 a g^{\mu\nu} \tag{3.3}$$

where,$c_1, c_2$ are pure constant, a is lorentz scalar, $u^\nu$ is the flow velocity which is Lorentz vector, $g^{\mu\nu}$ is a space-time matric of second-rank-tensor.

Using properties of stress-energy Tensor, Trace $(T_o^{\mu\nu}) = 0$, transformation of $T_o^{\mu\nu}$ back to local rest frame, we get,

$$T_o^{\mu\nu} = eu^\mu u^\nu - P\Delta^{\mu\nu} \tag{3.4}$$

$$N^\mu = nu^\mu, \tag{3.5}$$

$$\begin{aligned} \partial_\mu T_o^{\mu\nu} = 0, \partial_\mu N^\mu = 0 \\ u_\mu u^\mu = 1 \end{aligned} \tag{3.6}$$

where,

$$\Delta^{\mu\nu} = g^{\mu\nu} - u^\mu u^\nu \qquad g^{\mu\nu} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \tag{3.7}$$

here, e and P are Energy density and Pressure respectively. One can easily check $u^\mu$ is an eigen-vector of $T_o^{\mu\nu}$.

Note in eq: 3.6 we have 7 variables(e, P, n, $u^\mu$) and 6 eq's., we have to supply one more eq. to close our system of equations, that equation is the equation of state $P = P(e, n)$. Here the notion of equation of state is in complete thermodynamic sense. Such an equation of state allows (by definition) to determine thermodynamical variables. Once the equation of state is known along the initial conditions, the equations of motion uniquely determine the dynamics of the collision. Also, for ideal case,

$$\partial_\mu S^\mu = 0, \tag{3.8}$$

where $S^\mu = su^\mu$,it can be derived using $u_\mu(n_{bar}u^\mu) = 0$ and contracting $u^\mu$ with eq. 3.6, $\mu_c$ here is chemical potential corresponding to conserved number density.

$$\begin{aligned} u_\nu \partial_\mu T_o^{\mu\nu} &= u_\nu[\partial_\mu((e+p)u^\mu u^\nu - g^{\mu\nu}p)], \\ &= u_\nu[\partial_\mu((Ts+\mu n)u^\mu u^\nu - g^{\mu\nu}p)], \\ &= u_\nu[\partial_\mu(Ts+\mu n)u^\mu u^\nu - g^{\mu\nu}\partial_\mu p)] \end{aligned}$$

Using $e + p = Ts + \mu n, \quad dp = sdT + nd\mu$, Further simplifying,

$$\begin{aligned} u_\nu \partial_\mu T_o^{\mu\nu} &= T(u^\mu \partial_\mu s + s\partial_\mu u^\mu) + \mu(n\partial_\mu u^\mu + u^\mu \partial_\mu n), \\ 0 &= \partial_\mu(su^u) = \partial_\mu(S^u) \end{aligned}$$

We can also write thermodynamical quantities in form of Stress-Energy Tensor,

$$e = u_\mu T_o^{\mu\nu} u_\nu \tag{3.9}$$

$$p = \frac{-1}{3} T_o^{\mu\nu} \Delta_{\mu\nu} \tag{3.10}$$

### 3.2.1 Bjorken approximation

In 1982, 'Bjorken' modified Landau description of relativistic hydrodynamics[22]. He used boost invariant velocity flow, in longitudinal direction of high energy collision. He used uniform motion along longitudinal direction(z) so that $v_z = z/t$, all particles at a given z have the same $v_z$ , hence the fluid velocity. The region of highly excited matter is supposed to rapidly equilibrate locally within a time $\tau$.

$$u^\mu_{BJ} = \frac{\tilde{x}^\mu}{\tau} = \frac{(t,0,0,z)}{\tau} = (cosh\eta_s, 0, 0, sinh\eta_s) \tag{3.11}$$

where we introduced Milne coordinates,

$$\text{rapidity}(\eta_s) = \frac{1}{2}\log\frac{t+z}{t-z} \tag{3.12}$$

$$\text{proper time}(\tau) = \sqrt{t^2 - z^2} \tag{3.13}$$

Expanding eq: 3.6,

$$[(\partial_\mu u^\mu)u^\nu + u^\mu(\partial_\mu u^\nu)](e+P) + u^\mu u^\nu \partial_\mu(e+P) - g^{\mu\nu}\partial_\mu P = 0$$

Contracting this eq. with $u_\nu$ and using $u_\mu \partial_\nu u^\mu = 0$, we get,

$$(e+P)\partial_\mu u^\mu + u^\mu \partial_\mu e = 0 \tag{3.14}$$

Similarly multiplying eq: 3.6 with $\Delta_{\mu\nu}$ we get,

$$(e+P)u^\mu \partial_\mu u_\nu - \partial_\nu P + u_\nu u^\mu \partial_\mu P = 0 \tag{3.15}$$

Now transforming to Milne co-ordinate's and using eqs: 3.14, 3.15 and assuming initial densities are to depend on t and z only through the longitudinal proper time $\tau$, not on $\eta_s$ we get,

$$\frac{de}{d\tau} = -[\frac{e+P}{\tau}], \frac{ds}{d\tau} = \frac{-s}{\tau} \tag{3.16}$$

Using eq. of state, $P_s = \frac{e}{3}$ (ideal-relativistic case), we can plot variation of pressure with time of elapse, as it is differential eq. it needs initial conditions. Phenomenologically, Bjorken estimated initial condition on Energy density[22] in terms of rapidity distribution ($\frac{dN}{dy}$),

$$e = \frac{E}{V} \approx \frac{1}{\pi R_T^2}\frac{dE_T}{dz} \approx \frac{1}{\pi R_T^2 \tau}\frac{dE_T}{dy} \tag{3.17}$$

$$e_0^{BJ}(\tau) = \frac{< m_T >}{\pi R^2 \tau}\frac{dN}{dy} \tag{3.18}$$

Taking $\tau$ =1 fsec, nucleus radii(R) of Au = 6.98fm, as taking most central collisions for 200 GeV, calculated using eq. of state $P_s = \frac{e}{3}$, we get $P_0^{BJ} = 0.0086 GeV fm^{-3}$.

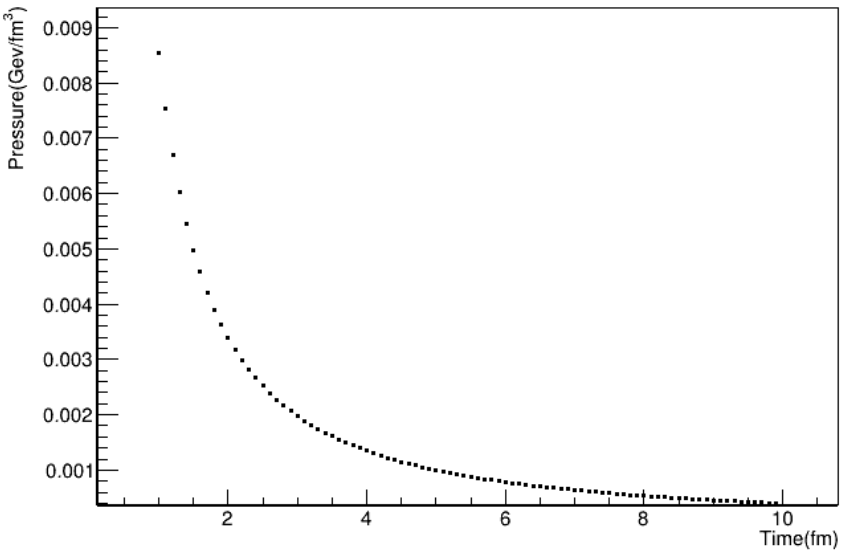



Figure 3.1: Bjorken Approximation for variation of pressure with proper time.

But Reality is not that simple. There are shortcomings of ideal fluid dynamics. As said being 'Ideal', which is never the case behaved by nature[9][23]. It is thus important to study the effects to viscosity (dissipation) on space-time evolution of fluid like QGP (viscosity could be small, $\frac{\eta}{s} \geq \frac{1}{4\pi}$, but nevertheless, it is non-zero).

## 3.3 Viscous hydrodynamics

The theory of viscous relativistic fluid was formulated in 1821 by Navier [24]. The original viscous (dissipative) relativistic-fluid equations were formulated by Landau and Lifshitz[20], Eckart[25] and named it the first-order viscous theories. The eq. of motion are derived keeping in mind the second law of thermodynamics, $\partial_\mu S^\mu \geq 0$, i.e entropy is not decreasing. Formally, in relativistic dissipative hydrodynamics, we add orders in terms of the dissipative fluxes(perturbations), in the expansion of entropy current. When addition of terms linear in dissipative quantities is done in entropy four-current we obtain first order viscous theory.

$$S^\mu = su^\mu + O(\delta T^{\mu\nu}) + O((\delta T^{\mu\nu})^2) + ...$$

Relativistic viscous hydrodynamics describes non-equilibrium processes of the system in consideration. When effects of dissipation is included into relativistic hydrodynamics, one is confronted with complicated situations. Bjorken's equation in First-Order Theory can be derived [18],

$$\frac{de}{d\tau} = -\frac{e+P}{\tau}(1 - \frac{4}{3\tau T}\frac{\eta}{s} - \frac{1}{\tau T}\frac{\zeta}{s}) \tag{3.19}$$

The eq: 3.19 describes time evolution of energy density as function of other macroscopic quantities. $\eta$ is the shear viscosity and $\zeta$ is the bulk viscosity.
First order theory approaches Navier-Stokes eq. in non-relativistic limit. It suffers from actuality due to parabolic equations[26](first order in time and second order in space it is not Lorentz-covarient theory). When second order dissipative terms are included the acasualty is removed by introducing relaxation times along with quadratic viscous terms. In 1970's Israel and Stewart gave description of second-order viscous hydrodynamics[27].

We are talking about dissipation in flow, but flow of what?
In Ideal-Hydrodynamics the flow is determined uniquely, as it is in direction of eigenvectors of stress-energy tensor, $T_o^{\mu\nu}u_\mu = eu^\nu$. In literature, in case of viscous-hydrodynamics there are two theories which describe the flow in local rest frame,
1) *Landau flow*: Flow of energy, the flow is taken in direction of total energy flux so dissipation of energy does not appear.

$$u_L^\mu = \frac{T^\mu_\nu u_L^\nu}{u_L^\alpha T_\alpha^\beta T_{\beta\gamma} u_L^\gamma} = \frac{1}{e} T^\mu_\nu u_L^\nu \tag{3.20}$$

2) *Eckart flow* : Flow of conserved charges(particle),here flow is chosen such that total conserved charge flux is diffution independent.

$$u_E^\mu = \frac{N^\mu}{\sqrt{N_\nu N^\nu}} \tag{3.21}$$

In high energy heavy-ion collisions, mostly Landau's definition is preferred as small number of baryon charges are expected in central collisions[18].

We have used simulators to solve these hydrodynamic equations. We have used UrQMD, model based on a hybrid approach based on intermediate hydrodynamic evolution with transport model. In it the partial differential equations are solved on 3-D grid using SHASTA algorithm. It uses SU(3) parity duality model based equation of state [28] to close the system for solving ideal-hydrodynamics equation. The hydro-dynamical evolution is stopped if the energy density drops below five times the initial nuclear energy density in all the cells in the grid, the hydrodynamic cells are mapped to hadrons using Cooper-Frye criteria[17]. These hadrons information is put back in UrQMD for re-scattering and resonance decay and final state particle are recorded.

# Chapter 4

# Approach by Statistical Mechanics

The main concern to high-energy physics is in studying the properties to QGP. Due to the asymptotic freedom and very nature of QCD coupling, the coupling strength is extremely strong at low energies making it almost impossible to apply perturbative calculations in this region. Thus,we rely on more fundamental techniques. Studying statistical Thermal models to explain experimental high energy was first proposed by Koppe in 1948[29]. Two years later, Fermi introduced a statistical framework to study the energy distribution of particles coming out of small volume formed when ions are collided at high energies[30].We rely on Relativistic kinetic theory to study our system under consideration. In this chapter we'll first discuss phenomenological Statistical Thermodynamical Models which agree with experiments and lattice QCD upto some limits[31]. Then we'll formulate Tsallis statistics[32] to study the thermodynamical quantities of our system.

## 4.1 Thermodynamical Models

### 4.1.1 MIT Model

A phenomenological description of patrons in hadrons is provided by the MIT Bag Model, which describes quarks being confined inside a hadron. Non-interacting massless partons are forced to move inside a Bag having external pressure 'B', which behaves as a confinement for the partons. The basic motivation behind it is that if quarks are placed in the QCD vacuum, the vacuum will expel the color field of the quarks isolating them into a bag.

Using statistical mechanics for particles at ultra high-relativistic energies, if we use Boltzmann-statistics in high temperature regime taking $m = 0, \mu = 0$ and expressing in natural units (Appendix: A),

Pressure of QGP is given by,

$$P_{QGP} = P_q + P_g - B \tag{4.1}$$

And,

Pressure of Quarks : $P_q = \frac{7}{8}\frac{\pi^2}{90}g_q T^4$,

Pressure of Gluons: $P_g = \frac{\pi^2}{90}g_g T^4$

So,

$$\begin{aligned} P_{QGP} =& \frac{\pi^2}{90}T^4[\frac{7}{8}g_q + g_g] - B \\ =& \frac{\pi^2}{90}T^4 g_{QGP} - B \end{aligned} \tag{4.2}$$

here, $g_g$ is degeneracy of gluons, $g_q$ is degeneracy of quarks, considering 3 flavour of quarks, each flavour have 2 spin-state, 3 colour-state and 2 charge-state, so $g_q$=2×2×3×3 = 36 and there are 8 types of gluons and their 2 helicity states, so $g_g$ = 8×2 = 16.

