Semiclassical Methods

By

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Certificate of Examination

This is to certify that the dissertation titled "**Semiclassical Methods**" submitted by **Bhim Sen** (Reg.No. MP18019) for the partial fulfilment of MSc programme of the Institute, has been examined by the thesis committee duly appointed by the Institute. The committee finds the work done by the candidate satisfactory and recommends that the report be accepted.

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Declaration

The work presented in this dissertation has been carried out by me under the guidance of Dr. K. P. Yogendran at the Indian Institute of Science Education and Research Mohali.

This work has not been submitted in part or in full for a degree, a diploma, or a fellowship to any other university or institute. Whenever contributions of others are involved, every effort is made to indicate this clearly, with due acknowledgement of collaborative research and discussions. This thesis is a bonafide record of original work done by me and all sources listed within have been detailed in the bibliography.

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In my capacity as the supervisor of the candidate's project work, I certify that the above statements by the candidate are true to the best of my knowledge.

Dr. K. P. Yogendran
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At the end, I would love to dedicate all of my work to my mother ”Padma”.

Abstract

An interesting question that can be asked of a quantum system is whether it can produce a classical effect. We can loosely phrase this as asking whether quantum evolution can produce a "tennis ball".

To study this question, we need to explore transition probabilities between an initial state and a final state which is chosen to model the classical effect. In textbook quantum mechanics, examples of such classical states are the "coherent states". These have the property that the associated probability is tightly peaked on the corresponding *classical trajectory* in position space.

A classical particle has energies and momenta that are expressed in Joules. In particular, since the quantum energies are usually proportional to $\hbar$, this implies the coherent states we are interested in have high occupation numbers $N \sim \frac{1}{\hbar}$.

The technical problem we face now is to calculate the transition amplitude between an initial state with small occupation numbers to a final state with very large occupation numbers $N \sim \frac{1}{\hbar}$. In our work, these transitions are driven by either external sources, which we model as operator insertions at time $t = 0$, or by additional interaction terms in the Hamiltonian.

Such transitions have been considered in the literature. For instance, the review [3] describes a procedure to calculate the transition amplitude for the process $few \to$ *many* particle production having high energy and large number of particles in the final state. The key aspect is the saddle point approximation to the path integral which describes this amplitude.

We therefore adapt the methods in that review to a final state having a single particle with high energy. Subsequently, this idea can be extended to quantum field theory. In this case, the final state will be chosen to be a suitable coherent state of the *field* theory.

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Chapter 1

Introduction

In this thesis, we try to explore questions regarding transition amplitudes from generic initial quantum states to final states that have a classical interpretation. We take these final states to be coherent states whose parameters are such that the energy and momenta of these states are classical (i.e, without explicit factors of $\hbar$).

This latter requirement means that these are very highly excited states of the quantum system and hence the transition amplitudes cannot be calculated in perturbation theory. Further, the source operator which triggers the transition itself must be capable of exciting the quantum state with required amount of energy.

In this thesis, we studied the review [3] [5] by Khoze and D.T.Son, which describes a path integral approach to calculating transition amplitudes between coherent states. In this review, the author uses the saddle point approximation to evaluate these amplitudes and explains how the parameters labelling the coherent states are fixed by the properties of the initial and final states.

We will begin this thesis by briefly discussing about coherent state and how it describes a classical state and then we will move forward with a summary of the article [3], outlining the important steps leading to the calculation of the transition amplitude. Subsequently, we describe our calculations which adapt the methods of the review to calculate transitions to final classical states.

We first study these transition probabilities for the simple harmonic oscillator for a few source operators. In this case, the exact answer is easily determined. However, our **focus** will be on the validity of the saddle point approximation for these amplitudes.

Next, we will apply this idea to the anharmonic oscillator. In this case, we will take the final state to be a coherent state of the SHO - with the idea that the anharmonicity adiabatically turns off in the far future (mimicking the ideas that go into the S-matrix in the quantum field theory).

Finally, we propose to apply these ideas to quantum field theory. Here the question of interest is if it is possible to produce a QFT transition which results in a final state which can have a classical effect. In this case, since we have a *field* theory, the state needs to chosen so that it is localized in position space

This thesis is divided into two parts. In the first part, we describe how to use a coherent state based approach to calculating S-matrix elements. Since the technique is relatively novel, we collect all details together in the first sections.

However, in actual calculations, one has to resort to approximate methods. In the review [3], the authors explain how to evaluate these S-matrix elements using saddle point techniques. We explain the salient steps in section [3].

We then describe how to modify the previous technique and evaluate transition amplitudes to final coherent states and apply it to systems of our interest.

In the appendix, we collect together various classical solutions of the $\lambda\phi^4$ theory which could be useful in evaluating these amplitudes in the QFT.

Chapter 2

Coherent States in Quantum Mechanics

In quantum mechanics of a single particle, we have the Heisenberg algebra,

$$[x, p] = i\hbar \qquad [a_-, a_+] = 1 \tag{2.1}$$

where we have also rewritten the algebra in terms of the ladder operators $a_- = (x + ip)/\sqrt{2}$ (in $\hbar = 1$ units). The *coherent state* is defined as the eigenstate of annihilation operator.

$$a_- |\alpha\rangle = \alpha |\alpha\rangle \tag{2.2}$$

where $|\alpha\rangle$ is the coherent state, which is the superposition of "Number states" of LHO.
Equation (1) can be written as

$$\sqrt{\frac{m\omega}{2\hbar}}(\hat{x} + \iota\frac{\hat{p}}{m\omega}) |\alpha\rangle = \alpha |\alpha\rangle \tag{2.3}$$

$$\sqrt{\frac{m\omega}{2\hbar}}\Big(\langle x| \hat{x} |\alpha\rangle + \frac{\hbar}{m\omega} \langle x| \frac{\partial}{\partial x} |\alpha\rangle \Big) = \alpha \langle x|\alpha\rangle \tag{2.4}$$

using

$$\langle x| \hat{p} |\beta\rangle = -\iota\hbar\frac{\partial}{\partial x} \langle x|\beta\rangle \tag{2.5}$$

$$\langle x|\alpha\rangle = \psi_\alpha(x) \tag{2.6}$$

$$\sqrt{\frac{\hbar}{m\omega}} = x_0 \tag{2.7}$$

Solving the differential equation (4), and normalizing the state we will get,

$$\psi_\alpha(x) = \Big(\frac{1}{\pi x_0^2}\Big)^{\frac{1}{4}} \exp\Big[- \frac{(x - \sqrt{2}\alpha x_0)^2}{2x_0^2}\Big] \tag{2.8}$$

Since we have not specified any Hamiltonian, this state can be thought of as a state in the Hilbert space of any 1D quantum system.

2.1 Coherent states as superposition of "Number states" of LHO

By using the **definition** we can write

$$|\alpha\rangle = \sum_{n=0}^{\infty} C_n |n\rangle \tag{2.9}$$

$$C_n = \langle n|\alpha\rangle \tag{2.10}$$

$$|n\rangle = \frac{(a_+)^n}{\sqrt{n!}} |0\rangle \tag{2.11}$$

therefore, we can write

$$C_n = \langle 0| \frac{(a)^n}{\sqrt{n!}} |\alpha\rangle \tag{2.12}$$

$$C_n = \frac{\alpha^n}{\sqrt{n!}} C_0 \tag{2.13}$$

after normalizing ,we will get

$$|\alpha\rangle = \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} e^{-\frac{|\alpha|^2}{2}} |n\rangle \tag{2.14}$$

We can get time dependent state by operating $e^{\frac{-\iota \hat{H} t}{\hbar}}$, where we now choose the Hamiltonian to be that of the simple harmonic oscillator. We then get,

$$|\alpha, t\rangle = e^{\frac{-\iota\omega t}{2}} \sum_{n=0}^{\infty} \frac{(\alpha e^{-\iota\omega t})^n}{\sqrt{n!}} e^{-\frac{|\alpha|^2}{2}} |n\rangle \tag{2.15}$$

now we will find the time dependent coherent wave-function

$$\langle x|\alpha, t\rangle = e^{\frac{-\iota\omega t}{2}} e^{-\frac{|\alpha|^2}{2}} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} e^{-\iota n\omega t} \langle x| (a_+)^n |0\rangle \tag{2.16}$$

where

$$a_\pm = \frac{1}{\sqrt{2} x_0} \left[\hat{x} \pm x_0^2 \frac{\partial}{\partial x}\right] \tag{2.17}$$

where x_0 is defined in eq.(6)
so by using eq.(4)

$$\langle x| a_+ |0\rangle = \frac{1}{\sqrt{2} x_0} \left[x - x_0^2 \frac{\partial}{\partial x}\right] \psi_0 \tag{2.18}$$

$$\psi_0 = \left(\frac{1}{\pi^{1/4}\sqrt{x_0}}\right) e^{-\frac{1}{2}(\frac{x}{x_0})^2} \tag{2.19}$$

by putting eq.(16)(17)(18) in eq.(15), we get

$$\psi_\alpha(x,t) = e^{\frac{-\iota wt}{2}} e^{-\frac{|\alpha|^2}{2}} \sum_{n=0}^{\infty} \frac{\left[\frac{\alpha e^{-\iota wt}}{\sqrt{2} x_0}(x - x_0^2 \frac{\partial}{\partial x})\right]^n}{n!} \frac{1}{\pi^{1/4}\sqrt{x_0}} e^{-\frac{1}{2}(\frac{x}{x_0})^2} \tag{2.20}$$

Now after expanding the series and solving up to second order term (A detailed solution is given in **Appendix-1**), we will get

$$\psi_\alpha(x,t) = \frac{1}{\pi^{1/4}\sqrt{x_0}} e^{\frac{-\iota wt}{2}} e^{-\frac{|\alpha|^2}{2}} e^{(\sqrt{2}\alpha\frac{x}{x_0}e^{-\iota wt} - \frac{\alpha^2}{2}e^{-2\iota wt})} e^{-\frac{1}{2}(\frac{x}{x_0})^2} \tag{2.21}$$

Since

$$\alpha = |\alpha| e^{\iota\sigma} \tag{2.22}$$

equation (20) can be written as

$$\psi_\alpha(x,t) = \frac{1}{\pi^{1/4}\sqrt{x_0}} e^{\frac{-\iota\omega t}{2}} e^{-\frac{|\alpha|^2}{2}} e^{(\sqrt{2}|\alpha|\frac{x}{x_0}e^{-\iota(\omega t-\sigma)} - \frac{|\alpha|^2}{2}e^{-2\iota(\omega t-\sigma)})} e^{-\frac{1}{2}(\frac{x}{x_0})^2} \tag{2.23}$$

by using eq.(20), we can also find

$$|\psi_\alpha(x,t)|^2 = \frac{1}{\sqrt{\pi}x_0} e^{-\frac{(x-\sqrt{2}|\alpha|x_0\cos(\omega t-\sigma)^2}{x_0^2}} \tag{2.24}$$

We observe that the probability is peaked above the the trajectory

$$x = \sqrt{2}|\alpha| x_0 \cos(\omega t - \sigma) \tag{2.25}$$

which is a solution of the classical equation of motion. The energy of the quantum state is $E = (|\alpha|^2 + \frac{1}{2})\hbar\omega$ which can be compared with energy of the classical solution $E = m\omega^2 x_0^2 |\alpha|^2$. For the state to have a classical meaning, the energy should be classical - that is to say, explicit factors of the Planck's constant present in x_0 must be cancelled by choosing $\alpha x_0 = A$ where A is a classical amplitude of the order of centimeters, say.

