Geometric Measure Theory

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Certificate of Examination

This is to certify that the dissertation titled "**Geometric Measure Theory**" submitted by **Mr. Satpute Ganesh Ashok** (Reg. number – MP17014) for the partial fulfillment of the M.S. degree program of the Institute, has been examined by the thesis committee duly appointed by the Institute. The committee finds the work done by the candidate satisfactory and recommends that the report be accepted.

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Declaration of Authorship

The work presented in this dissertation has been carried out by me under the guidance of Dr. Lingaraj Sahu at the Indian Institute of Science Education and Research Mohali.

This work has not been submitted in part or in full for a degree, a diploma, or a fellowship to any other university or institute. Whenever contributions of others are involved, every effort is made to indicate this clearly, with due acknowledgement of collaborative research and discussions. This thesis is a bonafide record of work done by me and all sources listed within have been detailed in the bibliography.

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In my capacity as the supervisor of the candidate's project work, I certify that the above statements by the candidate are true to the best of my knowledge.

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Contents

Abstract

The aim of this thesis is to study the Geometric measure theory. First part of this thesis mainly focuses on the Hausdorff measure and Hausdorff dimensions. In subsequent chapters Haar measure, covering theorems and the Area-Corea formulas and applications are discussed.

Chapter 1

Hausdorff Measure and Hausdorff Dimension

In this chapter, we shall study a Hausdorff measure on the real line and the Hausdorff dimension. This chapter is primarily taken from the [1].

1.1 Hausdorff measure on $\mathbb{R}$

Definition 1.1. A and B be the nonempty bounded subsets of $\mathbb{R}^N$. Define

$$\begin{aligned} HD\,(A, B) &= max\ \{\ sup_{a \in A}\ d(a, B),\ sup_{b \in B}\ d(A, b)\ \} \\ &= \sup_{x \in \mathbb{R}^N}\ |dist(x, A) - dist(x, B)| \end{aligned}$$

This function is known as Hausdorff Distance.

Definition 1.2 (Hausdorff measure function h)**.** It is monotone increasing function on $[0, \infty)$, $h(x) > 0$ for $x > 0$ and $\lim_{x \to 0^+} h(t) = h(0) = 0$.

Consider special case for $h(x) = x^p$. For this case we will construct Hausdorff measure and study its properties.

1.1.1 Definitions

Definition 1.3. The *approximating measure* $H^*_{p,\delta}$ of the set A is given by,

$$H^*_{p,\delta}\,(A) = \inf\left\{\sum l(I_k)^p : A \subset \bigcup I_k,\ \text{and } l(I_k) \leq \delta \text{ For each } I_k\right\}$$

.

Definition 1.4. The *Hausdorff outer measure* H^*_p is given by

$$H^*_p(A) = \lim_{\delta \to 0} H^*_{p,\delta}(A).$$

As δ decreases $H^*_{p,\delta}$ increases, therefore this limit must exist in the extended real valued system.

1.1.2 Properties of Hausdorff measure

In this section we will prove some properties of the Hausdorff measure.

Theorem 1.5. *Let $A, B \subset \mathbb{R}$ then*
(i) H^*_p *is non-negative i.e.* $H^*_p(A) \geq 0.$
(ii) *For an empty set* $\phi,\quad H^*_p(\phi) = 0.$
(iii) *If $A \subset B$ then* $H^*_p(A) \leq H^*_p(B)$
(iv) *For any $x \in \mathbb{R}$* $H^*_p(\{x\}) = 0$.

Proof. (i), (ii) and (iii) follows from the definition 1.4.
To prove (iv), for each $\delta > 0$, $\{x\} \subset \left(x - \frac{\delta}{2}, x + \frac{\delta}{2}\right)$. Therefore

$$0 \leq H^*_p(A) \leq (\delta)^p$$

since δ can be any positive number, $H^*_p(A) = 0$. ∎

We have defined outer measure which is different from lebesgue measure. Now the basic question arises, Do all open sets are are H^*-measurable? i.e. Is it Borel measure 1.6?

Definition 1.6 (Borel Measure)**.** Let X be the topological space. The measure μ on X is called the Borel measure if every open set is μ-measurable.

Following theorem gives characterization of Borel measure on the metric spaces.

Theorem 1.7 (Caratheodory's Criteria)**.** *, Let X be the metric space and μ be a measure on X, μ is Borel measurable if and only if*

$$\mu(A) + \mu(B) \leq \mu(A \cup B) \tag{1.1.1}$$

whenever $A, B \subset X$ and $0 < d(A, B)$.

Proof. Suppose μ is Borel measurable that is every open set in X is μ-measurable. Let $A, B \subset X$ with $d(A, B) = d > 0$ Then define V,

$$V = \bigcup_{a \in A} \left\{ x \in X : d(x, a) < \frac{d}{2} \right\}$$

$$i.e. \quad V = \left\{ \ x \in X : d(x, A) < \frac{d}{2} \right\}.$$

Because V is an open set, V is measurable. Therefore

$$\mu(A \cup B) = \mu((A \cup B) \cap V) + \mu((A \cup B) - V) = \mu(A) + \mu(B).$$

So the equation 1.1.1 holds.

Now assume conversely, Suppose equation 1.1.1 holds. We will prove that any close set is measurable. Let V be a closed subset of X. We need to prove that for any set $A \subset X$,

$$\mu(A) \geq \mu(A \cap V) + \mu(A \setminus V).$$

Define set $V_i = \{x \in X : d(x, V) \leq \frac{1}{i}\}$.

Now for $A \cap V$ and $A \setminus V_j$, $d((A \cap V), (A \setminus V_j)) > 0$, therefore

$$\mu(A) \geq \mu((A \cap V) \ \cup \ (A \setminus V_j)) = \mu(A \cap V) + \mu(A \setminus V_j).$$

We will show that $\mu(A \setminus V) \leq \lim_{j \to \infty} \mu(A \setminus V_j)$. Define $E_j = V_j \setminus V_{j+1}$. Therefore for every j, $A \setminus V = (A \setminus V_j) \cup \left(\bigcup_{k=j}^{\infty} A \cap E_k \right)$. Here we are using assumption V is closed. Subadditivity of μ will give us,

$$\mu\left(A \setminus V\right) \leq \mu\left(A \setminus V_j\right) + \sum_{k=1}^{\infty} \mu(A \setminus E_k).$$

Now for $|j-k| \geq 2$, we have $d(E_j, E_k) > 0$. Therefore for every N,

$$\begin{aligned}\sum_{k=1}^{N} \mu\left(A \cap E_k\right) &\leq \sum_{k=1}^{N} \mu(A \cap E_{2k}) + \sum_{k=1}^{N} \mu\left(A \cap E_{2k-1}\right) \\ &\leq \mu\left(\bigcup_{k=1}^{N}(A \cap E_{2k})\right) + \mu\left(\bigcup_{k=1}^{N}(A \cap E_{2k-1}\right) \\ &\leq 2\mu(A) < \infty.\end{aligned}$$

Therefore

$$\sum_{k=j}^{\infty} \mu(A \cap E_k) \leq \lim_{N \to \infty} \sum_{k=1}^{N} \mu(A \cap E_k) < \infty.$$

$$\text{Implies } \mu(A \setminus V) \leq \lim_{j \to \infty} \mu(A \setminus V_j).$$

■

Theorem 1.8. *Let I be an interval $(-\infty, a]$, then I is H_p^*- measurable.*

Proof. We will show that for any set A, $H_p^*(A) \geq H_p^*(A \cap I) + H_p^*(A \setminus I)$. If $H_p^*(A) = \infty$ then there is nothing to prove. Therefore assume $H_p^*(A) < \infty$.

Let $A_n = A \cap [a + \frac{1}{n}, \infty)$ Then $A_n \subset A_{n+1}$ and $\bigcup_{n=1}^{\infty} A_n = A \setminus I$. Also $\lim_{n \to \infty} H_p^*(A_n)$ exists and is finite. By theorem 1.7 we have $H_p^*(A) \geq H_P^*(A \cap I) + H_p^*(A_n)$.

Let $D_n = A_{n+1} \setminus A_n$, Then

$$\begin{aligned}A \setminus I &= A_{2n} \cup \bigcup_{k=2n}^{\infty} D_k \\ &= A_{2n} \cup \bigcup_{k=n}^{\infty} D_{2k} \bigcup_{k=n}^{\infty} \cup D_{2k+1}.\end{aligned}$$

Therefore $$H_p^*(A \setminus I) \leq H_p^*(A_{2n}) + \sum_{k=n}^{\infty} H_p^*(D_{2k}) + \sum_{k=n}^{\infty} H_p^*(D_{2k+1}). \tag{1.1.2}$$

All outer measures in the above equations are finite. Suppose both series in the above equation 1.1.2 converge.

Then letting $n \to \infty$, we get $H_p^*(A \setminus I) \leq \lim H_p^*(A_{2n}) = \lim H_p^*(A_n)$. But $A_n \subset A \setminus I$, Therefore

$$\begin{aligned} & H_p^*(A_n) = H_p^*(A \setminus I) \\ \text{implies} \quad & H_p^*(A) \geq H_p^*(A \cap I) + H_p^*(A \setminus I) \end{aligned}$$

We have assume that both series in equation 1.1.2 are convergent. Suppose the first series is divergent. Then we have $\bigcup_{k=1}^{n-1} D_{2k} \subset A_{2n}$. Also $d(D_{2k}, D_{2k+2}) > 0$. So from theorem 1.7,

$$H_p^*(A - 2n) \geq \sum_{k=1}^{N-1} H_p^*(D_{2k}) \to \infty.$$

. But $H_p^*(A)$ is finite. Therefore series must me convergent. Similarly for the second series in the equation 1.1.2.

■

Corollary 1.9. *All Borel sets are H_P^* measurable.*

Theorem 1.10. *Hausdorff outer measure is translation invariant, that is*

$$H_p^*(A + x) = H_p^*(A).$$

Proof. Let $\{I_n\}$ be the covering for the set A with length of each interval is at most δ, then $\{I_n + x\}$ will be covering for the set $A + x$, Also for each I_n, $l(I_n) = l(I_n + x)$

$$\begin{aligned} \therefore \quad & \sum l(I_n)^p = \sum l(I_n + x)^p \\ \implies \inf & \sum l(I_n)^p = \inf \sum l(I_n + x)^p \\ \implies & H_{p,\delta}(A) = H_{p,\delta}(A + x) \end{aligned}$$

Taking limit $\delta \to 0$, gives $H_p^*(A) = H_p^*(A + x)$. ■

Theorem 1.11. *For $k \in \mathbb{R}$, $k > 0$*

$$H_p^*(kA) = k^p H_p^*(A).$$

Proof. For cover I_n of A, kI_n will be cover of the the set kA with length at most $k\delta$. Also $\sum l(kA)^p = k^p \sum l(I_n)^p$, Now if we take infimum over all such covers, we will get, $H^*_{p,k\delta}(A) = k^p H^*_{p,\delta}(A)$, letting $\delta \to 0$ gives the result. ■

Theorem 1.12. *$H^*_p(A)$ is the same whether we stipulate that the intervals I_k in the Definition 1.4 are open closed or half-open.*

Proof. We will prove this for closed intervals, other cases can be done in similarly. For fixed p, let $H^{*^c}_\delta(A)$ and $H^{*^c}_\delta(A)$ be the corresponding set functions. Let I' denotes closure of the open interval I. Then,

$$\begin{aligned}
H^*_{p,\delta}(A) &= \inf \sum l(I_k)^p, \quad l(I_k) \le \delta,\ A \subset \cup I_k \\
&= \inf \sum l(I'_k)^p \\
&\ge \inf \sum l(J_k)^p, \quad l(J_k) \le \delta,\ A \subset \cup J_k,\ J_k \text{ closed interval} \\
&\ge H^{*^c}_\delta(A).
\end{aligned}$$

Now we have $H^*_p(A) \ge H^{*^c}(A)$. Now we will prove opposite inequality. let J be an interval of length $\epsilon > 0$, then we can find open interval J'' of length $\epsilon(1+\delta)$. Then

$$\begin{aligned}
H^{*^c}_\delta(A) &= \inf \sum l(J_k)^p\ , \quad l(J_k) \le \delta,\ J_k \text{ closed},\ A \subset \cup J_k \\
&= (1+\delta)^{-p} \inf \sum l(J''_k)^p \\
&\ge (1+\delta)^{-p} \inf \sum l(I_k)^p\ , \quad l(I_k) \le \delta + \delta^2,\ I_k \text{ open}\ ,\ A \subset \cup I_k, \\
&= (1+\delta)^{-p} H_{p,\delta+\delta^2}(A).
\end{aligned}$$

$\delta \to 0$ will give $H^{*^c}(A) \ge (A)$.

■

Theorem 1.13. *Let $\{E_i\}_{i\in\mathbb{N}} \subset \mathbb{R}$ then*

$$H^*_p\left(\bigcup_{i=1}^{\infty} E_i\right) \le \sum_{i=1}^{\infty} H^*_p(E_i).$$

Proof. For each E_i, given $\epsilon > 0$, we can find family of intervals $I_{i,j}$ with $l(I_{i,j}) < \delta$ such that $E \subset \bigcup_{j=1}^{\infty} I_{i,j}$ and

$$H^*_{p,\delta}(E_i) \geq \sum l(I_{i,j})^p - \frac{\epsilon}{2^i}.$$

Now, $\bigcup_{i=1}^{\infty} E_i \subset \bigcup_{i=1}^{\infty} \bigcup_{j=1}^{\infty} I_{i,j}$, therefore

$$\begin{aligned} H^*_{p,\delta}\left(\bigcup_{i=1}^{\infty}\right) &\leq \sum_{i,j} l(I_{i,j})^p \\ &\leq \sum H^*_{p,\delta}(E_i) + \epsilon \\ &\leq \sum H^*_p(E_i) + \epsilon\,, \quad \text{for all } \delta > 0. \end{aligned}$$

Now result follows. ■

1.2 Hausdorff measure on Metric space

Now we will define *Hausdorff measure* on a general metric space (X, d) analogous to the definition 1.4.

