Complete Hyperbolic Structure on Manifold

By

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Certificate of Examination

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Declaration

The work presented in this dissertation has been carried out by me under the guidance of Dr. Shane D'Mello at the Indian Institute of Science Education and Research Mohali. This work has not been submitted in part or in full for a degree, a diploma, or a fellowship to any other university or institute. This thesis is a bonafide record of expository work done by me and all sources listed within have been detailed in the bibliography.

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In my capacity as the supervisor of the candidate's project work, I certify that the above statements by the candidate are true to the best of my knowledge.

Dr. Shane D'Mello

(Supervisor)

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Abstract

In this thesis we have studied the way to determine if the hyperbolic structure on 3-manifold is complete. These tools can be to used to find complete hyperbolic structure on 3-manifolds using ideal triangulation. We will look at how to decompose the 8-knot complement. The algorithm thus developed can be applied to find ideal triangulation of knot. We study the relation between developing maps and completeness of the (G,X) structure on a manifold.

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1.1 Preliminaries

[5]In this section we will give some definitions which will help us throughout.

Definition 1.1.1. A *knot* K is a subset of S^3 such that it is PL homeomorphic to S^1. Alternatively, we can think of a knot as a PL embedding $K : S^1 \to S^3$, using the symbol K to refer to both the map K and $K(S^1)$. It will be clear from the context which one we are using.

Note: Under a PL homeomorphism S^1 is mapped to a finite number of line segments.

An advantage of using a PL homeomorphism is that it allows us to assume that the knot, $K \subset S^3$ has a regular tubular neighborhood. That is, it allows us to assume that there is an embedding of a solid torus $S^1 \times D^2$ into S^3 such that $S^1 \times \{0\}$ is mapped to $K \subset S^3$.

Remark 1.1.1. A knot which cannot be embedded piecewise linearly into S^3 is called a *wild knot*.

Definition 1.1.2. Two knots K_1 and K_2 are said to be *ambient isotopic* if there is (PL or smooth) homotopy $H : S^3 \times [0,1] \to S^3$ such that each $H(.,t) = H_t : S^3 \to S^3$ is a homeomorphism for each t and,

$$H(K_1, 0) = H_0(K_1) = K_1$$

and

$$H(K_2, 1) = H_1(K_1) = K_2$$

Such a map is called an *ambient isotopy*.

Note that ambient isotopy is an equivalence relation.

Definition 1.1.3. For a knot K, $N(K)$ denotes an *open regular neighborhood* of K in S^3 of knot K in S^3. The *exterior of the knot* is $S^3 \setminus N(K)$.

Note: The knot complement $S^3 \setminus K$ is a compact 3-manifold with boundary, whose boundary is homeomorphic to a torus.

Definition 1.1.4. A *knot invariant* is a function which maps the set of knots to some set whose value depends on the equivalence class of knots, i.e. the value of the function for two knots is the same if the knots are equivalent (i.e. ambient isotopic).

Definition 1.1.5. A link $L : \sqcup S^1 \to S^3$ is a PL embedding of a disjoint union of circles into the sphere. Once again we use L to refer to either the map or the image $L(\sqcup S^1)$.

Theorem 1.1.1 (Gordon-Luecke Theorem)**.** . *If there are two knots $K_1 and K_2$ such that the complement of their images in S^3 are homeomorphic to each other by an orientation preserving homeomorphism, then the knots are equivalent.*

From the above theorem we conclude that in variants of knots that are able to distinguish between their complements must be able to distinguish between the knots We get a counterexample when we consider links instead of knot in the hypothesis of Gordon-Luecke theorem.

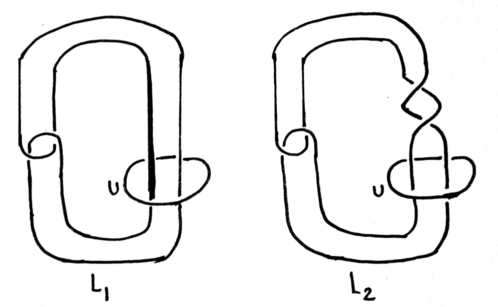


Figure 1.1.1: Two non-homeomorphic links whith homeomorphic exterior

Example 1.1.1. [4]Consider two links L_1 and L_2 as shown, now we see that these two links are not homeomorphic, since removing one component from L_1 gives us unknot whereas, removing unknot from L_2 gives us trefoil knot. Consider regular neighborhood $N(U)$ of unknot U in L_1. Now, consider $S^3\backslash N(U)$, we get a 3-manifold. We cut this 3-manifold along the seiffert surface that is disc of unknot U, then we make a 2π twist in the cut manifold. This twist is called Rolfsen twist. We glue back the manifold along the cut. Thus we obtain a homeomorphism between the exteriors of two links. This is a counterexample of the above theorem.

1.2 Overview

[2][7][5]In this Section. we will try to understand how to obtain a polyhedral decomposition of the figure 8 knot. Thurston was the first to give such a decomposition for the figure 8-knot.

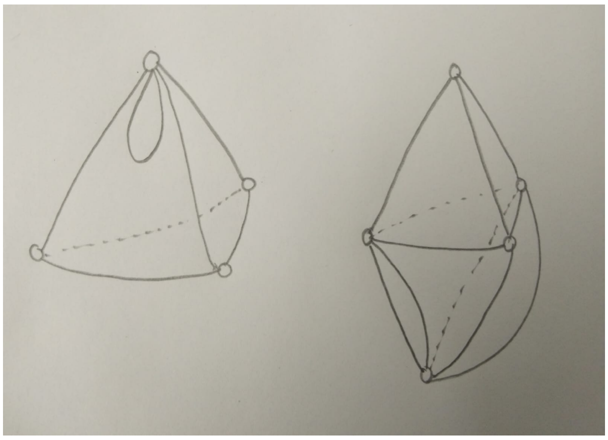


Figure 1.2.1: In the diagram on left we see that there's monogon on the top vertex and in right figure we have bigon.

By the term polyhedron we mean a 3-ball with finite number of vertices and edges labeled on its boundary by a finite graph such that the vertices and edges demarcate the faces which are simply connected and have disjoint interiors. This definition of polyhedra permits polyhedra to contain monogons and bigons. An ideal polyhedra is polyhedra with all the vertices removed. You can observe, how in the above diagram, the 3-ball is divided by the finite vertices and edges and all the region obtained are simply connected.

We will decompose the knot complement into two ideal tetrahedrons. We consider our knot to be embedded into S^3 which is identified as $R^3 \cup \infty$ and lying on the XY-plane. Wherever there is over-crossing our knot is slightly displaced above and wherever our knot has under-crossing it is slightly displaced below the XY-plane. Insert an ideal edge at each crossing. Here by an ideal edge, we mean that it have no end points.

We consider two balloons being expanded from the ∞, one below the XY-plane and other above the XY-plane. As they keep on expanding they bump into each other at the XY-plane. This divide the surface of balloons into faces and edges. These faces correspond to the region cut out by the graph of diagram. We label these faces as A,B, C, D, E and F. These faces meet in edges which correspond to the the edge at each crossing. These two balls will give us the knot complement when pair of edges are identified to each other in certain way.