We pause to remark that had we chosen a different Hamiltonian, the time evolved state will not remain Gaussian in general, and there may be no correspondence with a classical trajectory.

2.2 Coherent State using the Displacement operator

In this case, we will first define displacement operator and then we will give definition of coherent state using displacement operator then we will get that the coherent states are eigenstates of lowering operator as a byproduct.

Definition : Displacement operator is defined as

$$D(\alpha) = e^{\alpha\hat{a}_+ - \alpha^*\hat{a}_-} \tag{2.26}$$

$$\alpha = |\alpha| e^{\iota\sigma} \tag{2.27}$$

properties of displacement operator :

$$D^\dagger(\alpha) = D^{-1}(\alpha) = D(-\alpha) \tag{2.28}$$

$$D^\dagger(\alpha)\hat{a}_- D(\alpha) = \hat{a}_- + \alpha \tag{2.29}$$

$$D^\dagger(\alpha)\hat{a}_+ D(\alpha) = \hat{a}_+ + \alpha^* \tag{2.30}$$

$$D(\alpha+\beta) = D(\alpha)D(\beta)e^{-\iota Im(\alpha\beta^*)} \tag{2.31}$$

Definition of coherent state : A coherent state can also be generated from the ground state of the SHO by using displacement operator as

$$|\alpha\rangle = D(\alpha)\,|0\rangle \tag{2.32}$$

lemma :

$$\begin{aligned}\hat{a}_-\,|\alpha\rangle &= D(\alpha)D^\dagger(\alpha)\hat{a}_-D(\alpha)\,|0\rangle \\ &= D(\alpha)(\hat{a}_- + \alpha)\,|0\rangle = \alpha D(\alpha)\,|0\rangle \\ &= \alpha\,|\alpha\rangle\end{aligned}$$

Coherent state in number basis

$$|\alpha\rangle = \sum |n\rangle \langle n|\alpha\rangle \tag{2.33}$$

Now let us find $\langle n|\alpha\rangle$ using

$$\langle n|\,\hat{a}_-\,|\alpha\rangle = \alpha\,\langle n|\alpha\rangle \tag{2.34}$$

$$\hat{a}_+\,|n\rangle = \sqrt{n+1}\,|n+1\rangle \tag{2.35}$$

we can write equation (35) as

$$\sqrt{n+1}\,\langle n+1|\alpha\rangle = \alpha\,\langle n|\alpha\rangle \tag{2.36}$$

Now iterating this equation, we get

$$\langle n|\alpha\rangle = \frac{\alpha^n}{\sqrt{n!}}\,\langle 0|\alpha\rangle \tag{2.37}$$

This equation is similar to equation (12).
After normalozation we get

$$|\alpha\rangle = \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} e^{-\frac{|\alpha|^2}{2}}\,|n\rangle \tag{2.38}$$

and time dependent state are given by operating $e^{\frac{-\iota\hat{H}t}{\hbar}}$

$$|\alpha, t\rangle = e^{\frac{-\iota\omega t}{2}} \sum_{n=0}^{\infty} \frac{(\alpha e^{-\iota\omega t})^n}{\sqrt{n!}} e^{-\frac{|\alpha|^2}{2}}\,|n\rangle \tag{2.39}$$

Now apply $\langle x|$ to last two equation, we will get

$$\psi_\alpha(t,x) = \psi(x)_{\alpha(t)} e^{-\iota\omega t} \tag{2.40}$$

$$\alpha(t) = \alpha e^{-\iota\omega t} = |\alpha| e^{-\iota(\omega t - \sigma)} \tag{2.41}$$

Co-ordinate Space Representation

$$\psi_\alpha(x) = \langle x|\alpha\rangle = \langle x|\,D(\alpha)\,|0\rangle = \langle x|\,e^{\alpha\hat{a}_+ - \alpha^*\hat{a}_-}\,|0\rangle \tag{2.42}$$

We can write

$$\hat{a}_- = \frac{1}{\sqrt{2}}\Big(y + \frac{d}{dy}\Big) \quad \hat{a}_+ = \frac{1}{\sqrt{2}}\Big(y + \frac{d}{dy}\Big) \tag{2.43}$$

$$y = \frac{x}{x_0} \quad x_0 = \sqrt{\frac{\hbar}{m\omega}} \tag{2.44}$$

Therefore we can write

$$\psi_\alpha(x) = e^{\sqrt{2}\iota Im(\alpha)y - \sqrt{2}Re(\alpha)\frac{d}{dy}}\psi_0(y) \tag{2.45}$$

using equation (39)

$$\psi_\alpha(t,x) = e^{-\iota\omega t/2}e^{\sqrt{2}\iota Im(\alpha(t))y - \sqrt{2}Re(\alpha(t))\frac{d}{dy}}\psi_0(y) \tag{2.46}$$

A similar type of calculation which is given in **Appendix-1**.
After solving last equation, we will get

$$\psi_\alpha(t,x) = \frac{1}{\sqrt{x_0}\pi^{1/4}}e^{-\iota\omega t}e^{\sqrt{2}\alpha(t)y - \frac{y^2}{2} - Re(\alpha(t))\alpha(t)} \tag{2.47}$$

$$\psi_\alpha(t,x) = \frac{1}{\sqrt{x_0}\pi^{1/4}}e^{\sqrt{2}|\alpha|e^{-\iota(\omega t-\sigma)}\frac{x}{x_0} - \frac{1}{2}(\frac{x}{x_0})^2 - |\alpha|^2\cos(\omega t-\sigma)e^{-\iota(\omega t-\sigma)}} \tag{2.48}$$

Using this wave-function we can get same results as in equation (23)(24)(25)

2.3 Semi-Classical States in QM and Graph between $|\psi_\alpha(x,t)|^2$ and $< x(t) >$

If we study the time development of the harmonic oscillator, we will get Heisenberg equation of motion as

$$x(t) = x(0)\cos\omega t + \Big[\frac{p(0)}{m\omega}\sin wt\Big] \tag{2.49}$$

$$p(t) = -m\omega x(0)\sin\omega t + p(0)\cos\omega t \tag{2.50}$$

From these equations one may think that $< x >$ and $< p >$ always oscillate with angular frequency ω.
But this is not correct.
If we try to calculate $< x >$ and $< p >$ in any energy eigenstate, we will get zero. which means we do not have classical interpretation.
That is why we define a most generalised state i.e superposition of all energy eigenstates (**coherent state**).
Now if we try to calculate $< x >$, we will have get an oscillating function as same as we get in classical harmonic oscillator. so in this sense, coherent states can be thought of as the quantum analogs of classical states.

We can see in eq.(23) that, energy depends on $\hbar$.In order to make energy independent of $\hbar$ we need to choose $|\alpha|^2$ inversely proportional to $\hbar$. ($\frac{1}{2}\hbar w$ can be shifted.)
So let us choose.

$$|\alpha|^2 = \frac{E_{cl}}{\hbar\omega} \tag{2.51}$$

therefore we need to plot

$$< x >= \sqrt{\frac{2E_{cl}}{m\omega^2}} \cos(\omega t - \sigma) \tag{2.52}$$

$$|\psi_\alpha(x,t)|^2 = \frac{1}{\sqrt{\pi} x_0} e^{-\frac{1}{x_0^2}(x - \sqrt{\frac{2E_{cl}}{m\omega^2}} \cos(\omega t - \sigma))^2} = \frac{1}{\sqrt{\pi} x_0} e^{-\frac{(x - <x>_{cl})^2}{x_0^2}} \tag{2.53}$$

Now for $w = 1, \sigma = 0, \frac{2E_{cl}}{m} = 1$ and for different values of t we will find different values of $< x >$ represented on x-axis in the graph below (figure 1).
For the same values of w, t we will have $|\psi(x)|^2$ (Gaussian wave-function) as a function of x, which is plotted on y-axis (figure 1).
we will find that the blue line (classical trajectory) and the maximum of the Gaussian coincide that is why we say that the coherent state actually represent a classical partical behaviour.

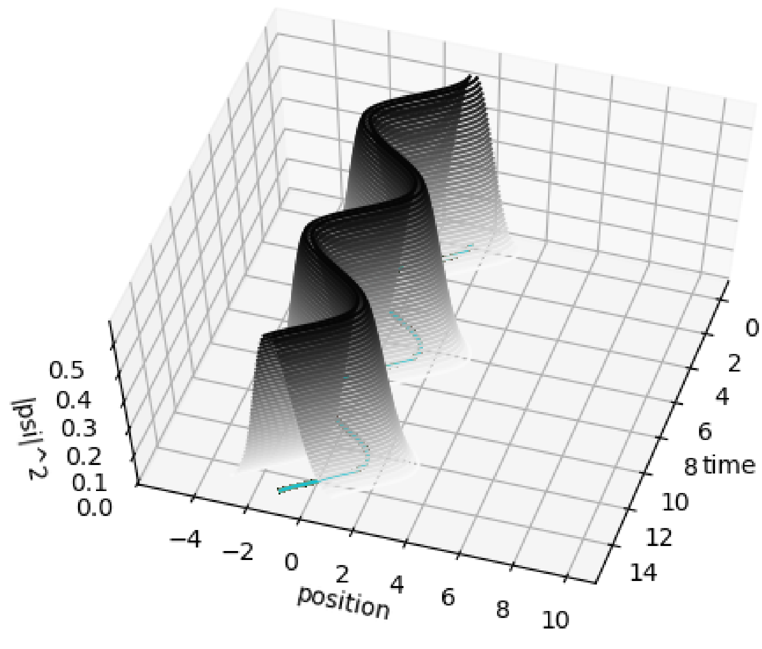

Figure 2.3.1

Chapter 3

Saddle Point methods for High Multiplicity Events

In this section we summarise the review article of Khoze which describes a saddle point approximation method to calculate transition amplitudes in QFT to final states which have a high particle number. The article can be divided into three parts which describe, respectively,

- How SHO Coherent states provide an easy way to get at S-matrix amplitudes ?
- How this technique also allows one to use saddle point approximation effectively ?
- Finally, the transition amplitude to final states with large number of particles is determined together with an estimate of quantum corrections.