Definition 1.14 (Diameter)**.** Let (X, d) be metric space and $A \subset X$, then diameter of A is given by,

$$diam(A) = \sup\{d(x, y) : x, y \in A\}.$$

Definition 1.15 (approximating measure on X)**.** For set $A \subset X$ approximating measure is given by,

$$H^*_{p,\delta}(A) = \inf\left\{\sum diam(C_i)^p : A \subset \bigcup U_i,\ \text{and } diam(C_i) \leq \delta \text{ For each } I_k\right\}.$$

Definition 1.16 (Hausdorff outer measure on X)**.** For $A \subset X$,

$$H^*_p(A) = \lim_{\delta \to 0} H_{p,\delta}(A).$$

1.3 Hausdorff measure on $\mathbb{R}^n$

We will define *Hausdorff measure on* $\mathbb{R}^n$ analogous to the definition 1.3 and 1.4. Let $A \subset \mathbb{R}^n,\ 0 \leq p < \infty,\ 0 < \delta < \infty$, Define

$$H^*_{p,\delta}(A) = \inf\left\{\sum_{i=1}^{\infty}(diam(C_i))^p :\ A \subset \bigcup_{i=1}^{\infty} C_i,\ diam(C_i) \leq \delta\right\}.$$

Now for A and p above define $p-$*dimensional Hausdorff measure*,

$$H^*_p(A) = \lim_{\delta\to 0} H_{p,\delta}(A) = \sup_{\delta>0} H_{p,\delta}(A).$$

Here we have used $h(x) = x^p$ as our *Hausdorff measure function.* Suppose we change our *Hausdorff measure function* to $h(x) = \alpha(p)\left(\frac{x}{2}\right)^p$ Then we can define measure as below,

$$H^*_{p,\delta}(A) = \inf\left\{\sum_{i=1}^{\infty}\alpha(p)\left(\frac{diam(C_i)}{2}\right)^p :\ A \subset \bigcup_{i=1}^{\infty} C_i,\ diam(C_i) \leq \delta\right\}$$

where

$$\alpha(p) = \frac{\pi^{\frac{p}{2}}}{\Gamma\left(\frac{p}{2}+1\right)}.$$

here $\Gamma(p) = \int_0^\infty e^{-x}x^{p-1}\,dx, \quad 0 < p < \infty$, is the usual *gamma function.*

This is our *approximating measure.*

Now for A and p above define *pdimensional Hausdorff measure*,

$$H_p(A) = \lim_{\delta\to 0} H_{p,\delta}(A) = \sup_{\delta>0} H_{p,\delta}(A).$$

1.4 Hausdorff measure and Lebesgue measure

Theorem 1.17. *For $p = 1$ Hausdorff outer measure (1.4) and Lebesgue outer measure are same, that is*

$$H^*_1(A) = m^*(A).$$

Proof. From the definition of the *Lebesgue measure* and the *Hausdorff measure*, we know that, $m^*(A) \leq H_1^*(A)$. Now If $m^*(A) = \infty$ then other inequality is trivial. Hence we are done. So assume that $m^*(A) < \infty$. We want to show that $H_1^*(A) \leq m^*(A)$, For given $\epsilon > 0$, we can find family of intervals $\{I_n\}_{n=1}^{\infty}$ such that $A \subset \bigcup_{i=1}^{\infty} I_n$ and $m^*(A) \geq \sum_{n=1}^{\infty} l(I_n) - \epsilon$.

Now if we let I to be finite interval then, For given $\epsilon' > 0$, we can choose an open intervals $J_i, i = 1, \ldots, m$ with $l(J_i) \leq \delta$, $I \subset \bigcup_{i=1}^{m} J_i$ and $\sum_{i=1}^{m} J_i < l(I) + \epsilon'$. Therefore $H_{1,\delta}^*(I) < l(I) + \epsilon'$ for all $\delta > 0$, and letting $\delta \to 0$ will give us $l(I) \geq H_1^*(I) - \epsilon'$.

now put $I = I_n$ and $\epsilon' = \frac{\epsilon}{2^n}$

$$m^*(A) \geq \sum_{n=1}^{\infty} l(I)_n - \epsilon$$

$$\begin{aligned}
\Longrightarrow \quad m^*(A) &\geq \sum_{n=1}^{\infty} H_1^*(I_n) - \sum_{n=11}^{\infty} \frac{\epsilon}{2^n} - \epsilon \\
&\geq \sum_{n=1}^{\infty} H_1^*(I_n) - 2\epsilon \\
&\geq H_1^*(A) - 2\epsilon.
\end{aligned}$$

Hence $m^*(A) = H_1^*(A)$. ■

Corollary 1.18. *For any interval $I \subset \mathbb{R}$, $H_P^*(I) = \infty$ for $0 < p < 1$, $H_1^*(I) = l(I)$ and $H_p^*(I) = 0$ for $p > 1$.*

Corollary 1.19. *$H_p^*(\mathbb{R}) = \infty$ for $0 < p \leq 1$, $H_p^*(\mathbb{R}) = 0$ for $p > 1$.*

Corollary 1.20. *For every nonempty open set $G \subset \mathbb{R}$, $H_p^*(G) = \infty$.*

1.5 Hausdorff Dimension

From the definition 1.4, we will note following following.

Theorem 1.21. *For $0 \leq p < q < \infty$, If $H_p^*(A) < \infty$ then $H_q^*(A) = 0$.*

Proof. for $\delta > 0$, let $\{I_k\}$ be the covering of the set A with $l(I_k) \leq \delta$ for each then,

$$\frac{l(I_k)^q}{l(I_k)^p} = l(I_k)^{q-p} \leq \delta^{q-p}.$$

$$\text{Therefore} \quad l(I_k)^q \leq \ \delta^{q-p}\, l(I_k)^p$$

$$\begin{aligned}\text{Now,} \quad H^*_{p,\delta}(A) &\leq \sum l(I_k)^q \\ &\leq \sum \delta^{q-p}\, l(I_k)^p \\ &= \delta^{q-p} \sum l(I_k)^p.\end{aligned}$$

taking infimum over all such coverings will give us,

$$H^*_{q,\delta}(A) \leq \delta^{q-p} H^*_{p,\delta}(A) \leq \delta^{q-p} H^*_p(A).$$

$\delta \to 0$ will give required result. ■

Corollary 1.22. *If $0 < H^*_p(A) < \infty$, then $H^*_q(A) = \infty$ for $q < p$.*

Corollary 1.23. *For a fixed set A, If we consider $H^*_p(A)$ as a function of p, $p > 0$, then either $H^*_P(A) = 0$ for all $p > 0$ or for some p_0 we have $H^*_p(A) = \infty$ for $0 < p < p_0$ and $H^*_p(A) = 0$ for all $p > p_0$.*

Definition 1.24. The **Hausdorff dimension** of the set A is defined as

$$\inf\{p : H^*_p(A) = 0\}.$$

From 1.22 and 1.5 Hausdorff Dimension is well defined number.

For any set A with $H^*_p(A) = 0$ for all p, we have that Hausdorff Dimension equals $\sup\{p : H^*_p(A) = \infty\}$. In the corollary 1.23, p_0 is the *Hausdorff dimension.*

1.6 Computing Hausdorff Dimension

Earlier, we have studied the topological dimension and vector space dimension. Under these concepts, a point has a dimension 0 while a line is a one dimensional. Similarly, the plane has 2 dimensions, and the cube has three dimensions. We know that, Euclidean plan $\mathbb{R}^n$ has a dimension n.

From previous section it is clear that *Hausdorff dimension* of the countable set is zero. Also *Hausdorff dimension* of the interval is one. In general $n-$dimensional cube will have *Hausdorff dimension* n.

Consider the Cantor set. Cantor set has *Cantor set* has *Lebesgue measure* zero but Its cardinality is same as line. Therefore we would expect its dimension to be 1. but it is just collection of some isolated points therefore its topological dimension is zero.

Definition 1.24 is the generalisation of the vector space dimension. Also from the example of the Cantor set we will see it will distinguish set with measure zero.

1.6.1 Cantor Set

We will show that Hausdorff dimension of the cantor set is $\frac{\log 2}{\log 3}$, Also $0 < \frac{\log 2}{\log 3} < 1$.

Let P be the cantor set. Then $C = \bigcap_{n=1}^{\infty} C_n$, Where each C_n is union of 2^n intervals with each interval has length $\frac{1}{3^n}$. Also let $L : [0,1] \to [0,1]$ be the lebesgue function, which is uniform limit of the functions $L_n : [0,1] \to [0,1]$. Where $L - n$is the monotone increasing function defined on $[0,1]$ which is linear and increasing by $\frac{1}{2^n}$ on each $J_{n,k}$ residual interval-interval remaining after removing middle one third.

For $p = \frac{\log 2}{\log 3}$,

$$
\begin{aligned}
H_{p,\frac{1}{3^n}}(C) \leq \sum l(C_n)^p &= 2^n \left(\frac{1}{3}\right)^P \\
&= \left(\frac{2}{3^p}\right)^n.
\end{aligned}
$$

Therefore $\frac{2}{3^p} = 1$, This implies $3^p = 2$.

For the Lebesgue function L, we have, $|L_{n+1}(x) - L_n(x)| \leq \frac{1}{2^n}$.

$$
\text{Therefore} \quad |L_m(x) - Ln(x)| \leq \sum_{j=0}^{m-n} \frac{1}{2^{n+j}} < \sum_{0}^{\infty} \frac{1}{n+j} < \frac{1}{2^{n-1}}.
$$

Letting $m \to \infty$, gives $|L(x) - L_n(x)| \leq \frac{1}{2^{n-1}}$. Also each L_n has slope equal to $\left(\frac{3}{2}\right)^n$. Therefore

$$
|L_n(x) - L_n(y)| \leq \left(\frac{3}{2}\right)^n |x - y|.
$$

$$\text{impliess} \quad |L(x) - L(y)| \leq |L(x) - L_n(x)| + |L_n(y) - L_n(y)|$$
$$\leq \left(\frac{3}{2}\right)^n |x - y| + \frac{4}{2^n}$$
$$= \frac{1}{2^n}(3^n|x - y| + 4) \qquad \forall n \geq 1.$$

Now for every $x, y \in [0, 1]$, there exists a $n \in \mathbb{N}$ such that $\frac{1}{3^n} \leq |x-y| < \frac{1}{3^{n-1}}$. This implies $1 \leq 3^n|x - y| < 3$ therefore $|L(x) - L(y)| \leq \frac{7}{2^n} = \frac{7}{3^{np}}$, Since $3^n|x - y| \geq 1$, $\frac{1}{3^{np}} \leq |x - y|^p$.

$$|L_n(x) - L_y(x)| \leq 7|x - y|^p.$$

1.6.2 Koch Curve

Koch curve is constructed as follows:

Step 1: We will begin with the line segment.

Step 2: Now we will divide line segment in three equal segments now replace middle line segment by two sides of equilateral triangle with the same length. As shown in the Figure 1.6.1.

Step 3: After step 2, we have total 4 line segments each has equal length. Now For each line segment we will again repeat step 2.

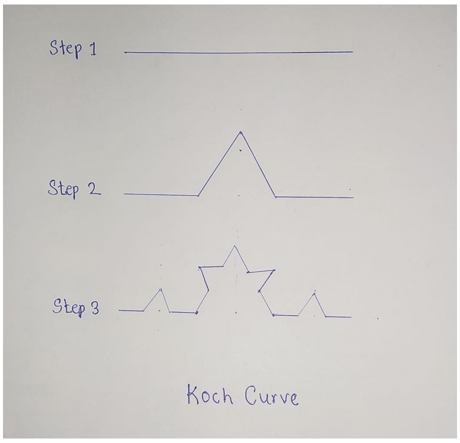

FIGURE 1.6.1: Koch Curve

Koch curve is obtained by limit of these transformations. We will use notation K for Koch curve. Now Koch curve is composed of lines, In $\mathbb{R}^2$ its area is equal to zero. Therefore its *Hausdorff measure* is zero i.e. $\mathcal{H}^2(K) = 0$. So *Hausdorff dimension* of K must be less than 2.

Now suppose the line segment in the step 1, in the construction has a length 1. Then in the second step we will get 4 line segments each of the length $\frac{1}{3}$. Therefore total length is $\frac{4}{3}$. In the step 3, each line segment is divided in 3 equal line segments resulting in the 4 line segments each of the length $\frac{1}{3} \times \frac{1}{3} = \frac{1}{3^2}$. There are four such segments therefore total length is $\frac{1}{3^2} \times 4 = \left(\frac{4}{3}\right)^2$. K is obtained by doing this transformation infinite times. Therefore total length of K is

$$\mathcal{H}^1(K) = \lim_{n\to\infty} \left(\frac{4}{3}\right)^{n-1} = \infty.$$

This implies *Hausdorff dimension* of the Koch curve is greater than 1. Therefore Hausdorff dimension of the Koch curve is non-integer. Also

$$1 < \text{Hausdorff Dimension}(K) < 2.$$

1.7 Hausdorff measure and Lipschitz function

In this section we will study the Hausdorff measure and lipschitz function. This section is primarily taken from [3].

