Arithmetic Geometric aspects of modular groups

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Certificate of Examination

This is to certify that the dissertation titled **Arithmetic Geometric aspects of modular groups** submitted by **Mr. Titiksh Gupta** (Reg. No. MS09131) for the partial fulfilment of BS-MS dual degree programme of the Institute, has been examined by the thesis committee duly appointed by the Institute. The committee finds the work done by the candidate satisfactory and recommends that the report be accepted.

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Declaration

The work presented in this dissertation has been carried out by me under the guidance of Dr. Krishnendu Gongopadhyay at the Indian Institute of Science Education and Research Mohali.
This work has not been submitted in part or in full for a degree, a diploma, or a fellowship to any other university or institute. Whenever contributions of others are involved, every effort is made to indicate this clearly, with due acknowledgement of collaborative research and discussions. This thesis is a bonafide record of original work done by me and all sources listed within have been detailed in the bibliography.

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In my capacity as the supervisor of the candidate's project work, I certify that the above statements by the candidate are true to the best of my knowledge.

Dr. Krishnendu Gongopadhyay
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Finally, I would like to acknowledge that the material presented in this thesis is based on other people's work. Every theorem in this thesis can be found elsewhere. At the beginning of every chapter, we have clearly mentioned the main texts that laid the foundation of the chapter. If I have made any contribution then it is the selection, presentation and elaboration of the materials from different sources those are listed in the bibliography.

Titiksh Gupta

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Notations

I	Identity Matrix
$\mathbb{C}$	Complex numbers
$\mathbb{R}$	Real numbers
$\mathbb{Z}$	Integers
$\mathbf{H}^2$	Upper Half Hyperbolic Space
$\partial\mathbf{H}^2$	Boundary of Upper Half Hyperbolic Space
$\mathbf{D}^2$	Poincaré Disc Model
$\partial\mathbf{D}^2$	Boundary of Poincaré Disc Model
ι	Iota

Contents

Abstract

The aim of this THESIS is to highlight the major developments in the arithmetic-geometric aspects of the modular group. After covering geometric aspects of Fuchsian groups, we study various variants of the Poincaré polygon theorem. Arithmetic methods like Farey Symbols have been used to describe the subgroups of $PSL(2, \mathbb{Z})$. Graph-theoretical approach has been used to study algorithm for generating all trivalent diagrams. Finally, we conclude by describing algorithms for testing membership of matrices in $PSL(2, \mathbb{Z})$ by using the concept of Farey Symbols.

Chapter 1

Hyperbolic Geometry

Hyperbolic geometry was created in the first half of the 18^{th} century because the main criticism of Euclid's fifth postulate was that unlike other postulates, it is not self-evident to be accepted without proof.

Euclid's Postulates

1. A straight line may be drawn from any point to any other point.
2. A finite straight line may be extended continuously in a straight line.
3. A circle may be drawn with any centre and any radius.
4. All right-angles are equal.
5. (Parallel Postulate) If a straight line falling on two straight lines makes the interior angles on the same side less than two right-angles, then the two straight lines, if extended indefinitely, meet on the side on which the angles are less than two right-angles.

The plane geometry can be developed without parallel postulate and hence suggesting that the parallel postulate is not necessary. A number of geometers like Proclus, Ibn al-Haytham, Nasir al-Din al-Tusi, Witelo, Gersonides, Alfonso, Saccheri, John Wallis, Lambert and Legendre made attempts to prove the parallel postulate by trying to derive a contradiction. Johann Heinrich Lambert introduced the concept of hyperbolic functions and computed area of a hyperbolic triangle in the 18^{th} century.

In 1824, Gauss found that *the assumption that the sum of the three sides (of a triangle) is smaller than 180 degrees leads to a geometry which is quite different from our (Euclidean) geometry which is completely consistent within itself.* This was the

first example of a non-Euclidean geometry. It was discovered again by Lobachevsky in 1829 in a paper entitled *on the principles of geometry* and by Bolyai in 1832 in a paper entitled *the absolute science in space.* we call non-Euclidean geometry of Gauss, Lobachevsky and Bolyai as hyperbolic geometry and any geometry which is not Euclidean is called non-Euclidean geometry. Although Gauss, Boyali and Lobachevsky developed non-Euclidean geometry, they didn't prove the consistency of the geometry. During the investigation of curved surfaces Euler, Gauss and Monge laid the Analytic study of hyperbolic non-Euclidean geometry. The term *hyperbolic geometry* was coined by Felix Klein in 1871.
There are several ways of modelling hyperbolic geometry. In this section we have discussed two models. The first one is the upper half model and the second is Poincaré Disc model. Material for studying this topic has been taken mainly from [CW12], [JA05] and [SK92].

1.1 Upper Half Plane

1.1.1 Hyperbolic Metric

Definition 1.1.1. *The upper half-plane is the set of complex numbers z with positive imaginary part*

$$\mathbf{H}^2 = \{z \in \mathbb{C} \mid Im(z) > 0\}.$$

Definition 1.1.2. *The boundary of* $\mathbf{H}^2$ *or the circle at infinity is defined to be the set*

$$\partial\mathbf{H}^2 = \{z \in \mathbb{C} \mid Im(z) = 0\} \cup \{\infty\}.$$

Definition 1.1.3. *Let* $\mathbb{C}$ *be the complex plane. Using the notations* $Re(z) = x$ *and* $Im(z) = y$*. The upper half plane model of hyperbolic plane is defined to be* $\mathbf{H}^2$ *equipped with the metric*

$$\partial s = \frac{\sqrt{\partial x^2 + \partial y^2}}{y}.$$

1.1.2 Path Integrals

By a path σ in $\mathbb{C}$ we mean the image of differentiable function $\sigma : [x, y] \to \mathbb{C}$ where $[x, y] \subset \mathbb{R}$ is an interval.
If f : $\mathbb{C} \to \mathbb{R}$ is a continuous function, then the integral f along the path σ is defined

as

$$\int_\sigma f = \int_x^y f(\sigma(t))|\sigma'(t)|dt.$$

In this equation $|.|$ signifies modulus of a complex number i.e.

$$|\sigma'(t)| = \sqrt{(Re\ \sigma'(t))^2 + (Im\ \sigma'(t))^2}.$$

Definition 1.1.4. *A path σ with parametrisation $\sigma : [x, y] \to \mathbb{C}$ is piecewise differentiable if σ is continuous and differentiable except at finitely many points.*

1.1.3 Distances

Definition 1.1.5. *Let $\sigma : [x, y] \to \mathbf{H}^2$ be a path in upper half-plane, then the hyperbolic length of σ is defined as*

$$Length_{\mathbf{H}^2}(\sigma) = \int_\sigma \frac{1}{Im(z)} = \int_x^y \frac{|\sigma'(t)|}{Im(\sigma(t))}dt.$$

Example In the path $\sigma(t) = x_1 + t(x_2 - x_1) + \iota y$, $0 \leq t \leq 1$, the length is given by

$$Length_{\mathbf{H}^2}(\sigma) = \int_0^1 \frac{|x_2 - x_1|}{y}dt = \frac{|x_2 - x_1|}{y}.$$

Definition 1.1.6. *Let $m_1, m_2 \in \mathbb{Z}$. The hyperbolic distance $d_{\mathbf{H}^2}(m_1, m_2)$ between 2 points m_1 and m_2 is defined to be equal to inf $Length_{\mathbf{H}^2}(\sigma)$, where σ is a piecewise differentiable path which has end points at m_1 and m_2.*

1.1.4 Lines and Circles

Euclidean line has the form $ax + by + c = 0$ for $a, b, c \in \mathbb{R}$. Writing $z = x + \iota y$ and $\overline{z} = x - \iota y$ we can write

$$x = \frac{1}{2}(z + \overline{z}), \quad y = \frac{1}{2}(z - \overline{z}).$$

If we put back the values of x and y in the equation of the line we will get

$$\frac{1}{2}(a - \iota b)z + \frac{1}{2}(a + \iota b)\overline{z} + c = 0.$$

Putting $\alpha = \frac{(a-\iota b)}{2}$, the equation of the line becomes

$$\alpha z + \overline{\alpha z} + c = 0. \tag{1.1}$$

Equation of the circle with centre (x_0, y_0) and radius r in the Euclidean plane is $(x - x_0)^2 + (y - y_0)^2 = r^2$. We can rewrite this equation as

$$(z - z_0)\overline{(z - z_0)} = r^2 \quad \text{where} \quad z = x + \iota y.$$

On expanding we get

$$z\overline{z} - \overline{z_0}z - z_0\overline{z} + z_0\overline{z_0} - r^2 = 0.$$

Now taking $\alpha = -\overline{z_0}$ and $\beta = z_0\overline{z_0} - r^2$, the equation of line becomes

$$z\overline{z} + \alpha z + \overline{\alpha z} + \beta = 0. \tag{1.2}$$

From the Equation 1.1 and Equation 1.2 the following result follows.

Proposition 1.1.7. *Let M be either circle or a straight line in $\mathbb{C}$. Then the equation of M is*

$$\gamma z\overline{z} + \alpha z + \overline{\alpha z} + \beta = 0. \tag{1.3}$$

where $\beta, \gamma \in \mathbb{R}$ and $\alpha \in \mathbb{C}$. We will denote all such paths by κ.

Proposition 1.1.8. *Let M be either circle or a straight line in $\mathbb{C}$ satisfying the equation $\gamma z\overline{z} + \alpha z + \overline{\alpha z} + \beta = 0$ with $\alpha, \beta, \gamma \in \mathbb{R}$, then M is either*

1. *circle with centre on Real-axis or ;*
2. *vertical straight line.*

Proposition 1.1.8. is the consequence of Proposition 1.1.7.

1.1.5 Möbius Transformation

Definition 1.1.9. *Let $p, q, r, s \in \mathbb{R}$ such that $ps - qr > 0$ and defines a map*

$$\gamma(z) = \frac{pz + q}{rz + s}$$

Transformations of $\mathbf{H^2}$ of this form are called Möbius Transformations of $\mathbf{H^2}$.

Notation Möb($\mathbf{H^2}$) denotes the set of all Möbius Transformations of $\mathbf{H^2}$.

Properties of Möbius Transformation

1. Möbius Transformation is a group under composition.

2. Dilations $z \mapsto mz$ (m > 0), Translations $z \mapsto z+a$ and Inversion $z \mapsto -\frac{1}{z}$ are Möbius Transformations.

3. Möbius Transformations maps $\mathbf{H^2}$ to itself bijectively.

4. $d_{\mathbf{H}^2}(\gamma(z_1), \gamma(z_2)) = d_{\mathbf{H}^2}(z_1, z_2)$.

5. Let $H \in \kappa$. Then there exists $\gamma \in$ Möb$(\mathbf{H}^2)$ such that γ maps H bijectively to the imaginary axis.

Proposition 1.1.10. *If M is either a semi-circle orthogonal to the real-axis or a vertical straight line and γ be a Möbius Transformation of $\mathbf{H}^2$, then $\gamma(M)$ is also a semi-circle orthogonal to the real-axis or a vertical straight line.*

Proof. Since, Möbius Transformations maps $\mathbf{H}^2$ to itself bijectively. Hence we show that γ maps vertical straight lines in $\mathbb{C}$ and the circles in $\mathbb{C}$ with real centres to vertical straight lines and circles with real centres.

$$w = \gamma(z) = \frac{pz+q}{rz+s},$$

then

$$z = \frac{sw-q}{-rw+p}.$$

Substituting this in Equation 1.3 we have $(\beta s^2 - 2\gamma rs + \alpha r^2)w\overline{w} + (-\beta qs + \gamma ps + \gamma qr - \alpha pr)w + (-\beta qs + \gamma ps + \gamma qr - \alpha pr)\overline{w} + (\beta q^2 - 2\gamma pq + \alpha p^2) = 0$ which is the equation of either a vertical line or a circle with real-axis.

Definition 1.1.11. *We define the shortest path between any 2 points in the space as geodesic.*

□

Proposition 1.1.12. *Let $p \leq q$. The hyperbolic distance between ιp and ιq is $\log \frac{q}{p}$ and the vertical line joining these points is the geodesic.*

Proof. Let $\sigma(z) = \iota z$, $p \leq z \leq q$. Here $\|\sigma'(z)\| = 1$. Then σ is a path from ιp to ιq. So

$$Length_{\mathbf{H}^2}(\sigma) = \int_p^q \frac{1}{z} dz = \log \frac{q}{p}$$

Now let $\sigma(z) = x(z) + \iota y(z) : [0;1] \to \mathbf{H}$ be any path from ιp to ιq. Then

$$Length_{\mathbf{H}^2}(\sigma) = \int_0^1 \frac{\sqrt{x'^2(z) + y'^2(z)}}{y(z)} dz \geq \frac{|y'(z)|}{y(z)} = \log \frac{q}{p}$$

Hence, the path joining ιp and ιq has minimum hyperbolic length $\log \frac{q}{p}$ with equality when $x'(z) = 0$ i.e. only when σ is a vertical line. □

Theorem 1.1.13. *The geodesics in $\mathbf{H}^2$ are the semi-circles orthogonal to the real axis and the vertical straight lines. Also given any 2 points, there exists a unique geodesic passing through them.*

Proof. Apply Proposition 1.1.12 and Property 5 of the Möbius transformation we can prove the first part of the theorem. By applying γ^{-1} we see that $\mathbf{H}^2$ is a unique geodesic passing through the two given points. □

1.1.6 Area and Angles

Definition 1.1.14. *Suppose we have 2 paths σ_1 and σ_2 which intersects at point $m \in \mathbf{H}^2$. By choosing a suitable parametrisation of paths, we assume that $m = \sigma_1(0) = \sigma_2(0)$. The angles between σ_1 and σ_2 is defined as the angle between their tangent vectors at the point of intersection.*
Möbius transformation preserves the angles. The transformation which preserves angles is called conformal.

Definition 1.1.15. *Let $M \subset \mathbf{H}^2$ be subset of the upper-half plane. Then the hyperbolic area is defined as*

$$Area_{\mathbf{H}^2}(M) = \int\int_M \frac{1}{y^2} \partial x \partial y = \int\int_M \frac{1}{Im(z)^2} dz.$$

Hyperbolic area is invariant under Möbius Transformation.

Proposition 1.1.16. *Let γ be a Möbius Transformation. Then*

1. *γ is conformal.*
2. *$Area_{\mathbf{H}^2}(\gamma(M)) = Area_{\mathbf{H}^2}(M)$.*

Proof. We get the proof using the concepts of Cauchy-Riemann Equations, Inner Product, Cauchy-Schwartz Inequality as given in [CW12] Chapter 5, Section 9. □

1.2 Poincaré Disc Model

Definition 1.2.1. *The disc* $\mathbf{D}^2 = \{z \in \mathbb{C} \mid |z| < 1\}$ *is called Poincaré Disc. The circle* $\partial\mathbf{D}^2 = \{z \in \mathbb{C} \mid |z| = 1\}$ *is called the circle at* ∞ *or boundary of* $\mathbf{D}^2$.

1.2.1 Distance and Möbius Transformations on $\mathbf{D}^2$

Consider the map

$$h(z) = \frac{z - \iota}{\iota z - 1}$$

h maps the upper-half plane $\mathbf{H}^2$ to Poincaré Disc $\mathbf{D}^2$. Also h maps $\partial\mathbf{H}^2$ to $\partial\mathbf{D}^2$ bijectively.
Let $g(z) = h^{-1}(z)$, then g maps $\mathbf{D}^2$ to $\mathbf{H}^2$ and is defined by

$$g(z) = \frac{-z + \iota}{-\iota z + 1}.$$

Let $\sigma : [x, y] \to \mathbf{D}^2$ be a path in $\mathbf{D}^2$, then $g \circ \sigma : [x, y] \to \mathbf{D}^2$ be a path in $\mathbf{H}^2$. The length of $g \circ \sigma$ is defined as

$$Length_{\mathbf{H}^2}(g \circ \sigma) = \int_x^y \frac{|(g \circ \sigma)'(t)|}{Im(g \circ \sigma(t))} dt.$$

Using the chain rule, we calculate the values of $g'(z)$ and $Im(g(z))$, we can calculate the value of $Length_{\mathbf{H}^2}(g \circ \sigma)$ and hence the value of $Length_{\mathbf{D}^2}(\sigma)$ which is found to be equal to

$$Length_{\mathbf{D}^2}(\sigma) = \int_x^y \frac{2}{1 - |\sigma(t)|^2} |\sigma'(t)| dt.$$

The difference between calculating distances in Upper half model and Poincaré disc model is that in upper-half model we integrate $\frac{1}{Im(z)}$ along the path to obtain the length whereas in Poincaré Disc we integrate $\frac{2}{1-|z|^2}$ along the path.

Definition 1.2.2. *We call a map of the form*

$$\gamma(z) = \frac{\alpha z + \beta}{\overline{\beta} z + \overline{\alpha}}, \quad \alpha, \beta \in \mathbb{C}, \quad |\alpha|^2 - |\beta|^2 > 0$$

as Möbius Transformation of $\mathbf{D}^2$.
For any $\gamma \in$ Möb$(\mathbf{H}^2)$ and $u, v \in \mathbf{D}^2$. We obtain an isometry of $\mathbf{D}^2$ by using the map $h\gamma h^{-1}$.

