Mobius transformations
and the related inequalities

Tamanna

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Certificate of Examination

This is to certify that the dissertation titled **"Mobius Transformations- and the related inequalities"** submitted by **Tamanna** (Reg. No. MS15141) for the partial fulfilment of BS-MS dual degree programme of the Institute, has been examined by the thesis committee duly appointed by the Institute. The committee finds the work done by the candidate satisfactory and recommends that the report be accepted.

Dr. Pranab Sardar   Dr. Jotsaroop Kaur   Dr. Krishnendu Gongopadhyay

(Supervisor)

Dated: December 3, 2020

Declaration

The work presented in this dissertation has been carried out by me under the guidance of Dr. Krishnendu Gongopadhyay at the Indian Institute of Science Education and Research Mohali.

This work has not been submitted in part or in full for a degree, a diploma, or a fellowship to any other university or institute. Whenever contributions of others are involved, every effort is made to indicate this clearly, with due acknowledgement of collaborative research and discussions. This thesis is a bonafide record of original work done by me and all sources listed within have been detailed in the bibliography.

Tamanna

(Candidate)

Dated: December 3, 2020

In my capacity as the supervisor of the candidate's project work, I certify that the above statements by the candidate are true to the best of my knowledge.

Dr. Krishnendu Gongopadhyay
(Supervisor)

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Notations

$\mathcal{M}$:	Group of Mobius Transformations
$\tilde{\mathbb{C}}$:	Extended complex plane $\mathbb{C} \cup \{\infty\}$
$tr(A)$:	trace of A
$[f, g]$:	commutator $fgf^{-1}g^{-1}$
$q(z_1, z_2)$:	chordal distance between z_1, z_2
$\mathbf{j}$:	(0,0,1)
$\tilde{H}$:	upper half-space
$\tilde{f}$:	Poincare extension of f in $\tilde{H}$
A^*:	Hermitian transpose of A

Contents

Abstract

The aim is to study the Möbius transformations and the various norms related to it. These norms are defined by using three parameters which are invariant under conjugation. Using these parameters we define a two generator group. We then derive analogous results when this group is an arbitrary discrete subgroup of $\mathcal{M}$. Lastly, continued fractions is studied by examining action of Möbius maps in hyperbolic space as continued fractions can be regarded as sequence of Möbius maps.

Chapter 1

Introduction

We associate with each Möbius transformation

$$f = \frac{az+b}{cz+d} \in \mathcal{M}, \quad ad - bc = 1, \tag{1.1}$$

the matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{C}) \tag{1.2}$$

and set tr(f)=tr(A).Now $\forall$ f,g$\in \mathcal{M}$, we define three complex numbers as

$$\beta(f) = tr^2(f) - 4, \quad \beta(g) = tr^2(g) - 4, \quad \gamma(f,g) = tr([f,g]) - 2 \tag{1.3}$$

$\beta(f)$,$\beta(g)$ and $\gamma(f,g)$ are called the parameters of the two generator subgroup $< f, g >$ and we write

$$par(< f, g >) = (\gamma(f,g), \beta(f), \beta(g))$$

$\forall\ f \in \mathcal{M}$, we can have two representation matrices say A and B. Then $tr^2(A) = tr^2(B)$. If A and B are representation matrices for Möbius transformations f and g then they are determined to within a factor of -1 and so trace of commutator is uniquely determined. Thus the parameters are uniquely defined and are independent of the choice of representations for f and g. Thus $< f, g >$ is uniquely determined upto conjugacy whenever $\gamma(f,g)$ is non-zero.

We define chordal distance between $z_1, z_2 \in \tilde{\mathbb{C}}$ as

$$q(z_1, z_2) = \frac{2|z_1 - z_2|}{(|z_1|^2 + 1)^{1/2}(|z_2|^2 + 1)^{1/2}} \tag{1.4}$$

and

$$q(z_1, \infty) = \frac{2}{(|z_1|^2 + 1)^{1/2}}$$

z_1 and z_2 are antipodal if $q(z_1, z_2) = 2$. We define a Mobius tranformation f as a chordal isometry if

$$q(f(z_1), f(z_2)) = q(z_1, z_2) \quad \forall z_1, z_2 \in \tilde{\mathbb{C}}$$

Define

$$d(f, g) = sup\{q(f(z), g(z)) : z \in \tilde{\mathbb{C}}\} \tag{1.5}$$

This is a metric on $\mathcal{M}$.

Theorem 1.0.1. *Let G be a subgroup of $\mathcal{M}$. Then G is discrete if $\exists$ a constant d=d(G)>0 st*

$$d(f, g) \geq d \tag{1.6}$$

for every distinct f and g in G. G is nonelementary if its limit set L(G) has cardinality ≥ 2, G is Fuchsian if some disk or half plane D is preserved by each element of G and G is purely elliptic if each element of G is elliptic or the identity.

Theorem 1.0.2. *If $< f, g >$ is nonelementary and discrete, then*

$$|\gamma(f, g)| + |\beta(f)| \geq 1, \quad |\gamma(f, g)| + |\beta(g)| \geq 1 \tag{1.7}$$

For f,g$\in \mathcal{M}$, if f or g is close to id, $\beta(f)$ *or* $\beta(g)$ will be small along with $\gamma(f, g)$. (1.7) will then imply that $< f, g >$ is nonelementary and discrete.

Now we define the three norms for the given Möbius Transformastion f :

(i) Matrix norm: If f is represented by the A as given in (1.2), we define matrix norm as follows,

$$m(f) = ||A - A^{-1}|| = (2|a - d|^2 + 4|b|^2 + 4|c|^2)^{\frac{1}{2}} \tag{1.8}$$

where $||A|| = tr(AA^*)^{\frac{1}{2}}$ denotes the Euclidean norm of matrix A$\in GL(2,\mathbb{C})$ and A^* is Hermitian transpose of A.

(ii) Chordal norm: Chordal norm is defined in terms of the metric given in (1.5) as

$$d(f) = d(f, id) = sup\{q(f(z), z) : z \in \tilde{\mathbb{C}}\} \tag{1.9}$$

(iii) Hyperbolic norm: Hyperbolic norm for f is given as

$$\varrho(f) = \varrho_{\tilde{H}}(\tilde{f}(\mathbf{j}), \mathbf{j}) \tag{1.10}$$

where $\varrho_{\tilde{H}}$ is the hyperbolic distance in upper half-space with curvature -1.

These norms remain invariant wrt conjugation by chordal isometries.

In its action on $\tilde{\mathbb{C}}$, a Möbius transformation g has exactly one fixed point, exactly two fixed points or is the identity. This classification is invariant under conjugation.

Certain normalized Möbius transformations are:

For every non-zero k in $\mathbb{C}$ we define

$$m_k(z) = kz \quad (k \neq 1) \tag{1.11}$$

and

$$m_1(z) = z + 1; \tag{1.12}$$

These are called the standard forms. $\forall$ k,

$$tr^2(m_k) = k + \frac{1}{k} + 2 \tag{1.13}$$

If g($\neq I$) is any Möbius transformation then either g has exactly two fixed points α and β in $\tilde{\mathbb{C}}$ or g has a unique fixed point α in $\tilde{\mathbb{C}}$. We choose β to be a distinct point other than α. Let h be a Möbius transformation st

$$h(\alpha) = \infty, \quad h(\beta) = 0, \quad h(g(\beta)) = 1 \quad if g(\beta) \neq \beta, \tag{1.14}$$

Then,

$$hgh^{-1}(\infty) = \infty, \quad hgh^{-1}(0) = 0 \quad if \quad g(\beta) = \beta, \quad hgh^{-1}(0) = 1 \quad if g(\beta) \neq \beta \tag{1.15}$$

If g fixes both α and β, then hgh^{-1} fixes 0 and ∞ and for some $k \neq 1$, we have $hgh^{-1} = m_k$. If g fixes α only then hgh^{-1} fixes ∞ only and we have $hgh^{-1}(0) = 1$; and hence $hgh^{-1} = m_1$. This proves that any Möbius transformation g($\neq I$) is conjugate to one of the standard forms m_k.