When considering Fermi-Dirac and Boson Statistics for quarks and gluons respectively, in high temperature regime and m = 0, $\mu \neq 0$ ,

Pressure of gluons: $P_g = \frac{g_g \pi^2}{90}T^4$,

Pressure of quarks: $P_q = \frac{g_q}{3}T^4\Big[\frac{7\pi^2}{120} + \frac{1}{4}(\frac{\mu}{T})^2 + \frac{1}{8\pi^2}(\frac{\mu}{T})^4\Big]$

So,

$$P_{QGP} = \frac{g_g \pi^2}{90}T^4 + \frac{g_q}{3}T^4\Big[\frac{7\pi^2}{120} + \frac{1}{4}\big(\frac{\mu}{T}\big)^2 + \frac{1}{8\pi^2}\big(\frac{\mu}{T}\big)^4\Big] - B \tag{4.3}$$

### 4.1.2 Hadron Gas Model

HRG Model assume all the particles are non-interacting point-like particles. According to ideal HRG model total thermodynamical pressure is the sum of pressure of constituent gas of different particle. It gives statistical description of hadrons in the grand canonical (G.C.) ensemble. The Partition function (Z) in G.C. ensemble follows,

$$\begin{aligned} \ln Z =& \frac{1}{\eta}\sum_i \ln\Big(1 + \eta \exp\big(-\frac{(E_i - \mu_i)}{T}\big)\Big), \\ =& \frac{gV}{\eta(2\pi)^3}\int dp^3 \ln\Big(1 + \eta \exp\big(-\frac{(E_i - \mu_i)}{T}\big)\Big) \end{aligned} \tag{4.4}$$

where, $E_i = \sqrt{p^2 + m_i^2}$, and we have used, $\sum_i \rightarrow \int \frac{gVd^3p}{(2\pi)^3}$

Pressure formula,

$$\begin{aligned} P_j &= T\frac{lnZ_j}{V} \\ &= \frac{g_j}{2\pi^2}\int_0^\infty \frac{p^4 dp}{3E_j\left[exp[(E_j - \mu_j)/T] + \eta\right]} \end{aligned} \tag{4.5}$$

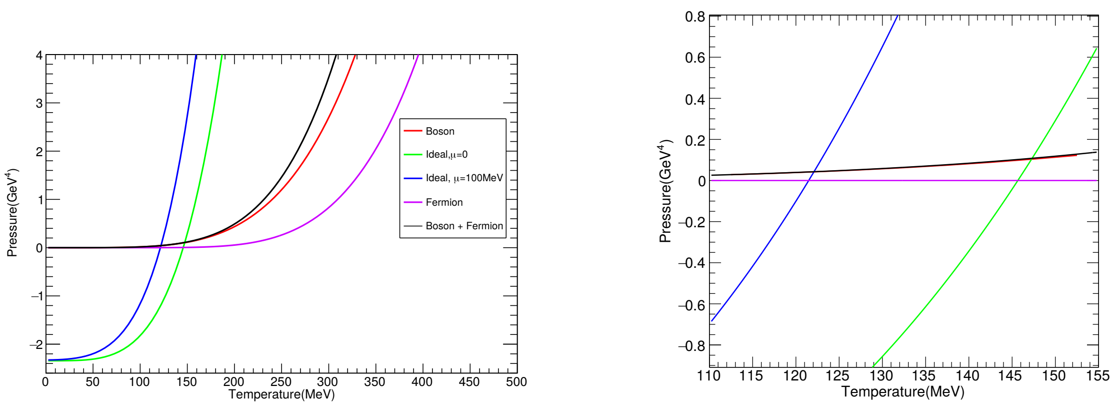
Once pressure of single kind of particle is known, according to HRG we'll have Total pressure as sum over all kinds of particle in the system.

$$P = \sum_{boson} P_b + \sum_{fermion} P_f \tag{4.6}$$

We can simplify this integral in terms of modified Bessel Function of Second kind see Appendix:B.

$$P_{HRG} = \sum_b \frac{g_b}{2\pi^2}\frac{m_b^2}{T^2}\sum_{n=1}^{\infty}\frac{1}{n^2}K_2(\frac{nm_b}{T})exp(\frac{n\mu_b}{T}) + \sum_f \frac{g_f}{2\pi^2}\frac{m_f^2}{T^2}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^2}K_2(\frac{nm_f}{T})exp(\frac{n\mu_f}{T}) \tag{4.7}$$

'Fig: 4.1' shows the variation of pressure with temperature for MIT Bag Model with taking B=$220^4 MeV^4$ [33][34](calculated by QCD calculation) and Ideal HRG model, plot is made using hadrons: pions, kaons, protons and neutrons with $\mu_b$ = $\mu_f$ = 0 (considering equal number of particles and anti-particles). One can see a change in pressure as one scans the plot Fig: 4.1b from lower temperature to higher value i.e. when one follows black line for hadrons and blue or green for quarks. Initially we have hadrons at low temperature and as the temperature is increased beyond hadronization temperature, $T_H$ ($P_{HRG}$= $P_{QGP}$ here, $T_H$ $\approx$ 120-150 MeV), there is an abrupt change in value of pressure ('black to blue' or 'black to green'), which tells the hint of occurrence of phase transition.




(a) MIT Model and HRG Model

(b) Magnified

Figure 4.1: Thermodynamical Models

## 4.2 Tsallis Framework

Boltzmann-Gibbs statistics is mainly used in approximation to study the systems where constituents are independent or weakly correlated. It fails to explain the strongly correlated systems where long-range correlations and interactions are significant. On the other hand, the memory effects and long-range color interactions may give rise to non-Markovian processes which in turn affect the dynamical evolution of the fireball produced in heavy-ion collision as described in[35]. Further, in a strongly correlated system, there might exist the case when entropy is non-extensive or non-additive. In '1988' Tsallis proposed generalization of Boltzmann entropy which gave birth to non-extensive statistical mechanics[32]. He introduce an extra parameter in entropy function the 'q-parameter' which under certain condition gives back standard entropy. There are systems which involve interactions and long term memory effects. This statistics takes into account long-range-interactions and complex microscopic dynamics. It is widely used in hydrodynamics turbulence, astrophysics, condense matter systems, Particle Physics (high-energy collisions). This statistics is appropriate for the study of complex systems with a certain degree of non-equilibrium as can be the case, with these nuclear collisions[36][37]. We will introduce the concept and will try derive the relevant formula used in the thesis. Tsallis tried attempt to modified standard statistical mechanics entropy formula, i.e. $(S_b) = -\sum_i k_B p_i ln(p_i)$ to,

$$S_q = k_B \frac{1 - \sum_i p_i^q}{q-1} = -k_B \sum_i p_i^q ln_q(p_i) \tag{4.8}$$

where,

$$ln_q(p_i) = \frac{p_i^{1-q} - 1}{1-q}$$

is named q-logarithmic. We, can define q-exponential,

$$exp_q(x) = \begin{cases} [1 + (1-q)x]^{\frac{1}{1-q}} = [1 - (q-1)x]^{\frac{-1}{q-1}} & \text{if, } x \leq 0, \\ [1 + (q-1)x]^{\frac{1}{q-1}} & \text{if, } x > 0 \end{cases} \tag{4.9}$$

Note, with $\xrightarrow{q\to1}$ we get back B-G entropy,

$$S_q \xrightarrow{q\to1} S_b, \;\; exp_q(x) \xrightarrow{q\to1} exp(x), \;\; log_q(x) \xrightarrow{q\to1} log(x), \tag{4.10}$$

Non-extensive parameter 'q' is related to temperature fluctuation by[38][39]

$$q = 1 + \frac{Var(T)}{< T^2 >} \tag{4.11}$$

We can drive back all the formulas in standard statistical mechanics using, $f_q \xrightarrow{q \to 1} f_b$. One can see non-extensive nature of Tsallis statistics, Consider two probabilistically independent systems A and B, this means

$$p_{ij}(A+B) = p_i(A) \times p_j(B) \tag{4.12}$$

Now, using eq: 4.8,

$$\begin{aligned}
S_q(A+B) &= k_B \frac{(1 - \sum_k p^q_{AB,k})}{q-1}, \\
&= k_B \frac{(1 - \sum_i \sum_j p^q_{A,i} p^q_{B,j})}{q-1}, \\
&= k_B \frac{2 - \sum_i p^q_{A,i} - \sum_j p^q_{B,j} - (1 - \sum_i p^q_{A,i})(1 - \sum_j p^q_{B,j})}{q-1}, \\
&= k_B \frac{(1 - \sum_i p^q_{A,i})}{q-1} + k_B \frac{(1 - \sum_j p^q_{B,j})}{q-1} - k_B(q-1) \frac{(1 - \sum_i p^q_{A,i})}{q-1} \frac{(1 - \sum_j p^q_{B,j})}{q-1}, \\
&= S_q(A) + S_q(B) - \frac{(q-1)}{k_B} S_q(A) S_q(B)
\end{aligned} \tag{4.13}$$

Note the third term which is giving birth to non-extensive nature to entropy. In standard statistical mechanics, using Grand-Canonical ensemble, we drive occupation number using Partition function (Z) which itself is derived by maximising the entropy and following constraints,

$$\begin{aligned}
\sum_i p_i = 1, \qquad N = \sum_i p_i N_i, \\
E = \sum_i p_i E_i, \qquad S = -k_B \sum_i p_i \ln p_i
\end{aligned} \tag{4.14}$$

Using these constrains and Lagrange multipliers for $p_i$ solving eq: 4.14, we get,

$$\begin{aligned}
p_i &= \frac{e^{\beta(\mu N_i - E_i)}}{Z}, \\
Z &= \sum_i e^{\beta(\mu N_i - E_i)}
\end{aligned} \tag{4.15}$$

here, $p_i$ is the probability that the 'system' has $N_i$ particles and $E_i$ energy, This Z can be written in different form, in terms of single partition function($\xi$),

$$Z = \prod_a \xi_a, \qquad \xi_a = \sum_\nu e^{\beta\nu(\mu - e_a)} \tag{4.16}$$

Occupation number($f_a$) i.e. average number of particles in an orbital 'a' of energy e, is given by,

$$f_a = <\nu(e_a)> = \frac{1}{\beta}(\frac{\partial}{\partial \mu} \ln \xi_a)$$

For different types of particles we have different $\xi_a$, hence Z and $f_a$, we can summarize Partition function in generalised(Gen.) form as,

$$\ln Z^{Gen.} = \frac{1}{\eta}\sum_a \ln(1+\eta e^{\beta(\mu-e)})$$
$$\eta = \begin{cases} +1 & \text{for Fermions,} \\ -1 & \text{for Boson,} \\ 0 & \text{for Boltzmann-Gibbs.} \end{cases} \tag{4.17}$$

Using Thermo-statistical relations, we can write entropy in the form of occupation number,

$$\begin{aligned} S^{B.G.} &= -k_B\sum_a \Big[f_a \ln f_a - f_a\Big] \\ S^{F.D.} &= -k_B\sum_a \Big[f_a \ln f_a + (1-f_a)\ln(1-f_a)\Big] \\ S^{B.E.} &= -k_B\sum_a \Big[f_a \ln f_a - (1+f_a)\ln(1+f_a)\Big] \\ S^{Gen.} &= -k_B\sum_a \Big[f_a \ln f_a + \frac{(1-\eta f_a)}{\eta}\ln(1-\eta f_a)\Big] \end{aligned} \tag{4.18}$$

Note also, we can drive the form of occupation number using eq: 4.18, and the constrains,

$$\begin{aligned} \delta S = 0, N = \sum_a f_a, E = \sum_a f_a e_a, \\ \delta S = 0, \sum_a \delta f_a = 0, \sum_a \delta f_a e_a = 0 \end{aligned} \tag{4.19}$$

Now, We generalize eq's: 4.19, to **Tsallis- statistics**, maintaining Thermodynamical consistency [37][40],

$$\begin{aligned} \delta S_q = 0, N = \sum_a f_a^q, E = \sum_a f_a^q e_a, \\ \delta S_q = 0, \sum_a \delta f_a^q = 0, \sum_a \delta f_a^q e_a = 0 \end{aligned} \tag{4.20}$$

And the generalized form of Tsallis-entropy in form of new $f_i$ is given as,

$$\begin{aligned} S_q^{B.G.} &= -k_B\sum_a \Big[f_a^q \ln_q f_a - f_a\Big] \\ S_q^{F.D.} &= -k_B\sum_a \Big[f_a^q \ln_q f_a + (1-f_a)^q\ln_q(1-f_a)\Big] \\ S_q^{B.E.} &= -k_B\sum_a \Big[f_a^q \ln_q f_a - (1+f_a)^q\ln_q(1+f_a)\Big] \\ S_q^{Gen.} &= -k_B\sum_a \Big[f_a^q \ln_q f_a + \frac{(1-\eta f_a)^q}{\eta}\ln_q(1-\eta f_a)\Big] \end{aligned} \tag{4.21}$$

We can solve eq: 4.20 and 4.21 for $f_a$ using method of Lagrange multipliers,

$$\delta[S_q(f_a) + \gamma(E - \sum_a f_a^q e_a) + \alpha(N - \sum_a f_a^q)] = 0 \tag{4.22}$$

Solving individual differentiation,

$$\begin{aligned}
\delta(N - \sum_a f_a^q) &= -\delta\Big(\sum_a f_a^q\Big) = -\sum_a \Big[q f_a^{q-1}\Big]\delta f_a, \\
\delta(E - \sum_a f_a^q e_a) &= -\delta\Big(\sum_a f_a^q e_a\Big) = -\sum_a \Big[q f_a^{q-1} e_a\Big]\delta f_a
\end{aligned} \tag{4.23}$$

and For B-G,

$$\begin{aligned}
\delta\Big(S_q^{B.G.}\Big) &= -k_B \sum_a \frac{\partial}{\partial f_a}\Big[f_a^q \ln_q f_a - f_a\Big]\delta f_a, \\
&= -k_B \sum_a \Big[q f_a^{q-1} \ln_q f_a + f_a^q \frac{\partial}{\partial f_a}(\ln_q f_a) - 1\Big]\delta f_a, \\
&= -k_B \sum_a \Big[q(\ln_q f_a) f_a^{q-1}\Big]\delta f_a \\
&= -k_B \sum_a \Big[\frac{q}{q-1}(f_a^{q-1} - 1)\Big]\delta f_a
\end{aligned} \tag{4.24}$$

For Boson,

$$\begin{aligned}
\delta\Big(S_q^{B.E.}\Big) &= -k_B \sum_a \frac{\partial}{\partial f_a}\Big[f_a^q \ln_q f_a - (1+f_a)^q \ln_q(1+f_a)\Big]\delta f_a, \\
&= -k_B \sum_a \Big[q f_a^{q-1} \ln_q f_a + 1 - q(1+f_a)^{q-1} \ln_q(1+f_a) - 1\Big]\delta f_a, \\
&= -k_B \sum_a \Big[q\Big(\frac{1 - f_a^{q-1}}{1-q}\Big) - q\Big(\frac{1 - (1+f_a)^{q-1}}{1-q}\Big)\Big]\delta f_a, \\
&= -k_B \sum_a \Big[\Big(\frac{q}{1-q}\Big) f_a^{q-1}\Big(\frac{(1+f_a)^{q-1}}{f_a^{q-1}} - 1\Big)\Big]\delta f_a
\end{aligned} \tag{4.25}$$

For Fermion,

$$\begin{aligned}
\delta\Big(S_q^{F.D.}\Big) &= -k_B \sum_a \frac{\partial}{\partial f_a}\Big[f_a^q \ln_q f_a + (1-f_a)^q \ln_q(1-f_a)\Big]\delta f_a, \\
&= -k_B \sum_a \Big[q f_a^{q-1} \ln_q f_a + 1 - q(1-f_a)^{q-1} \ln_q(1-f_a) - 1\Big]\delta f_a, \\
&= -k_B \sum_a \Big[q\Big(\frac{1 - f_a^{q-1}}{1-q}\Big) - q\Big(\frac{1 - (1-f_a)^{q-1}}{1-q}\Big)\Big]\delta f_a, \\
&= -k_B \sum_a \Big[\Big(\frac{q}{1-q}\Big) f_a^{q-1}\Big(\frac{(1-f_a)^{q-1}}{f_a^{q-1}} - 1\Big)\Big]\delta f_a
\end{aligned} \tag{4.26}$$

Solving by taking eq: 4.24 or eq: 4.25 or eq: 4.26 and eq: 4.23 and substituting in eq: 4.22, also here all $\delta f_a$ can be treated as independent so the coefficients of each $\delta f_a$ must vanish.This gives result,

$$\begin{aligned}
f_{a,q}^{B.G.} &= [1+(q-1)(\alpha+\gamma e_a)]^{\frac{-1}{q-1}}, \\
f_{a,q}^{B.E.} &= \frac{1}{[1+(q-1)(\alpha+\gamma e_a)]^{\frac{1}{q-1}}-1}, \\
f_{a,q}^{F.D.} &= \frac{1}{[1+(q-1)(\alpha+\gamma e_a)]^{\frac{1}{q-1}}+1}
\end{aligned} \tag{4.27}$$

Values of Lagrange multiplier($\alpha,\gamma$) are calculated by equating it to B-G limit ($q \to 1$), we get $\alpha = -\beta\mu$ , $\gamma = \beta$, where $\beta = \frac{1}{k_B T}$, Eq: 4.26 can be written in terms of q-exponential,

$$\begin{aligned}
\text{Tsallis-Boltzmann: } f_i^{B.G.} &= exp_q\Big(-\frac{e_i-\mu}{k_B T}\Big), \\
\text{Tsallis-Boson: } f_i^{B.E.} &= \frac{1}{exp_q(\frac{e_i-\mu}{k_B T})-1}, \\
\text{Tsallis-Fermion: } f_i^{F.D.} &= \frac{1}{exp_q(\frac{e_i-\mu}{k_B T})+1}, \\
\text{Tsallis-Gen.: } f_i^{Gen.} &= \frac{1}{exp_q(\frac{e_i-\mu}{k_B T})+\eta}
\end{aligned} \tag{4.28}$$

### 4.2.1 Thermodynamics Consistency

Since we have derived the formula for occupation number and hence entropy, we have to prove that the statistics build from these derived formulas is thermodynamically consistent. So in this section we'll verify the thermodynamical relations, which characterises classical thermodynamics using our statistics. In proving the relations we'll keep volume to be constant and use natural units-system (Appendix: A). The relations are:

$$\frac{\partial P}{\partial \mu}\Big|_T = \frac{N}{V} = n, \tag{4.29}$$

$$\frac{\partial P}{\partial T}\Big|_\mu = \frac{S}{V} = s, \tag{4.30}$$

$$\frac{\partial E}{\partial N}\Big|_S = \frac{\partial \epsilon}{\partial n}\Big|_s = \mu, \tag{4.31}$$

$$\frac{\partial E}{\partial S}\Big|_N = \frac{\partial \epsilon}{\partial s}\Big|_n = T \tag{4.32}$$

We'll use following thermodynamical equality in proving the above relations for Tsallis distributions.

$$S = \frac{E-\mu N+PV}{T}, \tag{4.33}$$

or in terms of densities,

$$\begin{aligned} s =& \frac{\epsilon - \mu n + P}{T}, \\ P =& sT + \mu n - \epsilon \end{aligned} \tag{4.34}$$

We will prove for Tsallis-Bolt and Tsallis-Boson, Tsallis-Fermion can be done similarly.

