Analyses of Traffic Flow through Lattice Hydrodynamic Approach

Himanshu Nagpal

A dissertation submitted for the partial fulfilment of BS-MS dual degree in Science

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Dedicated to Tina
and
my parents

Certificate of Examination

This is to certify that the dissertation titled **Analyses Of Traffic Flow through Lattice Hydrodynamic Approach** submitted by **Himanshu Nagpal** (Reg. No. MS09059) for the partial fulfillment of BS-MS dual degree programme of the Institute, has been examined by the thesis committee duly appointed by the Institute. The committee finds the work done by the candidate satisfactory and recommends that the report be accepted.

Prof. Kapil H. Pranjape

Dr. Chanchal Kumar

Dr. Alok Maharana
(Mentor)

Dated: April 25, 2014

Declaration

The work presented in this dissertation has been carried out by me under the guidance of Dr. Alok Maharana at the Indian Institute of Science Education & Research, Mohali.

This work has not been submitted in part or in full for a degree, a diploma, or a fellowship to any other university or institute. Whenever contributions of others are involved, every effort is made to indicate this clearly, with due acknowledgement of collaborative research and discussions. This thesis is a bonafide record of original work done by me and all sources listed within have been detailed in the bibliography.

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(Candidate)

Dated: April 25, 2014

In my capacity as the supervisor of the candidates project work, I certify that the above statements by the candidate are true to the best of my knowledge.

Dr. Alok Maharana
(Mentor)

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Abstract

The basics of traffic flow theory are studied briefly. Traffic modeling is classified and some earlier important work in the field is discussed concisely. Lattice hydrodynamic modeling is introduced. Various lattice hydrodynamic models are presented including the most basic Nagatani's model. Then discussion of traffic modeling is extended to two lane.
At last a new lattice hydrodynamic model is proposed to investigate the traffic flow properties on a circular road.

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Chapter 1

Traffic Flow

Transportation problem have been topic of concern for man long before the automobile invention. However, in recent years, the scientific analysis of traffic problems has become an attentive topic of deliberation and research for scientist and engineers. Traffic problem include: installation of traffic light or stop signs; cycle timing of traffic lights; where to construct flyovers, whether to change a two way street to a one way street, number of lanes on a road; where to construct overpasses, exits and entrances. In particular, the main aim is to study the traffic phenomena with an objective of eventually making decisions which may palliate congestion, reduce accidents, maximize traffic flow, minimize automobile exhaust pollution etc.

1.1 Traffic Flow

Traffic flow is the scientific study of movement of individual drivers, vehicles and infrastructure (highways, traffic signage and traffic control devices) and interaction they make with one another. The main purpose of the study is to develop an optimal road network with maximum flow of traffic and minimum traffic congestion.

1.2 Modeling of Traffic Flow

Modeling of traffic flow has been a key tool to simulate the behavior of transportation system. There are mainly three different types of methodology used in literature, namely microscopic car following model, macroscopic continuum model and mesoscopic gas kinetic model.

1.2.1 Microscopic Approach

Microscopic model describes both the space-time behavior of the systems entities as well their interaction at a high level of detail. For example: Car following model, Cellular Automata model etc

1.2.2 Mesoscopic Approach

Simulate individual vehicles, but describe their activities and interactions based on aggregate (macroscopic) relationships.

1.2.3 Macroscopic Approach

Instead of describing the individual behavior of each vehicles, in this model we look at the traffic flow from a global perspective. For example: Hydrodynamic model, Gas Boltzmann model etc.

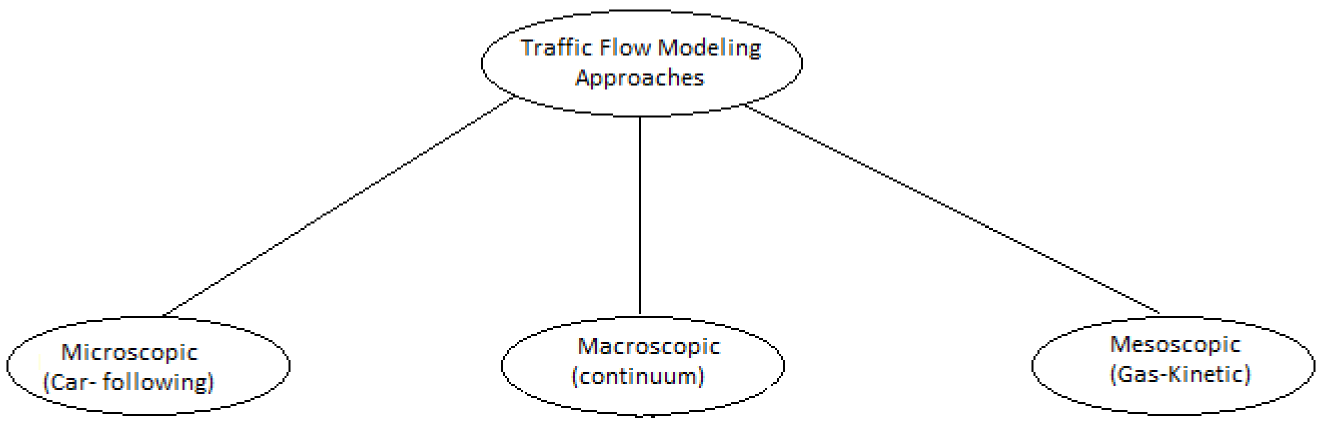

Figure 1.1: Classification of Traffic Flow Modeling Approaches.

Different Approaches are useful at different level of details. Our research will focus on continuum modeling approach so we will discuss some literature work on this modeling approach.

1.3 Traffic Stream Properties

There are three main variables to understand the traffic stream: Speed(v), Density(ρ) and Flow(q).

1.3.1 Speed(v)

Speed in traffic flow is as usual distance covered per unit time. It is impossible to record the speed of each individual vehicle. So average speed is based on taking all vehicles over a period of time or space. If time is kept as reference then it is *time mean speed* or otherwise it is *space mean speed.*

TimeMeanSpeed : It is defined as average speed of all the vehicles passing a reference point on a highway over a fixed period of time.

$$v_t = (1/m)\sum_{i=1}^{m}$$

where m represents number of vehicle which are passing through reference point.

SpaceMeanSpeed : It is defined as average speed of all the vehicles occupying a given segment of highway over some fixed period of time.

$$v_s = n(\sum_{i=1}^{n}(1/v_i))^{-1}$$

where n represents number of vehicle passing the given segment of highway.

1.3.2 Density(ρ)

Density is defined as number of vehicles per unit length of road. It is expressed as vehicles per km. Let n number of vehicles are occupied in l length of the road section. Then

$$\rho = \frac{n}{l}$$

1.3.3 Flow(q)

Flow is defined as average number of cars passing per unit time through reference point. All the fundamental variable discussed above depends on position of vehicles and time.

1.4 Fundamental Law

In the last section we have briefly discussed the fundamental traffic variables: velocity, density and flow. There is a close relationship among all three variables which is as

$$q = \rho v$$

Above relationship is called fundamental law of traffic flow.

1.5 Conservation Law of Cars

In this section we will see that on a unidirectional road with no entrance and no exit, the number of vehicles between two points are conserved. Assume on some section of road, between $x = \alpha$ and $x = \beta$ as shown in the Fig. 1.1, the number of cars are N.

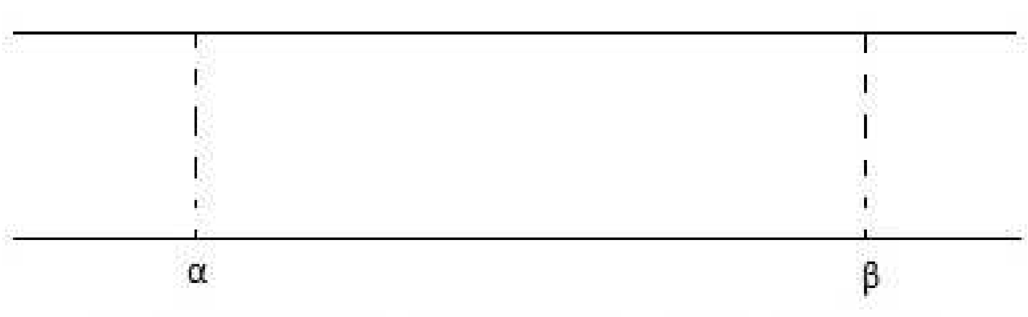

Figure 1.2: Cars entering and leaving the road segment

Then

$$N = \int_{\alpha}^{\beta} \rho(x, t) dx \tag{1.1}$$

If there are no entrances and exits on the road. The N decreases due to cars leaving from β and N increases due to cars entering from α. Then the rate of change of the number of cars is, equals the number of cars crossing $x = \alpha$ per unit time $q(\alpha, t)$ minus number of cars leaving $x = \beta$ per unit time $q(\beta, t)$, or

$$\frac{dN}{dT} = q(\alpha, t) - q(\beta, t) \tag{1.2}$$

So combining equations (1.1) and (1.2), results

$$\frac{d}{dt} \int_{\alpha}^{\beta} \rho(x, t) dx = q(\alpha, t) - q(\beta, t) \tag{1.3}$$

Above equation shows that changes in number of cars are only because of the flow across the boundary. The number of cars is conserved. This equation is called **integral conservation law**.