1.3 At crossings

[7][5]We see that when balloons starts expanding from above the plane and below the plane, it divide it into different regions. When the balloons hit an isolated strand as shown in the figure in wraps itself around the strand. It seems like that strand is passing through the tunnel. This happens because knot itself is not a part of knot complement. Now we insert

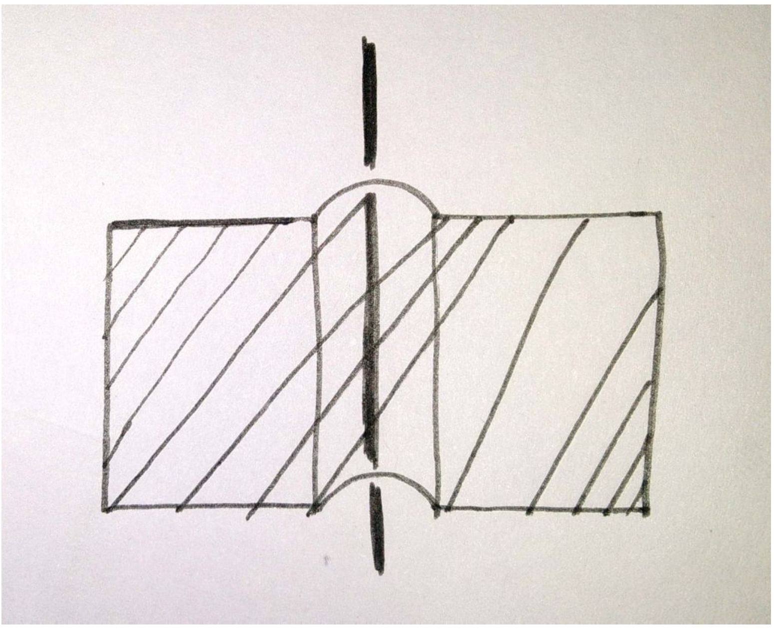


Figure 1.3.1: Balloon at an isolated arc

an ideal edge at every crossing in the knot complement, such that it goes from overstrand to understrand. Now we split this edge into two parts (it lies in the knot complement and thus on the surface of the balloon), later on we will be identifying them. Because to the crossing, the surface of balloon is divided into four parts. The faces S and V are adjacent to each other and, U and V are adjacent to each other. As you can see in the figure how this one edge is being separated into two parts. This figure gives us a clear picture of how adding strand divides the surface into four parts.

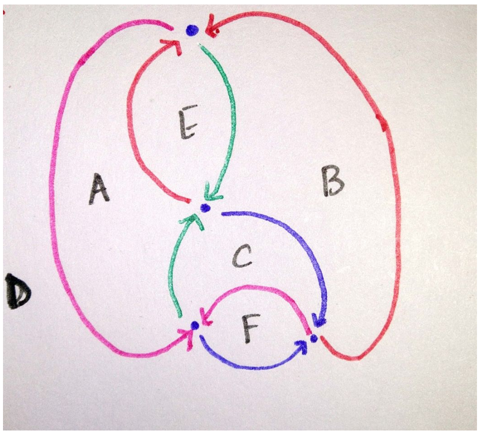
When we repeat this process from the bottom, our understrands become overstrands and overstrands becomes understrands. Earlier, the edge we inserted went from overstrand to understrand, now it will go the other way round. Our strand will be split from understrand to overstrand this time and again we have four faces, This time, S is adjacent to V and T is adjacent to U. We note here that we have four copies for each edge we have inserted, two copies comes form the top and two comes from the bottom. We repeat this procedure at each crossing, and we shrink the isolated arcs of the knots to ideal vertices. Retracting the edge to a single point can seem confusing at first, as the arc of the knot is not homeomorphic to the point. Here we do not consider the arc but complement of it. The complement of the arc on the surface of the balloon is homeomorphic to the complement of a point on it, and hence we are able to replace these edges by the ideal vertices. This gives rise to two polyhedra in the knot complement, and we identify the faces and edges appropriately to get the knot complement. In the top polyhedra we shrink each overstrand and in bottom polyhedra we shrink the understrand to the ideal vertex.

Note that we can choose the orientation for the edge inserted to be either from overstrand to understrand or vice versa, as long as it remains same throughout the whole process at a crossing.

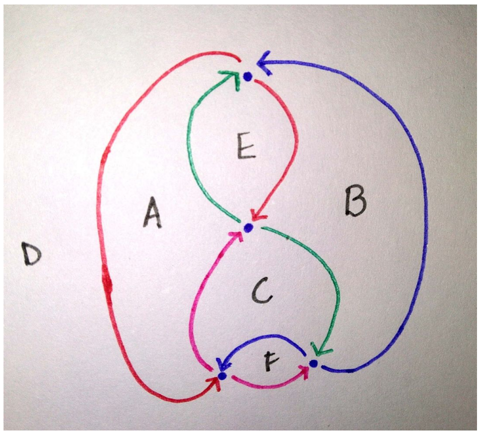
1.4 Figure 8-knot complement

[7][5]In this section we will apply the procedure discussed above to obtain the figure 8-knot complement. We will firstly discuss how we can obtain the top polyhedra and then work on the similar lines to get bottom polyhedra. We expand the balloon from ∞ from top. This divides our plain into 6 parts, and balloon into 6 regions namely, A,B,C,D,E and F. D is basically originating from the ∞. Now we insert an ideal edge at each crossing and split it into two which will be identified later on. One interesting, thing which can be done to visualize these regions in a better manner is to make a knot with a wire and dip it int the soap solution. You will be distinctively able to see the faces.

Now, visualize that our surface is stretching, since the arcs of knots does knot belong the knot complete, surface stretch in such a way that these arcs become shorter and shorter and collapse to an ideal vertex. We extend each edge which we inserted at crossing so that they meet at these ideal vertices. At the end of this procedure we will get the top polyhedra.



(a) top polyhedron



(b) bottom polyhedron

Figure 1.4.1: Knot diagrams for the top and bottom polyhedron

We will repeat this process by expanding balloon from the bottom of the plane. When we see our knot from the bottom the overstrands becomes understands and vice-versa. Again the surface of balloon is divided into A,B,C,D,E and F, where D originates from the ∞. The inserted edge now splits in the other way as can be seen in the diagram. Like above we shrink the edges of the knot to the ideal vertices as mentioned in the previous section and thus we get the bottom polyhedron.

In the end we identify the face A labeled on the top polyhedra with face labeled A on the bottom making sure that the corresponding edges are matched. We do it for each face. Using this process of gluing th faces and edges gives us the knot complement.

Chapter 2

Hyperbolic Geometry

In the later sections we will need to manipulate objects in 2 and 3-dimension hyperbolic space. This section provides a brief introduction to the tools which will be required. In this chapter we will be focusing mainly on isometries of the upper half-plane.