**Relation: 4.29**

Differentiating eq: 4.34, w.r.t $\mu$ keeping T constant,

$$V\frac{\partial P}{\partial \mu}\Big|_T = T\frac{\partial S}{\partial \mu}\Big|_T + N + \mu\frac{\partial N}{\partial \mu}\Big|_T - \frac{\partial E}{\partial \mu}\Big|_T \tag{4.35}$$

We'll first proof for ***Tsallis-Bolt***, so from eq: 4.24,

$$\begin{aligned} \frac{\partial S}{\partial \mu}\Big|_T =& -\sum_a \Big[\frac{q}{q-1}(f_a^{q-1} - 1)\Big]\frac{\partial f_a}{\partial \mu}, \\ =& -\sum_a \Big[\frac{q}{q-1}(f_a^{q-1} - 1)\Big]\frac{f_a^q}{T} \end{aligned} \tag{4.36}$$

$$\frac{\partial N}{\partial \mu}\Big|_T = \sum_a \frac{\partial f_a^q}{\partial \mu} = \sum_a q f_a^{q-1}\frac{\partial f_a}{\partial \mu} = \sum_a q\frac{f_a^{2q-1}}{T}, \tag{4.37}$$

$$\frac{\partial E}{\partial \mu}\Big|_T = \sum_a q e_a \frac{f_a^{2q-1}}{T} \tag{4.38}$$

Substituting these differentials in eq: 4.35,

$$\begin{aligned} V\frac{\partial P}{\partial \mu}\Big|_T =& \sum_a \Big( - \Big[\frac{q}{q-1}(f_a^{q-1} - 1)\Big] f_a^q + f_a^q + \mu q \frac{f_a^{2q-1}}{T} - q e_a \frac{f_a^{2q-1}}{T}\Big), \\ =& \sum_a \Big( - \frac{q}{q-1} f_a^{2q-1} + \frac{q}{q-1} f_a^q + f_a^q + \mu q \frac{f_a^{2q-1}}{T} - q e_a \frac{f_a^{2q-1}}{T}\Big), \\ =& \sum_a \Big( - \frac{q}{q-1}\big[1 + (q-1)\frac{e_a - \mu}{T}\big] f_a^{2q-1} + \frac{q}{q-1} f_a^q + f_a^q\Big), \\ =& \sum_a \Big( - \frac{q}{q-1}\big[f_a^{1-q}\big] f_a^{2q-1} + \frac{q}{q-1} f_a^q + f_a^q\Big), \\ =& \sum_a f_a^q = N \end{aligned} \tag{4.39}$$

For ***Tsallis-Boson***, so from eq: 4.25,

$$\begin{aligned} \frac{\partial S}{\partial \mu}\Big|_T =& -\sum_a \Big[\Big(\frac{q}{1-q}\Big) f_a^{q-1}\Big(\frac{(1+f_a)^{q-1}}{f_a^{q-1}} - 1\Big)\Big]\big(\frac{\partial f_a}{\partial \mu}\big), \\ =& -\sum_a \Big[\Big(\frac{q}{1-q}\Big) f_a^{q-1}\Big(\frac{(1+f_a)^{q-1}}{f_a^{q-1}} - 1\Big)\Big]\Big(\frac{f_a^2}{T}\Big[1 + (q-1)\frac{e_a - \mu}{T}\Big]^{\frac{2-q}{q-1}}\Big) \end{aligned} \tag{4.40}$$

$$
\begin{aligned}
\frac{\partial N}{\partial \mu}\Big|_T &= \sum_a \frac{\partial f_a^q}{\partial \mu} = \sum_a q f_a^{q-1} \frac{\partial f_a}{\partial \mu}, \\
&= \sum_a \Big[ q f_a^{q-1} \Big( \frac{f_a^2}{T} \Big[ 1 + (q-1) \frac{e_a - \mu}{T} \Big]^{\frac{2-q}{q-1}} \Big) \Big], \qquad (4.41) \\
&= \sum_a \Big[ q \frac{f_a^{q+1}}{T} \Big[ 1 + (q-1) \frac{e_a - \mu}{T} \Big]^{\frac{2-q}{q-1}} \Big] \\
\frac{\partial E}{\partial \mu}\Big|_T &= \sum_a \Big[ e_a q \frac{f_a^{q+1}}{T} \Big[ 1 + (q-1) \frac{e_a - \mu}{T} \Big]^{\frac{2-q}{q-1}} \Big] \qquad (4.42)
\end{aligned}
$$

Using these differentials and supplying to eq: 4.35,

$$
\begin{aligned}
V\frac{\partial P}{\partial \mu}\Big|_T &= \sum_a \Big[ - T \Big[ \Big( \frac{q}{1-q} \Big) f_a^{q-1} \Big( \frac{(1+f_a)^{q-1}}{f_a^{q-1}} - 1 \Big) \Big] \Big( \frac{f_a^2}{T} \Big[ 1 + (q-1) \frac{e_a - \mu}{T} \Big]^{\frac{2-q}{q-1}} \Big) + f_a^q \\
&\quad + \mu q \frac{f_a^{q+1}}{T} \Big[ 1 + (q-1) \frac{e_a - \mu}{T} \Big]^{\frac{2-q}{q-1}} - e_a q \frac{f_a^{q+1}}{T} \Big[ 1 + (q-1) \frac{e_a - \mu}{T} \Big]^{\frac{2-q}{q-1}} \Big], \\
&= \sum_a \Big[ - \frac{q}{1-q} (1+f_a)^{q-1} \Big( f_a^2 \Big[ 1 + (q-1) \frac{e_a - \mu}{T} \Big]^{\frac{2-q}{q-1}} \Big) \\
&\quad + \frac{q}{1-q} f_a^{q-1} \Big( f_a^2 \Big[ 1 + (q-1) \frac{e_a - \mu}{T} \Big]^{\frac{2-q}{q-1}} \Big) + f_a^q \\
&\quad + \Big( \frac{\mu - e_a}{T} \Big) q f_a^{q+1} \Big[ 1 + (q-1) \frac{e_a - \mu}{T} \Big]^{\frac{2-q}{q-1}} \Big]
\end{aligned}
\qquad (4.43)
$$

Using,

$$
\Big[ 1 + (q-1) \frac{e_a - \mu}{T} \Big]^{\frac{2-q}{q-1}} = \Big[ \frac{1+f_a}{f_a} \Big]^{2-q} \qquad (4.44)
$$

$$
\begin{aligned}
V\frac{\partial P}{\partial \mu}\Big|_T &= \sum_a \Big[ - \frac{q}{1-q} (1+f_a)^{q-1} \Big( f_a^2 \Big[ \frac{1+f_a}{f_a} \Big]^{2-q} \Big) + \frac{q}{1-q} f_a^{q-1} \Big( f_a^2 \Big[ \frac{1+f_a}{f_a} \Big]^{2-q} \Big) \\
&\quad + f_a^q + \Big( \frac{\mu - e_a}{T} \Big) q f_a^{q+1} \Big[ \frac{1+f_a}{f_a} \Big]^{2-q} \Big], \\
&= \sum_a \Big[ - \frac{q}{1-q} (1+f_a) f_a^q + \frac{q}{1-q} f_a^{q+1} \Big[ \frac{1+f_a}{f_a} \Big]^{2-q} \\
&\quad + f_a^q + \Big( \frac{\mu - e_a}{T} \Big) q f_a^{q+1} \Big[ \frac{1+f_a}{f_a} \Big]^{2-q} \Big], \\
&= \sum_a \Big[ - \frac{q}{1-q} (1+f_a) f_a^q + \frac{q}{1-q} [1 + (1-q) \frac{\mu - e_a}{T}] f_a^{q+1} \Big[ \frac{1+f_a}{f_a} \Big]^{2-q} + f_a^q \Big], \\
&= \sum_a \Big[ - \frac{q}{1-q} (1+f_a) f_a^q + \frac{q}{1-q} (1+f_a) f_a^q + f_a^q \Big], \\
&= \sum_a f_a^q = N
\end{aligned}
\qquad (4.45)
$$

**Relation: 4.30**

Differentiating eq: 4.34 w.r.t T keeping $\mu$ constant,

$$V\frac{\partial P}{\partial T}\Big|_{\mu} = S + T\frac{\partial S}{\partial T}\Big|_{\mu} + \mu\frac{\partial N}{\partial T}\Big|_{\mu} - \frac{\partial E}{\partial T}\Big|_{\mu} \tag{4.46}$$

We'll first proof for ***Tsallis-Bolt***, so from eq: 4.24,

$$\begin{aligned}\frac{\partial S}{\partial T}\Big|_{\mu} &= -\sum_a \Big[\frac{q}{q-1}(f_a^{q-1}-1)\Big]\frac{\partial f_a}{\partial T},\\ &= -\sum_a \Big[\frac{q}{q-1}(f_a^{q-1}-1)\Big][\frac{e_a-\mu}{T^2}]f_a^q\end{aligned} \tag{4.47}$$

$$\frac{\partial N}{\partial T}\Big|_{\mu} = \sum_a qf_a^{q-1}\frac{\partial f_a}{\partial T} = \sum_a qf_a^{2q-1}[\frac{e_a-\mu}{T^2}], \tag{4.48}$$

$$\frac{\partial E}{\partial T}\Big|_{\mu} = \sum_a qe_af_a^{2q-1}[\frac{e_a-\mu}{T^2}] \tag{4.49}$$

Substituting these differentials in eq: 4.46,

$$\begin{aligned}V\frac{\partial P}{\partial T}\Big|_{\mu} - S = T\frac{\partial S}{\partial T}\Big|_{\mu} + \mu\frac{\partial N}{\partial T}\Big|_{\mu} - \frac{\partial E}{\partial T}\Big|_{\mu},&\\ = \sum_a \Big[-\frac{q}{q-1}(f_a^{q-1}-1)[\frac{e_a-\mu}{T}]f_a^q + \mu\frac{q}{T^2}f_a^{2q-1}(e_a-\mu) - q\frac{e_a(e_a-\mu)}{T^2}f_a^{2q-1}\Big],&\\ = \sum_a \Big[\frac{q}{q-1}(f_a^{q-1}-1)[\frac{\mu-e_a}{T}]f_a^q + q\frac{(\mu-e_a)}{T^2}f_a^{2q-1}(e_a-\mu)\Big]&\end{aligned} \tag{4.50}$$

Using,

$$(e_a-\mu) = T\big(\frac{f_a^{1-q}-1}{q-1}\big) \tag{4.51}$$

Simplifying eq: 4.50,

$$\begin{aligned}V\frac{\partial P}{\partial T}\Big|_{\mu} - S =& \sum_a \Big[\frac{q}{q-1}(f_a^{q-1}-1)[\frac{\mu-e_a}{T}]f_a^q + q\frac{(\mu-e_a)}{T}f_a^{2q-1}\big(\frac{f_a^{1-q}-1}{q-1}\big)\Big],\\ =& \sum_a \Big[\frac{q}{q-1}(f_a^{q-1}-1)[\frac{\mu-e_a}{T}]f_a^q + \frac{q}{q-1}\frac{(\mu-e_a)}{T}f_a^{2q-1}(f_a^{1-q}-1)\Big],\\ =& \sum_a \Big[\frac{q}{q-1}(f_a^{q-1}-1)[\frac{\mu-e_a}{T}]f_a^q + \frac{q}{q-1}\frac{(\mu-e_a)}{T}(f_a^q - f^{2q-1})\Big],\\ =& 0\end{aligned} \tag{4.52}$$

For ***Tsallis-Boson***, So from eq: 4.25,

$$
\begin{aligned}
\frac{\partial S}{\partial T}\Big|_{\mu} &= -\sum_a \Big[\Big(\frac{q}{1-q}\Big) f_a^{q-1}\Big(\frac{(1+f_a)^{q-1}}{f_a^{q-1}}-1\Big)\Big]\Big(\frac{\partial f_a}{\partial T}\Big),\\
&= -\sum_a \Big[\Big(\frac{q}{1-q}\Big) f_a^{q-1}\Big(\frac{(1+f_a)^{q-1}}{f_a^{q-1}}-1\Big)\Big]\Big(\frac{f_a^2(e_a-\mu)}{T^2}\Big[1+(q-1)\frac{e_a-\mu}{T}\Big]^{\frac{2-q}{q-1}}\Big)
\end{aligned}
\tag{4.53}
$$

$$
\begin{aligned}
\frac{\partial N}{\partial T}\Big|_{\mu} &= \sum_a \frac{\partial f_a^q}{\partial T} = \sum_a q f_a^{q-1}\frac{\partial f_a}{\partial T},\\
&= \sum_a \Big[q f_a^{q-1}\Big(\frac{f_a^2(e_a-\mu)}{T^2}\Big[1+(q-1)\frac{e_a-\mu}{T}\Big]^{\frac{2-q}{q-1}}\Big)\Big], \qquad (4.54)\\
&= \sum_a \Big[q\frac{f_a^{q+1}(e_a-\mu)}{T^2}\Big[1+(q-1)\frac{e_a-\mu}{T}\Big]^{\frac{2-q}{q-1}}\Big]\\
\frac{\partial E}{\partial T}\Big|_{\mu} &= \sum_a \Big[q e_a(e_a-\mu)\frac{f_a^{q+1}}{T^2}\Big[1+(q-1)\frac{e_a-\mu}{T}\Big]^{\frac{2-q}{q-1}}\Big] \qquad (4.55)
\end{aligned}
$$

Using these differentials and supplying to eq: 4.46,

$$
\begin{aligned}
V\frac{\partial P}{\partial T}\Big|_{\mu} - S &= T\frac{\partial S}{\partial T}\Big|_{\mu} + \mu\frac{\partial N}{\partial T}\Big|_{\mu} - \frac{\partial E}{\partial T}\Big|_{\mu},\\
&= \sum_a \Big[-T\Big[\Big(\frac{q}{1-q}\Big) f_a^{q-1}\Big(\frac{(1+f_a)^{q-1}}{f_a^{q-1}}-1\Big)\Big]\Big(\frac{f_a^2(e_a-\mu)}{T^2}\Big[1+(q-1)\frac{e_a-\mu}{T}\Big]^{\frac{2-q}{q-1}}\Big)+\\
&\quad \mu\Big[q\frac{f_a^{q+1}(e_a-\mu)}{T^2}\Big[1+(q-1)\frac{e_a-\mu}{T}\Big]^{\frac{2-q}{q-1}}\Big] - \Big[q e_a(e_a-\mu)\frac{f_a^{q+1}}{T^2}\Big[1+(q-1)\frac{e_a-\mu}{T}\Big]^{\frac{2-q}{q-1}}\Big]\Big],\\
&= \sum_a \Big[-\Big[\Big(\frac{q}{1-q}\Big) f_a^{q+1}\Big(\frac{(1+f_a)^{q-1}}{f_a^{q-1}}-1\Big)\Big]\Big(\frac{(e_a-\mu)}{T}\Big[1+(q-1)\frac{e_a-\mu}{T}\Big]^{\frac{2-q}{q-1}}\Big)\\
&\qquad + (\mu-e_a)\Big[q\frac{f_a^{q+1}(e_a-\mu)}{T^2}\Big[1+(q-1)\frac{e_a-\mu}{T}\Big]^{\frac{2-q}{q-1}}\Big]\Big]
\end{aligned}
\tag{4.56}
$$