Consider the integral conservation of cars over a small interval of highway from $x = a$ to $x = a + \Delta a$. Thus from equation (1.3)

$$\frac{\partial}{\partial t}\int_{a}^{a+\Delta a}\rho(x,t)dx = q(a,t) - q(a+\Delta a,t)$$

Divide by $-\Delta a$ and take the limit as $\Delta a \to 0$:

$$\lim_{\Delta a \to 0}\frac{\partial}{\partial t}\frac{1}{-\Delta a}\int_{a}^{a+\Delta a}\rho(x,t)dx = \lim_{\Delta a \to 0}\frac{q(a,t) - q(a+\Delta a,t)}{-\Delta a}. \tag{1.4}$$

Introduce the function $N(x_1, t)$, the number of cars on the roadway between any fixed position x_0 and the variable position x_1.

$$N(x_1,t) = \int_{x_0}^{x_1}\rho(x,t)dx$$

Then, the average number of cars per mile between a and $a + \Delta a$ is

$$-\frac{1}{\Delta a}\int_{a}^{a+\Delta a}\rho(x,t)dx = \frac{N(a+\Delta a,t) - N(a,t)}{-\Delta a}$$

. When the limit $\Delta a \to 0$, the right-hand side is $\frac{\partial N(a,t)}{\partial a}$. Using the definition of $N(a,t)$, from the Fundamental Theorem of Calculus,

$$\frac{\partial N(a,t)}{\partial a} = \rho(a,t)$$

. Thus the left-hand side of equation (1.4) becomes $-\frac{\partial}{\partial t}\rho(a,t)$. Since equation (1.4) holds for every a, we can replace a by x and we get

$$\frac{\partial \rho(x,t)}{\partial t} + \frac{\partial q(x,t)}{\partial x} = 0 \tag{1.5}$$

Equation (1.5) is represent the conservation of car in mathematical form.

1.6 Velocity-Density Relationship

Traffic density and Car velocity are related by one equation, conservation of vehicles,

$$\frac{\partial \rho(x,t)}{\partial t} + \frac{\partial \rho(x,t)v(x,t)}{\partial x} = 0$$

If the velocity and initial density are known, above equation can be used to predict future density. So now we can think of velocity to be the function of density. The choice of such function depends on the different models. On basis of some experiments and observations, we assume that at any point along the road, the velocity of car only depends upon the density of cars.

$$v = v(\rho) \tag{1.6}$$

If there are no cars or the density of cars on the road is very low, then car can move at maximum speed v_{max}.

$$v(0) = v_{max} \tag{1.7}$$

Now, if the density on the road increases (it means there are more cars on the road) so due to presence of other cars vehicle will move slow. As the density keep on increasing velocity will keep on decreasing. So we can say

$$\frac{dq}{d\rho} = v'(\rho) \leq 0. \tag{1.8}$$

And at a certain density car will not move. This maximum density (ρ_{max}), is represent the bumper to bumper traffic,

$$v(\rho_{max}) = 0. \tag{1.9}$$

So, different scientist purposed there different models to represent the relationship between velocity and density. Here are some well known models given:

- Greenshield Linear Relationship(1934)

$$v = v_{max}(1 - \frac{\rho}{\rho_{max}})$$

- Greenberg Relation (1959)

$$v = v_{max} \log(\frac{\rho_{max}}{\rho})$$

- Underwood Relation(1961)

$$v = v_{max} \exp^{\frac{-\rho}{\rho_{max}}}$$

- The Eddie and Bell Curve Relationship

$$v = v_{max} \exp^{-\frac{1}{2}(\frac{\rho}{\rho_{max}})^2}$$

v_{max} and ρ_{max} are freeway velocity and jam density respectively.
It is inauspicious that as still there has been no precise mathematical description of the relationship between speed and density. Though a large amount of ideas are still being put into this problem. Still, observation has shown that, though imperfect, one or more of the above models can still be applied to the majority of situations, producing predictions that are, in most cases, highly accurate.

1.7 Fundamental Diagram

Assume a road is homogeneous such that the car velocity is dependent only on the density of vehicles on road. Since the traffic flow is equals to density time velocity, so the flow only depends upon the density of vehicles along the road.

$$q = \rho u(\rho) \tag{1.10}$$

This traffic flow has some certain properties. The traffic flow may be zero in two ways:

1. if there is no traffic on the road that means $\rho = 0$ and
2. if the velocity is zero it means no vehicle is moving that is the case of $\rho = \rho_{max}$

So for all other values of density between 0 and ρ_{max} traffic flow is positive. The relationship between traffic flow and density is shown in Fig. 1.2. This Flow-Density relationship is called **Fundamental Diagram of Traffic Flow.**

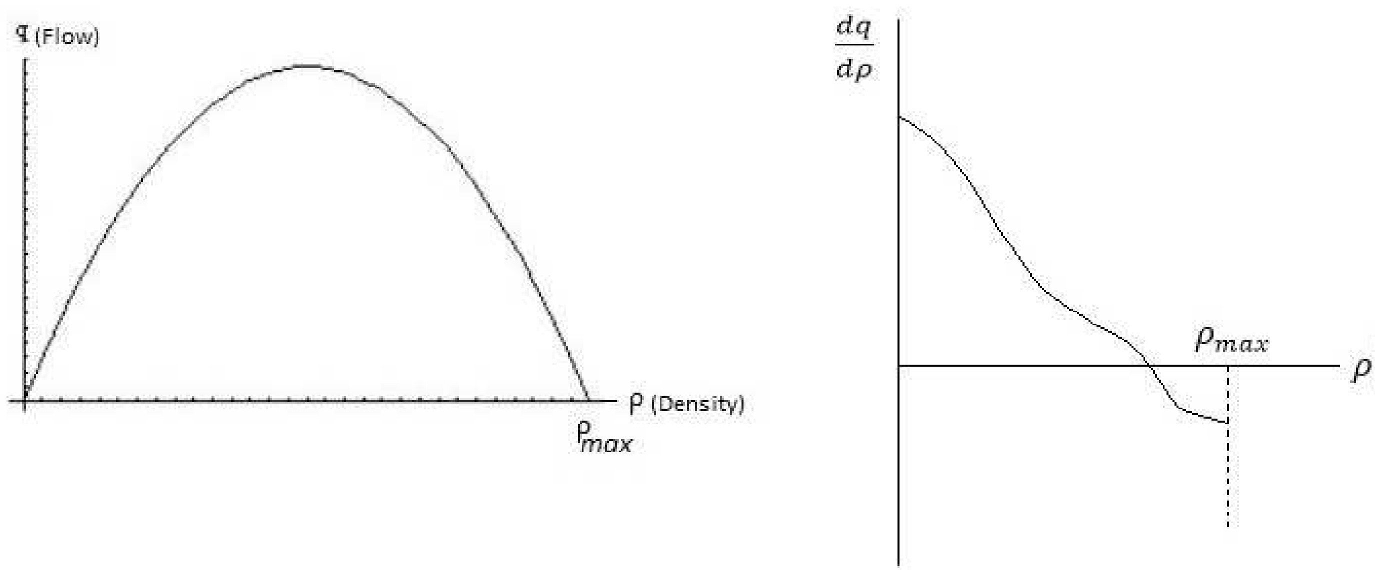

Figure 1.3: Fundamental Diagram of Road Traffic

The maximum traffic flow is called the capacity of the road. It is assumed that the flow-density relationship is concave downwards, $\frac{d^2q}{dq^2} < 0$. So as ρ increases $dq/d\rho$ decreases.