2.1 Hyperbolic geometry in two space

[5]We have various models for hyperbolic space in dimension two, like upper half space model and disk model . The one which we will be using is upper half space model. In this model, hyperbolic 2 space $\mathbb{H}$ is defined as the set of points in the upper half plane as:

$$\mathbb{H}^2 = \{x + iy \in \mathbb{C} | y > 0\}$$

equipped with the following riemannian metric:

$$ds^2 = \frac{dx^2 + dy^2}{y^2}$$

or in complex co-ordinate z as

$$ds = \frac{|dz|}{Imz}$$

We define length of the curve γ in $\mathbb{H}^2$ as:

$$|\gamma| = \int_a^b \frac{\sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2}}{y} dt. \tag{2.1}$$

Definition 2.1.1. $\mathbb{R}^2 \cup \{\infty\}$ is the boundary at infinity for $\mathbb{H}^2$

Definition 2.1.2. For every $z \neq 0$, we have $\frac{z}{0} = \infty$ and $\frac{z}{\infty} = 0$

2.2 Möbius Transformation

[3][5][1]Consider hyperbolic plane $\mathbb{H}^2$, fix a point in the plane say i. With the above metric we can observe that for any point z, distance between i and z becomes larger and it tends to ∞ as imaginary part of z tends to 0, or z becomes closer to ∞ in $\mathbb{H}^2$. Along vertical line $y = a$, we observe that distance between any two points on this line is same as Euclidean

distance divided by a.

With above observations in mind, some isometries of $\mathbb{H}^2$ are :

1. Translation : $t_a(z) = z + a$, where a is a real number.
2. Dilation : $d_p(z) = pz$, where p is a positive real number.
3. Reflection about y axis : $r(z) = -z$

We will now show that translation is an isometry. Clearly, t_a maps, $\mathbb{H}^2$ to $\mathbb{H}^2$ and its inverse is $t_{-a} = z - a$ again maps $\mathbb{H}^2$ to $\mathbb{H}^2$. Translation t_a is $t_a(z) = u = z + a$ where $z = x + \imath y$. Then as $dz = du$, we have

$$\frac{|dz|}{y} = \frac{|du|}{y}$$

We have dialation, $d_p(z) = u = pz$ where $z = x + \imath y$ and p is positive real number is a bijective map $\mathbb{H}^2$ to $\mathbb{H}^2$. Then as $dz = du/p$, we have

$$\begin{aligned}\frac{|du|}{Imu} &= \frac{|pdz|}{py} \\ &= \frac{|dz|}{y}\end{aligned}$$

We have reflection, $r(z) = u = -z$ where $z = x + \iota y$ is a bijective map from $\mathbb{H}^2$ to $\mathbb{H}^2$. Then as $du = -dz$, we have,

$$\begin{aligned}(\frac{|du|}{Imu})^2 &= (\frac{|-dz|}{-y})^2 \\ &= (\frac{|dz|}{y})^2\end{aligned}$$

By above calculations we see that, t_a, d_p, r are isometries of $\mathbb{H}^2$

We have another model for hyperbolic space in dimension 2 which is the disk model $\mathbb{D}^2$, $\mathbb{D}^2 = \{z \in \mathbb{C}^2 ||z| < 1\}$.

The metric on $\mathbb{D}^2$ is :

$$ds = \frac{|2dw|}{1 - |w^2|}, w \in \mathbb{D}^2 \tag{2.2}$$

From the above metric we can observe that as we move away from the origin the hyperbolic distance becomes larger and larger. This distance tends to infinity as we approach boundary of disk $\partial\mathbb{D}^2$, where $\partial\mathbb{D}^2 = \{z \in \mathbb{C}^2 ||z| = 1\}$.

We can interchangeably use the upper half plane model or the disk model. We can go from the upper half plane model to the disk model using the transformation $T : \mathbb{H}^2 \longrightarrow \mathbb{D}^2$, defined by

$$w = T(z) = \frac{iz+1}{z+\iota} \tag{2.3}$$

Let f be an isometry of $\mathbb{H}^2$, then we can get isometry of $\mathbb{D}^2$ as follows:

$$\begin{array}{ccc} \mathbb{H}^2 & \xrightarrow{f} & \mathbb{H}^2 \\ {\scriptstyle T^{-1}}\uparrow & & \downarrow{\scriptstyle T} \\ \mathbb{D}^2 & \xrightarrow[g]{} & \mathbb{D}^2 \end{array}$$

Here, $g = T^{-1}fT$. We observe isometries of $\mathbb{D}^2$ are basically $T-$conjugate of isometries of $\mathbb{H}^2$.

Some natural isometries of disk model are :

1. $r'(z) = e^{i\alpha}z$, which is a rotation and θ is any real number. r' corresponds to rotation. Consider,

$$f(z) = \frac{(cos\alpha)z + (sin\alpha)z}{(-sin\alpha)z + (cos\alpha)z}$$

, which is a rotation in upper half-plane. When we calculate, $T \circ f \circ T^{-1}$ we get,

$$T \circ f \circ T^{-1} = \exp \iota \frac{\alpha}{2}$$

2. $i(w) = \overline{w}$, in upper half plane this corresponds to inversion about unit circle as

$$T^{-1} i T(z) = \frac{1}{\overline{z}}$$

Now as in $\mathbb{H}^2$, inversion in unit circle is an isometry. We observe that inversion about any circle with center on real axis is an isometry as any such circle can be brought to unit circle using translation and dilation.

Theorem 2.2.1. *Any (euclidean) vertical line in $\mathbb{H}^2$ is a geodesic.*

Proof. consider y-axis, we will show that its a geodesic. Let a=(0,a) and b=(0,b) be two points on y=0. We will need to parametrize the curve to find the length of it.consider,

$$X(t) = (0, t), t \in [a, b]. \tag{2.4}$$

Then the arclength will be,

$$|\gamma| = \int_a^b \frac{\sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2}}{y} dt = \int_a^b \frac{1}{y} dt = ln(a) - ln(b) = \frac{ln(a)}{ln(b)} \tag{2.5}$$

Now consider any other piecewise smooth curve,

$$Y(t) = (x(t), y(t)) \tag{2.6}$$

passing through a and b such that $y(t)$ is an increasing function, as if we consider y to be any function then we can divide the the domain of y in two parts, one where y is increasing and other where y is decreasing. Using that we will get same result.

Then we have,

$$\begin{aligned} |Y| &= \int_a^b \frac{\sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2}}{y} dt \\ &\geq \int_a^b \frac{\sqrt{(\frac{dy}{dt})^2}}{y} dt \\ &\geq \int_{y(a)}^{y(b)} \frac{dy}{y} \\ &\geq ln(y(a)) - ln(y(b)) \end{aligned} \tag{2.7}$$

We see that length of $|Y|$ is greater than or equal to our segment of vertical line. Thus the shortest path that joins a and b is the segment of y-axis. $\square$

Before we proceed further we will see how inversion map takes vertical lines to circles or vertical lines.

Theorem 2.2.2 (Inversions). *1. They take vertical lines passing through the origin to itself.*

2. *They take vertical lines not passing through origin to circles passing through the origin.*
3. *They take circles passing through the origin to vertical lines not through the origin.*
4. *Circles which do not pass through the origin gets mapped to the circles not through the origin.*

Proof. Let $r(z) = \frac{1}{z}$.

1. We note that vertical lines passing through the origin will be of the form $z = x\iota$ with $x \in \mathbb{R}$. Then,

$$r(x\iota) = \frac{1}{x\iota} = \frac{-\iota}{x}$$

We have for each $x \in R, -1/x \in R, x \neq 0$. If $x = 0, r(x\iota) = \infty, r(\infty) = 0$. Thus, vertical lines through origin are mapped to vertical lines through origin.