Definition 1.2.3. *Using the h(z) and transferring this to definition of* $\mathbf{D}^2$ *we can define the area of* $M \subset \mathbf{D}^2$ *as*

$$Area_{\mathbf{H}^2}(\mathbf{D}^2) = \int\int_M \frac{4}{(1-|z|^2)} dz.$$

1.3 Trigonometry

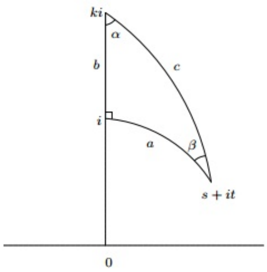

Figure 1.1: Right Angled Triangle

Theorem 1.3.1 (Pythagoras theorem for hyperbolic triangles)**.** *Let* $\triangle$ *be a right-angled triangle in* $\mathbf{H}^2$ *with internal angles* α*,* β*,* $\pi/2$ *and opposing sides with lengths* a*,* b*,* c*. Then*

$$\cosh c = \cosh a \ \cosh b.$$

Proof. Let $\triangle$ be a triangle which satisfies the given hypothesis. By applying a Möbius Transformation of $\mathbf{H}^2$, we assume that the vertex with internal angle $\pi/2$ is at ι and side of length b lies along the imaginary axis. So the side of length a lies along the geodesic which is given by the semi-circle centred at the origin and radii 1. (See Figure 1.1)

Other vertices of $\triangle$ can be taken to be kι for some k $>$ 0 and at s $+$ ιt. s $+$ ιt lies on the circle which is centred at the origin and has radii 1. Using the formula that for any $z, w \in \mathbf{H}^2$

$$\cosh \ d_{\mathbf{H}^2}(z,w) = 1 + \frac{| \ z - w \ |^2}{2Im(z)Im(w)}.$$

We can apply this formula to calculate the the three sides of $\triangle$

$\cosh a = \frac{1}{t}$

$\cosh b = \frac{1+k^2}{2k}$

$\cosh c = \frac{1+k^2}{2tk}$

Combining the above equations we see that

$$\cosh c = \cosh a \ \cosh b.$$

□

1.3.1 Gauss Bonnet Theorem

Theorem 1.3.2 (Gauss Bonnet theorem for hyperbolic triangle)**.** *Let* $\triangle$ *be a hyperbolic triangle with internal angles* α, β *and* γ*. Then*

$$Area(\triangle) = \pi - (\alpha + \beta + \gamma).$$

Proof. **Case 1** Triangle is ideal.
In this case all the vertices are at the boundary of the hyperbolic plane and we know that hyperbolic area of any A $\subset \mathbf{H}^2$ is given by

$$\begin{aligned} Area(\triangle) &= \int\int_T \frac{\mathrm{d}x\mathrm{d}y}{y^2} \\ &= \int_{-1}^{1}\int_{\sqrt{(1-x^2)}}^{\infty} \frac{\mathrm{d}y}{y^2}\mathrm{d}x \\ &= \int_{-1}^{1} [-\frac{1}{y}]_{\sqrt{1-x^2}}^{\infty}\mathrm{d}x \\ &= \int_{-1}^{1} \frac{\mathrm{d}x}{\sqrt{1-x^2}} \\ &= [Sin^{-1}x]_{-1}^{1} \\ &= \pi. \end{aligned}$$

Case 2 Two vertices of a triangle are at infinity and α be an angle of the triangle at the finite vertex. This can be done by a suitable transformation in PSL(2,$\mathbb{R}$). So the finite vertex will be $e^{\iota(\pi-\alpha)}$. The area of the triangle will be

$$\begin{aligned} Area(\triangle) &= \int\int_T \frac{\mathrm{d}x\mathrm{d}y}{y^2} \\ &= \int_{Cos(\pi-\alpha)}^{1} \int_{\sqrt{(1-x^2)}}^{\infty} \frac{\mathrm{d}y}{y^2}\mathrm{d}x \\ &= \int_{Cos(\pi-\alpha)}^{1} [-\frac{1}{y}]_{\sqrt{(1-x^2)}}^{\infty}\mathrm{d}x \\ &= \int_{Cos(\pi-\alpha)}^{1} \frac{\mathrm{d}x}{\sqrt{(1-x^2)}} \\ &= \pi - \alpha. \end{aligned}$$

Case 3 Two vertices are finite. Let v_1 and v_2 be the finite vertices and l be a geodesic joining them and α and β are the angles at the vertices. By a suitable transformation in PSL(2, $\mathbb{R}$) we can map the infinite vertex at ∞ and v_1 and v_2 on the unit circle. Now we say $v_1 = e^{\iota(\pi-\alpha)}$ and $v_2 = e^{\iota\beta}$. Therefore,

$$\begin{aligned} Area(\triangle) &= \int\int_T \frac{\mathrm{d}x\mathrm{d}y}{y^2} \\ &= \int_{Cos(\pi-\alpha)}^{Cos\beta} \int_{\sqrt{1-x^2}}^{\infty} \frac{\mathrm{d}y}{y^2}\mathrm{d}x \\ &= \int_{Cos(\pi-\alpha)}^{Cos\beta} [-\frac{1}{y^2}]_{\sqrt{(1-x^2)}}^{\infty}\mathrm{d}x \\ &= \int_{Cos(\pi-\alpha)}^{Cos\beta} \frac{\mathrm{d}x}{\sqrt{(1-x^2)}} \\ &= \pi - (\alpha+\beta). \end{aligned}$$

Case 4 All vertices are finite. We can express this as difference of 2 hyperbolic triangles each with 1 vertex at infinity and solve this as in Case 2 and Case 3. We get

$$Area(\triangle) = \pi - (\alpha+\beta+\gamma).$$

$\square$

Theorem 1.3.3 (Gauss Bonnet Theorem for a hyperbolic Polygon)**.** *Let Z be n-sided polygon with vertices $v_1, v_2, \cdots, v_n$ and internal angles $\alpha_1, \alpha_2, \cdots, \alpha_n$. Then*

$$Area_{\mathbf{H}^2}(Z) = (n-2)\pi - (\alpha_1+\alpha_2+\cdots+\alpha_n).$$

Theorem 1.3.4. *There exists a tessellation of the hyperbolic plane by regular hyperbolic n-gons with m polygons meeting at each vertex if*

$$\tfrac{1}{n} + \tfrac{1}{m} < \tfrac{1}{2}.$$

Proof. Let β denote the internal angle of a regular n-gon Z . Since m such polygons meet at each vertex so we must have $\beta = 2\pi/m$. Since the area of the polygon is positive hence on substituting $\beta = 2\pi/m$ into the area we have

$$\tfrac{1}{n} + \tfrac{1}{m} < \tfrac{1}{2}.$$

□

1.3.2 Relation between angles and sides

Proposition 1.3.5. *Let* $\triangle$ *be a right-angled triangle in* $\mathbf{H}^2$ *with internal angles* α, β *and* $\pi/2$ *and opposing sides of lengths a, b, c. Then*

1. $\sin\alpha = \sinh a/\sinh c$;
2. $\cos\alpha = \tanh b/\tanh c$;
3. $\tan\alpha = \tanh c/\sinh b$.

Proof. See [SK92] Chapter 1. □

1.3.3 Angle of Parallelism

We will consider a special case of a right-angled triangle with one ideal vertex. The internal angles of the triangle are α, $\pi/2$ and 0. The only side with finite length is between the vertices which have internal angles α and $\pi/2$. The angle of parallelism is a term for this angle expressed in terms of the side of finite length.

Proposition 1.3.6. *Let* $\triangle$ *be a hyperbolic triangle with angles* α*, 0 and* $\pi/2$*. Let* m *denote the length of the only finite side. Then*

1. $\sin\alpha = \frac{1}{\cosh m}$;
2. $\cos\alpha = \frac{1}{\coth m}$;
3. $\tan\alpha = \frac{1}{\sinh m}$.

Proof. See [JA05] Chapter 5. □

1.3.4 Non-right-angled triangles

1. **Sine Rule**
   Let $\triangle$ be a hyperbolic triangle with internal angles α, β and γ and side lengths x, y and z. Then
   $$\frac{\sin\alpha}{\sinh x} = \frac{\sin\beta}{\sinh y} = \frac{\sin\gamma}{\sinh z}.$$

2. **Cosine Rule**
   Let $\triangle$ be a hyperbolic triangle with internal angles α, β and γ and side lengths a, b, c. Then
   $$\cosh c = \cosh a \cosh b - \sinh a \sinh b \cos\gamma.$$

Proof. See [JA05] Chapter 5. □

1.4 Classification of Möbius Transformation

Note that it follows that every Möbius Transformation has atleast one fixed point in $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. This is because the equation

$$\frac{az+b}{cz+d} = z$$

is a qudratic equation and it must have a root in $\mathbb{C} \cup \{\infty\}$. One of them must be in $\mathbf{H}^2 \cup \partial\mathbf{H}^2$.
From the above equation there are 2 possibilities

1. 1 or 2 real solutions.

2. 2 complex conjugates (But only 1 of them lies in upper-half plane).

Definition 1.4.1. *Let γ be a Möbius Transformation of $\mathbf{H}^2$. Then*

1. *γ is hyperbolic if it has two fixed points in $\partial\mathbf{H}^2$ and none in $\mathbf{H}^2$.*

2. *γ is parabolic if it has one fixed point in $\partial\mathbf{H}^2$ and none in $\mathbf{H}^2$.*

3. *γ is elliptic if it has one fixed point in $\mathbf{H}^2$ and none in $\partial\mathbf{H}^2$.*

Definition 1.4.2. *Let γ_1, $\gamma_2 \in$ Möb($\mathbf{H}^2$) . We say that γ_1 and γ_2 are conjugates if there exists another Möbius Transformations $g \in$ Möb($\mathbf{H}^2$) such that $\gamma_1 = g^{-1} \circ \gamma_2 \circ g$. We define Trace of a Möbius Transformation with $\gamma(z) = \frac{az+b}{cz+d}$ to be $\tau(\gamma) = (a+d)^2$.*

Proposition 1.4.3. *Let γ be a Möbius Transformation of $\mathbf{H}^2$ and suppose that γ is not an identity. Then*

1. *γ is parabolic if and only if $\tau(\gamma) = 4$.*

2. *γ is elliptic if and only if $\tau(\gamma) \in [0, 4)$.*

3. *γ is hyperbolic if and only if $\tau(\gamma) \in (4, \infty)$.*

Proposition 1.4.4. *Let γ be a Möbius Transformation of $\mathbf{H}^2$ and suppose that γ is not an identity. Then the following are equivalent*

1. *γ is parabolic.*

2. *$\tau(\gamma) = 4$.*

3. *γ is conjugate to a translation.*

Proof. We only need to prove that $1 \implies 3$.
We suppose that γ has a unique fixed point at $\alpha \in \partial\mathbf{H}^2$. Let h be a Möbius Transformation which maps α to ∞, then $h\gamma h^{-1}$ is a translation. As discussed above $h\gamma h^{-1}$ will have a fixed point at $b/(d-a)$. As $h\gamma h^{-1}$ has only 1 fixed point so we have $d = a$. Thus, $h\gamma h^{-1} = z + m$ for some $m \in \mathbb{Z}$. Hence $1 \implies 3$. □

Proposition 1.4.5. *Let γ be a Möbius Transformation of $\mathbf{H}^2$ and suppose that γ is not an identity. Then the following are equivalent*

1. *γ is hyperbolic.*

2. *$\tau(\gamma) > 4$.*

3. *γ is conjugate to a dilation i.e. γ is conjugate to a Möbius transformation of $\mathbf{H^2}$ of the form $z \to kz$, for some $k > 0$.*

Proof. We only need to prove that $1 \implies 3$.
Since γ is hyperbolic transformation, so it has 2 fixed points say α_1 and α_2. We have 2 cases.

1. Suppose $\alpha_2 = \infty$ and $\alpha_1 \in \mathbb{R}$. Let $h(z) = z - \alpha_1$. Then the Möbius Transformation $h\gamma h^{-1}$ is conjugate to γ and has fixed points at 0 and ∞.

2. Suppose that $\alpha_1 \in \mathbb{R}$ and $\alpha_2 \in \mathbb{R}$. We assume that $\alpha_1 > \alpha_2$. Let h be a transformation then

$$h(z) = \frac{z - \alpha_1}{z - \alpha_2}$$

As $h(\alpha_2) = \infty$ and $h(\alpha_1) = 0$ we see that $h\gamma h^{-1}$ has fixed points at 0 and ∞.

Hence $1 \implies 3$. □

Proposition 1.4.6. *Let γ be a Möbius Transformation of $\mathbf{H}^2$ and suppose that γ is not an identity. Then the following are equivalent*

1. γ is elliptic.

2. $\tau(\gamma) \in [0, 4)$.

3. γ is conjugate to rotation $z \mapsto e^{\iota\theta} z$.

Proof. We only need to prove that $1 \implies 3$.
Suppose that γ is elliptic and has a unique fixed point at $\alpha \in \mathbf{D}^2$. Let h be a Möbius Transformation of $\mathbf{D}^2$ that maps α to origin 0. Then $h\gamma h^{-1}$ has a unique fixed point at 0. Suppose that

$$h\gamma h^{-1}(z) = \frac{\alpha z + \beta}{\overline{\beta} z + \overline{\alpha}}, \quad |\alpha|^2 - |\beta|^2 > 0.$$

As 0 is fixed point, we must have $\beta = 0$. Writing α in polar coordinate form we have $\alpha = re^{\iota\theta}$. Then

$$h\gamma h^{-1}(z) = \frac{\alpha}{\overline{\alpha}} z = e^{2\iota\theta} z.$$

Hence $1 \implies 3$. □

Chapter 2

Fuchsian groups

General Fuchsian groups were studied for the first time by Poincaré in 1882, who was inspired by *Ueber eine Klasse von Funktionen mehrerer Variablen, welche durch Umkehrung der Integrale von Lsungen der linearen Differentialgleichungen mit rationalen Coeffizienten entstehen* (*On a class of functions of several variables, which are produced by inverting the integrals of solutions of linear differential equations with rational coefficients*) by Lazarus Fuchs in 1880 and therefore named the theory after Lazarus Fuchs's name. In this section we have studied properties of Fuchsian Group and various variants of Poincaré Theorem. [SK92], [CW12] and [AB07] have been used as the main references for studying this chapter.
Discrete subgroup of $PSL(2,\mathbb{R})$ is known as Fuchsian group.

$$PSL(2,\mathbb{R}) = \{\gamma(z) = \tfrac{az+b}{cz+d} | a,b,c,d \in \mathbb{R}, ad-bc=1\}.$$

Examples

1. Any finite subgroup of Möb(H).
2. The subgroup $\Gamma = \{\gamma_n(z) = 2^n z \mid n \in \mathbb{Z}\}$ is a Fuchsian group.
3. PSL(2,$\mathbb{Z}$).

Generators of SL(2,$\mathbb{Z}$) We will prove that any matrix in SL(2, $\mathbb{Z}$) can be written as product of positive powers of T and S where T and S are given by the following

$$\mathrm{T} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

We will use the fact that all elements of SL(2, $\mathbb{Z}$) are invertible and hence can be brought down to the I (identity element) by a series of row and column operations. Also we know that $\mathbb{Z}$ is Euclidean.

Proof. If a matrix $B^{-1} \in SL(2, \mathbb{Z})$ is written as a product of some powers of S and T and since each $A \in SL(2, \mathbb{Z})$ is the inverse of $A^{-1} \in SL(2, \mathbb{Z})$ so all A can be expressed in a similar way. We will denote the required matrix by B and its inverse by A. The set of elementary operations namely interchanging rows, multiplying by a non-zero scalar or multiplying a row and adding it to another cannot be used as in the two first cases, the relevant elementary matrices will not be in SL(2, $\mathbb{Z}$). Hence we must allow only the third kind of operation along with multiplication by powers of S. For this task we require the matrix $T^z = \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix}$ where $z \in \mathbb{Z}$ and its transpose. On checking we get

$S^{-1} = S^3$

$T^{-1} = S^3TSTS$

$ST^{-1} = S^2STS$

$T^t = TST$

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and since $det(A) = ad - bc = 1$ and we know that gcd (a, b) = gcd (a, c) = gcd (b, c) = gcd (c, d) = 1 (because if x|a and x|b, then $x|(ad - bc) = 1$ and hence x = ±1).

Case 1 Suppose that c = 0, then A= $\begin{pmatrix} \pm1 & b \\ 0 & \pm1 \end{pmatrix}$. Then $\begin{pmatrix} 1 & \mp b \\ 0 & 1 \end{pmatrix} \times \begin{pmatrix} \pm1 & b \\ 0 & \pm1 \end{pmatrix} =$ ±1. We are done since $S^2 = -I$.

Case 2 Suppose c ≠ 0. Since $\mathbb{Z}$ is Euclidean and gcd(a,c) = 1, we have a list of integers $[p_1, \cdots, p_n, p_{n+1}]$ which gives us the above equations with last remainder r_n = 1. This procedure is actually the division algorithm carried out on a column of A.

$$T^{-p_1}A = \begin{pmatrix} r_1 & b_1 \\ c & d_1 \end{pmatrix}$$

where $r_1 = a - p_1c$ is the first remainder. Also b_1 and d_1 are the new entries. The rows remain relatively prime. Then

$$T^{-p_2^t}A = \begin{pmatrix} r_1 & b_2 \\ r_2 & d_2 \end{pmatrix}$$

where $r_2 = c - p_2r_1$. Since gcd (a, c) = 1 so this process must terminate with $(1, 0)^t$ or $(0, 1)^t$ in first column of A. In the last case, multiplication on the left by S^3 brings A to A' = $\begin{pmatrix} \pm1 & b'_n \\ 0 & \pm1 \end{pmatrix}$(because $det(A') = 1$.) It is clear that multiplication by S^2, and then T^{-b_n} gives I. □

2.1 Fundamental Domains

When a group acts on a set, it divides it in orbits. If there exists an element of the group which takes one element to another then those two points are in the same equivalence class. If H is a subgroup of Γ then we say that the two points m_1, $m_2 \in$ $\mathbf{H}^2$ are H-equivalent if there exists g $\in$ H such that $m_2 = gm_1$ i.e. they are in the same orbit class.

Definition 2.1.1. *Let Γ be a Fuchsian group. A fundamental domain F for Γ is an open subset of $\mathbf{H}^2$ such that*

1. $\cup_{\gamma\in\Gamma}\ \gamma(cl(F)) = \mathbf{H}^2$.
2. *The images $\gamma(F)$ are pairwise disjoint i.e. $\gamma_1(F) \cap \gamma_2(F) = \phi$ if $\gamma_1(F), \gamma_2(F) \in \Gamma, \gamma_1 \neq \gamma_2$.*

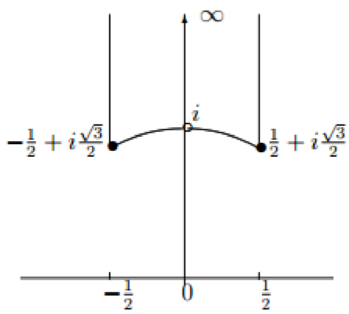

Figure 2.1: Fundamental Domain of $PSL(2, \mathbb{Z})$

Theorem 2.1.2. *The region F defined by*

$$F = \{z \in \mathbf{H}^2 \mid \frac{-1}{2} \leq Re\ \ z \leq \frac{1}{2} and \mid z \mid \geq 1\}$$

is a fundamental domain for $PSL(2, \mathbb{Z})$.

Idea of the proof

We will first prove that every z $\in \mathbf{H}^2$ is $PSL(2, \mathbb{Z})$-equivalent to a point in F. We will first use the translation T^{j_1} to move point inside the strip $\frac{-1}{2} \leq Rez \leq \frac{1}{2}$. If the point goes outside the circle then it is inside the Fundamental Domain, else we move the point outside the unit circle using S and we will then use the translation T^{j_2} to bring it back to the strip. We will continue this process till we get the point inside F. Notation for S and T are same as stated above.

