Derived categories in Algebraic Geometry and Motivic decomposition

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1 Introduction

This thesis is divided into three parts. In the first part I start with the notion of divisors and line bundles, where we work with codimension 1 subvarieties of a given scheme X. In the next section I generalize the notion of divisors to work with varieties of arbitrary codimension. I calculate various class groups, Chow groups which are important invariants of a scheme, we define the intersection product and at the end of this section we introduce intersection numbers on a surface. With the help of this we give an easy proof of the Bezout's theorem, prove the Riemann-Roch theorem on surfaces and calculate the intersection numbers of a few general types of divisors (curves). The last part of this thesis is dedicated to the study of Chow motives, in particular we are interested in various decompositions of the Chow motive of a space. This ends with proving the Chow-Kunneth decomposition of the motive of an Abelian variety. For this we give a brief overview of the theory of Abelian varieties in the first part. With the help of the theorem of Cube and the Seesaw theorem, I study line bundles on an abelian variety. We also give a sketch of the construction of the dual abelian variety. While constructing the dual $\hat{X}$ of an abelian variety X we define a line bundle on $X \times \hat{X}$, which plays a significant role in the theory of Fourier-Mukai transforms in derived categories. This is what the second part of this thesis is concerned with. In this part, I study the derived categories of Coherent sheaves thoroughly and give a few applications. I prove Serre duality in the derived category realm, study the derived category of a smooth projective curve, give a set of generators of $\mathrm{D}^b(\mathrm{Coh}(\mathbb{P}^1))$. Next I introduce Fourier-Mukai transforms and as an application prove that the derived categories of an abelian variety and its dual are equivalent. After this I define the Chow motives, give a few basic calculations and study the motivic decompositions of a few simple spaces. The thesis ends with the proof of the existence of Chow-Kunneth decomposition of an abelian variety.

Part I: Results from Algebraic Geometry

2 Divisors and Line bundles

In this section, I am going to give a quick review of the basics of divisors. I will introduce Weil and Cartier divisors, and discuss the correspondences between Weil divisors, Cartier divisors and line bundles. At the end of this section I will compute the divisor class groups of a few important schemes using some useful exact sequences.

2.1 Weil and Cartier divisors

Weil divisors: Weil divisors are the easiest to explain. On a scheme X of dimension n, a *Weil divisor* is just a formal sum of codimension 1 closed subvarieties of X. Thus the group of Weil divisors, $\operatorname{div} X$ is the free abelian group generated by the codimension one subvarieties of X over $\mathbb{Z}$. A Weil divisor $D \in \operatorname{div} X$ has the form

$$D = \sum_{i=0}^{k} n_i V_i,$$

where V_i are closed irreducible subsets of X. If $n_i \geq 0$, for all $i = 0, \ldots, k$, we say the divisor D is effective.

A *prime divisor* on X is a closed irreducible subset $Y \subset X$ of codimension one. Equivalently, a prime divisor is an integral closed subscheme of codimension one. Now we will associate a divisor to a rational function over X. For this we need X to be a nice enough scheme.

What do we mean by 'nice enough'? We will call X 'nice' if: X is Noetherian integral separated scheme which is regular in codimension one.

We say a scheme X is *regular in codimension one* if every local ring $\mathcal{O}_{X,x}$ of X of dimension one is regular.

If Y is a prime divisor on X, let $\eta \in Y$ be its generic point. Then $\dim \mathcal{O}_{X,\eta} = \operatorname{codim}(Y, X) = 1$, so $\mathcal{O}_{X,\eta}$ is a discrete valuation ring. We call the corresponding discrete valuation v_Y on K the valuation of Y. If $f \in K^*$ is a nonzero rational function on X then $v_Y(f)$ is an integer. If it is positive, we say that f has a zero along Y of that order; if it is negative, we say f has a pole along Y, of order $-v_Y(f)$.

Proposition 1. *Let X be a nice scheme, and let $f \in K(X)$, the field of rational functions on X. Then $v_Y(f) = 0$ for all but finitely many prime divisors Y.*

Proof. (Sketch) Let $U = \operatorname{Spec} A$ be an affine open subset of X. If $f|_U = \frac{g}{h}$, where $g, h \in A$, we see that f restricts to a unit on the complement of the closed subset $Z = (X - U) \cup V(g) \cup V(h)$ in X. Since X is Noetherian, Z contains only finitely many irreducible components, and all of them of codimension ≥ 1. This tells us that there are only finitely many prime divisors of X contained in Z. If Y is a prime divisor of X, and $Y \cap Z = \phi$, f restricts to a unit on D, and thus $v_D(f) = 0$. Hence the sum

$$(f) = \sum_{Y \in \mathrm{PD}(X)} v_Y(f) \cdot Y,$$

is well-defined. This is called the *principal divisor* associated to f. (Here $\mathrm{PD}(X)$ is the set of all prime divisors of X.) □

Note that if $f, g \in K^*$ then $(f/g) = (f) - (g)$ because of the properties of valuations. Therefore sending a function f to its divisor (f) gives a homomorphism of multiplicative group K^* to the additive group $\operatorname{div} X$, and the image, which consists of the principal divisors, is a subgroup of $\operatorname{div} X$.

Definition 2.1. *Let X be a nice scheme. Two divisors D and D' are said to be linearly equivalent, written $D \sim D'$, if $D - D'$ is a principal divisor. The group $\operatorname{div} X$ of all divisors divided by the subgroup of principal divisors is called the divisor class group of X and is denoted $\mathrm{Cl}X$.*

The following two easy propositions will be helpful in calculating the valuation.

Proposition 2. *Let $A = k[x_1, \ldots, x_n]$ for a field k and $n \geq 1$. Let f be an irreducible polynomial and $\mathfrak{p} = (f)$. The discrete valuation on K with valuation ring A_p is defined for nonzero $g, h \in A$ by $v_p(g/h) = v_f(g) - v_f(h)$.*

Proposition 3. *Let $S = k[x_0, \ldots, x_n]$ for a field k and $n \geq 1$. Let f be an irreducible homogeneous polynomial and $p = (f)$. The discrete valuation on the field $S_{((0))}$ with valuation ring $S_{(p)}$ is defined for nonzero homogeneous $g, h \in S$ of the same degree by $v_{(p)}(g/h) = v_f(g) - v_f(h)$.*

Example 2.1. *Let k be a field. Let us compute* $\operatorname{div}(x^3/(x+1))$ *on* A_k^1*. Using the previous proposition to calculate the valuation at the prime ideal* $(x+1)$*, we get that the valuation at* $x=-1$ *is* -1*. Similarly at* $x=0$*, the valuation is* 3*. In particular, the divisor is* $3[0]-[-1]$*.*

Example 2.2. *Now Let us compute the divisor of the rational section* $X^2/(X+Y)$ *on* $\mathbb{P}_k^1$*. There are two basic open sets* $D(X)$ *and* $D(Y)$*. On the former, the trivialization is by dividing by* X*, so we get* $X/(X+Y)=1/(1+(X/Y))$*, that is when we restrict the rational function on the affine pacth* $D(X)$*, we get the function* $X/(X+Y)$*. Then again using the previous proposition, we see that the valuation at* $X/Y=-1$ *is* -1*, in other words, it has a simple pole at* $X/Y=-1$*. So at the point* $[-1:1]$*, there is a simple pole of this divisor.*
The second trivialization is by dividing by Y*, in which case we get* $X^2/((X+Y)Y)$*, which has a zero of order 2 at* $X=0$*. So the total divisor is*

$$2[0:1]-[-1:1].$$

Cartier divisors: To define Cartier divisors on a scheme X, we will not be needing X to be nice. That is, a Cartier divisor can be defined on any Scheme. Thus when X is nice, we can define both Weil and Cartier divisor. One may ask, if there is some relationship between them, if given one can we go to the other. As we will see Cartier divisors are just locally principal Weil divisors. From any Cartier divisor we can define a Weil divisor, but for the other way we need X to be locally factorial. We will give an example when this correspondence fails.

Definition 2.2. *Let* X *be a scheme. For each open affine subset* $U=\operatorname{Spec} A$*, let* S *be the set of elements of* A *which are not zero divisors, and let* $K(U)$ *be the localization of* A *by the multiplicative system* S*. We call* $K(U)$ *the total quotient ring of* A*. For each open set* U*, let* $S(U)$ *denote the set of elements of* $\Gamma(U,\mathcal{O}_X)$ *which are not zero divisors in each local ring* $\mathcal{O}_x$ *for* $x\in U$*. Then the rings* $S(U)^{-1}\Gamma(U,\mathcal{O}_X)$ *form a presheaf, whose associated sheaf of rings* $\mathscr{K}$ *we call the sheaf of total quotient rings of* $\mathcal{C}$*. On an arbitrary scheme, the sheaf* $\mathscr{K}$ *replaces the concept of function field of an integral scheme. We denote by* $\mathscr{K}^*$ *the sheaf (of multiplicative groups) of invertible elements in the sheaf of rings* $\mathscr{K}$*. Similarly* $\mathcal{O}^*$ *is the sheaf of invertible elements in* $\mathcal{O}_X$*.*

Definition 2.3. *A Cartier dicisor on a scheme X is a global section of the sheaf $\mathscr{K}^*/\mathcal{O}^*$. Equivalently, we see that a Cartier divisor on X can be described by giving an open cover $\{U_i\}$ of X, and for each i an element $f_i \in \Gamma(U_i, \mathscr{K}^*)$, such that for each $i, j, f_i/f_j \in \Gamma(U_i \cap U_j, \mathcal{O}^*)$. A Cartier divisor is principal if it is in the image of the natural map $\Gamma(X, \mathscr{K}^*) \to \Gamma(X, \mathscr{K}^*/\mathcal{O}^*)$, that is given by the cover $\{(U, f)\}$. Two Cartier divisors are linearly equivalent if their difference is principal. The group of Cartier divisors modulo linear equivalence is denoted $CaCl(X)$.*

We now describe the relationship between the two notions of divisors. We will merely talk about the maps between $\operatorname{div} X$ and $\mathrm{CaDiv}X$, for the details see Proposition 6.11,[Hart].

Proposition 4. *Let X be an integral, separated Noetherian scheme, all of whose local rings are unique factorization domains (in which case we say X is locally factorial). Then the group Div X of Weil divisors on X is isomorphic to the group of Cartier divisors $\Gamma(X, \mathscr{K}^*/\mathcal{O}^*)$, and furthermore, the principal Weil divisors correspond to the principal Cartier divisors under this isomorphism.*

Proof. (Sketch) Since X is nice, we can talk about Weil divisors. Now let we have a Cartier divisors $\{(U_i, f_i)\}$. For a prime divisor $Y \in \mathrm{PD}(X)$, we take the coefficient of Y to be $v_Y(f_i)$, where i is any index such that $Y \cap U_i = \phi$. This is well-defined, because for some other such j, $f_i/f_j \in \mathcal{O}^*(U_i \cap U_j)$, thus $v_Y(f_i/f_j) = v_Y(f_i) - v_Y(f_j) = 0$. The sum is finite again the same way as before, since X is Noetherian.

Conversely, for $D \in \operatorname{div} X$, D_x is a Weil divisor of $\operatorname{Spec} \mathcal{O}_X, x$ for any point $x \in X$. Now since X is locally factorial $\mathcal{O}_X, x$ is a UFD, and hence $Cl(\operatorname{Spec} \mathcal{O}_X, x) = 0$. This implies that $D_x = (f_x)$, for some $f_x \in K(X)$. We get a Cartier divisor $\{(U_x, f_x)\}$ since such $\{U_x\}$ cover X.

This two constructions are inverses to each other and its clear from the construction that the principal divisors correspond to each other. Hence we have got the following:

$$\mathrm{Cl}X \cong \mathrm{CaCl}(X).$$

$\square$

In the last section I will give examples of Weil divisors which are not Cartier. From the proof we can say that for this to happen we need to find a scheme which is not locally factorial, because only then we will be able to get a divisor that is not locally principal. In the next section I will explain the correspondence between Cartier divisors and line bundles. For Integral schemes we will get isomorphism between $\mathrm{Cl}X$ and $\mathrm{Pic}(X)$. As a consequence in the last section we will prove that the only line bundles over $\mathbb{P}^n$ are $\mathcal{O}(l)$ for some $l \in \mathbb{Z}$.

2.2 Line bundles and Pic$X$

The set of all line bundles $\mathcal{L}$ over a scheme X form a group under the tensor product operation. The group is called the Picard group of X, Pic$X$.

Given a Cartier divisor $D = \{(U_i, f_i)\}$ we define a subsheaf $\mathcal{L}(D)$ of the sheaf of total quotient rings $\mathscr{K}$ by taking $\mathcal{L}(D)$ to be the sub-$\mathcal{O}_X$ module of $\mathscr{K}$ generated by f_i^{-1} on U_i. This is well-defined and $\mathcal{L}(D)$ is called the sheaf associated to D.

Proposition 5. *Let X be a scheme. Then:*
(a) for any Cartier divisor $D, \mathscr{L}(D)$ is an invertible sheaf on X. The map $D \mapsto \mathscr{L}(D)$ gives a 1-1 *correspondence between Cartier divisors on X and invertible subsheaves of $\mathscr{K}$;*
(b) $\mathscr{L}(D_1 - D_2) \cong \mathscr{L}(D_1) \otimes \mathscr{L}(D_2)^{-1}$;
(c) $D_1 \sim D_2$ if and only if $\mathscr{L}(D_1) \cong \mathscr{L}(D_2)$ as invertible sheaves.

Proof. (a) Clear.
(b) $\mathscr{L}(D_1 - D_2)$ is locally generated by $f_i^{-1}g_i$, so $\mathscr{L}(D_1 - D_2) \cong \mathscr{L}(D_1) \otimes \mathscr{L}(D_2)^{-1}$.
(c) If D is principal, then $\mathscr{L}(D)$ is globally generated by f^{-1}, and thus $1 \mapsto f^{-1}$ gives an isomorphism $\mathcal{O}_X \cong \mathscr{L}(D)$. Conversely, given such an isomorphism, we can take the image of 1 to be the global generator of D. Using (b), we see that $D = D_1 - D_2$ is principal if and only if $\mathscr{L}(D) \cong \mathcal{O}_X$. □

Corollary 2.1. *On any scheme X, the map $D \mapsto \mathscr{L}(D)$ gives an injective homomorphism of the group* $\mathrm{CaCl}X$ *of Cartier ditisors modulo linear equivalence to Pic X.*

Proposition 6. *If X is an integral scheme, CaCl $X \cong$ Pic$X$.*

Proof. Only surjectivity is remaining, that is we need to show that any invertible sheaf is isomorphic to a subsheaf of $\mathcal{K}$, which is the constant sheaf K in this case, where $K = K(X)$ is the function field of X. Let $\mathcal{L} \in \mathrm{Pic}(X)$. Then there is an open covering $\{U_i\}$ of X, such that $\mathcal{L}|_{U_i} = \mathcal{O}_X$. Then $(\mathcal{L} \otimes \mathcal{K})|_{U_i} \cong \mathcal{K}|U_i$. Since X is irreducible, this shows that $\mathcal{L} \otimes \mathcal{K}$ is isomorphic to the constant sheaf $\mathcal{K}$, and via the map $\mathcal{L} \to \mathcal{L} \otimes \mathcal{K} \cong \mathcal{K}$, we can view $\mathcal{L}$ as a subsheaf of $\mathcal{K}$. □

Corollary 2.2. *If X is a noetherian, integral, separated locally factorial scheme, then there is a natural isomorphism*

$$Cl(X) \cong CaCl(X) \cong Pic(X).$$

2.3 Computations

We shall now calculate a few class groups, and gives examples where the previously defined correspondences fail.

Proposition 7. *Let X be the projective space $\mathbb{P}^n_k$ over a field $k(n \geq 1)$. Let H be the hyperplane $x_0 = 0$. Then:*

(a) If D is any divisor of degree d, then $D \sim dH$;

(b) For any $f \in K^*$, $\deg(f) = 0$;

(c) The degree function gives an isomorphism of abelian groups $\deg : Cl(X) \longrightarrow \mathbb{Z}$.

Proof. Let $S = k[x_0, \ldots, x_n]$ and K the function field $\mathcal{O}_{X,0}$ of X. If $f \in K^*$ then f corresponds to a quotient g/h of two nonzero homogenous polynomials $g, h \in S$ of the same degree. If we factor g, h as $g = up_1^{n_1} \cdots p_r^{n_r}$ and $h = vp_1^{m_1} \cdots p_r^{m_r}$ for $u, v \in k$ and irreducible polynomials p_i, then the p_i must be homogenous (we allow some zero indices to get the occurring p_i the same in both cases) and by Proposition 11 the principal divisor (f) is defined by

$$(f) = \sum_{i=1}^{r} (n_i - m_i) \cdot Y_i, \quad Y_i = V(p_i)$$

Hence

$$\deg(f) = \sum_{i=1}^{r} (n_i - m_i) \deg(p_i) = \sum_{i=1}^{r} n_i \deg(p_i) - \sum_{i=1}^{r} m_i \deg(p_i) = \deg(g) - \deg(h) = 0$$

Which proves (b). To prove (a), let $D = \sum_{i=1}^{r} n_i \cdot D(p_i)$ be any nonzero effective divisor of degree d with $n_i > 0$ and the p_i homogenous irreducibles. Then $p_1^{n_1} \cdots p_r^{n_r}/x_0^d \in S_{((0))}$ and the corresponding principal divisor is $D - dH$, where $H = V(x_0)$, which shows that $D \sim dH$. It follows immediately that any divisor of degree zero is principal.

Taking degrees defines a morphism of abelian groups DivX $\longrightarrow \mathbb{Z}$, and we have just shown the kernel of this map consists of the principal divisors. Since $\deg(dH) = d$ for any $d \in \mathbb{Z}$ we obtain the required isomorphism $ClX \longrightarrow \mathbb{Z}$. $\square$

Proposition 8. *Let X be a nice scheme and let U be a nonempty open subset of X, and let $Z = X \backslash U$. Then:*

(a) There is a surjective morphism of groups $Cl(X) \longrightarrow Cl(X)$ defined by $\sum_i n_i \cdot Y_i \mapsto \sum_i n_i \cdot (Y_i \cap U)$ where we ignore those $Y_i \cap U$ which are empty;

(b) If $\operatorname{codim}(Z, X) \geq 2$ then $ClX \longrightarrow ClU$ is an isomorphism;

(c) If Z is an irreducible subset of codimension 1, then there is an exact sequence of abelian groups

$$\mathbb{Z} \longrightarrow Cl(X) \longrightarrow Cl(U) \longrightarrow 0$$

where the first map is defined by $1 \mapsto [Z]$.

Proof. (Sketch) (a) If X is nice, then so is U, so $\mathrm{Cl}(U)$ is defined. We have bijection between the prime divisors of X meeting U and the prime divisors of U given by $Y \mapsto Y \cap U$, and $\overline{Z} \leftarrow Z$. Since Z is a proper closed subset of the noetherian space X, we can write it as a union $Z_1 \cup \ldots Z_n$ of irreducible components, so there is only a finite number of prime divisors of X *not* meeting U. Hence the map $\varphi : \operatorname{div}(X) \to \operatorname{div}(U)$ is well-defined and surjective, and also sends principal divisors to principal divisors, thus extends to the surjection $\mathrm{Cl}(X) \to \mathrm{Cl}(U)$.
(b) This is clear since this inequality ensures that every prime divisor of X must meet U, so there is a bijection between the prime divisors of U and X, which proves that $\mathrm{Cl}(X) \longrightarrow \mathrm{Cl}(U)$ is an isomorphism.
(c) The surjectivity of the last map follows from (a). Since the map $\mathrm{Cl}(X) \longrightarrow \mathrm{Cl}(U)$ is given by $Y \mapsto Y \cap U$, the composition is clearly zero, because Z restricts to the trivial divisor on $\mathrm{Cl}(U)$. Conversely, suppose $Y \cap U = \operatorname{div}(f)$, where f is a rational function on U, then f is also a rational function on its closure Y, which shows that $Y = n[Z] + \operatorname{div}(f)$ in $\mathrm{Cl}(X)$.

This proves that Y is in the image of the first map, and hence the sequence is exact. $\square$

Example 2.3. *Given an irreducible curve Y of degree d in $\mathbb{P}^2$, using the above propositions, we have the following exact sequence*

$$\mathbb{Z} \longrightarrow Cl(X) \cong \mathbb{Z} \longrightarrow Cl(U) \longrightarrow 0,$$

where the first map is given by $1 \mapsto d$, this is because by proposition 7, via the isomorphism $\deg : Cl(\mathbb{P}^2) \to \mathbb{Z}$, *$Y$ corresponds to d in $\mathbb{Z}$. This proves that $Cl(\mathbb{P}^2 - Y) \cong \mathbb{Z}/d\mathbb{Z}$.*

Proposition 9. *Any invertible sheaf on $X = \mathbb{P}^n$ is isomorphic to some $\mathcal{O}(l)$, where $l \in \mathbb{Z}$.*

Proof. We know that $\mathrm{Cl}(X) \cong \mathrm{Pic}(X) \cong \mathbb{Z}$. By proposition 7, we see that the generator of $\mathrm{Cl}(X)$ is the hyperplane $H : x_0 = 0$ (there is nothing special about x_0, as a matter of fact this can be any x_i). Via the isomorphism of proposition 6, this corresponds to $\mathcal{O}(1)$. Hence $\mathrm{Pic}(X)$ is the free group generated by $\mathcal{O}(1)$, and thus any invertible sheaf on X is isomorphic to some $\mathcal{O}(l)$, where $l \in \mathbb{Z}$. $\square$

Proposition 10. *Let A be a Noetherian domain. Then A is a UFD if and only if $X = \operatorname{Spec} A$ is normal and $Cl(X) = 0$.*

Proof. See [4, Proposition 6.2]. $\square$

Example 2.4. *$Cl(\mathbb{A}^n) = Pic(\mathbb{A}^n) = 0$.*

Proposition 11. *Let X be a nice scheme. Then $Cl(X \times \mathbb{A}^n) \cong Cl(X)$.*

Proof. The map is just pull-back

$$\begin{aligned} \pi : \mathrm{Cl}(X) &\to \mathrm{Cl}(X \times \mathbb{A}^n) \\ D = \sum n_i Y_i &\mapsto \mathrm{pr}_1^* D = \sum n_i \mathrm{pr}_1^{-1}(Y_i). \end{aligned}$$

See [4, Proposition 6.6] for details. $\square$

Example 2.5. *Let Q be the nonsingular quadric surface $xy = zw$ in $\mathbb{P}^3_k$. We will show that $Cl(Q) \cong \mathbb{Z} \oplus \mathbb{Z}$. We use the fact that Q is isomorphic to $\mathbb{P}^1 \times_k \mathbb{P}^1$. Let p_1 and p_2 be the projections of Q onto the two factors.*

Then we have the pull-backs $p_1^*, p_2^* : Cl\mathbb{P}^1 \to Cl(Q)$. *First we show that* p_1^* *and* p_2^* *are injective. Let* $Y = \{0\} \times \mathbb{P}^1$. *Then* $Q - Y = \mathbf{A}^1 \times \mathbf{P}^1$, *and the composition*

$$Cl(\mathbb{P}^1) \xrightarrow{p_2^*} Cl(Q) \to Cl\left(\mathbb{A}^1 \times \mathbb{P}^1\right)$$

is isomorphism. Hence p_2^* *(and similarly* p_1^* *) is injective. Now consider the exact sequence for* Y *:*

$$\mathbf{Z} \to \mathrm{Cl}Q \to \mathrm{Cl}\left(\mathbf{A}^1 \times \mathbf{P}^1\right) \to 0.$$

$1 \mapsto Y$. *But if we identify* $Cl\mathbb{P}^1$ *with* $\mathbb{Z}$ *by letting* 1 *be the class of the point* $\{0\}$, *then this first map is just* p_i^*, *hence is injective. Since the image of* p_2^* *is isomorphic to* $Cl(\mathbb{A}^1 \times \mathbb{P}^1)$ *as we have just seen, we conclude that this sequence splits and hence* $Cl(Q) \cong \mathrm{Im}\, p_1^* \oplus \mathrm{Im}\, p_2^* = \mathbf{Z} \oplus \mathbf{Z}$. *If* D *is any divisor on* Q, *let* (a, b) *be the ordered pair of integers in* $\mathbf{Z} \oplus \mathbf{Z}$ *corresponding to the class of* D *under this isomorphism. Then we say* D *is of type* (a, b) *on* Q.