Definition 1.25 (Lipschitz function)**.** A function $f : A \to \mathbb{R}^m$, $A \subset \mathbb{R}^n$ is called *Lipschitz function* if there exists a constant C such that,

$$\text{for all } x, y \in A, \quad |f(x) - f(y)| \leq C|x - y|. \tag{1.7.1}$$

The smallest such a C in the definition 1.25 is denoted by,

$$Lip(f) = \sup\left\{\frac{|f(x) - f(y)|}{|x - y|} \ : \ x, y \in A, x \neq y\right\}.$$

Theorem 1.26. *Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be Lipschitz, $0 \leq p < \infty$, $A \subset \mathbb{R}^n$, Then*

$$H_p(f(A)) \leq (Lip(A))^p \ H_p(A).$$

Proof. For a fix $\delta > 0$, choose a cover $\bigcup_i C_i$ of A such that $diam(C_i) < \delta$.

Then, $\quad diam(f(C_i) \leq Lip(f)\ diam(C_i) \leq Lip(f)\ \delta \quad$ and $\quad f(A) \subset \bigcup_{i=1}^{\infty} f(C_i)$

$$H_{p,Lip(f)\ \delta}\ (A) \leq \sum_{i=1}^{\infty} \alpha(p) \left(\frac{diam\ f(C_i)}{2} \right)^p$$
$$\leq (Lip(f))^p \sum_{i=1}^{\infty} \alpha(p)\ (diam\ (C_i))^p.$$

Now taking infimum over all such covers $\bigcup_{i=1}^{\infty} C_i$,

$$H_{p,Lip(f)\ \delta}\ (A) \leq (Lip(f))^p\ H_{p,\delta}(A).$$

If we let $\delta \to 0$, We will get required result. ■

Corollary 1.27. *Let $n > k$, $A \subset \mathbb{R}^n$,For the usual projection map $P : \mathbb{R}^n \to \mathbb{R}^k$, $Lip\ (P) = 1$ therefore $H_p(P(A)) \leq H_p(A)$.*

Theorem 1.28. *If $O : \mathbb{R}^n \to \mathbb{R}^m$ is orthogonal map, then for any $A \subset \mathbb{R}^n$, $H_p(O(A)) = H_p(A)$.*

Orthogonal maps on $\mathbb{R}^n$ are distance preserving. Therefore Hausdorff measure will be invariant under orthogonal transformation.

Remarks 1.7.2. Suppose A is lebesgue measurable set in $\mathbb{R}^n$ then its image under orthogonal map is Hausdorff measurable. Because we can write A countable union of compact sets and sets with measure zero. Therefore its orthogonal image is countable union of compact sets and set with zero lebesgue measure. From the above theorem these zero measure sets will have Hausdorff measure zero. Therefore this image will be Hausdorff measurable.

In the next chapter we will discuss more about Invariant measures.

Chapter 2

Invariant Measures

In this chapter, we will construct a left inavariant Haar measure on a locally compact metrizable topological group by using the Caretheodary's construction. This construction is discussed in detail in [5]. This chapter is primarily taken from here.

2.1 Topological Group

Definition 2.1 (Topological Group)**.** Let G be the group which is also topological space. If the multiplication map $(g, h) \to gh$ from $G \times G$ to G and the inverse map $g \to g^{-1}$ are continuous, then G is called topological group.

Here we require that the group operations be continuous in the given topology. Some examples of topological group are as below.

1. $(\mathbb{R}^n, +)$ n-dimentional Euclidean space under addition.
2. $(\mathbb{R}^*, \cdot), (\mathbb{C}^*, \cdot)$ Nonzero Real numbers and nonzero Complex numbers under multiplication are topological groups.
3. $(\mathbb{T}, \cdot)$ the set of complex numbers of modulus 1 is topological group under multiplication operation.
4. $(O(n), \cdot)$ $n \times n$ orthogonal matrices under operation of matrix multiplication.

5. $(SO(n), \cdot)$ the special orthogonal matrices equivalently $n \times n$ orthogonal matrices with determinant 1 under the operation of multiplication.

Definition 2.2 (Compact topological group)**.** Topological group G is called compact topological group if G is compact set in the given topology.

2.2 Haar measure

For a Group G, lets define following sets:

1. $gB = \{ gb : b \in B\}$.
2. $Bg = \{bg : b \in B\}$.
3. $B^{-1} = \{b^{-1} : b \in B\}$.

Definition 2.3 (Left-invariant Haar measure)**.** Let G be a topological group. A measure λ on its Borel σ-field is said to be a left-invariant Haar measure on G if,

1. For $K \subset G$ compact , $\lambda(K) < \infty$.
2. For $U \subset G$ open, if $U \neq \phi$ then $\lambda(U) > 0$.
3. For every Borel set $B \subset G$ and $g \in G$, $\lambda(gB) = \lambda(B)$.

Similarly, we can define $right - invariant\ Haar\ measure$ on G, by replacing $\lambda(gB)$ by $\lambda(Bg)$. Also left-multiplication by g is a homeomorphism; therefore, the image of a Borel set is Borel set.

We have natural one-one correspondence between left-invariant Haar measure and right-invariant Haar measure on a given topological space. For a left-invariant measure λ defining right invariant measure λ' as $\lambda'(B) = \lambda(B^{-1})$.

Now we will prove that every locally compact metrizable topological space has left-invariant Haar measure.

2.2.1 Ultrafilters

Definition 2.4 (Ultrafilter). $\mathcal{U} \subset 2^{\mathbb{N}}$ is said to be ultrafilter if

1. For all $i \in \mathbb{N}$, $\mathbb{N} \setminus \{i\} \in \mathcal{U}$.
2. if $A \in \mathcal{U}$, then every set containing A is also in $\mathcal{U}$.
3. $A, B \subset \mathcal{U} \implies A \cap B \in \mathcal{U}$.
4. For all $A \subset \mathbb{N}$, exactly one of A or $\mathbb{N} \setminus A$ belongs to $\mathcal{U}$.

Ultrafilter $\mathcal{U}$ includes complements of singletons set and is closed under finite intersection; therefore, it contains all co-finite sets. By condition 4, It does not contain singleton set, Every element of $\mathcal{U}$ is infinite, in particular ϕ does not belong to $\mathcal{U}$.

Lemma 2.5. *Suppose* $\{x_n\}_{n=1}^{\infty}$ *is a bounded sequence of real numbers. Then*

$$\bigcap_{U \in \mathcal{U}} \overline{\{x_n : n \in U\}}$$

has exactly one element.

Proof. Each set $\overline{\{x_n \,:\, n \in U\}}$ is compact, because it is close and bounded. Therefore by finite intersection property, intersection of finitely many of them is nonempty. Let $U_1, U_2, U_k \in \mathcal{U}$, then there intersection also belongs to $\mathcal{U}$, so it is nonempty. Suppose $p \in U_1 \cap \cdots \cap U_k$ Then

$$x_p \in \bigcap_{i=1}^{k} \{x_n \,:\, x_n \in U\} \subset \bigcap_{i=1}^{k} \overline{\{x_n \,:\, n \in U\}}$$

Finite intersection property guaranty that this intersection is nonempty. Therefore following intersection will be nonempty.

$$\bigcap_{U \in \mathcal{U}} \overline{\{x_n \,:\, n \in U\}} \neq \phi$$

To prove that this intersection has exactly one element, consider $\alpha, \beta \in \mathbb{R}$ such that $\alpha < \beta$. Define

$$S = \left\{ n \,:\, x_n > \frac{\alpha + \beta}{2} \right\} \quad \textit{and} \quad \mathbb{N} \setminus S = \left\{ n \,:\, x_n \leq \frac{\alpha + \beta}{2} \right\}.$$

Now exactly one of S and $\mathbb{N} \setminus S$ belongs to $\mathcal{U}$, If $S \in \mathcal{U}$ then $\alpha \notin \overline{\{x_n \ : \ n \in S\}}$. Similarly if $\mathbb{N} \setminus \mathbb{S} \in \mathcal{U}$ then $\beta \notin \overline{\{x_n \ : \ n \in \mathbb{N} \setminus S\}}$. In either case both α and β cannot belong to $\bigcap_{U \in \mathcal{U}} \overline{\{x_n \ : \ n \in U\}}$.

■

Definition 2.6 (convergence along ultrafilter)**.** For bounded sequence $\{x_n\}$ of the reals, let x be the unique element belonging to $\bigcap_{U \in \mathcal{U}} \overline{\{x_n : n \in U\}}$, then we say that sequence $\{x_n\}$ converges along the ultrafilter $\mathcal{U}$ to the x.

In upcoming lemma we will compare convergence along ultrafilter with our usual convergence. Let $\mathcal{F} \subset 2^{\mathbb{N}}$ consisting of all co-finite subsets of $\mathbb{N}$ and let $\mathcal{F}$ satisfies first three conditions in the Definition 2.4, and $\mathcal{F} \subset \mathcal{U}$.

Lemma 2.7. *Suppose* $\{x_n\}_{n=1}^{\infty}$ *be the bounded sequence of real numbers, and* x *be real number.*

1. $\{x_n\}_{n=1}^{\infty}$ *converges to* x *in usual convergence if and only if given* $\epsilon > 0$ *there exists* $A \in \mathcal{F}$ *such that* $\{x_n : n \in A\} \subset (x - \epsilon, x + \epsilon)$.
2. $\{x_n\}_{n=1}^{\infty}$ *converges to* x *along* $\mathcal{U}$ *if and only for given* $\epsilon > 0$ *there exists* $A \in \mathcal{U}$ *such that* $\{x_n : n \in A\} \subset (x - \epsilon, x + \epsilon)$.

Theorem 2.8. *There exists* $\mathcal{U} \subset 2^{\mathbb{N}}$ *which is an ultrafilter.*

2.2.2 Construction of Haar measure

We will construct left-invariant Haar measure on locally compact metrizable topological group G. Let $\mathcal{U}_o$ be set of nonempty open subsets of G whose closure is compact. G is locally compact topological space which means every element of G has a neighbourhood whose closure is compact therefore $\mathcal{U}_o$ has lots of set.

For $U, V \in \mathcal{U}_0$, define $\left[\frac{U}{V}\right]$ as the least number of left translates of V needed to cover $\overline{U}$. Left-translate of V means set of the form gV, for $g \in G$. From now whenever we will use translate it means left-translate. Each translate of V is an open set and set of all translates will form an open cover for a compact set $\overline{U}$, therefore finitely many translates of V will cover $\overline{U}$. So $\left[\frac{U}{V}\right]$ is well defined.

Fix a metric d on G. Fix $U_0 \in \mathcal{U}_0$. Fix a sequence $\{B_n\}_{n=1}^{\infty}$ decreasing to $\{e\}$ and $\{diam(B_n)\}_{n=1}^{\infty}$ decreases to 0. This is possible because G is locally compact. We will also fix ultrafilter on $\mathbb{N}$, which will be anonymous.

Step 1

In this step we will define approximating measure analogous to the the definition 1.3.

Define $\lambda_n : \mathcal{U}_0 \to [0, \infty]$ and $\lambda : \mathcal{U}_0 \to \infty$ as follows.

$$\lambda_n(U) = \left(\left[\frac{U}{B_n}\right] \Big/ \left[\frac{U_0}{B_n}\right] \right).$$

$$\text{Therefore} \qquad \lambda(U) = \lim_n \lambda_n(U).$$

Where the limit is along the chosen ultrafilter. We will prove that sequence $\{\lambda_n(U_n)\}_{n=1}^{\infty}$ is bounded. so the limit is well-defined. Then We will define ultrafilter $\mathcal{U} = \mathcal{U}_0 \cup \{\phi, G\}$, Then extend λ to $\mathcal{U}$ by setting $\lambda(\phi) = 0$ and $\lambda(G) = \infty$, if G is not compact. Note that $G \in \mathcal{U}_0$ iff G is compact.

For $U, V, W \in \mathcal{U}_o$

$$1 \leq \left[\frac{U}{V}\right] \leq \left[\frac{U}{W}\right]\left[\frac{W}{V}\right].$$

Since $\overline{U}$ is nonempty, $1 \leq \left[\frac{U}{V}\right]$. let $k = \frac{W}{V}$ and $l = \frac{U}{W}$ then suppose $g_1V,\ g_2V,\ \dots, g_kV$ covers $\overline{W}$ and $h_1W,\ h_2W,\ \dots,\ h_lW$ covers $\overline{U}$. For $x \in \overline{U}$ choose i such that $x \in h_iW$. Then $h_i^{-1}x \in W \subset \overline{W}$, so pick j such that $h_j^{-1}x \in g_jV$. $\therefore x \in h_ig_jW$ So $\left[\frac{U}{V}\right] \leq kl$.

For all $n \in \mathbb{N}$ and for all $U \in \mathcal{U}_0$, from above the above inequality, we have $\left[\frac{U}{B_n}\right] \leq \left[\frac{U}{U_0}\right]\left[\frac{U_0}{B_n}\right]$. This gives $\lambda_n(U) \leq \left[\frac{U}{U_0}\right]$. Similarly , $\left[\frac{U_0}{B_n}\right] \leq \left[\frac{U_0}{U}\right]\left[\frac{U}{B_n}\right]$ This gives $1 \Big/ \left[\frac{U_0}{V}\right] \leq \lambda_n(U)$. Therefore

$$1 \Big/ \left[\frac{U_0}{V}\right] \leq \lambda_n(U) \leq \left[\frac{U}{U_0}\right].$$

So sequence $\{\lambda_n(U)\}_{n=1}^{\infty}$ is bounded so $\lambda(U)$ is well defined. Now limit along the ultrafilter of a bounded sequence also lies within the bound of a sequence.

Therefore

$$1/\left[\frac{U_0}{V}\right] \leq \lambda(U) \leq \left[\frac{U}{U_0}\right].$$

Now we will show that for all $U_1, U_2 \in \mathcal{U},\ \lambda(U_1 \cup U_2) \leq \lambda(U_1) + \lambda(U_2)$.