Proof. Let $z \in \mathbf{H^2}$ be fixed. If g = $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PSL(2, \mathbb{Z})$. Then we will have $Im\ g(z) = \frac{Im\ z}{|cz+d|^2}$. If |cz+d| $\geq$ 1 then $Im\ g(z) \leq Im\ z$. Now since, c and d vary through integers so the complex numbers cz + d run through lattices which are generated by 1 and z whereas inside the unit circle there are finitely many lattice points. As a result, there are finitely many complex numbers y which are of the form $y = g\ (z)$ with $Im\ y \geq Im\ z$. Hence, there is some g = $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PSL(2, \mathbb{Z})$ such that $Im\ g(z)$ is maximal. We will now replace g by T^jg for some suitable j and assume that g(z) is inside $\frac{-1}{2} \leq Rez \leq \frac{1}{2}$. Then if $g(z) \notin F$ then we would have $Im\ S(g(z)) = \frac{Im\ g(z)}{|g(z)|^2} > Im\ g(z)$, which will contradict the maximality of $Im\ g(z)$. Hence, there is atleast one g $\in PSL(2, \mathbb{Z})$ such that g(z) $\in$ F.

It remains to prove that no two points in the interior of F are $PSL(2, \mathbb{Z})$-equivalent. Let z_1 and $z_2 \in$ F be $PSL(2, \mathbb{Z})$-equivalent and g = $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PSL(2, \mathbb{Z})$ be a matrix with $z_2 =$ g(z_1). Next we will assume that $Im\ z_2 = Im\ g(z_1) \geq Im\ z_1$. Since $Im\ z_2 = Im\ g(z_1) = \frac{Im\ z_1}{|cz+d|^2} \geq Im\ z_1$ so we have $|\ cz_1 + d\ | \leq 1$. If $z_1 = a_1 + b_1\iota$, then $|\ cz_1 + d\ | \leq 1$ implies $|\ ca_1 + d + cb_1\iota\ | \leq 1$ or $(ca_1 + d)^2 + (cb_1)^2 \leq 1$. Thus, $|\ cb_1\ | \leq 1$ and since b_1 = Im $z_1 \geq \frac{\sqrt{3}}{2}$ we have $|\ c\ | \leq \frac{1}{b_1} \leq \frac{2}{\sqrt{3}} < 2$ that is | c | $\leq$ 1. If we add d to cz_1 it will translate cz_1 to left or right by | d |. Hence we need |d| $\leq$ 1 for cz_1 + d to be inside circle. Thus, we are left with 4 cases.

Case 1

d = $\pm$1, c = 0.

Here g is a translation g = T^j, where j $\geq$ 0. g will take a point in F to another point only if it is identity or if j = $\pm$ 1 and the points are on the two boundary lines Re(z) = $\pm\ \frac{1}{2}$.

Case 2

d = 0, c = $\pm$1 with z_1 on the circle.

Here g must be of the form g = T^mS with m = 0. Also z_1 and z_2 lie on the unit circle located evenly with respect to the imaginary axis or m = $\pm$1 and $z_1 = z_2 = \pm \frac{-1}{2} + \frac{\sqrt{3}}{2}\iota$.

Case 3

c = d = $\pm$1 and $z_1 = \frac{-1}{2} + \frac{\sqrt{3}\iota}{2}$.

Here we have $|\ z_1 + 1\ | = |\ cz_1 + d\ | \leq 1$. Since on adding 1 will translate a point in F to the right so the lone possibility for z_1 is $z_1 = \frac{-1}{2} + \frac{\sqrt{3}}{2}\iota$.

Case 4

c = −d = ±1 and $z_1 = \frac{1}{2} + \frac{\sqrt{3}\iota}{2}$.
Here we must have $| z_1 - 1 | = | cz_1 + d | \leq 1$. The only possibility is that $z_1 = \frac{1}{2} + \frac{\sqrt{3}}{2}\iota$.
So we have z_1 and z_2 in the interior of F if and only if $g = I$ and $z_1 = z_2$. □

Proposition 2.1.3. *Let F_1 and F_2 be two fundamental domains for a Fuchsian group Γ with $Area_{\mathbf{H}^2}(F_1) < \infty$. Then $Area_{\mathbf{H}^2}(F_1) = Area_{\mathbf{H}^2}(F_2)$.*

2.2 Dirichlet Polygon

Definition 2.2.1. *Geodesics divide $\mathbf{H}^2$ into 2 compartments and we call those formed compartments as half planes.*
A convex hyperbolic polygon is the intersection of a finite number of half planes.

Dirichlet region is defined as

$$D(p) = \{z \in \mathbf{H}^2 \mid d_{\mathbf{H}^2}(z, p) < d_{\mathbf{H}^2}(z, \gamma(p))\}.$$

The fundamental domain that we construct for a given Fuchsian group is known as Dirichlet polygon.

2.2.1 Construction of Dirichlet Polygon

For constructing Dirichlet Polygon we follow the following procedure

1. Choose p $\in \mathbf{H}^2$ such that $\gamma(p) \neq$ p for all $\gamma \in \Gamma \setminus \{Id\}$.
2. For a given $\gamma \in \Gamma \setminus \{Id\}$ construct the geodesic segment [p, $\gamma(p)$].
3. Take $L_p(\gamma)$ to be the perpendicular bisector of [p, $\gamma(p)$].
4. Let $H_p(\gamma)$ be the half-plane determined by $L_p(\gamma)$ that contains p.
5. D(p) $= \cap_{\gamma \in \Gamma \setminus \{Id\}} H_p(\gamma)$.

Example Dirichlet polygon for Fuchsian group $\Gamma = \{\gamma_n(z) \mid \gamma_n(z) = 2^n z,\ n \in \mathbb{Z}\}$ is given by

$$D(p) = \{z \in \mathbf{H}^2 \mid 1/\sqrt{2} < |z| < \sqrt{2}\}.$$

Let $\Gamma = \{\gamma_n(z) | \gamma_n(z) = 2^n z\}$. Let $p = \iota$ and also $\gamma_n(p) = 2^n \iota \neq p$ unless n = 0. For each n, [p, $\gamma_n(p)$] is an arc of imaginary axis from ι to $2^n\iota$. Suppose that $n > 0$. We know by Proposition 1.1.12 that $2^{n/2}\iota$ is the midpoint. Hence $L_p(\gamma_n)$ is the semicircle of radius $2^{n/2}$ centred at origin and

$$H_p(\gamma_n) = \{z \in \mathbf{H}^2 \mid |z| < 2^{n/2}\}.$$

For $n < 0$ we see

$$H_p(\gamma_n) = \{z \in \mathbf{H}^2 \mid |z| > 2^{n/2}\}.$$

Hence

$$D(p) = \cap_{\gamma_n \in \Gamma \setminus \{Id\}} H_p(\gamma_n) = \{z \in \mathbf{H}^2 \mid 1/\sqrt{2} < |z| < \sqrt{2}\}.$$

2.3 Side Pairing Transformation

Let D be a hyperbolic polygon and the side s $\subset \mathbf{H}^2$ of D is an edge of D in $\mathbf{H}^2$ equipped with an orientation namely, a side of D is an edge starting at one vertex and ending at another vertex.
Let Γ be a Fuchsian group and let D(p) be a Dirichlet polygon for Γ, which has finitely many sides. Suppose that for some $\gamma \in \Gamma \setminus \{Id\}$, we have $\gamma(s)$ as side of D(p).

Definition 2.3.1. *The sides s and $\gamma(s)$ are paired and γ is called as side-pairing transformation.*

2.3.1 Construction of Elliptic Cycles

We know that each vertex v of D is mapped to another vertex under the side-pairing transformation associated to a side with end point at v. Each vertex v of D has two sides say s and $*$s of D which has end points at v.
Notation We denote the pair (v, s) as vertex v of D and a side s of D with an endpoint at v.
We construct elliptic cycles by the following procedure

1. Let v = v_0 be a vertex of D and let s_0 be a side with an endpoint at v_0. Let γ_1 be the side-pairing transformation associated to the side s_0. Thus γ_1 maps s_0 to another side s_1 of D.

2. Let $s_1 = \gamma_1(s_0)$ and let $v_1 = \gamma_1(v_0)$. This gives a new pair (v_1,s_1).

3. Now consider the pair $*(v_1,s_1)$. This is the pair consisting of the vertex v_1 and the side $*s_1$.

4. Let γ_2 be the side-pairing transformation associated to the side $*s_1$. Then $\gamma_2(*s_1)$ is a side s_2 of D and $\gamma_2(v_1) = v_1$, a vertex of D.

5. Repeat this process inductively.

Definition 2.3.2. *The sequence of vertices* $\varepsilon = v_0 \longrightarrow v_1 \longrightarrow v_2 \longrightarrow \cdots \longrightarrow v_{n-1}$ *is called an elliptic cycle and the transformation* $\gamma_n\gamma_{n-1}\cdots\gamma_2\gamma_1$ *is called an elliptic cycle transformation.*

Definition 2.3.3. *Let* $\angle v$ *denote the internal angle of D at the vertex v. Considering the elliptic cycle as defined above i.e.* $\varepsilon = v_0 \longrightarrow v_1 \longrightarrow v_2 \longrightarrow \cdots \longrightarrow v_{n-1}$ *of the vertex* $v = v_0$. *The angle sum is defined as*

$$sum(\varepsilon) = \angle v_0 + \angle v_1 + \cdots + \angle v_{n-1}.$$

Proposition 2.3.4. *Let* Γ *be a Fuchsian group with Dirichlet polygon D with all vertices in* $\mathbf{H}^2$ *and let* ε *be an elliptic cycle. Then there exists an integer* $m_\varepsilon \geq 1$ *such that* $m_\varepsilon sum(\varepsilon) = 2\pi$.

Definition 2.3.5. *Elliptic cycle is said to be accidental if the elliptic curve transformation is equal to identity, i.e. we have* $m_\varepsilon = 1$ *and hence* $sum(\varepsilon) = 2\pi$.

2.4 Signature of Fuchsian Groups

Definition 2.4.1. *Let* ε *be an elliptic cycle and suppose that* $sum(\varepsilon) \neq 2\pi$. *On gluing together the vertices on this elliptic cycle gives a point on* $\mathbf{H}^2/\Gamma$ *with total angle less than* 2π. *This point is called a marked point.*

Definition 2.4.2. *Let X be a 2-dimensional surface. Then the genus g of X is given by*

$$\chi(X) = 2 - 2g.$$

2.4.1 Existence of Cocompact group with signatures

Definition 2.4.3. *Let* Γ *be a Fuchsian group. We say that* Γ *is cocompact if it has a finite-sided Dirichlet polygon D(p) with all vertices in* $\mathbf{H}^2$ *and none in* $\partial\mathbf{H}^2$.

Definition 2.4.4. *Let Γ be a cocompact Fuchsian group. Let g be the genus of $\mathbf{H}^2/\Gamma$. Suppose that there are k elliptic cycles $\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_k$. and if ε_j has order $m\varepsilon_j = m_j$ so that $m_j\varepsilon_j = 2\pi$. Also if $\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_r$ are non-accidental cycles and $\varepsilon_{r+1}, \varepsilon_{r+2}, \cdots, \varepsilon_k$ are accidental cycles. The signature of Γ is defined to be*

$$sig(\Gamma) = (g; m_1, \cdots m_r).$$

Let v = v_0 be a boundary vertex of D and let s = s_0 be a side with an end-point at v_0. We can then repeat the procedure described (as in Section 2.3) starting at the pair (v_0,s_0) to obtain a finite sequence of boundary vertices P = $v_0 \longrightarrow v_1 \longrightarrow v_2 \longrightarrow \cdots \longrightarrow v_{n-1}$ and an associated Möbius transformation $\gamma_{v,s} = \gamma_n\gamma_{n-1}\cdots\gamma_2\gamma_1$.
We say that P = $v_0 \longrightarrow v_1 \longrightarrow v_2 \longrightarrow \cdots \longrightarrow v_{n-1}$ is a parabolic cycle with associated parabolic cycle transformation $\gamma_{v,s} = \gamma_n\gamma_{n-1}\cdots\gamma_2\gamma_1$.

Poincaré's Theorem : The case of no boundary vertices

Theorem 2.4.5. *Let D be a convex hyperbolic polygon with finitely many sides. Suppose that all vertices lie inside $\mathbf{H}^2$ and D is equipped with a collection G of side-pairing Möbius transformations and no side of D is paired with itself. Let the elliptic cycles be $\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_r$. Suppose that each elliptic cycle ε_j of D satisfies the elliptic cycle condition that for each ε_j there exists and integer $m_j \geq 1$ such that*

$$m_j sum(\varepsilon_j) = 2\,\pi.$$

Then

1. *The subgroup $\Gamma = \langle G\rangle$ generated by G is a Fuchsian group.*
2. *The Fuchsian group Γ has D as a fundamental domain.*
3. *The Fuchsian group Γ can be written in terms of generators and relations as follows. For each elliptic cycle ε_j, choose a corresponding elliptic cycle transformation $\gamma_j = \gamma_{v,s}$. Then Γ is isomorphic to the group with generators $\gamma_s \in G$ and relations $\gamma_j^{m_j}$*

$$\Gamma = \langle \gamma_s \in G | \gamma_1^{m_1} = \gamma_2^{m_2} = \cdots \gamma_r^{m_r} = e\rangle.$$

Poincaré's Theorem : The case of boundary vertices

Theorem 2.4.6. *Let D be a convex hyperbolic polygon with finitely many sides, and boundary vertices (but with no free edge). Suppose D is equipped with a collection G of side-pairing Möbius transformations and no side of D is paired with itself.*
Let the elliptic cycles be $\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_r$ *and the parabolic cycles be* $P_1, P_2, \cdots, P_s$*. Suppose that*

1. *each elliptic cycle* ε_j *of D satisfies the elliptic cycle condition.*

2. *each parabolic cycle* P_j *of D satisfies the parabolic cycle condition.*

Then

1. *The subgroup* $\Gamma = \langle G \rangle$ *generated by G is a Fuchsian group.*

2. *The Fuchsian group* Γ *has D as a fundamental domain.*

3. *The Fuchsian group* Γ *can be written in terms of generators and relations as follows.*
   For each elliptic cycle ε_j*, choose a corresponding elliptic cycle transformation* $\gamma_j = \gamma_{v,s}$*. Then* Γ *is isomorphic to the group with generators* $\gamma_s \in G$ *and relations* $\gamma_j^{m_j}$ *where* $m_j sum(\varepsilon_j) = 2\,\pi$

$$\Gamma = \langle \gamma_s \in G | \gamma_1^{m_1} = \gamma_2^{m_2} = \cdots \gamma_r^{m_r} = e \rangle.$$

2.4.2 Area of a Dirichlet polygon

Let Γ be a cocompact Fuchsian group with signature sig(Γ) = (g; $m_1, \cdots, m_r$). Let D be a fundamental domain for Γ. Then

$$\text{Area}_{\mathbf{H}^2}(D) = 2\pi((2g-2) + \textstyle\sum_{j=1}^{r}(1 - \frac{1}{m_j})).$$

Proof. We first suppose that there are s accidental cycles. The internal angle sum along an accidental cycle is 2π. Hence the internal angle sum along all accidental cycles is 2πs. Suppose that we have n non-accidental cycles. As each vertex belongs to some elliptic cycle so the sum of all internal angles of D is given by

$$2\pi(\sum_{j=1}^{r} \frac{1}{m_j} + s).$$

By Gauss Bonnet Theorem for hyperbolic polygons (Theorem 1.3.2) we have

$$Area_{\mathbf{H}^2}(D) = (n-2)\pi - 2\pi(\sum_{j=1}^{r} \frac{1}{m_j} + s).$$

Now considering the space $\mathbf{H}^2/\Gamma$ which is formed by taking D and gluing together paired sides. Since the vertices along each elliptic cycle are glued together hence each elliptic cycle in D gives one vertex in the triangulation of $\mathbf{H}^2/\Gamma$. Since all paired sides are glued together so there are E = n/2 edges. Moreover F = 1 as we need the single polygon D.

$$2 - 2g = \chi(\mathbf{H}/\Gamma) = V - E + F = r + s - n/2 + v + 1.$$

Substituting this value in the above equation we get

$$Area_{\mathbf{H}^2}(D) = 2\pi((2g-2) + \sum_{j=1}^{r}(1 - \frac{1}{m_j})).$$

□

Now we will calculate the lower bound of this area.

Proposition 2.4.7. *Let* Γ *be a cocompact Fuchsian group. Then*

$$Area_{\mathbf{H}^2}(D) \geq \frac{\pi}{21}.$$

Proof. Taking above theorem into consideration it remains to prove that

$$(2g-2) + \sum_{j=1}^{r}(1 - \frac{1}{m_j}) \geq \frac{1}{42}.$$

Firstly we see that $1 - \frac{1}{m_j} > 0$.
If $g > 1$ then $2g - 2 > 1$. Hence the above result holds.
If $g = 1$ then $2g - 2 = 0$ and since $m_1 > 2$ so that $1 - \frac{1}{m_1} \geq 1/2$. Hence this result holds.
If $g = 0$ then $2g - 2 = -2$, we have the following cases
Case 1 When $r > 5$.
As $m_i > 2$ so that $1 - \frac{1}{m_i} > 1/2$. Hence the result holds.
Case 2 When $r = 4$.

The area is positive if and only if $(m_1, m_2, m_3, m_4) \neq (2, 2, 2, 2)$. Now the area is minimised when $(m_1, m_2, m_3, m_4) = (2, 2, 2, 3)$. Hence on computing we see that the above result holds.

Case 3 When r = 3.

For the L.H.S to be positive we need

$$\frac{1}{m_1} + \frac{1}{m_2} + \frac{1}{m_3} < 1.$$

This means that the minimal triplets (m_1, m_2, m_3) are (2,3,7), (2,4,5) or (3,3,3) and on computing we see that above condition holds.

Case 4 When r $\leq$ 2.

Then there is no hyperbolic orbifold.