Chapter 2

Traces and the various norms

2.1 Matrix Norm

The following lemma checks how far A and B commute:

Lemma 2.1.1. *For A,B∈* $SL(2,\mathbb{C})$*, we have*

$$\|AB - BA\|^2 \leqq \frac{1}{8}\left\|A - A^{-1}\right\|^2 \left\|B - B^{-1}\right\|^2 \tag{2.1}$$

This is sharp inequality.

Proof. Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $B = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$. Then

$\|AB - BA\|^2 = 2|b\gamma - c\beta|^2 + |b(\alpha - \delta) - \beta(a - d)|^2 + |c(\alpha - \delta) - \gamma(a - d)|^2$

$= (|b|^2 + |c|^2)\,|\alpha - \delta|^2 + (|\beta|^2 + |\gamma|^2)\,|a - d|^2 + u$

where we can look u as

$u = 2|b\gamma - c\beta|^2 - 2\,\mathrm{Re}((\bar{a} - \bar{d})(\alpha - \delta)(b\bar{\beta} + c\bar{\gamma}))$

and further, $\frac{1}{8}\left\|A - A^{-1}\right\|^2 \left\|B - B^{-1}\right\|^2 = 2\left(\frac{1}{2}|a - d|^2 + |b|^2 + |c|^2\right)\left(\frac{1}{2}|\alpha - \delta|^2 + |\beta|^2 + |\gamma|^2\right)$

$= (|b|^2 + |c|^2)\,|\alpha - \delta|^2 + (|\beta|^2 + |\gamma|^2)\,|a - d|^2 + v$

where v can be, $v = \frac{1}{2}|a - d|^2|\alpha - \delta|^2 + 2\,(|b|^2 + |c|^2)\,(|\beta|^2 + |\gamma|^2)$

$= \frac{1}{2}|a - d|^2|\alpha - \delta|^2 + 2|b\bar{\beta} + c\bar{\gamma}|^2 + 2|b\gamma - c\beta|^2$

So,

$$v - u = \frac{1}{2}|a-d|^2|\alpha-\delta|^2 + 2\,\mathrm{Re}((\bar{a}-\bar{d})(\alpha-\delta)(b\bar{\beta}+c\bar{\gamma})) + 2|b\bar{\beta}+c\bar{\gamma}|^2 \tag{2.2}$$

therefore,

$$\begin{aligned} &\tfrac{1}{8}\left\|A-A^{-1}\right\|^2\left\|B-B^{-1}\right\|^2 - \|AB-BA\|^2 \\ &= 2\left|\tfrac{1}{2}(\bar{a}-\bar{d})(\alpha-\delta) + (\bar{b}\beta+\bar{c}\gamma)\right|^2 \geqq 0 \end{aligned} \tag{2.3}$$

For b=c=0 and $\alpha = \delta$, the right hand sides of above eqaution vanishes and we obtain equality. □

Lemma 2.1.2. *If C is in $GL(2,\mathbb{C})$, then we can write,*

$$|\det(C)| \leqq \frac{1}{2}\|C\|^2 \tag{2.4}$$

For $C \in$ SL(2,C), we have

$$\det(C-I) = 2 - \mathrm{tr}(C) \tag{2.5}$$

Lemma 2.1.3. *For f and g belonging to $\mathcal{M}$,we have:*

$$|\beta(f)| \leqq \frac{1}{2}m(f)^2 \tag{2.6}$$

and,

$$|\gamma(f,g)| \leqq \frac{1}{16}m(f)^2m(g)^2 \tag{2.7}$$

Both of above inequalities are sharp.

Proof. If f and g are represented by matrices A and B in $SL(2,\mathbb{C})$, then by setting $C = A^2$ in 2.5, we get

$$\begin{aligned} |\beta(f)| &= \left|\mathrm{tr}^2(A) - 4\right| = \left|\mathrm{tr}\left(A^2\right) - 2\right| = \left|\det\left(A^2 - I\right)\right| \\ &= \left|\det\left(A - A^{-1}\right)\right| \leqq \frac{1}{2}\left\|A - A^{-1}\right\|^2 = \frac{1}{2}m(f)^2 \end{aligned} \tag{2.8}$$

Now considering C matrix as commutator of A and B, we get

$$
\begin{aligned}
|\gamma(f,g)| &= |\operatorname{tr}([A,B])-2| = \left|\det\left((AB)(BA)^{-1}-I\right)\right| = |\det(AB-BA)| \\
&\leqq \frac{1}{2}\|AB-BA\|^2 \leqq \frac{1}{16}\left\|A-A^{-1}\right\|^2\left\|B-B^{-1}\right\|^2 = \frac{1}{16}m(f)^2m(g)^2
\end{aligned} \tag{2.9}
$$

So if

$$
f = a^2 z \text{ and } g = \frac{bz+c}{cz+b}
$$

then the equality holds.

□

Lemma 2.1.4. *If $f \in \mathcal{M}$ with fix $(f) = \{z_1, z_2\}$,*

$$
|\beta(f)| = \frac{1}{2}\frac{q\left(z_1,z_2\right)^2}{8-q\left(z_1,z_2\right)^2}m(f)^2 \tag{2.10}
$$

Proof. For $z_1 = -r$ and $z_2 = r$. f can be represented by,

$$
A = \begin{pmatrix} a & br \\ br^{-1} & a \end{pmatrix}, \quad b^2 = a^2 - 1
$$

$$
\beta(f) = \operatorname{tr}(A)^2 - 4 = 4b^2, \quad q(-r,r) = \frac{4r}{r^2+1} \tag{2.11}
$$

and so,

$$
m(f)^2 = \left\|A-A^{-1}\right\|^2 = \left(r^2+r^{-2}\right)|2b|^2 = 2\frac{8-q(-r,r)^2}{q(-r,r)^2}|\beta(f)| \tag{2.12}
$$

□

Lemma 2.1.5. *If f and g are in $\mathcal{M}$ with fix$(f) = \{z_1, z_2\}$ and fix $(g) = \{w_1, w_2\}$, then*

$$
\gamma(f,g) = \beta(f)\beta(g)R \tag{2.13}
$$

where

$$R=\frac{\left(z_1-w_1\right)\left(z_2-w_1\right)\left(z_1-w_2\right)\left(z_2-w_2\right)}{\left(z_1-z_2\right)^2\left(w_1-w_2\right)^2} \tag{2.14}$$

Proof. Since

$$A=\begin{pmatrix} a & b \\ b & a \end{pmatrix}, \quad B=\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \quad \gamma \neq 0$$

so for

$$z_1=-1, z_2=1, w_1 \neq \infty, w_2 \neq \infty$$

and $\gamma(f,g)$ can be expressed as,

$$\begin{aligned}\gamma(f,g)&=b^2\left((\gamma-\beta)^2-(\alpha-\delta)^2\right)=b^2\gamma^2\left(\left(1+w_1w_2\right)^2-\left(w_1+w_2\right)^2\right)\\&=b^2\gamma^2\left(z_1-w_1\right)\left(z_2-w_1\right)\left(z_1-w_2\right)\left(z_2-w_2\right)=\beta(f)\beta(g)R\end{aligned} \tag{2.15}$$