2. A line which does not pass through the origin have form $z = x + \iota y, x, y \in \mathbb{R}$ such that $ax + by = c, a, b, c \in \mathbb{R}$. Let $r(z) = u + \iota v$. Then,

$$r(x + \iota y) = u + \iota v = \frac{1}{x + \iota y}$$

i.e.,

$$x + \imath y = \frac{u - \imath v}{u^2 + v^2}$$

Thus,

$$x = \frac{u}{u^2 + v^2}, y = \frac{u - \imath v}{u^2 + v^2}$$

as $ax + by = c$ we have

$$\frac{au}{u^2 + v^2} - \frac{bv}{u^2 + v^2} = c$$

$$au - bv = c(u^2 + v^2)$$

$$u^2 + v^2 - \frac{au}{c} + \frac{bv}{c} = 0$$

Thus lines not passing through origin are mapped to circles through origin.

3. Since $r(z) = \frac{1}{z}$ is its own inverse, we observe that circle not passing through origin maps to lines not through origin by above.

4. Consider, $z = x + \imath y, x, y \in R$. Then, circle not through origin have equation, $x^2 + y^2 + ax + by = c$ where $a, b, c \in \mathbb{R}$ are fixed real numbers and $c \neq 0$. Then let $r(z) = u + \imath v$ We have,

$$x = \frac{u}{u^2 + v^2}, y = \frac{-v}{u^2 + y^2}$$

We get,

$$(\frac{u}{u^2+v^2}) + (\frac{-v}{u^2+v^2}) = \frac{au}{u^2+v^2} - \frac{bv}{u^2+v^2} = c$$

Thus,

$$\frac{u^2+v^2}{(u^2+v^2)^2} + \frac{au-bv}{u^2+v^2} = c$$

So,

$$1 + au - bv = cu^2 + cv^2$$

and,

$$u^2 + v^2 - \frac{au}{c} + \frac{bv}{c} = \frac{1}{c}$$

which is the equation of circle not passing through origin.

□

Theorem 2.2.3. *Vertical lines and circles having centre on real line are lines in upper half plane model.*

Proof. Let a and b be two points in $\mathbb{H}^2$. Let C be the circle passing through a and b such that its centre lies on $\mathbb{R}$ and it intersects the real line at P and Q. Then translate the circle by -P on the x-axis, So that P lies on the origin. Now compose this translation with the inversion through the unit circle. We see that after the inversion P is sent to the ∞. As the above translation and inversion are isometries, their composition is also an isometry and we know that isometries preserves angles. As C intersects with $\mathbb{R}$ at right angles, we see that the image of C is a vertical line. We have already shown that, vertical lines are geodesics. Since isometries preserves the arclength, C is a line in $\mathbb{H}^2$ $\square$

2.3 Hyperbolic geometry in three dimension

[5]In the last section we dealt with hyperbolic geometry in 2-dimension. In this section we will give some results for hyperbolic geometry in dimension 3. Hyperbolic 3-space $\mathbb{H}^3$ is defined as:

$$\mathbb{H}^3 = \{(x + \iota y, t) \in \mathbb{C} \times \mathbb{R} | t > 0\}$$

with the metric:

$$ds^2 = \frac{dx^2 + dy^2 + dz^2}{t^2}$$

. Here we will mention few results without proof in $\mathbb{H}^3$.

Theorem 2.3.1. *Vertical lines and semi-circles in $\mathbb{H}^3$ with their centre on the boundary $\partial\mathbb{H}^3 = \mathbb{C} \cup \infty$ are the geodesics in $\mathbb{H}^3$. hemispheres centered on $\mathbb{C}$ and vertical planes are the geodesic planes.*

2.3.1 Isometries in $\mathbb{H}^3$

Theorem 2.3.2. *Reflection in geodesic planes in $\mathbb{H}^3$ generate the group of isometries of $\mathbb{H}^3$.*

2.4 Triangles and horocycles

[5]

Lemma 2.4.1. *Given any three distinct points z_1, z_2, z_3, in the boundary of upper half plane, there exists an orientation preserving isometry which take them to $0, 1, \infty$ respectively, i.e. there exist an orientation preserving isometry such that any three points in the boundary of $\mathbb{H}^2$ can be taken to any three points of the boundary.*

Definition 2.4.1. An *ideal triangle* in $\mathbb{H}$ is a triangle with three geodesic edges and the three vertices as the points on boundary at infinity.

Definition 2.4.2. An ideal tetrahedron in $\mathbb{H}^3$ is a tetrahedron with ideal vertices that is it's vertices lies on the boundary of $\mathbb{H}^3$.

Definition 2.4.3. A horocycle is curve whose centre p, is an ideal point on the boundary of $\mathbb{H}^2$ such that it is perpendicular to the geodesics which pass through p. If p lies on $\mathbb{R}$ then the horocycle is the euclidean circle tangent to $\mathbb{R}$ at p. When p is ∞ the horocycle is a horizontal euclidean line. A horoball is the region enclosed by the horocycle.

Definition 2.4.4. A horosphere about $\infty \in \partial\mathbb{H}^3$ is a plane $\{(x+\iota y, a) \in \mathbb{C}\times\mathbb{R}\}$ where $a > 0$ is constant. For $p \in \mathbb{C}$ the euclidean sphere tangent to p is also a horosphere.

Chapter 3

Geometric structure on Manifolds

We want to study knots by studying the geometric properties of the knot complement. Till now we have studied an outline of algorithm how to do polyhedral decomposition of knot complement and some properties of hyperbolic plane. In this chapter we will be studying how to endow manifold with geometric structure and under which conditions it will be complete.

3.1 Geometric structure on manifolds and examples

[5]We will begin by giving some definitions.

Definition 3.1.1. Let M be a 2-manifold, then *topological polygonal decomposition* of M is defined to be a combinatorial way of gluing the polygons together so that the result manifold is homeomorphic to M.

Definition 3.1.2. A *geometric polygonal decomposition* is topological decomposition such that each polygon has metric on them, gluing is done by isometrics and the resulting manifold is smooth with a complete metric

Definition 3.1.3. Let X be a manifold and G be the group acting on it. We say that manifold M have a (G,X) structure if for every point x∈M, there is a chart $(U, \phi) : U \to \phi(U) \subset X$, such that U is an open neighborhood of x and ϕ maps U onto its image $\phi(U) \subset X$. It should satisfy the following condition: If (U, ϕ) and (V, ψ) are two charts such that $U \cap V$ is non-empty, then the transition map $\psi \circ \phi^{-1} : \phi(U \cap V \to \psi(U \cap V)$ is given by the restriction of elements of G on each connected component of $U \cap V$.

We will be considering X to be simply connected, X and M to be real analytic manifolds and G to be the group of real analytic diffeomorphisms acting on X. As real analytic diffeomorphisms are uniquely determined by their restriction on any connected open set, we will get that transition maps on each connected components of $U \cap V$ will be given by the restriction of some element of G.
Also here our manifold X will generally have a metric, and G will be group of isometrics of X. M will inherit metric from X. Under such conditions we will say that M has geometric structure.

Here we will describe a geometric structure on torus. Let T be the torus and $X = \mathbb{R}^2$ and G be the group of isometries of Euclidean space. Now we know that $\mathbb{R}^2$ is universal cover of T and covering transformations of $\mathbb{R}^2$ are Euclidean transformations. The tiling of square of length 1 gives us $\mathbb{R}^2$.