The above cases completes our proof. $\square$

Proposition 2.4.8. *Suppose* $\mathbf{H}^2/\Gamma$ *has genus g and r marked points of the order* $m_1, \cdots, m_r$ *and c cusps. Then signature is defined as* sig(Γ) = (g; $m_1, \cdots m_r$;c) *and the area is defined as*

$$Area_{\mathbf{H}^2}(D) = 2\pi((2g-2) + \sum_{j=1}^{r}(1 - \frac{1}{m_j}) + c).$$

Theorem 2.4.9. *Let* $g \geq 0$ *and* $m_j \geq 2$, $1 \leq j \leq r$ *be integers. Suppose that*

$$(2g-2) + \textstyle\sum_{j=1}^{r}(1 - \frac{1}{m_j}) > 0.$$

Then there exists a cocompact Fuchsian group Γ *with signature*

$$sig(\Gamma) = (g; m_1, \cdots, m_r).$$

Proof. We will work in the Poincaré disc D. We consider the origin $0 \in$ D. Let us denote the angle by θ where

$$\theta = \tfrac{2\pi}{4g+r}.$$

Next we draw 4g + r radius each of which is separated by angle θ. We fix the value of t between 0 and 1. On each radius we choose a point at an Euclidean distance t from the origin. We join successive points with a hyperbolic geodesic and by doing this we get a regular hyperbolic polygon say P(t) with 4g + r vertices. We start at an arbitrary point and label the vertices clockwise direction as

$$v_1, v_2, \cdots v_r, v_{1,1}, v_{1,2} v_{1,3}, v_{1,4}, v_{2,1} \cdots v_{g,1}, \cdots v_{g,4}$$

On first r sides of P(t) we construct isosceles triangles which are external to P(t). We now label the vertex at the tip of the j^{th} isosceles triangle by w_j and construct triangle in such a way that the internal angle at w_j is $2\pi/m_j$. Let us denote this polygon by N(t). Now we will consider the vertices v_j,w_j,v_{j+1} ($1 \leq$ j $\leq$ r). We now pair the

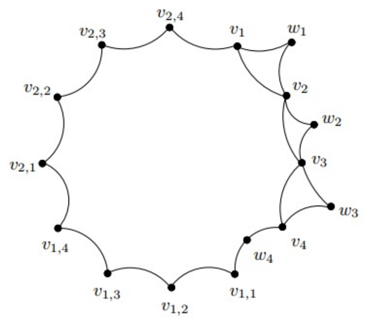

Figure 2.2: Illustrating N(t) in the case of g=2, n=4

sides and call the side-pairing transformation γ_j(γ_j is a rotation about w_j through an angle $2\pi/m_j$) (as shown in Fig 2.3). For each l = 1,2, $\cdots$, g we consider the vertices $v_{l,1}, v_{l,2}v_{l,3}, v_{l,4}$ and pair these sides and call them as side pairing transformation of $\gamma_{l,1}$, $\gamma_{l,2}$. We label the sides of N(t) by $s(v_j), s(v_{l,j}), s(w_j)$ where s(v) is immediately clockwise of vertex v. We now apply Poincaré's Theorem to the polygon N(t).
First task is to calculate elliptic cycles. For each j = 1,2, $\cdots$, r consider the pair (w_j,$s(v_j)$). Then

$$\begin{pmatrix} w_j \\ s(v_j) \end{pmatrix} \xrightarrow{\gamma_j} \begin{pmatrix} w_j \\ s(w_j) \end{pmatrix} \xrightarrow{*} \begin{pmatrix} w_j \\ s(v_j) \end{pmatrix}.$$

Hence we have an elliptic cycle w_j with corresponding elliptic cycle transformation γ_j. Sum(w_j) = $2\pi/m_j$ gives us angle sum. So that the elliptic cycle condition holds. Consider the pair ($v_{l,1}$,s($v_{l,1}$)). We get the following elliptic cycle

$$\cdots \to v_{l,1} \to v_{l,4} \to v_{l,3} \to v_{l,2} \to v_{l+1,1} \to \cdots$$

with corresponding elliptic cycle transformation

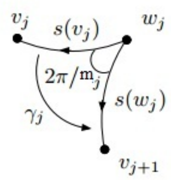

Figure 2.3: Pairing the sides between vertices v_j, w_j, v_{j+1}

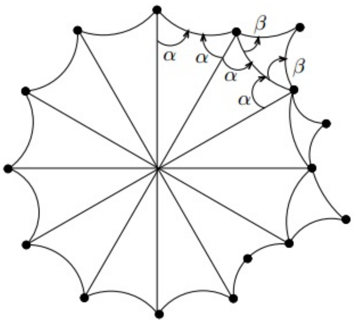

Figure 2.4: Labelling the angles $\alpha(t), \beta_j(t)$ in the polygon N(t)

$$\cdots \gamma_{l,2}^{-1}\gamma_{l,1}^{-1}\gamma_{l,2}\gamma_{l,1}\cdots$$

which we will denote by $[\gamma_{l,1}\gamma_{l,2}]$. Now consider the pair $(v_j,\ (s(v_j)))$. The elliptic cycle through this pair contains

$$\cdots \to \begin{pmatrix} v_j \\ s(v_j) \end{pmatrix} \xrightarrow{\gamma_j} \begin{pmatrix} v_{j+1} \\ s(w_j) \end{pmatrix} \xrightarrow{*} \begin{pmatrix} v_{j+1} \\ s(v_{j+1}) \end{pmatrix} \to \cdots.$$

Starting at the pair $v_{1,1}, s(v_{1,1}))$, we have the elliptic cycle ε

$$v_{1,1} \to v_{1,4} \to v_{1,3} \to v_{1,2} \to \cdots \to v_{g-1,2} \to v_{g,1} \to v_{g,4} \to v_{g,3} \cdots \to v_1 \to \cdots \to v_r$$

with corresponding elliptic cycle transformation

$$\gamma_r\gamma_{r-1}\cdots\gamma_1[\gamma_{g,1},\gamma_{g,2}]\cdots[\gamma_{1,1},\gamma_{2,2}]$$

Now lets denote the internal angle of each vertex in the polygon P(t) by $2\alpha(t)$. Lets denote $\beta_j(t)$ the internal angle at each vertex at the base of the j^{th} isosceles triangle which is added to the polygon P(t) and results in formation of polygon N(t). Then the angle sum along the elliptic cycle ε is given by

$$sum(\varepsilon) = 8g\alpha(t) + 2\Sigma_{j=1}^{r}(\alpha(t) + \beta_j(t)).$$

It remains to show that t can be chosen so that sum(ε) = 2π. In this situation the elliptic cycle condition holds, and we can apply Poincaré's Theorem.
Using hyperbolic trigonometry we know that
$lim_{t\to 1}\ \alpha(t) = 0,$
$lim_{t\to 1}\ \beta(t) = 0,$
$lim_{t\to 0}\ \alpha(t) = \frac{\pi}{2}$ - $\frac{2\pi}{2(4g+r)}$,
$lim_{t\to 0}\ \beta(t) = \frac{\pi}{2}$ - $\frac{\pi}{m_j}$.

Now

$$lim_{t\to 1}8g\alpha(t) + 2\Sigma_{j=1}^{r}(\alpha(t) + \beta_j(t)) = 0.$$

and after rearrangement we get

$$lim_{t\to 0}8g\alpha(t) + 2\Sigma_{j=1}^{r}(\alpha(t) + \beta_j(t)) = 2\pi((2g-2) + \sum_{j=1}^{r}(1 - \frac{1}{m_j}) + 2\pi.$$

Since the first term in the right-hand side is positive, hence

$$lim_{t\to 0}\ 8g\alpha(t)\ +\ 2\Sigma_{j=1}^{r}(\alpha(t) + \beta_j(t)) > 2\pi.$$

So there exists $t_0 \in$ (0,1) such that sum(ε) = 2π.
Hence the elliptic cycle condition holds for $N(t_0)$. By Poincaré's Theorem, the side-pairing transformations generates Fuchsian group Γ.
It remains to show that the group Γ has the required signature. Corresponding to each of the w_j group Γ has r elliptic cycles . The elliptic cycle transformation associated to the elliptic cycle w_j has order m_j. Consider the space $\mathbf{H}^2$ /Γ which is formed by taking $N(t_0)$ and gluing together the paired sides. Thus $\mathbf{H}^2/\Gamma$ has a triangulation using a single polygon (so F = 1) with V = r + 1 vertices (as there are r + 1 elliptic cycles) and E = 2g + r edges. Let h denote the genus of $\mathbf{H}^2/\Gamma$. Then by the Euler formula,

$$2 - 2h = V - E + F = (r + 1) - (2g + r) + 1 = 2 - 2g.$$

Hence h = g. Hence Γ has signature sig(Γ) = (g; $m_j, \cdots, m_r$). □

2.5 Limit set of Fuchsian groups

Definition 2.5.1. *Let Γ be a Fuchsian group acting on the Poincaré disc* $\mathbf{D}^2$ *and let* $z \in \mathbf{D}^2$. $\wedge(\Gamma)(z)$ *denote the set of limit points in* $\mathbf{D}^2 \cup \partial\mathbf{D}^2$ *of the orbit* $\Gamma(z)$.
The point $\zeta \in \partial\mathbf{D}^2$ *is an element of* $\wedge(\Gamma)(z)$ *if there exists* $\gamma_n \in \Gamma$ *such that* $\gamma_n(z) \to \zeta$ *as* $n \to \infty$.

Proposition 2.5.2. *Let Γ be a Fuchsian group and let* z_1, $z_2 \in \mathbf{D}^2$. *Then*

$$\wedge(\Gamma)(z_1) = \wedge(\Gamma)(z_2)$$

Proposition 2.5.3. *Let Γ be Fuchsian group . Then limit set* $\wedge(\Gamma)$ *of Γ is a closed subset of* $\partial\mathbf{D}^2$. *The limit set of a Fuchsian group is compact.*

Proposition 2.5.4. *The limit set $\wedge(\Gamma)$ is invariant under Γ, namely $\gamma(\wedge(\Gamma)) = \wedge(\Gamma)$.*

Proof. Let $\alpha \in \wedge(\Gamma)$, then there exists $\gamma_n \in \Gamma$ such that $\gamma_n(z) \to \alpha$. Now since Γ is a group we have $\gamma\gamma_n \in \gamma(\alpha)$. Hence $\gamma(\alpha) \in \wedge(\Gamma)$. Thus $\gamma(\wedge(\Gamma)) \subset \wedge(\Gamma)$. Replacing γ by γ^{-1} we conclude that $\wedge(\Gamma) \subset \gamma(\wedge(\Gamma))$ and hence $\gamma(\wedge(\Gamma)) = \wedge(\Gamma)$. □

Proposition 2.5.5. *Let Γ be a Fuchsian group.*

1. *Suppose that $\gamma \in \Gamma$ is a parabolic Möbius transformation of $\mathbf{D}^2$. Then the fixed point $\zeta \in \partial\mathbf{D}^2$ of γ is an element of $\wedge(\Gamma)$.*

2. *Suppose that $\gamma \in \Gamma$ is a hyperbolic Möbius transformation of $\mathbf{D}^2$ the two fixed point ζ_1, $\zeta_1 \in \partial\mathbf{D}^2$ of γ are elements of $\wedge(\Gamma)$.*

Proposition 2.5.6. *Let γ be a Fuchsian group and let $\wedge(\Gamma)$ be its limit set. Then $\wedge(\Gamma)$ has either 0,1,2 or infinitely many elements.*

Proof. We prove this by contradiction. Lets assume that $\wedge(\Gamma)$ is finite but has atleast 3 elements. Let κ be finite collection of geodesics which have end points in $\wedge(\Gamma)$. Since $\wedge(\Gamma)$ is Γ invariant and so is κ. Let κ(M) denotes all the set of points which are within the distance M ($M > 0$) of every geodesic in κ

$$\kappa(M) = \{d \in \mathbf{D}^2 | d_{\mathbf{D}^2}(d, x) \leq M\}.$$

Since κ is Γ invariant, so is $\kappa(M)$. We choose M to be sufficiently large and choose a point $d_0 \in \kappa$(M). As $\wedge(\Gamma)$ has atleast 3 elements for any $\alpha \in \kappa(M) \cap$ $\mathbf{D}^2$ there exists an geodesic L $\in \kappa$ which does not have α as the end point. Let $d \to \alpha$, then we have $d_{\mathbf{D}^2}(z, L) = inf_{x \in L} d_{\mathbf{D}^2}(d, x) \to \infty$. Hence d $\notin \kappa$(M) if d is sufficiently close to α. Hence κ(M) is a bounded set. Now we have orbit $\Gamma(d_0)$ of d_0 lying in $\kappa(M)$ which is bounded away from $\partial\mathbf{D}^2$. So the orbit cannot have limit points on the boundary, hence contradicting the fact that $\wedge(\Gamma)$ has atleast 3 points. □

Definition 2.5.7. *A Fuchsian group Γ is called elementary if $\wedge(\Gamma)$ has finitely many elements. It is called non-elementary if $\wedge(\Gamma)$ has infinitely many elements.*

Case 1: When $\wedge(\Gamma)$ has 0 elements.

Proposition 2.5.8. *Let Γ be a Fuchsian group. Suppose that all elements of Γ, other than the identity are elliptic. Then all the elliptic transformations have a common fixed point and the limit set $\wedge(\Gamma)$ is empty.*

Case 2: When $\wedge(\Gamma)$ has 1 element.

Proposition 2.5.9. *Let Γ be a Fuchsian group and suppose that $\wedge(\Gamma) = \{\zeta\}$. Then Γ is of the form*

$$\{\Gamma = \gamma^n \mid n \in Z\},$$

for some parabolic $\gamma \in \Gamma$, i.e. Γ is an infinite cyclic group generated by a parabolic transformation.

Case 3: When $\wedge(\gamma)$ has 2 elements.

Proposition 2.5.10. *Let Γ be a Fuchsian group and suppose that $\wedge(\Gamma)$ has 2 points. Then either*

1. *Γ is an infinite cyclic group generated by a hyperbolic transformation or ;*

2. *Γ is conjugate to a Fuchsian group generated by*

$$z \mapsto kz \ , \ z \mapsto -\frac{1}{z}$$

   for some $k > 1$.

Case 4: When limit set contains infinitely many elements.

Proposition 2.5.11. *Let Γ be a Fuchsian group and suppose that $|\wedge(\gamma)| = \infty$. Then*

1. *$\wedge(\gamma) = \partial\mathbf{D}^2$ or,*

2. *$\wedge(\gamma)$ is a perfect, nowhere dense subset of $\partial\mathbf{D}^2$.*

Fuchsian groups can also be classified into 2 categories namely

1. Fuchsian groups of the first kind.

2. Fuchsian groups of the second kind.

Fuchsian group is of first kind if $\wedge(\Gamma) = \partial\mathbf{D}^2$. Otherwise it is of second kind.
There are 2 possibilities in case of Fuchsian group of second kind. These are

1. $\wedge(\Gamma)$ is a finite set.

2. $\wedge(\Gamma)$ is a Cantor set, namely a perfect set.

The following theorem given below gives a method for calculating limit sets and also helps us to distinguish between Fuchsian groups of first kind and second kind.

Theorem 2.5.12. *Let Γ be a Fuchsian group. Γ is a Fuchsian group of first kind if and only if there exists a fundamental domain with finite hyperbolic area.*

Chapter 3

Subgroups of A Modular Group

Some finite subgroups of the modular group that can be described by congruence relations are called congruence subgroups of a modular group. Computational methods for working with modular form mostly work for congruence subgroup. The tool that is used for working with non-congruence subgroup is the method of Farey symbols which was in given by Ravi Kulkarni in [RK91]. Later, Kurth and Ling Long computed finite index subgroups using the methods used by Kulkarni in their paper [KL05]. and Lang, Lim and Tan gave algorithm for determining the membership of a matrix to a subgroup Γ using the concepts of Farey Symbols in their paper [LLT95]. All the above mentioned references have been discussed in this chapter.

3.1 Special Polygon

Here we will study by taking the case of upper-half plane model. We let D be a hyperbolic triangle with vertices at ι, $\rho = exp(\frac{\pi\iota}{3})$ and ∞. The group generated by reflections on the edges of hyperbolic triangle is called extended modular group Γ. The Γ-translate of D defines the extended modular tessellation τ of $\mathbf{H}^2$.
The elements in the Γ-orbit of ι are called even vertices of τ.
The elements in the Γ-orbit of ρ are called odd vertices of τ.
The Γ-orbit of ∞ consists of rational numbers which are called cusps of τ.
The complete geodesics which are unions of two even edges are called even lines.
The complete geodesics which are unions of two odd edges and two f-edges are called odd lines.

Theorem 3.1.1. *[RK91] The following properties hold*

1. *The even edges come in pairs, each pair forming a complete hyperbolic geodesic. These geodesics are precisely the ones with end-points $\frac{a}{c}$, $\frac{b}{d}$ satisfying $\mid ad - bc \mid = 1$. Each of these geodesics contains an even vertex. Γ acts transitively on these geodesics and the stabilizer subgroup of Γ preserving any one of these geodesics is isomorphic to $\mathbb{Z}_2$ which fixes the even vertex.*

2. *A pair of odd edges and a pair of f-edges form a complete hyperbolic geodesic. The geodesics obtained in this way are precisely the ones which have end-points $\frac{a}{c}$, $\frac{b}{d}$ satisfying $\mid ad - bc \mid = 2$. Each of these geodesics contains an even vertex Γ and a pair of odd vertices. Again Γ acts transitively on these geodesics and the stabilizer subgroup of Γ preserving any one of these geodesics is isomorphic to $\mathbb{Z}_2$ which fixes the even vertex.*

Proof. 1. We will first consider the geodesic which joins ∞ to 0 which has two even edges (∞, ι) and $(\iota, 0)$ and an even vertex ι. It satisfies $\mid a_0 d_0 - b_0 c_0 \mid = 1$, where $\frac{a_0}{c_0} = \infty = \frac{1}{0}$ and $\frac{b_0}{d_0} = 0 = \frac{0}{1}$. If g = $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ or Γ^* then the geodesic $(\frac{a}{c}, \frac{b}{d})$ is translate of the geodesic $(\infty, 0)$, so $(\frac{a}{c}, \frac{b}{d})$ retains all properties of $(\infty, 0)$. Also S = $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ sends the edge (∞, ι) to $(0, \iota)$ and also fixes even vertex ι. But $S^2 = $ I in Γ^* so $< S > \simeq \mathbb{Z}_2$. Same is the case for translated edge $(\frac{a}{c}, \frac{b}{d})$ for which we find that the stabilizer is isomorphic to $\mathbb{Z}_2$.
2. We will consider the geodesic joining -1 to 1. This geodesic is basically an odd line since it consists of paired odd edges $(1,\rho)$ and $(-1, \rho^2)$ and paired f-edges (ι,ρ) and (ι, ρ^2). The side pairing matrix is B = $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$. Since B fixes ι and $A^2 = $ I in Γ^*, we have $< B > \simeq \mathbb{Z}^2$. Any Γ or Γ^*-translate of $(-1, 1)$ has the same property. $\square$

Let P be a convex hyperbolic polygon with boundary ∂P which is a union of even and odd edges. The following is assumed.

Definition 3.1.2. *1. The even edges in ∂P come in pairs, each pair forming an even line.*

2. *The odd edges in ∂P come in pairs. The edges in each pair meet at an odd vertex making an internal angle of $\frac{2\pi}{3}$.*

3. *An odd edge a is paired to the odd edge b which makes an internal angle of $\frac{2\pi}{3}$ with a.*

4. *Let e, f be two even edges in ∂P forming an even line. Then either e is paired to f, or e and f form a free side of P and this free side is paired to another free side of P.*

5. *0 and 1 are two of the vertices of P.*

A convex hyperbolic polygon P satisfying all the above conditions is called a special polygon.

Notation Let P be a special polygon which has canonical orientation on each of its side. Then there exists a unique element of Γ which carries one paired side to another. We denote the subgroup of Γ which is generated by these side pairing transformations by Φ_P.