Also,

$$|R|=\frac{q\left(z_1,w_1\right)q\left(z_2,w_1\right)q\left(z_1,w_2\right)q\left(z_2,w_2\right)}{q\left(z_1,z_2\right)^2 q\left(w_1,w_2\right)^2} \tag{2.16}$$

□

Lemma 2.1.6. *If f_1 and f_2 are in $\mathcal{M}$ with fix $(f_1) \cap$ fix $(f_2) \neq \emptyset$, so*

$$\left|\beta\left(f_1\right)-\beta\left(f_2\right)\right| \leqq \frac{1}{2}m\left(f_1f_2\right)m\left(f_1f_2^{-1}\right) \tag{2.17}$$

If f_1, f_2 and g are in $\mathcal{M}$ with fix $(f_1)=\operatorname{fix}(f_2)$, then

$$\left|\gamma\left(f_1,g\right)-\gamma\left(f_2,g\right)\right| \leqq \frac{1}{16}m\left(f_1f_2\right)m\left(f_1f_2^{-1}\right)m(g)^2 \tag{2.18}$$

both of above two inequalities hold sharp.

Proof. Since

$$\operatorname{tr}\left(\left[A_1,A_2\right]\right)-2=\gamma\left(f_1,f_2\right)=0$$

$$(\mathrm{tr}\,(A_1) - \mathrm{tr}\,(A_2))^2 = (\mathrm{tr}\,(A_1A_2) - 2)\left(\mathrm{tr}\left(A_1A_2^{-1}\right) - 2\right)$$

replacing A_2 by $-A_2$, we get

$$(\mathrm{tr}\,(A_1) + \mathrm{tr}\,(A_2))^2 = (\mathrm{tr}\,(A_1A_2) + 2)\left(\mathrm{tr}\left(A_1A_2^{-1}\right) + 2\right)$$

therefore,

$$\begin{cases}(\beta(f_1) - \beta(f_2))^2 = \left(\mathrm{tr}^2(A_1) - \mathrm{tr}^2(A_2)\right)^2 \\ \qquad = \left(\mathrm{tr}^2(A_1A_2) - 4\right)\left(\mathrm{tr}^2\left(A_1A_2^{-1}\right) - 4\right) \\ \qquad = \beta(f_1f_2)\,\beta\left(f_1f_2^{-1}\right)\end{cases} \tag{2.19}$$

Thus our first required inequality follows. Now

$$\gamma(f_1, g) - \gamma(f_2, g) = (\beta(f_1) - \beta(f_2))\,\beta(g)R$$

Thus,

$$\begin{aligned}|\gamma(f_1, g) - \gamma(f_2, g)| &= |\beta(f_1f_2)|^{1/2}\left|\beta\left(f_1f_2^{-1}\right)\right|^{1/2}|\beta(g)||R| \\ &= \frac{1}{4}Sm(f_1f_2)\,m\left(f_1f_2^{-1}\right)m(g)^2 \\ &\leqq \frac{1}{16}m(f_1f_2)\,m\left(f_1f_2^{-1}\right)m(g)^2\end{aligned} \tag{2.20}$$

For

$$f_1 = a^2z, f_2 = b^2z \text{ and } g = \frac{cz+d}{dz+c}$$

the above equality holds.

□

Lemma 2.1.7. *For z and w in H, we have*

$$\cosh^2(\varrho_H(z, w)) \leqq \frac{8 - q(z, \bar{z})^2}{q(z, \bar{z})^2}\frac{8 - q(w, \bar{w})^2}{q(w, \bar{w})^2} \tag{2.21}$$

where e_H denotes the hyperbolic distance in H of curvature -1. So the equality can *hold, if* $|z| = |w| = 1$ *and* $\mathrm{Re}(z) + \mathrm{Re}(w) = 0$

Lemma 2.1.8. *Let* $f, g \in \mathcal{M}$ *and f(H)=g(H)=H.If* $\gamma(f,g) < 0$ *and both the transformations are hyperbolic, then* $\exists h \in \mathcal{M}$ *st h(H)=H and*

$$\beta\left(f_1\right) = \frac{1}{2} m\left(f_1\right)^2, \quad \beta\left(g_1\right) = \frac{1}{2} m\left(g_1\right)^2 \tag{2.22}$$

where $f_1 = hfh^{-1}$ *and* $g_1 = hgh^{-1}$. *If f is hyperbolic and g is elliptic, then*

$$\left|\gamma\left(f_2, g_2\right)\right| = \frac{1}{16} m\left(f_2\right)^2 m\left(g_2\right)^2 \tag{2.23}$$

where $f_2 = hfh^{-1}$ *and* $g_2 = hgh^{-1}$.

Proof. Let z_1, z_2 be the fixed points of f and w_1, w_2 be the fixed points of g.

Case1: Both f and g are hyperbolic. Then $z_1, z_2, w_1, w_2 \in \mathbb{R}$ and $\neq 0$ st

$$\frac{\left(z_1 - w_1\right)\left(z_2 - w_2\right)}{\left(z_1 - w_2\right)\left(z_2 - w_1\right)} = \frac{\gamma(f,g)}{\beta(f)\beta(g)} \frac{\left(z_1 - z_2\right)^2\left(w_1 - w_2\right)^2}{\left(z_1 - w_2\right)^2\left(z_2 - w_1\right)^2} = -t^2 \tag{2.24}$$

For mapping z_1, z_2, w_1, w_2 onto $0, \infty, t, -t^{-1}$ and H onto itself, we can obtain $h \in \mathcal{M}$ by replacing t with -t, if necessary. And thus we get $fix(f_1) = \{0, \infty\}$ and $fix(g_1) = \{t, -t^{-1}\}$,

$$q(0, \infty) = q\left(t, -t^{-1}\right) = 2$$

and thus first two equalities of lemma follow.

Case2: f is hyperbolic and g is elliptic and $w_2 \in H$. $z_1, z_2 \in \mathbb{R}$ and $w_1 = \bar{w_2}$, choose $t \in (0, \infty$ so that

$$\frac{\left(z_1 - w_1\right)\left(z_2 - w_2\right)}{\left(z_1 - w_2\right)\left(z_2 - w_1\right)} = \frac{\left(-t + it^{-1}\right)\left(t - it^{-1}\right)}{\left(-t - it^{-1}\right)\left(t + it^{-1}\right)} \tag{2.25}$$

Thus an $h \in \mathcal{M}$ maps z_1, z_2, w_1, w_2 onto $-t, t, -it^{-1}, it^{-1}$ and H onto itself. Thus $fix(f_2) = \{-t, t\}$ and $fix(g_2) = \{-it^{-1}, it^{-1}\}$,

$$2q\left(\pm t, \pm it^{-1}\right)^2 = 8 - q(-t, t)^2 = 8 - q\left(-it^{-1}, it^{-1}\right)^2 \tag{2.26}$$

and our third equality of lemma follows. □

2.2 The chordal and hyperbolic norm

Lemma 2.2.1. *Suppose that f in $\mathcal{M}$ has fixed points $-r, r$ and multiplier a^2 where $0 < r \leqq 1$ and* $\operatorname{Re}(a) \geqq 0$. *Then*

$$d(f) = \begin{cases} 4\left(r\left|\frac{a+1}{a-1}\right| + \frac{1}{r}\left|\frac{a-1}{a+1}\right|\right)^{-1} & \text{if } r\left|\frac{a+1}{a-1}\right| \geq 1 \\ 2 & \text{if } r\left|\frac{a+1}{a-1}\right| < 1 \end{cases} \tag{2.27}$$