Using, eq: 4.44,

$$
\begin{aligned}
V\frac{\partial P}{\partial T}\Big|_{\mu} - S &= \sum_a \Big[-\Big[\Big(\frac{q}{1-q}\Big) f_a^{q+1}\Big(\frac{(1+f_a)^{q-1}}{f_a^{q-1}}-1\Big)\Big]\Big(\frac{(e_a-\mu)}{T}\Big[\frac{1+f_a}{f_a}\Big]^{2-q}\Big)\\
&\qquad + (\mu-e_a)\Big[q\frac{f_a^{q+1}(e_a-\mu)}{T^2}\Big[\frac{1+f_a}{f_a}\Big]^{2-q}\Big]\Big],
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_a \left[\left[\Big(\frac{q}{q-1}\Big) f_a^{2q-1}\Big(\frac{(1+f_a)^{q-1}}{f_a^{q-1}} - 1\Big)\right]\Big(\frac{(e_a-\mu)}{T}\Big[1+f_a\Big]^{2-q}\Big) \right.\\
&\qquad\qquad\qquad \left. + (\mu - e_a)\left[q\frac{f_a^{2q-1}(e_a-\mu)}{T^2}\Big[1+f_a\Big]^{2-q}\right]\right],\\
&= \sum_a \Big(\frac{q}{q-1}\Big) f_a^{2q-1}\frac{(e_a-\mu)}{T}\Big[1+f_a\Big]^{2-q}\left[\Big(\frac{(1+f_a)^{q-1}}{f_a^{q-1}} - 1\Big) + (q-1)\Big[\frac{\mu - e_a}{T}\Big]\right],\\
&= \sum_a \Big(\frac{q}{q-1}\Big) f_a^{2q-1}\frac{(e_a-\mu)}{T}\Big[1+f_a\Big]^{2-q}\left[\frac{(1+f_a)^{q-1}}{f_a^{q-1}} - \Big[1 + (q-1)\frac{e_a-\mu}{T}\Big]\right],\\
&= \sum_a \Big(\frac{q}{q-1}\Big) f_a^{2q-1}\frac{(e_a-\mu)}{T}\Big[1+f_a\Big]^{2-q}\left[\frac{(1+f_a)^{q-1}}{f_a^{q-1}} - \Big[\frac{(1+f_a)^{q-1}}{f_a^{q-1}}\Big]\right],\\
&= 0
\end{aligned}
\tag{4.57}
$$

**<u>Relation: 4.31</u>**

$$
\begin{aligned}
\frac{\partial E}{\partial N}\Big|_S &= \frac{\frac{\partial E}{\partial T}\big|_\mu dT + \frac{\partial E}{\partial \mu}\big|_T d\mu}{\frac{\partial N}{\partial T}\big|_\mu dT + \frac{\partial N}{\partial \mu}\big|_T d\mu},\\
&= \frac{\frac{\partial E}{\partial T}\big|_\mu + \frac{\partial E}{\partial \mu}\big|_T \frac{d\mu}{dT}}{\frac{\partial N}{\partial T}\big|_\mu + \frac{\partial N}{\partial \mu}\big|_T \frac{d\mu}{dT}}
\end{aligned}
\tag{4.58}
$$

Further, since S is kept constant so, dS = 0,

$$
\begin{aligned}
dS &= \frac{\partial S}{\partial T}\Big|_\mu dT + \frac{\partial S}{\partial \mu}\Big|_T d\mu = 0,\\
\text{so,}\qquad \frac{d\mu}{dT} &= -\left[\frac{\frac{\partial S}{\partial T}\big|_\mu}{\frac{\partial S}{\partial \mu}\big|_T}\right]
\end{aligned}
\tag{4.59}
$$

$$
\frac{\partial E}{\partial N}\Big|_S = \frac{\frac{\partial E}{\partial T}\big|_\mu - \frac{\partial E}{\partial \mu}\big|_T\Big[\frac{\frac{\partial S}{\partial T}|_\mu}{\frac{\partial S}{\partial \mu}|_T}\Big]}{\frac{\partial N}{\partial T}\big|_\mu - \frac{\partial N}{\partial \mu}\big|_T\Big[\frac{\frac{\partial S}{\partial T}|_\mu}{\frac{\partial S}{\partial \mu}|_T}\Big]}
\tag{4.60}
$$

We can explicitly solve this eq. by substituting the corresponding derivatives. For ***Tsallis-Bolt***, using eqs: 4.36, 4.37, 4.38, 4.47, 4.48, 4.49, we can write the numerator ($N'$) as,

$$
N' = \sum_a q e_a f_a^{2q-1}\Big[\frac{e_a-\mu}{T^2}\Big] - \Big[\sum_a q e_a \frac{f_a^{2q-1}}{T}\Big]\left[\frac{-\sum_b\Big[\frac{q}{q-1}(f_b^{q-1}-1)\Big]\Big[\frac{e_b-\mu}{T^2}\Big]f_b^q}{-\sum_b\Big[\frac{q}{q-1}(f_b^{q-1}-1)\Big]\frac{f_b^q}{T}}\right],
$$

$$=\sum_a qe_a f_a^{2q-1}[\frac{e_a-\mu}{T^2}] - \Big[\sum_a qe_a \frac{f_a^{2q-1}}{T}\Big]\Big[\frac{\sum_b (f_b^{q-1}-1)[\frac{e_b-\mu}{T}]f_b^q}{\sum_b (f_b^{q-1}-1)f_b^q}\Big],$$

$$=\frac{\Big[\Big[\sum_a qe_a f_a^{2q-1}[\frac{e_a-\mu}{T^2}]\Big]\Big[\sum_b (f_b^{q-1}-1)f_b^q\Big] - \Big[\sum_a qe_a \frac{f_a^{2q-1}}{T}\Big]\Big[\sum_b (f_b^{q-1}-1)[\frac{e_b-\mu}{T}]f_b^q\Big]\Big]}{\sum_b (f_b^{q-1}-1)f_b^q},$$

$$=\frac{\sum_a\sum_b \Big[\frac{q}{T^2} f_a^{2q-1}(f_b^{q-1}-1)f_b^q\Big]\Big(e_a\big[e_a-\mu\big] - e_a\big[e_b-\mu\big]\Big)}{\sum_b (f_b^{q-1}-1)f_b^q},$$

$$=\frac{\sum_a\sum_b \Big[\frac{q}{T^2} f_a^{2q-1}(f_b^{q-1}-1)f_b^q\Big]\Big(e_a^2 - e_a e_b\Big)}{\sum_b (f_b^{q-1}-1)f_b^q} \tag{4.61}$$

Using,

$$(f_b^{q-1}-1) = -f_b^{q-1}\Big[(q-1)\frac{e_b-\mu}{T}\Big] \tag{4.62}$$

Substituting in eq: 4.61,

$$N^{'} = \frac{\sum_a\sum_b \Big[\frac{-q}{T^2} f_a^{2q-1}(f_b^{2q-1}\big[(q-1)\frac{e_b-\mu}{T}\big])\Big]\Big(e_a^2 - e_a e_b\Big)}{\sum_b (f_b^{q-1}-1)f_b^q},$$

$$= \frac{\sum_a\sum_b \Big[\frac{-q}{T^3}(q-1) f_a^{2q-1} f_b^{2q-1}\Big]\Big((e_b-\mu)(e_a^2 - e_a e_b)\Big)}{\sum_b (f_b^{q-1}-1)f_b^q},$$

$$= \frac{\sum_a\sum_b \Big[\frac{-q}{T^3}(q-1) f_a^{2q-1} f_b^{2q-1}\Big]\Big(e_b e_a^2 - e_a e_b^2\Big) - \mu\sum_a\sum_b \Big[\frac{-q}{T^3}(q-1) f_a^{2q-1} f_b^{2q-1}\Big]\Big(e_a^2 - e_a e_b\Big)}{\sum_b (f_b^{q-1}-1)f_b^q} \tag{4.63}$$

Similarly, we can write denominator ($D^{'}$) of 4.60 as,

$$D^{'} = \frac{\sum_a\sum_b \Big[\frac{-q}{T^3}(q-1) f_a^{2q-1} f_b^{2q-1}\Big]\Big(e_b e_a - e_b^2\Big) - \mu\sum_a\sum_b \Big[\frac{-q}{T^3}(q-1) f_a^{2q-1} f_b^{2q-1}\Big]\Big(e_a - e_b\Big)}{\sum_b (f_b^{q-1}-1)f_b^q} \tag{4.64}$$

Note, the first term in the numerator (4.63) and second term in denominator (4.64) are 'zero', So,

$$\frac{\partial E}{\partial N}\Big|_S = \frac{-\mu\sum_a\sum_b \Big[\frac{-q}{T^3}(q-1) f_a^{2q-1} f_b^{2q-1}\Big]\Big[e_a^2 - e_a e_b\Big]}{\sum_a\sum_b \Big[\frac{-q}{T^3}(q-1) f_a^{2q-1} f_b^{2q-1}\Big]\Big[e_b e_a - e_b^2\Big]} \tag{4.65}$$

Since exchanging labels a,b won't affect the terms, so the numerator is equal to $\mu$ times the denominator,

$$\frac{\partial E}{\partial N}\Big|_S = \mu \tag{4.66}$$

For ***Tsallis-Boson***, using eq: 4.40, 4.41, 4.42, 4.53, 4.54, 4.55 and 4.44 in eq: 4.60, we can write the numerator ($N^{'}$) as,

$$\begin{aligned}
N^{'} &= \sum_a q e_a f_a^{q+1}\big[\frac{e_a-\mu}{T^2}\big]\big[\frac{1+f_a}{f_a}\big]^{2-q} - \Bigg[\Big[\sum_a q e_a \frac{f_a^{q+1}}{T}\big[\frac{1+f_a}{f_a}\big]^{2-q}\Big] \\
&\qquad \times \Bigg[\frac{-\sum_b \big[\frac{q}{1-q} f_b^{q+1}\big]\big[\frac{e_b-\mu}{T^2}\big]\big[(\frac{1+f_b}{f_b})^{q-1}-1\big]\big[\frac{1+f_b}{f_b}\big]^{2-q}}{-\sum_b \big[\frac{q}{1-q}\frac{f_b^{q+1}}{T}\big]\big[(\frac{1+f_b}{f_b})^{q-1}-1\big]\big[\frac{1+f_b}{f_b}\big]^{2-q}}\Bigg]\Bigg], \\
&= \sum_a q e_a f_a^{q+1}\big[\frac{e_a-\mu}{T^2}\big]\big[\frac{1+f_a}{f_a}\big]^{2-q} - \Bigg[\Big[\sum_a q e_a \frac{f_a^{q+1}}{T}\big[\frac{1+f_a}{f_a}\big]^{2-q}\Big] \\
&\qquad \times \Bigg[\frac{\sum_b f_b^{q+1}\big[\frac{e_b-\mu}{T}\big]\big[(\frac{1+f_b}{f_b})^{q-1}-1\big]\big[\frac{1+f_b}{f_b}\big]^{2-q}}{\sum_b f_b^{q+1}\big[(\frac{1+f_b}{f_b})^{q-1}-1\big]\big[\frac{1+f_b}{f_b}\big]^{2-q}}\Bigg]\Bigg], \\
&= \frac{\sum_a q e_a f_a^{q+1}\big[\frac{e_a-\mu}{T^2}\big]\big[\frac{1+f_a}{f_a}\big]^{2-q}[\sum_b f_b^{q+1}\big[(\frac{1+f_b}{f_b})^{q-1}-1\big]\big[\frac{1+f_b}{f_b}\big]^{2-q}]}{\sum_b f_b^{q+1}\big[(\frac{1+f_b}{f_b})^{q-1}-1\big]\big[\frac{1+f_b}{f_b}\big]^{2-q}} \\
&\qquad - \frac{\Big[\sum_a q e_a \frac{f_a^{q+1}}{T}\big[\frac{1+f_a}{f_a}\big]^{2-q}\Big]\Big[\sum_b f_b^{q+1}\big[\frac{e_b-\mu}{T}\big]\big[(\frac{1+f_b}{f_b})^{q-1}-1\big]\big[\frac{1+f_b}{f_b}\big]^{2-q}\Big]}{\sum_b f_b^{q+1}\big[(\frac{1+f_b}{f_b})^{q-1}-1\big]\big[\frac{1+f_b}{f_b}\big]^{2-q}}, \\
&= \frac{\sum_a\sum_b \Big[\frac{q}{T^2} f_a^{q+1} f_b^{q+1}\big[\frac{1+f_a}{f_a}\big]^{2-q}\big[\frac{1+f_b}{f_b}\big]^{2-q}\big[(\frac{1+f_b}{f_b})^{q-1}-1\big]\Big]\Big(e_a(e_a-\mu)-e_a(e_b-\mu)\Big)}{\sum_b f_b^{q+1}\big[(\frac{1+f_b}{f_b})^{q-1}-1\big]\big[\frac{1+f_b}{f_b}\big]^{2-q}}, \\
&= \frac{\sum_a\sum_b \Big[\frac{q}{T^2}(f_a f_b)^{q+1}\Big[\big[\frac{1+f_a}{f_a}\big]\big[\frac{1+f_b}{f_b}\big]\Big]^{2-q}\big[(\frac{1+f_b}{f_b})^{q-1}-1\big]\Big]\Big(e_a^2-e_a e_b\Big)}{\sum_b f_b^{q+1}\big[(\frac{1+f_b}{f_b})^{q-1}-1\big]\big[\frac{1+f_b}{f_b}\big]^{2-q}}
\end{aligned} \tag{4.67}$$

Using,

$$\Big[(\frac{1+f_b}{f_b})^{q-1}-1\Big] = \Big[(q-1)\frac{e_b-\mu}{T}\Big] \tag{4.68}$$

Simplifying eq: 4.67,

$$\begin{aligned}
N^{'} &= \frac{\sum_a\sum_b \Big[\frac{q}{T^2}(f_a f_b)^{q+1}\Big[\big[\frac{1+f_a}{f_a}\big]\big[\frac{1+f_b}{f_b}\big]\Big]^{2-q}\Big[(q-1)\frac{e_b-\mu}{T}\Big]\Big]\Big(e_a^2-e_a e_b\Big)}{\sum_b f_b^{q+1}\big[(\frac{1+f_b}{f_b})^{q-1}-1\big]\big[\frac{1+f_b}{f_b}\big]^{2-q}}, \\
&= \frac{\sum_a\sum_b \Big[\frac{q(q-1)}{T^3}(f_a f_b)^{q+1}\Big[\big[\frac{1+f_a}{f_a}\big]\big[\frac{1+f_b}{f_b}\big]\Big]^{2-q}\Big]\Big((e_b-\mu)(e_a^2-e_a e_b)\Big)}{\sum_b f_b^{q+1}\big[(\frac{1+f_b}{f_b})^{q-1}-1\big]\big[\frac{1+f_b}{f_b}\big]^{2-q}},
\end{aligned}$$

$$= \frac{\sum_a \sum_b \left[\frac{q(q-1)}{T^3}(f_a f_b)^{q+1}\left[\left[\frac{1+f_a}{f_a}\right]\left[\frac{1+f_b}{f_b}\right]\right]^{2-q}\right]\Big(e_b(e_a^2 - e_a e_b)\Big)}{\sum_b f_b^{q+1}\left[(\frac{1+f_b}{f_b})^{q-1} - 1\right]\left[\frac{1+f_b}{f_b}\right]^{2-q}}$$
$$- \frac{\sum_a \sum_b \left[\frac{q(q-1)}{T^3}(f_a f_b)^{q+1}\left[\left[\frac{1+f_a}{f_a}\right]\left[\frac{1+f_b}{f_b}\right]\right]^{2-q}\Big(\mu(e_a^2 - e_a e_b)\Big)\right]}{\sum_b f_b^{q+1}\left[(\frac{1+f_b}{f_b})^{q-1} - 1\right]\left[\frac{1+f_b}{f_b}\right]^{2-q}} \tag{4.69}$$