Definition 3.1.3. *Fundamental domain whose side pairing transformations form an independent set of generators is called admissible fundamental domain*

Theorem 3.1.4. *Let P be a special polygon and Φ_P be an associated subgroup of Γ. Then P is an admissible fundamental domain for Φ_P. Also Φ_P is free if and only if P has only free sides.*

Proof. P is fundamental domain of Φ_P. Space obtained by identifying sides of P by side-pairing transformations associated to P has a complete hyperbolic metric. Also we know that the singularities corresponds to the branch points. So the condition of Poincaré theorem on fundamental polygons is fulfilled. This allows us to see a complete set of relation among the generators which are given by side pairing transformation $x^3 = 1$ corresponding to the side-pairing of type 3 (as in the definition of special polygons) and $x^2 = 1$ corresponding to first alternative in the side pairing of type 4. These relations appear if and only if P has a non free side. □

Theorem 3.1.5. *Every subgroup Φ of a finite index admits an admissible domain which is a special polygon P, so that $\Phi = \Phi_P$.*

The interesting property of an admissible domain is that in counting the sides of a fundamental domain we follow the convention that if an even line is contained in the boundary of the fundamental domain and the even edges contained in this even line are paired then this line counts as two sides of the fundamental domain.

Proposition 3.1.6. *Let Φ be a subgroup of finite index in Γ. Among all the fundamental polygons for Φ whose Φ-translates form a locally finite tessellation of $\mathbf{H}^2$ an admissible fundamental domain has the least number of sides. If Φ is isomorphic to a* *free product of p copies of $\mathbb{Z}_2$, q copies of $\mathbb{Z}_3$ and r copies of $\mathbb{Z}$ then this least number is $2(r+p+q)$.*

Proof. If P is a fundamental polygon for Φ whose Φ-translates form a locally finite tessellation of $\mathbf{H}^2$ then the side pairing transformation generates Φ (Theorem 3.1.8). If Φ is a free product decomposition then by Grushko's Theorem (Theorem 3.1.7.) the least number of generators for Φ is (r + p + q). So P has atleast twice the number of sides. Also if P is admissible then its side pairing transformation are independent so it has exactly 2(r + p + q) sides. □

Theorem 3.1.7 (Grushko's Theorem)**.** *Let X and Y be finitely generated groups and let $X * Y$ be free product of X and Y. Then*

$$Rank(X * Y) = Rank\ X + Rank\ Y.$$

Theorem 3.1.8. *Let P be any locally finite Fundamental Domain for a Fuchsian group Φ then*

$$\Phi_0 = \{g \in \Phi : g(\tilde{P}) \cap \tilde{P} \neq \Phi\} \text{ generates } \Phi.$$

Poincaré Theorem on Fundamental Polygons

Let U be the unit disk in Complex plane and D a polygon in U bounded by finite sides. We assume that to each side s there is a side s' and an isometry A = A(s, s') of U such that A(s) = s' and the following conditions are satisfied i.e. to each s there is a neighbourhood V of s such that A(V ∩ D) does not meet D. The isometries A are called *identifying generators* and we D is a polygon with identification. To each such D there is an identified polygon $D^* = \pi(\overline{D})$ obtained by identifying the sides of D. We set $\rho^*(x,y) = inf \sum_{i=1}^{n} \rho(z_i, z_i')$ where the inf being taken over all n and all 2n-tuples of points of $\overline{D}$ where $\pi(z_1)$ = x, $\pi(z_i') = \pi(z_{i+1})$ and $\pi(z_n')$ = y. D is said to be *complete* if for x ∈ D^*, π^{-1}(x) is a finite set (then ρ^* is a metric) and if D^* is complete in this metric. To each vertex in U there corresponds a cycle of equivalent vertices in D and is assumed that the sum of the angles at the vertices of the cycle is a sub-multiple of 2π. If D is a complete polygon with an identification and satisfies the cycle condition then we say that D is a Poincaré polygon.

Theorem 3.1.9. *[BM71] Let D be a Poincaré polygon and G the group generated by the identifying generators then G is discontinuous, D is a fundamental polygon for G, and the cycle relations together with the reflection relations form a complete system of relations for G.*

3.2 Farey Symbols

We introduce this concept as it is a useful way to represent special polygons and also a very important way of representation of subgroup of a modular group. The n^{th} Farey Sequence (which will be denoted by F_n) is a finite sequence of all rational numbers between 0 and 1 placed in an increasing order in a way that the denominators have maximum value n.

Definition 3.2.1. *A generalised Farey Sequence (g.F.S.) is an expression of the form*

$$\{-\infty, x_1, x_2, \cdots, x_n, \infty\} \tag{3.1}$$

where

1. x_1 *and* x_n *are integers, and some* $x_j = 0$.
2. $x_j = \frac{a_j}{b_j}$ *are rational numbers in their reduced form and ordered in increasing order, such that*

$$|a_{j+1}b_j - a_j b_{i+1}| = 1, \quad j = 1, 2, 3, \cdots, n-1.$$

Note First and Last term are basically the same.

Lemma 3.2.2. *If* $\gamma \in PSL(2, \mathbb{Z})$ *and* $\frac{p_1}{q_1}, \frac{p_2}{q_2}, \frac{p_1'}{q_1'}, \frac{p_2'}{q_2'}$ *are rational numbers in simplest form such that*

$$\gamma(\frac{p_1}{q_1}) = \frac{p_1'}{q_1'}, \quad \text{and} \quad \gamma(\tfrac{p_2}{q_2}) = \tfrac{p_2'}{q_2'}$$

then

$$p_2q_1 - p_1q_2 = p_2'q_1' - p_1'q_2'.$$

Proof. If $\gamma = \begin{pmatrix} P & Q \\ R & S \end{pmatrix}$ then

$$\gamma(\frac{p_1}{q_1}) = \frac{Pp_1 + Qq_1}{Rr_1 + Ss_1} = \frac{p_1'}{q_1'} \quad \text{and} \quad \gamma(\frac{p_2}{q_2}) = \frac{Pp_2 + Qq_2}{Rr_2 + Ss_2} = \frac{p_2'}{q_2'}$$

$$\begin{aligned} p_1'q_2' - p_2'q_1' &= (Pp_1 + Qq_1)(Rr_2 + Ss_2) - (Pp_2 + Qq_2)(Rr_1 + Ss_1) \\ &= (PS - QR)(p_1q_2 - p_2q_1) \\ &= (p_1q_2 - p_2q_1). \end{aligned}$$

□

Proposition 3.2.3. *[RK91] The set of Farey symbols are in natural 1-1 correspondence between the set of special polygons. In particular a Farey Symbol determines a subgroup of finite index in* Γ *and every subgroup of finite index in* Γ *arises in this way. The map*

$$\{\textit{Farey Symbols}\} \to \{\textit{Subgroups of finite index in } \Gamma\ \}$$

is finite-to-one.

Proof. **Special polygon to g.F.S.**
We suppose that P is a special polygon and the Equation 3.1 is the g.F.S. formed by its vertices in $\mathbb{R} \cup \{\infty\}$. If a complete hyperbolic geodesic which joins x_j to x_{j+1} consists of two paired even edges then we denote this by

$$x_j \overbrace{\ \circ\ } x_{j+1}$$

We will call the above notation as the even interval of g.F.S.
If x_j and x_{j+1} are the end points of two paired odd edges then we denote this by

$$x_j \overbrace{\ \bullet\ } x_{j+1}$$

We will call the above notation as the odd interval of g.F.S.
If x_j and x_{j+1} are the end points of a free side s of P which paired to free side t which has end points x_k and x_{k+1} then we denote this by

$$x_j \overbrace{\ m\ } x_{j+1} \qquad\qquad x_k \overbrace{\ m\ } x_{k+1}$$

Here m is a numerical symbol (It has no significance). We will call each pair of the above notation as a free interval of g.F.S.
A g.F.S. with an extra structure on each pair of consecutive terms of above 3 types is called Farey Symbols.
g.F.S. to Special polygons
Let P_0 be a convex hull of the x_j's, Now we assume that we have an odd interval. Then, the complete geodesic which joins x_j to x_{j+1} together with the odd edges which are situated inside the hyperbolic convex hull form a hyperbolic triangle with angles 0, $\frac{2\pi}{3}$ and 0. If we adjoin all such triangles for each odd interval in the Farey symbol we obtain convex hyperbolic polygons. The side pairing transformation is defined by reversing the above process. □

There are several restrictions on the elements in the g.F.S.

Example In a g.F.S. the first and the last finite element on a real axis are integers. The other restrictions are the two propositions which are given below.

Proposition 3.2.4. *Let F be a g.F.S. Let m be an integer and q_i's be the denominators of those x_i's which lie in $[m, m+1)$. Then q_i's determine x_i's uniquely.*

Proof. We will use the concept of Diophantine equations.
Considering the equation $uq_i - vq_{i+1} = 1$, to be solved for u and v in the integers. From the definition of Farey Sequence we can easily see that one of the solution will be of the form u $= p_{i+1}$ and v $= p_i$, where p_i's are numerators of the x_i's. Other solutions will be of the form u $= p_{i+1} + zq_{i+1}$ and $v = p_i + zq_i$ where $z \in \mathbb{Z}$. It is clear that for any $z \neq 0$ the corresponding $\frac{u}{q_{i+1}}$, $\frac{v}{q_i}$ lie outside the given interval $[m, m+1)$. □

Proposition 3.2.5. *Let x be a non integer element of g.F.S. and m be an integer such that $m < x < m+1$. Let $y < x < z$ be the three consecutive terms in the g.F.S. then $m \leq y < x < z \leq m+1$. Also if $y = \frac{t}{u}$, $x = \frac{q}{s}$ and $z = \frac{p}{r}$ with u, s, r positive, then $p + t = \lambda q$, $r + u = \lambda s$ where $\lambda = pu - rt$ is a positive integer.*

Proof. Since distinct even lines do not intersect and as we know that m, m+1 are end points of an even line so y, z lie in the interval [m, m+1). Now, since u, s, r are positive integers so ps − qr, uq − ts, pu − rt are also positive integers and since x, z and y, z are end points of an even line we have $ps - qr = 1$ and $uq - st = 1$. In the matrix form these equations are

$$\begin{pmatrix} p & -r \\ -t & u \end{pmatrix} \begin{pmatrix} s \\ q \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

or

$$\begin{pmatrix} s \\ q \end{pmatrix} = \lambda \begin{pmatrix} u & r \\ t & p \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

This gives $p + t = \lambda q$, $r + u = \lambda s$ where $\lambda = pu - rt$ is a positive integer. □

Geometrical Interpretation of λ

If P is a convex hull of the g.F.S. in $\mathbf{H}^2$ then λ is the number of tiles which are contained in P and incident with x.

3.3 Side Pairing Matrices

Here we find the matrices associated to the side-pairings of a special polygon.

Theorem 3.3.1. *[RK91] Let F be a Farey symbol and let $\Phi \subseteq \Gamma$. Let $x_i = \frac{a_i}{b_i}$ (reduced forms) with b_i positive. Then*

1. *For each even interval $x_i \overbrace{\ \circ\ } x_{i+1}$, i= $i_1, i_2, \cdots i_a$ in F the side pairing matrices are given by*

$$A_i = \begin{pmatrix} a_{i+1}b_{i+1} + a_i b_i & -a_i^2 - a_{i+1}^2 \\ b_i^2 + b_{i+1}^2 & -a_{i+1}b_{i+1} - a_i b_i \end{pmatrix}.$$

2. *For each odd interval $x_j \overbrace{\ \bullet\ } x_{j+1}$, $j = j_1, j_2, \cdots, j_b$ in F the side pairing matrices are given by*

$$B_j = \begin{pmatrix} a_{j+1}b_{j+1} + a_j b_j & -a_j^2 - a_{j+1}^2 - a_j a_{j+1} \\ b_j^2 + b_{i+1}^2 + b_j b_{j+1} & -a_{j+1}b_{j+1} - a_{j+1}b_{j+1} - a_j b_j \end{pmatrix}.$$

3. *For each pair of free intervals $x_k \overbrace{\ a\ } x_{k+1}$, $x'_k \overbrace{\ a\ } x_{k'+1}$, $k = k_1, k_2, \cdots k_r$ in F the side pairing matrices are given by*

$$C_k = \begin{pmatrix} a_{k'+1}b_{k+1} + a_{k'}b_k & -a_{k'}a_k - a_{k'+1}a_{k+1} \\ b_{k'}b_k + b_{k+1}b_{k'+1} & -a_{k+1}b_{k'+1} - a_k b_{k'} \end{pmatrix}.$$

Proof. If $(x_i,\, x_{i+1})$ is an even line, then the even line $(\infty,\, 0)$ is paired to $(x_i,\, x_{i+1})$ in an orientation-reversing manner which is given by the matrix

$$\mathrm{A}(x_i,\, x_{i+1}) = \begin{pmatrix} a_{i+1} & a_i \\ b_{i+1} & b_i \end{pmatrix}$$

which sends ∞ to $x_{i+1} = \frac{a_{i+1}}{b_{i+1}}$ and 0 to $x_i = \frac{a_i}{b_i}$. Now the matrix $\mathrm{S} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ just reverses the orientation of $(\infty,\, 0)$. For the cases 1 and 3, the side pairing transformations can be obtained by sending the first side to $(0,\, \infty)$ and then reversing the orientation by S. Finally, we send $(\infty,0)$ to the second side.

Case 1 For an even interval we obtain the side-pairing matrix given by

$$A_i = A(x_i, x_{i+1})\mathrm{S}A(x_i, x_{i+1})^{-1} = \begin{pmatrix} a_{i+1} & a_i \\ b_{i+1} & b_i \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_{i+1} & a_i \\ b_{i+1} & b_i \end{pmatrix}^{-1}.$$

Hence on calculation we get

$$A_i = \begin{pmatrix} a_{i+1}b_{i+1} + a_i b_i & -a_i^2 - a_{i+1}^2 \\ b_i^2 + b_{i+1}^2 & -a_{i+1}b_{i+1} - a_i b_i \end{pmatrix}.$$

The only task that is done by reversing orientation is that the side pairing matrices switches the end points i.e. it sends x_i to x_{i+1}.

Case 3 For pair of two free sides $(x_k,\, x_{k+1})$ and $(x_{k'},\, x_{k'+1})$ we get side-pairing matrix given by

$$C_k = \mathrm{A}(x_{k'},\, x_{k'+1})\mathrm{S}\mathrm{A}(x_k,\, x_{k+1})^{-1} = \begin{pmatrix} a_{k'+1} & a_{k'} \\ b_{k'+1} & b_{k'} \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_{k+1} & a_k \\ b_{k+1} & b_k \end{pmatrix}^{-1}.$$

Hence on calculation we get

$$C_k = \begin{pmatrix} a_{k'+1}b_{k+1} + a_{k'}b_k & -a_{k'}a_k - a_{k'+1}a_{k+1} \\ b_{k'}b_k + b_{k+1}b_{k'+1} & -a_{k+1}b_{k'+1} - a_k b_{k'} \end{pmatrix}.$$

The only task that is done by reversing orientation is that the side pairing matrices switch the end points i.e. it sends x_k to $x_{k'+1}$ and $x_{k'}$ to x_{k+1}.

Case 2 This case is slightly different and complicated then the above 2 cases. Here we will use matrix of order 3 in Γ that will fix $\rho = \exp^{\frac{\iota\pi}{3}}$ and also it will pair the odd edges $(\infty,\, \rho)$ and $(\rho,\, 0)$. So our matrix will be $\mathrm{Z} = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}$.

The side pairing matrix corresponding to the odd edge is given by

$$B_j = A(x_i, x_{i+1})\mathrm{Z}A(x_i, x_{i+1})^{-1} = \begin{pmatrix} a_{i+1} & a_i \\ b_{i+1} & b_i \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} a_{i+1} & a_i \\ b_{i+1} & b_i \end{pmatrix}^{-1}.$$

Hence on calculation we get

$$B_j = \begin{pmatrix} a_{j+1}b_{j+1} + a_j b_j & -a_j^2 - a_{j+1}^2 - a_j a_{j+1} \\ b_j^2 + b_{i+1}^2 + b_j b_{j+1} & -a_{j+1}b_{j+1} - a_{j+1}b_{j+1} - a_j b_j \end{pmatrix}.$$

□

3.4 Geometric Interpretation of Continued fractions And Their Properties

Let x be a rational number and let

$$x = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \cfrac{1}{a_m}}}}$$

be its continued fraction expansion.
We can see that a_0 =$[x_0]$, a_1 =$[\frac{1}{x-a_0}]$ etc.
The other way of writing the above equation is $[a_0; a_1, a_2, a_3, \cdots, a_k]$. We can see that if x is an integer then all a_i (i ≥ 1) are zero.

Definition 3.4.1. *Depth, denoted by $\triangle(x)$ of any rational number is given by $a_1 + a_2 + a_3 + \cdots + a_k$. Hence the depth is defined to be zero if x is an integer. It can be easily seen that $\triangle(x)$ depends only on congruence class of x (mod 1).*

Notation The even lines tile $\mathbf{H}^2$ into ideal triangles. We denote this by λ.
The vertices of any hyperbolic polygon forms a General Farey Structures of the form

$$\{\infty,\ 0 = x_1,\ \cdots, x_n = 1,\ \infty\}. \tag{3.2}$$

Among the given values, x_i is one of the value x. We know that two even lines do not intersect and intersection of any polygons of such kind is also a polygon of same kind and hence we can conclude that there will exist a unique polygon say $Z_0(x)$ of same kind which will be intersection of all such polygons. In terms of the General Farey Structure it implies that among all the g.F.S. listed in the above equation containing x, there is a unique minimal one and will consist of vertices of $Z_0(x)$.