1.8 Earlier Important Work in Continuum Modeling Approach

In continuum model traffic flow is described in terms of average density, average velocity and average flow.

1.8.1 LWR Model

The research on traffic flow was started in 1950's. To describe the dynamical properties on a homogeneous and unidirectional highway, Lighthill(1955), Whitham(1955) and Richards(1956) independently proposed a continuum model, which is known as LWR model. In LWR model, the relationship between fundamental variables: flow, speed and density is supplemented by the continuity equation

$$\partial_\rho + \partial_x(\rho v) = 0 \tag{1.11}$$

where $\rho(x,t)$ be the density and $v(x,t)$ the velocity of the traffic flow. Above equation is not consistent in self but needs an additional relation which is supplemented by the equation of traffic flow

$$q = \rho v, \tag{1.12}$$

and the relation ship between the mean velocity and the traffic density under steady-state uniform flow.

$$v = v_e(\rho) \tag{1.13}$$

Where $v_e(\rho)$ is the equilibrium velocity; x and t represent the space and time respectively. Using LWR modeling approach exhibits a wide range of phenomena such as shock formation and rarefaction waves.

Drawbacks: LWR model could not explain amplification of small disturbances on heavy traffic. Model failed to explain traffic flow instabilities, such as cluster formation in initially homogeneous traffic conditions. This was only because of the fact that all vehicles travels at v_e, equilibrium velocity. Therefore non-equilibrium models were needed.

1.8.2 Payne-Whitham Model

The first non equilibrium model was given by Payne and Whitham known as, PW model. Which was

$$\partial_t \rho + \partial_x(\rho v) = 0, \tag{1.14}$$

$$\partial_t v + v\partial_x v + \frac{c^2(\rho)}{\rho}\partial_x \rho = \frac{V_*(\rho) - v}{\tau} \tag{1.15}$$

Where V_* is the equilibrium velocity, τ is the relaxation time and c is the sound speed. The relaxation time is the time taken by driver to adjust it's velocity due to front stimuli. This model is able to explain the cluster formation.

Drawbacks: In this model there exist a characteristic speed which is greater than the flow velocity. This means the future condition of traffic flow will be affected by the traffic conditions behind the flow. But in traffic flow vehicles only travels in one direction and respond only to front stimuli. This is known as 'wrong way' travel problem.

1.8.3 Zhang Model

The wrong way problem was the major flaw in the PW model, that is why new higher model was developed by Zhang.

$$\partial_t \rho + \partial_x(\rho v) = 0, \tag{1.16}$$

$$\partial_t + v\partial_x v + 2\beta c(\rho)\partial_x v + \frac{c^2(\rho)}{\rho}\partial_x \rho = \frac{V_*(\rho) - v}{\tau} + \mu(\rho)\partial_{xx} v \tag{1.17}$$

where

$$\mu(\rho) = 2\beta\tau c^2(\rho) \tag{1.18}$$

$$c(\rho) = \rho V_*'(\rho) \tag{1.19}$$

μ is the viscosity coefficient and β is the dimensionless parameter. Different models are used for the equilibrium velocity.
Continuum modeling approach can be further divided into modeling approaches as shown in the Fig. 1.3. In next chapter we will talk about traffic models using lattice hydrodynamic

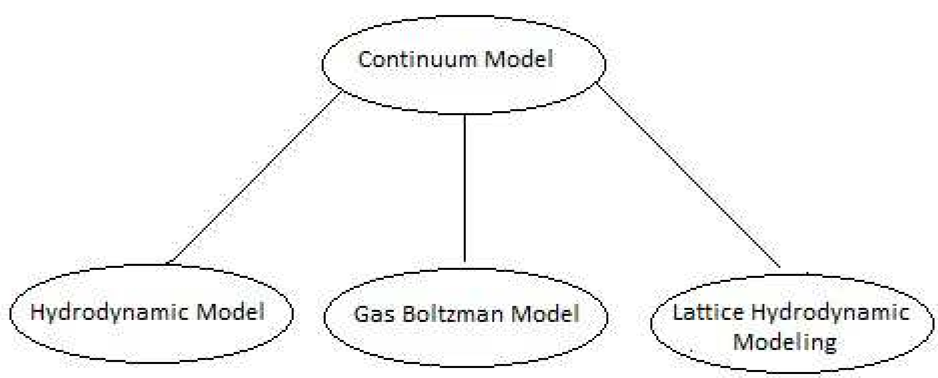

Figure 1.4: Classification of Continuum Models

approach and focus our research work on the topic.

Chapter 2

Introduction to Lattice Hydrodynamic Model

Lattice hydrodynamic model is the simplified version of continuum model by making space variable to be dimensionless,which means that density is locally described at each point. Lattice hydrodynamic modeling is very helpful to describe traffic jam in term of kink-antikink density waves. The main motivation behind the lattice hydrodynamic approach is to check analytically that kink-antikink types of density waves exit in the traffic flow or not.

2.1 Modified KdV equation for jamming transition in the continuum models of traffic (1998)

In 1998, Nagatani introduced a lattice hydrodynamic model which is the simplified version of continuum model and also incorporates some ideas of car following model.

2.1.1 Model

Model is

$$\partial_t \rho + \rho_0 \partial_x(\rho v) = 0, \tag{2.1}$$

$$\partial_t(\rho v) = a[\rho_0 V(\rho(x+\delta)) - \rho v] \tag{2.2}$$

where ρ_0 is the average density; a is the sensitivity of drivers; $V(.)$ is the optimal velocity function; $\rho(x+\delta)$ represents the density at position $x+\delta$ at time t; v and $\delta = 1/\rho_0$ are the velocity and the average headway, respectively. The above simplified hydrodynamic model is further modified with dimensionless space x (let $x^* = x/\delta$ and x^* indicated as x hereafter) and expressed as

$$\partial_t \rho_j + \rho_0(\rho_j(t)v_j(t) - \rho_{j-1}(t)v_{j-1}(t)) = 0, \tag{2.3}$$

$$\partial_t(\rho_j(t)v_j(t)) = a[\rho_0 V(\rho_{j+1}) - \rho_j v_j], \tag{2.4}$$

Optimal velocity function has these properties:

- it is monotonically decreasing function;
- It has an upper bound (V_{max}) and
- It has an inflection point at safety distance h_c or $\rho = \rho_0$ ($h_c = \frac{1}{\rho_0}$).

j indicates the j^{th} site on the one-dimensional lattice,The basic idea behind such models is that drivers adjust their velocity according to the observed headway. In this model, he derived the kdV and mkdV equation from with the help of non-linear analysis to describe the traffic jam because solition and kink-antikink density wave is appearing in the traffic jam which is explained by kdV and mkdV equation.

2.1.2 Stability Analysis

Linear stability results

To do linear analysis, initially uniform traffic flow is considered and then a small perturbation is given to the uniform stable state of traffic flow. By expanding the perturbation in Fourier modes and analyzing we get following condition for stable traffic flow:

$$\boxed{a > -2{\rho_0}^2 V'} \tag{2.5}$$

Non-Linear stability results

To do non linear analysis reduction perturbation method is used. To extracting space and time on slow scale, slow variables X and T is defined as:

$$X = \epsilon(j + bt) \qquad T = \epsilon^3 t$$

$$\rho_j(t) = \rho_c + \epsilon R(X, T)$$

By applying non linear stability analysis on the model following mKdV is obtained (by ignoring the ϵ term):

$$\partial_{T'} R' = \partial^3_{X'} R' - \partial_{X'} {R'}^3 - \epsilon[3\partial^2_{X'} R' + (3/4)\partial^4_{X'} R' - (1/2)\partial^2_{X'} R^3] \tag{2.6}$$

Where
$T' = (-\rho_0{}^2 V'/6)T$ and $R = (-\rho_0{}^2 V'/\rho_c{}^2 V''')^{\frac{1}{2}} R'$

$$\boxed{\rho_j = \rho_c + \epsilon[(-5\rho_c{}^2 V')/(\rho_c{}^2 V''')]^{1/2} tanh(5/2)^{1/2}(X - 5T/6)} \tag{2.7}$$

2.1.3 Results and Discussion

The modified KdV equation is derived near the critical points from the continuum models. Coexisting curves are derived from the modified KdV equation. It was shown that the modified KdV equation obtained by the lattice model agrees with that of the optimal velocity model.