A more visual proof goes like this: Suppose $h_1 = \{0\} \times \mathbb{P}^1$, *and* $h_2 = \mathbb{P}^1 \times \{0\}$, *then* $U = X - h_1 - h_2 \cong \mathbb{A}^2$. *Suppose* $D \in \operatorname{div} X$, *then* $D|_U = \operatorname{div}(\varphi)$, *where* φ *is a rational function on* $\mathbb{A}^2$. *This shows that*

$$D = \operatorname{div}(\varphi) + m_1 h_1 + m_2 h_2 \implies D \sim m_1 h_1 + m_2 h_2.$$

Thus h_1, h_2 *generate* $Cl(X)$ *and any divisor is of type* (m, n). *This shows that* $Cl(X) \cong \mathbb{Z} \oplus \mathbb{Z}$.

Example 2.6. *This gives us a proof of the fact that* $\mathbb{P}^1 \times \mathbb{P}^1 \neq \mathbb{P}^2$, *since* $Cl(\mathbb{P}^1 \times \mathbb{P}^1) \cong \mathbb{Z}$, *but by the previous proposition* $Cl(\mathbb{P}^2) \cong \mathbb{Z} \oplus \mathbb{Z}$.

2.3.1 Counterexamples

Example 2.7. *Let* $X = \{x_1 x_2 = 0\} \subset \mathbb{P}^2$. X *is the union of two projective lines that meet in a point,* $X = X_1 \cup X_2$. *Any divisor on* X *is just sum of points. Since any two points on* X_1 *and* X_2 *can be joined by a line, any two points on* X *can also be joined by a line, a line through the intersecting point. This shows that given two points* $P_1, P_2 \in C$, *there exists a rational function on* f *on* X, *such that* $[P_1] - [P_2] = \operatorname{div} f$, $\implies$ $[P_1] = [P_2]$. *Hence* $Cl(X)$ *is generated by any class of point* $[P]$. *Thus* $Cl(X) \cong \mathbb{Z}$.

Now let $P_1 \in X_1 \backslash X_2$ and $P_2 \in X_2 \backslash X_1$ be two points. Note that the line bundles $O_X(P_1)$ and $O_X(P_2)$ (defined in the obvious way: $O_X(P_i)$ is the sheaf of rational functions that are regular away from P_i and have at most a simple pole at P_i) are not isomorphic: if $i : X_1 \to X$ is the inclusion map of the first component, then $i^* O_X(P_1) \cong O_{\mathbb{P}^1}(1)$, whereas $i^* O_X(P_2) \cong O_{pi}$, (roughly) this is because X_2 does not contain the pole P_1. This shows that the one-to-one correspondence between Cl(X) and Pic(X) no longer holds.

Example 2.8. *(CaDiv$(X) \to \operatorname{div}(X)$ is not injective) Take the previous example. Let $P \in X_1 \cap X_2, Q \in X_1 - X_2$. Define a Cartier divisor $D = \{(U, 1), (V, f)\}$, where $U = X - \{Q\}, V = X_1 - \{P\}$, f is a rational function on V that has a simple pole at Q. Since quotients of these functions are regular on $U \cap V$, D is well-defined. The associated Weil divisor is just the point $[Q]$.*
By symmetry define another Cartier divisor D' associated to the Weil divisor $[Q']$, where $Q' \in X_2 - X_1$.
Since $D - D'$ is not the divisor of a rational function, the two Cartier divisors are different. But on the other hand we have shown that Cl$(X) = \mathbb{Z}$, thus $[Q] = [Q']$.

Example 2.9. *(CaDiv$(X) \to \operatorname{div}(X)$ is not surjective) To construct a Weil divisor that is not Cartier, we need to work in a space that is not locally factorial. In other words, a space where codimension 1 subsets are not cut by single function, this tells us that the space should not be smooth.*
Consider the cone $X = \operatorname{Spec} A$, where $A = k[x, y, z]/(xy - z^2)$. Take $Z = \{x = z = 0\}$ to be the closed subscheme of X.
Z is of codimension one. So we get the exact sequence

$$\mathbb{Z} \stackrel{1 \to [Z]}{\longrightarrow} Cl(X) \to Cl(X - Z) \to 0.$$

Here $X - Z$ is the locus where $x \neq 0$ (if $x = 0$, then $z = 0$). So it is equivalently

$$\operatorname{Spec} k\left[x, y, y^{-1}, z\right] / \left(xy - z^2\right) \cong k[y]_y[t, u] / \left(t - u^2\right) \cong k[y, u]_y,$$

so it is a UFD. Thus its class group is zero. We find that $[Z]$ generates Cl(X). We will now show that Z is not principal at the origin, which will prove that Z is not Cartier.

The local ring at the origin $(k[x,y,z]/(xy-z^2))_{(x,y,z)}$ *is not principal, which tells us that* Z *is not a Cartier divisor.*

3 Intersection theory

In this section, we collect a few results from basic Intersection theory, work through some examples, which will be necessary for our last part 'Motives'.

In the last section we defined divisors as the formal sum of codimension 1 subarieties, and then with an equivalence relation we defined the divisor class group. We now extend this notion to higher codimension subvarieties.
Throughout this section we will be working with schemes over a field. *Variety* is a Noetherian, integral, separated scheme of finite type over k, and a *subvariety* is a closed subscheme which is a variety. We now define *cycles* of dimension k.

Definition 3.1. *Let X be a scheme. For $k \geq 0$ denote by $Z_k(X)$ the free Abelian group generated by the k-dimensional subvarieties of X. In other words, the elements of $Z_k(X)$ are finite formal sums $\sum_i n_i [V_i]$, where $n_i \in \mathbb{Z}$ and the V_i are k-dimensional (closed) subvarieties of X. The elements of $Z_k(X)$ are called cycles of dimension k.*

To define *Chow groups* we need to define an equivalence relation on $Z_k(X)$. So what does it mean for two cycles to be *equivalent*?
To answer this question, we define the order of a rational function,

Definition 3.2. *(Order) Let X be a variety, and let $V \subset X$ be a subvariety of codimension 1 , and set $R = O_{X,V}$. For every non-zero $f \in R \subset K(X)$ we define the order of f at V to be the integer ord $\mathrm{o}_V(f) := l_R(R/(f))$. If $\varphi \in K(X)$ is a non-zero rational function we write $\varphi = \frac{f}{g}$ with $f, g \in R$ and define the order of φ at V to be*

$$\mathrm{ord}_V(\varphi) := \mathrm{ord}_V(f) - \mathrm{ord}_V(g).$$

To show that this is well-defined, i. e. that ord $_V \frac{f}{g} = \mathrm{ord}_V \frac{f'}{g'}$ whenever $fg' = gf'$, use the exact sequence

$$0 \to R/(a) \xrightarrow{\cdot b} R/(ab) \to R/(b) \to 0$$

and the fact that the length of modules is additive on exact sequences. From this it also follows that the order function is a homomorphism of groups ord $\mathrm{gin}_V : K(X)^* := K(X)\backslash\{0\} \to \mathbb{Z}$.

(Divisor of a rational function) For any $(k+1)$-dimensional subvariety W of X and any non-zero rational function φ on W we define a cycle of dimension k on X by

$$\operatorname{div}(\varphi) = \sum_V \operatorname{ord}_V(\varphi)[V] \in Z_k(X),$$

called the divisor of φ, where the sum is taken over all codimension-1 subvarieties V of W.

Since for any $\varphi \in K(W)^*$ we have $\operatorname{div}(\varphi^{-1}) = -\operatorname{div}(\varphi)$, cycles of the form $\operatorname{div}(\varphi)$ for all $k+1$ dimensional subvarieties $W \subset X$ generate a subgroup $B_k(X) \subset Z_k(X)$.

Definition 3.3. *(Chow groups) We define the Chow group of k-cycles to be $A_k(X) = Z_k(X)/B_k(X)$. Two cycles in $Z_k(X)$ that determine the same element in $A_k(X)$ are said to be rationally equivalent. We set $Z_*(X) = \bigoplus_{k\geq 0} Z_k(X)$ and $A_*(X) = \bigoplus_{k\geq 0} A_k(X)$.*

Example 3.1. *Let X be a scheme of pure dimension n. Then $B_n(X)$ is trivially zero, and thus $A_n(X) = Z_n(X)$ is the free Abelian group generated by the irreducible components of X. In particular, if X is an n-dimensional variety then $A_n(X) \cong \mathbb{Z}$ with $[X]$ as a generator. In the same way, $Z_k(X)$ and $A_k(X)$ are trivially zero if $k > n$.*

Example 3.2. *Let $X = \{x_1x_2 = 0\} \subset \mathbb{P}^2$ be the union of two projective lines $X = X_1 \cup X_2$ that meet in a point. Then $A_1(X) = \mathbb{Z}\,[X_1] \oplus \mathbb{Z}\,[X_2]$ since X_1 and X_2 are the irreducible components of the 1-dimensional variety X. Moreover, $A_0(X) \cong \mathbb{Z}$ is generated by the class of any point in X. In fact, any two points on X_1 are rationally equivalent since in this case $A_0(X) \cong Pic(X) \cong \mathbb{Z}$, and the same is true for X_2. As both X_1 and X_2 contain the intersection point $X_1 \cap X_2$ we conclude that all points in X are rationally equivalent. So $A_0(X) \cong \mathbb{Z}$.*

Example 3.3. *Let $X = \mathbb{A}^n$. If $P \in X$ is any point, pick a line $W \cong \mathbb{A}^1 \subset \mathbb{A}^n$ through P and a linear function φ on W that vanishes precisely at P. Then $\operatorname{div}(\varphi) = [P]$. It follows that the class of any point is zero in $A_0(X)$. Therefore $A_0(X) = 0$.*

Example 3.4. *Now we show that $A_0(\mathbb{P}^n) \cong \mathbb{Z}$. For any two points $P, Q \in X = \mathbb{P}^n$, let $W \cong \mathbb{P}^1 \subset \mathbb{P}^n$ be the line through P and Q, and let φ be a rational function on W that has a simple zero at P and a simple pole at Q.* Then $\operatorname{div}(\varphi) = [P] - [Q]$, *i.e. the classes in $A_0(X)$*

of any two points in X are the same. It follows that $A_0(X)$ is generated by the class $[P]$ of any point in X.
On the other hand, if $W \subset X = \mathbb{P}^n$ is any curve and φ a rational function on W then the degree of the divisor of φ is always zero (this follows from the fact that φ is of the form f/g where $degf = degg$). This shows that $n \cdot [P] \in A_0(X) \implies n = 0$. We conclude that $A_0(X) \cong \mathbb{Z}$ with the class of any point as a generator.

Let $f : X \to Y$ be a morphism of schemes. Then it is important to study whether there exits push-forward $f_* : A_*(X) \to A_*(Y)$ or pull-back $F^* : A_*(Y) \to A_*(X)$. These do not exist in general, take the following two examples:

(i) Let X be a scheme $Y \subset X$ is a closed suscheme, the inclusion morphism $i : Y \to X$. Then there are canonical push-forward maps $i_* : A_k(Y) \to A_k(X)$ for any k, given by $[Z] \mapsto [Z]$ for any k-dimensional subvariety $Z \subset Y$. But a subvariety of X is in general not a subvariety of Y, so there is no pull-back morphism $i^* : A_*(X) \to A_*(Y)$ sending $[V]$ to $[V]$ for any subvariety $V \subset X$.

(ii) Now suppose $U \subset X$ is an open subset, with $i : U \to X$. Then there are no push-forward maps $i_* : A_*(U) \to A_*(X)$: if $U = \mathbb{A}^1$ and $X = \mathbb{P}^1$ then the class of a point is zero in $A_*(U)$ but non-zero in $A_*(\mathbb{P}^1)$.

3.1 Pull-back

Let us first recall a few basic properties.

Definition 3.4. *A morphism of schemes $f : X \to Y$ is flat of relative dimension n if every fiber $X_y = X \otimes_Y k(y)$ is of pure dimension n.*

Proposition 12. *If $f : X \to Y$ is a morphism of varieties over k. Then the following are equivalent:*
(I) Every irreducible component of X has dimension equal to $\dim Y + n$.
(II) For any point $y \in Y$, X_y is of pure dimension n.

Proof. See [4, Corollary 9.6]. $\square$

For a flat morphism (of relative dimension n) $f : X \to Y$, and any subvariety V of Y, set

$$f^*[V] = \left[f^{-1}(V)\right]$$

Here $f^{-1}(V)$ is the inverse image scheme, a subscheme of X of pure dimension $\dim(V) + n$, and $[f^{-1}(V)]$ is its cycle. This extends by linearity to pull-back homomorphisms

$$f^* : Z_k Y \to Z_{k+n} X.$$

We now have the following theorem,

Theorem 3.1. *Let $f : X \to Y$ be a flat morphism of relative dimension n, and α a k-cycle on Y which is rationally equivalent to zero. Then $f^*\alpha$ is rationally equivalent to zero in $Z_{k+n}X$.*

Hence we have the induced homomorphisms, the flat pull-backs,

$$f^* : A_k(Y) \to A_{k+n}(X).$$

Now suppose X is a variety of pure dimension, then for a flat morphism $f : X \to Y$ (by the previous proposition), we get the following flat pull-back,

$$f^* : A_k(Y) \to A_{k+\dim(X)-\dim(Y)}(X).$$

We now list some important flat morphisms (of relative dimension n), i.e. morphisms for which pull-backs exist,
(I) an open imbedding ($n = 0$).
(II) the projection of a vector bundle or a projective bundle, to its base.
(III) the projection from a Cartesian product $X = Y \times Z$ to the first factor, where Z is a purely n-dimensional scheme.

Proposition 13. *(An important exact sequence) Let X be a scheme, let $Y \subset X$ be a closed subset, and let $U = X \backslash Y$. Denote the inclusion maps by $i : Y \to X$ and $j : U \to X$. Then the sequence*

$$A_k(Y) \xrightarrow{i_*} A_k(X) \xrightarrow{j^*} A_k(U) \to 0$$

is exact for all $k \geq 0$. The homomorphism i_* is in general not injective however.

Proof. This follows more or less from the definitions. If $Z \subset U$ is any k-dimensional subvariety then the closure Z of Z in X is a k-dimensional subvariety of X with $j^*[Z] = [Z]$. So j^* is surjective.

If $Z \subset Y$ then $Z \cap U = 0$, so $j^* \circ i_* = 0$. Conversely, assume that we have a cycle $\sum a_r [V_r] \in A_k(X)$ whose image in $A_k(U)$ is zero. This means that there are rational functions φ_s on $(k+1)$-dimensional subvarieties W_s of U such that $\sum \operatorname{div}(\varphi_s) = \sum a_r [V_r \cap U]$ on U. Now the φ_s are also rational functions on the closures of W_s in X, and as such their divisors can only differ from the old ones by subvarieties V_r' that are contained in $X \backslash U = Y$. We conclude that $\sum \operatorname{div}(\varphi_s) = \sum a_r [V_r] - \sum b_r [V_r']$ on X for some b_r. So $\sum a_r [V_r] = i_* \sum b_r [V_r']$.

Now we will give an example where i_* is not injective. Let Y be a smooth cubic curve in $X = \mathbb{P}^2$. Then by the degree-genus formula we can see that Y is an elliptic curve. Then we have a one-one correspondence $P \leftrightarrow \mathcal{L}(P - P_0)$, where P_0 is a fixed point (See Example 1.3.7, Hart). Thus if P and Q are two distinct points on Y then $[P] - [Q] \neq 0 \in A_0(Y)$, but $[P] - [Q] = 0 \in A_0(X) \cong \operatorname{Pic}(X) \cong \mathbb{Z}$. Thus i_* is not injective. □

Proposition 14. *Let X be a scheme, and let $\pi : E \to X \mid$ be a vector bundle of rank r on X. Then the flat pull-back $\pi^* : A_k(X) \to A_{k+r}(E)$ given on cycles by $\pi^*[V] = [\pi^{-1}(V)]$ is surjective.*

Proof. See [3, Proposition 1.9]. □

Corollary 3.1. *The Chow groups of affine spaces are given by*

$$A_k(\mathbb{A}^n) = \begin{cases} \mathbb{Z} & \textit{for } k = n \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. The statement for $k \geq n$ follows from example 2.1. For $k < n$ note that the homomorphism $A_0\left(\mathbb{A}^{n-k}\right) \to A_k(\mathbb{A}^n)$ is surjective by proposition 8, so the statement of the corollary follows from the fact that $A_0(\mathbb{A}^1) \cong \operatorname{Pic}(\mathbb{A}^1) = 0$. □

Corollary 3.2. *The Chow groups of projective spaces are $A_k(\mathbb{P}n) \cong \mathbb{Z}$ for all $0 \leq k \leq n$ with an isomorphism given by $[V] \mapsto \deg V$ for all k-dimensional subvarieties $V \subset \mathbb{P}^n$.*

Proof. The statement for $k \geq n$ is trivial, so let us assume that $k < n$. We prove the statement by induction on n. By proposition 14, there is an exact sequence

$$A_k\left(\mathbb{P}^{n-1}\right) \to A_k\left(\mathbb{P}^n\right) \to A_k\left(\mathbb{A}^n\right) \to 0$$

We have $A_k\left(\mathbb{A}^n\right) = 0$ by corollary 9.1.16, so we conclude that $A_k\left(\mathbb{P}^{n-1}\right) \to A_k\left(\mathbb{P}^n\right)$ is surjective. By the induction hypothesis this means that $A_k\left(\mathbb{P}^n\right)$ is generated by the class of a k-dimensional linear subspace. As the morphism $Z_k\left(\mathbb{P}^n - 1\right) \to Z_k\left(\mathbb{P}^n\right)$ trivially preserves degrees it only remains to be shown that any cycle $\sum a_i\left[V_i\right]$ that is zero in $A_k(\mathbb{P}n)$ must satisfy $\sum a_i \deg V_i = 0$. But this is clear as $\deg \operatorname{div}(\varphi) = 0$ for all rational functions on any subvariety of $\mathbb{P}^n$. $\square$

3.2 Push-forward

It turns out that the condition of properness of a morphism $f : X \to Y$ is enough to guarantee the existence of well-defined push-forward maps $f_* : A_k(X) \to A_k(Y)$.
Let $f : X \to Y$ be a proper morphism. For any subvariety V of X, the image $W = f(V)$ is then a closed subvariety of Y. We define the degree as follows,

$$\deg(V/W) = \begin{cases} [R(V) : R(W)] & \text{if} \quad \dim(W) = \dim(V) \\ 0 & \text{if } \dim(W) < \dim(V) \end{cases}$$

where $[R(V) : R(W)]$ denotes the degree of the field extension induced by f. Define

$$f_*[V] = \deg(V/W)[W].$$

This extends linearly to a homomorphism

$$f_* : Z_k X \to Z_k Y.$$

By the multiplicativity of degrees of field extensions it follows that the push-forwards are functorial, i.e. $(g \circ f)_* = g_* f_*$ for any two morphisms $f : X \to Y$ and $g : Y \to Z$. Now we want to show that these homomorphisms pass to the Chow groups, i.e. give rise to well-defined homomorphisms $f_* : A_k(X) \to A_k(Y)$. For this we have to show by definition that divisors of rational functions are pushed forward to divisors of rational functions. The following theorem fulfills our purpose.

Theorem 3.2. *Let $f : X \to Y$ be a proper surjective morphism of varieties, and let $\varphi \in K(X)^*$ be a non-zero rational function on X. Then*

$$f_* \operatorname{div}(\varphi) = \begin{cases} 0 & \text{if } \dim Y < \dim X \\ \operatorname{div}(N(\varphi)) & \text{if } \dim Y = \dim X \end{cases}$$

in $Z_*(Y)$, where $N(\varphi) \in K(Y)$ denotes the determinant of the endomorphism of the $K(Y)$ vector space $K(X)$ given by multiplication by φ (this is usually called the norm of φ).

Proof. See [3, Proposition 1.4]. □

Corollary 3.3. *Let $f : X \to Y$ be a proper morphism of schemes. Then there are well-defined push-forward maps $f_* : A_k(X) \to A_k(Y)$ for all $k \geq 0$ given on cycles by*

$$f_*[Z] = \begin{cases} [K(Z) : K(f(Z))] \cdot [f(Z)] & \text{if } \dim(f(Z)) = \dim Z, \\ 0 & \text{if } \dim(f(Z)) < \dim Z. \end{cases}$$

3.3 Divisors

The goal of intersection theory is to describe intersections on the level of Chow groups. In this section we will explain the easiest case, namely with the intersection of a variety with a subset of codimension 1. Given a subvariety $V \subset X$ of dimension k and a divisor D, that is a subvariety of codimension 1, we want to construct an *intersection cycle* $[V] \cdot [D] \in A_{k-1}(X)$. Let us first recall the definition of Weil and Cartier divisors,

Definition 3.5. *Let X be a scheme.*

(I) If X has pure dimension n, a Weil divisor on X is an element of $Zn - 1(X)$. Two Weil divisors are called linearly equivalent if they define the same class in $A_{n-1}(X)$. The quotient group $A_{n-1}(X)$ is called the group of Weil divisor classes.

(II) A Cartier divisor on X is a global section of $\mathcal{K}_X^*/\mathcal{O}_X^*$.

We now state some properties we already had proved in the section of divisors,

(I) Let X be a purely n-dimensional scheme. Then there is a homomorphism $\operatorname{Div}(X) \to Z_{n-1}(X)$, which extends to $\operatorname{Pic}(X) \to A_{n-1}(X)$. In other words, every Cartier divisor (class) determines a Weil divisor (class).

(II) Since smoothness implies locally factorial, we have the following theorem,

Theorem 3.3. *Let X be a smooth n-dimensional scheme. Then Div $X \cong Z_{n-1}(X)$ and* $\operatorname{Pic} X \cong A_{n-1}(X)$.

(III) For a scheme X, we have the following correspondence (remember we needed X to be integral for this),

$$\{\text{Cartier divisors on } X\} \longleftrightarrow \{\text{Line bundles on } X\}.$$

3.3.1 Intersections with Cartier divisors

We will now define the intersection products of Chow cycles with Cartier divisors.

Definition 3.6. *Let X be a scheme, let $V \subset X$ be a k-dimensional subvariety with inclusion morphism $i : V \to X$, and let D be a Cartier divisor on X. We define the intersection product $D \cdot V \in A_{k-1}(X)$ to be*

$$D \cdot V = i_* \left[i^* \mathcal{O}_{\mathrm{X}}(D) \right],$$

where $\mathcal{O}_X(D)$ is the line bundle on X associated to the Cartier divisor D.