Proposition 3.4.2. *Let x be a rational number in $(0,1)$. Then there exists a hyperbolic polygon which is a union of finitely many tiles of λ whose boundary contains 0 and ∞ as vertices and which is contained in the vertical strip bounded by the geodesic joining 0 to ∞ and 1 to ∞.*

Proposition 3.4.3. *Let x_0 be a rational number in $(0,1)$. Then there exists uniquely determined rationals a_0 and b_0 in $[0,1]$, $a_0 < x_0 < b_0$ with the following properties*

1. *a_0, x_0, b_0 form the vertices of a tile of λ, say $\mu_0(x_0)$.*

2. *Any even line incident with x_0 has its other point lying either in $[a_0, x_0)$ or in $(x_0, b_0]$.*

3. *The tile $\mu_0(x_0)$ is contained in $Z_0(x_0)$.*

Proof. **Proof of Part 2**
We will first consider the point ∞. The even lines which are incident to ∞ are the *vertical half lines* i.e. x = m and y > 0, where m is an integer. The end point of these lines are $\mathbb{Z} \cup \{\infty\}$ and these have ∞ as their limit point. Since we know that distinct even lines do not intersect and if we translate this situation at x_0 we will see that the end points of even lines incident at x_0 have x_0 as their limit point and they are contained in [0, 1]. Let a_0 and b_0 be respectively the smallest and largest of these end points. Hence we can conclude that even line incident with x_0 has its other point lying either in $[a_0, x_0)$ or in $(x_0, b_0]$.
Proof of Part 1
Lets assume that p_0 and q_0 are any endpoints of some even line which is incident to x_0 such that $p_0 < x_0 < q_0$. Then there exists a convex hyperbolic M which satisfies

1. It is union of finitely many tiles.

2. It has even lines joining p_0 to x_0 and q_0 to x_0 as sides.

Let S be an element of Γ which carries x_0 to ∞. Then $Sp_0 = \beta$ and $Sq_0 = \alpha$ must be integers. Also since S preserves the orientation of the circle $\mathbb{R} \cup \{\infty\}$ we see that $\alpha < l$. If M' is the convex hull of $\{\infty, \alpha, \alpha + 1, \cdots, \beta, \infty\}$ in $\mathbf{H}^2$ then M $= S^{-1}M'$ fulfils our requirements. Since S is determined upto left multiplication of the elements of the form $\varphi_m : q \mapsto q + m$, where m is an integer it is clear that M' is determined upto a translation by φ_m. In these cases M is determined uniquely. Among the hyperbolic polygons M" satisfying the above 2 conditions, our construction produces the smallest one which is contained in all such M". Also every vertex of M is an endpoint of some even line incident to x_0. Let a'_0 and b'_0 be respectively the smallest and the largest vertices of M. Then clearly $a'_0 \leq p_0 < x_0 < q_0 \leq b'_0$ and a'_0, x_0, b'_0 form the vertices of a tile of λ. If we apply this construction specially to $p_0 = a_0$ and $q_0 = b_0$, we observe that $b'_0 = b_0$ and $a'_0 = a_0$.
Proof of Part 3
Let Z be a polygon containing x_0 as a vertex. Let p_0 and q_0 be the vertices of Z adjacent to x_0 such that $p_0 < x_0 < q_0$ and Z be the corresponding polygon. Then M is contained in Z. So the tile $\lambda(x_0)$ which has vertices a_0, x_0, v_0 is also contained in Z. Since Z is arbitrary we see that this tile is contained $Z_0(x_0)$. □

Let x = $[0; a_1, a_2, \cdots, a_k]$ be a rational number in (0, 1). Set

$$y_i = [0; a_1, a_2, \cdots, a_i], \quad 1 \le i \le k$$

be the convergent of x. So $y_k = x$. It is easy to put

$$p_{-1} = 1, \quad q_{-1} = 0, \quad p_0 = 0, \quad q_0 = 1,$$

$$p_i = a_i p_{i-1} + p_{i-2}, \qquad q_i = a_i q_{i-1} + q_{i-2}, \qquad 1 \le i \le k.$$

$$y_{-1} = \infty, \qquad y_0 = 0.$$

Some of the properties are

1. $p_i q_{i-1} - q_i p_{i-1} = (-1)^{i-1}$.
2. $p_i q_{i-2} - q_i p_{i-2} = (-1)^i a_i$.
3. $0 < y_2 < y_4 < \cdots < x < \cdots < y_3 < y_1$.

Proof. 1. Substituting the values of p_i and q_i in the above equation we have

$$\begin{aligned} p_i q_{i-1} - q_i p_{i-1} &= (a_i p_{i-1} + p_{i-2}) q_{i-1} - (a_i q_{i-1} + q_{i-2}) p_{i-1} \\ &= -(p_{i-1} q_{i-2} - p_{i-2} q_{i-1}). \end{aligned}$$

Now we repeat this argument with $i-1, i-2, \cdots, 2$ in place of i and hence obtain

$$\begin{aligned} p_i q_{i-1} - q_i p_{i-1} &= (-1)^i (p_1 q_0 - q_1 p_0) \\ &= (-1)^{i-1}. \end{aligned}$$

2. Again substituting the values of p_i and q_i in the above equation we have

$$\begin{aligned} p_i q_{i-2} - q_i p_{i-2} &= (a_i p_{i-1} + p_{i-2}) q_{i-2} - p_{i-2}(a_i q_{i-1} + q_{i-2}) \\ &= a_i (p_{i-1} q_{i-2} - p_{i-2} q_{i-1}) \\ &= (-1)^n a_i. \end{aligned}$$

3. See [HW].

□

Proposition 3.4.4. *With the above notation all y_i's are among the vertices of $Z_0(x)$.*

Proof. We write $x = \frac{a}{b}$, and let $u_0 = \frac{q}{s}$ and $v_0 = \frac{p}{r}$ be the rationals in (0, 1) which satisfies the properties given in the above proposition. We take b, s, r to be positive integers. Now taking x = $y_k = \frac{p_k}{q_k}$ and y_k and y_{k-1} are the end vertices of an even line. Lets suppose that $y_{k-1} \in (x, v_o]$. Our claim is that $y_{k-1} = v_0$. So we have

$$|x - v_0| = \frac{1}{br} \geq |x - y_{k-1}| = \frac{1}{bq_{k-1}}.$$

So $r \leq q_{k-1} < q_k$. Also $\{p, r\}$ and $\{p_{k-1}, q_{k-1}\}$ are solutions of $\{\alpha, \beta\}$ of the equation $b\alpha - a\beta = 1$. Since r and q_{k-1} lie in (0, q_k) therefore they must be equal. Hence $v_0 = y_{k-1}$ is the vertex of $Z_0(x)$. It follows that $Z_0(y_{k-1}) \subset Z_0(x)$. So y_{k-2} is a vertex of $Z_o(x)$. Continuing this process we see that all y_i's, $i \geq 1$, are the vertices of $Z_0(x)$. □

Proposition 3.4.5. *Between y_{i-2} and y_i, $i \geq 1$ there are $a_i - 1$ vertices of $Z_0(x)$.*

Proof. By the property $p_iq_{i-1} - q_ip_{i-1} = (-1)^{i-1}$, we know that y_{i-1} is joined to both y_i and y_{i-2} by the even lines, say t_1 and t_2. If n is the number of tiles of λ which are incident with y_{i-1} and lie in the circular sector made by t_1 and t_2, then there are $n-1$ vertices of $Z_0(x)$ lying between y_{i-2} and y_i. We know that there are $2|p_iq_{i-2} - q_ip_{i-2}|$ tiles of λ^* which come into this circular sector. Since at each vertex of a tile there are 2 tiles of λ^* it follows that there are $|p_iq_{i-2} - q_ip_{i-2}|$ tiles of λ in this sector which are incident with y_{i-1} and since $p_iq_{i-2} - q_ip_{i-2} = (-1)^ia_i$, this number is equal to a_i. □

Corollary 3.4.6. *The number of vertices of $Z_0(x)$ is precisely $\triangle(x) + 2$, and so the number of tiles of λ of $Z_0(x)$ is $\triangle(x)$.*

Proof. The y_i's are $k + 2$ vertices of $Z_0(x)$ and from above we have

$$\infty,\ y_0 = 0,\ y_2,\ y_4,\ \cdots,\ y_k = x,\ \cdots,\ y_3,\ y_1 = \infty$$

in the cyclic order. These form k + 2 intervals of which k have end-points of the form $\{y_i, y_{i-2}\}$. By the above proposition, these k intervals contain $\Sigma_{i=1}^{k}(a_i - 1)$ vertices. The remaining 2 intervals are $\{\infty, 0\}$ and $\{y_k, y_{k-1}\}$. These end points of each of these 2 intervals are also end-points of an even line, so these intervals contains no other vertex of $Z_0(x)$. So $Z_0(x)$ has

$$k + 2 + \Sigma_{i=1}^{k}(a_i + 1) = \triangle(x) + 2$$

vertices. The last assertion follows by induction on the number of vertices. □

Example Let x $= \frac{4}{11} = [0;\ 2,1,3]$. A g.F.S. containing $\frac{4}{11}$ must contain $6+2=8$ terms. These includes $\infty, 0$ and the convergents $\frac{1}{2}$, $\frac{1}{3}$ and $\frac{4}{11}$. The interval $\frac{4}{11}$ and $\frac{1}{2}$ contains two extra vertices. The interval $\frac{1}{2}$ and ∞ contains one extra vertex.

Proposition 3.4.7. *Let x be a rational number in $(0,1)$. The shortest path in the cubic tree of f-edge from $\rho = \exp^{\frac{\pi\iota}{3}}$ leading into the tile of λ incident with x has length $\triangle(x) - 1$.*

Proof. We first built $Z_0(x)$. We have $y_1 = \frac{1}{a_1}$. The convex hull of 0, $\frac{1}{a_1}$, $\frac{1}{a_1-1}$, $\frac{1}{a_1-2}$, $\cdots$, $\frac{1}{2}$, 1, ∞ consists of a_1 tiles of λ and is contained in $Z_0(x)$. To this region there is attached along the even line connecting y_0 to y_1, the convex hull of y_0, y_1, y_2 and the a_2 vertices lying in the (y_0, y_2). This region contains a_2 tiles of λ. The path in the tree of f-edges starting from ρ and leading into a tile of λ incident with x is the one which successively connects the barycentres of these $a_1 + a_2 + \cdots + a_k$ tiles and has length

$$\Sigma_{i=1}^{k} a_i - 1 = \triangle(x) - 1.$$

□

Application of the above Proposition

The x_1 must be of the form $\frac{1}{m}$ for some $m \in \mathbb{N}$. If we choose $x_1 = \frac{1}{k}$ then we choose between $\frac{1}{k}$ and ∞ the $k-1$ terms $\frac{1}{k-1}$, $\frac{1}{k-2}$, $\cdots$, 1. Similarly x_{n-1} must be of the form $\frac{l-1}{l}$ for some natural number l. If we choose $x_{n-1} = \frac{l-1}{l} = [0;1,l-1]$ then we have $y_1 = 1$ and $y_2 = \frac{l-1}{l}$ and $y_0 = 0$ the following $l-2$ terms $\frac{l-2}{l-1}, \frac{l-3}{l-2}, \cdots \frac{1}{2}$.

Definition 3.4.8. *Let Φ be a subgroup of finite index in Γ and P be a special polygon of Φ then the Φ orbits of the free vertices (cusps) of P are called inequivalent cusps of P.*

Definition 3.4.9. *If α is the cusp of Φ and given $\gamma \in PSL(2,\mathbb{Z})$ such that $\gamma(\infty) = \alpha$. The smallest positive integer r such that $\begin{pmatrix} 1 & r \\ 0 & 1 \end{pmatrix} \in \gamma^{-1}\Phi\gamma$ is called width of the cusp.*

Geometrical Interpretation

It corresponds to the number of ideal triangles which meet at α.

Definition 3.4.10. *We define width of an inequivalent cusp to be the sum of width of the cusps in a given Φ-orbit and half the number of special triangles which intersect the cusps in Φ-orbit.*

Geometrical Interpretation

It corresponds to the number of even lines which meet at x in $\Phi \setminus \mathbf{H}$.

Application in Farey Symbols

Width of a cup can be calculated by using the Farey Symbols. Let

$$F = \{-\infty, x_1, x_2, \cdots, x_n, \infty\}$$

be a Farey symbol for Φ and P be its associated Special Polygon. We suppose that $x_i = \frac{a_i}{b_i}$ be in reduced form i.e. g.c.d $(a_i, b_i) = 1$. We consider x_i, i = 0, 1, 2, $\cdots$, n+1 that the x_i's are in cyclic order and $x_0 = -\infty = \frac{-1}{0}$ and $x_{n+1} = \infty = \frac{1}{0}$ are identified. Let

$$d(x_i) = |a_{i-1}b_{i+1} - a_{i+1}b_{i-1}| \quad for \quad i = 1, 2, \cdots, n.$$

and

$$d(x_0) = d(-\infty) = |a_1 b_n - a_n b_1|.$$

We define $w(x_i)$ of x_i to be $d(x_i)$, $d(x_i) + \frac{1}{2}$ and $d(x_i) + 1$ respectively if x_i's are incident to 0, 1 and 2 odd intervals, respectively. If C denotes inequivalent cusp of Φ, then the width of the cusp is given by $w(C) = \sum w(x_i)$ (where x_i runs over the cusp vertices in the equivalence class of C).

Definition 3.4.11. *The geometric level of Φ is the lcm of the inequivalent cusp.*

3.5 Bipartite Cuboid Graphs and Tree Diagrams

The purpose of this section is to describe a graph theoretical approach to find all the subgroups of a given index. I will provide algorithm for obtaining the Farey symbols which corresponds to a marked trivalent diagram. We will first see the correspondence between the special polygons, bipartite graphs and the tree diagrams. We will also find subgroups corresponding to the bipartite cuboid graph.

Definition 3.5.1. *A bipartite cuboid graph (also called trivalent diagram) is a finite connected graph whose vertex set is partitioned into two disjoint subsets V_0 and V_1 such that*

1. *Every vertex in V_0 has degree (valence) 1 or 2.*
2. *Every vertex in V_1 has degree (valence) 1 or 3.*

3. *Every edge joins a vertex in V_0 with a vertex in V_1.*

4. *There is a prescribed cyclic order on the edges incident at each vertex of degree 3 in V_1.*

The trivalent diagrams may have multiple edges between the two vertices so it need not to be simple. We will assume the cyclic orientation around the bipartite cuboid graph to be anticlockwise orientation of the plane.

Definition 3.5.2. *A marked trivalent diagram is a trivalent diagram with a distinguished edge called the marked edge.*

Definition 3.5.3. *A cuboid tree diagram or a tree diagram is a finite tree $\boldsymbol{T}$ which contains atleast one edge and which satisfies the following*

1. *All internal vertices have degree 3.*

2. *There is a prescribed cyclic order on the edges incident at each internal vertex.*

3. *The terminal vertices are partitioned into two possible empty subsets W and B where the vertices in W are called white vertices and those in B are called black vertices.*

4. *There is an involution Ω on W.*

The tree $\boldsymbol{T}$ can be embedded in the plane in such a way that the cyclic order on the edges at each internal vertex coincides with the cyclic order which is induced by orientation of the plane.

Notation We will represent the black vertices as $\bullet$ and white vertices as $\circ$.
The distinct white vertices which are related by Ω are given the same numerical labels and different pairs of distinct white vertices which are related by Ω carry different labels. The unlabelled white vertices are actually those which are fixed by Ω.

Correspondence between tree diagrams and the bipartite cuboid graphs
Tree diagram to Bipartite cuboid graph

Let $\boldsymbol{T}$ be a tree diagram. To convert it into trivalent diagram, say $\boldsymbol{G}$, we identify all the white vertices v by $\Omega(v)$. Now on all the edges which joins two internal vertices or the edges joining an internal vertex to a black vertex we introduce a new vertex of degree 2. Now, these newly formed vertices along with the white vertices constitutes V_0 while the black vertices and the vertices of degree 3 constitutes V_1. The cyclic orders of the trivalent vertices in V_1 comes from the cyclic orders on the edges which are incident at each internal vertex of $\boldsymbol{T}$. This turns tree into a bipartite graph.

Bipartite cuboid graph to Tree diagram

Let $\boldsymbol{G}$ be a bipartite cuboid graph. If its cycle rank is r then we can choose r vertices of degree 2 in V_0 such that if we cut $\boldsymbol{G}$ along these vertices we obtain a tree $\boldsymbol{T}$. These r cuts corresponds to 2r terminal vertices in $\boldsymbol{T}$. The terminal vertices of degree 1 in V_0 and the 2r terminal vertices obtained above corresponds to white vertices of $\boldsymbol{T}$. We will not count the remaining vertices of degree 2 as vertices of $\boldsymbol{T}$. Now we set involution Ω as fixing the terminal vertices of degree 1 in V_0 and interchanging the two vertices obtained at all the r cuts. The vertices of degree 1 in the V_1 corresponds to the black vertices. The cyclic order on the edges incident to the vertices of valency 3 in $\boldsymbol{T}$ corresponds directly to the cyclic order in $\boldsymbol{G}$.

From this we conclude that the tree diagram $\boldsymbol{T}$ depends on the choice of r cuts.

Thus we have a well defined, finite-to-one map from the isomorphism class of $\boldsymbol{T}$ onto $\boldsymbol{G}$.

Correspondence between special polygons and the tree diagrams

Special polygon to Tree diagram

Let $\mathbf{P}$ be a special polygon and $\boldsymbol{T}$ be union of all f edges in $\mathbf{P}$. We do not count the even vertices in int $\mathbf{P}$ as vertices. The white vertices are the even vertices on the boundary and similarly the black vertices are the odd vertices on the boundary, say $\partial\mathbf{P}$. The cyclic order on the edges incident to vertices of degree 3 is induced by orientation of $\partial\mathbf{P}$ and the involution(Ω) on the white vertices is given by side-pairings of $\partial\mathbf{P}$. This process turns $\boldsymbol{T}$ into a tree diagram.

Tree diagram to Special Polygon

Let $\boldsymbol{T}$ be a tree diagram. On all the edges which join two internal vertices or an internal vertex with a black vertex we introduce a new vertex of degree 2. Now the tree $\boldsymbol{T}$ must have atleast one vertex(black or white). Let us suppose that it has a white vertex v. We first identify the edge which is incident to v with the *f-edge* joining ι to ρ identifying v with ι. Then $\boldsymbol{T}$ can be developed into a tree of *f-edge* so that the cyclic orders on the edges incident at vertices of degree 3 in $\boldsymbol{T}$ matches with the ones induced by the orientation of $\mathbf{H}^2$. In case we have a black vertex we identify that with ρ. Now at the image of the white vertex v we assign the even line which passes through v. The even edges are paired if v is fixed by the involution Ω. Otherwise we will consider this complete geodesic as a free side and it will be paired with other free side constructed at $\Omega(v)$. Similarly, at the black image of a black vertex of degree 1

incident to unique edge say f, we assign those 2 edges which make angle $\frac{\pi}{3}$ with the image of f. These odd edges are paired. It is clear that these even sides, odd sides and free sides together with their pairing define a special polygon.
From this we conclude that special polygon $\mathbf{P}$ associated to a tree diagram $\boldsymbol{T}$ depends on initial choice of the vertex.
Thus we have a well defined, finite-to-one map between $\mathbf{P}$ onto isomorphism classes of $\boldsymbol{T}$.

Method for finding out trivalent diagrams of size m

We assume that all the diagrams of size 1 to $m-1$ are given. We have two operations to apply to the diagrams.

1. Connect a new edge say n_e to an edge say e_e of the diagram of size $m-1$. Since the connection is made at the white vertex of degree 1 so e_e must be of this kind. We apply the above procedure to all possible e_e and to all the diagrams of size $m-1$.

2. We can add 3 adjacent edges to diagrams of size $m-3$. For that we have 2 choices

   (a) add an edge joined to a $2-$cycle,

   (b) add a tripod(b_t).

   In the former case, there is only one possibility for the connection points that is a white vertex of degree 1. So for each white vertex of degree 1 of each diagram of the given size we get a new diagram of size m.
   In the latter case, we have multiple choices for the connection points.