As mentioned in the introduction, the review details a procedure which was successfully used to calculate the transition amplitude to a final state which had a large number of particles which the authors termed "Higgsplosion".

We propose to adapt the procedure to calculate transitions to final states which have large energies but are also localized in position as befits a classical particle.

3.1 Review of LHO

Note: that the reason to review SHO again in this section is because we will use different operator (other than the displacement operator defined above) to define coherent state.

Consider the Hamiltonian and **Note** that in this section we will use $\hat{\alpha}$ symbol for lowering and raising operators.

$$\hat{H}_0 = \frac{\hat{p^2}}{2m} + \frac{1}{2}\omega^2\hat{q^2} \tag{3.1}$$

$$\hat{\alpha} = \sqrt{\frac{\omega}{2}}(\hat{q} + \iota\frac{\hat{p}}{\omega}) \tag{3.2}$$

$$\hat{\alpha}^\dagger = \sqrt{\frac{\omega}{2}}(\hat{q} - \iota\frac{\hat{p}}{\omega}) \tag{3.3}$$

$$[\hat{\alpha}, \hat{\alpha}^\dagger] = 1 \tag{3.4}$$

$$|n\rangle = \frac{(\hat{\alpha}^\dagger)^n}{\sqrt{n!}}|0\rangle \tag{3.5}$$

3.2 Coherent states

They are eigenstates of lowering operator.
since $|n\rangle$ forms complete set,any state $|\psi\rangle$ can be written as,

$$|\psi\rangle = \sum_{n=0}^{\infty}\psi_n|n\rangle \tag{3.6}$$

$$|\psi\rangle = \psi_0\sum_{n=0}^{\infty}\frac{(a\hat{\alpha}^\dagger)^n}{n!}|0\rangle = \psi_0 e^{a\alpha^\dagger}|0\rangle \tag{3.7}$$

we will set $\psi_0 = 1$.
Now We can define coherent state as

$$|a\rangle = e^{a\hat{\alpha}^\dagger}|0\rangle \tag{3.8}$$

$$\hat{\alpha}|a\rangle = a|a\rangle \tag{3.9}$$

$$\langle a| = \langle 0| e^{a^*\hat{\alpha}} \tag{3.10}$$

$$\langle a|\hat{\alpha}^\dagger = a^*\langle a| \tag{3.11}$$

where a is the complex number, which is the eigenvalue of operator α when acted on state $|a\rangle$, and a^* which is the complex conjugate of a.
keeping this in mind, we can introduce any number of coherent states $|b\rangle, |c\rangle$, whose eigenvalues will be b, c under the same single set of operators $\hat{\alpha}$ and $\hat{\alpha^\dagger}$. Example

$$|b\rangle = e^{b\hat{\alpha}^\dagger}|0\rangle \tag{3.12}$$

$$\alpha^\dagger|b\rangle = b|b\rangle \tag{3.13}$$

Therefore we can established a one-to-one correspondence between a complex number z and a coherent state $|z\rangle$, defined via,

$$|z\rangle = e^{z\hat{\alpha}^\dagger}|0\rangle \tag{3.14}$$

Definition: The set of coherent states $\{|z\rangle\}$ obtained by the complex number z spanning the entire complex plane is known to be an **over-complete set**. Mathematically,

$$1 = \int\frac{dz^*dz}{2\pi\iota}e^{-z^*z}|z\rangle\langle z| \tag{3.15}$$

In this equation we have two real dimentional integral over complex plane z.

3.3 Properties of coherent states in quantum mechanics

From the definition of coherent states, we can get

$$\alpha^\dagger |a\rangle = \frac{\partial}{\partial a} |a\rangle \tag{3.16}$$

$$\langle b|a\rangle = e^{b^* a} \tag{3.17}$$

The last equation shows the inner product of two coherent states and we can see that they are **Not-Orthogonal**.
The analogue of the completeness relation for coherent states is,

$$1 = \int \frac{da^* da}{2\pi\iota} e^{-a^* a} |a\rangle \langle a| = \int d(a^*, a) e^{-a^* a} |a\rangle \langle a| \tag{3.18}$$

This identity is called the **over-completeness relation** due to the non-trivial exponential factor ($e^{-a^* a}$) in the integral.
Now, Let us **define** the coherent state representation of the state $|\psi\rangle$ as

$$\langle a|\psi\rangle = \psi(a^*) \tag{3.19}$$

From this we can find the inner product of two states as,

$$\langle \psi_A|\psi_B\rangle = \int d(a^*, a) e^{-a^* a} \psi_A^*(a) \psi_B(a^*) \tag{3.20}$$

$$\psi_A^*(a) = [\psi_A(a^*)]^* \tag{3.21}$$

Now in the same way, we **define** the matrix element of an operator $\hat{A}$

$$\langle b| \hat{A} |a\rangle = A(b^*, a) \tag{3.22}$$

and the of the operator $\hat{A}$ on $|\psi\rangle$ can be written as,

$$(\hat{A}\psi)(b^*) = \int d(a^*, a) e^{-a^* a} A(b^*, a) \psi(a^*) \tag{3.23}$$

and we can write the matrix element of the product of two operators as

$$(AB)(b^*, a) = \int d(c^*, c) e^{-c^* c} A(b^*, c) B(c^*, a) \tag{3.24}$$

Now, let us try to find the coherent state representation of position eigenstate

$$\langle q|a\rangle = e^{a\sqrt{\frac{\omega}{2}}(q - \frac{1}{\omega}\frac{d}{dq})} \langle q|0\rangle \tag{3.25}$$

$$\langle q|0\rangle = Ne\frac{-\omega q^2}{2} \tag{3.26}$$

$$\langle q|a\rangle = Nexp(-\frac{1}{2}a^2 - \frac{1}{2}\omega q^2 + \sqrt{2\omega} a q) \tag{3.27}$$

A detailed calculation of equation (80) is given [3] and we can also find the action of a time evolution operator on the coherent state $|a\rangle$ as,

$$\hat{U}_0(t) |a\rangle = e^{-\iota \hat{H} t} |a\rangle = \left|a e^{-\iota\omega t}\right\rangle \tag{3.28}$$

Here we should **Note** that the time evolution operator shift the phase of coherent state variable i.e, Now if we apply the lowering operator on $|ae^{-\iota wt}\rangle$ we will get an eigenvalue $ae^{-\iota wt}$.

3.4 Coherent state formalism in QFT and the S-matrix

Consider a free real scalar QFT in d+1 dimensions.
Quantum SHO and real scalar field has so much similarities, that is why we are discussing free real scalar fields (It can also be thought as superposition of LHO as show in Appendix 3).
Consider the hamiltonian of free real scalar field theory

$$\mathcal{H}_0 = \frac{1}{2}\pi^2 + \frac{1}{2}|\nabla\phi|^2 + \frac{1}{2}m^2\phi^2 \tag{3.29}$$

The normal ordered hamiltonian can be written as

$$\hat{H}_0 = \int d^d k \omega_k \hat{a}_k^\dagger \hat{a}_k \tag{3.30}$$

$$\omega_k^2 = m^2 + k^2 \tag{3.31}$$

where $\hat{a}_k^\dagger$ and $\hat{a}_k$ are the creation (raising) and annihilation (lowering) operators of QFT.
It has commution relation

$$[\hat{a}_k, \hat{a}_p^\dagger] = (2\pi)^d/2\delta(k-p) \tag{3.32}$$

We can write scalar field operator in terms of Fourier modes as,

$$\hat{\phi}(x) = \int \frac{d^d k}{(2\pi)^{d/2}} \frac{1}{\sqrt{2w_k}} (\hat{a}e^{-\iota k.x} + \hat{a}^\dagger e^{\iota k.x}) \tag{3.33}$$

Definition : In analogy with Quantum Mechanics, a coherent state in QFT is a common eigenstate of all annihilation operators.
We label the coherent state as $|\{a\}\rangle$, and denoting its eigenvalue under operator $\hat{a}_k$ as a_k,

$$\hat{a}_k |\{a\}\rangle = a_k |\{a\}\rangle \tag{3.34}$$

we have added an index k because we have infinite set of harmonic oscillator in QFT.
Now, the coherent state in QFT can be written as,

$$|\{a\}\rangle = e^{\int dk a_k \hat{a}_k^\dagger} |0\rangle \tag{3.35}$$

The fourier transformation of the field operator is defined as,

$$\bar{\hat{\phi}}(k) = \bar{\hat{\phi}}(t,k) = \int \frac{d^d x}{(2\pi)^{d/2}} e^{-\iota k.x} \hat{\phi}(t,x) \tag{3.36}$$

So the fourier transform of our free real scalar field can be written as,

$$\bar{\hat{\phi}}(k) = \frac{1}{\sqrt{2\omega_k}} (\hat{a}_k e^{-\iota\omega_k t} + \hat{a}_{-k}^\dagger e^{\iota\omega_k t}) \tag{3.37}$$

Now, with the same analogy we can **define** the Fourier transforms of the complex-valued scalar field as,

$$\bar{\phi}(k) = \frac{1}{\sqrt{2\omega_k}}(a_k e^{-\iota\omega_k t} + a^*_{-k} e^{\iota\omega_k t}) \tag{3.38}$$

where a_k and a^*_k are complex-valued eigenfunction as per equation (87).
As in QM, we have inner product of coherent states in QFT as,

$$\langle\{b\}|\{a\}\rangle = e^{\int dk b^*_k a_k} \tag{3.39}$$

$$\tag{3.40}$$

and over-completeness relation is given as,

$$1 = \int d(\{a^*\}, \{a\}) e^{\int dk b^*_k a_k} |\{a\}\rangle \langle\{a\}| \tag{3.41}$$

and with the same analogy of QM case, we can also find the coherent state representation of field $\bar{\phi}_k$ as, (a detailed calculation is shown in [3])

$$\langle\phi|\{a\}\rangle = N \ \exp\Big(-\frac{1}{2}\int dk a_k \ a_{-k} - \frac{1}{2}\int dk\omega_k \tilde{\phi}(k)\tilde{\phi}(-k) + \int dk\sqrt{2\omega_k} a_k \phi(k)\Big) \tag{3.42}$$