Similarly, we can write denominator ($D'$) of eq: 4.60 as,

$$D' = \frac{\sum_a \sum_b \left[\frac{q(q-1)}{T^3}(f_a f_b)^{q+1}\left[\left[\frac{1+f_a}{f_a}\right]\left[\frac{1+f_b}{f_b}\right]\right]^{2-q}\right]\Big(e_b(e_a - e_b)\Big)}{\sum_b f_b^{q+1}\left[(\frac{1+f_b}{f_b})^{q-1} - 1\right]\left[\frac{1+f_b}{f_b}\right]^{2-q}}$$
$$- \frac{\sum_a \sum_b \left[\frac{q(q-1)}{T^3}(f_a f_b)^{q+1}\left[\left[\frac{1+f_a}{f_a}\right]\left[\frac{1+f_b}{f_b}\right]\right]^{2-q}\Big(\mu(e_a - e_b)\Big)\right]}{\sum_b f_b^{q+1}\left[(\frac{1+f_b}{f_b})^{q-1} - 1\right]\left[\frac{1+f_b}{f_b}\right]^{2-q}} \tag{4.70}$$

Note, the first term in the numerator (4.69) and second term in denominator (4.70) are 'zero', So,

$$\frac{\partial E}{\partial N}\Big|_S = \frac{-\sum_a \sum_b \left[\frac{q(q-1)}{T^3}(f_a f_b)^{q+1}\left[\left[\frac{1+f_a}{f_a}\right]\left[\frac{1+f_b}{f_b}\right]\right]^{2-q}\right]\Big(\mu(e_a^2 - e_a e_b)\Big)}{\sum_a \sum_b \left[\frac{q(q-1)}{T^3}(f_a f_b)^{q+1}\left[\left[\frac{1+f_a}{f_a}\right]\left[\frac{1+f_b}{f_b}\right]\right]^{2-q}\right]\Big(e_b(e_a - e_b)\Big)} \tag{4.71}$$

It can be noticed that numerator is equal to $\mu$ multiplied by the denominator,

$$\frac{\partial E}{\partial N}\Big|_S = \mu \tag{4.72}$$

**<u>Relation: 4.32</u>**

$$\begin{aligned}\frac{\partial E}{\partial S}\Big|_N &= \frac{\frac{\partial E}{\partial T}\big|_\mu dT + \frac{\partial E}{\partial \mu}\big|_T d\mu}{\frac{\partial S}{\partial T}\big|_\mu dT + \frac{\partial S}{\partial \mu}\big|_T d\mu}, \\ &= \frac{\frac{\partial E}{\partial T}\big|_\mu + \frac{\partial E}{\partial \mu}\big|_T \frac{d\mu}{dT}}{\frac{\partial S}{\partial T}\big|_\mu + \frac{\partial S}{\partial \mu}\big|_T \frac{d\mu}{dT}}\end{aligned} \tag{4.73}$$

Further, since $N$ is kept constant so, $dN$ = 0,

$$\begin{aligned} dN =& \frac{\partial N}{\partial T}\Big|_\mu dT + \frac{\partial N}{\partial \mu}\Big|_T d\mu = 0, \\ \text{so,} \qquad \frac{d\mu}{dT} =& -\left[\frac{\frac{\partial N}{\partial T}\big|_\mu}{\frac{\partial N}{\partial \mu}\big|_T}\right]\end{aligned} \tag{4.74}$$

$$\frac{\partial E}{\partial S}\Big|_N = \frac{\frac{\partial E}{\partial T}\Big|_\mu - \frac{\partial E}{\partial \mu}\Big|_T \Big[\frac{\frac{\partial N}{\partial T}|_\mu}{\frac{\partial N}{\partial \mu}|_T}\Big]}{\frac{\partial S}{\partial T}\Big|_\mu - \frac{\partial S}{\partial \mu}\Big|_T \Big[\frac{\frac{\partial N}{\partial T}|_\mu}{\frac{\partial N}{\partial \mu}|_T}\Big]} \tag{4.75}$$

For ***Tsallis-Bolt***, using eqs: 4.36, 4.37, 4.38, 4.47, 4.48, 4.49, we can write the numerator ($N'$) as,

$$\begin{aligned}
N' &= \sum_a q e_a f_a^{2q-1}\Big[\frac{e_a-\mu}{T^2}\Big] - \Big[\sum_a q e_a \frac{f_a^{2q-1}}{T}\Big]\Bigg[\frac{\sum_b q f_b^{2q-1}[\frac{e_b-\mu}{T^2}]}{\sum_b q \frac{f_b^{2q-1}}{T}}\Bigg],\\
&= \frac{\Big[\sum_a q e_a f_a^{2q-1}[\frac{e_a-\mu}{T^2}]\Big]\sum_b f_b^{2q-1} - \Big[\sum_a q e_a \frac{f_a^{2q-1}}{T}\Big]\sum_b f_b^{2q-1}[\frac{e_b-\mu}{T}]}{\sum_b f_b^{2q-1}},\\
&= \frac{\Big[\sum_a\sum_b q e_a \frac{f_a^{2q-1}}{T^2} f_b^{2q-1}[e_a-\mu]\Big] - \Big[\sum_a\sum_b q e_a \frac{f_a^{2q-1}}{T^2} f_b^{2q-1}[e_b-\mu]\Big]}{\sum_b f_b^{2q-1}},\\
&= \frac{\sum_a\sum_b \Big[q e_a \frac{f_a^{2q-1}}{T^2} f_b^{2q-1}[e_a-\mu-(e_b-\mu)]\Big]}{\sum_b f_b^{2q-1}},\\
&= \frac{\sum_a\sum_b \Big[q e_a \frac{f_a^{2q-1}}{T^2} f_b^{2q-1}(e_a-e_b)\Big]}{\sum_b f_b^{2q-1}}
\end{aligned} \tag{4.76}$$

Similarly, we can write denominator ($D'$) of eq: 4.75 as,

$$D' = \frac{\sum_a\sum_b \Big[\frac{q}{q-1}(f_a^{q-1}-1)f_a^q\Big]\frac{f_b^{2q-1}}{T^2}(-e_a+e_b)}{\sum_b f_b^{2q-1}} \tag{4.77}$$

Using eq: 4.62,

$$(f_a^{q-1}-1)f_a^q = -f_a^{2q-1}\Big[(q-1)\frac{e_a-\mu}{T}\Big] \tag{4.78}$$

Substituting this in eq: 4.77,

$$\begin{aligned}
D' &= \frac{\sum_a\sum_b \Big[\frac{q}{q-1}\big[-f_a^{2q-1}(q-1)\frac{e_a-\mu}{T}\big]\Big]\frac{f_b^{2q-1}}{T^2}(-e_a+e_b)}{\sum_b f_b^{2q-1}},\\
&= \frac{\sum_a\sum_b \Big[q\big[f_a^{2q-1}\frac{e_a}{T}\big]\Big]\frac{f_b^{2q-1}}{T^2}(e_a-e_b) - \sum_a\sum_b \Big[q\big[f_b^{2q-1}\frac{\mu}{T}\big]\Big]\frac{f_b^{2q-1}}{T^2}(e_a-e_b)}{\sum_b f_b^{2q-1}}
\end{aligned} \tag{4.79}$$

Note that second term in $D'$ is zero as its just the matter of labels, hence denominator is $\frac{1}{T}$ times that of numerator, so

$$\frac{\partial E}{\partial S}\Big|_N = T \tag{4.80}$$

For ***Tsallis-Boson***, using eq: 4.40, 4.41, 4.42, 4.53, 4.54, 4.55 and 4.44 in eq: 4.75, we can write the numerator ($N'$) as,

$$
\begin{aligned}
N' &= \sum_a q e_a f_a^{q+1}\big[\frac{e_a-\mu}{T^2}\big]\big[\frac{1+f_a}{f_a}\big]^{2-q} - \Big[\sum_a q e_a \frac{f_a^{q+1}}{T}\big[\frac{1+f_a}{f_a}\big]^{2-q}\Big]\Big[\frac{\sum_b q f_b^{q+1}\big[\frac{e_b-\mu}{T^2}\big](\frac{1+f_b}{f_b})^{2-q}}{\sum_b q\frac{f_b^{q+1}}{T}(\frac{1+f_b}{f_b})^{2-q}}\Big], \\
&= \frac{\sum_a q e_a f_a^{q+1}\big[\frac{e_a-\mu}{T^2}\big]\big[\frac{1+f_a}{f_a}\big]^{2-q}\Big[\sum_b f_b^{q+1}\big[(\frac{1+f_b}{f_b})^{2-q}\big]\Big] - \Big[\sum_a q e_a\frac{f_a^{q+1}}{T}\big[\frac{1+f_a}{f_a}\big]^{2-q}\Big]\Big[\sum_b f_b^{q+1}\big[\frac{e_b-\mu}{T}\big](\frac{1+f_b}{f_b})^{2-q}\Big]}{\sum_b f_b^{q+1}(\frac{1+f_b}{f_b})^{2-q}}, \\
&= \frac{\sum_a\sum_b\Big[\frac{q}{T^2}f_a^{q+1}f_b^{q+1}\big[\frac{1+f_a}{f_a}\big]^{2-q}(\frac{1+f_b}{f_b})^{2-q}\Big[e_a(e_a-\mu)-e_a(e_b-\mu)\Big]\Big]}{\sum_b f_b^{q+1}(\frac{1+f_b}{f_b})^{2-q}}, \\
&= \frac{\sum_a\sum_b\Big[\frac{q}{T^2}f_a^{q+1}f_b^{q+1}\big[\frac{1+f_a}{f_a}\big]^{2-q}(\frac{1+f_b}{f_b})^{2-q}\Big[e_a(e_a-e_b)\Big]\Big]}{\sum_b f_b^{q+1}(\frac{1+f_b}{f_b})^{2-q}}
\end{aligned}
\tag{4.81}
$$

Similarly, we can write denominator ($D'$) of eq: 4.75 as,

$$
D' = \frac{\sum_a\sum_b\Big[\Big(\frac{q}{1-q}\Big)\frac{f_a^{q+1}}{T^2}\Big((\frac{1+f_a}{f_a})^{q-1}-1\Big)\Big(\frac{1+f_a}{f_a}\Big)^{2-q}\Big]f_b^{q+1}\Big(\frac{1+f_a}{f_b}\Big)^{2-q}(-e_a+e_b)}{\sum_b f_b^{q+1}(\frac{1+f_b}{f_b})^{2-q}}
\tag{4.82}
$$

From eq: 4.44,

$$
\Big((\frac{1+f_a}{f_a})^{q-1}-1\Big) = (q-1)\frac{e_a-\mu}{T}
\tag{4.83}
$$

Simplifying eq: 4.82,

$$
\begin{aligned}
D' &= \frac{\sum_a\sum_b\Big[\Big(\frac{q}{1-q}\Big)\frac{f_a^{q+1}}{T^2}\Big((q-1)\frac{e_a-\mu}{T}\Big)\Big(\frac{1+f_a}{f_a}\Big)^{2-q}\Big]f_b^{q+1}\Big(\frac{1+f_a}{f_b}\Big)^{2-q}(-e_a+e_b)}{\sum_b f_b^{q+1}(\frac{1+f_b}{f_b})^{2-q}}, \\
&= \frac{\sum_a\sum_b\Big[\frac{q}{T^2}f_a^{q+1}f_b^{q+1}\Big(\frac{1+f_a}{f_a}\Big)^{2-q}\Big(\frac{1+f_a}{f_b}\Big)^{2-q}\Big]\Big(\frac{e_a-\mu}{T}\Big)(e_a-e_b)}{\sum_b f_b^{q+1}\big[(\frac{1+f_b}{f_b})^{2-q}}, \\
&= \frac{\sum_a\sum_b\Big[\frac{q}{T^2}f_a^{q+1}f_b^{q+1}\Big(\frac{1+f_a}{f_a}\Big)^{2-q}\Big(\frac{1+f_a}{f_b}\Big)^{2-q}\Big]\Big(\frac{e_a}{T}\Big)(e_a-e_b)}{\sum_b f_b^{q+1}(\frac{1+f_b}{f_b})^{2-q}} \\
&\quad - \frac{\sum_a\sum_b\Big[\frac{q}{T^2}f_a^{q+1}f_b^{q+1}\Big(\frac{1+f_a}{f_a}\Big)^{2-q}\Big(\frac{1+f_a}{f_b}\Big)^{2-q}\Big]\Big(\frac{\mu}{T}\Big)(e_a-e_b)}{\sum_b f_b^{q+1}(\frac{1+f_b}{f_b})^{2-q}}
\end{aligned}
\tag{4.84}
$$

It can be easily observe that the second term in above equation is zero and the first term is $\frac{1}{T}$ times the eq: 4.81, So

$$
\frac{\partial E}{\partial S}\Big|_N = T
\tag{4.85}
$$

Since, we have proved all the thermodynamical relations, so our new form of statistics based on Tsallis formalism is thermodynamically consistent. In next section we will be using multiplicity formula based on this statistics for fitting $p_T$ spectra and will drive the formula for Pressure.