Lemma 2.2.2. *For f in $\mathcal{M}$ having a fixed point,*

$$d(f) = \begin{cases} 4\left(\frac{4}{m(f)} + \frac{m(f)}{4}\right)^{-1} & \text{if } \frac{4}{m(f)} \geqq 1 \\ 2 & \text{if } \frac{4}{m(f)} < 1 \end{cases} \tag{2.28}$$

Theorem 2.2.3. *If f is in $\mathcal{M}\backslash\{ id \}$ with $d(f) < 2.$ and f has one fixed point,*

$$m(f) = \frac{4d(f)}{2 + \left(4 - d(f)^2\right)^{1/2}} \tag{2.29}$$

Corollary 2.2.3.1. *If f is in $\mathcal{M}$, then*

$$m(f)^2 \leqq \frac{8d(f)^2}{4 - d(f)^2}$$

with equality if and only if f is either the identity or hyperbolic with antipodal fixed points.

2.3 Lower bounds for the chordal and matrix norms

Lemma 2.3.1. *Suppose that D is a disk or half plane in $\mathbb{C}$ Then there exists h in $\mathcal{M}$ such that $h(D) = H$ or $h(D) = B$ and such that for each f in $\mathcal{M}$ with $f(D) = D$,*

$$m(f) \geqq m(f_1), \quad d(f) \geqq d(f_1), \quad e(f) \geqq e(f_1)$$

where $f_1 = hfh^{-1}$

Proof. We need only consider the case where h(D)=B as we have the chordal isometry $g = \frac{z-i}{z+i}$ mapping H onto B. Now let Set $h(z) = \frac{z}{r}$. Then $f_1(B) = B, f$ and f_1 are represented by

$$A = \begin{pmatrix} a & br \\ br^{-1} & \bar{a} \end{pmatrix}, \quad A_1 = \begin{pmatrix} a & b \\ b & \bar{a} \end{pmatrix}, \quad |a|^2 - |b|^2 = 1$$

and

$$m(f)^2 - m\left(f_1\right)^2 = 4|b|^2 \left(r - r^{-1}\right)^2 \geqq 0$$

and our first and third inequalities follow. The second part follows from equation, when f has one fixed point. If f has two fixed points z_1 and z_2, then $|z_1|\,|z_2| = r^2$

$$q\left(z_1, z_2\right) \leqq q\left(h\left(z_1\right), h\left(z_2\right)\right)$$

and again second inequality follows since d(f) is nonincreasing function of $q = q(z_1, z_2)$.

□

[Remarks] There exists no positive universal lower bound for the chordal norms of both generators of a nonelementary discrete subgroup $< f, g >$ of $\mathcal{M}$. for if

$$f = a^2 z, \quad g = \frac{(b^2+1)z + 2b}{2bz + (b^2+1)},$$

where $1 < b < a < \infty$, then $< f, g >$ is a nonelementary discrete Fuchsian group and

$$d(f) = 2\frac{a^2-1}{a^2+1} \longrightarrow 0, \quad d(g) = \frac{4b}{b^2+1} \longrightarrow 2$$

as a$\longrightarrow$ 1 and hence b$\longrightarrow$ 1.

Chapter 3

Lower Bounds for Hyperbolic norms

If $< f, g >$ is nonelementary and discrete $\exists$ an absolute constant ϱ st

$$max(\varrho(f), \varrho(g)) \geq \varrho \tag{3.1}$$

Lemma 3.0.1. *For f,g$\in \mathcal{M}$, we have*

$$\varrho(fg) \leq \varrho(f) + \varrho(g), \quad \varrho(f^{-1}) = \varrho(f) \tag{3.2}$$

Proof. The above result comes from using the triangle inequality and the facts that $\tilde{f}$ is a hyperbolic isometry and that $\tilde{fg}$=$\tilde{f}\tilde{g}$ □

Lemma 3.0.2. *Suppose that $< f, g >$ is a discrete subgroup of $\mathcal{M}$ and that f and g have no common fixed points and are not both of order 2. If*

$$\varrho(f) + \varrho(g) < 0.1145, \tag{3.3}$$

then fg and fg^{-1} are elliptic.

Proof. Suppose that fg is not elliptic. Then gf is also not elliptic, $< fg, gf >$ is a discrete subgroup of $\mathcal{M}$ and

$$max(\varrho(fg), \varrho(gf)) \leq \varrho(f) + \varrho(g) < 0.1145$$

Thus $< fg, gf >$ is elementary and as fg and gf are both parabolic or both loxodromic, we have

$$fix(fg) = fix(gf) \tag{3.4}$$

Since f and g have no common fixed points, there exists a ϕ of order 2 in M, the Lie product of f and g, st $f = \phi f^{-1} \phi^{-1}$ and $g = \phi g^{-1} \phi^{-1}$. Then

$$gf = \phi g^{-1} f^{-1} \phi^{-1} = \phi (fg)^{-1} \phi^{-1} \tag{3.5}$$

Thus

$$fix(fg) = fix(gf) = \phi(fix(fg)) \tag{3.6}$$

By conjugation we may assume that $\{\infty\} \subset fix(fg) \subset \{0, \infty\}$. If $fix(fg) = \{\infty\}$, then $\phi(\infty) = \infty$ by 3.6, fg=z+a and $\phi = -z + b$ where a,b $\in (C)$ and thus by using 3.5 we have

$$gf = \phi(fg)^{-1}\phi^{-1} = -((-z + b) - a) + b = z + a = fg$$

Hence f and g commute and

$$fix(f) = fix(g) \tag{3.7}$$

contradicting the hypothesis that f and g have no common fixed points. If $fix(fg) = \{0, \infty\}$, then 3.6 implies that ϕ interchanges or fixes 0 and ∞. In the first case, fg=az and $\phi = bz^{-1}$ where $a, b \in \mathbb{C}\ \{0\}$,

$$gf = \phi(fg)^{-1}\phi^{-1} = b(a^{-1}(bz^{-1}))^{-1} = az = fg$$

and 3.7 follows. In the second case $(fg)^{-1}$ and ϕ commute,

$$gf = \phi(fg)^{-1}\phi^{-1} = (fg)^{-1}, \quad f^2 = g^{-2} \neq id$$

and we again obtain 3.7. Thus fg must be elliptic.