Since for any two linearly equivalent Cartier divisors D, D' on X $\mathcal{O}_X(D) \cong \mathcal{O}_X(D')$, from the definition we can see that the intersection product depends only on the divisor class of D, not on D itself. not on D itself. So using our definition we can construct bilinear intersection products

$$\operatorname{Pic} X \times Z_k(X) \to A_{k-1}(X), \quad \left(D, \sum a_i \left[V_i\right]\right) \mapsto \sum a_i \left(D \cdot V_i\right).$$

If X is smooth and pure-dimensional, i.e. Weil and Cartier divisors agree, then for a codimension-1 subvariety $W \subset X$, we denote by $W \cdot V \in A_{k-1}(X)$ the intersection product $D \cdot V$, where D is the Cartier divisor corresponding to the Weil divisor $[W]$.

We first state a few properties of this intersection product and after that we will work through a few examples.

Proposition 15. *, (Commutativity of the intersection product) Let X be an n-dimensional variety, and let D_1, D_2 be Cartier divisors on X with associated Weil divisors $[D_1], [D_2]$. Then $D_1 \cdot [D_2] = D_2 \cdot [D_1] \in A_{n-2}(X)$.*

Proof. See [3, Theorem 2.4]. □

Corollary 3.4. *The intersection product passes to rational equivalence, i. e. there are well-defined bilinear intersection maps* $\operatorname{Pic} X \times A_k(X) \to A_{k-1}(X)$ *determined by* $D \cdot [V] = [D \cdot V]$ *for all* $D \in PicX$ *and all* k*-dimensional subvarieties* V *of* X*.*

Proof. All that remains to be shown is that $D \cdot \alpha = 0$ for any Cartier divisor D if the cycle α is zero in the Chow group $A_k(X)$. But this follows from the previous proposition, as for any rational function φ on a $(k+1)$-dimensional subvariety W of X we have

$$D \cdot [\operatorname{div}(\varphi)] = \operatorname{div}(\varphi) \cdot [D] = 0$$

(note that $\operatorname{div}(\varphi)$ is a Cartier divisor on W that is linearly equivalent to zero). □

We can continue this process for finitely Cartier divisors. Suppose $D_1, D_2, \ldots, D_m \in \operatorname{Div}(X)$ for a scheme X, and $\alpha \in A_k(X)$ is a k-cycle, then

$$D_1 \cdot D_2 \ldots D_m \cdot \alpha \in A_{k-m}(X).$$

For an n-dimensional variety X, we can take $\alpha = [X]$, and then

$$D_1 \cdot D_2 \ldots D_{m-1} \cdot D_m \in A_{n-m}(X).$$

If $m = n$, $D_1 \cdot D_2 \ldots D_{m-1} \cdot D_m \in A_0(X)$ is a 0-cycle.

3.4 Intersections on a surface

Here we will talk about the intersection product on a smooth projective surface over an algebraically closed field. We will also get to see the Riemann-Roch theorem for surfaces and few of its consequences. Although I have provided sketches of most of the proofs, I have tried to include quite a few interesting applications to make this section more illuminating.

Throughout this section we will talk about smooth projective surfaces over an algebraically closed field.

Given two smooth curves $C, D \subset X$, we first define the *intersection multiplicity* of C and D at $x \in C \cap D$ as follows,

Definition 3.7. *Let C, D be two distinct irreducible curves on a surface $S, x \in C \cap D, \mathcal{O}_x$ the local ring of S at x. If f (resp. g) is an equation of C (resp. D) in $\mathcal{O}_x$, the intersection multiplicity of C and D at x is defined to be*

$$m_x (C \cap D) = \dim_k \mathcal{O}_x/(f, g)$$

We see that $m_x (C \cap D) = 1$ if and only if f and g generate the maximal ideal x, i.e. form a system of local coordinates in a neighbourhood of $x; C$ and D are then said to be transverse at x.

Now we define the intersection number as follows,

Definition 3.8. *If C, D are two distinct irreducible curves on S, the intersection number $(C \cdot D)$ is defined by:*

$$(C.D) = \sum_{x \in C \cap D} m_x (C \cap D) .$$

Recall that the ideal sheaf defining C (resp. C') is just the invertible sheaf $\mathcal{O}_S(-C)$ (resp. $\mathcal{O}_S (-C')$); define

$$\mathcal{O}_{C \cap C'} = \mathcal{O}_S / (\mathcal{O}_S(-C) + \mathcal{O}_S (-D)) .$$

Now suppose C, D are two curves who meet transversally. Then at each point $x \in C \cap D$, we have $m_x = \dim_k \mathcal{O}_x/(f, g) = \dim_k k = 1$.

Also since $C \cap D$ is a proper closed subset of C, it is a finite set of points. This shows that the sheaf $\mathcal{O}_{C \cap D}$ is a skyscraper sheaf supported at the finite set $C \cap D$, and at each of these points we have $(\mathcal{O}_{C \cap D})_x = \mathcal{O}_x/(f, g)$. Thus we see that

$$(C \cdot D) = h^0(S, \mathcal{O}_{C \cap D}) = \chi(\mathcal{O}_{C \cap D}),$$

since $\mathcal{O}_{C \cap D}$ is a sky-scraper sheaf, $h^i(S, \mathcal{O}_{C \cap D}) = 0$, for $i \geq 1$. We will now try to understand this number with the help of the following short exact sequences

$$0 \longrightarrow \mathcal{O}_C(-D) \longrightarrow \mathcal{O}_C \longrightarrow \mathcal{O}_{C \cap D} \longrightarrow 0$$

$$0 \longrightarrow \mathcal{O}_X(-C) \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_C \longrightarrow 0$$

$$0 \longrightarrow \mathcal{O}_X(-C - D) \longrightarrow \mathcal{O}_X(-D) \longrightarrow \mathcal{O}_C(-D) \longrightarrow 0$$

Hence by the additivity of the Euler-characteristic we have

$$\begin{aligned}\chi(\mathcal{O}_{C\cap D}) &= -\chi(C, \mathcal{O}_C(-D)) + \chi(C, \mathcal{O}_C) \\ &= \chi(\mathcal{O}_X(-C-D)) - \chi(\mathcal{O}_X(-D)) + \chi(C, \mathcal{O}_C) \\ &= \chi(\mathcal{O}_X(-C-D)) - \chi(\mathcal{O}_X(-D)) - \chi(\mathcal{O}_X(-C)) + \chi(\mathcal{O}_X)\end{aligned}$$

This suggests the following definition for general divisors D_1, D_2,

Definition 3.9. *(Intersection product) For divisors D_1, D_2, we define the intersection product to be*

$$D_1 \cdot D_2 = \chi(\mathcal{O}_X) + \chi(-D_1 - D_2) - \chi(-D_1) - \chi(-D_2)$$

Proposition 16. *This number enjoys certain properties,*

(i) The product $D_1 \cdot D_2$ depends only on the classes of D_1, D_2 in $\mathrm{Pic}(X)$.

(ii) $D_1 \cdot D_2 = D_2 \cdot D_1$.

(iii) $D_1 \cdot D_2 = |D_1 \cap D_2|$ if D_1 and D_2 are curves intersecting transversely.

(iv) The intersection product is bilinear.

Proof. By the definition, (i)-(iii) are clear. Lets prove (iv). We will need the following lemma,

Lemma 3.4. *Let C be a non-singular irreducible curve on S. Then for any divisor D on S, we have*

$$(\mathcal{O}_S(C) \cdot \mathcal{L}) = \deg\left(\mathcal{L}_{|C}\right)$$

Proof. We have the following exact sequences

$$0 \longrightarrow \mathcal{O}_S(-C) \longrightarrow \mathcal{O}_S \longrightarrow \mathcal{O}_C \longrightarrow 0$$

$$0 \longrightarrow \mathcal{L}^{-1}(-C) \longrightarrow \mathcal{L}^{-1} \longrightarrow \mathcal{L}^{-1} \otimes \mathcal{O}_C \longrightarrow 0$$

give the following relations between Euler characteristics:

$$\chi(\mathcal{O}_S) - \chi(\mathcal{O}_S(-C)) = \chi(\mathcal{O}_C)$$
$$\chi\left(\mathcal{L}^{-1}\right) - \chi\left(\mathcal{L}^{-1}(-C)\right) = \chi\left(\mathcal{L}^{-1}_{|C}\right)$$

For $\mathcal{L} \in \text{Pic}C$, using the Riemann-Roch theorem on C we have,

$$\begin{aligned}
\chi(\mathcal{L}) = h^0(C,\mathcal{L}) - h^1(C,\mathcal{L}) &= \deg \mathcal{L} - g + 1 \\
&= \deg \mathcal{L} - h^1(C,\mathcal{O}_C) + h^0(C,\mathcal{O}_C) \\
&= \deg \mathcal{L} + \chi(\mathcal{O}_C) \\
\implies \chi(\mathcal{O}_C) - \chi(\mathcal{L}) &= -\deg \mathcal{L}.
\end{aligned}$$

Hence we have the following,

$$\begin{aligned}
(\mathcal{O}_S(C)\cdot\mathcal{L}) &= \chi(\mathcal{O}_C) - \chi\left(\mathcal{L}^{-1}_{|C}\right) \\
&= -\deg \mathcal{L}^{-1}_{|C} \text{ by the Riemann-Roch theorem on } C \\
&= \deg \mathcal{L}_{|C}.
\end{aligned}$$

This proves the lemma. □

Fact: Let D be a divisor on S, and H a hyperplane section of S (for a given embedding). There exists $n \geq 0$ such that $D + nH$ is a hyperplane section (for another embedding). In particular we can write $D \equiv A - B$, where A and B are smooth curves on S, with $A \equiv D + nH$ and $B \equiv nH$.

Lets come back to our main proof. For $\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3 \in \text{Pic } S$, consider the expression

$$s(\mathcal{L}_1,\mathcal{L}_2,\mathcal{L}_3) = (\mathcal{L}_1\cdot\mathcal{L}_2\otimes\mathcal{L}_3) - (\mathcal{L}_1\cdot\mathcal{L}_2) - (\mathcal{L}_1\cdot\mathcal{L}_3).$$

Now let $\mathcal{L}, \mathcal{L}'$ be any two invertible sheaves. We can write $\mathcal{L}' = \mathcal{O}_S(A - B)$, where A and B are two smooth curves on S. Noting that $s(\mathcal{L},\mathcal{L}',\mathcal{O}_S(B)) = 0$, we get

$$(\mathcal{L},\mathcal{L}') = (\mathcal{L}\cdot\mathcal{O}_S(A)) - (\mathcal{L}\cdot\mathcal{O}_S(B)).$$

□

Example 3.5. *(Bezout's theorem) Let $X = \mathbb{P}^2$. C, C' are two curves of degree d, d' respectively. We know that $PicX = \mathbb{Z}$, where the generator is any hyperplane. Then we have $C \sim dL$, and $C' \sim d'L'$, where L, L' are two distinct lines on X. Now since L, L' meet at one* *point, we have* $(L \cdot L') = |L \cap L'| = 1$.

Then by bilinearity we have

$$(C \cdot C') = (dL \cdot d'L') = dd'(L \cdot L') = dd'.$$

Example 3.6. *Now let* $X = \mathbb{P}^1 \times \mathbb{P}^1$. *Then* $Pic(X) \cong \mathbb{Z} \oplus \mathbb{Z}$, *where we can take the generators as* $l = \mathbb{P}^1 \times \{0\}, m = \{0\} \times \mathbb{P}^1$. *Any divisor* $D \in PicX$ *is of the form,*

$$D \sim al + bm.$$

We can see that $(l \cdot m) = 1$, *and by taking* $l' = \mathbb{P}^1 \times \{\infty\} \sim l$, *we see that*

$$(l \cdot l) = (l \cdot l') = 0.$$

Similarly $(m \cdot m') = 0$.

Thus for any two curves $D_1 \sim a_1 l + b_1 m, D_2 \sim a_1 l + b_2 m$, *we have*

$$(D_1 \cdot D_2) = a_1 b_2 + a_2 b_1.$$

Theorem 3.5. *(Riemann-Roch) Let* $D \in \operatorname{div}(X)$. *Then*

$$\chi\left(X, \mathcal{O}_X(D)\right) = \frac{D \cdot (D - K_X)}{2} + \chi\left(\mathcal{O}_X\right).$$

Proof. Write $\chi\left(X, \mathcal{O}_X(D)\right) = \chi(D)$. Then let us compute $-D \cdot (D - K)$. By the definition of the intersection product

$$-D \cdot (D - K) = \chi(\mathcal{O}_X) - \chi(D) - \chi(K - D) + \chi(K).$$

By Serre duality we have $h^i(D) = h^{2-i}(K - D)$. Hence

$$\begin{aligned}
\chi(K) &= h^2(X, \omega_X) - h^1(X, \omega_X) + h^0(X, \omega_X) \\
&= h^2(X, \mathcal{O}_X) - h^1(X, \mathcal{O}_X) + h^0(X, \mathcal{O}_X) \\
&= \chi(\mathcal{O}_X).
\end{aligned}$$

Similarly, $\chi(K_D) = \chi(D)$. Putting these two in the equation, we have

$$\begin{aligned}
-D \cdot (D - K) &= 2(\chi(\mathcal{O}_X) - \chi(D)) \\
\implies \chi\left(X, \mathcal{O}_X(D)\right) &= \frac{D \cdot (D - K_X)}{2} + \chi\left(\mathcal{O}_X\right) \quad \text{(using bilinearity).}
\end{aligned}$$

□

Corollary 3.5. *(The genus formula) Let C be an irreducible curve on a surface X of genus g.Then we have*

$$g = 1 + \frac{1}{2}\left(C^2 + C \cdot K\right).$$

Proof. Take the exact sequence

$$0 \to \mathcal{O}_X(-C) \to \mathcal{O}_X \to \mathcal{O}_C \to 0$$

from this we get $\chi(\mathcal{O}_C) = 1 - g = \chi(\mathcal{O}_X) - \chi(\mathcal{O}_X(-C))$. Taking $D = -C$ in the Riemann-Roch proves our statement. $\square$

Notice that we can write this formula as $2g - 2 = C \cdot (C + K)$, hence alternatively this says that for a genus g curve on a surface X, we have

$$K_Y = C \cdot (C + K_X).$$

Example 3.7. *(The degree-genus formula) Let C be a curve of degree d in $\mathbb{P}^2$. Then since* $\deg \omega_{\mathbb{P}^2} = \deg \mathcal{O}_{\mathbb{P}^2}(-3) = -3$, *we have (by applying Bezout's theorem)*

$$2g - 2 = d(d-3) \implies g = \frac{1}{2}(d-1)(d-2).$$

Example 3.8. *Let C be a curve on the quadric surface ($\cong \mathbb{P}^1 \times \mathbb{P}^1$). Hence $C \sim al + bm$, thus $C + K \sim (a-2)l + (b-2)m$. We call C a curve of type (a, b). Then as calculated in example* 2.7, *we have*

$$2g - 2 = C \cdot (C + K) = a(b-2) + (a-2)b \implies g = (a-1)(b-1).$$

4 Abelian Varieties

In this section I will give a brief account of Abelian Varieties over an algebraically closed field k.

Definition 4.1. *An abelian variety X is a proper variety over k with a group law $m : X \times X \to X$ such that m and the inverse map are both morphisms of varieties.*
Here by a variety over we mean an integral, separated scheme of finite type over k.

First we note that Abelian varieties are smooth, in fact any group variety is smooth. Indeed, suppose x_0 is a smooth point of X, then for any $x \in X$ we have the translation morphism

$$T_{xx_0^{-1}}(y) = xx_0^{-1}y,$$
$$x_0 \longrightarrow x,$$

taking x_0 to x, which is an automorphism of X. This shows that x is also a smooth point of X.
From now on, we write the group law in X additively. Moreover, we will use the following notations: for $x \in X$, we denote by $T_x : X \to X$ the translation morphism $T_x(y) = x + y$; and the map $x \to n \cdot x$ will be denoted by n_X.

Abelian varieties are commutative: To show that an abelian variety is commutative, we will need an important lemma:

Lemma 4.1. *(Rigidity lemma) Suppose X, Y, Z are varieties with X complete, and $y_0 \in Y$ such that for the morphism $f : X\times Y \to Z$ we have $f(X \times \{y_0\}) = z_0$, where $z_0 \in Z$. Then there is a morphism $g : Y \to Z$ so that the following diagram*

$$\begin{tikzcd} X \times Y \arrow[r, "p"] \arrow[d, "f"'] & Y \arrow[dl, "g"] \\ Z & \end{tikzcd}$$

commutes, where p is projection onto the second factor.

Proof. Define $g : Y \to Z$ by $g(y) = f(x_0, y)$, where x_0 is a fixed point of X. To show that $f = g \circ p_2$, we need to show that these two morphisms coincide on an open subset of $X \times Y$, because the subset where they are equal is a closed subset. Let U is an open affine neighborhood of $z_0 \in Z$, take $F = Z - U$ and $G = p_2(f^{-1}(F))$. Since X is complete and p_2 is closed, G is a closed subset of Y.

Suppose $y_0 \in G$, this would imply that $y_0 = p_2(x', y_0) \in p_2(f^{-1}(F)) \implies f(x', y_0) \neq z_0$, which is a contradiction. Thus $y_0 \notin G$, and thus $V = Y - G$ is a nonempty open subset of Y. For each $y \in V$, the complete variety $X \times \{y\}$ gets mapped by f into the affine variety U, and since X is complete, it must be sent to a point. This means for any $x \in X, y \in V$, $f(x, y) = f(x_0, y) = g \circ p_2(x, y)$, and this proves our assertion. □

Corollary 4.1. *If X, Y are abelian varieties and $f : X \to Y$ is any morphism, then $f(x) = h(x) + a$, where h is a group homomorphism of X into Y and $a \in Y$.*

Proof. Replacing f by $f - f(0)$ we may assume $f(0) = 0$. Now define $\varphi : X \times X \to Y$, by $\varphi(x_1, x_2) = f(x_1 + x_2) - f(x_1) - f(x_2)$. Then $\varphi(X \times \{0\}) = 0$, and hence by the Rigidity lemma we get $\varphi = f \circ p_2 \equiv 0$. This completes our proof. □

Corollary 4.2. *Abelian variety X is a commutative group.*

Proof. Apply the previous corollary to the inverse morphism $e : X \to X$, $x \mapsto x^{-1}$. Since $e(0) = 0$, we get that e is a group homomorphism. This proves that X is a commutative group. □

4.1 Line bundles on Abelian Varieties

We will not prove the following two important theorems. For the proofs see [7, Pages 51,52].

Theorem 4.2. *(Seesaw Theorem) Let X be a complete variety, T be any variety, and L a line bundle on $X \times T$. Then the set*

$$T_1 = \{t \in T \mid L|_{X \times \{t\}} \text{ is trivial on } X \times \{t\}\}$$

is closed in T, and if $p_2 : X \times T_1 \to T_1$ is the projection, then $L|_{X \times T_1} \cong p_2^* M$ for some line bundle M on T_1.

Theorem 4.3. *(Theorem of the cube) Let X, Y are complete varieties, Z any variety and x_0, y_0, z_0 are base points on X, Y, Z resp. If $L \in Pic(X \times Y \times Z)$ whose restrictions to each $\{x_0\} \times Y \times Z, X \times \{y_0\} \times Z, X \times Y \times \{z_0\}$ are trivial, then L is trivial.*

Corollary 4.3. *(Corollary of the proof) Let X, Y are complete varieties, Z any variety. Then any line bundle $X \times Y \times Z$ is isomorphic to $p_{12}^*(L) \otimes p_{13}^*(M) \otimes p_{23}^*(N)$, where p_{ij} is projection of $X \times Y \times Z$ onto the product of ith and jth factors, and L, M, N are line bundles on $X \times Y, X \times Z$ and $Y \times Z$ respectively.*

Corollary 4.4. *Let X be any variety, Y an Abelian Variety, and $f, g, h : X \to Y$ morphisms. Then for all line bundles $L \in \mathrm{Pic}(Y)$ we have: $(f+g+h)^*L \cong (f+g)^*L \otimes (f+h)^*L \otimes (g+h)^*L \otimes f^*L^{-1} \otimes g^*L^{-1} \otimes h^*L^{-1}$.*

Proof. Let $p_i : Y \times Y \times Y \to Y$ be the projection onto the i-th factor, put $m_{ij} := p_i + p_j : Y \times Y \times Y \to Y$ and $m := p_1 + p_2 + p_3 : Y \times Y \times Y \to Y$. Consider the line bundle

$$M := m^*L \otimes m_{12}^*L^{-1} \otimes m_{13}^*L^{-1} \otimes m_{23}^*L^{-1} \otimes p_1^*L \otimes p_2^*L \otimes p_3^*L$$

on $Y \times Y \times Y$. If $q : Y \times Y \to Y \times Y \times Y$ is the map given by $q(y, y') = (0, y, y')$, we have

$$\begin{aligned} q^*M &= (m \circ q)^*L \otimes (m_{12} \circ q)^*L^{-1} \otimes (m_{13} \circ q)^*L^{-1} \otimes (m_{23} \circ q)^*L^{-1} \otimes (p_1 \circ q)^*L \\ &\otimes (p_2 \circ q)^*L \otimes (p_3 \circ q)^*L \\ &= n^*L \otimes q_1^*L^{-1} \otimes q_2^*L^{-1} \otimes n^*L^{-1} \otimes 0^*L \otimes q_1^*L \otimes q_2^*L \end{aligned}$$

where $0, q_1, q_2, n : Y \times Y \to Y$ are respectively the 0 map, the projections, and the addition. Therefore q^*M is trivial. By symmetry, M is trivial on $Y \times \{0\} \times Y$ and $Y \times Y \times \{0\}$ too. By Theorem of cube M must be trivial on $Y \times Y \times Y$. Now if we pull back M by the map $(f, g, h) : X \to Y \times Y \times Y$, we get our desired identity. $\square$

Corollary 4.5. *(Theorem of the square) Let X be a Complex Abelian Variety. Then for all line bundles L and $x, y \in X$ we have: Therefore if we define the map*

$$\phi_L(x) := \text{ isom. class of } T_x^*L \otimes L^{-1} \text{ in } \mathrm{Pic}(X),$$

then ϕ_L is a homomorphism from X to $Pic(X)$.

Proof. We get the desired identity by applying the previous corollary with $X = Y, f$ and g constant maps with images x, y respectively, and $h =$ identity.
We now show that ϕ_L is indeed a homomorphism $X \to \mathrm{Pic}(X)$,

$$\begin{aligned}\phi_L(x_1 + x_2) &\cong T^*_{x_1+x_2}L \ \otimes L^{-1} \\ &\cong T^*_{x_1}L \otimes T^*_{x_2}L \otimes L^{-1} \otimes L^{-1} \\ &\cong T^*_{x_1}L \otimes L^{-1} \otimes T^*_{x_2}L \otimes L^{-1} \\ &\cong \phi_L(x_1) \otimes \phi_L(x_2).\end{aligned}$$

□

We will see that the kernel of the map ϕ_L is going to give us a very efficient way to check whether a divisor on an abelian variety is ample or not. Because of this, we are introducing the following definition.

Definition 4.2. $K(L) = \ker(\phi_L) = \{x \in X \mid T^*_x L \cong L\}$.