   (a) Only 1 connection point i.e. a white vertex of degree 1. We will connect the tripod(b_t) to v and applying this to all vertices of degree 1 and all diagrams of size $m-3$.

   (b) 2 connection points say v_1 and v_2 both of which are white vertices of degree 1. We will connect two white vertices of the tripod say t_1 and t_2 with v_1 and v_2 in any order and not joining the third vertex of the tripod. Since black vertex of the b_t has cyclic orientation around it, we have 2 diagrams.

      i. Cyclic around b_t as (b_t, t_1), (b_t, t_3), (b_t, t_2) or

      ii. Cyclic around b_t as (b_t, t_1), (b_t, t_2), (b_t, t_3).

We apply these connections for all possible pairs (v_1, v_2) and for all diagrams of size $m-3$.

(c) 3 connection points say v_1, v_2, v_3 all three white vertices of degree 1. We have 2 diagrams which are same as that in Case(b) with the only. difference that t_3 is also connected to vertex v_3. Apply this to all possible triplets (v_1, v_2, v_3) and to all diagrams of size 3 .

For constructing diagrams of size m we will start with diagrams of size 1, 2 and 3. Constructing the diagrams of size 4 can be done by adding an edge to diagrams of size 3 and adding 3 edges of diagram of size 1. If we get isomorphic diagrams (duplicate) we keep one representation from each class and removes all duplicates. By this process we get all diagrams of size 4.
To construct the diagrams of size 5, we will add edge to the diagram of size 4 and 3 edges to diagrams of size 2. Finally we will remove duplicates to get all diagrams of size 5.
Continuing this process we get all diagrams of size m.

Theorem 3.5.4 (Vidal's Theorem). *1. There is a bijective correspondence between isomorphism classes of bipartite cuboid graphs and conjugacy classes of subgroups of* Γ.

2. There is a bijective correspondence between the marked trivalent diagrams and subgroups of Γ.

I will give an example by showing that subgroup of $PSL(2,\mathbb{Z})$ produces a bipartite cuboid group and bipartite group gives rise to conjugacy class of subgroup of Γ. We need to find special polygon and its correspondence tree diagram and we will then convert it to a bipartite cuboid graph. We have already proved that the Farey Symbol corresponds to bipartite cuboid graph. We take the following example
Example Let $\Phi \subseteq \Gamma$ be a subgroup with Farey Symbols given by g.F.S. $\{\infty, 0, \frac{1}{3}, \frac{1}{2}, 1, \infty\}$ and the labels $\{1, \bullet, 2, 2, 1\}$. For constructing the special polygons of Φ we use Γ-translates of the triangle which has vertices 0, ρ and ∞. We can also take the even edges between any cusp $x_i = \frac{a_i}{b_i}$ and $x_j = \frac{a_j}{b_j}$ which satisfies the definition of Farey symbols $|a_i b_j - a_j b_i| = 1$ and odd edges inside the ideal triangle with vertices 0, 1, ∞. Our notation is the same that is even vertices are represented by $\circ$, odd by $\bullet$ and we mark one of the edge joining ι to ρ by edge say e_1 (As shown in the Figure 3.1 [CC09]). To see the tree diagram we just ignore all the cusps. After rotating this

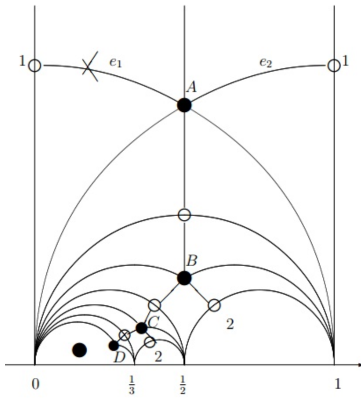

Figure 3.1: Tessellation inside a polygon.

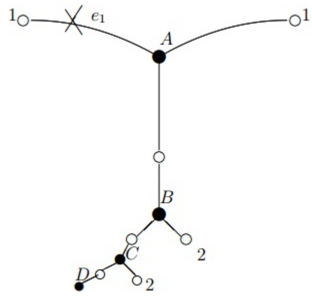

Figure 3.2: Tree diagram

tree diagram we obtain Figure 3.3 [CC09]. Now to obtain the corresponding bipartite cuboid graph, we identify vertices with same numerical labels and will make cycles. Now lets denote the bipartite cuboid graph by T_Φ (We do not consider the marked edge). Now by Vidal's Theorem T_Φ corresponds to the conjugacy class, say Γ_c of subgroups of Γ containing Φ. The index $[\Gamma : \Phi]$ is given by the size of T_Φ. In this case $[\Gamma : \Phi] = 10$. We know that marking a different edge of graph will produce another subgroup of Γ_c. If we mark a different edge of T_Φ and identify it with the edge joining ι to ρ, then by reversing the above process we get a subgroup Φ' conjugate to Φ. If we cut these cycles at different points then we get same subgroup but different Farey symbols. This is because despite the cut at different place, the bipartite cuboid graph

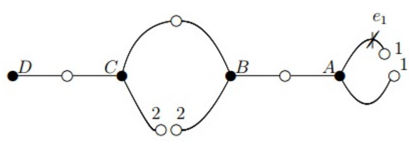

Figure 3.3: Rotated-tree diagram

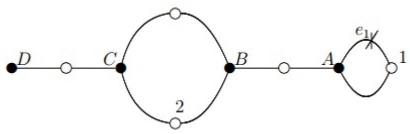

Figure 3.4: Trivalent diagram

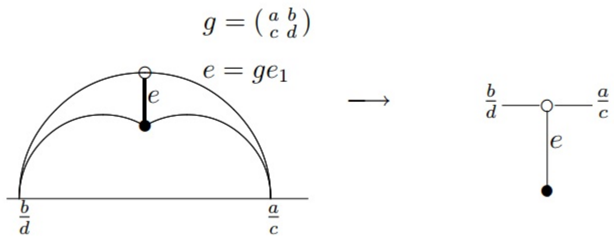

Figure 3.5: f-edge and its cusp

remains the same and for the farey graphs we start counting or reading the labels from the marked point.

Hence to find the conjugacy classes of subgroups of Γ which has index d, it is sufficient to find all bipartite cuboid graphs of size d. By reversing the process given above we can obtain a special polygon of the corresponding subgroup. We just need to find out the Farey Symbols of the corresponding subgroup. For finding it we need labels of degree 1 vertices. We start with this vertex of marked edge and then read the labels in anti-clockwise direction. Looking at Figure 3.3 and reading the labels in anti-clockwise direction we retrieve the sequence which is $\{1, \bullet, 2, 2, 1\}$.

Our next task is to find cusps corresponding to each vertex of degree 1 so that we are able to get the Farey Sequence. This is based on the fact that any f-edge e is image of the edge $e_1 = (\iota, \rho)$: e = ge_1 for some $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PSL(2, \mathbb{Z})$.

We start with a triangle with 0, ∞, ρ as vertices. Edge edge has cusps $\frac{b}{d}$ and $\frac{a}{c}$ and these must be added to an even vertex. Foe e.g. the edge e_1= (ι, ρ) has cusps 0 and ∞. Now our task is to find the cusp at even vertex since on knowing that we can easily compute the cusps at nearby even vertices. Now let us suppose that cusps of edge e are $\frac{a}{b}$ and $\frac{c}{d}$ with $\frac{a}{b} < \frac{c}{d}$. then around the edge e we can have three types of configurations

1. Adding a cusp between two given cusps.

2. Adding a cusp to the right.

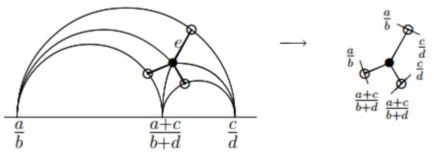

Figure 3.6: Adding cusp between 2 given cusps

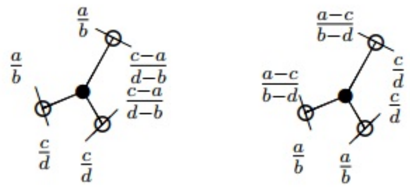

Figure 3.7: Adding cusp to right and left

3. Adding a cusp to the left.

Case 2 and 3 are same only since $\frac{a-c}{b-d} = \frac{c-a}{d-b}$ and hence these 2 will have same cusps. Hence for calculating the cusps of nearby vertices the we have

1. If $\frac{a}{b} < \frac{c}{d}$ then the cusp of vertices are as shown in Figure 3.6 [CC09].

2. If $\frac{a}{b} > \frac{c}{d}$ then the cusp of vertices are as shown in Figure 3.7 [CC09].

Now we will use the above method to determine the Farey sequence.
Now we see that the Farey Label is $\{1, 1, \bullet, 2, 2\}$. Using the above 2 ways of calculating the cusps we will start evaluating the cusp from the edge e_1 (as shown in Figure 3.8 [CC09]). First we calculate the cusps adjacent to cusp B followed by A and C. Now D is black vertex of degree 1, so we copy the cusps of adjacent white vertex and the result is that we get Figure 3.9 [CC09]. We start reading the cusps in anti-clockwise direction starting from the marked edge. We do not count the repeated entries and

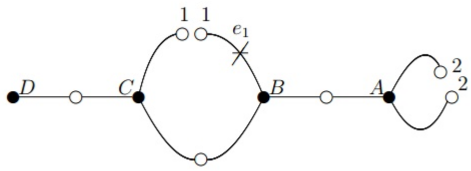

Figure 3.8: Finding the cusp

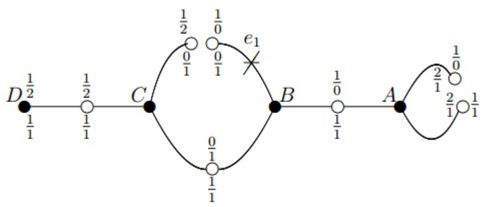

Figure 3.9: All the cusps

hence we get Farey Sequence $\{\infty, \frac{0}{1}, \frac{1}{2}, \frac{1}{1}, \frac{2}{1}, \infty\}$ which is same as $\{\infty, 0, \frac{1}{2}, 1, 2, \infty\}$. Farey symbol is just the combination of Farey Sequence and Farey labels.

$$\{\infty \overbrace{\ 1\ } 0 \overbrace{\ 1\ } \frac{1}{2} \overbrace{\ \bullet\ } 1 \overbrace{\ 2\ } 2 \overbrace{\ 2\ } \infty\}$$

We use the Theorem 3.3.1 to find out the generators of the Farey symbols, one corresponding to free sides labelled 1 free side labelled 2 and labelled $\bullet$.
For the sides labelled 1 we have $x_j = \frac{-1}{0}$, $x_{j+1} = \frac{0}{1}$, $x_k = \frac{0}{1}$, $x_{k+1} = \frac{1}{2}$ we get

$$g_1 = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}.$$

For the sides labelled 2 we have $x_{j'} = \frac{1}{1}$, $x_{j'+1} = \frac{2}{1}$, $x_{k'} = \frac{2}{1}$, $x_{k'+1} = \frac{1}{0}$ we get

$$g_2 = \begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix}.$$

For the odd side between $\frac{1}{2}$ and $\frac{1}{1}$ we have

$$g_3 = \begin{pmatrix} 4 & -3 \\ 7 & -5 \end{pmatrix}.$$

So the generators of the corresponding trivalent diagram are g_1, g_2 and g_3.

3.6 General Algorithms for finding special polygons and Farey Symbols

These algorithms have been studied in [CL05]. Before studying these algorithm we will give the definition of adjoinable.

Definition 3.6.1. *A T-tile is adjoinable to P(Polygon) if T is adjacent to a tile of P and if $P \cup T$ is contained in some fundamental domain of Γ*

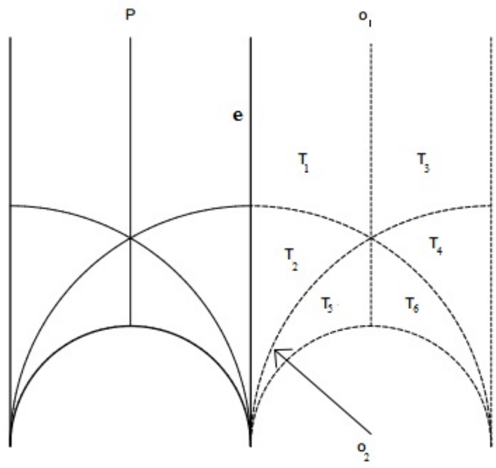

Figure 3.10: Special Polygon

Algorithm for calculating special polygon

1. If $\Gamma = PSL(2,\mathbb{Z})$. Let P be a special polygon with the given Farey symbol

$$-\infty \overbrace{\ \circ\ } 0 \overbrace{\ \bullet\ } \infty$$

   In that case return P and terminate.

2. If $\begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix} \notin \Gamma$, then let P be a hyperbolic polygon with 0, 1 and ∞ as vertices. Else, let P be the hyperbolic polygon with edges -1, 0 and ∞.

3. If any of the three sides of P map to each other by some $\gamma \in \Gamma$ then we assign that pairing to the side.

4. Now P is a polygon in which every side is either

   (a) even and unpaired,

   (b) even and already paired,

   (c) odd and already paired.

5. We now pick an unpaired even side say e (As shown in the Figure 3.10). All cases are similar as everything can be translated by some $\gamma \in PSL(2,\mathbb{Z})$. Now since e is unpaired so T_1 and T_2 are adjoinable. If now o_1 and o_2 are the new edges of P after we add T_1 and T_2 to P. Then either $\gamma o_1 = o_2$ for some $\gamma \in \Gamma$ or no such pair exists. If there exists such a γ then pair the edges. Go to Step 3.

6. If o_1 and o_2 doesn't pair with each other, then it doesn't pair with any other side as other unpaired sides are odd. So tiles T_3 and T_4 are adjoinable. Each of these tiles has a free edge and these free edges cannot pair with each other, so T_5 and T_6 are adjoinable.

7. We have now added 6 T-tiles to P. If either of the new even edges pair with any of the old unpaired even edges then assign that pairing.

8. If all the sides of P are paired then our algorithm is complete, otherwise we will go to Step 4.

The output is a special polygon.

Algorithm for calculating Farey Symbol

1. If $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ are Γ then $\Gamma = PSL(2,\mathbb{Z})$, so return

$$-\infty \overset{\circ}{\frown} 0 \overset{\bullet}{\frown} \infty$$

and terminate. If $\begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$ and $\begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}$ are in Γ then $\Gamma = \Gamma_2$ so return

$$-\infty \overset{\bullet}{\frown} 0 \overset{\bullet}{\frown} \infty$$

and terminate.

2. If $\begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix} \notin \Gamma$ then let F be the (partial) Farey symbol

$$-\frac{1}{0} \frown \frac{0}{1} \frown \frac{1}{1} \frown \frac{1}{0}$$

Otherwise F would be

$$-\frac{1}{0} \frown \frac{-1}{1} \frown \frac{0}{1} \frown \frac{1}{0}$$

3. For each i with $0 \leq i \leq n+1$, if the pairing between x_{i-1} and x_i is not fulfilled in then check whether it can be paired with itself (even or odd pairing), or if it can be paired with another unpaired edge. Whenever pairing is possible we assign that pairing.

4. If all the edges are paired then return F and terminate.

5. If there still exists an unpaired edge, say between $\frac{p_i}{q_i}$ and $\frac{p_{i+1}}{q_{i+1}}$ make a new vertex $\frac{p_i+p_{i+1}}{q_i+q_{i+1}}$ with no pairing transformation on the edges adjacent to it then go back to Step 3.

The output is the Farey Symbol for Γ.

3.7 Membership Test for Matrices in $PSL(2, \mathbb{Z})$

In this section we will study the Lang, Lim, Tan algorithm for determining whether a given matrix is member of subgroup $\Phi \subseteq \Gamma$. The basic concept that is used in this algorithm is that we use the line $(0, \infty)$ and its image under matrices to check whether the given matrix belongs to the subgroup or not.

Proposition 3.7.1. *[LLT95] If P is a special polygon in $\mathbf{H}^2$ and l is an even line then either $l \cap P = \emptyset$ or $l \subseteq P$.*

Proof. Let A (elliptic point of order 2) be the even vertex of l. In the tessellation of $\mathbf{H}^2$, A is a point of degree 2. If $l \cap P = \emptyset$, then $A \in P$ and then A belongs to even line $l' \subseteq P$. Hence the end points of l must coincide with end points of l'. These are actually the vertices which are adjacent to A in the tessellation of $\mathbf{H}^2$. Hence $l = l' \subseteq P$. □

Definition 3.7.2. *Let l be an even line, P a special polygon and $\boldsymbol{T}$ the $\mathbb{Z}$-tree of all f-edges in $\mathbf{H}^2$. We define the distance $d(l, P)$ between P and l to be the distance along the tree $\boldsymbol{T}$ between the sub-tree $P \cap \boldsymbol{T}$ and the vertex $l \cap \boldsymbol{T}$.*

Algorithm*[LLT95]* Let $A_k = \begin{pmatrix} a_k & b_k \\ c_k & d_k \end{pmatrix} \in \Gamma$ and $\Phi \subseteq \Gamma$ with special polygon P such that the even line $l_k = (\frac{b_k}{d_k}, \frac{a_k}{c_k}) \subseteq$ P. Then $A_k \in \Phi$ if and only if exactly one of the following holds

1. $\begin{pmatrix} a_k & b_k \\ c_k & d_k \end{pmatrix} = \pm$ I,

2. $(\frac{b_k}{d_k}, \frac{a_k}{c_k})$ is a free side of P paired to $(0, \infty)$,

3. $\begin{pmatrix} a_k & b_k \\ c_k & d_k \end{pmatrix} = \pm \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $(0, \infty)$ consists of two even edges which are paired.

Proof. Let $\Phi \subseteq \Gamma$ be a subgroup of the modular group and P be a special polygon of Φ with Farey symbol which have g.F.S. $\{x_0, x_1, x_2, \cdots, x_n\}$ where $x_0 = -\infty$, $x_{n+1} = \infty$ and $x_t = 0$ for some $1 \leq t \leq n$. Let $\{g_i\}_{i\in I}$ be the set of generators of Φ which corresponds to the side pairings of P and let g $= \begin{pmatrix} a_0 & b_0 \\ c_0 & d_0 \end{pmatrix} \in PSL(2, \mathbb{Z})$. The algorithm decides whether g belongs to the subgroup or not and how it gives decomposition of g as a reduced word in g_i's. g actually sends even line $(0, \infty)$ to even line $l_0 = (\frac{b_0}{d_0}, \frac{a_0}{c_0})$. If $d(l_0, P) > 0$ then we find an element $g_{i_1} \in \{g_i\}_{i\in I}$ such that

$$g_{i_1}^{p_1}.g = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} (where \;\; p = \pm 1)$$

and the even line $l_1 = (\frac{b_1}{d_1}, \frac{a_1}{c_1})$ has the property

$$d(l_1, P) < d(l_0, P).$$

Continuing this for finite number of steps, we get an even line l_k such that d(l_k, P) = 0 and we can determine whether corresponding element $g_{i_k}^{p_k} \cdots g_{i_k}^{p_k}.g$ is in Φ or not. Now we see how to choose the matrices $g_{i_t}^{p_t}$, t = 1, $\cdots$, k. Suppose

$$d(l_0, P) > 0 \;\; and \;\; l_o = (\frac{b_0}{d_0}, \frac{a_0}{c_0})$$

is such that $l_0 \cap P = \emptyset$. We now assume that $\frac{b_0}{d_0} < \frac{a_0}{c_0}$, then we must have

$$x_i \leq \tfrac{b_0}{d_0} < \tfrac{a_0}{c_0} \leq x_{i+1}$$

for some x_i and x_{i+1} of the g.F.S. of P. We have 3 cases

1. (x_i, x_{i+1}) is a free side paired to free side (x_j, x_{j+1}),

2. (x_i, x_{i+1}) is an even side,

3. (x_i, x_{i+1}) is an odd side.