If any of U_1, U_2 is empty, then inequality holds trivially. Similarly if any of U_1, U_2 equals to G then inequality is trivial.

Otherwise for $U_1, U_2 \in \mathcal{U}_0$ $U_1 \cup U_2 \in \mathcal{U}_0$,

$\left[\frac{U_1 \cup U_2}{B_n}\right] \leq \left[\frac{U_1}{U_2}\right] + \left[\frac{U_2}{B_n}\right]$. Dividing both sides by $\left[\frac{U_0}{B_n}\right]$, gives $\lambda_n(U_1 \cup U_2) \leq \lambda_n(U_1) + \lambda_n(U_2)$.

Taking limit along ultrafilter will give us required inequality. For $U_1, U_2 \in \mathcal{U}$, if $d(U_1, U_2) > 0$ then $\lambda(U_1 \cup U_2) = \lambda(U_1) + \lambda(U_2)$

For all $g \in G$ and $U \in \mathcal{U}, \lambda(gU) = \lambda(U)$. This is trivial if $U = \phi$ or $U = G$; otherwise $\left[\frac{gU}{B_n}\right] = \left[\frac{U}{B_n}\right]$ bacause if we pre-multiply translates of B_n by g, We will get cover for gU, similarly left multiplying cover of gU by g^{-1} will give us cover for U. So $\lambda_n(gU) = \lambda_n(U)$.. This holds for every n. Therefore $\lambda(gU) = \lambda(U)$.

Step 2

Now we have approximating measure, by using this we will define outer measure $\mu^* : 2^G \to [0, \infty]$, Since $G \in \mathcal{U}$, any subset E of G can be covered by by at most countably many subsets from $\mathcal{U}$. Now define,

$$\mu^*(E) = \inf\left\{\sum_{S \in \mathcal{S}} \lambda(S) : \mathcal{S} \subset \mathcal{U},\ \mathcal{S} \text{ is at most countable, } E \subset \bigcup S\right\}.$$

Clearly $\mu^*(\phi) = 0$, μ^* is monotonically increasing and μ^* is countably subadditive. Threfore μ^* is countably subadditive. This μ^* will give us left-invariant Haar measure.

For all $g \in G$ and $E \subset G$ we have $\mu^*(gE) = \mu^*(E)$ because similar holds for the λ. For $E_1, E_2 \subset G$ if $d(E_1, E_2) > 0$ then $\mu^*(E_1 \cup E_2) = \mu^*(E_1) + \mu^*(E_2)$.

Step 3

let $\mathscr{A} = \{E\ :\ \ E \subset G, \forall A \subset G(\mu^*(A) = \mu^*(A \cap E) + \mu^*(A \cap (G \setminus E))\}$.

If $E \in \mathscr{A}$ then clearly $G \setminus E \in \mathscr{A}$. Now we can see easily that $\phi \in \mathscr{A}$, This implies $G \in \mathscr{A}$. Therefore $\mathscr{A} \neq \phi$. $\mathscr{A}$ is a $\sigma-$field. This σ-filed contains all open subsets of G, Hence it contains Borel $\sigma-$field of G.

We have constructed left-invariant measure μ^* on $\mathscr{A}$. Now we will prove that for every compact set μ^* is finite and for every open set μ^* is positive.

Let $K \subset G$ be a compact set and $\mathcal{U}_0$ be a cover for K. K being a compact set we have a finite subcover, say $U_1, U_2, \ldots, U_n$ covers K. Also for each of this set U_i, $\lambda(U_i) < \infty$. Therefore

$$\mu^*(K) \leq \sum_{i=1}^{n} \lambda(U_i) < \infty.$$

Therefore for compact set μ^* is finite. Now let U be a nonempty open subset of G, Since G is a locally compact space, there is a open set H such that $H \subset \overline{H} \subset U$ and $\overline{H}$ is compact.

Now let $\epsilon > 0$ be given. We will get $U_1, U_2, \ldots, U_n \subset \mathcal{U}_0$ covering $\overline{H}$ such that $\sum_{i=1} n\lambda(U_i) < \mu^*(\overline{H}) + \epsilon$. Here we have chosen only finitely many sets U_i because $\overline{H}$ is a compacts set and finite subcover will give the better estimate. Note that λ is only defined for $\mathcal{U}$.

$$\begin{aligned}
\lambda(H) &\leq \lambda\left(\bigcup_{i=1}^{n} U_i\right) \\
&\leq \sum_{i=1}^{n} \lambda(U_i) \\
&\leq \mu^*(\overline{H} + \epsilon \\
&\leq \mu^*(U) + \epsilon
\end{aligned}$$

$$\text{Therefore} \quad 0 < \lambda(H) \leq \mu^*(U)$$

Hence for any nonempty open set μ^* is positive.

We have constructed *Haar measure* as defined in the definition 2.3. *Haar measure* constructed above is unique up to scalar multiplication.

2.3 Haar measure on $\mathbb{R}^n$

On $\mathbb{R}^n$, $\mathcal{L}^n$ $n-$dimensional Lebesgue measure is Haar measure. Also $\mathbb{R}^n$ is locally compact Hausdorff space and Hausdorff measure is invariant under the isometries of $\mathbb{R}^n$, therefore it is Haar measure, but Haar measure is unique upto scalar multiplication.

$$\therefore \mathcal{L}^n = \gamma_n \mathcal{H}^n \quad \gamma_n > 0. \tag{2.3.1}$$

This gamma is nothing but $n-$dimensional volume of the euclidean unit ball in the usual norm. i.e.

$$\gamma_n = \frac{\pi^{\frac{2}{n}}}{2^n \Gamma\left(\frac{n}{2}+1\right)}.$$

2.4 Examples of the Haar measure

1. Consider the set of nonzero real numbers under multiplication. Then it is topological group. $\lambda = \frac{dx}{|x|}$ is the Haar measure on the $\mathbb{R} \setminus \{0\}$ for the Borel subsets of the this group.

   For example, Let $I = [a, b]$ then $\lambda(I) = \int_I \frac{1}{|x|} dx = \log\left(\frac{b}{a}\right)$. This is translation invariant.

2. $G = GL(n, \mathbb{R}$ is a topological group For $A \subset G$, μ defined by the following

   $$\mu(A) = \int_A \frac{1}{|det(X)|^n} dX.$$

   Where dX is the Lebesgue measure on the $\mathbb{R}^{n^2}$ is left invariant Haar measure on the G.

Chapter 3

Covering Theorems and Differentiation of Integrals

In this chapter we will study the technique of covering lemmas. By using these techniques, we can prove many analytic results by using an elementary arguments from Euclidean geometry. This chapter is primarily taken from [6].

3.1 Wiener's Covering Lemma

3.1.1 Basic definitions

let $S \subset \mathbb{R}^n$ be a set. Then

Definition 3.1 (covering)**.** A collection $\mathcal{U} = \{U_\alpha\}_{\alpha \in \mathcal{A}}$ satisfying $S \subset \cup_{\alpha \in \mathcal{A}} U_\alpha$ is called covering for the set S.

Definition 3.2 (open covering)**.** If all the sets of $\mathcal{U}$ in definition 3.1, are open, then we called $\mathcal{U}$ an open covering of S.

Definition 3.3 (subcovering)**.** Subcovering of the of the covering $\mathcal{U}$ is a covering $\mathcal{V} = \{V_\beta\}_{\beta \in \mathcal{B}}$ such that V_β is one of the U_α.

Definition 3.4 (refinement)**.** A refinement of the covering $\mathcal{U}$ is the collection $\mathcal{W} = \{W_\gamma\}_{\gamma \in \mathcal{G}}$.

Definition 3.5 (valence)**.** If $\mathcal{U}$ is a covering of a set S then valence of $\mathcal{U}$ is the least positive integer M such that no point of S lies in more than M of the sets in $\mathcal{U}$.

For any open cover of $S \subset \mathbb{R}^N$, we always have countable subcover for S. For a compact set we have finite subcover, sets in this subcover may or may not be disjoint.

Lemma 3.6 (Wiener's Lemma)**.** *Let $K \subset \mathbb{R}^n$ be the compact set with the covering $\mathcal{U} = \{B_\alpha\}_{\alpha \in A}$, $B_\alpha = \mathbb{B}(c_\alpha, r_\alpha)$, by open balls. Then there is a subcollection $B_{\alpha_1}, B_{\alpha_2}, \ldots, B_{\alpha_m}$, consisting of pairwise disjoint balls, such that*

$$\bigcup_{j=1}^{m} \mathbb{B}(c_{\alpha_j}, 3r_{\alpha_j}) \supset K.$$

Proof. Here K is a compact set; therefore we can cover K by finitely many balls B_α. Choose B_{α_1} a ball with the largest radius among these finitely many balls. Now choose B_{α_2} disjoint from B_{α_1} with the greatest radius among other balls. At j-th step choose the ball disjoint with $B_{\alpha_1}, B_{\alpha_2}, \ldots, B_{\alpha_{j-1}}$ that has largest radius among those balls that are disjoint from $B_{\alpha_1}, B_{\alpha_2}, \ldots, B_{\alpha_{j-1}}$. This process will terminate in finitely many steps as we have only finitely many balls. Note that at each step choice of ball is not unique.

We will prove that $B_\alpha \subset \bigcup_j \mathbb{B}(c_{\alpha_j}, 3r_{\alpha_j}$ for every α. If $\alpha = \alpha_j$ for some j then we are done. If $\alpha \neq \{\alpha_j\}$, let j_0 be the first index j with $B_{\alpha_j} \cap B_\alpha \neq \phi$, such j_0 exist because process terminated after finite steps. Now from the above process $r_{\alpha_{j_0}} \geq r_\alpha$, But from the triangle in equality $\mathbb{B}(c_{\alpha_{j_0}}, 3r_{\alpha_{j_0}}) \supset \mathbb{B}(c_\alpha, r_\alpha)$. ■

Theorem 3.7. *Let $A \subset \mathbb{R}^n$ and $\mathcal{B}$ be the family of open balls. Suppose for each point of A is contained in arbitrarily small balls belonging to $\mathcal{B}$. Then there exist pairwise disjoint balls $B_j \in \mathcal{B}$ such that*

$$\mathcal{L}^N\left(A \setminus \bigcup_j B_j\right) = 0.$$

Furthermore, for any $\epsilon > 0$, we may choose balls B_j in such a way that

$$\sum_j \mathcal{L}^N(B_j) \leq \mathcal{L}^N(A) + \epsilon.$$

Proof. Assume that $A \equiv A_0$ is bounded. Select a bounded set U_0 so that $\overline{A_0} \subset U_0$ and $\mathcal{L}^N(U_0) \leq (1 + 5^{-N})\mathcal{L}^N(A_0)$. Now focus attention on those balls that lie in U_0, By Wiener's lemma 3.6, we may select a finite, pairwise disjoint collection $B_j = \mathbb{B}(x_j, r_j) \in \mathcal{B}, j = 1, 2, \ldots, K_1$, such that $B_j \subset U_0$ and $\overline{A} \subset \bigcup_{j=1}^{k_1} \mathbb{B}(x_j, 3r_j)$

Now,

$$\begin{aligned}\frac{1}{3^N}\mathcal{L}^N(A_0) &\leq \frac{1}{3^N}\sum_j \mathcal{L}^N\left(\mathbb{B}(x_j, 3r_j)\right)\\ &= \frac{1}{3^N}\sum_j 3^N\mathcal{L}(B_j)\\ &= \sum_j \mathcal{L}^N(B_j).\end{aligned}$$

Let $A_1 = A_0 \setminus \bigcup_{j=1}^{k_1} B_j$. Then

$$\begin{aligned}\mathcal{L}^N(A_1) &\leq \mathcal{L}^N\left(U_0 \setminus \bigcup_{j=1}^{k_1} B_j\right)\\ &= \mathcal{L}^N(U_0) - \sum_{j=1}^{k_1}\mathcal{L}^N(B_j)\\ &\leq \left(1 + \frac{1}{5^N} - \frac{1}{3^N}\right)\mathcal{L}^N(A_0)\\ &= u \cdot \mathcal{L}^N(A_0).\end{aligned}$$

where $u = 1 + \frac{1}{5^N} - \frac{1}{3^N} < 1$. Now $A_1 \subset \mathbb{R}^N \setminus \bigcup_{j=1}^{k_1} B_j$, and this later set is bounded. Therefore we can find bounded, open set U_1 such that $A_1 \subset U_1 \subset \mathbb{R}^N \setminus \bigcup_{j=1}^{k_1} B_j$ and $\mathcal{L}^N(U_1) \leq \left(1 + \frac{1}{5^N}\right)\mathcal{L}^N(A_1)$.

By repeating above procedure, we may find disjoint balls $B_j, j = k_1 + 1, \ldots, k_2$ for which $B_j \subset U_1$ and $\mathcal{L}^N(A_2) \leq u \cdot \mathcal{L}^N(A_1) \leq u^2\mathcal{L}^N(A_0)$; Here

$$A_2 = A_1 \setminus \bigcup_{j=k_1+1}^{k_2} B_j = A_0 \setminus \bigcup_{j=1}^{k_2} B_j.$$

By our assumption all the balls $B_1, B_2, \ldots, B_{k_2}$ are disjoint. After m repetitions of this procedure, we find that we have balls $B_1, B_2, \ldots, B_{k_m}$ such that

$$\mathcal{L}^N\left(A_0 \setminus \bigcup_{j=1}^{k_m} B_j\right) \leq u^m \mathcal{L}^N(A_0).$$

Since $u < 1$ result follows. At the beginning we have made additional assumption of boundedness. For the general case simply decompose $\mathbb{R}^N$ into closed unit cubes Q_l with disjoint interiors and the sides parallel to axes and apply the result just proved to each $A \cap Q_l$. ■

3.2 Vitali's Covering Lemma

Definition 3.8 (Vitali cover)**.** Let E be a subset of $\mathbb{R}$ a collection of intervals $\mathcal{C}$ in $\mathbb{R}$ is said to be *vitali cover* of set E if for all $x \in E$, for any $\epsilon > 0$, there is an interval $I \in \mathcal{C}$, such that $x \in I$ and $l(I) < \epsilon$.