Proposition 17. *$K(L)$ is a Zariski-closed subgroup of X.*

Proof. Consider the line bundle line bundle $m^*L \otimes p_2^*L^{-1}$ on $X \times X$, where $m : X \times X \to X$ is the addition. Then applying the Seesaw theorem on this line bundle the following set

$$\begin{aligned}X_1 &= \{x \in X \mid m^*L \otimes p_2^*L^{-1}|_{\{x\}\times X} \text{ is trivial}\} \\ &= \{x \in X \mid T^*_x L \otimes L^{-1}|_{\{x\}\times X} \text{ is trivial}\} \\ &= K(L)\end{aligned}$$

is closed. Hence $K(L)$ is closed. □

Next we will prove that Abelian Varieties are projective. We go about finding an ample line bundle on an abelian variety X. To find such a divisor is easy, but to prove that the divisor is ample we need a theorem,

Theorem 4.4. *Let X be a Complex Abelian Variety and let D be an effective divisor on X and $L = L(D)$ the associated line bundle. The following conditions are equivalent:*
(i) The complete linear system $|2D|$ has no base points, and defines a finite morphism $\phi_{|2D|}$:

$X \to \mathbb{P}^N$.

(ii) L is ample on X.

(iii) $K(L)$ is finite.

(iv) The subgroup $H := \{x \in X \mid T_x^*(D) = D\}$ of X is finite (equality of divisors, not divisor classes).

The proof is going to be a bit long, but as it is quite important we want to give a complete detailed proof.

Proof. $(i) \implies (ii)$ Clear.

$(ii) \implies (iii)$ If $K(L)$ is not finite, let Y be the connected component of $K(L)$ containing 0 (clearly, $0 \in K(L)$). We show that Y is an abelian variety of positive dimension.

Indeed, it is a group variety since X is a group variety by definition and it is closed by the previous Proposition. Furthermore, for every variety Z the projection $Y \times Z \to Z$ is closed since it is the composition of the closed maps $Y \times Z \hookrightarrow X \times Z$ and $X \times Z \to Z$, where the first map is the closed immersion of $Y \times Z$ in $X \times Z$ and the second map is the projection of $X \times Z$ on Z, and the latter is a closed map since X is complete. Thus Y is complete, hence it is an abelian variety.

Now, by assumption L is ample on X, then restricting to a closed subset $L|_Y = L_Y$ is also ample on Y. Moreover, since $Y \subseteq K(L) = \ker(\phi_L)$, we have $T_y^*(L_Y) \cong L_Y$ for all $y \in Y$. Let $m_Y : Y \times Y \to Y$ be the addition on Y and $p_i : Y \times Y \to Y$ be the projections for $i = 1, 2$.

Lemma 4.5. *The line bundle $m_Y^*(L_Y) \otimes p_1^*\left(L_Y^{-1}\right) \otimes p_2^*\left(L_Y^{-1}\right)$ is trivial on $Y \times Y$.*

Proof. We apply the Seesaw Theorem. Consider the line bundle $m_Y^* L_Y \otimes p_2^* L_Y^{-1}$ on $Y \times Y$. Clearly for every $y \in Y$ we have:

$$\left(m_Y^* L_Y \otimes p_2^* L_Y^{-1}\right)_{\{y\} \times Y} \cong \left(p_1^* L_Y\right)_{\{y\} \times Y}$$

since the first line bundle is isomorphic to $T_y^* L_Y \otimes L_Y^{-1}$, which is isomorphic by definition of Y to $L_Y \otimes L_Y^{-1}$, hence trivial, and the second line bundle is obviously trivial. Hence by the Seesaw Theorem we get some line bundle M on Y:

$$m_Y^* L_Y \otimes p_2^* L_Y^{-1} \cong p_1^* M.$$

To conclude we want to show that $M = L_Y$. Observe that:

$$\left(m_Y^* L_Y \otimes p_2^* L_Y^{-1}\right)_{|Y\times\{y\}} \cong \left(p_1^* L_Y\right)_{|Y\times\{y\}}$$

since both are isomorphic to L_Y, hence by the last part of Seesaw Theorem we have $M = L_Y$.

Now from $m_Y^* L_Y \otimes p_2^* L_Y^{-1} \cong p_1^* L_Y$ we have that $m_Y^* L_Y \otimes p_1^* L_Y^{-1} \otimes p_2^* L_Y^{-1}$ is trivial on $Y \times Y$. □

Consider now the composition

$$Y \overset{g:=(\mathrm{id},-\mathrm{id})}{\longrightarrow} Y \times Y \overset{m_Y}{\longrightarrow} Y.$$

By the lemma we have that $m_Y^* L_Y^{-1} \otimes p_1^* L_Y \otimes p_2^* L_Y$ is also trivial on $Y \times Y$, hence

$$\begin{aligned} g^*\mathcal{O}_{Y\times Y} &\cong g^*\left(m_Y^* L_Y^{-1} \otimes p_1^* L_Y \otimes p_2^* L_Y\right) \\ &\cong \left(m_Y \circ g\right)^* L_Y^{-1} \otimes \left(p_1 \circ g\right)^* L_Y \otimes \left(p_2 \circ g\right)^* L_Y \\ &\cong L_Y \otimes (-\mathrm{id})^* L_Y. \end{aligned}$$

and this is trivial on Y, because g^* is a group homomorphism. But we have seen that L_Y is ample on Y, and since $(-\mathrm{id})$ is an automorphism of Y, $L_Y \otimes (-\mathrm{id})^*(L_Y) \cong g^*\mathcal{O}_{Y\times Y}$ is ample. So $L_Y \otimes (-\mathrm{id})^*(L_Y)$ is both ample and trivial, and this is a contradiction since $\dim Y > 0$. Hence $K(L)$ must be finite.

$(iii) \implies (iv)$ Clear, since $H \subset K(L)$.

$(iv) \implies (i)$ See [7, Application 1, Page 57]. □

Corollary 4.6. *Abelian varieties are projective.*

Proof. Let U be an open affine subset of X containing the point 0.

Fact: Let X be a Noetherian separated scheme. Let $U \subset X$ be a dense affine open. If $\mathcal{O}_{X,x}$ is a UFD for all $x \in X \setminus U$, then there exists an effective Cartier divisor $D \subset X$ with $U = X \setminus D$.

Let $D_1, \ldots, D_t$ be the irreducible components of $X \setminus U$ and let $D = \sum D_i \in \mathrm{div}(X)$. We will show that D satisfies (iv) of the above theorem.

Consider the set $H = \{x \in X \mid T_x^* D = D\}$. Clearly, for any $x \in H$ we have that $T_x(U) = U$. Since $0 \in U$, it follows that $H \subseteq U$. On the other hand, H is a closed set as it can be seen as $f^{-1}(D)$ where $f : K(L) \to |D|$ which is defined as $f(x) = T_x^* D$. H is clearly proper and being a closed subset of an affine scheme U, it is affine. Therefore H is finite. This follows from the following,

Fact: For a morphism $f : X \to Y$ of schemes, the following assertions are equivalent:
(I) f is finite.
(II)f is quasi-finite and proper.
(III) f is affine and proper. □

4.2 The dual abelian variety

First we recall that for a line bundle $L \in \mathrm{Pic}X$, we associated the map $\phi_L : X \to \mathrm{Pic}X$, which sent an element $x \in X$ to the isomorphism class of line bundles $T_x^*(L) \otimes L^{-1} \in \mathrm{Pic}X$. We showed that this is a group homomorphism and we denoted the kernel of ϕ_L to be $K(L)$.

Definition 4.3. *We define the subgroup $Pic^0(X)$ of $Pic(X)$ consisting of the line bundles L for whom ϕ_L is identically zero.*

$$\begin{aligned} Pic^0(X) &= \{L \in Pic(X) \mid \phi_L \textit{ is identically zero}\} \\ &= \{L \in Pic(X) \mid K(L) = X\} \\ &= \{L \in Pic(X) \mid T_x^* L \cong L, \textit{ for all } x \in X\}. \end{aligned}$$

The main purpose of this section is to show (in Char 0) that $\mathrm{Pic}^0(X)$ is naturally isomorphic to another abelian variety $\hat{X}$, called the dual of X. First we will collect some properties of $\mathrm{Pic}^0(X)$.

(I) Image of ϕ_L is contained in $\mathrm{Pic}^0(X)$.

Proof. Suppose $x, y \in X$. Then using the Seesaw theorem we get

$$\begin{aligned} T_x^*(\phi_L(y)) &= T_x^*(T_y^*(L) \otimes L^{-1}) \\ &= T_{x+y}^*(L) \otimes T_x^*(L^{-1}) \\ &= T_x^*(L) \otimes T_y^*(L) \otimes L^{-1} \otimes T_x^*(L^{-1}) \\ &= T_y^*(L) \otimes L^{-1} \\ &= \phi_L(y). \end{aligned}$$

Since this is true for any $x, y \in X$, it follows that $\mathrm{Im}(\phi_L) \subset K(L)$. □

(II) For any $L \in \mathrm{Pic}^0(X)$, we get $m^*L \cong p_1^*L \otimes p_2^*L$ on $X \times X$.

Proof. Let $x \in X(k)$. Then

$$(m^*\mathcal{L} \otimes p_1^*\mathcal{L}^\vee \otimes p_2^*\mathcal{L}^\vee)_{X \times \{x\}} \cong t_x^*\mathcal{L} \otimes \mathcal{L}^\vee \cong \mathcal{O}_X$$
$$(m^*\mathcal{L} \otimes p_1^*\mathcal{L}^\vee \otimes p_2^*\mathcal{L}^\vee)\,|_{\{0\} \times X} \cong \mathcal{L} \otimes \mathcal{L}^\vee \cong \mathcal{O}_X$$

we can see this as the maps in question are given explicitly by

$$m, p_1, p_2 : X \cong \{x\} \times X \hookrightarrow X \times X \to X, \quad m(y) = x + y, \quad p_1(y) = x, \quad p_2(y) = y,$$
$$m, p_1, p_2 : X \cong X \times \{x\} \hookrightarrow X \times X \to X, \quad m(y) = x + y, \quad p_1(y) = y, \quad p_2(y) = x.$$

Then by the seesaw principle, this implies $m^*\mathcal{L} \otimes p_1^*\mathcal{L}^\vee \otimes p_2^*\mathcal{L}^\vee \cong \mathcal{O}_{X \times X} \implies m^*L \cong p_1^*L \otimes p_2^*L$. □

(III) If $L \in \mathrm{Pic}^0(X)$, then for all schemes S and all morphisms $S \to X$, $(f+g)^*L \cong f^*L \otimes g^*L$.

Proof. Consider the last isomorphism in (II) and pull it back to S by $(f, g) : S \to X \times X$, then clearly we have

$$\begin{aligned} m^*L &\cong p_1^*L \otimes p_2^*L \\ \implies (f,g)^*m^*L &\cong (f,g)^*p_1^*L \otimes (f,g)^*p_2^*L \\ \implies (f+g)^*L &\cong f^*L \otimes g^*L. \end{aligned}$$

□

(IV) If $L \in \mathrm{Pic}^0(X)$, then $n_X^* L \cong L^n$.

Proof. We can use induction to (III). □

(V) For all $\mathrm{Pic}^0(X)$, $n_X^* L \cong L^{n^2} \otimes \{\text{something in } \mathrm{Pic}^0(X)\}$.

Proof. In the last section we proved that

$$\begin{aligned} n_X^* L &\cong L^{\left(\frac{n^2+n}{2}\right)} \otimes (-1_X)^* L^{\left(\frac{n^2-n}{2}\right)} \\ &\cong L^{n^2} \otimes L^{\left(\frac{n-n^2}{2}\right)} (-1_X)^* L^{\left(\frac{n^2-n}{2}\right)} \\ &= L^{n^2} \otimes [L \otimes (-1_X)^* L^{-1}]^{\left(\frac{n-n^2}{2}\right)}. \end{aligned}$$

We will show that $L \otimes (-1_X)^* L^{-1} \in \mathrm{Pic}^0(X)$. Indeed for any $x \in X$, we have

$$\begin{aligned} T_x^*(L \otimes (-1_X)^* L^{-1}) &\cong T_x^* L \otimes (-1_X)^* T_{-x}^* L^{-1}, \quad (\text{since } T_x \circ -1_X = -1_X \circ T_{-x}) \\ &\cong T_x^* L \otimes (-1_X)^* [L \otimes T_{-x}^* L^{-1}] \otimes (-1_X)^* L^{-1} \end{aligned}$$

Now for any $x \in X$, we have

$$\begin{aligned} T_x^*(L \otimes T_{-x}^* L^{-1}) \otimes (L \otimes T_{-x}^* L^{-1})^{-1} &\cong T_x^* L \otimes L^{-1} \otimes L^{-1} \otimes T_{-x}^* L \\ &\cong L^2 \otimes L^{-2}, \quad (\text{by the theorem of square}) \\ &\cong \mathcal{O}_X. \end{aligned}$$

Hence $L \otimes T_{-x}^* L^{-1} \in \mathrm{Pic}^0(X)$. Then using (IV) we have

$$\begin{aligned} T_x^*(L \otimes (-1_X)^* L^{-1}) &\cong T_x^* L \otimes (-1_X)^* T_{-x}^* L^{-1}, \quad (\text{since } T_x \circ -1_X = -1_X \circ T_{-x}) \\ &\cong T_x^* L \otimes (-1_X)^* [L \otimes T_{-x}^* L^{-1}] \otimes (-1_X)^* L^{-1} \\ &\cong T_x^* L \otimes L^{-1} \otimes T_{-x}^* L \otimes (-1_X)^* L^{-1} \\ &\cong L \otimes (-1_X)^* L^{-1}, \quad (\text{by the theorem of square}). \end{aligned}$$

This completes the proof. □

(VI) If $L \in \mathrm{Pic}^0(X)$, and L is not trivial, then $H^i(X, L) = 0$, for all $i \geq 0$.

Proof. If $H^0(L) \neq 0$, then L has a non-trivial section s, so $(-1_X)^* L$ has the non-trivial section $(-1_X)^* s$. But by (IV), $(-1_X)^* L \cong L^{-1}$, so both L and L^{-1} have a non-trivial section. Hence $L \cong \mathcal{O}_X$, contradiction.

Now let $i > 0$ be the smallest positive integer such that $H^i(X, L) \neq 0$. The maps

$$X \overset{\mathrm{id}\times 0}{\to} X \times X \overset{m}{\to} X, \quad x \mapsto (x, 0) \mapsto x$$

from this we have the following sequence

$$H^i(X, L) \to H^i(X \times X, m^* L) \to H^i(X, L),$$

where the composition is the identity. Now we have $m^* L \cong p_1^* L \otimes p_2^* L$. Then using the Künneth formula we get

$$H^i(X \times X, m^* L) \cong H^i(X \times X, p_1^* L \otimes p_2^* L) \cong \bigoplus_{j=0}^{i} H^j(X, L) \otimes H^{i-j}(X, L)$$

Since $H^0(L) = 0$ by the first argument and $H^{i-j}(X, L) = 0$ for $j \geq 1$ by the choice of i, this yields $H^i(X \times X, m^* L) = 0$. So the identity of $H^i(X, L)$ factors through 0 and thus $H^i(X, L) = 0$. $\square$

4.2.1 Construction of the dual (Sketch)

We first state an important theorem

Theorem 4.6. *Let L be ample, that is $K(\mathcal{L})$ is finite and $M \in \mathrm{Pic}^0(X)$. Then for some $x \in X$,*

$$M \cong T_x^* \mathcal{L} \otimes L^{-1},$$

i.e. the map $\phi_{\mathcal{L}} : X \longrightarrow \mathrm{Pic}^0(X)$ is surjective.

Proof. See [7, Theorem 1, Page 73,]. $\square$

On the level of abelian groups, this gives an isomorphism

$$X(k)/K(\mathcal{L})(k) \cong \mathrm{Pic}^0(X),$$

The aim is now to construct a variety $K(\mathcal{L})$ so that the quotient $X/K(\mathcal{L})$ becomes an abelian variety, namely the dual abelian variety of X.

Mumford line bundle: On $X \times X$, define the Mumford line bundle

$$\Lambda(\mathcal{L}) := m^*\mathcal{L} \otimes p_1^*\mathcal{L}^\vee \otimes p_2^*\mathcal{L}^\vee,$$

where m denotes the addition and p_1, p_2 are the natural projections. Put $\mathcal{N} := \Lambda(\mathcal{L}) \otimes p_1^*\mathcal{M}^\vee$ and let $x \in X$. Then, we see that

$$\mathcal{N}|_{\{x\}\times X} \cong T_x^*\mathcal{L} \otimes \mathcal{L}^\vee \quad \text{and} \quad \mathcal{N}|_{X\times\{x\}} \cong T_x^*\mathcal{L} \otimes \mathcal{L}^\vee \otimes \mathcal{M}^\vee.$$

Proposition 18. *Let X be a complete variety, Y an arbitrary scheme, $\mathcal{L}$ a line bundle on $X \times Y$. Then there exists a unique closed subscheme $Y_1 \subseteq Y$ such that for every scheme Z, a morphism $f : Z \to Y$ factors through Y_1 if and only if the line bundle $(\mathrm{id} \times f)^*\mathcal{L}$ on $X \times Z$ is the pullback of some line bundle on Z via the projection onto the second factor.*

Proof. See [8, Proposition 9.3]. □

Let X be an abelian variety over k, and let $\mathcal{L}$ be a line bundle on X. Apply the previous roposition to the Mumford line bundle $\Lambda(\mathcal{L})$ on $X \times X$. This yields a closed subscheme $X_1 \subseteq X$ with the universal property as described above. For each $x \in X(k)$, by definition of the Mumford bundle, $\Lambda(\mathcal{L})|_{X\times\{x\}} \cong t_x^*\mathcal{L} \otimes \mathcal{L}^\vee$. Thus $K(\mathcal{L}) = \big\{x \in X(k) | \Lambda(\mathcal{L})|_{X\times\{x\}}$ is trivial $\} = X_1$, and we can view $K(\mathcal{L})$ as a closed subscheme of X.

Proposition 19. *$K(\mathcal{L})$ is a subgroup scheme of X.*

Proof. Let $f' : Z \to K(\mathcal{L})$ be a morphism of schemes. Composing with $K(\mathcal{L}) \hookrightarrow X$ gives a morphism $f : Z \to X$. By the previous Proposition, $(\mathrm{id} \times f)^*\Lambda(\mathcal{L}) = q_2^*\mathcal{M}$, where $q_2 : X \times Z \to Z$ is the natural projection onto Z. Let $\mathcal{L}_Z := q_1^*\mathcal{L}$, where $q_1 : X \times Z \to X$. Let further

$$t_f : X \times Z \to X \times Z, \quad (x, z) \mapsto (x + f(z), z)$$

be the translation by f. Then

$$\begin{aligned} t_f^*\mathcal{L}_Z = (\mathrm{id} \times f)^*m^*\mathcal{L} &= (\mathrm{id} \times f)^*\Lambda(\mathcal{L}) \otimes (\mathrm{id} \times f)^*p_1^*\mathcal{L} \otimes (\mathrm{id} \times f)^*p_2^*\mathcal{L} \\ &\cong q_2^*\mathcal{M} \otimes q_1^*\mathcal{L} \otimes q_2^*f^*\mathcal{L} \\ &= q_2^*\left(\mathcal{M} \otimes f^*\mathcal{L}\right) \otimes \mathcal{L}_Z. \end{aligned}$$

Conversely, if $f : Z \to X$ is any morphism such that $t_f^*\mathcal{L}_Z \otimes \mathcal{L}_Z^\vee$ is the pullback of a line bundle on Z via q_2, the previous Proposition states that f factors through $K(\mathcal{L})$.

Now let $f, g : Z \to X$ be morphisms of schemes such that $t_f^*\mathcal{L}_Z \otimes \mathcal{L}_Z^\vee$ and $t_g^*\mathcal{L}_Z \otimes \mathcal{L}^\vee$ are pullbacks of line bundles on Z via q_2. That is, f, g are points of $X(Z)$ that lie in $K(\mathcal{L})(Z)$. By the theorem of the square,

$$t_{f+g}^*\mathcal{L}_Z \otimes \mathcal{L}_Z^\vee \cong t_f^*\mathcal{L}_Z \otimes \mathcal{L}_Z^\vee \otimes t_g^*\mathcal{L}_Z \otimes \mathcal{L}_Z^\vee,$$

so $f + g$ lies in $K(\mathcal{L})(Z)$ as well. As a consequence, $K(\mathcal{L})(Z)$ is a subgroup of $X(Z)$. $\square$

Theorem 4.7. *Let X be an abelian variety over a field $k, \mathcal{L}$ an ample line bundle on X. Then the quotient scheme $X/K(\mathcal{L})$ exists and is an abelian variety over k with the same dimension as X.*

Proof. (Sketch) X is an (abelian) group scheme, $K(\mathcal{L})$ a finite subgroup scheme. This implies that $X/K(\mathcal{L})$ is a group scheme of the same dimension as X. If char $k = 0$, then X is automatically a variety, as group schemes in char 0 are smooth. $\square$

Definition 4.4. *(Dual abelian variety) This quotient is called the Dual abelian variety $X^\vee$ of X.*

Theorem 4.8. *(Universal property of the dual abelian variety). Let X be an abelian variety over k. Then there is a uniquely determined line bundle $\mathcal{P}$ on $X \times X^\vee$, called the Poincaré bundle, such that*
(a) $\mathcal{P}|_{X\times\{y\}} \in \mathrm{Pic}^0(X \times \{y\})$ for all $y \in X^\vee$,
(b) $\mathcal{P}|_{\{0\}\times X^\vee}$ is trivial,
and if Z is a scheme with a line bundle $\mathcal{R}$ on $X \times Z$ such that $\mathcal{R}|_{X\times\{z\}} \in \mathrm{Pic}^0(X \times \{z\})$ for all $z \in Z$ and $\mathcal{R}|_{\{0\}\times Z}$ is trivial, then there is a unique morphism $f : Z \to X^\vee$ such that $(\mathrm{id} \times f)^*\mathcal{P} = \mathcal{R}$.
In other words, $(X^\vee, \mathcal{P})$ represents the functor

$$Z \mapsto \left\{ \mathcal{L} \in \mathrm{Pic}(X \times Z) | \mathcal{L}|_{X\times\{z\}} \in \mathrm{Pic}^0(X \times \{z\}) \textit{ for all } z \in Z \textit{ and } \mathcal{L}|_{\{0\}\times Z} \textit{ is trivial} \right\},$$

and the Poincaré bundle $\mathcal{P}$ corresponds to id_{X^v}.

For proof see [7, Page 74-75].

Let $f : X \to Y$ be a homomorphism of abelian varieties. Denote by $\mathcal{P}_X$ and $\mathcal{P}_Y$ the Poincaré bundles on $X \times X^\vee$ and $Y \times Y^\vee$, respectively. Consider the line bundle $\mathcal{M} := (f \times \mathrm{id}_{Y^\vee})^* \mathcal{P}_Y$ on $X \times Y^\vee$. By the properties of the Poincaré bundle, $\mathcal{M}|_{X\times\{y\}} \in \mathrm{Pic}^0(X \times \{y\})$ and $\mathcal{M}|_{\{0\}\times Y^\vee}$ is trivial. Hence by the previous Theorem $\mathcal{M}$ defines a unique morphism $f^\vee : Y^\vee \to X^\vee$ with the propery that $(\mathrm{id}_X \times f^\vee)^* \mathcal{P}_X \cong (f \times \mathrm{id}_{Y^v})^* \mathcal{P}_Y$.

Definition 4.5. *If $f : X \to Y$ is a homomorphism of abelian varieties, then $f^\vee : Y^\vee \to X^\vee$ is called the dual morphism of f.*

Remark. If a point in $Y^\vee$ corresponds to a line bundle $\mathcal{L} \in \mathrm{Pic}^0(Y)$, then its image under $f^\vee$ is given by the pullback $f^*\mathcal{L}$.