3.5 Application to path integrals and amplitude calculation

If we have a initial state $|\phi_i(t_i)\rangle$ then, the S-matrix **defines** the probability amplitude to arrive at final state, $|\phi_f(t_f)\rangle$.
Consider the interaction picture, in which we split the Hamiltonian into the free part H_0, and the interacting part V ,

$$\hat{H} = \hat{H}_0 + \hat{V} \tag{3.43}$$

and the S-matrix is **defined** as,

$$S_{fi} = \langle\phi_f| \hat{S} |\phi_i\rangle = \lim_{t_f, t_i \to \pm\infty} \langle\phi_f| e^{\iota \hat{H}_0 t_f} \hat{U}(t_f, t_i) e^{-\iota \hat{H}_0 t_i} |\phi_i\rangle \tag{3.44}$$

where $|\phi_f\rangle$ and $|\phi_i\rangle$ are free states and prepared at times t_f and t_i respectively.
$\hat{S}$ can be thought of as a time evolution operator in interaction picture.
The $U(t_f, t_i)$ is the time-evolution operator for Heisenberg fields.

$$\hat{U}(t_f, t_i) = \mathcal{T} exp\Big(-\iota \int_{t_i}^{t_f} \hat{H} dt\Big) \tag{3.45}$$

where $\mathcal{T}$ is the time ordered product.
Generally, we write S-matrix as,

$$\hat{S} = 1 + \iota\hat{T} \tag{3.46}$$

are we **define** matrix element $\mathcal{M}$ as,

$$\langle\phi_f|\,\iota\hat{T}\,|\phi_i\rangle = (2\pi)^4\delta^{d+1}\Big(\sum k_f - \sum k_i\Big)\iota\mathcal{M} \tag{3.47}$$

We now express the S-matrix in the coherent states basis.
This is the **kernel of the S-matrix**,

$$S(b^*,a) = \lim_{t_f,t_i\to\pm\infty}\langle\{b\}|\,e^{\iota\hat{H}_0t_f}\hat{U}(t_f,t_i)e^{-\iota\hat{H}_0t_i}\,|\{a\}\rangle \tag{3.48}$$

Since we know from equation (81) that the time evolution operator shift the phase of coherent state variable, we can write

$$S(b^*,a) = \lim_{t_f,t_i\to\pm\infty}\left\langle\{be^{-\iota wt_f}\}\right|\hat{U}(t_f,t_i)\left|\{ae^{-\iota wt_i}\}\right\rangle \tag{3.49}$$

by using completness relation , we will get

$$S(b^*,a) = \lim_{t_f,t_i\to\pm\infty}\int d\phi_f d\phi_i\left\langle\{be^{-\iota wt_f}\}\middle|\phi_f\right\rangle\langle\phi_f|\,\hat{U}(t_f,t_i)\,|\phi_i\rangle\left\langle\phi_i\middle|\{ae^{-\iota wt_i}\}\right\rangle \tag{3.50}$$

We find that $\langle\phi_f|\,\hat{U}(t_f,t_i)\,|\phi_i\rangle$ as the Feynman path integral

$$\langle\phi_f|\,\hat{U}(t_f,t_i)\,|\phi_i\rangle = \int D\phi e^{\iota S[\phi]_{t_i}^{t_f}} \tag{3.51}$$

$$\phi(t_i) = \phi_i \tag{3.52}$$

$$\phi(t_i) = \phi_i \tag{3.53}$$

$$S[\phi]_{t_i}^{t_f} = \int_{t_i}^{t_f} dt\int d^dx\mathcal{L}(\phi) \tag{3.54}$$

Therefore we finally arrive at,

$$S(b^*,a) = \lim_{t_f,t_i\to\pm\infty}\int d\phi_f d\phi_i e^{B_i(\phi_i;a)+(\phi_f;b^*)}\int D\phi e^{\iota S[\phi]_{t_i}^{t_f}} \tag{3.55}$$

where

$$\begin{aligned}B_i(\phi_i;a) = \langle\phi_i|\{ae^{-\iota wt_i}\}\rangle = -\tfrac{1}{2}\textstyle\int dka_ka_{-k}e^{-2\iota w_kt_i}\\ -\tfrac{1}{2}\textstyle\int dkw_k\tilde{\phi}_i(k)\tilde{\phi}_i(-k) + \int dk\sqrt{2w_k}a_k\tilde{\phi}_i(k)e^{-\iota w_kt_i}\end{aligned} \tag{3.56}$$

$$\begin{aligned}B_f(\phi_f;b^*) = -\tfrac{1}{2}\textstyle\int dkb_k^*b_{-k}^*e^{2\iota w_kt_f} - \tfrac{1}{2}\int dkw_k\tilde{\phi}_f(k)\tilde{\phi}_f(-k)\\ +\textstyle\int dk\sqrt{2w_k}b_k^*\tilde{\phi}_f(-k)e^{\iota w_kt_f}\end{aligned} \tag{3.57}$$

In last two expresssions, $\tilde{\phi}_i(k)$ and $\tilde{\phi}_f(k)$ are the Fourier transforms of the boundary fields $\phi_i(x) = \phi(t_i,x)$ and $\phi_f(x) = \phi(t_f,x)$

$$\phi_i(x) = \int d^dxe^{-\iota k.x}\phi(t_i,x) \tag{3.58}$$

$$\phi_f(x) = \int d^dxe^{-\iota k.x}\phi(t_f,x) \tag{3.59}$$

There is useful property of the coherent state basis for scattering theory by which we can avoid the LSZ reduction formulae.

Definition : The kernel, $A(b^*;a) = \langle b| \hat{A} |a\rangle$, of any operator $\hat{A}$ in the coherent state representation is the **generating functional** for the same operator in the Fock space,

$$\langle q_1...q_m| \hat{A} |p_1...p_n\rangle = \frac{\partial}{\partial b^*_{q_1}}...\frac{\partial}{\partial b^*_{q_m}}\frac{\partial}{\partial a_{p_1}}...\frac{\partial}{\partial a_{p_n}}A(b^*,a)|_{a=b^*=0} \tag{3.60}$$

last equation can be obtained from the definition of coherent state

$$\frac{\partial}{\partial a_{p_1}}...\frac{\partial}{\partial a_{p_n}}e^{\int dk a_k \hat{a}^\dagger_k}|0\rangle\,|_{a=0} = |p_1...p_n\rangle \tag{3.61}$$

Simillarly, for S-matrix

$$\langle q_1...q_m| \hat{S} |p_1...p_n\rangle = \frac{\partial}{\partial b^*_{q_1}}...\frac{\partial}{\partial b^*_{q_m}}\frac{\partial}{\partial a_{p_1}}...\frac{\partial}{\partial a_{p_n}}S(b^*,a)|_{a=b^*=0} \tag{3.62}$$

Hence, just differentiating with respect to coherent state variables, we can calculate any scattering amplitudes directly from the kernel of the S-matrix.

3.6 The semiclassical method for multi-particle production

In this section we review the semiclassical method for calculating probabilistic rates or cross sections for processes $few \to many$ particle production processes.
In this article we are interested in a process

$$Resonance \quad decay: \qquad \left|X(\sqrt{s})\right\rangle = |1^*\rangle \to |n\rangle \Longrightarrow partial \quad width \quad \Gamma_n(s) \tag{3.63}$$

where $|1^*\rangle$ denotes a highly virtual particle. The authors show that the saddle point approximation is valid in the regime of high multiplicity ($n >> 1$) in a weakly coupled theory ($\lambda << 1$) and we will keep λn to a fixed and large value.

3.6.1 Setting up the problem

Consider a real scalar field $\phi(x)$ in d+1 dimensions with

$$\mathcal{L}(\phi) = \frac{1}{2}(\partial_\mu\phi)^2 - \frac{1}{2}m^2\phi^2 - \mathcal{L}_{int} \tag{3.64}$$

The two simplest examples are the ϕ^4 model in the **unbroken phase**, with $L_{int} = \frac{\lambda}{4}\phi^4$ and the theory with the **spontaneously broken Z_2 symmetry**

$$L = \frac{1}{2}\partial^\mu h\partial_\mu h - \frac{\lambda}{4}(h^2 - v^2)^2 \tag{3.65}$$

The theory has a non-zero vacuum expectation value $< h >= v$ and we introduce the shifted field of mass $m = \sqrt{2\lambda}v$,

$$\phi(x) = h(x) - v \tag{3.66}$$

Our main goal is to derive the probability rate for the process in which we have a single highly virtual off-shell particle which is produced as an intermediate state in a high energy collision, or we can think of a few energetic on-shell particles in the initial state ϕ_i, which produces an n-particle final state with $n >> 1$.
We want to write this probability rate in a suitable form for a semi-classical treatment.

Let us begin by specifying the initial state. we assume that the initial state is prepared by acting a local operator $\hat{\mathcal{O}}(x)$ on the vacuum,

$$|\phi_i\rangle = \hat{\mathcal{O}}(x)\,|0\rangle \tag{3.67}$$

We will see that the operator $\hat{\mathcal{O}}(x)$ will act as a local injection of energy into the vacuum state at space-time point x. we will tale this x=0 for further discussion.
For deep and clear understanding of how we get an highly excited state by applying some operator ? one has to understand about **"operator-smearing"**given in references [4].

$$\mathcal{O}_g(x) = \int d^4x' g(x'-x)\mathcal{O}(x') \tag{3.68}$$

$$|\phi_i\rangle = \mathcal{O}_g(0)\,|0\rangle = \int d^4x' g(x')\mathcal{O}(x')\,|0\rangle \tag{3.69}$$

For our case, the averages taken in above equation is not important.
That should be done in order to get a well defined state in Hilbert space.
The probability rate $R_n(E)$ is given by the square of the matrix element of the S-matrix with the projection operators P_E and P_n,

$$\langle\phi_f|\,\hat{P}_E\hat{P}_n\hat{S}\,|\phi_i\rangle = \langle\phi_f|\,\hat{P}_E\hat{P}_n\hat{S}\mathcal{O}\,|0\rangle \tag{3.70}$$

integrated over the final states phase space i.e to sum over all such final states which has energy E and and particle number n.