If we replace g by g^{-1}, we obtain fg^{-1} is elliptic. $\square$

Lemma 3.0.3. *Suppose that f and g in $\mathcal{M}$ have no common fixed points, that tr(f), tr(fg), tr(fg^{-1}) are real and that either f is not of order 2 or tr(g) is also real. Then $< f, g >$ is a Fuchsian or a purely elliptic group.*

Proof. Suppose that f has two fixed points. By means of preliminary conjugation assume that

$$f = \alpha^2 z, \quad g = \frac{az+b}{cz+d} \tag{3.8}$$

where $\alpha^2 \neq 1$, ad-bc=1 and $bc \neq 0$. If f is hyperbolic, then the hypothesis that

$$tr(fg) = \alpha a + \frac{d}{\alpha}, \quad tr(fg^{-1}) = \frac{a}{\alpha} + \alpha d$$

are real implies that a and d are real. Hence bc is real,

$$Im(cg(z)) = \frac{Im((acz+bc)(\bar{c}\bar{z}+d))}{|cz+d|^2} = \frac{Im(cz)}{|cz+d|^2}$$

and f and g map $D = \{z \in \mathbb{C} : Im(cz) > 0\}$ onto itself. If f is elliptic, then $|\alpha| = 1$,

$$tr(fg) = \alpha a + \bar{\alpha} d, \quad tr(fg^{-1}) = \bar{\alpha} a + \alpha d$$

are real and the hypothesis that $\alpha^2 \neq -1$ or a+d is real implies that $a = \bar{d}$ whence $\alpha^2 - 1 = bc \neq 0$. when $|a| > 1$, we can choose $r > 0$ so that $r^2|c|^2 = |a|^2 - 1$; then

$$g = \frac{az + r^2\bar{c}}{cz + \bar{a}}$$

and hence f and g map $D = \{z : |z| < r\}$ onto itself. When $|a| < 1$, g is elliptic with fixed points given by

$$z = \frac{i}{c}(Im(a) \pm (1 - (Re(a))^2)^{\frac{1}{2}})$$

and the segment joining these points contains the origin. Hence the axes of f and g intersect in $\tilde{H}$ and $< f, g >$ is purely elliptic.

Suppose that f has one fixed point. By conjugation assume that

$$f = z + \alpha, \quad g = \frac{az}{cz + d} \tag{3.9}$$

where $\alpha c \neq 0$ since $fix(f) \cap fix(g) = \emptyset$. Then

$$tr(fg) = a + d + \alpha c, \quad tr(fg^{-1}) = a + d - \alpha c,$$

and tr(g) and αc are real. If g is hyperbolic or elliptic, the desired conclusion follows by interchanging f and g. If g is parabolic, then a=d and f and g both map $D = \{z \in \mathbb{C} : Im(\bar{\alpha} z) > 0\}$ onto itself. $\square$

Lemma 3.0.4. *Suppose that $< f, g >$ is a nonelementary discrete subgroup of $\mathcal{M}$. Then*

$$\varrho(f) + \varrho(g) \geq 0.1145 \tag{3.10}$$

Proof. Suppose that the above inequality doesn't hold. Then f or g, say f, is elliptic. Suppose that f is not of order 2. Then fg and fg^{-1} are elliptic and thus tr(f), tr(fg), $tr(fg^{-1})$ are real. Hence $< f, g >$ is Fuchsian and

$$\varrho(f) + \varrho(g) \geq max(\varrho(f), \varrho(g)) \geq 0.262$$

, which is a contradiction.

Now suppose that f is of order 2, let δ denote the hyperbolic distance from j to the axis of f and choose x on axis(f) so that $\delta = \varrho_{\tilde{H}}(j, x)$. Next choose h in M so that $\tilde{h}(\tilde{H}) = \tilde{H}$ and $\tilde{h}(x) = j$, and set $f_1 = hfh^{-1}$ and $g_1 = hgh^{-1}$. Then $\tilde{f}_1(j) = j$ whence $\varrho(f_1) = 0$,

$$sinh(\frac{\varrho(f)}{2}) = sinh\delta = sinh(\delta(h))$$

and thus

$$\varrho(g_1 f_1) + \varrho(f_1) = \varrho(g_1) \leq 2\varrho(h) + \varrho(g) = \varrho(f) + \varrho(g) < 0.1145$$

Since $< g_1 f_1, f_1 >$ is nonelementary and discrete, $g_1 f_1$ is not of order 2 and also g_1 is elliptic. Hence g is elliptic with $ord(g) > 2$ and the above argument interchanging f and g results in contradiction. $\square$

Chapter 4

Continued fractions

4.1 Introduction

Definition. An infinite **continued fraction** is defined as an expression of the form

$$\mathbf{K}(a_n \,|\, b_n) = \cfrac{a_1}{b_1 + \frac{a_2}{b_2 + \frac{a_3}{b_3 + \cdots}}}, \tag{4.1}$$

where the a_i and b_j are infinite sequences of complex numbers with each $a_i \neq 0$.

The continued fraction 4.1 is said to converge to, or to have value, **K** if the sequence of truncated continued fractions

$$\frac{a_1}{b_1}, \quad \frac{a_1}{b_1 + \frac{a_2}{b_2}}, \quad \frac{a_1}{b_1 + \frac{a_2}{b_2 + \frac{a_3}{b_3}}} \cdots \tag{4.2}$$

converges to **K**.

We can consider the continued fraction 4.1 as simply the pair of complex sequences a_n and b_n in which no a_i is zero. Given two such sequences we can form the sequence s_n of Möbius maps defined by

$$s_n(z) = \frac{a_n}{b_n + z}, \tag{4.3}$$

and $s_n(\infty) = 0$. Conversely, as any Möbius map s with $s(\infty) = 0$ can be expressed uniquely in the form $s(z) = a/(b + z)$, we can identify the class of continued fractions with the class of sequences s_n of Möbius maps with $s_n(\infty) = 0$ for all n. With this identification the truncated continued fractions in 4.2 are $s_1(0)$, $s_1 s_2(0)$, $\ldots$, where, in general, fg denotes the map $z \to f(g(z))$, so that the convergence of the continued fraction is equivalent to the convergence of the complex sequence $s_1 \cdots s_n(0)$.

4.2 Continued fractions

Continued fraction 4.1 can be viewed as the sequence of Möbius maps s_n in 4.3. We shall refer to this continued fraction as the continued fraction generated by s_n and use the notation $[s_1, s_2, \cdots]$ for the continued fraction 4.1. We write $S_n = s_1 \cdots s_n$ and say s_j generates S_j.
In general, given a sequence of maps f_j of a set into itself, we can construct inner and outer composition sequences as $F_n = f_1 \cdots f_n$ and $G_n = f_n \cdots f_1$, respectively.

Theorem 4.2.1 (The Stern-Stolz Theorem)**.** *If $\sum_n |b_n|$ converges then $\boldsymbol{K}(1|b_n)$ diverges.*

For eg., if s(z)=1/z, then the continued fraction $[s, s, \cdots]$ has approximants $\infty, 0, \infty, 0, \cdots$ and so diverges. In the Stern-Stolz theorem, $s_n(z) = 1/(b_n + z)$ and $b_n \longrightarrow 0$, so that $s_n \longrightarrow s$, and we might expect that $[s_1, s_2, \cdots]$ diverges at least if $b_n \longrightarrow 0$ sufficiently quickly.

Remark. *(i) To measure the rate of convergence one uses the quantities equivalent to using the norms. For eg., if one intends to measure the rate in terms of coefficients, then matrix norm is geometrically significant.*
(ii) If $s_n(z) = 1/(b_n + z)$ then $|b_n|$ shows the distance that s_n moves the distinguished point in our model of hyperbolic 3-space. The proof of the Stern-Stolz Theorem is then nothing more than an application of the triangle inequality between hyperbolic distances.