Part II: Derived Categories in Algebraic Geometry

5 Derived and Triangulated Categories

5.1 Triangulated Categories

Definition 5.1. *(**Additive Category**) A category $\mathcal{C}$ is an additive category if for every two objects $A, B \in \mathcal{C}$ the set* $\mathrm{Hom}(A, B)$ *is endowed with a structure of abelian group and the following three conditions are satisfied:*

1. The composition $\mathrm{Hom}(A, B) \times \mathrm{Hom}(B, C) \to \mathrm{Hom}(A, C)$ *is bilinear, i.e. we have* $(f + g) \circ h = f \circ h + g \circ h$ *and* $f \circ (h + l) = f \circ h + f \circ l$.

2. There exist an object $0 \in \mathcal{C}$ *which is both inital and terminal, i.e. for all objects* $A \in \mathcal{C}$, *we have* $0 = \mathrm{Hom}(A, 0) \simeq \mathrm{Hom}(0, A)$.

3. For any two objects $A_1, A_2 \in \mathcal{C}$, *there exist an object* B, *called the biproduct of* A_1 *and* A_2, *and arrows* $j_i : A_i \to B$ *and* $p_i : B \to A_i, i = 1, 2$, *verifying the following properties:*

(I) For every object $D \in \mathcal{C}$ *and arrows* $l_i : A_i \to D$, *there exist a unique arrow* $l : B \to D$ *such that* $l_i = l \circ j_i$.
(II) For every object $D \in \mathcal{C}$ *and arrows* $q_i : D \to A_i$, *there exist a unique arrow* $q : D \to B$ *such that* $q_i = p_i \circ q$.
Such an object is unique and is denoted $A_1 \oplus A_2$.

Remark: Given a field k, one can define similarly a **k-linear category** asking the Hom sets to be k-vector spaces and the composition to be k-bilinear.

Definition 5.2. *(**Additive functor**) A functor* $F : \mathcal{C} \to \mathcal{D}$ *between additive categories (resp. k-linear categories) is said to be additive if the induced maps* $\mathrm{Hom}(A, B) \to \mathrm{Hom}(F(A), F(B))$ *are group homomorphisms (resp. k-linear maps).*

Definition 5.3. *(**Serre functor**) Let $\mathcal{C}$ be a k-linear category. A Serre functor is a k-linear equivalence* $S : \mathcal{C} \to \mathcal{C}$ *such that for any two objects* $A, B \in \mathcal{C}$ *there exists an isomorphism* *of k-vector spaces*

$$\eta_{A,B} : \mathrm{Hom}(A, B) \xrightarrow{\simeq} \mathrm{Hom}(B, S(A))^*$$

which is functorial in A and B, where the * denotes the dual vector space.

We will later use properties of this functor to extend the Serre duality to derived categories.

Definition 5.4. ***(Abelian category)*** *Definition 1.6. An additive category $\mathcal{A}$ is called abelian if every arrow $f : A \to B$ in $\mathcal{A}$ admits a kernel and a cokernel and the natural arrow* $\mathrm{Coim}(f) \to \mathrm{Im}(f)$ *is an isomorphism.*

we will now define the triangulated categories, the kind of categories we will be interested throughout.

Definition 5.5. *(**Triangulated categories**) Let $\mathcal{A}$ be an additive category. Then $\mathcal{A}$ can be given the structure of a triangulated category by an additive equivalence $T : \mathcal{A} \to \mathcal{A}$, the shift functor, and a set of distinguished triangles*

$$A \to B \to C \to T(A)$$

which satisfy four axioms $\boldsymbol{TR}1 - \boldsymbol{TR}4$ described below.
Before describing the axioms $\boldsymbol{TR}$, we will first introduce the notation $A[1] := T(A)$ for any object A in $\mathcal{A}$ and $f[1] := T(f) \in \mathrm{Hom}_{\mathcal{A}}(A[1], B[1])$ for any morphism $f \in \mathrm{Hom}_{\mathcal{A}}(A, B)$. Similarly, $A[n] := T^n(A)$ and $f[n] := T^n(f)$ for any $n \in \mathbb{Z}$. Then using these notations, a triangle can also be denoted by $A \to B \to C \to A[1]$.
A morphism between two triangles is given by the following commutative diagram:

$$\begin{array}{ccccccc}
A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & A[1] \\
\downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle f[1]} \\
A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & A'[1]
\end{array}$$

It is an isomorphism if f, g, h are isomorphisms.
Now we will explain the axioms for a triangulated category:

***TR**1: i) Any triangle of the form*

$$A \xrightarrow{\mathrm{id}} A \longrightarrow 0 \longrightarrow A[1]$$

is distinguished.

ii) Any triangle which is isomorphic to a distinguished triangle is distinguished.

(iii) Any morphism $f : A \to B$ can be completed to a distinguished triangle

$$A \xrightarrow{f} B \longrightarrow C \longrightarrow A[1]$$

***TR**2: The triangle*

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} A[1]$$

is a distinguished triangle if and only if

$$B \xrightarrow{g} C \xrightarrow{h} A[1] \xrightarrow{-f[1]} B[1]$$

is a distinguished triangle.

***TR**3: Assume that there exists a commutative diagram of distinguished triangles with vertical arrows f and g as follows:*

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "f"] & B \arrow[r] \arrow[d, "g"] & C \arrow[r] \arrow[d, dotted, "h"] & A[1] \arrow[d, "f[1]"] \\
A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & A'[1]
\end{tikzcd}$$

Then the above diagram can be completed to a commutative diagram, i.e. to a morphism of triangles, by a morphism $h : C \to C'$. Note that, the morphism h may not be unique.

***TR**4: This is the axiom (octahedron axiom) that is most complicated to state, and we will not need it throught this text, so we will omit it.*

Remark : (I) Since T is an equivalence, any object A in T is isomorphic to the object $(A[-1])[1]$ (i.e. an object in the image of T) and is also isomorphic to the object $(A[1])[-1]$, thus, using the axiom that any triangle isomorphic to a distinguished triangle is distinguished, we can extend the axiom TR-2 to the following: a triangle

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} T(A)$$

is distinguished if and only if any triangle extracted from the sequence

$$\cdots \to B[-1] \xrightarrow{-g[-1]} C[-1] \xrightarrow{-h[-1]} A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} A[1] \xrightarrow{-f[1]} B[1] \longrightarrow$$

is distinguished.

(II) In the same vein, one can prove from the axiom TR-1 that the triangles

$$A \xrightarrow{-\mathrm{Id}} A \longrightarrow 0 \longrightarrow T(A)$$

and

$$0 \longrightarrow A \xrightarrow{\pm\mathrm{Id}} A \longrightarrow 0$$

are also distinguished.

Proposition 20. *Let $A \to B \to C \to A[1]$ be a distinguished triangle in a triangulated category $\mathcal{D}$. Then for any object $A_0 \in \mathcal{D}$ the following induced sequences are exact sequences of abelian groups:*

$$\mathrm{Hom}\,(A_0, A) \longrightarrow \mathrm{Hom}\,(A_0, B) \longrightarrow \mathrm{Hom}\,(A_0, C)$$

$$\mathrm{Hom}\,(C, A_0) \longrightarrow \mathrm{Hom}\,(B, A_0) \longrightarrow \mathrm{Hom}\,(A, A_0)$$

Proof. Let $f : A_0 \to B$ composed with $B \to C$ is $0 : A_0 \to C$. Then applying the axioms we have the following diagram:

$$\begin{tikzcd}
A_0 \arrow[r, "Id"] \arrow[d, dashed] & A_0 \arrow[r] \arrow[d, "f"] & 0 \arrow[d] \\
A \arrow[r] & B \arrow[r] & C
\end{tikzcd}$$

we obtain a lift of f to an arrow $A_0 \to A$. $\square$

Proposition 21. *Let $A \to B \to C \to A[1]$ be a distinguished triangle.*

1. $A \to B$ is an isomorphism if and only if $C \cong 0$.

2. If $C \to A[1]$ is trivial, then the triangle splits, i.e. is given by a decomposition $B \cong A \oplus C$.

3. Consider a morphism of distinguished triangles

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "f"] & B \arrow[r] \arrow[d, "g"] & C \arrow[r] \arrow[d, "h"] & A[1] \arrow[d, "f[1]"] \\
A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & A'[1]
\end{tikzcd}$$

If two of the vertical arrows f, g, h are isomorphisms then so is the third.

Proof. 1. Consider the following sequence:

$$\mathrm{Hom}\,(A_0, A) \longrightarrow \mathrm{Hom}\,(A_0, B) \longrightarrow \mathrm{Hom}\,(A_0, C) \longrightarrow \mathrm{Hom}\,(A_0, A[1]) \longrightarrow \mathrm{Hom}\,(A_0, B[1])\,.$$

Since the morphisms are functorial, the result follows from the Yoneda lemma.

2.

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "id"] & B \arrow[r] \arrow[d, dashed, "\exists f"] & C \arrow[r] \arrow[d, "id"] & A[1] \arrow[d, "id"] \\
A \arrow[r] \arrow[d, "id"] & A \oplus C \arrow[r] \arrow[d, dashed, "\exists g"] & C \arrow[r] \arrow[d, "id"] & A[1] \arrow[d, "id"] \\
A \arrow[r] & B \arrow[r] & C \arrow[r] & A[1]
\end{tikzcd}$$

Maps f, g exists because of TR3, where we are actually using the triviality of $C \to A[1]$, for the commutativity of the right most squares. Also, $fg = id|_B$, hence we have

$$B \xrightarrow{\sim} A \oplus C.$$

3. This can be proved applying $\mathrm{Hom}(A_0, -)$ on the diagram and using the five lemma. □

Definition 5.6. *An additive functor $F : \mathcal{D} \to \mathcal{D}'$ between triangulated categories is called exact if the following conditions are satisfied:*

(I) There exist a natural isomorphism $F \circ T_{\mathcal{D}} \simeq T_{\mathcal{D}'} \circ F$.

(II) Any distinguished triangle $A \to B \to C \to A[1]$ in $\mathcal{D}$ is mapped to a distinguished triangle $F(A) \to F(B) \to F(C) \to F(A)[1] \simeq F(A[1])$ in $\mathcal{D}'$.

Proposition 22. *Let $F : \mathcal{D} \to \mathcal{D}'$ be an exact functor between triangulated categories. If $F \dashv H$, then $H : \mathcal{D}' \to \mathcal{D}$ is exact.*

Proof. See [6, Proposition 1.41]. □

Definition 5.7. *(**Triangulated subcategory**) A subcategory $\mathcal{D}' \subseteq \mathcal{D}$ of a triangulated category is a triangulated subcategory if $\mathcal{D}'$ admits a structure of triangulated category such that the inclusion is exact.*

Proposition 23. *If $\mathcal{D}' \subseteq \mathcal{D}$ is a full subcategory of a triangulated category $\mathcal{D}$, then it is a triangulated subcategory if and only if it is invariant by the shift functor and for any distinguished triangle $A \to B \to C \to A[1]$ in $\mathcal{D}$ with $A, B \in \mathcal{D}'$, the object C is isomorphic to an object in $\mathcal{D}'$.*

Proof. If $\mathcal{D}'$ is a triangulated subcategory, then it is invariant by the shift functor, and considering a distinguished triangle $A \to B \to C \to A[1]$ in $\mathcal{D}$ with $A, B \in \mathcal{D}'$, the arrow $A \to B$ can be completed to a distinguished triangle $A \to B \to C_0 \to A[1]$ in $\mathcal{D}'$ which is also distinguished in $\mathcal{D}$ since the inclusion is exact. Thus we get the commutative diagram:

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[r] \arrow[d, dashed] & A[1] \arrow[d] \\
A \arrow[r] & B \arrow[r] & C_0 \arrow[r] & A[1]
\end{tikzcd}$$

Using the axiom TR-3 one can complete the diagram to a morphism of distinguished triangle, and since all vertical arrows are isomorphism, so is $C \to C_0$. Conversely, the second hypothesis tells exactly that the third TR-1 axiom hold, and all other axioms follow from the fact that $\mathcal{D}'$ is full and invariant under the shift functor. □

5.2 Derived Categories

In this section we will describe the construction of Derived categories and then state some properties. We will see that the objects of the Derived category are easy to explain, the tricky part is how to comprehend the morphisms in this category. In the end of this section we will describe the idea of derived functors, and define Ext functor.

For a category $\mathcal{A}$, we define Kom$\mathcal{A}$ to be the category whose objects are complexes, and morphisms are morphisms of complexes.

Definition 5.8. *Let $A^\bullet \in \mathrm{Kom}(\mathcal{A})$ be a given complex. Then $A^\bullet[1]$ is the complex with $(A^\bullet[1])^i := A^{i+1}$ and differential $d^i_{A[1]} := -d^{i+1}_A$.*

The shift $f[1]$ of a morphism of complexes $f : A^\bullet \longrightarrow B^\bullet$ is the complex morphism $A^\bullet[1] \longrightarrow B^\bullet[1]$ given by $f[1]^i := f^{i+1}$.

Remark : (I) If $\mathcal{A}$ is an abelian category, then so is $\mathrm{Kom}(\mathcal{A})$.
(II) The shift functor $T : \mathrm{Kom}(\mathcal{A}) \to \mathrm{Kom}(\mathcal{A})$, $A^\bullet \to A^\bullet[1]$ defines an equivalence of abelian categories.
We have

$$A^\bullet \xrightarrow{T^{-1}} A^\bullet[-1],$$

also we have

$$A^\bullet[k]^i = A^{k+i}, \text{ and } d^i_{A[k]} = (-1)^k d^{i+k}_A, \text{ for any } k \in \mathbb{Z}.$$

(III) However $\mathrm{Kom}(\mathcal{A})$ is endowed with the shift functor T does not define a triangulated category. Indeed, we would also have to give the class of distinguished triangles, and the canonical choices, like short exact sequences or mapping cones, do not work. More precisely,the short exact sequences $0 \to A^\bullet \to B^\bullet \to C^\bullet \to 0$, which can be viewed as triangles with trivial $C^\bullet \to A^\bullet[1]$, do not in general, satisfy the conditions imposed on distinguished triangles in a triangulated category.

The central idea for the definition of the derived category is this: quasiisomorphic complexes should become isomorphic objects in the derived category. We shall begin our discussion with the following existence theorem. Details of the construction are provided by the subsequent discussion.

Definition 5.9. *Let $\mathcal{A}$ be an abelian category and let $\mathrm{Kom}(\mathcal{A})$ be its category of complexes. Then there exists a category $\mathrm{D}(\mathcal{A})$, the derived category of $\mathcal{A}$, and a functor*

$$Q : \mathrm{Kom}(\mathcal{A}) \longrightarrow \mathrm{D}(\mathcal{A})$$

such that:

i) If $f : A^\bullet \longrightarrow B^\bullet$ *is a quasi-isomorphism, then* $Q(f)$ *is an isomorphism in* $\mathrm{D}(\mathcal{A})$.

ii) Any functor $F : \mathrm{Kom}(\mathcal{A}) \longrightarrow \mathcal{D}$ *satisfying property i) factorizes uniquely over* $Q : \mathrm{Kom}(\mathcal{A}) \longrightarrow \mathrm{D}(\mathcal{A})$, *i.e. there exists a unique functor (up to isomorphism)* $G : \mathrm{D}(\mathcal{A}) \longrightarrow \mathcal{D}$ *with* $F \simeq G \circ \mathrm{Q}$:

$$\begin{tikzcd} \mathrm{Kom}(\mathcal{A}) \arrow[rr, "Q"] \arrow[dr, "F"'] & & D(\mathcal{A}) \arrow[dl, dotted, "G"] \\ & \mathcal{D} & \end{tikzcd}$$

As stated, the theorem is a pure existence result. In order to be able to work with the derived category, we have to understand which objects become isomorphic under $Q : \mathrm{Kom}\,\mathcal{A} \to \mathrm{D}(\mathcal{A})$ and, more complicated, how to represent morphisms in the derived category. Explaining this, will at the same time provide a proof for the above theorem. Moreover, we shall observe the following facts

Corollary 5.1. *(I) Under the functor* $Q : \mathrm{Kom}(\mathcal{A}) \longrightarrow \mathrm{D}(\mathcal{A})$ *the objects of the two categories* $\mathrm{Kom}(\mathcal{A})$ *and* $\mathrm{D}(\mathcal{A})$ *are identified.*
(II) The cohomology objects $H^i(A^\bullet)$ *of an object* $A^\bullet \in \mathrm{D}(\mathcal{A})$ *are well-defined objects of the abelian category* $\mathcal{A}$.
(III) Viewing any object in $\mathcal{A}$ *as a complex concentrated in degree zero yields an equivalence between* $\mathcal{A}$ *and the full subcategory of* $\mathrm{D}(\mathcal{A})$ *that consists of all complexes* $A^\bullet$ *with* $H^i(A^\bullet) = 0$ *for* $i \neq 0$.

Contrary to the category of complexes $\mathrm{Kom}(\mathcal{A})$, the derived category $\mathrm{D}(\mathcal{A})$ is in general not abelian, but it is always triangulated. The shift functor indeed descends to $D(\mathcal{A})$ and a natural class of distinguished triangles can be found, as will be explained shortly.

Definition 5.10. *A morphism of complexes* $f : A^\bullet \to B^\bullet$ *is a quasi-isomorphism (or qis for short) if for all* $n \in \mathbb{Z}$ *the induced arrow* $H^n(f) : H^n(A^\bullet) \to H^n(B^\bullet)$ *is an isomorphism.*

Definition 5.11. *(I) Two morphisms of complexes*

$$f, g : A^\bullet \to B^\bullet$$

are called homotopically equivalent, denoted $f \sim g$, if there exists a collection of homomorphisms $h^n : A^n \to B^{n-1}$ for all $n \in \mathbb{Z}$ such that

$$f^n - g^n = h^{n+1} \circ d_A^n + d_B^{n-1} \circ h^n.$$

Such a family $(h^n)_{n\in\mathbb{Z}}$ is called a homotopy between f and g. If $f \sim g$ then $H^n(f) = H^n(g)$ for all $n \in \mathbb{Z}$.

(II) The homotopy category of complexes $\mathrm{K}(\mathcal{A})$ is the category whose objects are the objects of $\mathrm{Kom}(\mathcal{A})$ and for all $A^\bullet, B^\bullet \in \mathrm{K}(\mathcal{A})$ we have

$$\mathrm{Hom}_{K(\mathcal{A})}\left(A^\bullet, B^\bullet\right) := \mathrm{Hom}_{\mathrm{Kom}(\mathcal{A})}\left(A^\bullet, B^\bullet\right) / \sim .$$

We next formally give the definition of the derived category of $\mathcal{A}$, we will see that the objects are just complexes, but the morphisms here are different, they are called roofs. From the definition it is not at all clear whether the roofs are well-defined or whether they form a category. The checkings come after the definition.

Definition 5.12. *Let $\mathcal{A}$ be an abelian category. Then we define the derived category of $\mathcal{A}$, denoted $\mathrm{D}(\mathcal{A})$ to be the category whose objects are the ones of $\mathrm{Kom}(\mathcal{A})$, i.e.*

$$\mathrm{Ob}(\mathrm{D}(\mathcal{A})) = \mathrm{Ob}(K(\mathcal{A})) = \mathrm{Ob}(\mathrm{Kom}(\mathcal{A})),$$

and arrows are defined as follows. Let $A^\bullet, B^\bullet$ be two objects in $\mathrm{D}(\mathcal{A})$. The set of morphisms $\mathrm{Hom}_{\mathrm{D}(\mathcal{A})}\left(A^\bullet, B^\bullet\right)$ is defined as the set of equivalent classes of diagrams (called roofs) of the form

$$\begin{tikzcd} & C^\bullet \arrow[dl, "qis"'] \arrow[dr, "f"] & \\ A^\bullet & & B^\bullet \end{tikzcd}$$

where $C^\bullet$ is another object in $\mathrm{Kom}(\mathcal{A}), s$ is a quasi-isomorphism and f is a morphism. Two such diagrams are equivalent if they are dominated in $\mathrm{K}(\mathcal{A})$ by a third one of the same *sort, i.e. if there exists a commutative diagram in* $\mathrm{K}(\mathcal{A})$ *of the form*

$$
\begin{tikzcd}
 & D^\bullet \arrow[dl] \arrow[dr] \arrow[ddl, dashed, bend left, "qis"] & \\
C^\bullet \arrow[d] \arrow[drr] & & C'^\bullet \arrow[d] \arrow[dll] \\
A^\bullet & & B^\bullet
\end{tikzcd}
$$

The composition of two morphisms

$$
\begin{tikzcd}
 & C_1^\bullet \arrow[dl, "qis"'] \arrow[dr] & & & C_2^\bullet \arrow[dl, "qis"'] \arrow[dr] & \\
A^\bullet & & B^\bullet & B^\bullet & & C^\bullet
\end{tikzcd}
$$

is given by a commutative diagram in $K(\mathcal{A})$ *of the form*

$$
\begin{tikzcd}
 & & C_0^\bullet \arrow[dl] \arrow[dr] \arrow[ddll, dotted, bend right, "qis"'] & & \\
 & C_1^\bullet \arrow[dl, "qis"'] \arrow[dr] & & C_2^\bullet \arrow[dl, "qis"] \arrow[dr] & \\
A^\bullet & & B^\bullet & & C^\bullet
\end{tikzcd}
$$

Our goal now is to check that these definitions really define a category, in particular that the composition exists and is unique up to equivalence. To do so, we need to introduce the mapping cone which plays a central role in the definition of triangulated structures on $\mathrm{K}(\mathcal{A})$ and $D(\mathcal{A})$.

Definition 5.13. *Let* $f : A^\bullet \to B^\bullet$ *be a morphism of complexes. Its mapping cone is the complex* $\mathrm{C}(f)$ *defined by*

$$
\mathrm{C}(f)^n = A^{n+1} \oplus B^n \text{ and } d^n_{\mathrm{C}(f)} := \begin{pmatrix} -d_A^{n+1} & 0 \\ f^{n+1} & d_B^n \end{pmatrix}
$$

Remark : (I) The mapping cone $\mathrm{C}(f)$ is indeed a complex: $d_B^{n+1} \circ f^{n+1} = f^{n+2} \circ d_A^{n+1}$ since f is a morphism of complexes.

(II) We have natural morphisms of complexes

$$
\tau : B^\bullet \to \mathrm{C}(f) \text{ and } \pi : \mathrm{C}(f) \to A^\bullet[1]
$$

given by the natural injection $B^n \to A^{n+1} \oplus B^n$ and the natural projection $A^{n+1} \oplus B^n \to A^{n+1}$.

(III) The composition $A^\bullet \to B^\bullet \to \mathrm{C}(f)$ is nullhomotopic (i.e. homotopic to the trivial map), such an homotopy is given by $\left(\iota_n : A^n \to A^n \oplus B^{n-1}\right)_{n\in\mathbb{Z}}$. Indeed, we have

$$\begin{tikzcd}
\dots \arrow[r] & A^{n-1} \arrow[r] & A^n \arrow[r, "d"] \arrow[dl, "\iota_n"'] & A^{n+1} \arrow[r] \arrow[dl, "\iota_{n+1}"'] & \dots \\
\dots \arrow[r] & A^n \oplus B^{n-1} \arrow[r, "d'"'] & A^{n+1} \oplus B^n \arrow[r] & A^{n+2} \oplus B^{n+1} \arrow[r] & \dots
\end{tikzcd}$$

and we have $\iota_{n+1} \circ d = \left(d_A^{n+1}, 0\right)$ and $d' \circ \iota_n = \left(-d_A^{n+1}, f^n\right)$.