### 4.2.2 $p_T$ Spectra

In Experiments we record $p_T$ spectra for particles detected, formula of $p_T$ can be derived from multiplicity formula as shown in section: 2.2, The modified form of multiplicity distribution in Tsallis statistics is given by eq: 4.20,

$$\begin{aligned} N_q^{B.G.} &= \frac{gV}{(2\pi)^3}\int d^3p\Big[1+(q-1)\frac{(E(p)-\mu)}{T}\Big]^{\frac{-q}{q-1}}, \\ N_q^{B.E/F.D} &= \frac{gV}{(2\pi)^3}\int d^3p\Bigg[\frac{1}{\big[1+(q-1)\frac{(E(p)-\mu)}{T}\big]^{\frac{1}{q-1}}+\eta}\Bigg]^q \end{aligned} \tag{4.86}$$

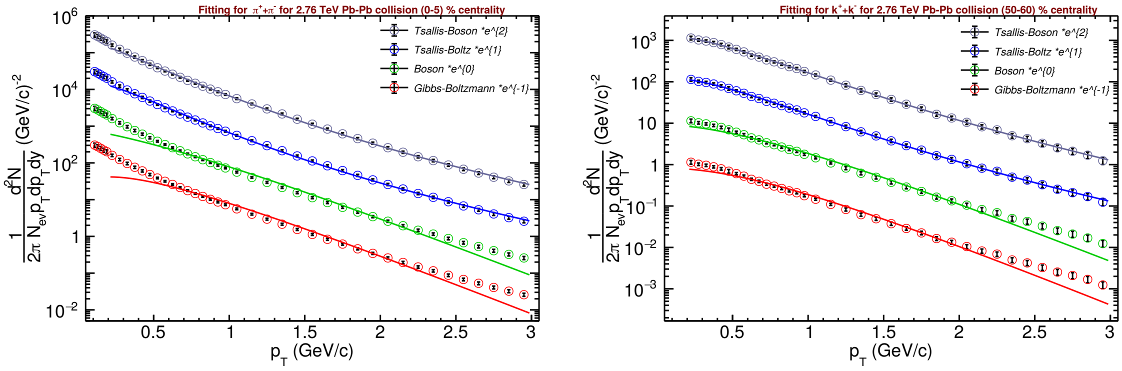
One can note, under limit $q \to 1$, we get back our standard statistical form (2.8),

$$\begin{aligned} N_q^{B.G.} &\xrightarrow{q\to 1} \frac{gV}{(2\pi)^3}\int d^3p\exp\Big(-\frac{(E(p)-\mu)}{T}\Big) = N^{B.G.}, \\ N_q^{B.E/F.D} &\xrightarrow{q\to 1} \frac{gV}{(2\pi)^3}\int d^3p\frac{1}{\exp\big(\frac{E(p)-\mu}{T}\big)+\eta} = N^{B.E/F.D} \end{aligned} \tag{4.87}$$

corresponding Yield's formula from eq: 2.13, in mid-rapidity (y $\approx$ 0) is given by,

$$\frac{d^2N}{2\pi p_T dp_T dy}\Bigg|_{y=0} = \begin{cases} \frac{gVm_T}{(2\pi)^3}\Big[1+(q-1)\frac{(m_T-\mu)}{T}\Big]^{\frac{-q}{q-1}} & \text{for Tsallis-Boltzmann,} \\ \frac{gVm_T}{(2\pi)^3}\Big[\frac{1}{[1+(q-1)\frac{m_T-\mu}{T}]^{\frac{1}{q-1}}+\eta}\Big]^q & \text{for Tsallis-(B.E/F.D)} \end{cases} \tag{4.88}$$

In Fig: 4.2 we have shown the goodness of fitting, data used is of 2.76 TeV Pb-Pb collision[41]. In the plots, small circles represent the experimental data of yield for charged pions and charged kaons and solid lines represents fitting by B-G, Boson, their Tsallis-variant distribution functions. We used chi-square test, to determine the goodness of fit note in Table 4.1. Chi-square goodness of fit is used to find out how the observed value of given phenomena significantly different from the expected value and to compare the observed sample distribution with expected/theoretical probability distribution. Lower the value of Chi-sq/NDF better is the fit.




(a) Charged $\pi$ (b) Charged $k$

Figure 4.2: Comparison between different Distributions by fitting to Experimental data

| Particle | Statistics | Chi-sq/NDF |
|---|---|---|
| $\pi^+ + \pi^-$ | Tsallis-Boson | 26.6022/33 = 0.806128 |
| | Tsallis-Boltz | 43.6842/33 = 1.332376 |
| | Boson | 587.804/34 = 17.2883 |
| | Boltz | 751.133/34 = 22.0921 |
| $k^+ + k^-$ | Tsallis-Boson | 4.69683/33 = 0.142328 |
| | Tsallis-Boltz | 5.70484/33 = 0.172874 |
| | Boson | 155.242/34 = 4.56594 |
| | Boltz | 196.348/34 = 5.77494 |

Table 4.1: Chi-Sq values

By fitting eq: 4.88 to $p_T$ spectra for particular system of particles one can get Temperature, q, Volume of the system. These latent variables that we get by fitting are the kinetic-freezeout Temperature, non-extensive parameter and volume of the system. These fitting parameters can be used to extract further thermodynamical properties of system. Since we are dealing in LHC energies we took chemical potential $\mu = 0$ in our analysis, as at LHC energies there is particle-antiparticle symmetry (approximately equal production of particles and anti-particles[4]).

[4]It is found even at LHC energies, particle to antiparticle ratio not equal to 1, so $\mu$ may not be taken to be zero[42].

### 4.2.3 Pressure Formulation

We know Tsallis statistics is themodynamically consistent. Formula of pressure can be easily derivable using thermodynamical relations,

$$P = \frac{ST - E + \mu N}{V} \tag{4.89}$$

Since we know the form of S, E, N we can derive the corresponding formula for Pressure, For **Tsallis-Boltz**,

$$\begin{aligned}
P =& \frac{T\Big(-\sum_a [f_a^q \ln_q f_a - f_a]\Big) - \sum_a f_a^q e_a + \mu \sum_a f_a^q}{V}, \\
=& \sum_a \Big[\frac{-T f_a^q \ln_q f_a + T f_a - f_a^q e_a + \mu f_a^q}{V}\Big], \\
=& \sum_a \Bigg[\frac{-T[1+(q-1)\frac{(-\mu+e_a)}{T}]^{\frac{-q}{q-1}} \Big[\frac{[1+(q-1)\frac{(-\mu+e_a)}{T}]^{\frac{-(1-q)}{q-1}} - 1}{1-q}\Big] + T f_a + f_a^q(-e_a+\mu)}{V}\Bigg], \\
=& \sum_a \Big[\frac{f_a^q (e_a - \mu) + T f_a + f_a^q(-e_a+\mu)}{V}\Big], \\
=& \sum_a \Big[\frac{T[1+(q-1)\frac{(e_a-\mu)}{T}]^{\frac{-1}{q-1}}}{V}\Big]
\end{aligned} \tag{4.90}$$

For **Tsallis-Boson**,

$$\begin{aligned}
P =& \frac{T\Big(-\sum_a \Big[f_a^q \ln_q f_a - (1+f_a)^q \ln_q(1+f_a)\Big]\Big) - \sum_a f_a^q e_a + \mu \sum_a f_a^q}{V}, \\
=& \frac{-\sum_a f_a^q \Big[T \ln_q f_a + e_a - \mu\Big] + \sum_a T(1+f_a)^q \ln_q(1+f_a)}{V}, \\
=& \frac{-\sum_a \frac{T}{1-q} f_a^q \Big[(f_a^{1-q} - 1) + (1-q)(e_a-\mu)\Big] + \sum_a T(1+f_a)^q \ln_q(1+f_a)}{V}, \\
=& \frac{-\sum_a \frac{T}{1-q} f_a^q \Big[f_a^{1-q} - \big(1+(q-1)(e_a-\mu)\big)\Big] + \sum_a T(1+f_a)^q \ln_q(1+f_a)}{V}, \\
=& \frac{-\sum_a \frac{T}{1-q} f_a^q \Big[f_a^{1-q} - \big(\frac{1+f_a}{f}\big)^{q-1}\big)\Big] + \sum_a T(1+f_a)^q \ln_q(1+f_a)}{V}, \\
=& \frac{-\sum_a \frac{T}{1-q} f_a \Big[1 - \big(1+f_a\big)^{q-1}\big)\Big] + \sum_a T(1+f_a)^q \ln_q(1+f_a)}{V}, \\
=& \frac{-\sum_a T f_a (1+f_a)^{q-1} \ln_q(1+f_a) + \sum_a T(1+f_a)^q \ln_q(1+f_a)}{V}, \\
=& \sum_a \frac{T(1+f_a)^{q-1} \ln_q(1+f_a)}{V}
\end{aligned} \tag{4.91}$$

Similarly we can derive the formula of pressure for **Tsallis-Fermion**. Now expressing Pressure in terms of Transverse momentum ($p_T$) and rapidity ($y$) and we take following limit,

$$\sum_a \rightarrow \int \frac{gVd^3p}{(2\pi)^3}, \quad \text{and} \quad e = m_T \cosh y, \quad \frac{dp_z}{e} = dy \tag{4.92}$$

Expression of Pressure in differential form,

$$\frac{d^3P}{dp^3} = \frac{d^3P}{(dp_x dp_y)(dp_z)} = \frac{d^2P}{2\pi p_T dp_T (edy)} \tag{4.93}$$

For eq: 4.90 i.e. for Tsallis-Boltz,

$$\begin{aligned}
\frac{d^2P}{2\pi p_T dp_T(edy)} &= \frac{g}{(2\pi)^3}\Big[T[1+(q-1)\frac{(e-\mu)}{T}]^{\frac{-1}{q-1}}\Big], \\
\frac{d^2P}{dp_T dy} &= \frac{g(p_T)e}{(2\pi)^2}\Big[T[1+(q-1)\frac{(e-\mu)}{T}]^{\frac{-1}{q-1}}\Big], \\
P = \iint dp_T dy \frac{g(p_T)m_T\cosh y}{(2\pi)^2}&\Big[T[1+(q-1)\frac{(m_T\cosh y-\mu)}{T}]^{\frac{-1}{q-1}}\Big]
\end{aligned} \tag{4.94}$$

Similarly for eq: 4.91 i.e. for Tsallis-Boson,

$$\begin{aligned}
\frac{d^2P}{2\pi p_T dp_T(edy)} &= \frac{g}{(2\pi)^3}\Big[T(1+f_a)^{q-1}\ln_q(1+f_a)\Big], \\
\frac{d^2P}{dp_T dy} &= \frac{gp_Te}{(2\pi)^2}\Big[T(1+f_a)^{q-1}\ln_q(1+f_a)\Big],
\end{aligned} \tag{4.95}$$

$$P = \iint dp_T dy \frac{g(p_T)m_T\cosh y}{(2\pi)^2}\Big[T(1+f_a)^{q-1}\ln_q(1+f_a)\Big] \tag{4.96}$$

In the next chapter we have calculated Pressure using these relation for different system of particles formed in heavy-ion collision. We have done analysis on pions($\pi^0, \pi^+, \pi^-$) and kaons($k^+, k^-$), and taking chemical potential, $\mu$ =0 for all the particles and considering the value of g(degeneracy factor) for $\pi^0$ to be 2, $\pi^+ + \pi^-$ to be 2 and for $k^+ + k^-$ to be 2. We have used the limits of integration, for $p_T$ from 0 to the maximum value of $p_T$ considered, and for $y$ from -0.5 to 0.5.

# Chapter 5

# Analysis

In this chapter we will discuss the analysis and calculation done using the synthetic data made using UrQMD[16] for Pb-Pb collision at 2.76 TeV for different centralities and for experimental data for Au-Au collision at 200 GeV by PHENIX collaboration [43],Au-Au collision at 200 GeV by STAR collaboration [44], Pb-Pb collision at 2.76 TeV[41] and 5.02 TeV [45]. The invariant yields of various partiles are fitted using Tsallis distributions and the extracted parameters are used to calculate Pressure. We have used ROOT, CERN data analysis framework for fitting [46]. In Fig. 5.1 we have shown that production of charged pions yield is relatively higher than other hadrons particles. This multiplicity analysis is done on final state particles for 52k events generated using UrQMD in hydro mode[47][48] for Pb-Pb collision at 2.76 TeV. So in our analysis we have used system of charged pions ($\pi^{+}$+ $\pi^{-}$) and neutral pions ($\pi^{0}$) for calculating thermodynamical quantities.

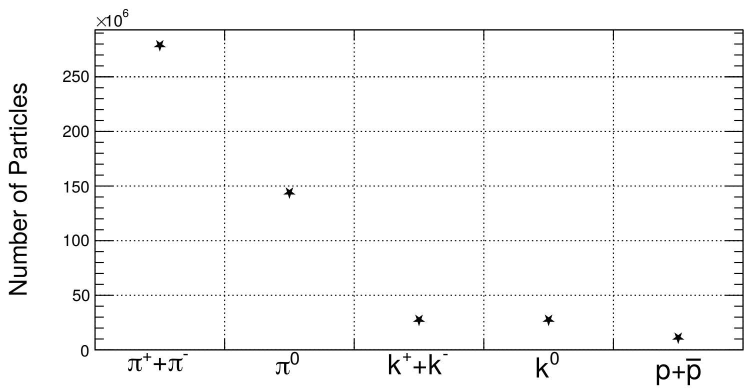



Figure 5.1: Number of Particles in (0-5)% centrality for Pb-Pb collision at 2.76 TeV

## 5.1 Hydrodynamics evolution

Hydrodynamics gives dynamical evolution of various thermodynamical quantities with one important assumption of local thermal equillibrium. It stands on pillar of conservation laws. Given the initial input parameter along with the equation of state to be used for the system in considration, we can extract space-time evolution of thermodynamical quantities. We have used UrQMD for solving hydrodynamic equations. UrQMD uses ideal hydrodynamics along with SU(3) parity duality model based equation of state[28]. We have plotted variation of thermodynamical quantities with time, calculated by fitting $p_T$ spectra using Tsallis-Boson statistics for pions, in different centralities at 2.76 TeV for Pb-Pb collision generated by UrQMD in hydro mode. Data is generated for time-steps 10 fmsec to 150 fmsec with freezeout at time 150 fmsec.

In Fig 5.2 and Fig 5.3 we have calculated dynamical variation of Pressure, Temperature, non-extensive parameter(q) and the normalizing constant,Volume for charged pions and neutral pions formed in different centrality for Pb-Pb collision at 2.76 TeV. As one can note that Pressure for charged pions is more than that of neutral pions due to more abundance of charged pions formed. Also with time, volume of either particles, is increasing and the Temperature is decreasing, this is due to expansion and cooling of system with increase in number of particles in the system with time. Note that non-extensive parameter(q) is almost constant with time. One can quantify from value of q, how much system deviates from its equilibrium. when q value approach unity it is close to equilibrium. The geometric information i.e. centrality dependence is also important to understand. Also with centrality the value of Pressure, Temperature, Volume is decreasing as expected, as centrality decides the overlap region of two nucleus colliding, less centrality, more is the overlap and more volume of the system formed. There is decrease in Temperature, it indicates the energy deposition is more in central then peripheral collisions.