Now consider the basic square with corners $(0,0)$, $(0,1)$,$(1,0)$ and $(1,1)$. Let P be any point on torus. Then when we lift P under the covering map, the set of points obtained is infinite in $\mathbb{R}^2$ which will give an integer lattice on $\mathbb{R}^2$. Each point lifted belongs to a copy of unit square. Let there be an open disc of radius less than $1/2$, in general we can take radius as $1/4$. Now when we project this neighbourhood under covering map in T, we get neighbourhood U of P. Thus, we obtain a chart (U,ϕ) around P where ϕ map U to a disk of radius $1/4$, to any disc around lift of P.

And if there's another chart say (U,φ) such tha φ maps U to another disc of radius 1/4 around P to another lift of P in some other square, we can see that the transition map from $\varphi(U)$ to $\phi(U)$ is given by Euclidean translation. Thus $\phi \circ \varphi^{-1}$ is a Euclidean isometry. The torus obtained by giving opposite edges of basic square can be given $(Iso(\mathbb{E}^2),\mathbb{E}^2)$ by above process. A manifold with $(Iso(\mathbb{E}^2),\mathbb{E}^2)$ is said to have Euclidean structure.

Example 3.1.1 (Torus)**.** Now the above process gives us a metric on T induced by the euclidean metric. The induced metric obtained by the pullback on T is complete. Below we will give another geometric structure on T. We will observe that the structure we will obtain will not have a complete metric. Let $X = \mathbb{R}^2, G = Isom(\mathbb{R}^2)$. Any affine transformation is of the form $x \mapsto Ax + b$.Any manifold with this (G,X) structure is said to have an affine structure. This time we not only consider translations, but rotation and scaling. Take any quadrilateral. Then we can glue the sides of quadrilateral using these affine transformation. When we glue the copies of this quadrilateral to obtain an universal cover, we have to rotate, shrink and expand the quadrilateral and such a tiling is not a plane. Rather its a plane which is missing a point. By above gluing of quadrilateral we obtain an affine structure on T.

As we have missed a point in the plane the metric thus obtained by the pullback won't be complete. As we have seen above it is a tiresome process to work with charts to obtain geometric structure on manifolds. It can be more easily obtained by working with the qoutient spaces.

3.2 Hyperbolic structure

[5]Our main focus throughout will be to study hyperbolic structure on the manifolds. In this section we define hyperbolic structure on manifolds.

Definition 3.2.1. Let $X = \mathbb{H}^2$ and $G = Isom(\mathbb{H}^2)$. If M is a manifold such that it has (G, X) structure it is said to admit *hyperbolic structure*. In general any manifold with $(Isom(\mathbb{H}^n, \mathbb{H}^n)$-structure have *hyperbolic structure*.

Hyperbolic structure on manifolds will useful to study knot complements. Consider a hyperbolic 2-manifold, which is obtained from the geometric polygonal decomposition. Let the polygons be hyperbolic triangles, vertices either be finite (inside hyperbolic plane) or ideal(on the boundary of hyperbolic plane) and the edges to be the segments of hyperbolic geodesics. Under certain nice conditions such as when one edge of a polygon is identified with exactly one edge of the edge of other polygon and edges are identified with edges and vertices are identified with vertices we get a manifold. (Massey, Algebraic Topology) So we know that gluing of polygons with these nice conditions will be a manifold. But under what conditions gluing of hyperbolic polygons will result in manifold with hyperbolic structure? In the following lemma we will try to understand the conditions to answer this question.

Note: In the following lemma by gluing of polyhedra we will mean a collection of polyhedra with geodesic edges, with faces being identified by the isometries of $\mathbb{H}^n$.

Lemma 3.2.1. *A gluing of hyperbolic polygons gives us a 2-manifold with* *hyperbolic structure that agrees with the structure inside polygons if and only if, for every point in the gluing, that is quotient manifold M we have (in the manifold obtained) a neighborhood of the point isometric to the hyperbolic disk in* $\mathbb{H}^2$.

Proof. By gluing here we mean that, the quotient space which is obtained by identifying on faces of hyperbolic polyhedras under consideration using isometry or gluing maps. Each face is identified to some other using these gluing maps. Also, by hyperbolic structure agreeing with the structure inside polygon we mean that for every point inside the interior of polygon there a neighborhood U of that point such that, there is a map from U to $U \subset \mathbb{H}$, and this map provides chart for the point.

Suppose the gluing of polygon has yielded manifold M, such that hyperbolic structure on it agrees with the structure inside the polygon. Then, p in M have a chart (U, ϕ) such that $\phi : U \to \phi(U)$, $\phi(U) \subset \mathbb{H}^2$. Consider $U \cap P$, where P is polygon. Then composing ϕ with the identity map on $U \cap P$ gives us a transition map from $U \cap P$ to $\phi(U \cap P)$, which is an isometry in the interior of P. Since U is made up by gluing these polygons, we can think of $U \cap P$ as the intersection of open in $\mathbb{H}^2$ with P.

Let each point in M have neighborhood isometric to a ball in $\mathbb{H}^2$ by using the isometry ϕ. Then these isometries gives us the charts on M

$(U, \phi), (V, \psi)$. Let (U, ϕ) and (V, ψ) such that their interaction is non-empty. Then we get that the transitions maps $\phi \circ \psi^{-1}$. These transitions maps are isometries. Thus they will give an isometry of $\mathbb{H}^2$. Hence, the manifold thus obtained will have a hyperbolic structure.

□

Lemma 3.2.2. *The gluing of hyperbolic polygons gives us a manifold such that every point have a neighborhood isometric to hyperbolic disk if and only if the interior angle sum at each finite vertex is* 2π

Proof. Consider a gluing of hyperbolic polygons, then if at each point we have neighborhood isometric to hyperbolic disk. Since isometries preserve the angle, we will get that at each finite vertex interior angle sum is 2π.

Let the gluing be done in such manner that for a finite vertex the interior angle sum is 2π. From the above lemma, we know that for any point which is in the interior of the polygon or is on the edge of the polygon will have a neighborhood isometric to a disk in $\mathbb{H}^2$. Now, we can use isometries to map the vertex under consideration to ι. Now we place the polygons adjacent to each other in a cyclic order, while gluing the edge of one polygon to the other till we reach the last polygon. Notice that the edge of the last polygon gets glued to the edge of first polygon (as the interior angle sum around vertex is 2π. Hence, the finite vertex will have neighborhood which is isometric to a disk in $\mathbb{H}^2$. $\square$

3.3 Developing Map

[5][6]We saw in the above section conditions under which gluing of hyperbolic polygons gives us a manifold with hyperbolic structure. But under what conditions will this hyperbolic structure be complete? In the following section we will discuss developing map and holonomy which will help us understand this.

Let M be a real analytic manifold having (G, X)- structure where X is a connected real analytic diffeomorphisms acting transitively on X. Let $p \in M$ and (U, P) be a chart of P. Then (U, ϕ) map U homeomorphically onto $\phi(U)$ in X. Now we know that if P have two charts say (U, ϕ) and (V, φ) such that the intersection is non empty, then the transition map $\phi \circ \varphi^{-1}$ is uniquely determined by the restriction of $g(p)$ in neighbourhood of p on each connected component of $U \cap V$ where $p \in U \in V$ and

$$g = \phi \circ \varphi^{-1} : \varphi(U \cap V) \longrightarrow \phi(U \cap V)$$

Let $y \in U \cap V$. Consider a map $y \longrightarrow g$, we observe that this map is locally constant. Let $g(y)$ denotes the element of G and we define a map $\Phi : U \cup V \longrightarrow X$ as following :

$$\Phi(x) = \begin{cases} \phi(x) & if \;\; x \in U \\ g(y)\varphi(x) & if \;\; x \in V \end{cases}$$

Then $\Phi(x)$ is a well defined map if $U \cap V$ is connected. If $x \in U \cap V$,

$$\begin{aligned} \phi(x) &= g(y)\varphi(x) \\ &= \phi\varphi^{-1}\varphi(x) \\ &= \phi(x) \end{aligned}$$

Φ can be seen as extension of ϕ.