Based on the Nagatani's above model, different scientist incorporated new factors in the evolution equation and purposed new models.
Following are some extensions of Nagatani's Model:

- **The backward looking effect in the lattice hydrodynamic model (2008)**-
  Later many extended lattice models were developed which stabilized the traffic flow, but all their properties were related to only 'forward looking effect', are they the same if 'backward looking effect' of driver is considered. To check this conception some car following and lattice models were purposed. Once the backward looking effect is considered, the larger of the stable region. The traffic jam could be suppressed effectively. Due to the backward looking effect, large cluster disappears gradually, and the KdV soliton density waves occur, which could be derived from the KdV equation.

- **Nonlinear analysis of lattice model with consideration of optimal current difference (2011)**-
  In this paper, a new lattice model is proposed with the consideration of optimal current difference on a single lane highway. The factor about the optimal current difference information has an important influence on traffic flow. By introducing the optimal current difference information, not only the stability of traffic flow is improved, but also the appearance of traffic jams is suppressed effectively.

- **A new lattice model of traffic flow with the consideration of the drivers forecast effects(2011)**-
  The intelligent transportation system (ITS) can forecast the future traffic situation based on the current traffic status, so ITS will produce guidance information to driver. In order to study the effects of the guidance information on traffic flow, Peng. et al[20]. the stability of traffic flow is improved with the consideration of forecast effect.

Up to now we were just analyzing the traffic flow on a single lane road. But in reality most roads are made up of either two lane or higher. So now we will discuss some earlier work done in the two lane traffic flow modeling.

2.2 Jamming transitions and the modified Korteweg de Vries equation in a two-lane traffic flow (1999)

In this paper, Nagtani presented the lattice models of continuum traffic flow on a two lane highway and derived MKdV equation for the two lane traffic flow.

Modified Model

we are proposing a new lattice hydrodynamic model in a two-lane traffic flow. The lane

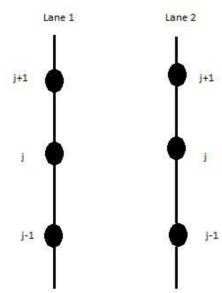

Figure 2.1: Schematic diagram of two-lane traffic flow

change occurs only in the following possibilities:

1. If the density at site $j-1$ on the first lane is higher than that at site j on the second lane, the lane changing occurs from the first lane to the second lane and will be proportional to their density difference as follows: $\gamma|\rho_0^2 V'(\rho_0)|\,(\rho_{1,j-1}(t) - \rho_{2,j}(t))$.

2. If the density at site j on the second lane is higher than that at site $j+1$ on the firstlane, the lane changing occurs from the second lane to the first lane and will be proportional to their density difference as follows: $\gamma|\rho_0^2 V'(\rho_0)|\,(\rho_{2,j}(t) - \rho_{2,j+1}(t))$.

Here, $\rho_{1,j}(t)$ and $\rho_{2,j}(t)$ are the densities on the first and second lane, respectively. The proportionality constant $\left(\gamma|\rho_0^2 V'(\rho_0)|\right)$ is chosen in such a way that it becomes dimensionless. Model is

$$\partial_t \rho_j + \rho_0(\rho_j v_j - \rho_{j-1} v_{j-1}) = \gamma|\rho_0^2 V'(\rho_0)|(\rho_{j+1} - 2\rho_j + \rho_{j-1}), \tag{2.8}$$

$$\partial_t(\rho_j v_j) = a[\rho_0 V(\rho_{j+1}) - \rho_j v_j] \tag{2.9}$$

where $\rho_j = \frac{\rho_{1,j}+\rho_{2,j}}{2}$ and $\rho_j v_j = \frac{\rho_{1,j}v_{1,j}+\rho_{2,j}v_{2,j}}{2}$. where ρ_0 is the average density, ρ and v denotes the local density and velocity, respectively. j indicates the j^{th} site on the one-dimensional lattice, $V(.)$ is the optimal velocity function.

2.2.1 Stability Analysis

Linear stability Results

Condition for stable traffic flow is:

$$\boxed{a > -\frac{2{\rho_0}^2 V'}{(1+2\gamma)}}. \tag{2.10}$$

Non-Linear stability Results

$$X = \epsilon(j+bt) \qquad T = \epsilon^3 t$$

$$\rho_j(t) = \rho_c + \epsilon R(X,T)$$

By applying non linear stability analysis on the model following mKdV is obtained (by ignoring the ϵ term):

$$\partial_{T'} R' = \partial_X^3 R' - \partial_X {R'}^3 - \epsilon[\frac{27(1+2\gamma)}{2(1-5\gamma+4\gamma^2)}\partial_X^2 R' + \frac{(1-6\gamma-9\gamma^2)+14\gamma^3}{2(1-5\gamma+4\gamma^2)}\partial_X^4 R' - \frac{(1+4\gamma)}{2}\partial_X^2 {R'}^3] \tag{2.11}$$

where $T' = \frac{(1+12\gamma^2)(-{\rho_c}^2 V')}{27} T$ and $R = (\frac{-2(1-5\gamma+4\gamma^2){\rho_c}^2 V'}{9{\rho_c}^2 V'''})(\frac{1}{2}) R'$

$$R(X,T) = \sqrt{\frac{2(1-5\gamma+4\gamma^2 c(-{\rho_c}^2 V'))}{9{\rho_c}^2 V'''}} tanh \sqrt{\frac{c}{2}\left(X - \frac{(1-5\gamma+4\gamma^2)c(-{\rho_c}^2 V')}{27} T\right)} \tag{2.12}$$

2.2.2 Results and Discussion

We set τ as the unit time step. We obtain from Eq. (2.29) and (2.30):

$$\rho_j(t+2) - \rho_j(t+1) + \tau{\rho_0}^2[V(\rho_{j+1}(t)) - V(\rho_j(t))] - \tau\gamma|{\rho_0}^2 V' \rho_0|[\rho_{j+1}(t+1) - 2\rho_j(t+1) + \rho_{j-1}(t+1)] = 0 \tag{2.13}$$

Periodic boundary conditions are adopted and the initial density profile is set as follows:

$$\rho_j(1) = \rho_j(0) = \begin{cases} \rho_0; & j \neq \frac{M}{2}, \frac{M}{2}+1 \\ \rho_0 - \sigma; & j = \frac{M}{2} \\ \rho_0 + \sigma; & j = \frac{M}{2}+1 \end{cases}$$

where, σ is the initial disturbance, the total number of sites M is taken as 100 and other parameters are set as follows: $\sigma = 0.05, a = 2.5, \rho_0 = 0.2$.
Results of the numerical simulation are:

- The jam propagates from right to left where cars move from left to right.
- With increasing rate γ of the lane changing the critical point, the coexisting curve and the neutral stability line decreases.

Following are some extension of Nagatani's two lane model:

- **Flow difference effect in the two-lane lattice hydrodynamic model (2012)**-
Under an advanced traveler information system (ATIS) environment, vehicle information, such as velocity, position, and flow, for the drivers themselves as well as the others, is available. Nevertheless, the existing studies only factor either the velocity or the position into models. Flow information may be very comprehensive since it contains both the position (density) and velocity information indirectly, and so little analytical work for the two-lane traffic flow has been implemented up to now. In this paper Wang Tao et.al[21] presented a new lattice model considering the flow difference equation for a two lane highway. He concluded that FDE consideration stable the traffic flow.
Analyses of a modified two-lane lattice model by considering the density difference effect (2014)-
The density difference effect play an important role in stabilizing the traffic flow and this effect is studied by J.F. Tian (2012) on a single lane. But the effect of density difference was not studied on a two-lane traffic flow. In 2014 Gupta et al.[19] incorporated the density difference effect and found that the inclusion of the effect increase the stability of traffic flow.

So as we see that scientist create new traffic models by including various factors taking their model to more realistic situation and study the efficiency of their models. They may find ways to calibrate models to empirical observational data or determine where a model has difficulty providing realistic results.

Chapter 3

A New Lattice Hydrodynamic Model for Circular Road

As we discussed, The very first lattice hydrodynamic model was presented by Nagatani[8] comprises the idea of car following model and as well as macroscopic model to analyze the density wave of traffic flow on a unidirectional simple road and is given by

$$\partial_t \rho + \rho_0(\rho_j v_j - \rho_{j-1} v_{j-1}) = 0. \tag{3.1}$$

$$\partial_t(\rho_j v_j) = a[\rho_0 V(\rho_{j+1}) - \rho_j v_j]. \tag{3.2}$$

where j represent the j_{th} site on one dimensional lattice. ρ_j and v_j represent the local density and velocity respectively at the j_{th} site at time t. ρ_0 is the average density; a is the sensitivity of driver.