4.3 Complex M$\ddot{o}$bius maps

Let φ be the map from $\tilde{\mathbb{C}}$ onto the unit sphere in $\mathbb{R}^3$ by stereographic projection. This map provides us with the chordal metric on $\tilde{\mathbb{C}}$, namely

$$q(z_1, z_2) = \frac{2|z_1 - z_2|}{(|z_1|^2 + 1)^{1/2}(|z_2|^2 + 1)^{1/2}} = |\varphi(z_1) - \varphi(z_2)| \tag{4.4}$$

$(\tilde{\mathbb{C}}, q)$ is a compact metric space.
A complex M$\ddot{o}$bius map is a map g of the form

$$g(z) = \frac{az + b}{cz + d}, \quad ad - bc \neq 0$$

Each complex M$\ddot{o}$bius map g is a homeomorphism of $\tilde{\mathbb{C}}$ onto itself. Every chordal isometry is a M$\ddot{o}$bius map (of z or $\bar{z}$).Note that $z \mapsto 1/z$ is a chordal isometry as $q(1/z_1, 1/z_2) = q(z_1, z_2)$.

Each M$\ddot{o}$bius map is a Lipschitz map wrt chordal metric. Indeed, if g(z)=(az+b)/(cz+d), where ad-bc=1, then

$$\frac{q(g(z), g(w))}{q(z, w)} = \left(\frac{|g(z) - g(w)|}{|z - w|}\right)\sqrt{\frac{(1 + |z|^2)(1 + |w|^2)}{(1 + |g(z)|^2)(1 + |g(w)|^2)}}$$

$$= \frac{1}{|(cz + d)(cw + d)|}\sqrt{\frac{(1 + |z|^2)(1 + |w|^2)}{(1 + |g(z)|^2)(1 + |g(w)|^2)}}$$

$$= \frac{\sqrt{1 + |z|^2}\sqrt{1 + |w|^2}}{\sqrt{|az + b|^2 + |cz + d|^2}\sqrt{|aw + b|^2 + |cw + d|^2}}$$

Now applying the Cauchy-Schwarz inequality to each term, we get

$$|az + b|^2 + |cz + d|^2 \leq (|a|^2 + |b|^2 + |c|^2 + |d|^2)(|z|^2 + 1)$$

and define the norm $||g||^2$ of g as

$$||g||^2 = |a|^2 + |b|^2 + |c|^2 + |d|^2$$

Thus

$$\frac{q(g(z), g(w))}{q(z, w)} \geq \frac{1}{||g||^2}$$

Since ad-bc=1, $||g|| = ||g^{-1}||$. After putting u=g(z) and v=g(w), we get

$$\frac{q(g^{-1}(u), g^{-1}(v))}{q(u, v)} = \frac{q(z, w)}{q(g(z), g(w))} \leq ||g||^2 = ||g^{-1}||^2$$

As this is true for all u,v and g^{-1}, after again interchanging the variables we get $\forall$ z,w and g that

$$\frac{q(g(z), g(w))}{q(z, w)} \leq ||g||^2$$

The best Lipschitz constant of a Möius map g is given as

$$L(g) = \sup_{z \neq w} \frac{q(g(z), g(w))}{q(z, w)};$$

Thus

$$L(g) = ||g||^2 \tag{4.5}$$

The vector space of all 2 $\times$ 2 matrices is equipped with the norm

$$||A||^2 = |a|^2 + |b|^2 + |c|^2 + |d|^2, \quad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

and $||AB|| \leq ||A|| \, ||B||$ for all A and B. Consider the following homomorphism

$$\phi : A \mapsto g_A, \quad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad g_A(z) = \frac{az + b}{cz + d} \tag{4.6}$$

from the group $GL(2, \mathbb{C})$ onto the group $\mathcal{M}$.

Theorem 4.3.1. *Let $\phi : SL(2, \mathbb{C}) \longrightarrow \mathcal{M}$ be the map given in 4.6, where these spaces have the norm $||.||$ and metric q_0, respectively. Then ϕ is an open, continuous,* *surjective map and the restriction of ϕ to any open ball of radius $\sqrt{2}$ is injective (and hence a homeomorphism).*

Theorem 4.3.2. *Suppose that a Möbius map g is represented by a unimodular matrix G. Then $q_0(g, I) \leq \sqrt{6}||G - I||$.*

Theorem 4.3.3. *Any Möbius map g can be represented by a unimodular matrix G st*

$$||G - I||^2 \leq \frac{4q_0(g, I)^2}{4 - q_0(g, I)^2}$$

Theorem 4.3.4. *For a 2×2 unitary matrix A, conjugation $X \mapsto AXA^{-1}$ is an isometry of vector space of 2×2 complex matrices wrt the metric $||X - Y||$. If f is the Möbius transformation $\phi(A)$; then f is the chordal isometry. In other words, $\forall$ g,h $\in \mathcal{M}$, $q_0(fgf^{-1}, fhf^{-1}) = q_0(g, h)$, so that the conjugation $g \mapsto fgf^{-1}$ is an isometry of $(\mathcal{M}, q_0)$. Hence, every chordal isometry is of the form $\phi(A)$ for some unitary matrix A.*

Theorem 4.3.5. *If a sequence g_n of Möbius maps converges at three distinct points of $\tilde{\mathbb{C}}$ to three distinct values, then the sequence g_n converges uniformly on $\tilde{\mathbb{C}}$ to some Möbius map g.*

Definition. Let g($\neq I$) be any Möbius transformation. Then
(i) g is parabolic iff g has a unique fixed point in $\tilde{\mathbb{C}}$;
(ii) g is loxodromic iff g has exactly two fixed points in $\tilde{\mathbb{R}}^3$;
(iii) g is elliptic iff g has infinitely many fixed points in $\tilde{\mathbb{R}}^3$.

Definition. Let g be a loxodromic transformation. We say that g is hyperbolic if g(D)=D for some open disc(or half-plane) D in $\tilde{\mathbb{C}}$; otherwise g is said to be strictly loxodromic.

Theorem 4.3.6. *A möbius map g is*
(1) parabolic if it is conjugate to $z \mapsto z + 1$;
(2) elliptic if it is conjugate to $z \mapsto kz$ for some k with $|k| = 1$, $k \neq 1$;
(3) hyperbolic if it is conjugate to $z \mapsto kz$ for some $k > 0$ and $k \neq 1$

Proof. We know that any Möbius transformation is conjugate to any one of the standard forms. Let us observe the fixed points of the standard forms. The action of m_k

in $\tilde{\mathbb{R}}^3$ is given as

$$m_k(z + tj) = kz + |k|tj \quad (k \neq 1); \tag{4.7}$$

$$m_1(z + tj) = z + 1 + tj, \tag{4.8}$$

These equations enable us to find the fixed points of each m_k. And we have:
(i) m_1 fixes ∞ but no other point in $\tilde{\mathbb{R}}^3$;
(ii) if $|k| \neq 1$, then m_k fixes 0 and ∞ but no other points in $\tilde{\mathbb{R}}^3$;
(iii) if $|k| = 1$, $k \neq 1$, then the set of fixed points of m_k is given by

$$\{tj : t \in \mathbb{R}\} \bigcup \{\infty\} \tag{4.9}$$

Using the above definition we get our required result. □

Theorem 4.3.7. *For a given Möbius g we have*
(1) g is parabolic iff $tr^2(g) = 4$*;*
(2) g is elliptic iff $tr^2(g) \in [0, 4)$*;*
(3) g is hyperbolic iff $tr^2(g) \in (4, +\infty)$*;*
(4) g is strictly loxodromic iff $tr^2 \notin [0, \infty)$*.*

Proof. Suppose that g is conjugate to the standard form m_p so that we have

$$tr^2(g) = p + \frac{1}{p} + 2 \tag{4.10}$$

We have g conjugate to m_p and $m_{\frac{1}{p}}$ and to no other m_q.
If g is parabolic, then g is conjugate to m_1 only: so p=1 and we get $tr^2(g) = 4$. Conversely, let $tr^2(g) = 4$. This implies p=1 and we get g is parabolic. This proves (1).
Now suppose that g is elliptic, then $p = e^{i\theta}$ where θ is real and $cos\theta \neq 1$. Then

$$tr^2(g) = 2 + 2cos\theta \tag{4.11}$$

And we get $tr^2(g) \in [0, 4)$. Now, suppose that $tr^2(g) \in [0, 4)$. Then we can write $tr^2(g)$ in the form 4.11 with $cos\theta \neq 1$ and then we get $p = e^{i\theta}, e^{-i\theta}$. Thus $|p| = 1, p \neq 1$ and we deduce that g is elliptic. Hence, this proves (2).