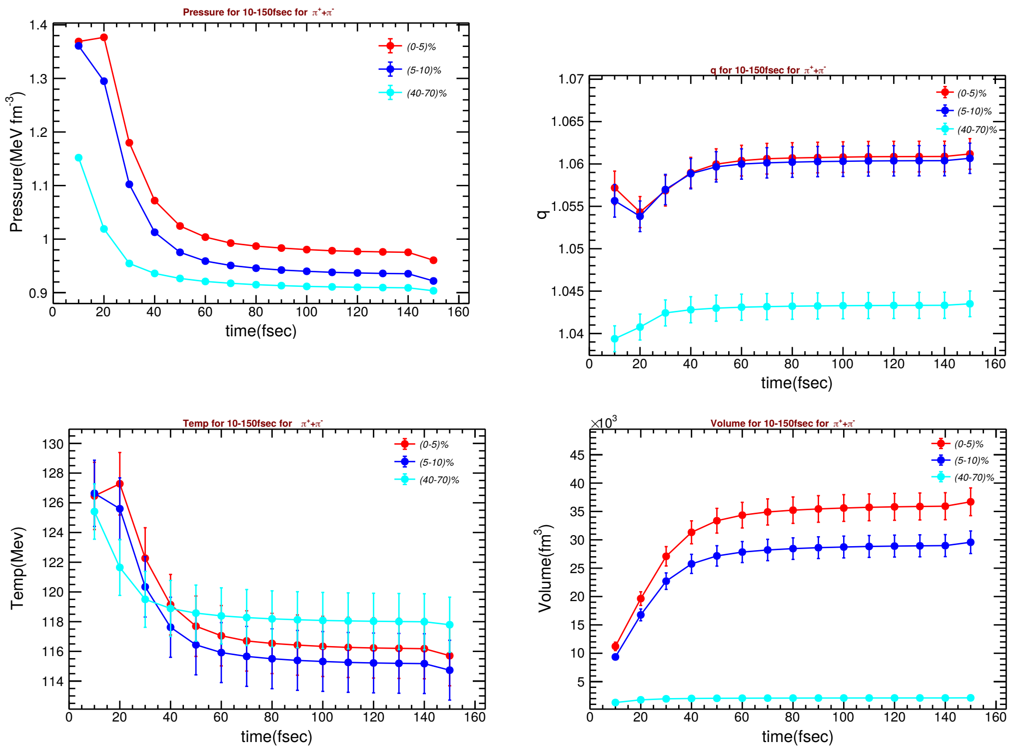



Figure 5.2: Thermodynamical quantities for $\pi^{+}$+ $\pi^{-}$ in Pb-Pb collision at 2.76 TeV

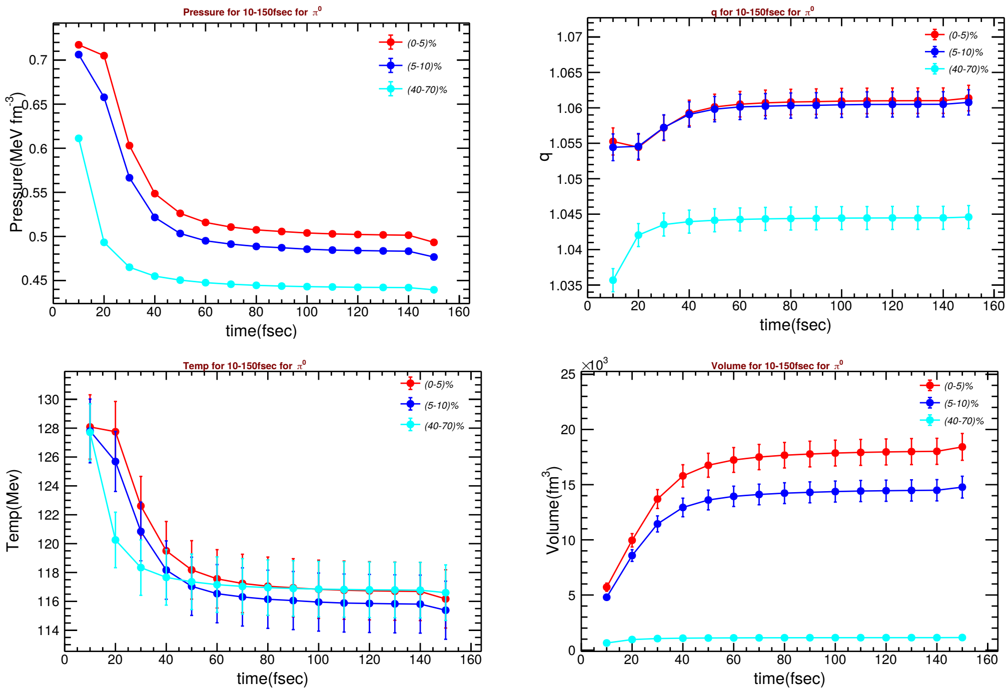



Figure 5.3: Thermodynamical quantities for $\pi^{0}$ in Pb-Pb collision at 2.76 TeV

### 5.1.1 Final state particles

In this section comparison of different distribution function used to fit the $p_T$ spectra for final state particles formed after freeze-out from UrQMD data i.e. at 150 fmsec. $p_T$ spectra of pions is fitted with B-G, Boson, Tsallis-Boltz and Tsallis-Boson. In the following table, goodness of fit, Chi-sq/NDF and Thermodynamical quantities are listed for system of charge pions and system of neutral pions. One can see from value of Chi-sq/NDF, measure of goodness of fit, that Tsallis-Boson fits better than other used distributions. Fig 5.4 shows the visualization of fitting for system of charge pions.

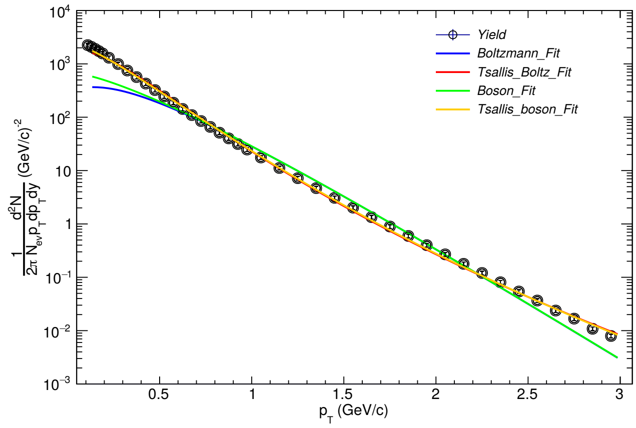



Figure 5.4: $p_T$ spectra for $\pi^+ + \pi^-$ for (0-5)% centrality Pb-Pb 2.76 TeV

| Particle | Statistics | Chi-sq/NDF | T (MeV) | q | V ($fm^3$) | P (MeV$fm^{-3}$) |
|---|---|---|---|---|---|---|
| $\pi^+ + \pi^-$ | Boltz | 1051.53/38= 27.6720 | 193.862 | - | 4898.954 | - |
| | Boson | 879.346/38= 23.1407 | 193.978 | - | 4882.749 | - |
| | Tsallis-Boltz | 98.5430/37=2.66332 | 109.249 | 1.06600 | 47312.825 | 0.753839 |
| | Tsallis-Boson | 66.9183/37=1.80860 | 115.710 | 1.06118 | 36701.636 | 0.960561 |
| $\pi^0$ | Boltz | 1064.03/38= 28.0007 | 194.590 | - | 2469.730 | - |
| | Boson | 888.353/38= 23.3777 | 194.679 | - | 2464.414 | - |
| | Tsallis-Boltz | 101.977/37=2.75614 | 109.519 | 1.06633 | 23896.530 | 0.384938 |
| | Tsallis-Boson | 69.5178/37=1.87886 | 116.171 | 1.06138 | 18421.526 | 0.493278 |

# 5.2 Experimental data

In previous section we studied the system generated using event generator,UrQMD. In real experiment we don't have time-wise information of whats happening inside the collider, we get information of final stage particles free-streaming to detectors after from the freeze-out surface. We have used $p_T$ of these particles as our kinematic observable for probing thermodynamical properties of the system produced. We have used Au-Au collision at 200 GeV by PHENIX collaboration [43], Au-Au collision at 200 GeV by STAR collaboration [44] and Pb-Pb collision at 2.76 TeV [41] and 5.02 TeV [45] for the analysis.

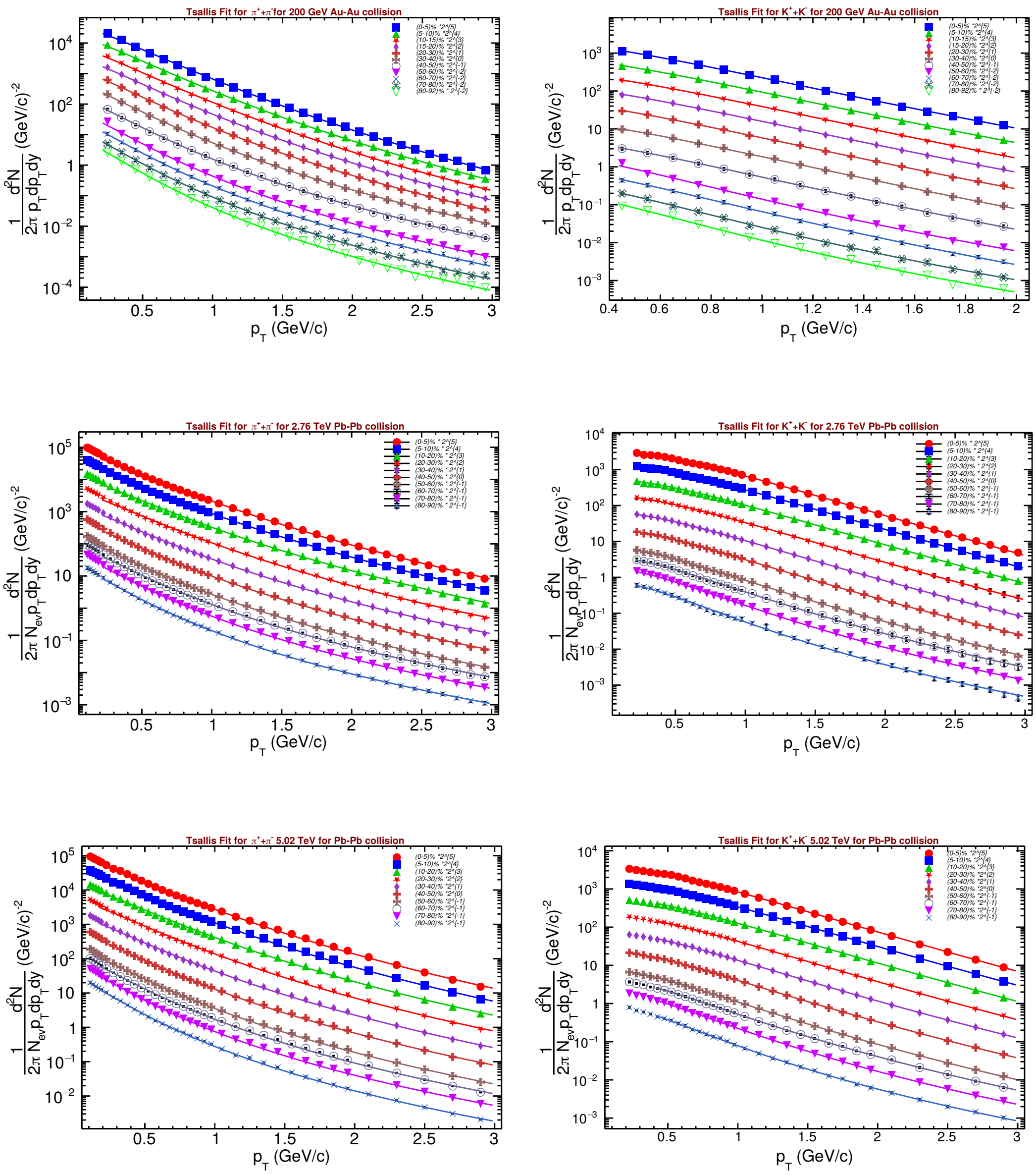



Figure 5.5: $p_T$ spectra fit with Tsallis-Boson for different Centralities and Beam-energies

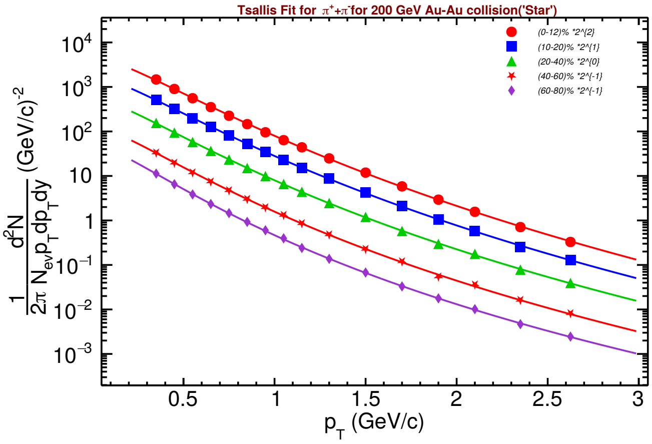



Figure 5.6: $p_T$ spectra fit with Tsallis-Boson for Au-Au collision at 200 GeV

Fig. 5.5 shows the $p_T$ fit of charged pions and charged kaons for different centralities and energies for low $p_T$ value i.e. upto 3-GeV, as for large $p_T$ values hard process are dominated. Fitting is done for Au-Au collision at 200 GeV by PHENIX collaboration [43] and Pb-Pb collision at 2.76 TeV [41] and 5.02 TeV [45]. Fig. 5.6 is for Au-Au collision at 200 GeV by STAR collaboration [44]. Systematic errors are used for the error bars and for composite system the errors of individual particle kind are added in quadrature. By Chi-square goodness of fit method, being a good fit to the spectra [4.2.2], we have used Tsallis-Boson for fitting the $p_T$ spectra with constrain chemical potential $\mu$ = 0.

Fig. 5.7 shows the variation of thermodynamical quantities for charged kaons with $N_{Part}$ or centrality at different energies, where $N_{Part}$ is the average number of participating nucleons in the collision. More the value of $N_{Part}$ more central is the collision. Centrality table is shown in Table: 5.1 for Pb-Pb coliision at 2.76 TeV and 5.02 TeV.

It is observed that value of Volume, Temperature and Pressure decreases when going from central to peripheral collisions i.e towards lower value of $N_{Part}$. Also the values of these thermodynamical quantities decreases with collision energy. Similarly in Fig. 5.8, system of charged pions is studied for Au-Au collision at 200 GeV (STAR), Pb-Pb at 2.76 TeV and 5.02 TeV. It can be noted that Temperature of system of charged kaons is more then that of charged pions, this can be attributed due to earlier freezing-out of heavier particles. Also massive the particles is, smaller the kinetic freeze-out volume, suggesting freeze-out surfaces are different for different mass hadrons. The non-extensive parameter ‘q’ approaches to 1 for large value of $N_{Part}$ i.e. it moves towards equilibrium for most-central collisions.

| Centrality(%) | $N_{Part}$ | |
|---|---|---|
| | 2.76 TeV | 5.02 TeV |
| 0 to 5 | 381.184 | 383.852 |
| 5 to 10 | 327.837 | 331.695 |
| 10 to 20 | 258.549 | 262.698 |
| 20 to 30 | 182.308 | 188.181 |
| 30 to 40 | 127.181 | 130.501 |
| 40 to 50 | 83.8042 | 86.5005 |
| 50 to 60 | 51.7977 | 53.8448 |
| 60 to 70 | 29.364 | 30.7842 |
| 70 to 80 | 15.1838 | 16.0166 |
| 80 to 90 | 7.81635 | 8.2189 |

Table 5.1: Centrality Table

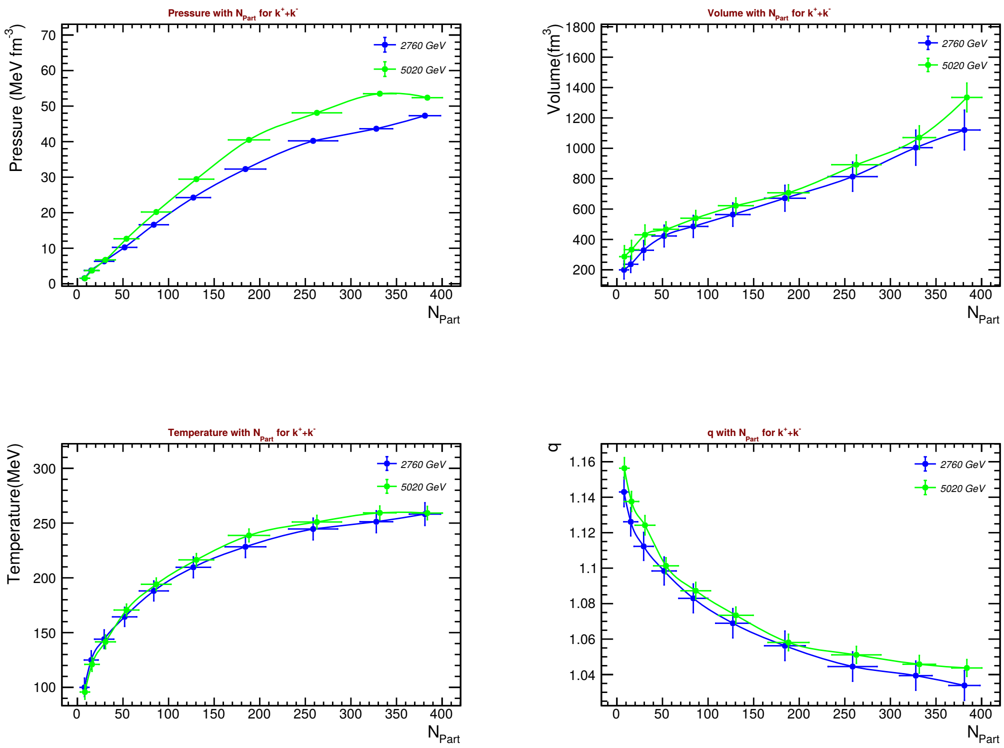



Figure 5.7: Thermodynamical Parameters for $k^{+} + k^{-}$

In Table 5.2, thermodynamical parameters are listed, which are extracted using Tsallis-Boltz and Tsallis-Boson for different beam energies in most central collision.