$V(.)$ is the optimal velocity function. Subsequently, many people has extended the Nagatani's Model[12],[13] by incorporating different factors and has given new lattice models. Furthermore, an extended lattice model is given by Peng et al[11]. The continuity equation remains same in it although evolution equation is altered by looking at the optimal current difference at site-$j+2$ and $j+1$. This modification plays an important role to stabilize the traffic flow and reduce efficaciously traffic jams. The altered equation is given by

$$\partial_t(\rho_j v_j) = a[\rho_0 V(\rho_{j+1}) - \rho_j v_j] + a\lambda[\rho_0 V(\rho_{j+2}) - \rho_0 V(\rho_{j+1})]. \tag{3.3}$$

Where λ is the reaction coefficient of optimal current difference and it's value is less than 1. Certainly, all the melioration in lattice hydrodynamic theory are remarkable but they have the same feature that lattice is divided along the straight road representing the normal road without curved section.

3.1 Model

The traffic flow on a curved road section is different from unidirectional non curved road. Keeping this conception in mind, we consider a situation in which vehicles run on a single-lane curved road under closed boundary conditions, as shown in the Fig 3.1. A central force $f = \mu mg$ acts on the running vehicles. Where μ is the friction coefficient, m is the mass of vehicle and g is the acceleration due to gravity. The total length of the curved section is $S = r\theta$. Where θ is angular distance and r is the radius of curved road section.

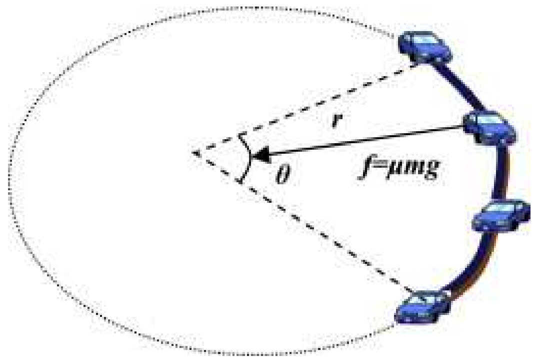

Figure 3.1: Illustration of vehicles running on curved road.

Based on the lattice optimal current difference model, we propose a new mathematical model to analyze the traffic flow on a curved section road as follows:

$$\partial_t \rho_j + \frac{1}{\eta r}(\rho_j \omega_j - \rho_{j-1}\omega_{j-1}) = 0. \tag{3.4}$$

Where η is average angular headway and is in radian ($\frac{1}{\eta r} = \rho_0$) and ω_j is the angular velocity at site j. r has the same meaning as before.

$$\partial_t(\rho_j \omega_j) = a[\rho_0 V(\rho_{j+1}) - \rho_j \omega_j] + a\lambda\rho_0[V(\rho_{j+2}) - V(\rho_{j+1})]. \tag{3.5}$$

Where $V(\rho)$ is of the following form:

$$V(\rho) = \frac{r\omega_{max}}{2}(tanh(\frac{1}{\rho} - h_c) + tanh(h_c)). \tag{3.6}$$

Where ω_{max} is the maximum angular velocity and h_c is the safe arc length.
According to the central force formula, w_{max} is related with friction coefficient

$$m\omega^2 r = \mu mg. \tag{3.7}$$

So the maximum angular velocity is given as follows

$$w_{max} = \sqrt{\frac{\mu g}{r}}. \tag{3.8}$$

The maximum angular velocity is less than the theoretical values in the real traffic situation so we introduce a constant coefficient k ($0 < k \leq 1$ for simplicity $k = 1$) and the optimal velocity function is rewritten as

$$V(\rho) = k\frac{\sqrt{\mu gr}}{2}(tanh(\frac{1}{\rho} - h_c) + tanh(h_c)). \tag{3.9}$$

Clearly, from Eq.3.9 we can see that maximum angular velocity is dependent on friction coefficient and radius of the curved section. The variation of maximum angular velocity with radius and friction coefficient is given below in Table 3.1 and Table 3.2 respectively.

Table 3.1: *Maximum velocity variation with the friction coefficient ($g = 9.8m/s^2$ and $r = 100m$)*

Coefficient μ	0.2	0.4	0.6	0.8	1.0	1.2
Maximum velocity v (m/s^2)	14	20	24	28	31	34

Table 3.2: *Maximum velocity variation with the radii of curvature($g = 9.8m/s^2$ and $\mu = 0.5$)*

Radius r (m)	30	60	90	120	150	180
Maximum velocity v (m/s^2)	12	17	21	24	27	30

Now eliminating ω_j from Eqs 3.3 and 3.5 we get,

$$\partial_t^2\rho_j + a\partial_t\rho_j + a{\rho_0}^2(V(\rho_{j+1}) - V(\rho_j)) + a\lambda{\rho_0}^2(V(\rho_{j+2}) - 2V(\rho_{j+1}) + V(\rho_j)) = 0. \tag{3.10}$$

3.2 Linear Stability

Let us discuss now Linear stability of our new traffic model to investigate the effect of friction coefficient and radius on traffic flow of circular road.
Assume traffic flow is uniform on the curved road, under these conditions density and velocity are taken as ρ_0 and $V(\rho_0)$. Hence, the steady-state solution of the homogeneous uniform traffic flow is given by

$$\rho = \rho_0 \qquad v = V(\rho_0). \tag{3.11}$$

Now we give small perturbation $y_j(t)$ to the steady state on site-j. Then

$$\rho_j(t) = \rho_0 + y_j(t). \tag{3.12}$$

Now we put this perturbed density profile into Eq. 3.10 and we get,

$$\partial_t^2 y_j + a\partial_t y_j + a{\rho_0}^2 V'(\rho_0)(y_{j+1} - y_j) + a\lambda{\rho_0}^2 V'(\rho_0)(y_{j+2} - 2y_{j+1} + y_j) = 0. \tag{3.13}$$

Where $V(\rho_0)$ is $dV(\rho)/d\rho$ at $\rho = \rho_0$.
By expanding $y_j(t)$ in the Fourier modes, i.e., $y_j(t) = Ae^{\iota kj+zt}$, we obtain

$$z^2 + az + a{\rho_0}^2 V'(\rho_0)(e^{\iota k} - 1) + a\lambda{\rho_0}^2(e^{2\iota k}) - 2e^{\iota k} - 1) = 0. \tag{3.14}$$

By expanding $z_1(\iota k) + z_2(\iota k^2) + ...$, we obtain first and second order terms of the coefficients of (ιk) and ιk^2, respectively, as

$$z_1 = -{\rho_0}^2 V'(\rho_0), \tag{3.15}$$

$$z_2 = \frac{(-{\rho_0}^2 V'(\rho_0))^2}{a} + \frac{{\rho_0}^2 V'(\rho_0)}{2} + {\rho_0}^2 V'(\rho_0)\lambda \tag{3.16}$$

If z_2 is a negative value, the uniform steady-state ow becomes unstable for long wavelengths. When z_2 is a positive value, the uniform flow is stable. The neutral stability conditions is given as

$$a = \frac{-2{\rho_0}^2 V'(\rho_0)}{1 + 2\lambda}. \tag{3.17}$$

The stability condition for homogeneous uniform traffic flow is

$$a > \frac{-2{\rho_0}^2 V'(\rho_0)}{1 + 2\lambda}. \tag{3.18}$$

The instability condition for homogeneous uniform traffic flow is

$$a < \frac{-2{\rho_0}^2 V'(\rho_0)}{1 + 2\lambda}. \tag{3.19}$$

Eq.(3.17) clearly shows that friction coefficient μ and radius of curvature of road r plays an important role on stability of traffic flow on a circular road.
The neutral stability lines are plotted in Fig 3.2 for different radii of curvature and friction coefficients of the curved road.
In Fig.3.2 the apex of the neutral stability curve is the critical point (ρ_c, a_c). The areas above the neutral stability lines are stable regions, and below these curves are unstable regions. From the two patterns one can find that with an increase in the radii of curvature and the friction coefficients the stable area decreases. Traffic flow is more stable for a shorter radius of curvature and smaller friction coefficient. Due to the increase of the friction coefficient