Now let $tr^2(g) \in (4, +\infty)$. Then we have solutions $p = k, \frac{1}{k}$ say, where $k > 0$. As both solutions are positive; m_p necessarily preserves upper half plane and is thus hyperbolic. Thus g is also hyperbolic. Conversely suppose that g and hence m_p is hyperbolic and let D be a disc invariant under m_p. $\forall z \in D$, the images of z under the iterates of m_p are in D and so

$$\{p^n(z) : n \in \mathbb{Z}\} \subset D \tag{4.12}$$

Because $|p| \neq 1$, this shows that 0 and ∞ are in the closure of D. The same argument, but with z chosen in the exterior of D, leads to the conclusion that 0 and ∞ lie on the boundary of D. Thus D is a half-plane and in order to preserve D, it is important that m_p leaves invariant each of the half-lines from 0 to ∞ on the boundary of D. Thus $p > 0$ and hence $tr^2(g) > 4$. This proves (3). □

Obviously, $\mathcal{M}$ is the disjoint union of $\{I\}$ and the three classes of parabolic, elliptic and loxodromic maps.

Theorem 4.3.8. *Let f and g be two Möbius maps neither of which is the identity. Then f and g are conjugate in $\mathcal{M}$ iff $tr^2(f) = tr^2(g)$.*

Proof. We know that if f and g are conjugate then $tr^2(f) = tr^2(g)$.
Now suppose that $tr^2(f) = tr^2(g)$. Also for some p and q, f and g are conjugate to the standard forms m_p and m_q respectively. Hence

$$tr^2(m_p) = tr^2(f) = tr^2(g) = tr^2(m_q) \tag{4.13}$$

From 1.13, we get p=q or $p = \frac{1}{q}$. Now we claim that m_p is conjugate to $m_{\frac{1}{p}}$, which is obvious for p=1. Now for $p \neq 1$ we have

$$hm_ph^{-1} = m_{\frac{1}{p}}, \quad h(z) = -\frac{1}{z} \tag{4.14}$$

Thus we have f is conjugate to m_p, g is conjugate to m_q and m_p is conjugate to m_q as p=q or $p = \frac{1}{q}$. Since conjugacy is an equivalence relation, we come to the conclusion that f is conjugate to g.

□

Class of loxodromic mpas is hyperbolic iff $tr^2 \in (4, +\infty)$, else it is called strictly loxodromic. A loxodromic map is hyperbolic iff it has an invariant disc and also if it can be written as the composition of exactly two inversions whereas most Mӧbius maps require four inversions.

Theorem 4.3.9. *The periodic continued fraction generated by* $s(z) = a/(b+z)$ *converges iff* $-b^2/a \notin [0, 4)$.

4.4 Mӧbius maps and hyperbolic geometry

We take upper-half plane $\mathbb{H}$ (given by $y > 0$, where z=x+iy) to be the hyperbolic plane. The hyperbolic metric ρ on $\mathbb{H}$ is derived from the line element $ds = |dz|/Im[z]$, and the Mӧbius maps that map $\mathbb{H}$ onto itself are the conformal isometries of $\mathbb{H}$. The geodesics are the semi-circles in $\mathbb{H}$ whose centres lie in $\mathbb{R}$, together with the 'vertical' half-lines whose initial point lies in $\mathbb{R}$.

Theorem 4.4.1. *The Mӧbius group* $\mathcal{M}$ *is the group of all conformal orientation-preserving isometries of* $\mathbb{H}^3$.

Theorem 4.4.2. *Let* g *be any complex Mӧbius map, and let* I *be the identity map. Then*

$$||g||^2 = ||I||^2 + 4sinh^2\frac{1}{2}\rho(\boldsymbol{j}, g(\boldsymbol{j})) \tag{4.15}$$

As $||I||^2 = 2$ we can also write 4.15 in the form $||g||^2 = 2cosh\rho(\mathbf{j}, g(\mathbf{j}))$.

Theorem 4.4.3. *For all* g *in* $\mathcal{M}$, *we have*

$$||g||^2 = 2cosh\rho(\boldsymbol{j}, g(\boldsymbol{j})) \tag{4.16}$$

Proof. We have

$$g(z) = \frac{az+b}{cz+d}, \quad ad - bc = 1; \tag{4.17}$$

Poincare extension of g is given by

$$g(z+tj) = \frac{(az+b)\overline{(cz+d)} + a\bar{c}t^2 + |ad-bc|tj}{|cz+d|^2 + |c|^2t^2} \tag{4.18}$$

Putting z=0 and t=1, we get

$$g(j) = \frac{(b\bar{d} + a\bar{c}) + j}{|c|^2 + |d|^2} \tag{4.19}$$

For $\zeta_1 = z_1 + t_1 j$ and $\zeta_2 = z_2 + t_2 j$, we have

$$cosh\rho(\zeta_1, \zeta_2) = 1 + \frac{|z_1 - z_2|^2 + (t_1 - t_2)^2}{2t_1t_2} \tag{4.20}$$

We have

$$|b\bar{d} + a\bar{c}|^2 + 1 = |b\bar{d} + a\bar{c}|^2 + |ad - bc|^2 = (|a|^2 + |b|^2)(|c|^2 + |d|^2) \tag{4.21}$$

Using the identity 4.21 and substituting $z_1 = 0, t_1 = 0$ (so $\zeta_1 = j, \zeta_2 = g(j)$) in 4.20, we get our required result. □

Theorem 4.4.4. *Let g be any complex Möbius map. Then the best Lipschitz constant (relative the chordal metric) for g is given by*

$$L(g) = exp[\rho(\boldsymbol{j}, g(\boldsymbol{j}))] \tag{4.22}$$

From 4.15 and 4.22, we get

$$L(g) = \frac{1}{2}(||g||^2 + \sqrt{||g||^4 - 4}) \tag{4.23}$$

Thus,

$$\frac{1}{2}||g||^2 \leq L(g) \leq ||g||^2 \tag{4.24}$$

and $L(g) \longrightarrow ||g||^2$ as $||g|| \longrightarrow +\infty$. Now, since g^{-1} is an isometry, $\rho(\mathbf{j}, g(\mathbf{j})) = \rho(g^{-1}(\mathbf{j}), \mathbf{j})$, we have from 4.22 that $\forall$ g,

$$L(g^{-1}) = L(g) \geq 1 \tag{4.25}$$

Theorem 4.4.5. *For any Möbius g represented by A the following are equivalent:*

(a) $A \in SU(2,\mathbb{C})$*;*

(b) $||g||^2 = 2$*;*

(c) $L(g)=1$*;*

(d) $\rho(\boldsymbol{j}, g(\boldsymbol{j})) = 0$*;*

(e) g is a chordal isometry.