Lemma 3.9 (Vitali)**.** *Let $E \subset \mathbb{R}$ with $m^*(E) < \infty$ and $\mathcal{V}$ be a Vitali cover for E, then for given $\epsilon > 0$, there exists a finite disjoint sub-collection $\{I_1, I_2, \ldots, I_n\} \subset \mathcal{V}$ such that,*

$$m^*\left(E \setminus \bigcup_{i=1}^{n} I_n\right) < \epsilon.$$

Proof. It is sufficient to prove the lemma for closed interval, for otherwise we will replace each interval $I_1, \ldots, I_n$ by its closure and observe that the set of end points has measure zero.

Here $m^*(E) < \infty$, therefore from the definition of *Lebesgue measure*, we can find an open set O containing E such that, $m^*(E) < m^*(O) < \infty$ Without loss of generality, we can assume that each interval $I_1, I_2, \ldots, I_n$ is contained in O. Now we can choose sequence of intervals I_n as follows

Let I_1 be any interval in $\mathcal{V}$ and suppose $I_1, I_2, \ldots, I_n$ are already been chosen. And let k_n be the supremum of the lengths of intervals which do not intersect with any of the $I_1, I_2, \ldots, I_n$.

$$i.e. \quad k_n = \sup\left\{ l(I) \ : \ I \cap \left(\bigcup_{i=1}^{n} I_i\right) = \phi \right\}.$$

Since I is contained in O we have $k_n \leq m^*(O) < \infty$. Unless $E \subset \bigcup_{i=1}^{n} I_i$, we can find $I_{n+1} \in \mathcal{V}$ such that $l(I_{n+1} > l(k_n)$ and I_{n+1} is disjoint from $I_1, I_2, \ldots, I_n$.

Thus we have sequence of intervals $\{I_n\}_{n=1}^{\infty}$ such that $\bigcup_{n=1}^{\infty} I_n \subset O$. Which implies,

$$\sum_{n=1}^{\infty} l\left(I_n\right) < m^*(O).$$

Therefore for given $\epsilon > 0$, there exists a $N \in \mathbb{N}$ such that,

$$\sum_{N+1}^{\infty} I_i < \frac{\epsilon}{5}.$$

Let $R = E \backslash \bigcup_{n=1}^{N} I_n$, The lemma will be established if we can show that $m^*(R) < \epsilon$. Let $x \in R$ be arbitrary point. Since $\bigcup_{n=1}^{N}$ is a closed set not containing x, we can find a interval $I \in \mathcal{V}$ such that $I \cap \left(\bigcup_{n=1}^{N} I_n\right) = \phi$. Now if, $I \cap I_i = \phi$ for $i \leq n$, we must have $l(I) \leq k_n < 2l(I_{n+1})$. Since $\lim\, l(I_n) = 0$, The interval I must meet at least one of the intervals I_n. Let n_0 be the smallest integer such that $I \cap I_{n_0}) \neq \phi$. Then we have, $n_0 > N$ and $l(I) \leq k_{n_0-1} \leq 2l(I_{n_0})$. Since $x \in I$ and $I \cap I_{n_0} \neq \phi$, It follows that distance from x to the midpoint of I_{n_0} is at most $l(I) + \frac{1}{2}l(I_{n_0}) \leq \frac{5}{2}l(I_n)$. Thus $x \in J_{n_0}$ having same midpoint as I_{n_0} and five times a the length. Thus we have shown that

$$R \subset \bigcup_{n_0=N+1}^{\infty} J_{n_0}.$$

Hence

$$m^*(R) \leq \sum_{n+1}^{\infty} l(J_{n_0}) = 5 \sum_{N+1}^{\infty} l(I_{n_0}) < \epsilon.$$

■

3.3 The Maximal Function

3.3.1 Fundamental theorem of calculus

From calculus, we know that differentiation and integration are inverse processes.

Theorem 3.10 (Fundamental theorem of calculus)**.** *Let $f(x)$ be continuous function on $[0,1]$, for $x \in [a,b]$ define $F(x) = \int_a^x f(t)dt$ then $F(x)$ is cotinuos on $[0,1]$ and differentiable on $(0,1)$ and $F'(x) = f(x)$.*

Now suppose f is integrable on $[0,1]$ and F is same as defined above in 3.10. Does this imply that F is differentiable? at least for almost every $x \in (a,b)$. If yes, then what is derivative of F ? Is it $F' = f$?

3.3.2 Differentiation of the Integral

Let f be defined on $[0,1]$ and integrable on $[0,1]$, define

$$F(x) = \int_a^x f(t)dt, \quad a \leq x \leq b.$$

We know that derivative is the limit of the quotient,

$$\frac{F(x+h) - F(x)}{h} \quad \text{when } h \to 0.$$

For $h > 0$, This fraction is,

$$\frac{1}{h}\int_x^{x+h} f(t)dt = \frac{1}{|I|}\int_I f(t)dt.$$

Where $I = (x, x+h)$ and $|I|$ is the length of the interval I, Now we want to know limit $|I| \to 0$ at point x. We are interested in knowing,

$$\lim_{|I| \to 0} \frac{1}{I}\int_I f(t)dt = f(x), \quad \text{for a.e. } x \in I.$$

holds for suitable points. In $\mathbb{R}^n$ we define similar problem with changing length by measure or volume of balls containing point x.

Suppose f is integrable on $\mathbb{R}^n$, Is it true that,

$$\lim_{m(B) \to 0} \int_B f(t)dt = f(x) \quad \text{for a.e. } x \in B.$$

The limit is taken as volume of open ball B containing x goes to zero.

3.3.3 Hardy-Littlewood maximal function

Definition 3.11. If f is locally integrable function on $\mathbb{R}^n$, we let

$$Mf(x) = \sup_{R>0} \frac{1}{\mathcal{L}^N[\mathbb{B}(x,R)]} \int_{\mathbb{B}(x,r)} |f(t)| \, d\mathcal{L}^N(t).$$

This operator M is called *Hardy-Littlewood maximal operator*. The fraction to which M is applied may be real or complex valued.

Definition 3.12 (Hardy-Littlewood maximal function)**.** If f is integrable on $\mathbb{R}^n$, we define its *maximal function* f^* by $f^*(x) = Mf(x)$

$$f^*(x) = \sup_{x\in B} \frac{1}{m(B)} \int_B |f(y)| \, dy, \quad x \in \mathbb{R}^n.$$

Where supremum is taken over all the balls containing the point x.

The main properties of f^* are summarized in the below theorem.

Theorem 3.13. *Suppose f is integrable on $\mathbb{R}^n$, Then:*

1. *f^* is measurable.*
2. *$f^*(x) < \infty$ for a.e. x.*
3. *f^* satisfies*
$$m(\{x \in \mathbb{R}^n : f^*(x) > \alpha\}) \leq \frac{A}{\alpha} ||f||_{L^1(\mathbb{R}^n)} \tag{3.3.1}$$
for all $\alpha > 0$, where $A = 3^n$, and $||f||_{L^1(\mathbb{R}^n} = \int_{\mathbb{R}^n} |f(x)| dx$.

Proof. (1) To show that f^* is measurable it is enough to show that the set $E_\alpha = \{x \in \mathbb{R}^n \ : \ f^*(x) > \alpha\}$ is open. For any $x \in B$ there exists a open ball B such that $x \in B$ and
$$\frac{1}{m(B)} \int_B |f(x)| dy > \alpha.$$
Now if $\bar{x}$ is any point close enough to x then x also belong to B; hence $\bar{x}\ in E_\alpha$ as well. Hence

$$\sup_{\bar{x}\in B'} \frac{1}{m(B')} \int_{B'} |f(y)| \, dy \ \geq \ \frac{1}{m(B)} \int_B |f(x)| dy > \alpha.$$

Since B belongs to the family on which supremum is taken, It means $\bar{x} \in E_\alpha$. That is small ball surrounding x is contained in E_α.

(2) To prove that $f^*(x)$ is finite a.e. x, we will assume (3) for time being. We observe that,

$$\{x : f^*(x) = \infty\} \subset \{x : f^*(x) > \alpha\} \qquad \text{for all } \alpha > 0$$

$$\text{Therefore} \qquad m\left(\{x : f^*(x) = \infty\}\right) \leq \frac{A}{\alpha} ||f||_{L^1}.$$

Taking limit as α goes to ∞ will give us required result.

(3) Inequality 3.3.1 is called **weak-type** inequality. To prove this we will use *Wiener's Lemma* 3.6.

Let $B = \{B_1, B_2, \ldots, B_N\}$ be a finite collection of balls in $\mathbb{R}^n$, Then there exists a disjoint sub-collection $\{B_{i_1}, B_{i_2}, \ldots, B_{i_k}\}$ of B such that

$$m\left(\bigcup_{l=1}^{N}\right) \leq 3^n \sum_{j=1}^{k} m\left(B_{i_j}\right).$$

Now we will prove (3) using above covering theorem. let $E_\alpha = \{x \mid f^*(x) > \alpha\}$, then for each $x \in E_\alpha$, there exists a ball B_x that contains x and such that

$$\frac{1}{m(B_x)} \int_{B_x} |f(y)|\, dy > \alpha.$$

Therefore for each ball B_x we have

$$m\left(B_x\right) < \frac{1}{\alpha} \int_{B_x} |f(y)| dy. \tag{3.3.2}$$

Since m is inner regular

$$m(E_\alpha) = \sup_{K \subset E_\alpha} m(K).$$

Fix a compact subset K of E_α. Since K is covered by $\bigcup_{x \in E_\alpha} B_x$, From *Wiener's Covering Lemma 3.6*, We may select a finite subcover of K, say $K \subset \bigcup_{l=1}^{N} B_l$ and a sub-collection $B_{i_1}, B_{i_2}, \ldots, B_{i_k}$ of disjoint balls with

$$m\left(\bigcup_{l=1}^{N} B_l\right) \leq 3^n \sum_{j=1}^{k} m(B_{i_j}). \tag{3.3.3}$$

Since the balls $B_{i_1}, \ldots, B_{i_k}$ are disjoint and satisfies equation 3.3.2 and 3.3.3, we can find that,

$$\begin{aligned} m(K) &\leq m\left(\bigcup_{l=1}^{N} B_l\right) \\ &\leq 3^n \sum_{j=1}^{k} m\left(B_{i_j}\right) \\ &\leq \frac{3^n}{\alpha} \sum_{j=1} k \int_{B_{i_j}} |f(y)|dy \\ &= \frac{3^n}{\alpha} \int_{\bigcup_{j=1}^{k} B_{i_j}} |f(y)|dy \\ &\leq \frac{3^n}{\alpha} \int_{\mathbb{R}^n} |f(y)|dy. \end{aligned}$$

This inequality is true for all compact subsets K of E_α, The proof of the weak type inequality for the maximal operator is complete. ■

3.4 The Besicovitch Covering Theorem

Theorem 3.14 (The Besicovitch covering theorem)**.** *Let N be the positive integer. There is constant $K = K(N)$ with the following property. Let $\mathcal{B} = \{B_j\}_{j=1}^{M}$ be any finite collection of open balls in $\mathbb{R}^N$ with the property that no ball contains the center of any other. Then we may write*

$$\mathcal{B} = \mathcal{B}_1 \cup \cdots \cup \mathcal{B}_K.$$

So that each $B_j, j = 1, 2, \ldots, K$, is a collection of pairwise disjoint balls.

To prove this theorem we need to prove the following lemma about the balls.

Lemma 3.15. *There is a constant $K = K(N)$, depending only on the dimension of our space $\mathbb{R}^n$, with the following property: Let $B_0 = \mathbb{B}(x_0, r_0)$ be a ball of fixed radius. Let $B_1 = \mathbb{B}(x_1, r_1)$, $B_2 = \mathbb{B}(x_2, r_2), \ldots, B_p = \mathbb{B}(x_p, r_p)$ be balls such that,* *(1) Each B_j has nonempty intersection with $B_0, j = 1, \dots, p$;*

(2) The radii $r_j \geq r_0$ for all $j = 1, 2, \dots, p$;

(3) No balls B_j contains the center of any other B_k for $j, k \in \{0, 1, \dots, p\}$ with $j \neq k$

Then $p \leq K$.

In simple terms this lemma says that , For a fixed ball B_0, at most K pairwise disjoint balls of (at least) the same size can touch B_0. We will prove this lemma by using some trigonometry. Proof the theorem 3.14 depends on this lemma. We will prove both of these without using any measure theory.

Proof. We will prove this lemma for all balls having the single radius r_0 then this will imply the general case. So we assume that all balls have the same radius. With the balls as given, replace each ball by $\frac{1}{2}B_j$- same center but radius $\frac{r_0}{2}$. we will denote shrunken balls by $B_j = \mathbb{B}(x_j, \frac{r_0}{2}$. Then each ball is contained in $\mathbb{B}(x_0, 3r_0)$.

We calculate that

$$p = \frac{\mathcal{L}^N \left(\bigcup_{j=1}^{p} B_j \right)}{\Omega_N \left(\frac{r_0}{2} \right)^N} = 6^N.$$

■

Where Ω_N is the volume of the unit ball in $\mathbb{R}^N$.

Therefore, $K(N)$ exists and does not depend on exceed 6^N.