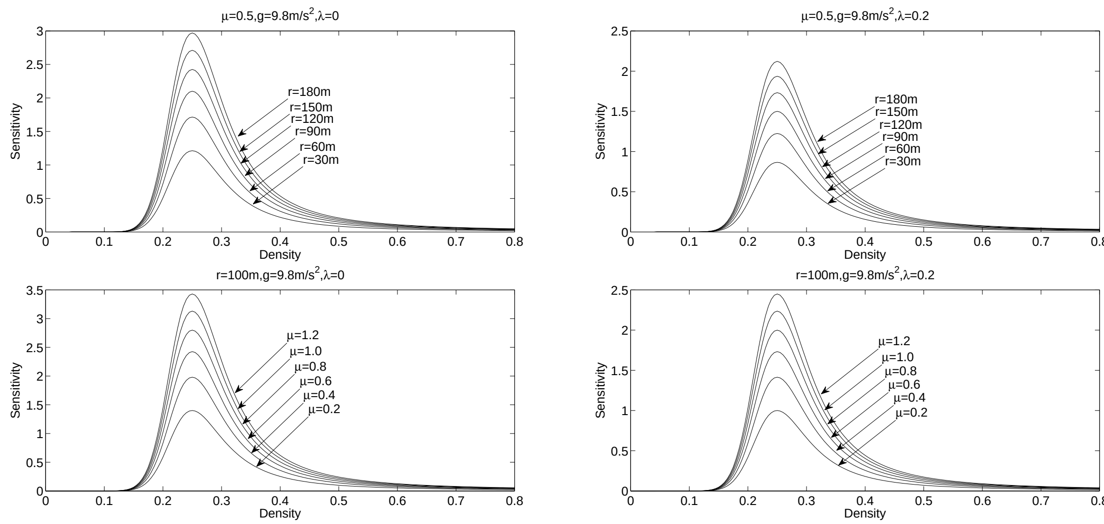

Figure 3.2: The neutral stability lines for different radii of curvature and friction coefficients.

and radii of curvature the maximum velocity becomes higher. The increase of maximum velocity of running vehicles leads to a decreasing stability of the traffic flow.

3.3 Non-Linear Stability Analysis

Define the slow variables X and T to derive the mKdV equation for a small positive scaling parameter near the critical point (ρ_c, a_c) as

$$X = \epsilon(j + bt), \quad T = \epsilon^3 t \tag{3.20}$$

$$\rho_j = \rho_c + \epsilon R(X, T) \tag{3.21}$$

Where b is a constant. By substituting Eq.3.20 and 3.21 in Eq. 3.10 and expanding each term in Eq. 3.10 to the fifth order of ϵ one obtains:

$$\epsilon^2[ab + a{\rho_c}^2V']\partial_X R + \epsilon^3[b^2 + \frac{a{\rho_c}^2V'}{2} + a\lambda{\rho_c}^2V']\partial_X^2 R + \epsilon^4[a\partial_T R + (\frac{a{\rho_c}^2V'}{6} + a\lambda{\rho_c}^2V')\partial_X^3 R + \frac{a{\rho_c}^2V'''}{6}\partial_X R^3] + \epsilon^5[2b\partial_T\partial_X R + (\frac{a{\rho_c}^2V'}{24} + \frac{7a\lambda{\rho_c}^2V'}{12})\partial_X^4 R + (\frac{a{\rho_c}^2V'''}{12} + \frac{a\lambda{\rho_c}^2V'''}{6})\partial_X^2 R^3] = 0 \tag{3.22}$$

Where $V' = dV(\rho)/d(\rho)$ at $\rho = \rho_c$ and $V''' = d^3(\rho)/d\rho^3$ at $\rho = \rho_c$. Near the critical point (ρ_c, a_c), $a_c = a(1 + \epsilon^2)$. By taking $b = -{\rho_c}^2V'$ and eliminating the second order and third order terms of ϵ in Eq. 3.22, one will obtain

$$\epsilon^4[\partial_T R - g_1\partial_X^3 R + g_2\partial_X^{R^3}] + \epsilon^5[g_3\partial_X^2 R + g_4\partial_X^4 R + g_5\partial_X^2 R^3] = 0 \tag{3.23}$$

The coefcients $g_i(i = 1, 2, ...5)$ are exhibited in Table 3.3. By making use of the method described by Ge et al., one will obtain the propagation velocity c for the kinkantikink soliton solution as follows:

$$c = \frac{5g_2g_3}{2g_2g_4 - 3g_1g_5} \tag{3.24}$$

Therefore, the Kink-Antikink solution can be obtained as

$$\rho_j(t) = \rho_c + \sqrt{\frac{g_1c}{g_2}(\frac{\tau}{\tau_c} - 1)}tanh\sqrt{\frac{c}{2}(\frac{\tau}{\tau_c} - 1)}[j + (1 - cg_1(\frac{\tau}{\tau_c} - 1))t] \tag{3.25}$$

Table 3.3: *Coefficient g_i of the model*

g_1	g_2	g_3	g_4	g_5
$-\frac{\rho_c{}^2V'}{6}(1+6\lambda)$	$\frac{\rho_c{}^2V'''}{6}$	$-\frac{\rho_c{}^2}{6}(1+2\lambda)$	$\frac{\rho_c{}^2V'}{24}(1+14\lambda) - \frac{b\rho_c{}^2V'}{3a_c}(1+6\lambda)$	$\frac{\rho_c{}^2V'''}{12}(1+2\lambda-\frac{4b}{a_c})$

Thus, the amplitude A of kink-antikink solitin solution can be derived as

$$A = \sqrt{\frac{g_1c}{g_2}(\frac{\tau}{\tau_c} - 1)} \tag{3.26}$$

The kinkantikink soliton solution represents the coexisting phase consisting of the freely moving phase at low density and the jammed (or congested) phase at high density, which can be described by $\rho_j = \rho_c + A$ in the space (ρ, a) respectively.

3.4 Numerical Simulation

A series of simulation is carried out under periodic boundary conditions for the new model to check whether the proposed model is capable of describing the effect of friction coefficient and radius on circular road traffic flow dynamics. The total number of vehicles taken are 200. The friction coefficients and radii of curvature are taken as follows $\mu = 0.2, 0.4, 0.6, 0.8$ and $r = 30, 60, 90, 120m$ respectively. The sensitivity is taken $a = 1.5$. Time step is taken $\Delta t = 1s$. The duration of traffic flow is long enough for traffic flow to evolve and reach the steady state.

The results of simulation are plotted in Figs. 3.3-3.10. Fig. 3.3, 3.4,3.7 and 3.8 shows the density profile and waves for different friction coefficient and different values of λ at $t = 18800s$. Fig. 3.5,3.6,3.9 and 3.10 shows the density profile and waves for different radii of curvature and different values of λ. From these figures we can clearly see that amplitude of density wave increases with an increase of friction coefficient and radii of curvature. But increase in λ stabilizes the density profile and decrease the amplitude of density waves. The numerical results are in good agreement with the results we get analytically.

3.5 Conclusion

Vehicles running on a curved road have different behaviors from those on a normal road. We investigate the characteristics of traffic flow on a curved road analytically and numerically. The maximum flux of the traffic flow increases with an increase of the radii of curvature and friction coefficients. But the stability decreases with an increase of these two parameters. Although increase in the value of λ stabilizes the traffic flow. Numerical results show that the analytical results are in good agreement with the simulation result.