Proof. (b) directly follows from (a) as A is unitary.

Putting $||g||^2 = 2$ in 4.23, we get L(g)=1. Hence (b)⇒(c).

Now (d) follows from (c) by putting L(g)=1 in 4.22.

Now we prove equivalence of (a) and (e). Observe that g will be isometry iff ∀ z,

$$\frac{|g^{(1)}(z)|}{1+|g(z)|^2} = \frac{1}{1+|z|^2} \tag{4.26}$$

Thus (e) holds iff ∀ z,

$$1 + |z|^2 = |az + b|^2 + |cz + d|^2 \tag{4.27}$$

or equivalenttly,

$$1 + |z|^2 = (|a|^2 + |c|^2)|z|^2 + (|b|^2 + |d|^2) + 2Re(a\bar{b} + c\bar{d})z \tag{4.28}$$

This is equivalent to

$$|a|^2 + |c|^2 = |b|^2 + |d|^2 = 1 \tag{4.29}$$

and

$$a\bar{b} + c\bar{d} = 0 \tag{4.30}$$

which in turn is equivalent to $\bar{A}^t A = I$ and this is (a). □

Theorem 4.4.6. *Suppose that the Möbius maps f and g satisfy* $q_0(f, g) < \sqrt{2}$*. Then* $L(g) \leq 18L(f)$*.*

4.5 Convergence

Definition. The continued fraction 4.1 is said to converge strongly to a value α in $\mathbb{C}$ if

(i) there are distinct u and v in $\tilde{\mathbb{C}}$ st $S_n(u) \to \alpha$ and $S_n(v) \to \alpha$ as n$\to \infty$, and

(ii) the set $\{S_n^{-1}(\infty) : n = 1, 2, 3, \cdots\}$ is not dense in $\tilde{\mathbb{C}}$.

Definition. The continued fraction 4.1 converges generally, or is generally convergent, to a value α in $\tilde{\mathbb{C}}$ if $\exists$ sequences u_n and v_n in $\tilde{\mathbb{C}}$ st

$$\lim_{n\to\infty} S_n(u_n) = \lim_{n\to\infty} S_n(v_n) = \alpha, \quad \lim_{n\to\infty} q(u_n, v_n) > 0$$

Definition. A sequence g_n of Möbius maps converges generally, or is generally convergent, to a value α in $\tilde{\mathbb{C}}$ if $\exists$ sequences u_n and v_n st

$$\lim_{n\to\infty} g_n(u_n) = \lim_{n\to\infty} g_n(v_n) = \alpha, \quad \lim_{n\to\infty} q(u_n, v_n) > 0 \tag{4.31}$$

Theorem 4.5.1. *A sequence g_n of Möius maps converges generally to α in $\tilde{\mathbb{C}}$ iff $g_n \longrightarrow \alpha$ on $\mathbb{H}^3$.*

We know that a sequence of Möius maps converges pointwise on $\mathbb{H}^3$ to a point α in $\tilde{\mathbb{C}}$ iff it converges to α uniformly on compact subsets of $\mathbb{H}^3$. This comes out of the facts that Each Moöbius map is an isometry of $\mathbb{H}^3$ and that the Euclidean length of a 'ruler' of fixed hyperbolic length shrinks to zero as the ruler approaches the boundary of hyperbolic space (as demonstrated by Escher's tesselations of unit disc in two-dimensions). The proof of the above theorem makes use of the following geometric lemma.

Lemma 4.5.2. *Let γ be a geodesic in $\mathbb{H}^3$ whose endpoints u and v in $\tilde{\mathbb{C}}$ are at a chordal distance δ apart. Then γ passes within a hyperbolic distance $cosh^{-1}(2/\delta)$ of the point **j**.*

4.6 Strong divergence

While convergence (whether classical or general) includes convergence to ∞, the divergence of a sequence of Mӧbius maps implies an oscillatory behaviour. Stronger notion of divergence can only be understood by considering the action of Mӧbius maps on $\mathbb{H}^3$. An extreme case in which notion of general convergence fails is when the sequence $g_n(\mathbf{j})$ does not even escape to the boundary of hyperbolic space.

Definition. A sequence g_n of complex Mӧbius maps is strongly divergent if for one (and hence all) x in $\mathbb{H}^3$, the sequence $g_n(x)$ lies in a compact subset of $\mathbb{H}^3$. A strongly divergent sequence cannot converge generally.

We have

$$||g_n||^2 = ||I||^2 + 4sinh^2\frac{1}{2}\rho(\mathbf{j}, g_n(\mathbf{j})) \tag{4.32}$$

Theorem 4.6.1. *A sequence g_n of complex Mӧbius maps is strongly divergent iff the sequence $||g_n||$ is bounded.*

Theorem 4.6.2. *Let $g_1, g_2, \cdots$ be Mӧbius maps. If $\sum_n(||g_n||^2 - 2)^2$ converges, then the sequence $g_1 \cdots g_n$ is strongly divergent.*

Proof. As $x \longrightarrow 0$, sinh x $\longrightarrow$ x and also from 4.32, we know that the following three series

$$\sum_n(||g_n||^2 - 2)^2, \quad \sum_n sinh\frac{1}{2}\rho(\mathbf{j}, g_n(\mathbf{j})), \quad \sum_n \rho(\mathbf{j}, g_n(\mathbf{j}))$$

converge or diverge together. Assume that $\rho(\mathbf{j}, g_n(\mathbf{j})) = M$, say where $M < \infty$. Let $G_0 = I$, and $G_n = g_1 \cdots g_n$. $g_j's$ being the isometries of $\mathbb{H}^3$, we have

$$\rho(\mathbf{j}, G_n(\mathbf{j})) \leq \sum_{m=0}^{n-1} \rho(G_m(\mathbf{j}), G_{m+1}(\mathbf{j}) = \sum_{m=0}^{n-1} \rho(\mathbf{j}, g_{m+1}(\mathbf{j})) \leq M$$

Thus $G_n(\mathbf{j})$ lie in some compact subset of $\mathbb{H}^3$ and hence the sequence G_n diverges strongly. □

Unitary maps fix $\mathbf{j}$; that is, $g(\mathbf{j}) = \mathbf{j}$ iff $||g||^2 = 2$. Thus Theorem 4.6.1 says that the inner composition sequence $g_1 \cdots g_n$ is strongly divergent iff (in some sense) the g_n approach the subgroup of unitary maps sufficiently rapidly. Thus, in this result we do not require that g_n converge, but merely that each g_n looks increasingly like some unitary map.

The following two results make use of matrix norms.

Theorem 4.6.3. *Suppose that s(z)=a/(b+z) is elliptic,* $s_n(z) = a_n/(b_n + z)$ *and* $\sum_n ||s_n - s||$ *converges. Then* $\boldsymbol{K}(a_n|b_n)$ *diverges. Also, if* z_0 *is a fixed point of* s*, then* $S_n(z_0)$ *converges to some limit, say* w*, and then* $S_n^{-1}(w) \longrightarrow z_0$.

Theorem 4.6.4. *Suppose that* $t_n \longrightarrow t$*, where each* t_n *and* t *is elliptic. If* $\sum_n ||t_n - t||$ *converges then* $t_1 \cdots t_n$ *converges to distinct values at the two fixed points of* t*, and diverges elsewhere.*

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