$$R_n(E) = \int d\phi_f\,\langle 0|\,\hat{\mathcal{O}}^\dagger\hat{S}^\dagger\hat{P}_E\hat{P}_n\hat{S}\,|\phi_f\rangle\,\langle\phi_f|\,\hat{P}_E\hat{P}_n\hat{S}\mathcal{O}\,|0\rangle \tag{3.71}$$

$$= \langle 0|\,\hat{\mathcal{O}}^\dagger\hat{S}^\dagger\hat{P}_E\hat{P}_n\hat{S}\mathcal{O}\,|0\rangle \tag{3.72}$$

Let us choose the operator of the form

$$\hat{\mathcal{O}} = j^{-1}e^{j\phi(0)} \tag{3.73}$$

Now, to determine $R_n(E)$ we need $\hat{P}_E$ and $\hat{P}_n$.

$$P_E(b^*,a) = \langle\{b\}|\,\hat{P}_E\,|\{a\}\rangle = \int \frac{d\xi}{2\pi}exp\Big[-\iota E\xi + \int dk b_k^* a_k e^{\iota w_k\xi}\Big] \tag{3.74}$$

$$P_n(b^*,a) = \langle\{b\}|\,\hat{P}_n\,|\{a\}\rangle = \frac{d\eta}{2\pi}exp\Big[-\iota E\eta + \int dk b_k^* a_k e^{\iota\eta}\Big] \tag{3.75}$$

Now using equation (), we can write

$$P_EP_n(b^*,a) = \int d(\{c^*\},\{c\})e^{-\int dk c_k^* c_k}P_E(b^*,c)P_n(c^*,a) \tag{3.76}$$

$$= \int \frac{d\eta}{2\pi}\frac{d\xi}{2\pi}exp\Big[-\iota E\xi - \iota n\eta + \int dk b_k^* a_k e^{\iota w_k\xi+\iota\eta}\Big] \tag{3.77}$$

Now by using overcompleteness relation, we can write

$$R_n(E) = \int d(\{b^*\}, \{b\}) e^{\int dk b_k^* b_k} \times S\mathcal{O}[(b^*, 0)]^* \times P_E P_n S\mathcal{O}(b^*, 0) \tag{3.78}$$

given that $\hat{\mathcal{O}} = \hat{\mathcal{O}}[\hat{\phi}(0)]$

$$S\mathcal{O}(b^*, a) = \lim_{t_f, t_i \to \pm\infty} \int d\phi_f d\phi_i e^{B_i(\phi_i; a) + (\phi_f; b)} \int D\phi \mathcal{O}[\phi] e^{\iota S[\phi]_{t_i}^{t_f}} \tag{3.79}$$

Now again using equation (3.24)

$$P_E P_n S\mathcal{O}(b^*, a) = \int d(\{c^*\}, \{c\}) e^{-\int dk c_k^* c_k} P_E P_n(b^*, c) S\mathcal{O}(c^*, a) \tag{3.80}$$

$$= \int \frac{d\eta}{2\pi} \frac{d\xi}{2\pi} exp\Big[- \iota E\xi - \iota n\eta \Big] \times S\mathcal{O}(b^* e^{\iota w_k \xi + \iota\eta}, a) \tag{3.81}$$

So now by combining last equation and equation (3.78), we can write

$$\begin{aligned} R_n(E) = \int \tfrac{d\eta}{2\pi} \tfrac{d\xi}{2\pi} d(\{b^*\}, \{b\}) exp\Big[- \iota E\xi - \iota n\eta - \int dk b_k^* b_k \Big] \\ \times [S\mathcal{O}]^*(b, 0) \times S\mathcal{O}(b^* e^{\iota w_k \xi + \iota\eta}, 0) \end{aligned} \tag{3.82}$$

Making changes of variable,

$$b^* \to b^* e^{-\iota w_k \xi - \iota\eta}, \qquad \eta \to -\eta \qquad \xi \to -\xi \tag{3.83}$$

$$\begin{aligned} R_n(E) = \int \tfrac{d\eta}{2\pi} \tfrac{d\xi}{2\pi} d(\{b^*\}, \{b\}) exp\Big[\iota E\xi + \iota n\eta - \int dk b_k^* b_k e^{\iota w_k \xi + \iota\eta} \Big] \\ \times [S\mathcal{O}]^*(b, 0) \times S\mathcal{O}(b^*, 0) \end{aligned} \tag{3.84}$$

by combining all the ingrediants we will finally arrive at,

$$\begin{aligned} R_n(E) = \lim_{t_f, t_i \to \pm\infty} \int d\eta d\xi db_k^* db_k d\phi_i(x) d\phi_f(x) D\phi(x,t) d\varphi_i(x) d\varphi_f(x) D\varphi(x,t) \\ \times exp\Big[\iota E\xi + \iota n\eta - \int dk b_k^* b_k e^{\iota w_k \xi + \iota\eta} + \Xi \Big] \end{aligned} \tag{3.85}$$

$$\begin{aligned} \Xi = B_i(\phi_i; 0) + B_f(\phi_i; b^*) + [B_i(\varphi_i, 0)]^* + [B_f(\varphi_f, b)]^* + \\ \iota S[\phi]_{t_i}^{t_f} - \iota S[\varphi]_{t_i}^{t_f} + j\phi(0) + j\varphi(0) \end{aligned} \tag{3.86}$$

3.6.2 Finding the saddle-point and Probability Amplitude

Note that the saddle-point in the steepest descent method allows $\phi(x)$ to be complex.
We now apply steepest descent approach in the last expression and search for extremum of,

$$W = \iota E\xi + \iota n\eta - \int dk b_k^* b_k e^{\iota\omega_k \xi + \iota\eta} + \Xi(\phi_i, \phi_f, \phi, \varphi_i.\varphi_f, \varphi, b_k^*, b_k) \tag{3.87}$$

We will select those extrema which will give maximum contribution to

$$R_n(E) \propto e^W \tag{3.88}$$

So we will get the stationary point, therefore the saddle points are solutions of

$$\delta_\chi W = 0 \tag{3.89}$$

where χ denotes all integration variables.
Now following D.T. Son [5] we will find those for a saddle-point for which ξ and η are purely imaginary.
Now we change the variables,

$$\xi = -\iota T, \quad \eta = \iota\theta \tag{3.90}$$

where T and θ are real variables.
Let us now vary W,

$$W = ET - n\theta - \int dk b_k^* b_k e^{\omega_k T - \theta} + \Xi(\phi_i, \phi_f, \phi, \varphi_i . \varphi_f, \varphi, b_k^*, b_k) \tag{3.91}$$

Now we will vary W with respect to integration variables. we will get,

$$\frac{\delta W}{\delta T} = 0 \quad \rightarrow \quad E = \int dk \omega_k b_k^* b_k e^{\omega_k T - \theta} \tag{3.92}$$

$$\frac{\delta W}{\delta \theta} = 0 \quad \rightarrow \quad n = \int dk b_k^* b_k e^{\omega_k T - \theta} \tag{3.93}$$

we have obtained equations for the T and θ variables.
Now let us obtain the saddle-point equations for $\phi, \tilde{\phi}_i, \tilde{\phi}_f, b_k^*$

$$\frac{\delta W}{\delta \phi(x)} = 0 \quad \rightarrow \quad \frac{\partial S}{\partial \phi(x)} = \iota j \delta^{d+1}(x) \tag{3.94}$$

$$\frac{\delta W}{\delta \tilde{\phi}_i(-k)} = 0 \quad \rightarrow \quad \iota \partial_{t_i} \tilde{\phi}_i(k) + \omega_k \tilde{\phi}_i(k) = 0 \tag{3.95}$$

$$\frac{\delta W}{\delta \tilde{\phi}_f(-k)} = 0 \quad \rightarrow \quad \iota \partial_{t_f} \tilde{\phi}_f(k) - \omega_k \tilde{\phi}_f(k) + \sqrt{2\omega_k} b_{-k}^* e^{\iota \omega_k t_f} = 0 \tag{3.96}$$

$$\frac{\delta W}{\delta b_k^*} = 0 \quad \rightarrow \quad -b_k e^{\omega_k T - \theta} - b_{-k}^* e^{2\iota \omega_k t_f} + \sqrt{2\omega_k} \tilde{\phi}_f(k) e^{\iota \omega_k t_f} = 0 \tag{3.97}$$

These equation are solved in detail in [3].
We have analogous equation to last four equations as,
Note there is no need for b_k and b_k^* to be complex conjugate, but Nevertheless, there exists a saddle-point for which $(b_k)^* = b_k^*$ and we will focus on this scenario as Son did.

$$\frac{\delta W}{\delta \varphi(x)} = 0 \quad \rightarrow \quad \frac{\partial S}{\partial \varphi(x)} = \iota j \delta^{d+1}(x) \tag{3.98}$$

$$\frac{\delta W}{\delta \tilde{\varphi}_i(-k)} = 0 \quad \rightarrow \quad -\iota \partial_{t_i} \tilde{\varphi}_i(k) + \omega_k \tilde{\varphi}_i(k) = 0 \tag{3.99}$$

$$\frac{\delta W}{\delta \tilde{\varphi}_f(-k)} = 0 \quad \rightarrow \quad -\iota \partial_{t_f} \tilde{\varphi}_f(k) - \omega_k \tilde{\varphi}_f(k) + \sqrt{2\omega_k} b_{-k}^* e^{-\iota \omega_k t_f} = 0 \tag{3.100}$$

$$\frac{\delta W}{\delta b_k} = 0 \quad \rightarrow \quad -b_k^* e^{\omega_k T - \theta} - b_{-k} e^{-2\iota \omega_k t_f} + \sqrt{2\omega_k} \tilde{\varphi}_f(-k) e^{-\iota \omega_k t_f} = 0 \tag{3.101}$$

Now let us understand the interpretation of the equation which we have obtained. Equation gives the classical field equations but, with a singular point-like source at the origin x = 0.
We are searching for classical solutions that become free fields at $t \to \pm\infty$ and therefore the classical field in these limits must be a superposition of plane waves.
solving (3.99), gives,

$$\tilde{\phi}_i(k) = \frac{1}{\sqrt{2\omega_k}} a^*_{-k} e^{\iota\omega t_i} \qquad t_i \to -\infty \tag{3.102}$$

and by solving (3.100), we will get

$$\tilde{\phi}_f(k) = \frac{1}{\sqrt{2\omega_k}} (b_k e^{\omega_k T - \theta - \iota\omega t_f + b^*_{-k} e^{\iota\omega t_f}}) \qquad t_i \to +\infty \tag{3.103}$$