3.6 Results

Numerical simulation results are as follows:

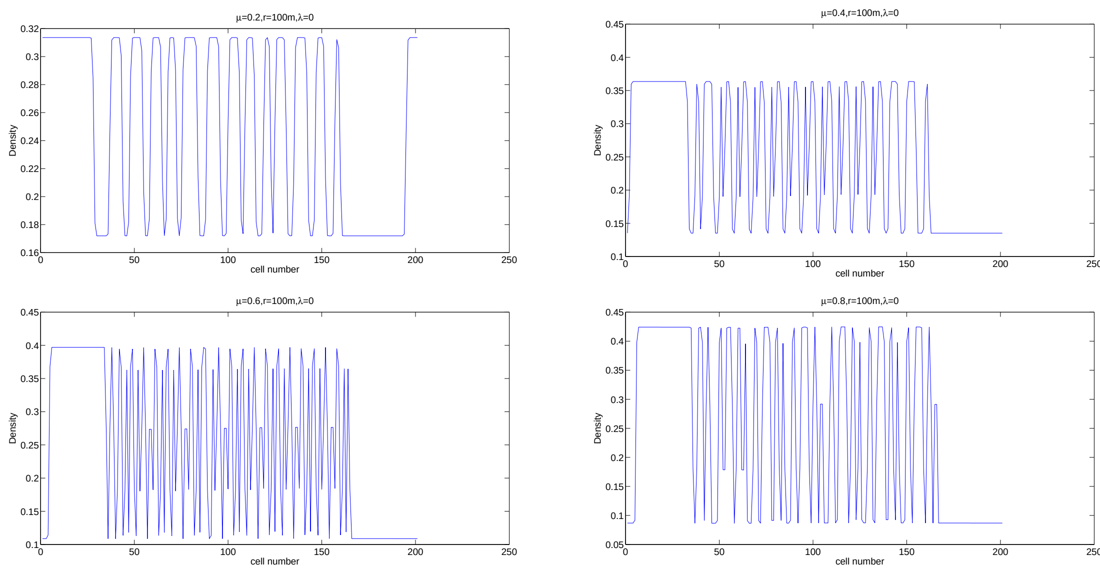

Figure 3.3: Density profiles for different friction coefficients at $t = 18800s$ when $\lambda = 0$

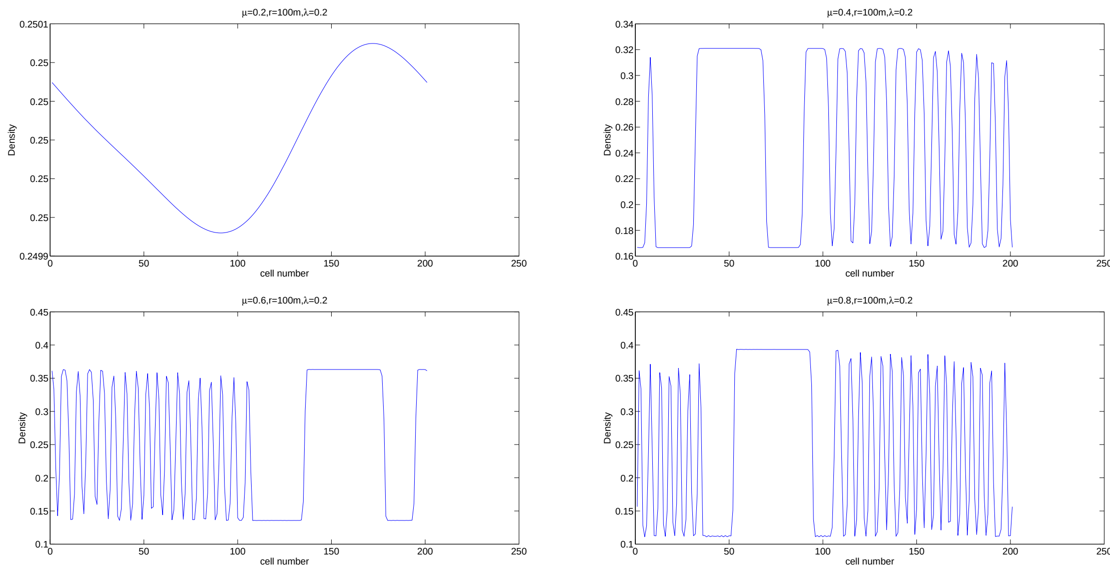

Figure 3.4: Density profiles for different friction coefficients at $t = 18800s$ when $\lambda = 0.2$

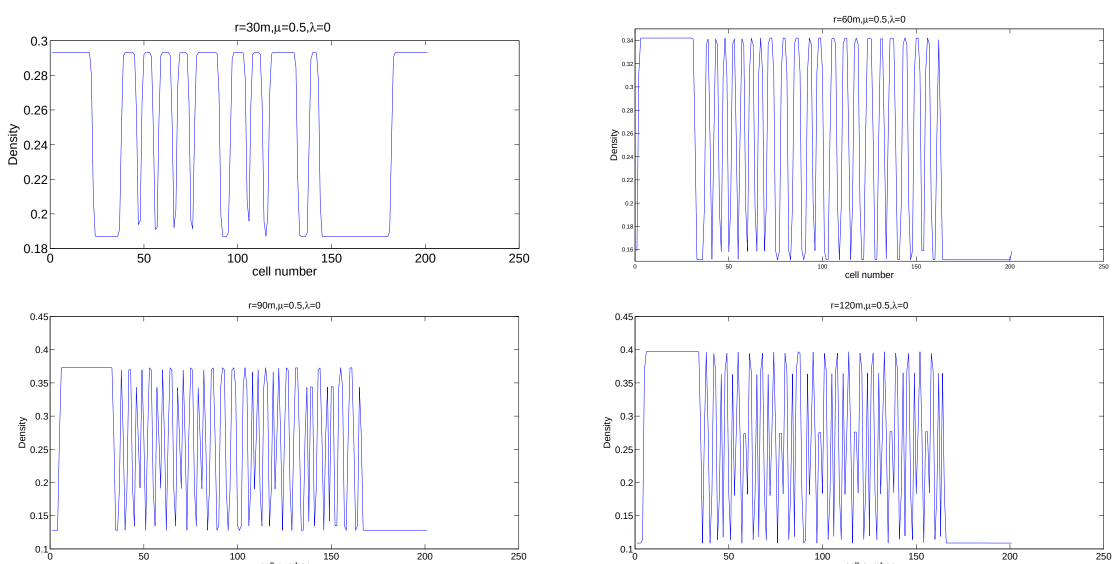

Figure 3.5: Density profiles for different radii of curvature at $t = 18800s$ when $\lambda = 0$

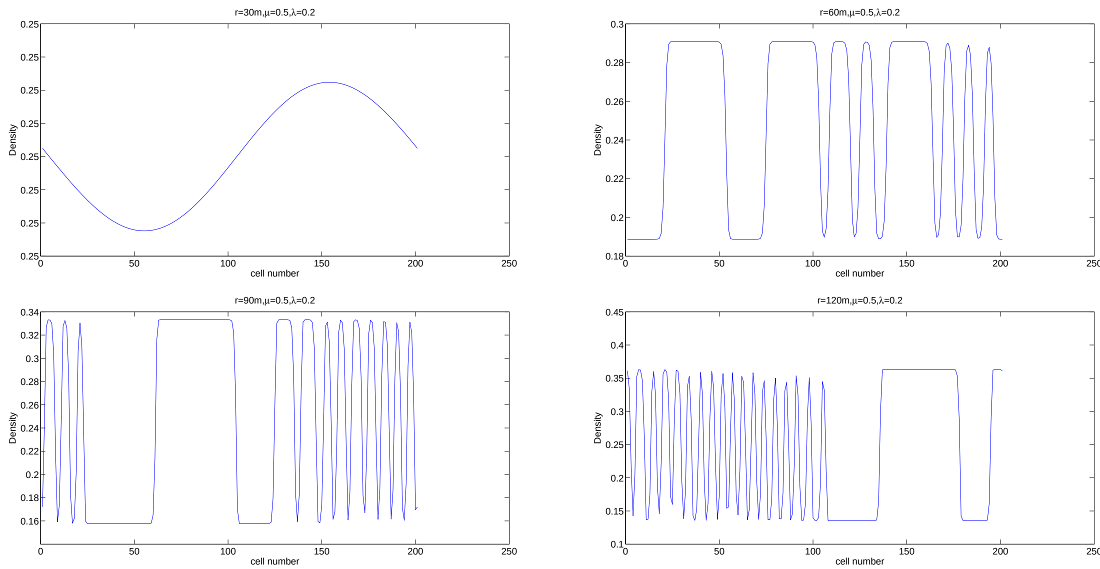

Figure 3.6: Density profiles for different radii of curvature at $t = 18800s$ when $\lambda = 0.2$