(IV) The sequence $0 \to B^\bullet \to \mathrm{C}(f) \to A^\bullet[1] \to 0$ is exact: it comes from the fact that the composition $M \to M \oplus N \to N$ is 0 in any additive category. In particular, we have a long exact sequence

$$\dots \to H^n\left(A^\bullet\right) \to H^n\left(B^\bullet\right) \to H^n(\mathrm{C}(f)) \to H^{n+1}\left(A^\bullet\right) \to \dots$$

(V) Using the previous long exact sequence, we obtain that f is a quasi-isomorphism if and only if $H^n(\mathrm{C}(f)) = 0$ for all $n \in \mathbb{Z}$.

Proposition 24. *Let $f : A^\bullet \to B^\bullet$ be a morphism of complexes and let $\mathrm{C}(f)$ be its mapping cone with its natural arrows $\tau : B^\bullet \to \mathrm{C}(f)$ and $\pi : \mathrm{C}(f) \to A^\bullet[1]$. Then there exists a morphism of complexes*

$$g : A^\bullet[1] \to \mathrm{C}(\tau)$$

which is an isomorphism in $\mathrm{K}(\mathcal{A})$ and such that the following diagram commutes in $\mathrm{K}(\mathcal{A})$:

$$\begin{tikzcd}
B^\bullet \arrow[r, "\tau"] \arrow[d, "\cong"] & C(f) \arrow[r, "\pi"] \arrow[d, "\cong"] & A^\bullet[1] \arrow[r, "-f"] \arrow[d, dotted, "g"] & B^\bullet[1] \arrow[d, "\cong"] \\
B^\bullet \arrow[r, "\tau"] & C(f) \arrow[r, "\tau_\tau"] & C(\tau) \arrow[r, "\pi_\tau"] & B^\bullet[1]
\end{tikzcd}$$

Proof. We construct g on degree n as the arrow

$$A^\bullet[1]^n = A^{n+1} \longrightarrow \mathrm{C}(\tau)^n = B^{n+1} \oplus A^{n+1} \oplus B^n$$

defined by $(-f^{n+1}, \mathrm{Id}, 0)$. It clearly define a morphism of complexes. The inverse g^{-1} in $\mathrm{K}(\mathcal{A})$ can be given by the projection onto the middle factor (note that $g \circ g^{-1}$ is homotopic to identity, but not equal to identity in general). From here checking that the desired diagram is indeed commutative up to homotopy is straightforward (the diagram does not commute in $\mathrm{Kom}(\mathcal{A})$, but it comuutes up to homotopy). $\square$

Proposition 25. *Let $f : A^\bullet \to B^\bullet$ be a quasi-isomorphism and $g : C^\bullet \to B^\bullet$ be an arbitrary morphism. Then there exists a commutative diagram in* $\mathrm{K}(\mathcal{A})$ *:*

$$\begin{tikzcd} C_0^\bullet \arrow[r, "qis"] \arrow[d] & C^\bullet \arrow[d, "g"] \\ A^\bullet \arrow[r, "f", "qis"'] & B^\bullet \end{tikzcd}$$

Proof. Consider the commutative diagram

$$\begin{tikzcd} C(\tau \circ g)[-1] \arrow[r] \arrow[d, dotted] & C^\bullet \arrow[r, "\tau\circ g"] \arrow[d, "g"'] & C(f) \arrow[r] \arrow[d, "\cong"'] & C(\tau \circ g) \arrow[d, dotted] \\ A^\bullet \arrow[r, "f"] & B^\bullet \arrow[r, "\tau"] & C(f) \arrow[r, "\pi"] & A^\bullet[1] \end{tikzcd}$$

By the previous proposition, we know that $B^\bullet \xrightarrow{\tau} \mathrm{C}(f) \longrightarrow A^\bullet[1]$ is isomorphic (in $\mathrm{K}(\mathcal{A})$) to $B^\bullet \xrightarrow{\tau} \mathrm{C}(f) \longrightarrow \mathrm{C}(\tau)$, and thus it suffices to use the natural morphism $\mathrm{C}(\tau \circ g) \to \mathrm{C}(\tau)$ given by the identity on the second factor of $C^{n+1} \oplus \mathrm{C}(f)^n \to B^{n+1} \oplus \mathrm{C}(f)^n$.

Now define $C_0^\bullet := \mathrm{C}(\tau \circ g)[-1]$. Notice that $C_0^\bullet \to C^\bullet$ is a quasi-isomorphism. Indeed, since $A^\bullet \to B^\bullet$ is a quasi-isomorphism, we have that $H^n(\mathrm{C}(f)) = 0$ for all $n \in \mathbb{Z}$, and then applying the long exact sequence in cohomology to $\tau \circ g$ we get:

$$\cdots \longrightarrow H^n\left(C^\bullet\right) \longrightarrow H^n(\mathrm{C}(f)) \longrightarrow H^n(\mathrm{C}(\tau \circ g)) \longrightarrow H^{n+1}\left(C^\bullet\right) \longrightarrow \ldots.$$

But since $H^n(\mathrm{C}(f)) = 0$ we have that $H^n(\mathrm{C}(\tau \circ g)) \simeq H^{n+1}\left(C^\bullet\right)$ for all $n \in \mathbb{Z}$. $\square$

Remark : (I) By construction, if g is also a quasi-isomorphism, so is $C_0^\bullet \to A^\bullet$.

(II) dual statement holds: assume that we have a quasi-isomorphism $f : B^\bullet \to A^\bullet$ and any morphism $B^\bullet \to C^\bullet$. Then we can construct a commutative diagram:

$$\begin{tikzcd} B^\bullet \arrow[r, "f", "qis"'] \arrow[d] & A^\bullet \arrow[d] \\ C^\bullet \arrow[r, "qis"'] & C_0^\bullet \end{tikzcd}$$

Corollary 5.2. *The composition of arrows exists and is well-defined.*

Proof. Applying the previous proposition to the following diagram gives us the result

$$\begin{tikzcd} C_1^\bullet \arrow[dr] & & C_2^\bullet \arrow[dl, "qis"'] \\ & B^\bullet & \end{tikzcd}$$

$\square$

Hence we can summarize construction of the derived category by the following diagram and proposition

$$\begin{tikzcd} \mathrm{Kom}(\mathcal{A}) \arrow[rr] \arrow[dr, bend right] & & \mathrm{D}(\mathcal{A}) \\ & \mathrm{K}(\mathcal{A}) \arrow[ur, bend right] & \end{tikzcd}$$

Proposition 26. *The composition of the functors:*

$$\mathcal{A} \to \mathrm{Kom}(\mathcal{A}) \to K(\mathcal{A}) \to D(\mathcal{A})$$

is fully faithful.

Remark : Recall that the functor $\mathcal{A} \to \mathrm{Kom}(\mathcal{A})$ sends an object A to the complex with A in degree zero and the zero object in all other degrees and the functor $K(\mathcal{A}) \to D(\mathcal{A})$ sends a $K(\mathcal{A})$-morphism $A \to B$ represented by $f \in \hom_{\mathrm{Kom}(\mathcal{A})}(A, B)$ to the $D(\mathcal{A})-$ morphism represented by $A \xleftarrow{id} A \xrightarrow{f} B$.

Proof. Let A and B be objects of $\mathcal{A}$. That $\mathcal{A} \to \mathrm{Kom}(\mathcal{A})$ is fully faithful is clear. By definition hom $_{\mathrm{Kom}(\mathcal{A})}(A, B) \to \mathrm{Hom}_{K(A)}(A, B)$ is surjective so consider a morphism $f \in \mathrm{Hom}_{\mathrm{Kom}(\mathcal{A})}(A, B)$ homotopically equivalent to zero (with A and B still in the image of $\mathcal{A} \to \mathrm{Kom}(\mathcal{A})$). That means there is a morphism $s \in$ Hom $_{\mathrm{Kom}(\mathcal{A})}(A[1], B)$ such that $ds - sd = f$. But all differentials in the complexes A and B are zero and so $f = 0$. Hence, $\hom_{\mathrm{Kom}(\mathcal{A})}(A, B) \cong \hom_{K(\mathcal{A})}(A, B)$. Now consider the morphism Hom $_{\mathcal{A}}(A, B) \to \mathrm{Hom}_{D(\mathcal{A})}(A, B)$. Surjectivity. Let $A \xleftarrow{s} C \xrightarrow{f} B$ represent a morphism in hom $_{D(\mathcal{A})}(A, B)$. So s is a quasi-isomorphism. Then

$$H^i(C) = \begin{cases} 0 & i \neq 0 \\ A & i = 0 \end{cases}$$

and so there exists a quasi-isomorphism $A \xrightarrow{t} C$. This means the composition $A \to C \to A$ is also a quasi-isomorphism and since A is concentrated in degree zero $st = id_A$. So the diagram

$$\begin{tikzcd}
 & C \arrow[dl, swap, "s"] \arrow[dr, "f"] & \\
A \arrow[r, "\cong"] \arrow[dr, swap, "\cong"] & A \arrow[u, swap, "t"] \arrow[r, "ft"] & B \\
 & A \arrow[u, "\cong"] \arrow[ur, swap, "ft"] &
\end{tikzcd}$$

commutes and therefore, $A \xleftarrow{s} C \xrightarrow{t} B$ is equivalent to $A \xleftarrow{id} A \xrightarrow{ft} B$ which is in the image of $\mathrm{Hom}\,_{K(\mathcal{A})}(A, B) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{Kom}(\mathcal{A})}(A, B) \to \mathrm{Hom}_{D(\mathcal{A})}(A, B)$.

Injectivity. A morphism $f \in \mathrm{Hom}_{\mathcal{A}}(A, B)$ gets sent to zero if and only if there is a commutative diagram:

$$\begin{tikzcd}
 & A \arrow[dl, swap, "\cong"] \arrow[dr, "f"] & \\
A \arrow[dr, swap, "\cong"] & C \arrow[u, swap, "g"] \arrow[l] \arrow[r] \arrow[d] & B \\
 & A \arrow[ur, swap, "0"] &
\end{tikzcd}$$

where the inside morphisms are quasi-equivalences. That is, if and only if there is a complex C and a quasi-equivalence $g : C \to A$ such that $fg = 0$ in $K(\mathcal{A})$. But this means that the morphism $H^0 C \cong A \to B$ induced on zeroth homology groups is also zero. Hence $A \to B$ is zero.

Remark : (I) The natural functor $\mathcal{Q}_{\mathcal{A}} : \mathrm{K}(\mathcal{A}) \to \mathrm{D}(\mathcal{A})$ is identity on objects and sends a (homotopy class of a) morphism $f : A^\bullet \to B^\bullet$ to the roof

$$\begin{tikzcd}
 & A^\bullet \arrow[dl, swap, "id"] \arrow[dr] & \\
A^\bullet & & B^\bullet
\end{tikzcd}$$

If f is a quasi-isomorphism, then $\mathcal{Q}_{\mathcal{A}}(f)$ is an isomorphism, which inverse is given by the roof

$$\begin{tikzcd}
 & A^\bullet \arrow[dl, swap, "f"] \arrow[dr] & \\
B^\bullet & & A^\bullet
\end{tikzcd}$$

But, the converse is not true, i.e. there can be complexes $A^\bullet, B^\bullet$ isomorphic in $\mathrm{D}(\mathcal{A})$, but there does not exist a quasi-isomorphism $A^\bullet \to B^\bullet$.
Let

Definition 5.14. *We say that a triangle*

$$A_1^\bullet \longrightarrow A_2^\bullet \longrightarrow A_3^\bullet \longrightarrow A_1^\bullet[1]$$

in $\mathrm{K}(\mathcal{A})$ *(resp.* $\mathrm{D}(\mathcal{A})$ *) is distinguished if it is isomorphic in* $\mathrm{K}(\mathcal{A})$ *(resp.* $\mathrm{D}(\mathcal{A})$ *) to a triangle of the form*

$$A^\bullet \xrightarrow{f} B\cdot \xrightarrow{\tau} \mathrm{C}(f) \xrightarrow{\pi} A^\bullet[1],$$

where $f : A^\bullet \to B^\bullet$ *is a morphism of complexes.*

Proposition 27. *The natural shift functor* $A^\bullet \to A^\bullet[1]$ *and distinguished triangles given as in Definition 1.14 make the homotopy category of complexes* $\mathrm{K}(\mathcal{A})$ *and the derived category* $\mathrm{D}(\mathcal{A})$ *of an abelian category* $\mathcal{A}$ *into triangulated categories.*

Proof. We are going to avoid this long and technical proof. See [2, Chapter I, 2]. $\square$

5.3 Properties of the Derived category

In the following propositions we will list some of the useful properties of the derived category

Proposition 28. *Let* $A, B, C \in \mathcal{A}$*. We identify an object in* $\mathcal{A}$ *with its image under the full embedding* $\mathcal{A} \to \mathrm{K}(\mathcal{A})$*, i.e. with the associated complex concentrated in degree 0 . If the sequence*

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

is exact in $\mathcal{A}$ *then the triangle*

$$A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow A[1]$$

is distinguished in $\mathrm{D}(\mathcal{A})$*.*

Proof. First, we need to define an arrow $C \to A[1]$. Notice that $\mathrm{C}(f)$ can be identified in this case with the complex $\cdots \longrightarrow 0 \longrightarrow A \xrightarrow{f} B \longrightarrow 0 \longrightarrow \cdots$, with A in degree -1 and B in degree 0 . Thus we can define the morphism of complexes $\mathrm{C}(f) \to C$ as :

$$\begin{tikzcd}
\dots \arrow[r] & 0 \arrow[r] \arrow[d] & A \arrow[r, "f"] \arrow[d] & B \arrow[r] \arrow[d, "g"] & 0 \arrow[r] \arrow[d] & \dots \\
\dots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & C \arrow[r] & 0 \arrow[r] & \dots
\end{tikzcd}$$

In particular, this morphism is a quasi-isomorphism by exactness of the initial short exact sequence, and thus there is a inverse $C \to \mathrm{C}(f)$ in $\mathrm{D}(\mathcal{A})$. Thus one can define the arrow $\delta : C \to A[1]$ by composing the arrows $C \to \mathrm{C}(f)$ and the natural morphism $\mathrm{C}(f) \to A[1]$. We obtain the isomorphism of triangles (in $\mathrm{D}(\mathcal{A})$) :

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "id"'] & B \arrow[r, "g"] \arrow[d, "id"'] & C \arrow[r] \arrow[d] & A[1] \arrow[d, "id"'] \\
A \arrow[r, "f"] & B \arrow[r] & C(f) \arrow[r] & A[1]
\end{tikzcd}$$

Note that these arrows are in $\mathrm{D}(\mathcal{A})$ so they should be thought of as roofs. □

Proposition 29. *Suppose $A^\bullet \to B^\bullet \to C^\bullet \to A^\bullet[1]$ is a distinguished triangle in $\mathrm{D}(\mathcal{A})$. Then there is a natural exact sequence*

$$\cdots \to H^n(A^\bullet) \to H^n(B^\bullet) \to H^n(C^\bullet) \to H^{n+1}(A^\bullet) \to \cdots$$

Proof. By definition of distinguished triangles, we have an isomorphism

$$\begin{tikzcd}
A^\bullet \arrow[r] \arrow[d, "\cong"] & B^\bullet \arrow[r] \arrow[d, "\cong"] & C^\bullet \arrow[r] \arrow[d, "\cong"] & A^\bullet[1] \arrow[d] \\
A_0^\bullet \arrow[r, "f"] & B_0^\bullet \arrow[r, "\tau"] & C(f) \arrow[r, "\pi"] & A_0^\bullet[1]
\end{tikzcd}$$

where f is a morphism of complexes, and the vertical arrows are isomorphisms in $\mathrm{D}(\mathcal{A})$, i.e. quasi-isomorphisms. The sequence with the mapping cone induces a long exact sequence in cohomology, and by the isomorphisms in cohomology we get:

$$\begin{tikzcd}
\dots \arrow[r] & H^n(A_0^\bullet) \arrow[r] \arrow[d, "\cong"] & H^n(B_0^\bullet) \arrow[r] \arrow[d, "\cong"] & H^n(C(f)) \arrow[r] \arrow[d, "\cong"] & H^n(A_0^\bullet[1]) \arrow[r] \arrow[d, "\cong"] & \dots \\
& H^n(A^\bullet) & H^n(B^\bullet) & H^n(C^\bullet) & H^{n+1}(A^\bullet) &
\end{tikzcd}$$

Since the first row is exact, up to composing with the isomorphisms, we get the desired natural exact sequence. □

In the following, we will consider the full subcategory $\mathrm{Kom}^*(\mathcal{A})$, with $* = +, -$ or b, consisting of complexes $A^\bullet$ with $A^n = 0$ for $n \ll 0, n \gg 0$ or $|n| \gg 0$ respectively. The same construction we performed before can be applied again to obtain the categories $\mathrm{K}^*(\mathcal{A})$ and $\mathrm{D}^*(\mathcal{A})$.

Proposition 30. *The natural functors* $\mathrm{D}^*(\mathcal{A}) \to \mathrm{D}(\mathcal{A})$*, where* $* = +, -$ *or* b*, define equivalences of* $\mathrm{D}^*(\mathcal{A})$ *with the full triangulated subcategories of all complexes* $A^\bullet \in \mathrm{D}(\mathcal{A})$ *with* $H^n(A^\bullet) = 0$ *for* $n \ll 0, n \gg 0$ *and* $|n| \gg 0$ *respectively.*

Proof. Suppose $H^n(A^\bullet) = 0$ for $n > n_0$. Then we have a quasi-isomorphism

$$\begin{tikzcd}
\dots \arrow[r] & A^{n_0-2} \arrow[r] \arrow[d, "\cong"] & A^{n_0-1} \arrow[r] \arrow[d, "\cong"] & \ker(d^{n_0}) \arrow[r] \arrow[d, hook] & 0 \arrow[r] \arrow[d] & \dots \\
\dots \arrow[r] & A^{n_0-2} \arrow[r] & A^{n_0-1} \arrow[r] & A^{n_0} \arrow[r] & A^{n_0+1} \arrow[r] & \dots
\end{tikzcd}$$

Thus $A^\bullet$ is isomorphic in $\mathrm{D}(\mathcal{A})$ to a complex bounded above, i.e. a complex in $\mathrm{D}^-(\mathcal{A})$. Similarly, if $H^n(A^\bullet) = 0$ for $n < n_0$ one considers:

$$\begin{tikzcd}
\dots \arrow[r] & A^{n_0-1} \arrow[r] \arrow[d] & A^{n_0} \arrow[r] \arrow[d] & A^{n_0+1} \arrow[r] \arrow[d, "\cong"] & A^{n_0+2} \arrow[r] \arrow[d, "\cong"] & \dots \\
\dots \arrow[r] & 0 \arrow[r] & \mathrm{coker}(d^{n_0-1}) \arrow[r] & A^{n_0+1} \arrow[r] & A^{n_0+2} \arrow[r] & \dots
\end{tikzcd}$$

In the case $* =$ b, one can use both sequences combined. These prove that the functors are essentially surjective, and they are clearly fully faithful, so the proof is finished. □

Before going to the next section, we need to define the notion of injective and projective resolutions, which will be useful when we will need to extend functors $F : \mathcal{A} \to \mathcal{B}$ into functors $RF : \mathrm{D}(\mathcal{A}) \to \mathrm{D}(\mathcal{B})$.

Definition 5.15. *Let* $\mathcal{A}$ *be an abelian category.*
(I) An object $I \in \mathcal{A}$ *(resp.* $P \in \mathcal{A}$ *) is said injective (resp. projective) if the functor* $\mathrm{Hom}(, I)$ *is exact (resp.* $\mathrm{Hom}(P, \quad)$ *is exact).*
(II) We say that the category $\mathcal{A}$ *contains enough injective (resp. enough projectives) objects if for any object* $A \in \mathcal{A}$ *there exists an injective morphism* $A \to I$ *with* I *injective (resp. a*

surjective morphism $P \to A$ with P projective).

(III) An injective resolution of an object $A \in \mathcal{A}$ is an exact sequence

$$0 \to A \to I^0 \to I^1 \to \cdots$$

with all I^n injective. Similarly, a projective resolution of A consists in an exact sequence

$$\cdots \to P^{-1} \to P^0 \to A \to 0$$

with all P^n projective.

Proposition 31. *Suppose that $\mathcal{A}$ is a category with enough injectives. For any $A^\bullet \in \mathrm{K}^+(\mathcal{A})$, there exist a complex $I^\bullet \in \mathrm{K}^+(\mathcal{A})$ with $I^n \in \mathcal{A}$ injective $\forall n \in \mathbb{Z}$ and a quasi-isomorphism $A^\bullet \to I^\bullet$.*

Proof. See [6, Proposition 2.35]. □

Corollary 5.3. *Let $\mathcal{A}$ be an abelian category with enough injectives. Any object $A^\bullet \in \mathrm{D}(\mathcal{A})$ with $H^n(A^\bullet) = 0$ for $n \ll 0$ is isomorphic in $\mathrm{D}(\mathcal{A})$ to a complex $I^\bullet$ of injective objects with $I^n = 0$ for $n \ll 0$.*

A dual statement of Proposition 12 is true in a category with enough projectives, considering $\mathrm{K}^-(\mathcal{A})$ instead of $\mathrm{K}^+(\mathcal{A})$: for any $A^\bullet \in \mathrm{K}^-(\mathcal{A})$ there exists a complex $P^\bullet \in \mathrm{K}^-(\mathcal{A})$ with $P^n \in \mathcal{A}$ projective objects and a quasi-isomorphism $P^\bullet \to A^\bullet$.

Proofs of the next two technical lemmas can be found in [6, Lemma 2.38] and [6, Lemma 2.39].

Lemma 5.1. *Suppose $A^\bullet \to B^\bullet$ is a quasi-isomorphism between two complexes $A^\bullet, B^\bullet \in \mathrm{K}^+(\mathcal{A})$. Then for any complex $I^\bullet$ of injectives objects with $I^n = 0$ for $n \ll 0$ the induced map*

$$\mathrm{Hom}_{\mathrm{K}(\mathcal{A})}(B^\bullet, I^\bullet) \xrightarrow{\simeq} \mathrm{Hom}_{\mathrm{K}(\mathcal{A})}(A^\bullet, I^\bullet)$$

is bijective.

Lemma 5.2. *Let $A^\bullet, I^\bullet \in \mathrm{K}^+(\mathcal{A})$ such that all I^n are injective. Then*

$$\mathrm{Hom}_{\mathrm{K}(\mathcal{A})}(A^\bullet, I^\bullet) = \mathrm{Hom}_{\mathrm{D}(\mathcal{A})}(A^\bullet, I^\bullet)$$

This lemma says something very important, it tells us that any morphism between $A^\bullet$ and $I^\bullet$ in the derived category can be lifted up to a morphism in the homotopoy category. For the next proposition, consider the full additive subcategory $\mathcal{I} \subset \mathcal{A}$ of all injectives of an abelian category $\mathcal{A}$: we can construct as before the homotopy category $\mathrm{K}^*(\mathcal{I})$ and the functor $\mathcal{Q}_{\mathcal{A}}$ induces a natural exact fundtor $\iota : \mathrm{K}^*(\mathcal{I}) \to \mathrm{D}^*(\mathcal{A})$.

Proposition 32. *Suppose that $\mathcal{A}$ contains enough injectives. Then the natural functor*

$$\iota : \mathrm{K}^+(\mathcal{I}) \to \mathrm{D}^+(\mathcal{A})$$

is an equivalence.