Proof of theorem 3.14. We have an iterative procedure for selecting balls.

Let B_1^1 be the ball with maximum radius, choose ball B_2^1 to be a ball of maximum radius that is disjoint from B_1^1. This process will terminate because we have only finitely many balls in total. Set $\mathcal{B}_1 = \{B_j^1\}$.

Now from the remaining balls B_1^2 ball with the greatest radius. Then select B_2^2 to be remaining ball with greatest radius, disjoint from B_1^2. This process will again terminate. Set $\mathcal{B}_2 = \{B_j^2\}$.

From the remaining balls we will produce the family $\mathcal{B}_3$ and so forth. since there are only finitely many balls this process will end too. We will have produced finitely many, say q-nonempty families of pairwise disjoint balls, $\mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_q$. Now it remains to show that how large this q can be.

Let $K(N)$ be as in the lemma and suppose $q > K(N) + 1$. The first ball in the family $\mathcal{B}_q$, B_1^q must have intersected a ball in the previous families; by our selection procedure each ball of those families must be at least as large in radius as B_1^q. Thus B_1^q is open ball with at least $K(N) + 1$ neighbors as in the lemma. But lemma says ball can have $K(N)$ such neighbors. That is a contradiction.

Therefore $q \leq K(N) + 1$. This proves the theorem.

■

Theorem 3.16. *Let μ be a radon measure on $\mathbb{R}^n$. Let $A \subset \mathbb{R}^n$ and let $\mathbb{B}$ be a family of closed balls such that each point A is the center of arbitrarily small balls in $\mathcal{B}$. Then there are disjoint balls $B_j \in \mathcal{B}$ such that*

$$\mu\left(A \setminus \bigcup_j B_j\right) = 0.$$

Proof. We will follow same strategy as for the theorem 3.7. We also assume the $\mu(A) > 0$, otherwise there is nothing to prove. We also suppose that A is bounded same as in the proof of 3.7. Let K be as in the theorem 3.6. From the Radon property of μ, there is an open set U such that $A \subset U$ and

$$\mu(U) \leq \left(1 + \frac{1}{4 \cdot K}\right) \mu(A).$$

Now from Wiener's lemma 3.6, there are subfamilies $\mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_K$ such that each $\mathcal{B}_j$ is a collection of pairwise disjoint balls and

$$\overline{A} = \bigcup_{j=1}^{K} \left(\bigcup_{B \in \mathcal{B}_j} B\right) \subset U.$$

Now it is clear that

$$\mu(A) \leq \sum_{j=1}^{K} \mu\left(\bigcup_{B \in \mathcal{B}_j} B\right).$$

Hence there is a index j_0 such that

$$\mu(A) \leq K \cdot \mu\left(\bigcup_{B \in \mathcal{B}_{j_0}} B\right).$$

Let $A_1 = A \setminus \bigcup_{B \in \mathcal{B}_{j_0}} B$, Then

$$\begin{aligned}
\mu(A_1) &= \mu\left(U \setminus \bigcup_{B \in \mathcal{B} - j_0} B\right) \\
&= \mu(U) - \mu\left(\bigcup_{B \in \mathcal{B}_{j_0}} B\right) \\
&\leq \left(1 + \frac{1}{4 \cdot K} - \frac{1}{K}\right) \cdot \mu(A) \\
&= u \cdot \mu(A).
\end{aligned}$$

Where $u = 1 - \frac{3}{4K}$, We will iterate this construction same as in proof of the 3.7, Also we may dispense the hypothesis that A is bounded just as in the proof of 3.7 by making an additional observation that the Radon measure μ can measure at most countably many hyperplanes parallel to the axes with positive measure.

■

3.5 Differentiation

Definition 3.17 (Radon Measure)**.** Let X be a locally compact Hausdorff topological space, measure μ on X is called Radon measure if the following conditions hold:

1. Every compact set in X has a finite measure.
2. Every open set is $\mu-$measurable and if $V \subset X$ is open then

$$\mu(V) = \sup\{\mu(K) : \ K \text{ is compact and } K \subset V\}.$$

3. For every $A \subset X$, $\mu(A) = \inf\{\mu(V) : V$ is open and $A \subset V\}$.

Theorem 3.18 (Lesbesgue differentiation theorem)**.** *Let $f \in \mathcal{L}^1(\mathbb{R}^n)$ then*

$$\lim_{m(B)\to 0,\, x\in B} \frac{1}{m(B)} \int_B f(y)dy = f(x) \qquad for\ a.e.\ x. \tag{3.5.1}$$

Proof. For each $\alpha > 0$, consider the set

$$E_\alpha = \left\{ x \ : \ \limsup_{m(B)\to 0,\, x\in B} \left| \frac{1}{m(B)} \int_B f(y)\, dy - f(x) \right| \right\}.$$

If we show that E_α has a measure zero, then $E = \bigcup_{n=1}^\infty E_{\frac{1}{n}}$ has measure zero and limit in the equation 3.5.1, will hold for all points of E^c.

Fix $\alpha > 0$ For each $\epsilon > 0$, we will choose a continuous function g of compact support with $||f - g||_{L^1(\mathbb{R}^n} < \epsilon$. Since g is a continuous function we have,

$$\lim_{m(B)\to 0,\, x\in B} \frac{1}{m(B)} \int_B g(y)dy = g(x), \quad \text{for all } x.$$

$$\frac{1}{m(B)} \int_B f(y)dy - f(x) = \frac{1}{m(B)} \int_B (f(y)-g(y))dy + \frac{1}{m(B)} \int_B g(y)dy - g(x) + g(x) - f(x).$$

We will get,

$$\limsup_{m(B)\to 0\, x\in B} \left| \frac{1}{m(B)} \int_B f(y)dy - f(x) \right| \leq (f-g)^*(x) + |g(x) - f(x)|.$$

where $(f - g)^*$is the maximal function. If $F_\alpha = \{x \ : \ (f - g0^*(x) > \alpha\}$ and $G_\alpha = \{x : |f(x) - g(x)| > \alpha\}$ then each $E_\alpha \subset (F_\alpha \cap G_\alpha)$, because if $u_1, u_2 > 0$, then $u_1 + u_2 > 2\alpha$ only if $u_i > \alpha$ for at least one u_i.

Also $m(G_\alpha) \leq \frac{1}{\alpha}||f - g||_{L^1(\mathbb{R}^n)}$ and from the weak type estimate for the maximal function we have $m(F_\alpha) \leq \frac{A}{\alpha}||f - g||_{L^1(\mathbb{R}^n})$ and from our earlier choice of g, $||f - g||_{L^1(\mathbb{R}^n}) < \epsilon$. Hence we get,

$$m(E_\alpha) \leq \frac{A}{\alpha}\epsilon + \frac{1}{\alpha}\epsilon.$$

Since ϵ is arbitrary, we must have $m(E_\alpha) = 0$. ∎

Chapter 4

Area and Coarea Formulas

Consider the Lipschitz map $f : \mathbb{R}^n \to \mathbb{R}^m$, In this chapter we will study this map and corresponding change of variable formulas. Depending on the sizes of n and m, we have two cases.

If $m \geq n$, the *Area formula* gives us n-dimensional measure of $f(A)$ counting multiplicity, can be calculated by integrating appropriate Jacobian of f over A.

If $n \geq m$, the *Coarea formula* states that the integral of $n - m$ dimensional measure of level sets of f is computed by integrating Jacobian of f over A.

This chapter is primarily taken from [3].

4.1 Lipschitz map and Rademacher's Theorem

4.1.1 Lipschitz map

We have seen some results on the Lipschitz functions in the section 1.7. Now we will study it in more details.

Definition 4.1 (locally Lipschitz map). A function $f : A \to \mathbb{R}^m$ is called locally Lipschitz map if for each *compact set* $K \subset A$ there exists a constant C_K such that $|f(x) - f(y)| \leq C_k|x - y| \quad \text{for all} \quad x, y \in K$.

.

Theorem 4.2 (Extension of Lipschitz function)**.** *Let $A \subset \mathbb{R}^n$ and $f : A \to \mathbb{R}^m$ be a Lipschitz map. Then there exists a Lipschitz function $\bar{f} : \mathbb{R}^n \to \mathbb{R}^m$ such that*

(i) $\bar{f} = f$ *on* A.

(ii) $Lip(\overline{f}) = \sqrt{m}\ Lip(f)$.

Proof. First we will prove this for $m = 1$, asssume $f : A \to \mathbb{R}$ a Lipschitz function. Define

$$\bar{f}(x) = \inf_{x \in A} \{f(a) + \ Lip(f)\ |x - a|\}.$$

Since f is a Lipschitz function, for all $a, b \in A$, $\quad |f(b) - f(a)| \leq Lip(f)\ |b - a|$.

therefore $f(b) - f(a) \leq Lip(f)\ |b - a|$ implies $f(b) \leq f(a) + Lip(f)\ |b - a|$.

Obviously $\bar{f}(b) \leq f(b)$, therefore we have $\bar{f}(b) = f(b)$ for $b \in A$.

If $x, y \in \mathbb{R}^n$ then

$$\begin{aligned}
\bar{f}(x) &\leq \inf_{a \in A} \{\ f(a) + \ Lip(f)\ |y - a| + |x - y|\} \\
&= \inf_{a \in A} \{\ f(a) + \ (Lip(f)\ |y - a|) + \ Lip(f)\ |x - y|\} \\
&= \inf_{a \in A} \{f(a) + \ Lip(f)\ |y - a|\} + Lip(f)\ |x - y| \\
&= \bar{f}(y) + Lip(f)\ |x - y|.
\end{aligned} \tag{4.1.1}$$

And similarly $\bar{f} \leq \bar{f}(x) + Lip(f)\ |x - y|$ implies $\quad |\bar{f}(x) - \bar{f}(y)| \leq Lip(f)\ |x - y|$.

Now for the general case $f : A \to \mathbb{R}^m$, $f = (f^1, f^2, \ldots, f^m)$ define $\bar{f} \equiv (\bar{f}^1, \bar{f}^2, \ldots, \bar{f}^m)$

$$|\bar{f}(x) - \bar{f}(y)|^2 = \sum_{i=1}^{m} |\bar{f^i}(x) - \bar{f^i}(y)|^2 \leq m\ (\ Lip(f)\ )^2 |x - y|^2$$

$$\text{implies} \quad |\bar{f}(x) - \bar{f}(y)| \leq \sqrt{m}\ (\ Lip(f)\ )\ |x - y|.$$

∎

4.1.2 Rademacher's Theorem

In this section we will prove that *Lipschitz function* is differentiable $\mathcal{L}^n$. Which is known as Rademacher's Theorem.

Definition 4.3. The function $f : \mathbb{R}^n \to \mathbb{R}^m$ is differentiable at $a \in R^n$ if there exist a linear mapping,

$$L : \mathbb{R}^n \to \mathbb{R}^m$$

such that

$$\lim_{x \to a} \frac{|f(x) - f(a) - L(x-a)|}{|x-y|} = 0$$

or equivalently,

$$f(x) = f(a) + L(x-a) + o(|x-a|) \quad as \quad x \to a.$$

Theorem 4.4 (Rademacher's Theorem)**.** *Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a locally Lipschitz function. Then f is a differentiable $\mathcal{L}^n$ a.e.*

Proof. Lets assume $m = 1$, since differentiability is local property we may assume f is Lipschitz.

Now fix $v \in \mathbb{R}^n$ and define

$$D_v(f(x)) = \lim_{t \to 0} \frac{f(x+tv) - f(x)}{t}$$

provided this limit exists.

First we will show that $D_v f(x)$ exists $\mathcal{L}^n$ a.e.

Since f is continuous,

$$\begin{aligned} \overline{D}_v f(x) &= \limsup_{t \to 0} \frac{f(x+tv) - f(x)}{t} \\ &= \lim_{k \to 0} \sup_{0<|t|<\frac{1}{k};\, t \in \mathbb{Q}} \frac{f(x+tv) - f(x)}{t} \end{aligned}$$

is the Borel measurable, as is the

$$\underline{D}_v f(x) = \liminf_{t \to 0} \frac{f(x+tv) - f(x)}{t}.$$

Then the set

$$\begin{aligned} A_v &= \{x \in \mathbb{R}^n \mid D_v f(x) \text{ does not exists}\} \\ &= \{x \in \mathbb{R}^n | \underline{D}_v f(x) < \overline{D}_v f(x)\} \end{aligned}$$

is Borel measurable. Now for each $x, v \in \mathbb{R}^n$, with $|v| = 1$, define $\phi : \mathbb{R} \to \mathbb{R}$ by

$$\phi(t) = f(x + tv)$$

. Then ϕ is Lipschitz, thus absolutely continuous and hence differentiable. $\mathcal{L}^1$ a.e. It follows that $\mathcal{H}^1(L_v \cap L) = 0$ for every line which is parallel to v. Then Fubini's theorem will give us

$$\begin{aligned} L^n(A_v) &= \int_{\{\langle a,v \rangle = 0\}} \mathcal{H}^1(A_v \cap L_x) dx \\ &= 0. \end{aligned}$$

Now we have shown that $D_v(f(x)$ a.e. Also

$$grad\, f(x) = \left(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n} \right).$$

Now we will show that $D_v(f(x) = v \cdot grad\, f(x)$ a.e..