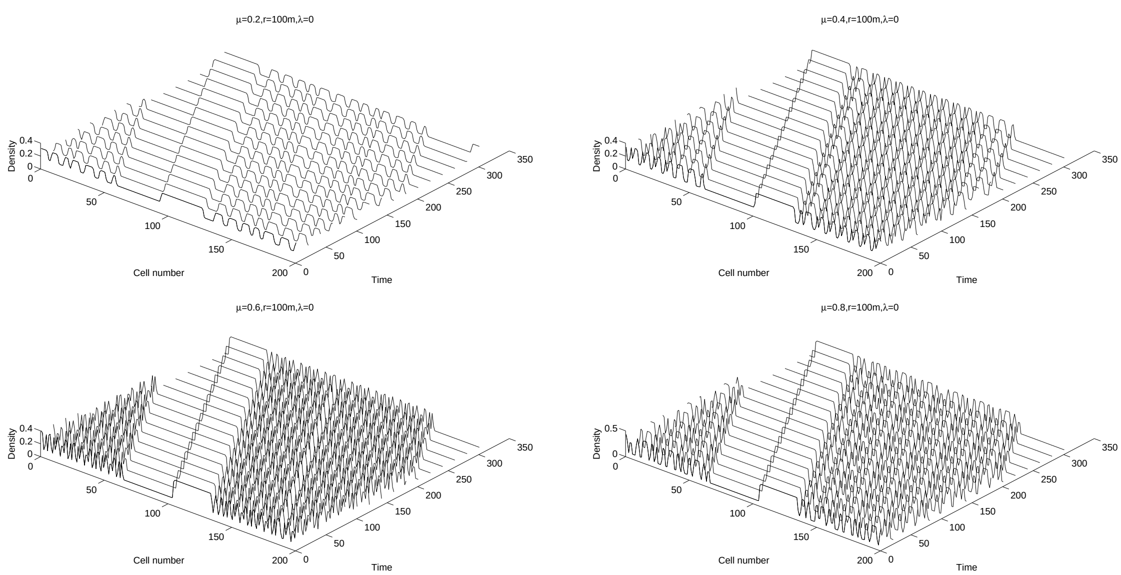

Figure 3.7: Density waves for different friction coefficient $t = 18800s$ when $\lambda = 0$

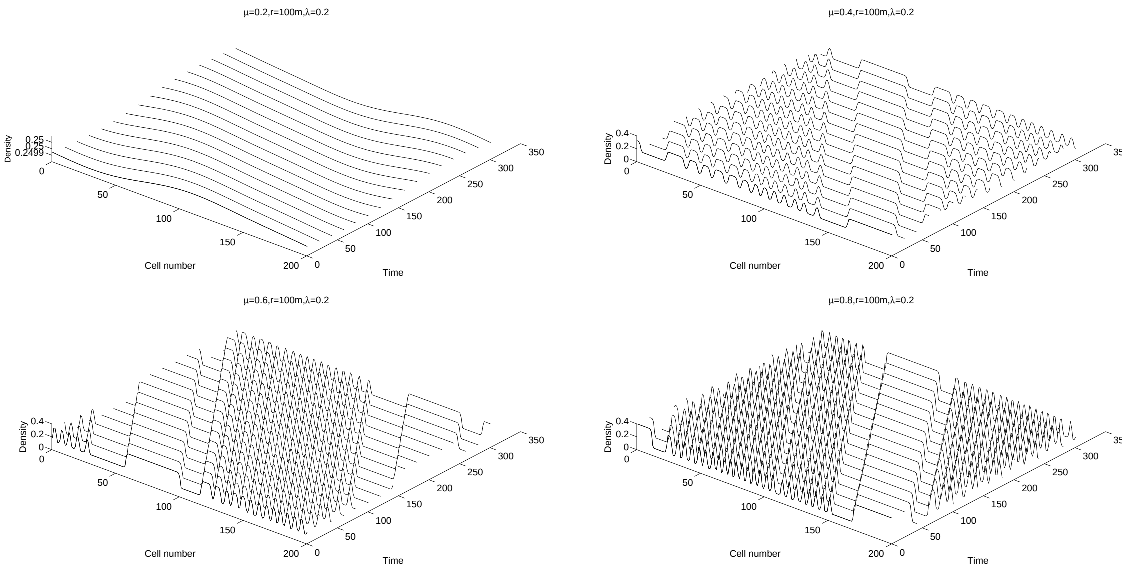

Figure 3.8: Density waves for different friction coefficient $t = 18800s$ when $\lambda = 0.2$

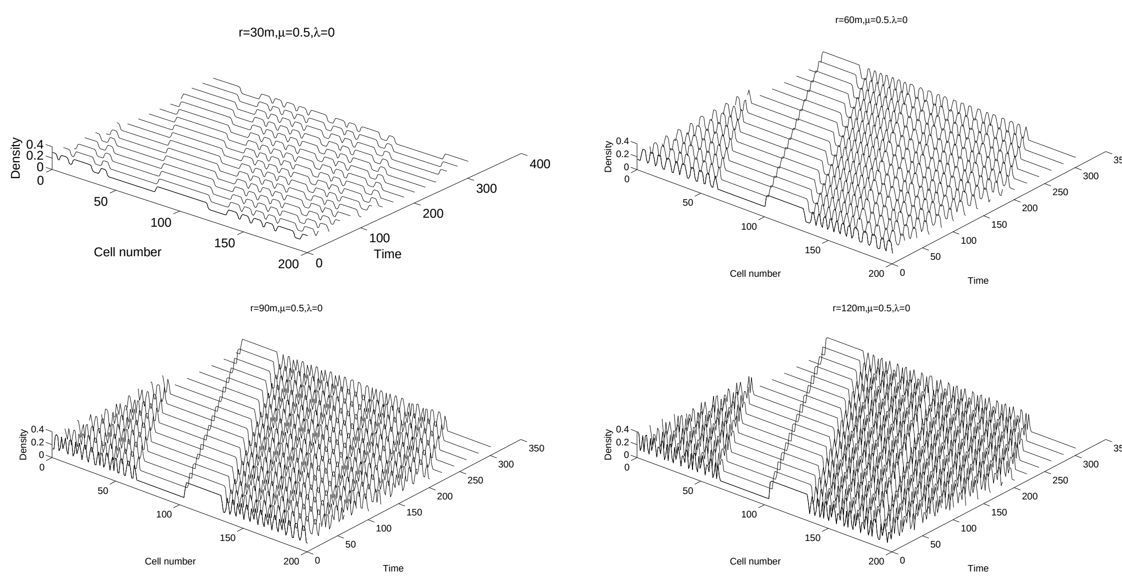

Figure 3.9: Density waves for different radii of curvature at $t = 18800s$ when $\lambda = 0$

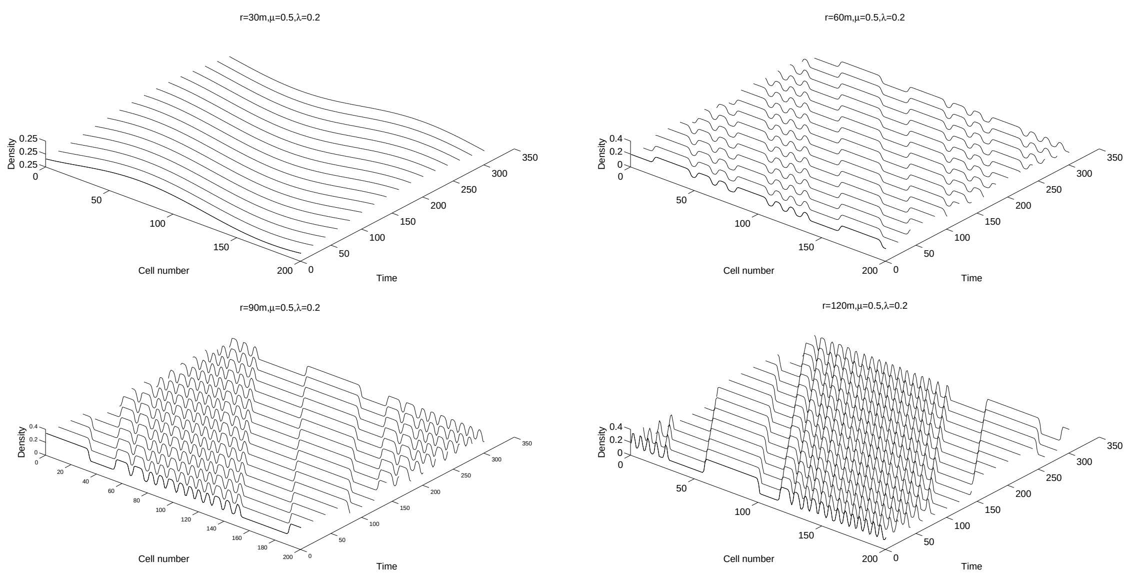

Figure 3.10: Density waves for different radii of curvature at $t = 18800s$ when $\lambda = 0.2$

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