Example 3.3.1. Consider the torus T obtained by gluing opposite edges of unit square in $\mathbb{R}^2$. Consider Euclidean structure on T. Consider two open sets U and V along the longitude of T as shown in the figure. Then we see that $U \cap V$ have connected components. Let $x, y \in T$ such that x lies in one connected component and y lies in other component. Then, there are two charts $(U, \phi), (V, \varphi)$ which maps y in the basic square. We observe that transitions map $g(y) = \phi \circ \varphi^{-1}$ is the identity map. Now we define $\Phi : U \cup V \longrightarrow X$ as above.

We see that Φ is not well defined in this case since $g(y)\varphi(x)$ lies in square bounded by $(1,0), (1,1), (2,0)$ and $(1,2)$. Thus, we arrive at a problem. To overcome this we will use universal cover of M. We fix a basepoint x_0. We will be developing charts along the path through x_0. Also, if M is not simply connected, then we have a non-trivial loop. We will face a problem here as well as when we will be developing the charts along this loop and when we will circle back to initial point we will have two different definitions of map at that point. Using universal cover helps tackling this problem as well.

To define developing map, we will use universal cover $\tilde{M}$ of M, in $\tilde{M}$ each point can be thought of as homotopy class of paths in M with a fixed base point x_0. We will define a function from $\tilde{M}$ to X. To do this we will extend the chart along the path in M corresponding to a point in $\tilde{M}$.

Let $\alpha : [0,1] \to M$ be a path in M representing a point $[\alpha] \in M$ with a fixed basepoint x_0. Let U_0, ϕ_0 be a chart around x_0. Now we can subdivide the interval $[0,1]$ in $[t_i, t_{i+1}]$ (where $i = 0, 1, ..., n-1$) such that each $\alpha[t_i, t_{i+1}]$ is contained in U_i for every i. Thus, we will get finitely many charts (U_i, ϕ_i)covering the path. Note that each $\alpha(t_i) \in U_i \cap U_{i-1}$, let x_i denote $\alpha(t_i)$ for each i. Here each x_i is contained in a connected component of $U_i \cap U_{i-1}$.

We know that transition map $\phi_{i-1} \circ \phi_i^{-1}$ is well defined on the connected component and is uniquely determined by an element of G in a neighbourhood of $x_i \in U_i \cap U_{i-1}$. Let us denote it by g_i.

Now, we will extend (U_0, ϕ_0) to a map from $[0, t_2]$ to X. Note that $\phi_0(\alpha(t))$ gives us a map from $[0, t_n]$ to X. We extend on $[0, t_2]$ by defining $\Phi : [0, t_2] \longrightarrow X$ as :-

$$\Phi_1(t) = \begin{cases} \phi_0(\alpha(t)) & \text{if} \quad t \in [0, t_1] \\ g_1\phi_1(\alpha(t)) & \text{if} \quad t \in [t_1, t_2] \end{cases}$$

$\Phi_1(t)$ is well defined on $[0, t_2]$ as $\phi_0(\alpha(t_1)) = g_1\phi_1(\alpha(t_1))$, $g_1 = \phi_0 \circ \phi_1^{-1}$ on the connected component. We can develop the chart further by inductively expanding $\Phi_i : [0, t_{i+1}] \longrightarrow X$ by defining :-

$$\Phi_1(t) = \begin{cases} \phi_{i-1}(t) & \text{if} \quad t \in [0, t_1] \\ g_1 g_2 .... g_i \phi_i(\alpha(t)) & \text{if} \quad t \in [t_i, t_{i+1}] \end{cases}$$

Let Φ_k are well defined for all $k = 1, .., i$. For $k = i + 1$, we can have Φ_{i+1} well defined as

$$\begin{aligned} \Phi_{i+1}(t_{i+1}) &= g_1 g_2 .... g_i g_{i+1} \phi_{i+1}(\alpha(t_{i+1}) \\ &= g_1 g_2 .... g_i \phi_i(\alpha(t_{i+1}) \\ &= \Phi(t_{i+1}) \end{aligned}$$

Hence, we get that Φ_{i+1} is well defined. By induction, Φ_i is defined for $i = 1, 2..., n-1$. Thus we get $\Phi_{n-1} : [0, 1] \longrightarrow X$, which is well defined. This map provides us another map in the small neighbourhood U of $\alpha(1)$,

$\Phi_{[\alpha]} : U \longrightarrow X$ defined as :-

$$\Phi_\alpha(x) = g_1 g_2 ... g_{n-1}\phi_{n-1}(x)$$

$\Phi_{[\alpha]}$ can be thought of as chart around $\alpha(1)$.

Definition 3.3.1. The developing map $D : \tilde{M} \longrightarrow X$ is defined as:

$$D[\alpha] = \Phi_{n-1}(1) = g_1 g_2 ... g_{n-1}\phi_{n-1}(\alpha(1))$$

Now we will show that D is well defined and independent of arbitrary choices we made while defining it.

Theorem 3.3.1. *The developing map $D : \tilde{M} \to X$ satisfies the following properties:*

1. *For a fixed basepoint x_0 and chart $(U_0, \phi_0$, such that $x_0 \in U_0$ the definition of developing map does not depend on the holonomy class $[\alpha]$, on the choice of charts and points chosen in the intersection of charts. The developing map is thus well-defined.*

2. *D is a local diffeomorphism.*

3. *If we begin with a different base point and develop a map along the path say D' then the developing map D' will be D composed with an unique element of G.*

Proof. 1. Firstly we will begin by showing that the definition of developing map does not depend on the choice of charts. Consider a chart (U, ϕ) between (U_{i-1}, ϕ_{i-1}) and (U_i, ϕ_i), the we will have two transitions maps $\phi \circ \phi_i^{-1}$ and $\phi_{i-1} \circ \phi^{-1}$. These will correspond to unique group elements f and k. Then our new developing map would be:

$$D'([\alpha]) = g_1 g_2 ... g_{i-1} k f g_{i+1} ... g_{n-1} \phi_{n-1}(\alpha(t))$$

Now as x_i belongs to U, U_{i-1}, U_i we have:

$$g_i = \phi_{i-1} \circ \phi_i^{-1} = \phi_{i-1} \circ \phi^{-1} \circ \phi \circ \phi_i^{-1} = kf$$

Thus we have,

$$\begin{aligned} D'([\alpha]) &= g_1 g_2 ... g_{i-1} g_i g_{i+1} ... g_{n-1} \phi_{n-1}(\alpha(t)) \\ &= D([\alpha]) \end{aligned}$$

In general, when we have two sets of charts, we can take the union of charts and we can use those charts to develop the map along the path. As from above we will see that we will get the same developing map with the refining.