To prove this, lets $\zeta \in C_0^\infty(\mathbb{R}^n$, Then

$$\int_{\mathbb{R}^n} \left[\frac{f(x+tv) - f(x)}{t} \right] \zeta(x) dx = - \int_{\mathbb{R}^n} \left[\frac{\zeta(x) - \zeta(x - tv)}{t} \right] f(x) dx.$$

Now let $t = \frac{1}{k}$ for $k = 1, 2, 3, \ldots$ in the above inequality and note that

$$\left| \frac{f(x + \frac{1}{k} v) - f(x)}{\frac{1}{k}} \right| \leq \mathrm{Lip}(f)|v| = \mathrm{Lip}(f).$$

Now by using Dominated Convergence Theorem to get that,

$$\begin{aligned} \int_{\mathbb{R}^n} D_v(f(x))\zeta(x) dx &= - \int_{\mathbb{R}^n} f(x) D_v \zeta(x) dx \\ &= - \sum_{i=1}^n v_i \int_{\mathbb{R}^n} f(x) \frac{\partial \zeta}{\partial x_i}(x) dx \\ &= \sum_{i=1}^n v_i \int_{\mathbb{R}^n} (v.grad\, f(x))\zeta(x) dx. \end{aligned}$$

where we used Fubini's Theorem and absolute continuity of f on lines. Since the above equality holds for any choice of $\zeta \in C_0^\infty(\mathbb{R}^n)$ implies $D_v(f) = v \cdot grad\ f\ \ \mathcal{L}^n$ a.e.

Let $\partial B(0,1)$ be a unit sphere. We choose $\{v_k\}_{k=1}^{\infty}$ to be countable, dense subset of $\partial B(0,1)$ and For $k \in \mathbb{N}$ set

$$A_k = \{x \in \mathbb{R}^n \mid D_{v_k} f(x),\ grad\ f(x) \text{ exist and } D_{v_k} f(x) = v_k \cdot grad\ f(x)\}$$

Then define

$$A = \bigcap_{k=1}^{\infty} A_k.$$

Observe $\mathcal{L}^n(\mathbb{R}^n \setminus A_k) = 0$ for each k and we have countably many k,

$$\text{Therefore } \mathcal{L}^n(\mathbb{R}^n \setminus A) = 0.$$

Finally we will show that f is differentiable at each point $x \in A$.

Fix any $x \in A$ and choose $v \partial B(0,1)$, $t \in \mathbb{R}$, $t \neq 0$ and write

$$Q(x,v,t) = \frac{f(x+tv) - f(x)}{t} - v \cdot grad\ f(x).$$

Then if $v' \in \partial B(0,1)$ we have

$$\begin{aligned}
|Q(x,v,t) - Q(x,v',t)| &\leq \left|\frac{f(x+tv) - f(x+tv'}{t}\right| + |(v-v') \cdot grad\ f| \\
&= \operatorname{Lip}(f)|v-v'| + |grad\ f(x)||v-v'| \\
&= (\sqrt{n}+1)\operatorname{Lip}(f)|v-v'|.
\end{aligned}$$

Now fix $\epsilon > 0$ and chose N large enough so that if $v \in \partial B(0,1)$, Then

$$|v - v_k| \leq \frac{\epsilon}{2(\sqrt{n}+1)\operatorname{Lip}(f)}, \qquad \text{for some } k \in \{1,2,\ldots,N\}.$$

We also have that,

$$\lim_{t \to 0} Q(x,v_k,t) = 0, \qquad \text{for } k = 1,2,\ldots,N.$$

and thus there exits a $\delta > 0$, so that

$$|Q(x,v_k,t)| \leq \frac{\epsilon}{2}, \quad \text{for all } 0 < |t| < \delta,\ k = 1,2,\ldots,N.$$

Therefore, it follows that for every $v \in \partial B(0,1)$, There is a $k \in \{1,2,\dots,N\}$, such that,

$$|Q(x,t,v)| \leq |Q(x,v_k,t)| + |Q(x,v,t) - Q(x,v_k,t)| < \epsilon.$$

If $0 < |t|\delta$, Choose any $y \in \mathbb{R}^n$, $y \neq x$. write $v = \frac{y-x}{|y-x|}$ so that $y = x + tv$ and $t = |x-y|$. Then

$$\begin{aligned}\frac{f(y) - f(x) - grad\ f(x)(y-x)}{|y-x|} &= \frac{f(x+tv) - f(x)}{t} - v \cdot grad\ f(x)\\ &= Q(x,v,t)\\ &\to 0\end{aligned}$$

as $t = |y-x| \to 0$.

Hence f is differntiable at x, with $Df(x) = grad\ f(x)$. ■

Corollary 4.5. *(i) If $f : \mathbb{R}^n \to \mathbb{R}^m$ is a locally Lipschitz function, and*

$$Z \equiv \{x \in \mathbb{R}^n \ : \ f(x) = 0\}.$$

Then $D\ f(x) = 0$ for $\mathcal{L}^n$ a.e. $x \in Z$.

(ii) If $f, g \in \mathbb{R}^n$ are locally Lipschitz, and

$$Y \equiv \{X \in \mathbb{R}^n \ : \ g(f(x)) = x\}$$

Then

$$D\ g(f(x))\ Df(x) = I \quad for \quad \mathcal{L}^n\ a.e. x \in Y.$$

4.2 Linear maps and Jacobian

4.2.1 Linear maps

Before the beginning of this section, we will revise some definitions from the Linear Algebra.

Definition 4.6. A linear map $O : \mathbb{R}^n \to \mathbb{R}^m$ is **orthogonal** if

$$O(x) \cdot O(y) = x \cdot y \quad \text{for all} \quad x, y \in \mathbb{R}^n.$$

Definition 4.7. A linear map $S : \mathbb{R}^n \to \mathbb{R}^n$ is **symmetric** if

$$x \cdot (Sy) = (Sx) \cdot y \quad \text{for all} \quad x, y \in \mathbb{R}^n.$$

Definition 4.8. A linear map $D : \mathbb{R}^n \to \mathbb{R}^n$ is **diagonal** if there exists $d_1, \ldots, d_n \in \mathbb{R}$ such that

$$Dx = (d_1 x_1, \ldots, d_n x_n) \quad \text{for all} \quad x \in \mathbb{R}^n.$$

Definition 4.9. Let $A : \mathbb{R}^n \to \mathbb{R}^m$ be linear. The **adjoint** of A is the linear map,

$$A^\star : \mathbb{R}^m \to \mathbb{R}^n \text{ defined by } \quad x \cdot (A^\star y) = (Ax) \cdot y \quad \text{for all} \quad x \in \mathbb{R}^n,\ y \in \mathbb{R}^m.$$

Theorem 4.10. *(1)* $(A^\star)^\star = A$.

(2) $(A \circ B)^\star = B^\star \circ A^\star$.

(3) If $O : \mathbb{R}^n \to \mathbb{R}^n$ is orthogonal then $O^\star = O^{-1}$.

(4) If $S : \mathbb{R}^n \to \mathbb{R}^n$ is symmetric then $S^\star = S$.

(5) If $S : \mathbb{R}^n \to \mathbb{R}^n$ is symmetric then, there exist a orthogonal map $O : \mathbb{R}^n \to \mathbb{R}^n$ and a diagonal map $D : \mathbb{R}^n \to \mathbb{R}^n$ such that,

$$S = O \circ D \circ O^{-1}.$$

(6) If $O : \mathbb{R}^n \to \mathbb{R}^m$, then $n \leq m$ and

$$O^\star \circ O = I \quad \text{on} \quad \mathbb{R}^n,$$

$$O \circ O^\star = I \quad \text{on} \quad O(\mathbb{R}^n).$$

Theorem 4.11 (Polar Decomposition)**.** *Let $L : \mathbb{R}^n \to \mathbb{R}^m$ be a linear mapping.*

(i) If $n \leq m$, then there exists a symmetric map $S : \mathbb{R}^n \to \mathbb{R}^m$ and a orthogonal map $O : \mathbb{R}^m \to \mathbb{R}^n$ such that

$$L = O \circ S.$$

(ii) If $n \geq m$, there exists a symmetric map $S : \mathbb{R}^m \to \mathbb{R}^m$ and a orthogonal map $O : \mathbb{R}^m \to \mathbb{R}^n$ such that,

$$L = S \circ O^\star.$$

Also S is positive semidefinite map means all eigenvalues of S are nonnegative.

Proof. (i) Suppose $n \leq m$, consider the map

$$C = L^* \circ L : \mathbb{R}^n \to \mathbb{R}^n.$$

Now

$$\begin{aligned}(Cx) \cdot y &= (L^* \circ Lx) \cdot y \\ &= Lx \cdot Ly \\ &= x \cdot L^* \circ Ly \\ &= x \cdot Cy.\end{aligned}$$

Therefore C is the symmetric map Also $C(x) \cdot x = Lx \cdot Lx = |Lx|^2 \geq 0$.

Therefore C is symmetric, Positive semidefinite, hence there exists an orthogonal basis $\{x_k\}_{k=1}^n$ of $\mathbb{R}^n$ and scalars $\mu_1, \mu_2, \ldots, \mu_n \geq 0$, such that

$$Cx_k = \mu_k x_k, \qquad k = 1, 2, \ldots, n.$$

Suppose $\mu_k = \lambda_k^2$, $\lambda_k \geq 0$, $k = 1, 2, \ldots, n$. We will show that there exits orthonormal set $\{z_k\}_{k=1}^n$ in $\mathbb{R}^m$ such that $Lx_k = \lambda_k z_k$, $k = 1, 2, \ldots, n$.

If $\lambda_k \neq 0$ then define $z_k = \frac{1}{\lambda_k} Lx_k$. Then if $\lambda_k, \lambda_l \neq= 0$ then,

$$\begin{aligned}z_k \cdot z_l &= \frac{1}{\lambda_k \lambda_l} Lx_k \cdot Lx_l \\ &= \frac{1}{\lambda_k \lambda_l} (Cx_k) \cdot x_l \\ &= \frac{\lambda_k^2}{\lambda_k \lambda_l} x_k \cdot x_l \\ &= \frac{\lambda_k}{\lambda_l} x_k \cdot x_l \\ &= \frac{\lambda_k}{\lambda_l} \delta_{kl}.\end{aligned}$$

Therefore the set $\{z_k \mid \lambda_k \neq 0\}$ is orthonormal. For $\lambda_k = 0$, choose z_k to be any nonzero vector such that $\{z_k\}_{k=1}^n$ is orthonormal.

Now define

$$S : \mathbb{R}^n \to \mathbb{R}^n \quad \text{by} \quad Sx_k = \lambda_k x_k \qquad k = 1, 2, \ldots, n.$$

and

$$O : \mathbb{R}^n \to \mathbb{R}^m \quad \text{by} \quad Ox_k = z_k \qquad k = 1, 2, \ldots, n.$$

Then $O \circ Sx_k = \lambda_k Ox_k = \lambda_k z_k = Lx_k$ therefore $L = O \circ S$. Here S is symmetric mapping and O is orthogonal mapping because $Ox_k \cdot Ox_l = z_k \cdot z_l = \delta_{kl}$.

To prove (ii), Apply (i) to L^*. ■

Remarks 4.2.1. If S is not invertible then then O is not unique. As in the last line for $\lambda_k = 0$ choices of z in not uique.

Lemma 4.12. $L : \mathbb{R}^n \to \mathbb{R}^m \quad (n \geq m)$ *be a linear map. Then L can be written as,* $L = S \circ P \circ Q$, *where* $P : \mathbb{R}^n \to \mathbb{R}^m$ *is a canonical projection map and* $Q : \mathbb{R}^n \to \mathbb{R}^n$ *is orthogonal and* $S : \mathbb{R}^m \to \mathbb{R}^m$ *is symmetric.*

Definition 4.13 (Jacobian)**.** Assume $L : \mathbb{R}^n \to \mathbb{R}^m$ is linear.

(i) If $n \leq m$, we write $L = O \circ S$ as above, and we define **Jacobian** of L to be

$$\mathbf{J}(L) = |\ det\ S\ |.$$

(ii) If $n \geq m$, We write $L = S \circ O^\star$ as above and we define **Jacobian** of L to be

$$\mathbf{J}(L) = |\ det\ S\ |.$$

Remark: $\mathbf{J}(L) = \mathbf{J}(L^\star)$.

Theorem 4.14. *1. If* $n \leq m$, *then* $\mathbf{J}(L)^2 = det\ (L^\star \circ L)$.

2. If $n \geq m$, *then* $\mathbf{J}(L)^2 = det\ (L \circ L^\star)$.

Proof. (1) Suppose $m \leq n$, L can be written as $L = O \circ S$ and $L^* = S^* \circ O^* = S \circ O^*$, Where O is orthogonal matrix and S is symmetric matrix. Therefore $O \circ O^* = I$ and $S^* = S$ Then

$$L^* \circ L = S \circ O^* \circ O \circ S = S^2.$$

Therefore $det(L^* \circ L) = |det(S)|^2$ and $\mathbf{J}(L) = |\ det S\ |$ Hence $\mathbf{J}(L)^2 =\ det\ (L^* \circ L)$.

Proof of (2) is similar. ■

Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a Lipschitz, Then by Rademacher's Theorem, f is differentiable L^n a.e., Therefore $Df(x)$ exists and can be regarded as linear mapping from $\mathbb{R}^n$ into $\mathbb{R}^m$ for $\mathcal{L}^n$ a.e. $x \in \mathbb{R}^n$.

NOTATION let $f : \mathbb{R}^n \to \mathbb{R}^m$, $f = (f^1, f^2, \dots, f^m)$, Then the gradient matrix is

$$\mathbf{D}f = \begin{bmatrix} \frac{\partial f^1}{\partial x^1} & \cdots & \frac{\partial f^1}{\partial x^n} \\ \vdots & & \vdots \\ \frac{\partial f^m}{\partial x^1} & \cdots & \frac{\partial f^m}{\partial x^n}. \end{bmatrix}$$

Definition 4.15 (Jacobian)**.** The *Jacobian* of f is

$$\mathcal{J}f(x) = \mathbf{J}(Df(x)) \qquad \text{for } \mathcal{L}^n \text{ } a.e.$$

.