These two equations provide the boundary conditions at t_i and t_f for the solution ϕ of (3.98).
We can now compute the energy and the particle number on the saddle-point solution from its $t \to \pm\infty$ asymptotics (3.102)(3.103). At $t \to -\infty$ the energy and the particle number are vanishing since the corresponding solution contains only the $e^{\omega_k} t$ harmonics.
On the other hand at $t \to +\infty$, using the solution(3.103), we will get,

$$E = \int dk\omega_k b^*_k b_k e^{\omega_k T - \theta} \quad n = \int dk b^*_k b_k e^{\omega_k T - \theta} \tag{3.104}$$

Note that energy is conserved in both $t < 0$ and $t > 0$ the regions. However, at t = 0, the point source will give a discontinuous jump in energy.

$$\delta E = \iota j \dot{\phi}(0) \tag{3.105}$$

3.6.3 The $j \to 0$ Limit

As describe in [3] we need to take $j \to 0$.
We got energy in $t > 0$ region as $E = \iota j \dot{\phi}(0)$. We want E to be fixed and non-vanishing, so for that we need to take,

$$\dot{\phi}(0)|_{x=0} \to \infty \tag{3.106}$$

3.6.4 Evaluation of integrand at saddle-point

We have found all the saddle point equations, so now we can solve for W as,

$$W = ET - n\theta - 2ImS[\phi] \tag{3.107}$$

where $\iota S[\varphi] = \iota S[\phi]^*$ on the saddle point solution.
It has been further calculated in [3],

$$W(E,n) = n\Big(log\frac{\lambda n}{4} + \frac{3}{2}log\frac{\epsilon}{3\pi} + \frac{1}{2}\Big) - 2nm\tau_\infty - 2ReS_E^{(1,2)}(\tau_0) \tag{3.108}$$

where

$$W(E,n)^{tree} = n\Big(log\frac{\lambda n}{4} - 1\Big) + \frac{3n}{2}(log\frac{\epsilon}{3\pi} + 1) \tag{3.109}$$

and the other term is the quantum fluctuation term.
The tree level amplitude is also verified by D.T. Son [5].
Finally, we get

$$\mathcal{R}_n(E) = e^{W(E,n)}, \quad W = n\Big(log\frac{\lambda n}{4} - 1\Big) + \frac{3n}{2}(log\frac{\epsilon}{3\pi} + 1) + 0.854n\sqrt{\lambda n} \tag{3.110}$$

in the limit

$$\lambda \to 0, \quad n \to \infty, \quad with \quad \lambda n = fixed >> 1, \quad \epsilon = fixed << 1 \tag{3.111}$$

Now let us come back to our aim and apply the techniques which we have learnt from Khoze's paper to Simple Harmonic Oscillator.

Chapter 4

Tennis Ball transitions

In this section, we try to explore questions regarding transition amplitudes from generic initial quantum states to final states that have a classical interpretation. We take these final states to be coherent states whose parameters are such that the energy and momenta of these states are classical (i.e, without explicit factors of $\hbar$).
We first study these transition probabilities for the simple harmonic oscillator for a few source operators. In this case, the exact answer is easily determined. However, our **focus** will be on the validity of the saddle point approximation for these amplitudes.

Next, we will apply this idea to the anharmonic oscillator. In this case, we will take the final state to be a coherent state of the SHO - with the idea that the anharmonicity adiabatically turns off in the far future (mimicking the ideas that go into the S-matrix in the quantum field theory).

4.1 SHO

The transition amplitude for a source operator $\mathcal{O}$ to produce a coherent state $|\alpha\rangle$ can be written

$$\langle\alpha|e^{-iHT}\mathcal{O}e^{-iHT}|0\rangle \tag{4.1}$$

This amplitude can be calculated directly by the operator formalism for simple cases

4.1.1 Simple operator calculations

For $\mathcal{O} = e^{j\hat{x}}$

$$\langle\alpha|e^{-iHT}\mathcal{O}e^{-iHT}|0\rangle = e^{-\frac{|\alpha|^2}{2}}e^{\frac{j^2\hbar}{4mw}}e^{\alpha^* j\sqrt{\frac{\hbar}{2mw}}e^{-\iota\omega T}}e^{-\iota wT} \tag{4.2}$$

The probability from this amplitude can exceed unity if j is sufficiently large. So, this cannot be interpreted as a transition amplitude. While the initial and final states are normalized, the operator produces, at time $t = 0$, the state

$$\mathcal{O}e^{-iHT}|0\rangle \tag{4.3}$$

which is not normalized. This is the reason for the problem. So, we must normalize properly to get a probability interpretation.

We hope to avoid this problem by smearing the source operator in time. That is, we let the operator act for a time Δ - but keep $j \sim O(1)$. Thus, the source pumps in energy over a duration t_0, and we will explore the conditions on t_0 which will lead to a finite transition amplitude.

Thus, let us turn on the operator as a part of the Hamiltonian for a range of time i.e., take our Hamiltonian to be

$$H = \hbar\omega a^\dagger a + f(t)(a + a^\dagger) \tag{4.4}$$

where $f(t)$ is nonzero in a "small" region around $t = 0$.

This is not quite equivalent to the insertion of the operator $\mathcal{O}$ - but will probably lead to the same effect.

4.1.2 Path Integral methods

Our goal is to calculate this transition amplitude using the coherent state and path integrals following the method described by Khoze. The reason for examining the SHO calculation itself is to understand how well the saddle point approximation reproduces the exact answer - and, if possible, to understand the validity of the saddle point approximation.

Inserting various position eigenstates, we can write

$$\int dq_1 dq_2 dq_3 \langle \alpha e^{-i\omega T}|q_1\rangle\langle q_1|\mathcal{O}|q_2\rangle\langle q_2|e^{-iHT}|q_3\rangle\langle q_3|0\rangle \tag{4.5}$$

Using

$$\psi_{\alpha e^{-i\omega T}}(x) = \langle x|\alpha, t\rangle = \frac{1}{\sqrt{x_0}\pi^{1/4}} e^{-\frac{1}{2}\iota wt} e^{\sqrt{2}\alpha(t)\frac{x}{x_0} - \frac{x^2}{2x_0^2} - Re(\alpha(t))\alpha(t)} \tag{4.6}$$

$$= \frac{1}{\sqrt{x_0}\pi^{1/4}} e^{-\frac{1}{2}\iota wt} e^{\sqrt{2}|\alpha|e^{-\iota(wt-\sigma)}\frac{x}{x_0} - \frac{x^2}{2x_0^2} - |\alpha|^2 cos(wt-\sigma)e^{-\iota(wt-\sigma)}} \tag{4.7}$$

where

$$x_0 = \sqrt{\frac{\hbar}{mw}} \tag{4.8}$$

and the ground state wavefunction is

$$\langle q_3|0\rangle = \frac{1}{\sqrt{x_0}\pi^{1/4}} e^{-\frac{q_3^2}{2x_0^2}} \tag{4.9}$$

The final transition amplitude is written

$$\int dq_1\ dq_3\ \psi^*_\alpha(q_1)\ \mathcal{O}(q_1) \int_{x(-T)=q_3}^{x(0)=q_1} \mathcal{D}x\ e^{i/\hbar S[x]} e^{-\frac{m\omega}{2\hbar}q_3^2} \frac{1}{\sqrt{x_0}\pi^{1/4}} \tag{4.10}$$

Here, I have assumed that $\mathcal{O}$ is a *position* operator, and integrated over q_2. At this stage, we can combine the various exponential terms to get

$$\int dq_1\ dq_3\ \psi^*_{\alpha e^{-i\omega T}}(q_1)\ \mathcal{O}(q_1) \int_{x(-T)=q_3}^{x(0)=q_1} \mathcal{D}x\ e^{i/\hbar S[x] - \frac{m\omega}{2\hbar}q_3^2} \frac{1}{\sqrt{x_0}\pi^{1/4}} \tag{4.11}$$

We know the answer for the path integral of the SHO

$$\int_{x(-T)=q_3}^{x(0)=q_1} \mathcal{D}x\ e^{iS[x]} = \sqrt{\frac{m\omega}{2\pi i\hbar \sin\omega T}} e^{\frac{im\omega}{2\hbar\sin\omega T}((q_3^2+q_1^2)\cos\omega T - 2q_3q_1)} \tag{4.12}$$

As we can see the integral over q_1, q_3 is not complicated - basically Gaussian integrals. So, it can be done directly.
Proceeding equation (14) and using equation (9)

$$\begin{aligned}\int dq_1 dq_3 \tfrac{1}{\sqrt{x_0}\pi^{1/4}} \sqrt{\tfrac{m\omega}{2\pi i\hbar\sin\omega T}} e^{\frac{1}{2}\iota wt} e^{\sqrt{2}|\alpha|e^{\iota(wt-\sigma)}\frac{x}{x_0} - \frac{x^2}{2x_0^2} - |\alpha|^2 cos(wt-\sigma)e^{\iota(wt-\sigma)}} \\ \times e^{jq_1} e^{-\frac{q_3^2}{2x_0^2}} e^{\frac{im\omega}{2\hbar\sin\omega T}((q_3^2+q_1^2)\cos\omega T - 2q_3q_1)}\end{aligned} \tag{4.13}$$

Now comparing this with

$$\int d^N x e^{-\frac{1}{2}x^T Ax + b^T x} = \frac{(2\pi)^{N/2}}{(detA)^{1/2}} e^{\frac{1}{2}b^T A^{-1} b} \tag{4.14}$$

we will get

$$= \frac{\sqrt{\pi}x_0}{\sqrt{x_0}\pi^{1/4}e^{\iota\omega T/2}} e^{\frac{1}{2}\iota\omega T} e^{-|\alpha|^2[cos^2(\omega T-\sigma)+\frac{\iota}{2}sin(2\omega T-2\sigma)]} e^{\frac{x_0^2}{4}[j+\frac{1}{x_0}\sqrt{2}|\alpha|e^{\iota(\omega T-\sigma)}]^2} \tag{4.15}$$

after solving this we will get

$$\left(\frac{\hbar\pi}{m\omega}\right)^{1/4} e^{-|\alpha|^2/2} e^{\frac{j^2\hbar}{4m\omega}} e^{j\alpha^*\sqrt{\frac{\hbar}{2m\omega}}e^{\iota\omega T}} \tag{4.16}$$

We see that the path integral reproduces the same amplitude as it must.