Now we will show that developing map is independent of the choice of the path in the holonomy class $[\alpha]$. Let $\alpha, \beta \in [\alpha]$. Then there's a path homotopy $H(t, y)$ from α to β such that $H(t, 0) = \alpha(t)$ and $H(t, 1) = \beta(t)$. Now we can divide the homotopy into smaller homotopies H_i such that (t_i, y) lies in a single chart for $y \in [0, 1]$. We notice that, throughout the process, $\alpha(t_i)$ remains in the same the same connected component of x_i.As the group elements g_i are uniquely determined by x_i, they will be same if we develop the map along the path β. Thus we will get the same developing map if we develop along the paths that are homotopic to each other.

Now, we will show that the definition of developing map is independent of the points we choose in the intersection of charts. Let $x_1, x_2, ..., x_{n-1}, xn$ be the point in the intersection of the charts. Then the group element g_i is uniquely determined on each connected component. If we choose different x_i in the connected component we will get the same g_i. Thus, we will get the same developing map developing along that path.

2. We know that the covering map from $\tilde{M}$ to M is a local diffeomorphism. The charts from M to X are local diffeomorphisms and all the g_i's are real analytic diffeomorphism. Thus, their composition is also a local diffeomorphism.

3. Let us start with a different basepoint say y_0, Let β be the path joining y_0 to x_0. Then, we develop our map along the path $\beta * \alpha$. Now along the path β we will have some charts and corresponding to them we will get group elements $h1_1, ..., h_m$, giving us a new developing map D' where:

$$D'([\beta * \alpha]) = h_1 h_2 ... h_m g_1 g_2 ... g_n \phi_{n-1}([\beta * \alpha])$$

Then,

$$\begin{aligned} D'([\alpha]) &= h_1 h_2 ... h_m g_1 g_2 ... g_n \phi_{n-1}([\alpha]) \\ &= h_1 h_2 ... h_m D([\alpha]) \end{aligned}$$

Let, $h_1 h_2 .... h_m = h$ then,

$$D'([\alpha]) = hD([\alpha])$$

. As h_j's are uniquely determined, there composition will also be unique and we get our desired result.

□

Consider $[\alpha] \in \tilde{M}$ such that $[\alpha] \in \pi(M)$. As we saw before this theorem we get a map $\Phi_{[\alpha]}$ which gives us a chart around small neighbourhood of $\alpha(1)$. Let $[\alpha] \in \tilde{M}$ such that $[\alpha] \in \pi_1(M)$ at basepoint x_0. Then $\Phi_{[\alpha]}$

gives us a chart around the basepoint. Thus, ϕ_0 and $\Phi_{[\alpha]}$ will differ by unique element of G. Let us call it $g_{[\alpha]}$, then

$$\Phi_{[\alpha]} = g_{[\alpha]}\phi_0$$

By using previous theorem we can choose initial chart (U_0, ϕ_0) around α to be same (U_{n-1}, ϕ_{n-1}). We observe that $g_{[\alpha]}$ gives us the difference between original chart and chart obtained by analytic continuation around the loop and $g_{[\alpha]}$ will be same as $g_1 g_2 ... g_{n-1}$ as group element is unaffected by the choice of charts.

Definition 3.3.2. The $g_{[\alpha}]$ is called holonomy of $[\alpha]$.

Let $T_{[\alpha]}$ be covering transformation of $\tilde{M}$ corresponding to $[\alpha] \in \pi_1(M)$. Then as covering transformation $T_{[\alpha]}$ acts on $\tilde{M}$ we will have :-

$$T_{[\alpha]}([\beta]) = [\alpha] * [\beta], \;\; [\beta] \in \tilde{M}$$

Let D be a developing map along α. Then,

$$\begin{aligned} D \circ T_{[\alpha]}([\beta]) &= D([\alpha] * [\beta]) \\ &= D[\alpha * \beta] \\ &= g_1 g_2 ... g_n h_1 .. h_{m-1} \phi(\beta(1)) \\ &= g_{[\alpha]} D([\beta])] \end{aligned}$$

As when we develop our chart along $[\alpha * \beta]$, we will first develop along

$[\alpha]$ and $[\beta]$. Let $g_1g_2...g_{n-1}$ and $h_1,,,,h_{m-1}$ be product of group elements corresponding to them respectively. Then as we saw above $g_1g_2...g_{n-1}$ will correspond to $g_[\alpha]$. The $g_[\alpha]$ we have defined is unique as if $g_[\alpha] \circ D = g \circ D$, then $g_[\alpha]$ and g agree on some open set in X. As group element of G is uniquely determined on open set of X, g will be unique.

Let $[\alpha], [\beta] \in \tilde{M}$, then

$$\begin{aligned} D \circ T_{[\alpha]} \circ T_{[\beta])} &= g_{[\alpha]} \circ D \circ T_{[\beta])} \\ &= g_{[\alpha]} \circ g_{[\beta]} \circ D \end{aligned}$$

Thus, the map $\rho : \pi_1(M) \longrightarrow G$ defined by $\rho([\alpha]) = g_{[\alpha]}$ is a group homomorphism.

Definition 3.3.3. The group homomorphism defined as the holonomy of M and image of ρ is called holonomy group of M.

Consider D' to be another developing map along α, then we know that $D' = gD$ for some $g \in G$. Corresponding to D', we will have ρ', from above discussion we can conclude that $\rho' = g\rho$.

3.4 Completeness of gluing of polygons

[5]As we have seen in section 3.2 the two theorems told us that in a gluing when at each finite vertex the angle sum is 2π we will get a hyperbolic

structure on the 2-manifold thus obtained. When we gluing ideal polyhedra, as they do not have any vertices those conditions do not give us the desired results. In this section we will explore how these polygons will be glued to give us complete hyperbolic structure on the resulting manifold.

Let M be an oriented manifold we get by the gluing of ideal hyperbolic polygons. In M we will get an equivalence class of ideal vertices based on the gluing of ideal edges of polygons. Consider an ideal vertex v in M (it is an equivalence class of ideal vertices identified with some vertex of polygons in gluing). Let $P_0, P_1, ...P_{n-1}$ be the polygons sharing this common vertex. Let these polygons be set out in counter-clockwise manner. Let h_0 be the hoorocycle around v_0 in P_0 counter-clockwise. Now it will meet an edge of P_1 which is glued to edge e_0 of P_0.