Case 1 Let g_{i_1}(given by the Theorem 3.3.1) be a side pairing matrix which sends the free sides (x_i, x_{i+1}) to its paired sides (x_j, x_{j+1}). Then $g \in \Phi \iff g_{i_1}.g \in \Phi$. So we can have the matrix

$$g_{i_1}.g = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}$$

and the even line

$$l_1 = (\frac{b_1}{d_1}, \frac{a_1}{c_1}), \quad where \quad l_1 = g_{i_1}.l_0.$$

To obtain the shortest path from l_0 to P, one has to go through $g_{i_1}^{-1}P$. Hence $d(l_0, P) > d(l_0, g_{i_1}^{-1}P)$. But $d(l_0, g_{i_1}^{-1}P) = d(l_1, P)$. Thus $d(l_0, P) > d(l_1, P)$. Hence we get an even line with a shorter distance to P. We proceed this inductively using $g_i.g$ instead of g and l_1 instead of l_o.

Case 2 This case is same as Case 1, with the only difference that g_{i_1} is given by the Theorem 3.3.1.

Case 3 In this case we take $x_i = \frac{a_i}{b_i}$ and $x_{i+1} = \frac{a_{i+1}}{b_{i+1}}$ to be those points which are paired and let y = $\frac{a_1+a_{i+1}}{b_i+b_{i+1}}$ and let z be the common end point of two paired odd sides. Now let g_{i_1} given by Theorem 3.3.1 be the side pairing transformation taking (x_i, z) to (z, x_{i+1}). We have the even lines (x_i, y), (y, x_{i+1}) and $(\frac{b_0}{d_0}, \frac{a_o}{c_o})$. Then either

$$x_i \leq \frac{b_0}{d_0} < \frac{a_o}{c_o} \leq y \quad or \quad y \leq \frac{b_0}{d_0} < \frac{a_o}{c_o} \leq x_{i+1}.$$

In the 1^{st} case $g_{i_1}.g = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}$ satisfies $d(l_o, P) > d(l_1, P)$.

In the 2^{nd} case $g_{i_1}^{-1}.g = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}$ satisfies $d(l_o, P) > d(l_1, P)$.

We can proceed inductively.

Thus there are finite number of generators g_{i_t}, t = 1, 2, $\cdots$, k of Φ such that

$$g_{i_k}^{p_k}.g_{i_{k-1}}^{p_{k-1}}.\cdots.g_{i_1}^{p_1}.g = \begin{pmatrix} a_k & b_k \\ c_k & d_k \end{pmatrix}, \quad (p_t = \pm 1)$$

has a property of $d(l_k, P) = 0$ where $l_k = (\frac{b_k}{d_k}, \frac{a_k}{c_k})$. This means that $l_k \subseteq P$.

The matrix $\begin{pmatrix} a_k & b_k \\ c_k & d_k \end{pmatrix}$ maps even lines $(0, \infty)$ to the even line $l_k = (\frac{b_k}{d_k}, \frac{a_k}{c_k})$ and both these even lines are contained in P. Then $\begin{pmatrix} a_k & b_k \\ c_k & d_k \end{pmatrix}$ is either identity or one of the side-pairing transformations $g_i^{\pm 1}$ of the special polygon. □

Example We will test whether the matrix g = $\begin{pmatrix} 1 & -1 \\ 9 & -8 \end{pmatrix}$ belongs to the subgroup

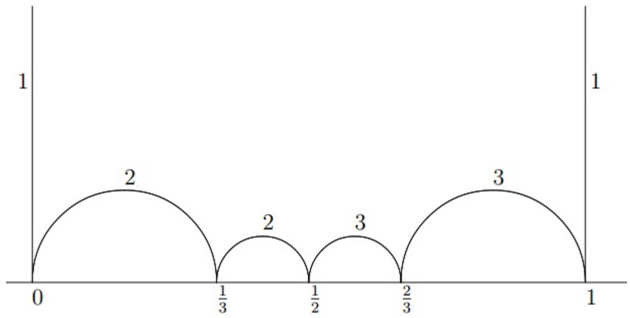

Figure 3.11: Fundamental Domain of $\Gamma_0(9)$

$\Phi = \Gamma_0(9)$ with the special polygon given by Figure 3.11
Using Theorem 3.3.1 or see Appendix we will first find the generators corresponding to free sides.

1. $g_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ sends the edge $(0, \infty)$ to $(1, \infty)$.

2. $g_2 = \begin{pmatrix} -4 & 1 \\ -9 & 2 \end{pmatrix}$ sends the edge $(0, \frac{1}{3})$ to the edge $(\frac{1}{3}, \frac{1}{2})$.

3. $g_3 = \begin{pmatrix} -7 & 4 \\ -9 & 5 \end{pmatrix}$ sends the edge $(\frac{1}{2}, \frac{2}{3})$ to the edge $(\frac{2}{3}, 1)$

Now for g = $\begin{pmatrix} 1 & -1 \\ 9 & -8 \end{pmatrix}$ we have first $l_1 = g.(0, \infty) = (\frac{1}{8}, \frac{1}{9})$. Since this edge lies inside(under) $(0, \frac{1}{3})$ we multiply g by g_2. Hence we obtain

$$g_2.g = \begin{pmatrix} -4 & 1 \\ -9 & 2 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 9 & -8 \end{pmatrix} = \begin{pmatrix} 5 & -4 \\ 9 & -7 \end{pmatrix}$$

The even line corresponds to this matrix is $l_2 = (\frac{5}{9}, \frac{4}{7})$. Now this l_2 is under the edge $(\frac{1}{2}, \frac{1}{3})$. Hence we will multiply this with g_3. Hence we obtain

$$g_3.g_2.g = \begin{pmatrix} -7 & 4 \\ -9 & 5 \end{pmatrix} \begin{pmatrix} 5 & -4 \\ 9 & -7 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$$

This corresponds to even line $l_3 = (0, \infty)$. Hence the conclusion is that g $\in \Phi$ and since $g_3.g_2.g =$ I, we have

$$g' = {g_2}^{-1}.g_3^{-1}.$$

Now, we will study a variant of the algorithm previously discussed. Algorithm will be to use the entire fundamental domain P as it is based on the way with which a fundamental domain and its Γ-translates tile the plane. Before giving the algorithm we prove a proposition which will be useful for the algorithm.

Proposition 3.7.3. *[CC09] Let F be a special polygon with side-pairing matrices Gl* $= \{g_1, g_2, \cdots, g_n\}$ *and* m_1*,* m_2 *be elements in* Γ

1. *If* $m_1F = m_2F$*, then* $m_1 = m_2$*,*

2. m_1F *is adjacent to* $m_2F \iff m_1 = m_2g_i$*, for some* $1 \leq i \leq n$*.*

Proof. 1. Suppose m_1F = m_2F and k ∈ m_1F_1. We assume that k is in the interior of m_1F. Since k ∈ m_1F, we have k = m_1z_1 for some $z_1 \in F$. Likewise, k = m_2z_2 for some $z_2 \in F$. So, $m_1z_1 = m_1z_2$ or $z_1 = m_1^{-1}m_2z_2$. Since there is a unique point z ∈ F with z_1 = gz for some g ∈ Γ, and we have z_1 = Iz_1, so we must have z_1=z_2 and hence $m_1^{-1}m_2$ must fix z_1. Points which are in the upper half plane are fixed by non-trivial matrix in SL(2, ℤ) only if they are elliptic of order 2 or 3, but these are isolated points. Since F is open, we can choose k to be some point in m_1F which is not elliptic and so z_1 is not elliptic. Then z_1 is fixed only by identity, so $m_1^{-1}m_2$ i.e. $m_1 = m_2$.
2. Since F is a fundamental domain, so either $m_1F = m_2F$ or these regions have no intersection points. Now lets suppose that these are adjacent. It implies that there exists a point x in the intersections of closure of these regions that is on the boundary of these 2 regions. Now we apply map m_1^{-1} to these domains. This is a continuous map on H, so F = $m_1^{-1}m_1F$ and $m_1^{-1}m_2F$ meet at a boundary point y = m_1^{-1}x. Now we suppose that y is a point on the edge say e_2(boundary edge) which is not included in the domain F. Then for some g_i and edge e_1, we have g_i which maps a point t to the edge e_1 to y which implies that $g_it = y$. But y is also an edge in $m_1^{-1}m_2F$. Now we know that y must be contained in some image and since its not in F so it must be contained in $m_1^{-1}m_2F$ because the the space between these domains along the edge where they can not meet is another domain. By uniqueness argument and not taking y as an elliptic point we have $t = p$ and $m_1^{-1}m_2 = g_i$ so $m_2 = m_1g_i$. Now supposing that the two domains have the form mF and mgF with g a side-pairing matrix, if we join edge e_1 with e_2 by g, then the region F has an adjoining region gF and these 2 regions join at e_2. The regions F and g^{-1}F meet each other along the edge e_1. Now taking any arbitrary matrix m we apply this matrix to both these regions. Since F and gF join along e_2 hence mF and mgF join along me_2. □

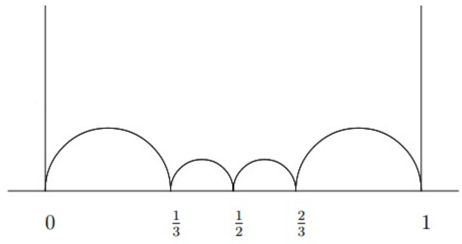

Figure 3.12: Fundamental domain of $\Phi = \Gamma_0(6)$

Algorithm This algorithm is based on existence of such a side pairing matrix g_e which enlarges the domain of mF at each step. We have a matrix and a subgroup $\Phi \subseteq \Gamma$ with special polygon F. We start the process by finding mF. We find the largest edge of mF, say me_2. Let e_2 be the preimage of me_2 in F. Then $g_e = g_1$ is the side pairing matrix which has destination e_2 i.e. g_e is a matrix that pairs e_1 with the edge e_2. We proceed by induction by taking mg_eF and mg_1 instead of mF and m. At each step we widen the domain. We have the following 2 cases

1. If we get overlapping at some stage then $m \notin \Phi$
2. If we get F exactly then $m.g_1.g_2 \cdots .g_kF = F$ for some side pairing matrices $g_1, g_2, \cdots, g_k$.

From the above proposition we get $m.g_1.g_2 \cdots .g_k =$ I so $m = g_k^{-1}g_{k-1}^{-1} \cdots g_2^{-1}g_1^{-1}$, which gives decomposition of m in terms of generators of Φ.

Example We consider the subgroup $\Phi = \Gamma_0(6)$ with a special polygon F (as shown in the Figure 3.12). Let the matrix be m = $\begin{pmatrix} 1 & 0 \\ -12 & 1 \end{pmatrix}$

Using Theorem 3.3.1 we will first find generators of the corresponding sides.
The generators which corresponds to the free sides are

$$g_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

which sends the edge (0, ∞) to (1, ∞)

$$g_2 = \begin{pmatrix} -5 & 1 \\ -6 & 1 \end{pmatrix}$$

which sends the edge $(0, \frac{1}{3})$ to $(\frac{2}{3}, 1)$

$$g_3 = \begin{pmatrix} -7 & 3 \\ -12 & 5 \end{pmatrix}$$

which sends the edge $(\frac{1}{3}, \frac{1}{2})$ to $(\frac{1}{2}, \frac{2}{3})$.
Now we use the algorithm to check whether the given matrix m is in Φ or not.
The domain mF has vertices

$$\{m.\infty,\ m.0,\ m.\tfrac{1}{3},\ m.\tfrac{1}{2},\ m.\tfrac{2}{3},\ m.1\}$$

or

$$\{\tfrac{-1}{12},\ 0,\ \tfrac{-1}{9},\ \tfrac{-1}{10},\ \tfrac{-2}{21},\ \tfrac{-1}{11}\}.$$

Now the largest edge of mF is $(\frac{-1}{9}$,0). The preimage of $(\frac{-1}{9}, 0)$ is the edge $(0, \frac{1}{3})$. We will first find which side-pairing matrix has the edge $(0, \frac{1}{3})$ as destination. Now we know that g_2^{-1} sends $(\frac{2}{3}, 1)$ to $(0, \frac{1}{3})$. The new matrix is m$g_2^{-1} = \begin{pmatrix} 1 & -1 \\ -6 & 7 \end{pmatrix}$ and the domain $mg_2^{-1}F$ whose vertices can be found in the same way as above. On calculation they are found to be

$$\{\frac{-1}{6},\ \frac{-1}{7},\ \frac{-2}{15},\ \frac{-1}{8},\ \frac{-1}{9},\ \frac{0}{1}\}.$$

Now the largest edge of $mg_2^{-1}F$ is $(\frac{-1}{6}$,0). The preimage of $(\frac{-1}{6}, 0)$ is the edge $(\infty, 1)$. We know that g_1 has destination $(\infty, 1)$. Then we apply g_1 to $mg_2^{-1}F$ to get the domain $mg_2^{-1}g_1F$. The new matrix is $mg_2^{-1}g_1 = \begin{pmatrix} 1 & 0 \\ -6 & 1 \end{pmatrix}$. Vertices of the domain $mg_2^{-1}g_1$ are found to be

$$\{\frac{-1}{6},\ 0,\ \frac{-1}{3},\ \frac{-1}{4},\ \frac{-2}{9},\ \frac{-1}{5}\}.$$

The largest edge of $mg_2^{-1}g_1F$ is $\frac{-1}{3}$ and it has an preimage edge $(0, \frac{1}{3})$. We know that g_2 has destination $(0, \frac{1}{3})$. Then we apply g_2 to $mg_2^{-1}g_1F$ to get the domain $mg_2^{-1}g_1g_2F$. The new matrix is $mg_2^{-1}g_1g_2 = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$. Vertices of the domain $mg_2^{-1}g_1g_2$ are found to be

$$\{\infty,\ -1,\ \frac{-2}{3}, \frac{-1}{2},\ \frac{-1}{3},\ 0\}.$$

The largest edge of $mg_2^{-1}g_1g_2$ is $(0, \infty)$ which has preimage edge $(1, \infty)$. So we will

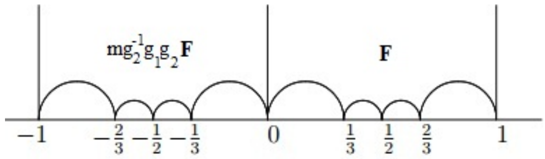

Figure 3.13: $mg_2^{-1}g_1g_2F$ and F

now apply the matrix $g_1^{-1} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. We get $mg_2^{-1}g_1g_2g_1^{-1}F = F$ and hence $m \in \Phi$ and the decomposition is

$$m = g_1g_2^{-1}g_1^{-1}g_2.$$

Lastly, we will prove that the algorithm ends in finite steps.
Suppose F be a fundamental domain and m be the initial matrix. If mF has an infinite edge, then algorithm stops. We assume that largest edge of mF, say (x_1, y_1) has a finite length. We will prove that there are infinitely many lengths of the form $|g(x_1) - g(y_1)| > |x_1 - y_1|$ where $g = \begin{pmatrix} p & q \\ r & s \end{pmatrix} \in \Gamma$.
But $g(x_1) - g(y_1) = \frac{x_1 - y_1}{(rx_1+s)(ry_1+s)}$
So, $|g(x_1) - g(y_1)| > |x_1 - y_1|$ implies that

$$\frac{1}{|rx_1 + s||ry_1 + s|} > 1.$$

Since x_1 and y_1 are fixed we will show that there are finitely many r and only finitely many s such that above equation is true. Since there are only finitely many lattice points of the form $ry_1 + s$ inside the circle. So, we deduce that

$$M_{y_1} = max_{r,s \in \mathbb{Z}}\{\frac{1}{|ry_1 + s|}, 1\}$$

is a finite number. Then, there are only finitely many lattices of the form $rx_1 + s$ which have the property $|rx_1 + s| < M_{y_1}$. This means that $|rx_1 + s||ry_1 + s| < 1$ only for finitely many r and s. Hence the algorithm ends in finite number of steps.

Appendix

In Sage, one can compute Farey symbols using computer directly. We have given some examples.
Our inputs are the Arithmetic Subgroups of $PSL(2,\mathbb{Z})$.
EXAMPLES

1. To find generators of $\Gamma_0(2)$.
   INPUT: sage: FareySymbol(Gamma0(2)).generators()
   OUTPUT : $\begin{pmatrix} 1 & -1 \\ 2 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

2. To find the cusps of $\Gamma_0(9)$.
   INPUT : sage: FareySymbol(Gamma0(9)).cusps()
   OUTPUT: $[0, 1/3, 1/2, 2/3, 1]$.

3. To calculate cusp width of $\Gamma_0(9)$.
   INPUT: sage: FareySymbol(Gamma0(6)).cusp widths()
   OUTPUT: $[6, 2, 3, 1]$

4. To calculate fraction of a farey symbol
   INPUT: sage: FareySymbol(Gamma0(9)).fractions()
   OUTPUT: $[0, 1/3, 1/2, 2/3, 1]$

5. To calculate the paired sides
   INPUT: sage: FareySymbol(Gamma0(9)).paired sides()
   OUTPUT : $[(0, 5), (1, 2), (3, 4)]$

6. To calculate the side pairing matrices of fundamental domain.
   INPUT: FareySymbol(Gamma0(6)).pairing matrices()
   OUTPUT: $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -5 & 1 \\ -6 & 1 \end{pmatrix}, \begin{pmatrix} -7 & 3 \\ -12 & 5 \end{pmatrix}, \begin{pmatrix} 5 & -3 \\ 12 & -7 \end{pmatrix}, \begin{pmatrix} 1 & -1 \\ 6 & -5 \end{pmatrix}, \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$.

7. To find the pairing sides in a fundamental domain.(Convention : We denote the even pairing by -2, the odd pairing by -3 while the free pairing by positive integer number)
   INPUT: sage: FareySymbol(Gamma0(13)).pairings()
   OUTPUT: $[1, -3, -2, -2, -3, 1]$

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