4.1.3 Saddle Point Method

We will now collect the terms in the exponential
by using equation

$$W = \frac{1}{2}\iota\omega T + \frac{\sqrt{2}}{x_0}|\alpha|e^{\iota(\omega T-\sigma)}q_1 - \frac{q_1^2}{2x_0^2} - |\alpha|^2\cos(\omega T-\sigma)e^{\iota(\omega T-\sigma)} \tag{4.17}$$

$$+\log\mathcal{O}(q_1) + \frac{\iota}{\hbar}S[x] - \frac{q_3^2}{2x_0^2} \tag{4.18}$$

where we have not used the known answer for the path integral.

We will approximate the integral over q_1, q_3 by a saddle point value. To do that, we need to first collect the $\hbar$ factors - this is needed to estimate the contributions from quantum fluctuations - which will tell us the validity of the saddle point method.

We then vary the above W with respect to q_1, q_3 to obtain equations of motion. The solutions of this equation of motion will give us values for q_1, q_3 in terms of T, α and will depend on the operator $\mathcal{O}$.

The equations determining the saddle point are

$$\frac{\sqrt{2}}{x_0}|\alpha|e^{\iota(\omega T-\sigma)} - \frac{q_1}{x_0^2} + j + \frac{\iota}{\hbar}\frac{\delta S[x]}{\delta q_1} = 0 \tag{4.19}$$

$$\frac{i}{\hbar}\frac{\partial S[x]}{\partial q_3} - \frac{m\omega}{\hbar}q_3 = 0 \tag{4.20}$$

$$\ddot{\mathrm{x}} + \omega^2\mathrm{x} = 0 \tag{4.21}$$

The correct boundary condition for the last equation whose general solution is $x(t) = A\sin(\omega t+\phi)$, are also obtained from the stationarity condition determining the saddle point

$$x(t) = A\sin(\omega t+\phi) \tag{4.22}$$

$$x(0) = q_1 \qquad x(-T) = q_3 \tag{4.23}$$

which will fix A and ϕ. These turn out to be

$$\tan\phi = \frac{\sin\omega T}{\cos\omega T - \frac{q_3}{q_1}} \tag{4.24}$$

$$A = \frac{\sqrt{q_1^2+q_3^2-2q_1q_3\cos\omega T}}{\sin\omega T} \tag{4.25}$$

Using this, we determine the values of $p_{1,3}$

$$p_1, -p_3 = m\omega\frac{q_{1,3}\cos\omega T - q_{3,1}}{\sin\omega T} \tag{4.26}$$

Substituting these, we find that

$$q_3 = q_1 e^{-i\omega T} \qquad q_1 = \frac{(jx_0^2+\sqrt{2}x_0|\alpha|e^{i(\omega T-\sigma)})}{2} \tag{4.27}$$

Note that the solution for q_3 is not real if q_1 is real. Using the solution, we can obtain the saddle point approximation to the transition amplitude.

The on-shell action for the SHO part in W is

$$S = \frac{1}{4}m\omega A^2\left(\sin(2\phi) - \sin\left(-2\omega T+2\phi\right)\right) = \frac{m\omega}{2sin\omega T}\left((q_1^2+q_3^2)cos\omega T - 2q_1q_3\right) \tag{4.28}$$

If we substitute for q_3, we get

$$S = -m\omega q_1^2\sin\omega T e^{-\iota\omega T} \tag{4.29}$$

and finally get

$$W = \frac{1}{2}\iota\omega T - \frac{|\alpha|^2}{2} + \frac{j^2\hbar}{4m\omega} + \alpha^* j\sqrt{\frac{\hbar}{2m\omega}}e^{\iota\omega T} \tag{4.30}$$

Therefore, the transition amplitude is

$$A = e^W = e^{\frac{1}{2}\iota\omega T - \frac{|\alpha|^2}{2} + \frac{j^2\hbar}{4m\omega} + \alpha^* j\sqrt{\frac{\hbar}{2m\omega}}e^{\iota\omega T}} \tag{4.31}$$

compare this equation with the amplitude calculated using the operator methods,

$$\langle\alpha|e^{-iHT}\mathcal{O}e^{-iHT}|0\rangle = e^{-\frac{|\alpha|^2}{2}}e^{\frac{j^2\hbar}{4mw}}e^{\alpha^* j\sqrt{\frac{\hbar}{2mw}}e^{-\iota\omega T}}e^{-\iota wT} \tag{4.32}$$

Note: We are getting exact answer except the sign in the exponential term. Which we need to fix.

It is a little surprising that the saddle point gives the correct answer, since we have not computed the quantum fluctuation contribution. Therefore, the quantum fluctuation must be absent in this case. It remains to be checked that this is indeed the case.

4.1.4 Other Choice of Operators

We can repeat this with other choices for the operator $\mathcal{O}$ and a couple of obvious choices are Take $\mathcal{O} = e^{j\hat{x}^2}$ and $\mathcal{O} = e^{-j\hat{p}}$.

Note:

For the latter $\mathcal{O} = e^{j\hat{x}^2}$,

$$q_1 = \frac{|\alpha| x_0 e^{\iota(\omega t - \sigma)}}{\sqrt{2}[1 - jx_o^2]} \tag{4.33}$$

and the final expression for W which determines the transition amplitude is

$$W = \frac{1}{2}\iota\omega T + \frac{|\alpha|^2 e^{2\iota(\omega t-\sigma)}}{2[1 - jx_0^2]} - |\alpha|^2 cos(\omega t - \sigma) e^{\iota(\omega T - \sigma)} \tag{4.34}$$

The exact answer (obtained by using wavefunction methods) also involves a quantum fluctuation prefactor $\sqrt{1 - jx_0^2}$ - and so after combining terms, we will get

$$\frac{1}{\sqrt{1 - jx_0^2}} e^{\frac{1}{2}\iota\omega T + \frac{|\alpha|^2 e^{2\iota(\omega t-\sigma)}}{2[1-jx_0^2]} - |\alpha|^2 cos(\omega t - \sigma) e^{\iota(\omega T - \sigma)}} \tag{4.35}$$

and thus in the limit we get a $\delta-$function. There are two unresolved puzzles here - firstly, the physical reason for the restriction $1 > jx_0^2$ is not clear - even in the operator formalism. And secondly, we need to understand the quantum fluctuations in the saddle point approximation which will contribute the sqrt prefactor.

For $\mathcal{O} = e^{-j\hat{p}}$,
from equation (4.5)

$$\int dq_1 dq_2 dq_3 dp_2 \langle \alpha e^{-i\omega T} | q_1 \rangle \langle q_1 | \mathcal{O} | p_2 \rangle \langle p_2 | q_2 \rangle \langle q_2 | e^{-iHT} | q_3 \rangle \langle q_3 | 0 \rangle \tag{4.36}$$

This can be further solved to,

$$\begin{gathered} \int dq_1 \ dq_2 \ dq_3 \ \psi^*_{\alpha e^{\iota\omega t}}(q_1) \left[\int dp_2 \tfrac{1}{2\pi} e^{-\iota(q_1 - q_2) \cdot p_2} \mathcal{O}(p_2) \right] \\ \int_{x(-T)=q_3}^{x(0)=q_1} \mathcal{D}x \ e^{iS[x]} e^{-\frac{m\omega}{2\hbar} q_3^2} \frac{1}{\sqrt{x_0}\pi^{1/4}} \end{gathered} \tag{4.37}$$

for $\mathcal{O} = e^{-jp}$ last equation can be solved, and we get,

$$\int dq_1 \ dq_3 \ \psi^*_{\alpha e^{\iota\omega t}}(q_1) \int_{x(-T)=q_3}^{x(0)=q_1 - \iota j} \mathcal{D}x \ e^{iS[x]} e^{-\frac{m\omega}{2\hbar} q_3^2} \frac{1}{\sqrt{x_0}\pi^{1/4}} \tag{4.38}$$

4.1.5 Modified Hamiltonian

As noted in the previous section, the operator needs to be large for a finite transition probability to a classical state. While this may not be a problem in the calculations for the SHO and possibly other systems, injection of a large amount of energy will typically lead to exciting the anharmonic modes which are always present in any physical system. This will of course modify the Hamiltonian drastically and hence make all the above calculations moot.

Thus, we try to excite the system in a different manner by addition a perturbation to the Hamiltonian which is turned on for a large time t_0 in quantum units (or a finite classical amount of time). In the case of the field theory, we could also smear the operator over position. We hope that this smearing of the operators over classical ranges obviate the difficulties arising from the $\hbar$ factors in a different way.

Thus, we take our Hamiltonian to be

$$H = \hbar\omega a^\dagger a + f(t)(a + a^\dagger) \quad or \quad H = \hbar\omega a^\dagger a + \iota f(t)(a - a^\dagger) \tag{4.39}$$

where $f(t)$ is nonzero in a "small" region around $t = 0$ i.e.,

$$f(t) = 1 \quad for \quad -t_0 < t < t_0, \qquad f(t) = 0 \text{ otherwise.} \tag{4.40}$$

The question we will focus on is how does the transition amplitude to a final coherent state depend on t_0.

We will approximate the answer using the saddle point approximation.

4.1.6 Special Case

In the particular case of the harmonic oscillator, there is a simpler way to get at the answer.

$$\int dq_1 dq_2 \langle \alpha e^{-i\omega T} | q_1 \rangle \langle q_1 | \mathcal{O} | q_2 \rangle \langle q_2 | e^{-iHT} | 0 \rangle \tag{4.41}$$

which is easily evaluated to be

$$\int dq_1 \psi_\alpha(q_1) \mathcal{O}(q_1) e^{-i\omega T/2} e^{-\frac{m\omega}{2\hbar} q_1^2} \frac{1}{\sqrt{x_0}\pi^{1/4}} \tag{4.42}$$

which is obtained without using the path integral. This is not possible in the more general situation involving anharmonic oscillators etc, so we did not try this route earlier.

All the integrals above can be done, and we should get the same answer as we got from the Hamiltonian formulation.

Chapter 5

Appendix

In the appendix, we detail a few calculations and also collect various classical solutions of the ϕ^4 theory. We also discuss a few of the properties of these classical solutions, which we hope will be relevant in the quantum field theory questions to be addressed in the future.