We notice that h_0 meet P_1 at right angle because,h_0 meets e_0 at right angle as horocycles meet geodesics emanating from centre at right angle. Now there exists an unique horocycle h_1 with centre as v_1 identified with v extending h_0 in P_1. We continue this process and keep extending the horocycles $h_1, h_2, ..., h_{n-1}$ at every edge. Now as we have finite number of polygons with finite number of vertices vertices, we reach our initial polygon P_0. As we extend h_{n-1} in P_0 h_n, it may not coincide with h_0. The hyperbolic distance between h_0 and h_n is denoted $d(v), d(v) > 0$ if h_n is near v_0 and $d(v) < 0$ if h_n away form v_0. Now we will show that d(v) is

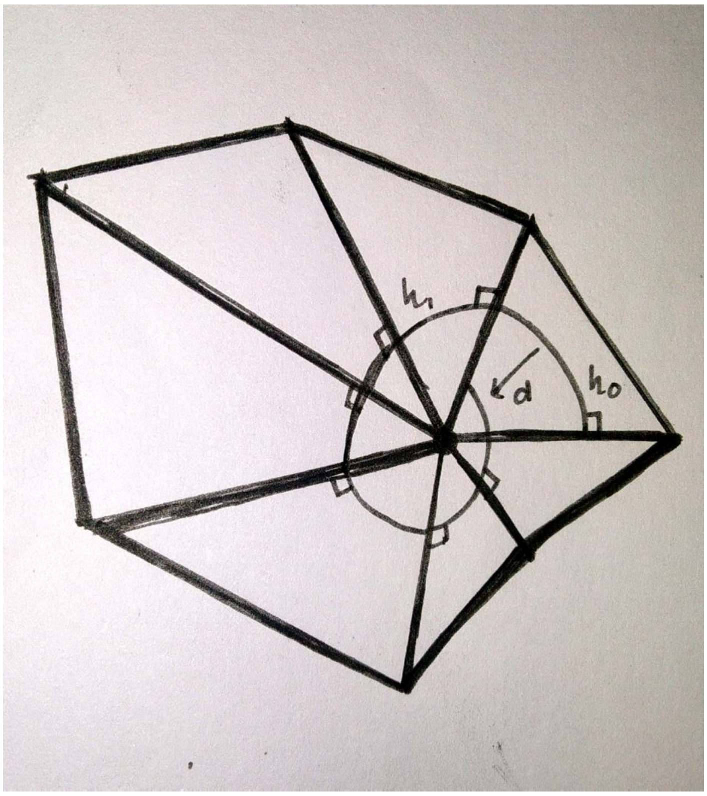


Figure 3.4.1: Extending a horocycle

well defined.

Lemma 3.4.1. *The parameter d(v) does not dependent on the initial choice of vertex and horocycle. It is well defined.*

Proof. Let us choose a different horocycle h_0' in P_0. Then h_0' will be at a constant distance from h_0. As we extend h_1' in P_1 the distance between h_1 and h_1' will be x as P_0 and P_1 are identified at the common edge isometrically. Similarly the distance between h_i' and h_i will be x. As h_{n-1}' is extended into P_0 the distance between h_n and h_n' will be x. As a result the distance $d(v)'$ between h_0' and h_n' will be same as $d(v)$.

Now let us choose any other polygon P_i with some other horocycle h'_i and let the constant distance between h_i and h'_i be y. Then as we keep on extending it as the previous case the distance between h_n and h'_n will also be y. Then again as keep extending that horocycle further in P_i, the distance between h_{n+i} and h'_{n+i} will be y. As h_{n+i} is at a distance $d(v)$ from h_i, the distance between h'_i from h'_{i+n} will also be $d(v)$. Thus $d(v)$ is well defined.

□

3.5 Developing map and completeness

[5]In this section we will discuss state two theorems which will tell us conditions for the gluing pof polyhedron to be a complete manifold and how it is linked with developing map.

Theorem 3.5.1. *Let M be the two manifold obtained by gluing hyperbolic polygons, then M has complete if and if $d(v) = 0$ for each ideal vertex.*

Proof. Let M be the 2-manifold obtained by the gluing of hyperbolic polygons. Let $d(v) > 0$. As $d(v) > 0$ we will get a sequence of points obtained by the intersection of horocycle with edges of polygons about v. We will show that above sequence is cauchy sequence and it doesn't converge. Let v be ∞. The the polygons lie in $\mathbb{H}^2$ and the horocycle would be horizontal line passing through them. We develop the polygons in $\mathbb{H}^2$ taking the path as horocycle. Then when the gluing of polygons is extended, we observe that second circuit, the horocycle moves up, and is at a distance d from first. We continue doing this process and each horocycle in the next circuit would be at a distance of d from the previous one. We also note that the length of horocycle in the first circuit is finite and the second circuit the length of horocycle is smaller than first by a constant factor. Similarly length of each horocycle in circuit is less that the length of previous horocycle by a constant factor. The length of these horocycles keeps on reducing by a factor of $1/d^2$. Then the sum of length of these horocycles converge and we get that the sequence of length is cauchy sequence. Thus the length of horocycle in a circuit will tend to 0. From here we infer that the distance of points also tend to zero, thus the sequence of points is also a cauchy sequence in M. But it doesn't converge as for any small neighborhood in the tail of sequence, there are infinitely many points that lie outside.

Let $d(v) = 0$ for every ideal vertex v. Then, there is some horocycle around each ideal vertex v which closes up around it. Now we remove the horoball contain in the interior of horocycle in each polygon. After removing the horoball we get a compact manifold with boundary. Let $t > 0, S_t$ be be the compact manifold obtained by removing the the interior of horocycle at a distance t from the original horocycle towards v. Then,

$M = \cup_{t \in \mathbb{R}^+} S_t$. We have $S_{t+\delta}$ as compact neighborhood of S_t. Then, any cauchy sequence in in M would be contained in S_t for t sufficiently large enough. Hence, our cauchy sequence will converge and M will be complete. □

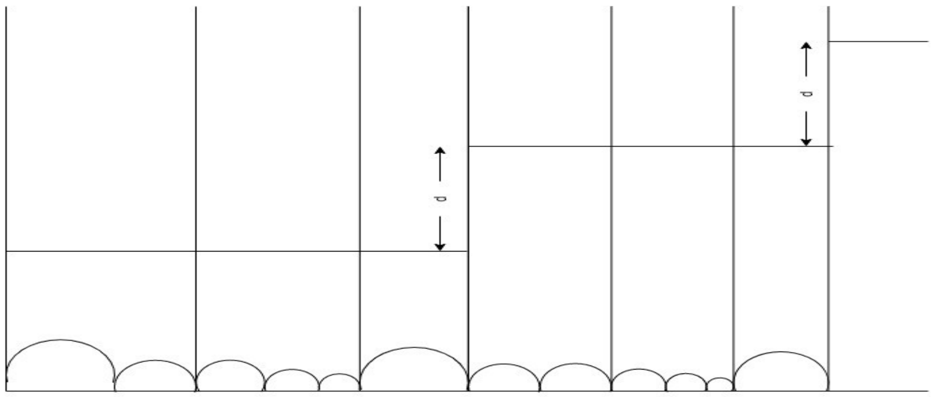


Figure 3.5.1: The length of horocycle keeps on reducing

Theorem 3.5.2. *Let M be a manifold with (G, X) structure, where G acts transitively on X and X is a complete Riemanian manifold. Then, M is complete with respect to the metric inherited from X iff developing map $D : \tilde{M} \to X$ is a covering map.*

3.6 Conclusion

In the first chapter we saw how the complement of figure-8 knot can be decomposed into polyhedrons, we can use the same procedure to to decompose the complement of a knot. In the chapter 2, we studied some isometries of upper half plane. In chapter 3, we defined geometric structure on manifolds and studied the conditions under which when we glue the hyperbolic polygons we will get a complete manifold with hyperbolic structure. All these tools can be further used to study gluing equations and Thurstons's consistency equations. These gluing and consistency equations can be used to determine the complete structure on 3-manifolds with torus boundaries. Further these can be used to study the hyperbolic volume of 3-manifolds. Using Mostow-Rigidity and Gorden-Luecke theorem one can hyperbolic volume is a knot invariant.

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