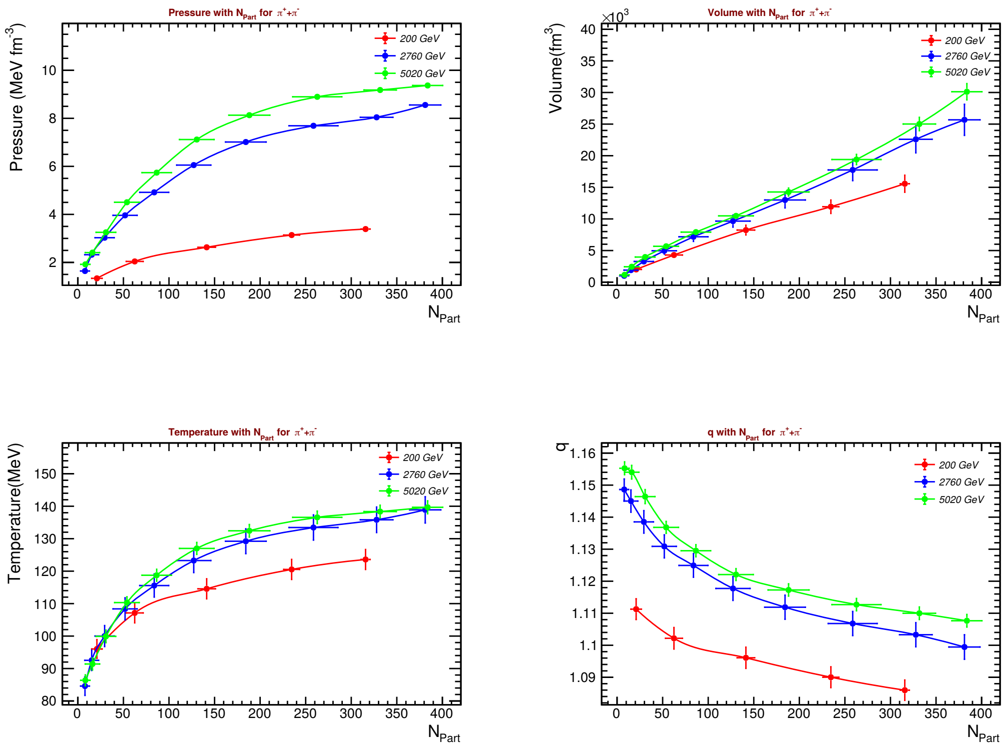



Figure 5.8: Thermodynamical Parameters for $\pi^+ + \pi^-$

| Particle | Statistics | Energy | T(MeV) | V($fm^3$) | q | P($MeV fm^{-3}$) |
|---|---|---|---|---|---|---|
| $k^+ + k^-$ | Tsallis-Boltz | 2.76 TeV | 241.175 | 1437.65 | 1.04445 | 36.8885 |
| | | 5.02 TeV | 245.078 | 1646.63 | 1.05193 | 42.5081 |
| | Tsallis-Boson | 2.76 TeV | 258.217 | 1120.53 | 1.03384 | 47.303 |
| | | 5.02 TeV | 259.158 | 1334.92 | 1.04372 | 52.3702 |
| $\pi^+ + \pi^-$ | Tsallis-Boltz | 200.0 GeV | 114.154 | 20652.8 | 1.09598 | 2.55433 |
| | | 2.76 TeV | 129.404 | 33401.3 | 1.10688 | 6.54471 |
| | | 5.02 TeV | 121.792 | 48000.6 | 1.12272 | 5.8307 |
| | Tsallis-Boson | 200.0 GeV | 123,59 | 15567.0 | 1.08597 | 3.38964 |
| | | 2.76 TeV | 138.881 | 25675.4 | 1.09945 | 8.55601 |
| | | 5.02 TeV | 139.678 | 30118.8 | 1.10765 | 9.36845 |

Table 5.2: Thermodynamical quantities for most-central collision at different energies

# Appendix A

# Natural Units

They are simple and have practical advantage for making equations less bulky.It is easy to convert cgs/MKS to natural units or vice-versa.They are natural in sense they provide scales appropriate in quantum or relativistic physics. One have to follow dimensional analysis for making sense of quantities in the eqs.

$$1Kg = 5.610 \times 1026\, GeV$$
$$1GeV = 1.160 \times 10^{13}\, K$$
$$1fm = 5.068\, GeV^{-1}$$
$$1\hbar c = 197.32\, MeV fm$$
$$\text{Pressure } = 8.878 \times 10^{-9} MeV fm^{-3} = 1.400 \times 10^{18}\, atm$$

Comparison between familiar values in both the unit systems,

- Temperature in the center of Sun, is around $1.571\times 10^{7}$K or $1.354\times 10^{-3}\, MeV$,
- Pressure in central region of sun is around $2.447\times 10^{11}$ atm or $1.552\times 10^{-15} MeV fm^{-3}$,
- Pressure in central region of Neutron star is around $1.579\times 10^{29}$ atm or $10.01\times 10^{2} MeV fm^{-3}$.

# Appendix B

# Modified Bessel Function of Second kind

Defination of $K_n(z)$:

$$K_n(z) = \frac{2^n n!}{(2n)!}\frac{1}{z^n}\int_z^\infty dx(x^2 - z^2)^{n-\frac{1}{2}}\exp(-x) \tag{B.1}$$

According to standard thermodynamics,

$$\frac{N}{V} = n = \frac{g}{(2\pi)^3}\int d^3p\exp(\frac{\mu - E}{T}) \tag{B.2}$$

$$P = \frac{g}{(2\pi)^3}\int d^3p\frac{|p|^2}{3E}\exp(\frac{\mu - E}{T}) \tag{B.3}$$

These formulas can be written in form of $K_n$, Let,

$$\begin{aligned} z = \frac{m}{T}, x = \frac{E}{T} &= \frac{\sqrt{|p|^2 + m^2}}{T} \\ |p| = T\sqrt{x^2 - z^2}, |p|d|p| &= T^2xdx \\ |p|^2d|p| &= T^3x\sqrt{x^2 - z^2}dx \end{aligned}$$

So,

$$\begin{aligned} n =& 4\pi\frac{gT^3}{(2\pi)^3}\exp\left(\frac{\mu}{T}\right)\int_z^\infty dx(x^2 - z^2)^{\frac{1}{2}}x\exp(-x) \\ =& \frac{gT^3}{2\pi^2}z^2K_2(z) = \frac{T^3}{2\pi^2}\left(\frac{m}{T}\right)^2K_2\left(\frac{m}{T}\right)\exp\frac{\mu}{T} \end{aligned} \tag{B.4}$$

$$\begin{aligned} P =& \frac{gT^4}{6\pi^2}\exp\left(\frac{\mu}{T}\right)\int_z^\infty dx(x^2 - z^2)^{\frac{3}{2}}\exp(-x) \\ =& \frac{gT^4}{2\pi^2}\left(\frac{m}{T}\right)^2K_2\left(\frac{m}{T}\right)\exp\left(\frac{\mu}{T}\right) \end{aligned} \tag{B.5}$$

# Bibliography


[1] Edmond Iancu. Theory summary for ISMD2000. In *40th International Symposium on Multiparticle Dynamics*, 5 2011.

[2] David Griffiths. *Introduction to elementary particles*. 2008.

[3] Mary K. Gaillard, Paul D. Grannis, and Frank J. Sciulli. The Standard model of particle physics. *Rev. Mod. Phys.*, 71:S96–S111, 1999.

[4] W. N. Cottingham and D. A. Greenwood. *An introduction to the standard model of particle physics*. Cambridge University Press, 4 2007.

[5] Frank Wilczek. Anyons. *Scientific American*, 264(5):58–65, 1991.

[6] Peter Jacobs and Xin-Nian Wang. Matter in extremis: Ultrarelativistic nuclear collisions at RHIC. *Prog. Part. Nucl. Phys.*, 54:443–534, 2005.

[7] LHC Machine. *JINST*, 3:S08001, 2008.

[8] M. Harrison, T. Ludlam, and S. Ozaki. RHIC project overview. *Nucl. Instrum. Meth. A*, 499:235–244, 2003.

[9] A. K. Chaudhuri. *A short course on Relativistic Heavy Ion Collisions*. IOPP, 2014.

[10] Wit Busza, Krishna Rajagopal, and Wilke van der Schee. Heavy Ion Collisions: The Big Picture, and the Big Questions. *Ann. Rev. Nucl. Part. Sci.*, 68:339–376, 2018.

[11] Raghunath Sahoo. Relativistic Kinematics. 4 2016.

[12] Michael L. Miller, Klaus Reygers, Stephen J. Sanders, and Peter Steinberg. Glauber modeling in high energy nuclear collisions. *Ann. Rev. Nucl. Part. Sci.*, 57:205–243, 2007.

[13] Ed Daw. Lecture 7-rapidity and pseudorapidity. *URL http://www. hep. shef. ac. uk/edaw/PHY206/Site/2012_course_files/phy206rlec7. pdf*, 2012.

[14] Kardar M. *Statistical Physics of Fields*. Cambridge University Press, 2007.

[15] Bjoern Schenke, Sangyong Jeon, and Charles Gale. (3+1)D hydrodynamic simulation of relativistic heavy-ion collisions. *Phys. Rev. C*, 82:014903, 2010.

[16] M. Bleicher et al. Relativistic hadron hadron collisions in the ultrarelativistic quantum molecular dynamics model. *J. Phys. G*, 25:1859–1896, 1999.

[17] Hannah Petersen, Jan Steinheimer, Gerhard Burau, Marcus Bleicher, and Horst Stöcker. A Fully Integrated Transport Approach to Heavy Ion Reactions with an Intermediate Hydrodynamic Stage. *Phys. Rev. C*, 78:044901, 2008.

[18] Sourav Sarkar, Helmut Satz, and Bikash Sinha, editors. *The physics of the quark-gluon plasma*, volume 785. 2010.

[19] Sangyong Jeon and Ulrich Heinz. Introduction to Hydrodynamics. *Int. J. Mod. Phys. E*, 24(10):1530010, 2015.

[20] L. D. Landau and E. M. Lifshitz. *Fluid Mechanics, Second Edition: Volume 6 (Course of Theoretical Physics)*. Course of theoretical physics / by L. D. Landau and E. M. Lifshitz, Vol. 6. Butterworth-Heinemann, 2 edition, January 1987.

[21] Luciano Rezzolla and Olindo Zanotti. *Relativistic Hydrodynamics*. 2013.

[22] J. D. Bjorken. Highly Relativistic Nucleus-Nucleus Collisions: The Central Rapidity Region. *Phys. Rev. D*, 27:140–151, 1983.

[23] Paul Romatschke. New Developments in Relativistic Viscous Hydrodynamics. *Int. J. Mod. Phys. E*, 19:1–53, 2010.

[24] Olivier Darrigol. Between hydrodynamics and elasticity theory: the first five births of the navier-stokes equation. *Archive for History of Exact Sciences*, 56(2):95–150, 2002.

[25] Carl Eckart. The Thermodynamics of irreversible processes. 3.. Relativistic theory of the simple fluid. *Phys. Rev.*, 58:919–924, 1940.

[26] G. S. Denicol, T. Kodama, T. Koide, and Ph. Mota. Stability and Causality in relativistic dissipative hydrodynamics. *J. Phys. G*, 35:115102, 2008.

[27] W. Israel and J. M. Stewart. Transient relativistic thermodynamics and kinetic theory. *Annals Phys.*, 118:341–372, 1979.

[28] J. Steinheimer, S. Schramm, and H. Stocker. The hadronic SU(3) Parity Doublet Model for Dense Matter, its extension to quarks and the strange equation of state. *Phys. Rev. C*, 84:045208, 2011.

[29] Abdel Nasser Tawfik. Koppe's Work of 1948: A fundamental for non-equilibrium rate of particle production. *Z. Naturforsch. A*, 69:106–107, 2014.

[30] Enrico Fermi. High-energy nuclear events. *Prog. Theor. Phys.*, 5:570–583, 1950.

[31] Abdel Nasser Tawfik. Equilibrium statistical-thermal models in high-energy physics. *Int. J. Mod. Phys. A*, 29(17):1430021, 2014.

[32] Constantino Tsallis. Possible Generalization of Boltzmann-Gibbs Statistics. *J. Statist. Phys.*, 52:479–487, 1988.

[33] C. E. Carlson, T. H. Hansson, and C. Peterson. Meson, Baryon and Glueball Masses in the MIT Bag Model. *Phys. Rev. D*, 27:1556–1564, 1983.

[34] A. Chodos, R. L. Jaffe, K. Johnson, and Charles B. Thorn. Baryon Structure in the Bag Theory. *Phys. Rev. D*, 10:2599, 1974.

[35] W. M. Alberico, A. Lavagno, and P. Quarati. Nonextensive statistics, fluctuations and correlations in high-energy nuclear collisions. *Eur. Phys. J. C*, 12:499–506, 2000.

[36] Constantino Tsallis and Ernesto P. Borges. Nonextensive statistical mechanics: Applications to nuclear and high-energy physics. In *10th International Workshop on Multiparticle Production: Correlations and Fluctuations in QCD (CF2002)*, 1 2003.

[37] J. Cleymans and D. Worku. Relativistic Thermodynamics: Transverse Momentum Distributions in High-Energy Physics. *Eur. Phys. J. A*, 48:160, 2012.

[38] G. Wilk and Z. Wlodarczyk. On the interpretation of nonextensive parameter q in Tsallis statistics and Levy distributions. *Phys. Rev. Lett.*, 84:2770, 2000.

[39] T. S. Biró, G. G. Barnaföldi, P. Ván, and K. Ürmössy. Statistical Power-Law Spectra due to Reservoir Fluctuations. 4 2014.

[40] J. Cleymans and D. Worku. The Tsallis Distribution in Proton-Proton Collisions at $\sqrt{s}$ = 0.9 TeV at the LHC. *J. Phys. G*, 39:025006, 2012.

[41] Betty Abelev et al. Centrality dependence of $\pi$, K, p production in Pb-Pb collisions at $\sqrt{s_{NN}}$ = 2.76 TeV. *Phys. Rev.*, C88:044910, 2013.

[42] J. Cleymans and M. W. Paradza. Determination of the chemical potential in the Tsallis distribution at LHC energies. 10 2020.

[43] S. S. Adler et al. Identified charged particle spectra and yields in Au+Au collisions at S(NN)**1/2 = 200-GeV. *Phys. Rev. C*, 69:034909, 2004.

[44] B. I. Abelev et al. Identified baryon and meson distributions at large transverse momenta from Au+Au collisions at s(NN)**(1/2) = 200-GeV. *Phys. Rev. Lett.*, 97:152301, 2006.

[45] Shreyasi Acharya et al. Production of charged pions, kaons, and (anti-)protons in Pb-Pb and inelastic $pp$ collisions at $\sqrt{s_{NN}}$ = 5.02 TeV. *Phys. Rev. C*, 101(4):044907, 2020.

[46] R. Brun and F. Rademakers. ROOT: An object oriented data analysis framework. *Nucl. Instrum. Meth. A*, 389:81–86, 1997.

[47] Dirk H. Rischke, Stefan Bernard, and Joachim A. Maruhn. Relativistic hydrodynamics for heavy ion collisions. 1. General aspects and expansion into vacuum. *Nucl. Phys. A*, 595:346–382, 1995.

[48] Dirk H. Rischke, Yaris Pursun, and Joachim A. Maruhn. Relativistic hydrodynamics for heavy ion collisions. 2. Compression of nuclear matter and the phase transition to the quark - gluon plasma. *Nucl. Phys. A*, 595:383–408, 1995. [Erratum: Nucl.Phys.A 596, 717–717 (1996)].