4.3 The area formula

As discussed at the beginning of this chapter, we will now consider our case with $n \leq m$ and find the area formula.

Theorem 4.16. *Let $L : \mathbb{R}^n \to \mathbb{R}^m$ be a linear map, Then for each Lebesgue measurable set $A \subset \mathbb{R}^n$,*

$$\mathcal{H}^n(L(A)) = \mathbf{J}(L)\mathcal{L}^n(A) = \int_A \mathbf{J}L \, dx.$$

Proof. L be a linear map. Then from theorem 4.11, $L = O \circ S$, where O is an orthogonal map and S is symmetric positive semidefinite map. Also since A is Lebesgue measurable set from theorem 1.4, $\mathcal{H}^n(A) = \mathcal{L}^n(A)$ in $\mathbb{R}^n$ and from theorem 1.28, $\mathcal{H}^n(OS(A)) = \mathcal{H}^n(S(A))$. Therefore

$$\mathcal{H}^n(L(A)) = \mathcal{H}^n(O\ S(A)) = \mathcal{H}^n(S(A)) = \mathcal{L}^n(S(A)).$$

If S is diagonal and positive semi-definite then $\mathcal{L}^n(S(A)) = det\ S\ \mathcal{L}^n(A)$ and also $\mathbf{J}(L) = det\ S$. Therefore $\mathcal{H}^n(L(A)) = \mathbf{J}(L)\mathcal{L}^n(A)$. For the general case we will write $S = Q^*DQ$ where $Q : \mathbb{R}^n \to \mathbb{R}^n$ is orthogonal, and $D : \mathbb{R}^n \to \mathbb{R}^n$ is diagonal. Then we will again use Theorem 1.28 and follow above steps.

■

Theorem 4.17 (the Area Formula)**.** *Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a Lipschitz map. For each Lebesgue measurable set A, $A \subset \mathbb{R}^n$, and let* $\mathbf{J}f(x) = \sqrt{det\ (\nabla f^*(x)\nabla f(x))}$.

If f is injective then

$$\int_A \mathbf{J}f(x)d\mathcal{H}^n(x) = \mathcal{H}^n(f(A)).$$

More generally,

$$\int_A \mathbf{J}f(x)\ dx = \int_{\mathbb{R}^m} \mathcal{H}(A \cap f^{-1}\{y\})\ d\mathcal{H}^n(y).$$

Here $\mathcal{H}(A \cap f^{-1}\{y\}$ is the number of points in the intersection of A with the preimage of f. We will denote $\mathcal{H}(A \cap f^{-1}\{y\}$ by $N(f, A, y)$.

Note that $N(f, y, A) \neq 0$ if and only if $y \in f(A)$.

Change of variable formula is a corollary to above formula.

Theorem 4.18 (Change of variable Formula)**.** *Let $n \leq m$ and $f : \mathbb{R}^n \to \mathbb{R}^m$ be a Lipschitz map. $A \subset \mathbb{R}^n$ be a lebesgue measurable set. And define the $\mathbf{J}f$ as above.*

If $u : \mathbb{R}^n \to [0, \infty]$ is lebesgue measurable, then

$$y \in \mathbb{R}^m \mapsto \sum_{x \in f^{-1}(y)} u(x).$$

is $\mathcal{H}^n-$ measurable and

$$\int_{\mathbb{R}^n} u(x)\mathbf{J}f(x)d\mathcal{H}^n(x) = \int_{\mathbb{R}^m} \left(\sum_{x \in f^{-1}\{y\}} u(x) \right) d\mathcal{H}^n(y).$$

As a result $v : \mathbb{R}^n \to [0, \infty]$ and $A \subset \mathbb{R}^n$ are $\mathcal{H}^n$ measurable then

$$\int_A v \circ f(x)\mathbf{J}f(x)d\ \mathcal{H}^n(x) = \int_{\mathbb{R}^m} v(y)N(f, A, y)d\ \mathcal{H}^n(y).$$

Example 4.3.1. Let $f : \mathbb{R} \to \mathbb{R}^m$ be a Lipschitz curve, then $\mathbf{J}f(x) = |f'(x)|$ a.e. and From the area formula , for the interval I,

$$\mathcal{H}^1(f(I)) = \int_I |f'(x)|dx \qquad \text{if f is injective.}$$

For the f non-injective consider the function $f(x) = (\cos(x), \sin(x))$ and suppose $I = [0, 3\pi]$, then

$$3\pi = \int_I |f'(x)|dx = \int_{\mathbb{R}^2} N(f, A, y)\, dy.$$

Where the multiplicity function $N(f, A, y)$ is

$$N(f, I, y) = \begin{cases} 0, & \text{if } y \text{ is not in the unit circle} \\ 1, & \text{if } y = (y_1, y_2) \text{ belongs to the unit circle, and } y_2 < 0 \\ 2, & \text{if } y = (y_1, y_2) \text{ belongs to the unit circle, and } y_2 \geq 0. \end{cases}$$

4.4 The coarea formula

Throughout this section we will assume that $n \geq m$.

Lemma 4.19. *$f : \mathbb{R}^n \to \mathbb{R}^n$ is integratble and $S : \mathbb{R}^n \to \mathbb{R}^n$ is invertible linear map with det $S > 0$. Then*

$$\int f(y)\mathcal{L}^n(y)(dy) = det\ S \int f \circ S(z)\mathcal{L}^n(dz).$$

Proof. Suppose f is the characteristic function of measurable set then problem is reduced to $n = m$ case in the 4.16. Then it follows for the finite linear combination of the characteristic functions. Then standard approximation argument will give the result. ■

Theorem 4.20. *Assume that $L : \mathbb{R}^n \to \mathbb{R}^m$ is a linear map with $n \geq m$, Then for every measurable set $A \subset \mathbb{R}^n$,*

$$\mathbf{J}(L)\mathcal{L}^n(A) = \int_A \mathbf{J}L dx = \int_{\mathbb{R}^n} \mathcal{H}^{m-n}(A \cap L^{-1}\{y\})dy \tag{4.4.1}$$

Proof. We will write $L = S \circ P \circ Q$ as before.

1. Suppose $\mathbf{J}L = 0$, Then $det\ S = 0$. Therefore image of L must be contained inside subspace of $\mathbb{R}^m$ of dimension at most $m-1$. Therefore $L^{-1}(A \cap L^{-1}\{y\}$ is empty at $\mathcal{L}^n$ a.e. Therefore both sides in the equation 4.4.1 are zero.

2. Now assume that $\mathbf{J}L > 0$, therefore S is invertible.

We will change variables in the integral on the right hand side of the equation 4.4.1, by using Lemma 4.19,

$$\int_{\mathbb{R}^m} \mathcal{H}^{m-n}(A \cap L^{-1}\{y\})dy = \mathbf{J}L \int_{\mathbb{R}^m} \mathcal{H}^{m-n}(A \cap L^{-1}\{Sz\}dz \tag{4.4.2}$$

Now for every $y \in \mathbb{R}^m$, Since $L^{-1} = Q^{-1} \circ P^{-1} \circ S^{-1}$,

$$A \cap L^{-1}\{Sz\} = A \cap Q^{-1}(P^{-1}\{z\}) = Q^{-1}\left(Q(A) \cap P^{-1}\{z\}\right).$$

Now we will use rotational invariance of the Hausdorff measure.

$$\mathcal{H}^{m-n}\left(A \cap L^{-1}\{Sz\}\right) = \mathcal{H}^{m-n}\left(Q(A) \cap P^{-1}\{z\}\right)$$

For every $z \in \mathbb{R}^m$.

3. Now by using Fubini's Theorem $z \mapsto \mathcal{H}^{m-n}\left(Q(A) \cap P^{-1}\{z\}\right)$ is $\mathcal{L}^m$ measurable. Again using the rotational invariance of Hausdorff measure,

$$\int_{\mathbb{R}^m} \mathcal{H}^{m-n}\left(Q(A) \cap P^{-1}\{z\}\right) = L^n\left(Q(A)\right) = L^n(A).$$

By using equation 4.4.2, we will get desire result.

■

Theorem 4.21 (the Coarea formula). *Assume $n \geq m$, let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a Lipschitz map.*

Define $\mathbf{J}f(x) = \sqrt{det\left(\nabla f(x) \nabla f^*(x)\right)}$. *Then for every measurable $A \subset \mathbb{R}^n$,*

$$\int_A \mathbf{J}f(x)dx = \int_{\mathbb{R}^m} \mathcal{H}^{n-m}(A \cap f^{-1}\{y\}d\mathcal{H}^m(y). \tag{4.4.3}$$

And for every measurable $g : \mathbb{R}^n \to [0, \infty]$,

$$\int_{\mathbb{R}^n} g(x)\mathbf{J}f(x)dx = \int_{\mathbb{R}^m} \left(\int_{f^{-1}\{y\}} g(x)d\mathcal{H}^{m-n}(x)\right) d\mathcal{H}^n(y). \tag{4.4.4}$$

Proof. Suppose A is Lebesgue measurable set. Then under Lipschitz map, its image is also Lebesgue measurable. Since A is Lebesgue measurable, A can be written as the union of compact sets and sets of measure zero. Now Lipschitz map is continuous; therefore, it maps compact sets to compact sets. And set of zero measure to the set of zero measure. Therefore its image is Lebesgue measurable.

$\therefore$ If $A \subset \mathbb{R}^n$ is $\mathcal{L}^n$ measurable then $f(A)$ is also $\mathcal{L}^m$ measurable.

Let μ be any measure on $\mathbb{R}^m$ and $g : \mathbb{R}^m \to \mathbb{R}$ be any function, Then we will define,

$$\int_{\mathbb{R}^m}^{*} g(x)d\mu(y) = \inf\left\{\int_{\mathbb{R}^m} h(y)d\mu(y) \ : \quad h \text{ is } \mu \text{ measurable and } g \leq h \; a.e.\right\}.$$

Then
$$\int_{\mathbb{R}^m}^{*} \mathcal{H}^{n-m}(A \cap f^{-1}\{y\}d\mathcal{H}^m \leq \frac{\omega_{n-m}\omega_m}{\omega_n}(Lip(f))^m\mathcal{H}^n(A),$$
where ω_k is voulume of the $k-$dimensional ball with radius $\frac{1}{2}$.

Here we have approximated function by the linear functions and we know the formula for the linear functions. $y \in \mathbb{R}^m \mapsto \mathcal{H}^{m-n}(A \cap f^{-1}\{y\})$ is $\mathcal{H}^n-$measurable and therefore

$$\int_{\mathbb{R}^m} \mathcal{H}^{n-m}(A \cap f^{-1}\{y\}d\mathcal{H}^m \leq \frac{\omega_{n-m}\omega_m}{\omega_n}(Lip(f))^m\mathcal{H}^n(A).$$

Given a Lipschitz map f, we can find a C^1 map that agrees with f a.e., gradient also agree except set of measure zero. To control the approximation we will use above bound.

For each $\epsilon > 0$, for a Lipschitz function f there exist a function f_ϵ and a measurable set G_ϵ such that

$$f = f_\epsilon \text{ and } \nabla f = \nabla f_\epsilon \text{ in } G_\epsilon, \quad \mathcal{L}^n(\mathbb{R} \setminus G_\epsilon) < \epsilon.$$

Let $A_\epsilon = A \cap G_\epsilon$ then $0 \leq C \; Lip(f)$ for $\mathcal{L}^n$ a.e. x, for some C depending on n, m, and $Lip(f)$

$$\left|\int_A \mathbf{J}f(x)dx - \int_{A_\epsilon} \mathbf{J}f(x)dx\right| = \int_{A\setminus A_\epsilon} \mathbf{J}f(x)dx \leq C\epsilon.$$

Therefore we have,

$$\begin{aligned}\left|\int_{\mathbb{R}^m} \mathcal{H}^{n-m}(A \cap f^{-1}\{y\})dy - \int_{\mathbb{R}^m} \mathcal{H}^{n-m}(A_\epsilon \cap f^{-1}\{y\})dy\right| &= \int_{\mathbb{R}^m} \mathcal{H}^{n-m}((A \setminus A_\epsilon) \cap f^{-1}\{y\}) \\ &\leq C\mathcal{L}^n(A \setminus A_\epsilon) \\ &= C\epsilon.\end{aligned}$$

For a constant C depending on n, m and $Lip(f)$. If the Coarea formula holds for the C^1 functions then

$$\int_{A_\epsilon} \mathbf{J}f(x)dx = \int_{\mathbb{R}^n} \mathcal{H}^{n-m}(A_\epsilon \cap f^{-1}\{y\})dy.$$

for every $\epsilon > 0$, and from the above steps Coarea formula for the Lipschitz function holds. ■

Bibliography

[1] G. D. Barra. ***Measure theory and Integration***. New Age International, 1981.

[2] J. Briggs and T. Tyree. Haussdorff measure. URL https://sites.math.washington.edu/~farbod/teaching/cornell/math6210pdf/math6210Hausdorff.pdf.

[3] C. Evans and R. Gariepy. ***Measure Theory and Fine Properties of Functions***. CRC Press, 1992.

[4] R. Jerrard. Geometric measure theorey. URL http://www.math.toronto.edu/rjerrard/1501/gmt.html.

[5] P. Karandikar. Prof B V Rao Measure theory course notes. URL https://www.cmi.ac.in/~prateek/measure_theory/.

[6] S. G. Krantz and H. R. Parks. ***Geometric Integration Theory***. Springer, 2008.

[7] H. L. Royden. ***Real Analysis***. Pearson, 2007.

[8] E. M. Stein and R. Shakarchi. ***Real Analysis***. New Age International, 2010.

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