Proof. The functor is fully faithful. Indeed, let $I^\bullet, J^\bullet$ be two complexes in $\mathrm{K}^+(\mathcal{I})$. Since $\mathcal{I}$ is a full subcategory and by the previous lemma, we have

$$\mathrm{Hom}_{\mathrm{K}^+(\mathcal{I})}(I^\bullet, J^\bullet) \simeq \mathrm{Hom}_{\mathrm{K}^+(\mathcal{A})}(I^\bullet, J^\bullet) \simeq \mathrm{Hom}_{\mathrm{D}^+(\mathcal{A})}(I^\bullet, J^\bullet)$$

To see that the functor is also essentially surjective, one applies Proposition 12. □

5.4 Derived functors

In this section, the main goal will be to lift functors between abelian categories (or homotopy categories) to functors between the associated derived categories.

Lemma 5.3. *Let $\mathcal{A}, \mathcal{B}$ be abelian categories, let $F : \mathrm{K}^*(\mathcal{A}) \to \mathrm{K}^*(\mathcal{B})$ be an exact functor of triangulated categories. Then F naturally induces a commutative diagram:*

$$\begin{array}{ccc} K^*(\mathcal{A}) & \longrightarrow & K^*(\mathcal{B}) \\ \downarrow & & \downarrow \\ D^*(\mathcal{A}) & \longrightarrow & D^*(\mathcal{B}) \end{array}$$

if one of the following equivalent conditions holds true:

1. A quasi-isomorphism is mapped by F to a quasi-isomorphism.

2. The image of an acyclic complex is acyclic.

Proof. First we will show that these two conditions are equivalent. The step $1 \Rightarrow 2$ is obvious. To see $2 \Rightarrow 1$, consider a morphism of complexes $f : A^\bullet \to B^\bullet$, then the triangle

$$A^\bullet \to B^\bullet \to \mathrm{C}(f) \to A^\bullet[1]$$

is distinguished, and $\mathrm{C}(f)$ is acyclic if and only if f is a quasi-isomorphism (using the long exact sequence). Now since F is exact and additive, $F(f)$ is a quasi-isomorphism if and only if $\mathrm{C}(F(f)) = F(\mathrm{C}(f))$ is acyclic. Hence we're done.

Assume that 1 is satisfied. The functor F can easily be lifted up to the derived categories: an object $A^\bullet$ is mapped to $F(A^\bullet)$, viewed as objects in the derived categories, and a roof

$$\begin{tikzcd} & C^\bullet \arrow[dl, "qis"'] \arrow[dr] & \\ A^\bullet & & B^\bullet \end{tikzcd}$$

is mapped to the roof

$$\begin{tikzcd} & F(C^\bullet) \arrow[dl, "qis"'] \arrow[dr] & \\ F(A^\bullet) & & F(B^\bullet) \end{tikzcd}$$

This completes the proof. $\square$

Now let $F : \mathcal{A} \to \mathcal{B}$ be a left exact functor of abelian categories, and assume that $\mathcal{A}$ contains enough injectives. The functor F induces a functor $\mathrm{K}(F) : K^+(\mathcal{A}) \to K^+(\mathcal{B})$ sending a complex $(A^n)_{n\in\mathbb{Z}}$ to $(F(A^n))_{n\in\mathbb{Z}}$, and a morphism of complexes $(f^n)_{n\in\mathbb{Z}}$ to $(F(f^n))_{n\in\mathbb{Z}}$. The latter makes sense in the homotopy categories: if h is a homotopy between to morphisms of complexes f and g, then $F(h)$ is a homotopy between $F(f)$ and $F(g)$ since F is additive.

We have the equivalence $\iota : \mathrm{K}^+(\mathcal{I}_\mathcal{A}) \to \mathrm{D}^+(\mathcal{A})$, so we can consider a quasi-inverse ι^{-1} of ι by choosing a complex of injective objects quasi-isomorphic to any given complex that is bounded below. We obtain the diagram:

$$\begin{tikzcd} \mathrm{K}^+(\mathcal{I}_\mathcal{A}) \arrow[r, hook] \arrow[dr, shift left, "\iota"] & \mathrm{K}^+(\mathcal{A}) \arrow[r, "K(F)"] \arrow[d, "\mathcal{Q}_\mathcal{A}"] & \mathrm{K}^+(\mathcal{B}) \arrow[d, "\mathcal{Q}_\mathcal{B}"] \\ & \mathrm{D}^+(\mathcal{A}) \arrow[ul, shift left, "\iota^{-1}"] & \mathrm{D}^+(\mathcal{B}) \end{tikzcd}$$

Definition 5.16. *The right derived functor of F is the functor:*

$$RF := \mathcal{Q}_{\mathcal{B}} \circ \mathrm{K}(F) \circ \iota^{-1} : \mathrm{D}^{+}(\mathcal{A}) \longrightarrow \mathrm{D}^{+}(\mathcal{B}).$$

In other words, the right derived functor consists in replacing a complex by a complex of injectives, applying $K(F)$ and embedding it into the target derived category.

Proposition 33. *1. There exists a natural morphism of functors*

$$\mathcal{Q}_{\mathcal{B}} \circ \mathrm{K}(F) \longrightarrow RF \circ \mathcal{Q}_{\mathcal{A}}.$$

2. The right derived functor $RF : \mathrm{D}^{+}(\mathcal{A}) \to D^{+}(\mathcal{B})$ is an exact functor of triangulated categories.

Proof. 1) Let $A^\bullet \in \mathrm{D}^{+}(\mathcal{A})$ and $I^\bullet := \iota^{-1}(A^\bullet)$. The natural transformation $\mathrm{Id} \to \iota \circ \iota^{-1}$ yields a functorial morphism $A^\bullet \to I^\bullet$ in $D^{+}(\mathcal{A})$. This morphism is given by a roof $A^\bullet \leftarrow C^\bullet \to I^\bullet$, but since $I^\bullet$ is injective it yields to a unique morphism $A^\bullet \to I^\bullet$ in $\mathrm{K}(\mathcal{A})$ by Lemma 1.1. Notice that this morphism is independent on the choice of $C^\bullet :$ assume we have two equivalent roofs

$$\begin{tikzcd} & C^\bullet \arrow[dl] \arrow[dr] & \\ A^\bullet \arrow[rr, dotted] & & I^\bullet \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} & D^\bullet \arrow[dl] \arrow[dr] & \\ A^\bullet \arrow[rr, dotted, "t_D"'] & & I^\bullet \end{tikzcd}$$

then it means that we have the commutative diagram

$$\begin{tikzcd} & E^\bullet \arrow[dl] \arrow[dr] \arrow[ddl, bend right, "f"'] \arrow[ddr, bend left, "g"] & \\ C^\bullet \arrow[d] \arrow[drr] & & D^\bullet \arrow[d] \arrow[dll] \\ A^\bullet \arrow[rr, "t_C"] \arrow[rr, bend right, "t_D"'] & & I^\bullet \end{tikzcd}$$

We obtain the equalities $g = t_C \circ f$ and $g = t_D \circ f$. But f is a quasi-isomorphism, so again by Lemma 1.1 there is a unique map $j : A^\bullet \to I^\bullet$ such that $g = j \circ f$. Thus we get $t_C = t_D$. Finally, we obtain a functorial morphism

$$\mathrm{K}(F(A^\bullet)) \to \mathrm{K}(F(I^\bullet)) = RF(A^\bullet)$$

2) The category $K^+(\mathcal{I}_{\mathcal{A}})$ is triangulated: if $f : I^\bullet \to J^\bullet$ is a morphism of complexes between complexes of injective objects, then $\mathrm{C}(f)$ is also a complex of injective objects. The functor $\iota : \mathrm{K}^+(\mathcal{I}_{\mathcal{A}}) \to \mathrm{D}^+(\mathcal{A})$ is clearly an exact functor (between triangulated categories), and thus ι^{-1} is also exact (cf. Proposition 3). Moreover, $\mathrm{K}(F)$ is exact: F is additive, so F preserves mapping cones. Finally, since $\mathcal{Q}_{\mathcal{B}}$ is exact, we obtain that RF is the composition of three exact functors and, therefore, is itself exact. □

Definition 5.17. *Let $RF : \mathrm{D}^+(\mathcal{A}) \to \mathrm{D}^+(\mathcal{B})$ be the right derived functor of a left exact functor $F : \mathcal{A} \to \mathcal{B}$. Then for any complex $A^\bullet \in \mathrm{D}^+(\mathcal{A})$ we define:*

$$R^iF(A^\bullet) := H^i(RF(A^\bullet)) \in \mathcal{B}.$$

Remark : (I) If A is a complex concentrated in degree 0, then we can give a more precise description of $R^iF(A)$. Indeed, consider an injective resolution $I^\bullet$ of A, i.e. an exact sequence

$$0 \to A \to I^0 \to I^1 \to \cdots .$$

We obtain that $R^iF(A) = H^i(F(I^\bullet))$, and in particular we have $R^0F(A) = F(A)$.

(II) Any short exact sequence

$$0 \to A \to B \to C \to 0$$

in $\mathcal{A}$ gives rise to a long exact sequence

$$0 \to F(A) \to F(B) \to F(C) \to \cdots \to R^nF(B) \to R^nF(C) \to R^{n+1}F(A) \to \cdots .$$

Indeed, the exact sequence in $\mathcal{A}$ gives rise to a distinguished triangle $RF(A) \to RF(B) \to RF(C) \to RF(A)[1]$ then looking at the long exact sequence from this we get our desired sequence.

(III) All the constructions we made could have been performed in the dual way: if you consider a functor F which is right exact, $\mathrm{K}(F) : \mathrm{K}^-(\mathcal{A}) \to \mathrm{K}^-(\mathcal{B})$ and then define the left derived functor LF by applying $\mathrm{K}(F)$ to a complex $P^\bullet$ of projective objects quasi-isomorphic to $A^\bullet$.

Now we will give a generalization of the construction of the right derived functor.

Proposition 34. *Let $\mathcal{A}, \mathcal{B}$ be abelian categories, and $F : \mathrm{K}^+(\mathcal{A}) \to \mathrm{K}(\mathcal{B})$ an exact functor. Suppose there exists a triangulated subcategory $\mathcal{K}_F \subset \mathrm{K}^+(\mathcal{A})$ which is adapted to F, i.e. which satisfies the following two conditions:*
1. If $A^\bullet \in \mathcal{K}_F$ is acyclic, then $F(A^\bullet)$ is acyclic.
2. Given any $A^\bullet \in \mathrm{K}^+(\mathcal{A})$ there is an object $T_A \in \mathcal{K}_F$ and a quasi-isomorphism $A^\bullet \to T_A \bullet$
Then there exists a right derived functor $RF : \mathrm{D}^+(\mathcal{A}) \to \mathrm{D}(\mathcal{B})$ satisfying the properties of Proposition 14.

Proof. The functor RF is defined as follows:

• Let $A^\bullet \in \mathrm{D}^+(\mathcal{A})$. There is a quasi-isomorphism $A^\bullet \to T_{A^\bullet}$ for some $T_{A^\bullet} \in \mathcal{K}_F$. Then define $RF(A^\bullet) := F(T_{A^\bullet})$.

• Let $A^\bullet, B^\bullet \in \mathrm{D}^+(\mathcal{A})$. Consider an arrow $A^\bullet \to B^\bullet$ in $\mathrm{D}^+(\mathcal{A})$ given by a roof $A^\bullet \leftarrow C^\bullet \to B^\bullet$. In $\mathrm{K}^+(\mathcal{A})$ we have the diagram

$$\begin{tikzcd} C^\bullet \arrow[r, "qis"] \arrow[d, "qis"'] & T_{C^\bullet} \\ T_{A^\bullet} & \end{tikzcd}$$

and it can be completed in

$$\begin{tikzcd} C^\bullet \arrow[r, "qis"] \arrow[d, "qis"'] & T_{C^\bullet} \arrow[d, "qis"] \\ T_{A^\bullet} \arrow[r, "qis"'] & D_A^\bullet \end{tikzcd}$$

$\mathrm{K}^+(\mathcal{A})$ into the diagram Composing with the quasi-isomorphism $D_A^\bullet \to T_{D_A}$, and doing the same with $B^\bullet$, we obtain a roof

$$\begin{tikzcd} & T_{C^\bullet} \arrow[dl, "qis"'] \arrow[dr] & \\ T_{D_A^\bullet} & & T_{D_B^\bullet} \end{tikzcd}$$

Since $T_{D_A^\bullet}$ and $T_{A^\cdot}$ are quasi-isomorphic within $\mathcal{K}_F$, so are their images by the functor F (proof of Lemma 1.3), and the same holds with $B^\bullet$. Thus they define isomorphic objects in

$\mathrm{D}(\mathcal{B})$. We defined the image of our initial arrow $A^\bullet \to B^\bullet$ in $D^+(\mathcal{A})$ by RF as the arrow given by the roof

$$\begin{tikzcd} & F(T_{C^\bullet}) \arrow[dl, "qis"'] \arrow[dr] & \\ F(T_{A^\bullet}) & & F(T_{B^\bullet}) \end{tikzcd}$$

□

Corollary 5.4. *Let $F : \mathcal{A} \to \mathcal{B}$ be a left exact functor (here $\mathcal{A}$ might not contain enough injectives), and assume that there exists a subclass of objects $\mathcal{I}_F \subset \mathcal{A}$ which are F-adapted, i.e. which is stable by finite sum and such that:*

1. *If $A^\bullet \in \mathrm{K}^+(\mathcal{A})$ is acyclic with all $A^\bullet \in \mathcal{I}_F$, then $F(A^\bullet)$ is acyclic.*

2. *Any object in $\mathcal{A}$ can be embedded into an object of $\mathcal{I}_F$. Then there exists a right derived functor $RF : \mathrm{D}^+(\mathcal{A}) \to D(\mathcal{B})$ satisfying the properties of Proposition* 14.

Proof. It suffices to check that the subcategory $\mathcal{K}_F \subset \mathrm{K}^+(\mathcal{A})$ defined as the full subcategory of complexes of objects in $\mathcal{I}_F$ satisfies the hypothesis of Proposition 15.

First, since $\mathcal{I}_F$ is stable by finite sum, $\mathcal{K}_F$ contains all mapping cones of morphism between any complexes in it. Then, by the Proposition $4, \mathcal{K}_F$ is indeed a triangulated subcategory of $K^+(\mathcal{A})$. Now, we just need to check the two conditions of the theorem. The first condition is obvious, and the second conditions can be proved by the same proof given for Proposition 12. □

Definition 5.18. *Let $A \in \mathcal{A}$ be an object in an abelian category containing enough injectives. Then we defined*

$$\mathrm{Ext}^n(A, \quad) := H^n \circ R\,\mathrm{Hom}(A, \quad).$$

Proposition 35. *Suppose $A, B \in \mathcal{A}$ are objects of an abelian category containing enough injectives. Then for all $n \in \mathbb{Z}$ there is a natural isomorphism*

$$\mathrm{Ext}^n(A, B) \simeq \mathrm{Hom}_{\mathrm{D}(\mathcal{A})}(A, B[n]).$$

Proof. Notice that here we identify once again objects in $\mathcal{A}$ with complexes concentrated in degree 0. Consider an injective resolution $B \to I^0 \to I^1 \to \cdots$, then R Hom $(A, B) \simeq$

$(\operatorname{Hom}(A, I^n))_{n\in\mathbb{N}}$. Now $f \in \operatorname{Hom}(A, I^n)$ is the kernel of $\operatorname{Hom}(A, I^n) \to \operatorname{Hom}(A, I^{n+1})$ if and only if it defines a morphism of complexes $f : A \to I^\bullet[n]$. Such a morphism is (homotopically) trivial if and only if f is in the image of $\operatorname{Hom}(A, I^{n-1}) \to \operatorname{Hom}(A, I^n)$. These last claims reads on the diagram

$$\begin{tikzcd}
\dots \arrow[r] & 0 \arrow[r] & A \arrow[r] \arrow[d, "f"] \arrow[dl, dotted] & 0 \arrow[r] & \dots \\
\dots \arrow[r] & I^{n-1} \arrow[r] & I^n \arrow[r] & I^{n+1} \arrow[r] & \dots
\end{tikzcd}$$

Then $\operatorname{Ext}^n(A, B) \simeq \operatorname{Hom}_{K(\mathcal{A})}(A, I^\bullet[n]) \simeq \operatorname{Hom}_{\mathrm{D}(\mathcal{A})}(A, I^\bullet[n])$ since $I^\bullet$ is a complex of injectives. $\square$

Definition 5.19. *Let* $A^\bullet \in \operatorname{Kom}(\mathcal{A})$ *and* $B^\bullet \in \mathrm{K}^+(\mathcal{A})$. *We defined the inner hom* $\operatorname{Hom}^\bullet(A^\bullet, B^\bullet)$ *as the complex*

$$\operatorname{Hom}^n(A^\bullet, B^\bullet) := \bigoplus_{k\in\mathbb{Z}} \operatorname{Hom}\left(A^k, B^{k+n}\right)$$

with differentials $d\left((f_k)_{k\in\mathbb{Z}}\right) := d_B \circ f_k - (-1)^n f_{k+1} \circ d_A$.

Proposition 36. *Let* $A^\bullet \in \operatorname{Kom}(\mathcal{A})$ *be a complex of objects in a abelian category containing enough injectives. The the right derived functor*

$$R\operatorname{Hom}^\bullet(A^\bullet, \quad) : \mathrm{D}^+(\mathcal{A}) \to \mathrm{D}(\mathbf{Ab})$$

exists, and if we set $\operatorname{Ext}^n(A^\bullet, B^\bullet) := H^n(R\operatorname{Hom}^\bullet(A^\bullet, B^\bullet))$ *we have*

$$\operatorname{Ext}^n(A^\bullet, B^\bullet) \simeq \operatorname{Hom}_{\mathrm{D}(\mathcal{A})}(A^\bullet, B^\bullet[n]).$$

Proof. To prove the existence of $R\operatorname{Hom}^\bullet(A^\bullet, B^\bullet)$, one checks that the full triangulated subcategory of $\mathrm{K}^+(\mathcal{A})$ of complexes of injectives objects is adapted to this functor. The second statement follows from arguments of the proof of Proposition 16 adapted to this more general situation. $\square$

Proposition 37. *Let* $F_1 : \mathcal{A} \to \mathcal{B}$ *and* $F_2 : \mathcal{B} \to \mathcal{C}$ *be two left exact functors between abelian categories. Assume that there exist adapted classes* $\mathcal{I}_{F_1} \subset \mathcal{A}$ *and* $\mathcal{I}_{F_2} \subset \mathcal{B}$ *for* F_1 *and* F_2 *respectively such that* $F(\mathcal{I}_{F_1}) \subset \mathcal{I}_{F_2}$.
Then the derived functor $R(F_2 \circ F_2) : \mathrm{D}^+(\mathcal{A}) \to \mathrm{D}^+(\mathcal{C})$ *exists and there is a natural isomorphism*

$$R(F_2 \circ F_1) \simeq RF_2 \circ RF_1.$$

Proof. The existence of RF_1 and RF_2 are provided by the assumptions, and since $\mathcal{I}_{F_1}$ is adapted to $F_2 \circ F_1, R\left(F_2 \circ F_1\right)$ exists as well. The natural isomorphism is given by the following remark. Let $A^\bullet \in \mathrm{D}^+(\mathcal{A})$ be isomorphic to $I^\bullet \in \mathrm{K}^+\left(\mathcal{I}_{F_1}\right)$, then

$$\begin{aligned} R\left(F_2 \circ F_1\right)\left(A^\bullet\right) &\simeq \mathrm{K}\left(F_2 \circ F_1\right)\left(I^\bullet\right) \\ &\simeq \left(\mathrm{K}\left(F_2\right) \circ \mathrm{K}\left(F_1\right)\right)\left(I^\bullet\right) \\ &\simeq \mathrm{K}\left(F_2\right)\left(\mathrm{K}\left(F_1\right)\left(I^\bullet\right)\right) \\ &\simeq RF_2\left(\mathrm{K}\left(F_1\right)\left(I^\bullet\right)\right) \\ &\simeq RF_2\left(RF_1\left(A^\bullet\right)\right) \end{aligned}$$

$\square$

6 Derived Categories of Coherent Sheaves

Purpose : In this section we introduce the derived category of coherent sheaves. We show that it is indecomposable if and only if the scheme is connected. We then state a derived version of Serre duality and using this we show that on a smooth curve objects in the derived category can always be written as direct sums of shifted sheaves. In the end we describe the two most common spanning classes in $\mathrm{D}^b(X)$.

6.1 Derived Category of Coherent Sheaves

Definition 6.1. *Let X be a scheme. Its derived category $D^b(X)$ is by definition the bounded derived category of the abelian category $Coh(X)$, i.e.*

$$D^b(X) = D^b(Coh(X)$$

Definition 6.2. *Two schemes X and Y defined over a field k are called derived equivalent if there exists a k-linear exact equivalence $D^*(X) \cong D^*(Y)$, where $* = b, +, -$.*

Before starting this section we need the following proposition. Consider a full abelian subcategory $\mathcal{A} \subset \mathcal{B}$ of an abelian category $\mathcal{B}$. Then there are two derived categories $\mathrm{D}(\mathcal{A})$ and $\mathrm{D}(\mathcal{B})$ with an obvious exact functor $\mathrm{D}(\mathcal{A}) \longrightarrow \mathrm{D}(\mathcal{B})$ between them.

One might wonder whether this functor defines an equivalence between $\mathrm{D}(\mathcal{A})$ and the full subcategory of $\mathrm{D}(\mathcal{B})$ containing those complexes whose cohomology is in $\mathcal{A}$. This does not hold, as in general $\mathrm{D}(\mathcal{A}) \longrightarrow \mathrm{D}(\mathcal{B})$ is neither full nor faithful. Fortunately, in the geometric situation, e.g. passing from $\mathrm{D^b}(\mathrm{Sh}(X))$ to $\mathrm{D^b}(\mathrm{Qcoh}(X))$, things are better behaved, as shown by the next proposition. Recall that a thick subcategory $\mathcal{A}$ of an abelian category $\mathcal{B}$ is a full abelian subcategory such that any extension in $\mathcal{B}$ of objects in $\mathcal{A}$ is again in $\mathcal{A}$.

Proposition 38. *Let $\mathcal{A} \subset \mathcal{B}$ be a thick subcategory and suppose that any $A \in \mathcal{A}$ can be embedded in an object $A' \in \mathcal{A}$ which is injective as an object of $\mathcal{B}$. Then the natural functor $\mathrm{D}(\mathcal{A}) \longrightarrow \mathrm{D}(\mathcal{B})$ induces an equivalence*

$$\mathrm{D}^+(\mathcal{A}) \xrightarrow{\sim} \mathrm{D}^+_{\mathcal{A}}(\mathcal{B})$$

of $\mathrm{D}^+(\mathcal{A})$ and the full triangulated subcategory $\mathrm{D}^+_{\mathcal{A}}(\mathcal{B}) \subset \mathrm{D}^+(\mathcal{B})$ of complexes with cohomology in $\mathcal{A}$. Analogously, one has $\mathrm{D^b}(\mathcal{A}) \simeq \mathrm{D^b_{\mathcal{A}}}(\mathcal{B})$.

Proof. See [1]. $\square$

We will be primarily interested in $\mathrm{Coh}(X)$, but often we will need to leave the abelian category of Coherent sheaves and work with the category of Quasi-Coherent sheaves $\mathrm{Qcoh}(X)$. The reason is this : we want to replace coherent sheaves by their injective resolutions, but there are almost no coherent injective sheaves.