5.1 Appendix-1

We can write (19) as

$$\psi_\alpha(x,t) = e^{\frac{-\iota wt}{2}} e^{-\frac{|\alpha|^2}{2}} e^{\left[\frac{\alpha e^{-\iota wt}}{\sqrt{2}x_0}(x - x_0^2 \frac{\partial}{\partial x})\right]} \frac{1}{\pi^{1/4}\sqrt{x_0}} e^{-\frac{1}{2}(\frac{x}{x_0})^2} \tag{5.1}$$

writing

$$D = \frac{\alpha e^{-\iota wt}}{\sqrt{2}x_0}(x - x_0^2 \frac{\partial}{\partial x}) \tag{5.2}$$

$$De^{-\frac{1}{2}(\frac{x}{x_0})^2} = \frac{\sqrt{2}\alpha e^{-\iota wt}}{x_0}(xe^{-\frac{1}{2}(\frac{x}{x_0})^2}) \tag{5.3}$$

$$D^2 e^{-\frac{1}{2}(\frac{x}{x_0})^2} = \frac{1}{2!}\frac{\alpha^2 e^{-2\iota wt}}{x_0^2}[2x^2 - x_o^2](e^{-\frac{1}{2}(\frac{x}{x_0})^2}) \tag{5.4}$$

Now combining all the terms, we will get

$$\left[1 + \frac{\sqrt{2}\alpha e^{-\iota wt}}{x_0}x - \frac{\alpha^2 e^{-2\iota wt}}{2} + \frac{1}{2!}\frac{2\alpha^2 e^{-2\iota wt}}{x_0^2}x^2 + ...\right] e^{-\frac{1}{2}(\frac{x}{x_0})^2} \tag{5.5}$$

Hence we finally arrive at equation (19).

5.2 Appendix 2: Classical solutions

In this appendix, we collect various classical solutions of ϕ^4 type theories.

5.2.1 Frasca's Solution

consider the equation [2]

$$-\Box\phi + \mu_0^2\phi + \lambda\phi^3 = 0 \tag{5.6}$$

where, $\Box = -\partial_t^2 + \Delta^2$, μ_0 the mass of the field and λ the coupling.
The solution is given by

$$\phi(x) = \pm\sqrt{\frac{2\mu^2}{\mu_0^2 + \sqrt{\mu_0^2 + 2\lambda\mu^2}}} sn\Big(p.x + \theta, \sqrt{\frac{-\mu_0^2 + \sqrt{\mu_0^2 + 2\lambda\mu^2}}{-\mu_0^2 - \sqrt{\mu_0^2 + 2\lambda\mu^2}}}\Big) \tag{5.7}$$

$$p^2 = \mu_0^2 + \frac{\lambda\mu^4}{\mu_0^2 + \sqrt{\mu_0^4 + 2\lambda\mu^4}} \tag{5.8}$$

Let us verify this
Let the solution can be written as

$$\phi(x) = Asn(Et - \vec{p}.\vec{x} + \theta, k) \tag{5.9}$$

$$A = \pm\sqrt{\frac{2\mu^2}{\mu_0^2 + \sqrt{\mu_0^2 + 2\lambda\mu^2}}} \tag{5.10}$$

$$k = \sqrt{\frac{-\mu_0^2 + \sqrt{\mu_0^2 + 2\lambda\mu^2}}{-\mu_0^2 - \sqrt{\mu_0^2 + 2\lambda\mu^2}}} \tag{5.11}$$

$$\partial_t\phi = A.cn.dn.E \tag{5.12}$$

$$\partial_t^2\phi = AE^2[-k^2cn^2sn - sndn^2] \tag{5.13}$$

$$\partial_x\phi = Acndn(-\vec{p}) \tag{5.14}$$

$$\partial_x^2\phi = A(\vec{p})^2[-k^2cn^2sn - sndn^2] \tag{5.15}$$

$$\tag{5.16}$$

Now, form equation (58)

$$p^2[-K^2 - 1] + \mu_0^2 + [2p^2k^2 + \lambda A^2]sn^2 = 0 \tag{5.17}$$

So we get

$$p^2 = -\frac{\lambda A^2}{2k^2} \tag{5.18}$$

$$1 + k^2 = \frac{\mu_0^2}{p^2} \tag{5.19}$$

NOTE= equation (70) and (71) does not satisfy relation (60).

5.2.2 Khoze Solution

consider a real scalar field $\phi(x)$ in (d + 1)-dimensional space-time

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)^2 - \frac{1}{2}m^2\phi^2 - \mathcal{L}_{int} \tag{5.20}$$

where $\mathcal{L}_{int}$ is the interaction term.The two simplest examples are the $\phi^4(x)$ model in the unbroken phase, with $\mathcal{L}_{int} = \frac{\lambda}{4}\phi^4$, and the model with
the **spontaneously broken Z_2 symmetry**

$$\mathcal{L} = \frac{1}{2}\partial^\mu h\partial_\mu h - \frac{\lambda}{4}(h^2 - v^2)^2 \tag{5.21}$$

The classical equation for the model (73) is the familiar Euler-Lagrange equation

$$\mathcal{L} = \partial^\mu \partial_\mu h + \frac{\lambda}{4} h(h^2 - v^2) = 0 \tag{5.22}$$

The classical solution, which provides the generating function of tree-level amplitudes on multi-particle mass thresholds

$$h_0(z_0; t) = v\Big(\frac{1 + z_0 \frac{e^{imt}}{2v}}{1 - z_0 \frac{e^{imt}}{2v}}\Big) \tag{5.23}$$

$$m = \sqrt{2\lambda} v \tag{5.24}$$

where z_0 is an auxiliary variable. It is easy to check by direct substitution that the expression in (75) satisfies the the time-dependent ODE,

$$\partial_t^2 h + \frac{\lambda}{4} h(h^2 - v^2) = 0 \tag{5.25}$$

for any value of the parameter z_0. Let us verify this solution

$$\partial_t h = z_0 im \frac{e^{imt}}{[1 - \frac{z-0e^{imt}}{2v}]^2} \tag{5.26}$$

$$\partial_{t^2} h = z_0(-m^2)e^{imt} \frac{1 + \frac{z_0 e^{imt}}{2v}}{[1 - \frac{z_0 e^{imt}}{2v}]^3} \tag{5.27}$$

puting these two relation in equation (77). we will find that it safifies equation (77) **If and only if**

$$m = \sqrt{\frac{\lambda}{2}} v \tag{5.28}$$

5.2.3 Castell's EOM

Frasca's solution for massless scalar field [1]

$$\phi(x) = \pm \mu \Big(\frac{2}{\lambda}\Big)^4 sn(p.x + \theta, \iota) \tag{5.29}$$

and Castell's EOM is

$$\frac{\partial^2 \phi}{\partial x^2} - \frac{\partial^2 \phi}{\partial t^2} - \lambda \phi^* \phi \phi = 0 \tag{5.30}$$

So we get,

$$\lambda \phi^* \phi \phi = \lambda \Big[\mu \Big(\frac{2}{\lambda}\Big)^{1/4}\Big]^3 sn(p.x + \theta, -\iota) sn^2(p.x + \theta, \iota) \tag{5.31}$$

$$\partial_x \phi = \mu \Big(\frac{2}{\lambda}\Big)^{1/4} cn.dn.(\vec{p}) \tag{5.32}$$

$$\partial_x^2 \phi = \mu \Big(\frac{2}{\lambda}\Big)^{1/4} (|\vec{p}|^2) sn[cn^2 - dn^2] \tag{5.33}$$

$$\partial_t^2 \phi = E^2 \mu \Big(\frac{2}{\lambda}\Big)^{1/4} sn[cn^2 - dn^2] \tag{5.34}$$

After putting these relations in equation (82) and using the relation (70) and (71). We will find that, it satisfies the solution **If**

$$sn = sn^* \tag{5.35}$$

5.3 Appendix 3: Scalar field as Oscillators

We describe how a scalar field can be viewed as a superposition of oscillators. For real scalar field

$$\mathcal{L} = \frac{1}{2}\dot{\phi}^2 - \frac{1}{2}(\nabla)^2 - \frac{1}{2}m^2\phi^2 \tag{5.36}$$

$$\Pi = \dot{\phi} \tag{5.37}$$

$$H = \int d^3x\Big[\frac{1}{2}\Pi^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2\Big] \tag{5.38}$$

$$\phi(\vec{x},t) = \frac{1}{(2\pi)^3}\int d^3p e^{\iota\vec{p}.\vec{x}}\phi(\vec{p},t) \tag{5.39}$$

therefore

$$H = \int d^3p\frac{1}{2}\Big[\frac{d\phi(\vec{p},t)}{dt}\frac{d\phi(\vec{-p},t)}{dt} + p^2\phi(\vec{p},t)\phi(\vec{-p},t) + m^2\phi(\vec{p},t)\phi(\vec{-p},t)\Big] \tag{5.40}$$

Since $\phi(\vec{x},t)$ is real scalar field

$$\phi^*(x) = \phi(x) \tag{5.41}$$

Now, from eq.(31)

$$\frac{1}{(2\pi)^3}\int d^3p e^{-\iota\vec{p}.\vec{x}}\phi(\vec{p},t) = \frac{1}{(2\pi)^3}\int d^3p e^{\iota\vec{-p}.\vec{x}}\phi(\vec{-p},t) \tag{5.42}$$

implies

$$\phi(\vec{p},t) = \phi(\vec{-p},t) \tag{5.43}$$

therefore

$$H = \int d^3p\frac{1}{2}\Big[\Big(\frac{d\phi}{dt}\Big)^2 + p^2\phi^2 + m^2\phi^2\Big] \tag{5.44}$$

$$H = \int d^3p\frac{1}{2}\Big[\Big(\frac{d\phi}{dt}\Big)^2 + w_p^2\phi^2\Big] \tag{5.45}$$

$$w_p^2 = p^2 + m^2 \tag{5.46}$$

eq.(37) shows that scalar field is sum of infinite number of LHO.

5.4 Appendix 4: Jacobi Elliptic functions

They are defined as

$$t = \int_0^\phi \frac{d\phi}{\sqrt{1-k^2\sin^2\phi}}, (k^2 < 1) \tag{5.47}$$

The parameter k is known as the **modulus of the elliptic integral**.
The various **Jacobian elliptic functions** are defined as

$$sn(t,k) = sin\phi \tag{5.48}$$

$$cn(t,k) = \cos\phi \tag{5.49}$$

$$dn(t,k) = \sqrt{1 - k^2\sin^2\phi} \tag{5.50}$$

Some of the important relationships between the Jacobian elliptic functions

$$sn^2 + cn^2 = 1 \tag{5.51}$$

$$dn^2 - k^2cn^2 = 1 - k^2 \tag{5.52}$$

$$k^2sn^2 + dn^2 = 1 \tag{5.53}$$

$$\frac{d}{dt}sn(t,k) = cn.dn \tag{5.54}$$

$$\frac{d}{dt}cn = -sn.dn \tag{5.55}$$

$$\frac{d}{dt}dn = -k^2sn.cn \tag{5.56}$$

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