Proposition 39. *On a Noetherian scheme X any quasi-coherent sheaf $\mathcal{F}$ admits a resolution*

$$0 \to \mathcal{F} \to \mathcal{I}^0 \to \mathcal{I}^1 \to \dots$$

by quasi-coherent sheaves $\mathcal{I}^i$ which are injective $\mathcal{O}_X$-modules.

Thus in our case, when X is a smooth projective variety over a field k, the result applies. Thus from the previous lemma we can think of $\mathrm{D}^*(\mathrm{Qcoh}(X))$ as the full triangulated subcategory of $\mathrm{D}^*(\mathrm{Sh}(X))$ of bounded complexes with quasi-coherent cohomology. Thus for any Noetherian schemes there are natural equivalences:

$$\mathrm{D}^*(\mathrm{Qcoh}(X)) \cong \mathrm{D}^*_{\mathrm{qcoh}}(\mathrm{Sh}(X))$$

with $* = b, +$.

In particular, we obtain that $\mathrm{Q\,coh}(X)$ has enough injectives whenever X is at least noetherian. In particular, it permits us to use the spectral sequences defined in Proposition 2.21. Thus for any $\mathcal{F}^\bullet, \mathcal{G}^\bullet \in \mathrm{Q\,coh}(X)$ we have:

$$E_2^{p,q} = \mathrm{Ext}^p\left(\mathcal{F}^\bullet, \mathcal{H}^q\left(\mathcal{G}^\bullet\right)\right) \Rightarrow \mathrm{Ext}^{p+q}\left(\mathcal{F}^\bullet, \mathcal{G}^\bullet\right)$$
$$E_2^{p,q} = \mathrm{Ext}^p\left(\mathcal{H}^q\left(\mathcal{F}^\bullet\right), \mathcal{G}^\bullet\right) \Rightarrow \mathrm{Ext}^{p+q}\left(\mathcal{F}^\bullet, \mathcal{G}^\bullet\right)$$

The following lemma will be helpful in proving the next important proposition

Lemma 6.1. *If $\mathcal{G} \to \mathcal{F}$ is an $\mathcal{O}_X$-module homomorphism from a quasi-coherent sheaf $\mathcal{G}$ onto a coherent sheaf $\mathcal{F}$ on a noetherian scheme X, then there exists a coherent subsheaf $\mathcal{G}' \subset \mathcal{G}$ such that the composition $\mathcal{G}' \subset \mathcal{G} \to \mathcal{F}$ is still surjective.*

Proof. This lemma is clear locally: for any surjection $M \to N$ of modules with N finitely generated, there exists a finitely generated submodule $M' \subset M$ such that the restriction $M' \to N$ is still surjective. The step to the global case is not trivial and follows from the following proposition : $\square$

Proposition 40. *[4, Proposition 5.15] Let S be a graded ring, which is finitely generated by S_1 as an S_0-algebra. Let $X = Proj\ S$, and let $\mathcal{F}$ be a quasi-coherent sheaf on X. Then there is a natural isomorphism $\beta : \widetilde{\Gamma_*(\mathcal{F})} \to \mathcal{F}$.*

Now, apply this proposition to the surjections

$$d^j : \mathcal{G}^j \longrightarrow \operatorname{Im}\left(d^j\right) \quad \text{and} \quad \ker\left(d^j\right) \longrightarrow \mathcal{H}^j$$

which yield subsheaves $\mathcal{G}_1^j \subseteq \mathcal{G}^j$ and $\mathcal{G}_2^j \subseteq \ker(d^j)$. Now define $\widetilde{\mathcal{G}}^j \subseteq \mathcal{G}^j$ the coherent subsheaf generated by $\mathcal{G}_1^j$ and $\mathcal{G}_2^j$, and define $\widehat{\mathcal{G}^{j-1}}$ as the pre-image of $\widetilde{\mathcal{G}^j}$ under the morphism $\mathcal{G}^{j-1} \to \mathcal{G}^j$. We get the injective morphism of complexes:

$$\begin{tikzcd}
\dots \arrow[r] & \mathcal{G}^{j-2} \arrow[r] \arrow[d, "\cong"] & \widetilde{\mathcal{G}^{j-2}} \arrow[r, "\widetilde{d^{j-1}}"] \arrow[d, hook, "i_{j-1}"] & \widetilde{\mathcal{G}^j} \arrow[r] \arrow[d, hook, "i_j"] & \mathcal{G}^{j+1} \arrow[r] \arrow[d, "\cong"] & \dots \\
\dots \arrow[r] & \mathcal{G}^{j-2} \arrow[r] & \mathcal{G}^{j-1} \arrow[r] & \mathcal{G}^j \arrow[r] & \mathcal{G}^{j+1} \arrow[r] & \dots
\end{tikzcd}$$

Notice that i_j induces an isomorphism in cohomology by construction of $\mathcal{G}_2^j$, and the $(j+1)^{th}$ cohomology group of the first row is still $\mathcal{H}^{j+1}$ by construction of $\mathcal{G}_1^j$. Finally, $\widehat{\mathcal{G}}^{j-1}$ is constructed so that $\widetilde{d^{j-1}}$ is well defined. Thus this morphism of complexes is a quasi-isomorphism and $\widetilde{\mathcal{G}^j}$ is coherent.

Although we cannot hope to find an injective resolution of a coherent sheaf by coherent sheaves, we have the following result

Proposition 41. *Let X be a Noetherian scheme. Then the natural functor*

$$D^b(X) \to D^b(Qcoh(X))$$

defines an equivalence between the derived category $D^b(X)$ of X and the full triangulated subcategory $D^b_{Coh}(Qcoh(X))$ of bounded complexes of quasi-coherentsheaves with coherent cohomology.

Proof. Let $\mathcal{G}^\bullet$ be a bounded complex of quasi-coherent sheaves

$$\cdots \longrightarrow 0 \longrightarrow \mathcal{G}^n \longrightarrow \cdots \longrightarrow 0^m \longrightarrow 0$$

with coherent cohomology $\mathcal{H}^i$. Suppose $\mathcal{G}^i$ is coherent for $i > j$. Then apply Lemma 2.1 to the surjections

$$d^j : \mathcal{G}^j \longrightarrow \operatorname{Im}\left(d^j\right) \subset \mathcal{G}^{j+1} \quad \text{and} \quad \operatorname{Ker}\left(d^j\right) \longrightarrow \mathcal{H}^j$$

which yield subsheaves $\mathcal{G}_1^j \subset \mathcal{G}^j$ and $\mathcal{G}_2^j \subset \operatorname{Ker}(d^j) \subset \mathcal{G}^j$, respectively. We may now replace $\mathcal{G}^j$ by the coherent sheaf generated by $\mathcal{G}_i^j, i = 1, 2$, and $\mathcal{G}^{j-1}$ by the pre-image of the new $\mathcal{G}^j$ under $\mathcal{G}^{j-1} \to \mathcal{G}^j$. Clearly, the inclusion defines a quasi-isomorphism of the new complex to the old one and now $\mathcal{G}^i$ is coherent for $i \geq j$. □

Remark : If X is a projective variety over a field k, for any coherent sheaf $\mathcal{F}$ the groups $H^n(X, \mathcal{F})$ are finite-dimensional (Serre, See [4, Theorem 5.19]).

This result can be used to show (by induction) that for any two coherent sheaves $\mathcal{F}, \mathcal{G}$ the groups $\operatorname{Ext}^n(\mathcal{F}, \mathcal{G})$ are also finite-dimensional for all $n \in \mathbb{Z}$. Indeed, the case $n = 0$ comes from the identity $\operatorname{Hom}(\mathcal{F}, \mathcal{G}) = H^0(X, Hom(\mathcal{F}, \mathcal{G}))$. The case $\mathcal{F} = \bigoplus \mathcal{L}_j$ with $\mathcal{L}_j$ locally free sheaves of finite rank comes from the equality

$$\begin{aligned} \operatorname{Ext}^n\left(\bigoplus \mathcal{L}_j, \mathcal{G}\right) &\simeq \bigoplus \operatorname{Ext}^n\left(\mathcal{L}_j, \mathcal{G}\right), \\ &\simeq \bigoplus \operatorname{Ext}^n\left(\mathcal{O}_X, \mathcal{L}^\vee \otimes \mathcal{G}\right) \\ &\simeq \bigoplus H^n\left(X, \mathcal{L}^\vee \otimes \mathcal{G}\right) \end{aligned}$$

Finally, one conclude using that any coherent sheaf can be placed in an exact sequence

$$0 \to \mathcal{K} \to \bigoplus \mathcal{L}_j \to \mathcal{F} \to 0$$

with $\mathcal{L}_j$ locally free (Theorem 3.1). Applying $\operatorname{Hom}(, \mathcal{G})$ we obtain a long exact sequence

$$\cdots \to \operatorname{Ext}^n(\mathcal{K}, \mathcal{G}) \to \operatorname{Ext}^{n+1}(\mathcal{F}, \mathcal{G}) \to \operatorname{Ext}^{n+1}\left(\bigoplus \mathcal{L}_j, \mathcal{G}\right) \cdots$$

Now since the first term is finite-dimensional by induction and the last term is finite-dimensional by the previous case, the middle one is also finite-dimensional.

Eventually, using both spectral sequences, we obtain that $\operatorname{Ext}^n\left(\mathcal{F}^\bullet, \mathcal{G}^\bullet\right)$ is finitedimensional.

Definition 6.3. *The support of a complex $\mathcal{F}^\bullet \in \mathrm{D}^{\mathrm{b}}(X)$ is the union of the supports of all its cohomology sheaves, i.e. it is the closed subset*

$$\operatorname{supp}\left(\mathcal{F}^\bullet\right) := \bigcup \operatorname{supp}\left(\mathcal{H}^i\left(\mathcal{F}^\bullet\right)\right)$$

Lemma 6.2. *Suppose $\mathcal{F}^\bullet \in \mathrm{D^b}(X)$ and $\operatorname{supp}(\mathcal{F}\bullet) = Z_1 \sqcup Z_2$, where $Z_1, Z_2 \subset X$ are disjoint closed subsets. Then $\mathcal{F}\bullet \simeq \mathcal{F}_{\mathbf{i}} \oplus \mathcal{F}_2^\bullet$ with $\operatorname{supp}(\mathcal{F}_j) \subset Z_j$ for $j = 1, 2$.*

Proof. See [6, 3.9]. □

Next using this lemma we will show that the derived category of a scheme is indecomposable if and only if the scheme is connected. For this we will first define what it means for a triangulated category to be decomposed into two subcategories.

Definition 6.4. *A triangulated category $\mathcal{D}$ is decomposed into triangulated subcategories $\mathcal{D}_1, \mathcal{D}_2 \subset \mathcal{D}$ if the following three conditions are satisfied:*
i) Both categories $\mathcal{D}_1$ and $\mathcal{D}_2$ contain objects non-isomorphic to 0.
ii) For all $A \in \mathcal{D}$ there exists a distinguished triangle

$$B_1 \longrightarrow A \longrightarrow B_2 \longrightarrow B_1[1]$$

with $B_i \in \mathcal{D}_i, i = 1, 2$.
iii) $\operatorname{Hom}(B_1, B_2) = \operatorname{Hom}(B_2, B_1) = 0$ for all $B_1 \in \mathcal{D}_1$ and $B_2 \in \mathcal{D}_2$. A triangulated category that cannot be decomposed is called indecomposable.

Theorem 6.3. *Let X be a noetherian scheme and let $D^b(X)$ be its bounded derived category of coherent sheaves. Then $D^b(X)$ is an indecomposable triangulated category if and only if X is connected.*

Proof. See [6, Proposition 3.10]. □

6.2 Extending Serre Duality

One of the fundamental facts that was known already to the Italian school is that, without additional information, on a smooth variety X there are two distinguished line bundles: the trivial bundle $\mathcal{O}_X$, and the canonical bundle ω_X. It was Serre who made more precise the role of ω_X, through Serre duality: for a coherent sheaf $\mathcal{F}$ on a smooth projective variety we have

$$H^i(X, \mathcal{F})^\vee \cong \operatorname{Ext}_X^{n-i}(\mathcal{F}, \omega_X)$$

for all i.

Remark : If $\mathcal{F}$ is locally free, we have

$$\operatorname{Ext}^i(\mathcal{F}, \omega_X) \simeq \operatorname{Ext}(\mathcal{O}_X, \mathcal{F}^\vee \otimes \omega_X) \simeq H^i(X, \mathcal{F}^\vee \otimes \omega_X)$$

and then Serre duality gives

$$H^i(X, \mathcal{F}) \simeq H^{n-i}(X, \mathcal{F}^\vee \otimes \omega_X)^*$$

In this section we will first state the version of Serre duality given by Bondal and Orlov. After that we will extend the Serre duality in a different way by going about finding the right adjoint of a functor.

Theorem 6.4. ***(Serre duality)*** *Let X be a smooth projective variety over a field k. Then*

$$S_X : \mathrm{D}^b(X) \to \mathrm{D}^b(X)$$

which sends $\mathcal{F}^\bullet$ to $\mathcal{F}^\bullet \otimes \omega_X[n]$ is a Serre functor (see Definition 1.4), i.e. for any two complexes $\mathcal{E}^\bullet, \mathcal{F}^\bullet \in \mathrm{D}^b(X)$ there exists a functorial isomorphism

$$\eta : \operatorname{Ext}^i(\mathcal{E}^\bullet, \mathcal{F}^\bullet) \xrightarrow{\sim} \operatorname{Ext}^{n-i}(\mathcal{F}^\bullet, \mathcal{E}^\bullet \otimes \omega_X)^*$$

Proof. Recall that we have $\operatorname{Ext}^i(\mathcal{E}^\bullet, \mathcal{F}^\bullet) = H^i(R\operatorname{Hom}^\bullet(\mathcal{E}^\bullet, \mathcal{F}^\bullet))$. Now replacing $\mathcal{E}^\bullet$ by a complex of locally free sheaves and $\mathcal{F}^\bullet$ by a complex of injective sheaves, we have $R\operatorname{Hom}(\mathcal{E}^\bullet, \mathcal{F}^\bullet) = \operatorname{Hom}(\mathcal{E}\bullet, \mathcal{F}^\bullet)$. Moreover,

$$\begin{aligned}
\operatorname{Hom}^i(\mathcal{E}^\bullet, \mathcal{F}^\bullet) &= \bigoplus \operatorname{Hom}\left(\mathcal{E}^k, \mathcal{F}^{k+i}\right) \\
&\simeq \bigoplus \operatorname{Ext}^0\left(\mathcal{O}_X, \left(\mathcal{E}^k\right)^\vee \otimes \mathcal{F}^{k+i}\right), \\
&\simeq \bigoplus \operatorname{Ext}^n\left(\mathcal{F}^{k+i}, \mathcal{E}^k \otimes \omega_X\right)^*, \\
&\simeq \bigoplus \operatorname{Hom}\left(\mathcal{F}^{k+i}, \mathcal{E}^k \otimes \omega_X[n]\right)^*, \\
&\simeq \operatorname{Hom}^{n-i}(\mathcal{F}^\bullet, \mathcal{E}^\bullet \otimes \omega_X)^*,
\end{aligned}$$

and thus the desired isomorphism is obtained by replacing $\mathcal{E}^\bullet \otimes \omega_X$ by a complex of injective objects and taking cohomology. □

Now using the Serre duality we will prove that every element of the derived category of curves decomposes into direct sum of shifted cohomology sheaves. For that we will need a few lemmas, and we will give sketches of their proofs below.

Lemma 6.5. *Let $A \to B \to C \to A[1]$ be a distinguished in a triangulated category $\mathcal{D}$. Suppose $C \to A[1]$ is trivial. Then the triangle splits, i.e. $B \cong A \oplus C$.*

Lemma 6.6. *Suppose $H^i(A^\bullet) = 0$, for $i < i_0$. Then there exists a distinguised triangle*

$$H^i(A^\bullet)[-i_0] \to A^\bullet \xrightarrow{\varphi} B^\bullet \to H^i(A^\bullet)[1 - i_0]$$

in $D(\mathcal{A})$, with $H^i(B^\bullet) = 0$, for $i \leq i_0$ and φ inducing isomorphisms $H^i(A^\bullet) \cong H^i(B^\bullet)$ for $i > i_0$.

Proof. (Sketch): We will find a morphism in $D(\mathcal{A})$ from $H^i(A^\bullet)[-i_0] \to A^\bullet$. For that we need to find a roof

$$\begin{tikzcd} & C^\bullet \arrow[dl, "qis"'] \arrow[dr] & \\ H^m(A^\bullet)[-m] & & A^\bullet \end{tikzcd}$$

For this we first have the following quasi-isomorphism $C^\bullet \xrightarrow{qis} H^m(A^\bullet)[-m]$

$$\begin{tikzcd} \dots \arrow[r] & A^k \arrow[r] \arrow[d] & A^{k+1} \arrow[r] \arrow[d] & \dots \arrow[r] & A^{m-1} \arrow[r] \arrow[d] & ker(d_m) \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \dots \\ \dots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \dots \arrow[r] & 0 \arrow[r] & H^m(A^\bullet) = \frac{ker(d_m}{im(d_{m-1})} \arrow[r] & 0 \arrow[r] & \dots \end{tikzcd}$$

Next consider the following morphism $C^\bullet \to A^\bullet$

$$\begin{tikzcd} \dots \arrow[r] & A^k \arrow[r] \arrow[d, "id"] & A^{k+1} \arrow[r] \arrow[d, "id"] & \dots \arrow[r] & A^{m-1} \arrow[r] \arrow[d, "id"] & ker(d_m) \arrow[r] \arrow[d, hook] & 0 \arrow[r] \arrow[d] & \dots \\ \dots \arrow[r] & A^k \arrow[r] & A^{k+1} \arrow[r] & \dots \arrow[r] & A^{m-1} \arrow[r] & A^m \arrow[r] & A^{m+1} \arrow[r] & \dots \end{tikzcd}$$

Since $D(\mathcal{A})$ is a triangulated category, we can extend this morphism to a distinguished triangle of the following form (using cone construction)

$$H^i(A^\bullet)[-i_0] \to A^\bullet \xrightarrow{\varphi} B^\bullet \to H^i(A^\bullet)[1 - i_0]$$

The result follows immediately from here. $\square$

Lemma 6.7. *Let X be a smooth projective variety. Then the cohomological dimension of X, that is the cohomological dimension of Coh(X) is equal to dim$X$.*

Proof. Let $\dim X = n$, and $\mathcal{F}, \mathcal{G} \in \mathrm{Coh}(X)$. Now by Serre duality we have

$$\mathrm{Ext}^i(\mathcal{F}, \mathcal{G}) \cong \mathrm{Ext}^{n-i}(\mathcal{F}, \mathcal{G} \otimes w_X)^*$$

Hence for $i > n$, we have $n - i < 0$, and thus

$$\mathrm{Ext}^i(\mathcal{F}, \mathcal{G}) \cong \mathrm{Ext}^{n-i}(\mathcal{F}, \mathcal{G} \otimes w_X)^* \cong 0$$

This proves that cd $\mathrm{Coh}(X) \leq n$. On the other hand,

$$\mathrm{Ext}^n(\mathcal{O}_X, w_X) \cong \mathrm{Ext}^0(\mathcal{O}_X, \mathcal{O}_X)^* \cong H^0(X, \mathcal{O}_X)^* \neq 0$$

This shows that cd $\mathrm{Coh}(X) \geq n$. Hence, cd $\mathrm{Coh}(X) = n = \dim X$. □

Theorem 6.8. *Let C be a smooth projective curve. Then any object in $D^b(C)$ is isomorphic to the direct sum of its shifted cohomology sheaves.*

Proof. We will prove this via induction on the length of the complex.
Let $A^\bullet \in D^b(C)$, with length n, and $H^i(A^\bullet) = 0$, for $i < i_0$. Using Lemma 0.2 we have the distinguished triangle

$$H^i(A^\bullet)[-i_0] \to A^\bullet \to B^\bullet \to H^i(A^\bullet)[1 - i_0]$$

We will show that this triangle splits, i.e. $A^\bullet \cong H^i(A^\bullet)[-i_0] \oplus B^\bullet$, where $H^i(B^\bullet) = 0$, for $i \leq i_0$, and $H^i(B^\bullet) = H^i(A^\bullet)$, for $i > i_0$.
Using Lemma 0.1 if we show that $\mathrm{Hom}(B^\bullet, H^i(A^\bullet)[1 - i_0]) = 0$, it will be enough.
Hence via induction we would have

$$B^\bullet \cong \oplus_{i>i_0} H^i(B^\bullet)[-i_0] \cong \oplus_{i>i_0} H^i(A^\bullet)[-i_0], \text{ since } H^i(B^\bullet) = 0 \text{ for } i \leq i_0$$

Thus

$$\begin{aligned}
\mathrm{Hom}(B^\bullet, H^i(A^\bullet)[1 - i_0]) &= \mathrm{Hom}(\oplus_{i>i_0} H^i(B^\bullet)[-i_0], H^i(A^\bullet)[1 - i_0]) \\
&= \oplus_{i>i_0} \mathrm{Hom}(H^i(B^\bullet)[-i_0], H^i(A^\bullet)[1 - i_0]) \\
&= \oplus_{i>i_0} \mathrm{Ext}^{1+i-i_0}(H^i(B^\bullet), H^i(A^\bullet)) \\
&= 0.
\end{aligned}$$

The last equality comes from Lemma 0.3, since dimension of a curve is one, and $i > i_0 \implies 1 + i - i_0 > 1$. This proves the theorem. $\square$

6.2.1 Finding the right adjoint

Next we explain Grothendieck's approach to Serre duality, which was motivated by the search for a right adjoint to the push-forward functor f_* for a morphism $f : X \to Y$ of schemes.

Recall that two functors $F : \mathcal{C} \to \mathcal{D}$, and $G : \mathcal{D} \to \mathcal{C}$ are said to be adjoint (written as $F \dashv G$, and then F is a left adjoint to G, and G is a right adjoint to F) if there are isomorphisms

$$\mathrm{Hom}_{\mathcal{D}}(FA, B) \cong \mathrm{Hom}_{\mathcal{C}}(A, GB)$$

natural in both variables, for every $A \in Ob\mathcal{C}, B \in Ob\mathcal{D}$.

We know that for a projective morphism $f : X \to Y$, the the push-forward functor $f_* : \mathrm{Coh}(X) \to \mathrm{Coh}(Y)$, and the pullback functor $f^* : \mathrm{Coh}(Y) \to \mathrm{Coh}(X)$ are adjoint, $f^* \dashv f_*$.
But finding the right adjoint to the functor f_* is not easy. At first the search for a right adjoint to f_* seemed to be doomed for failure, as the following proposition proves:

Proposition 42. *Assume that $F \dashv G$, and that $\mathcal{C}, \mathcal{D}$ are abelian categories. Then F must be right exact, and G must be left exact.*

Now f_* is left exact, but not exact in general. Thus if f_* had a right adjoint, from the previous proposition we would have that f_* is exact. Hence in the general case f_* does not have a right adjoint.

The problem, however, appears to be with the fact that we are dealing with abelian categories, which are a bit too restrictive. The point is that in passing to the derived categories, we have replaced a left exact functor f_* by its right derived version Rf_* which is now exact. So in principle there may be hope to find a right adjoint to $Rf_* : \mathrm{D}^b(\mathrm{Coh}(X)) \to \mathrm{D}^b(\mathrm{Coh}(Y))$.

Here we identify a point, pt with $\operatorname{Spec} k$, where k is a field.
Let $f : X \to \mathrm{pt}$ be the structure morphism of a smooth projective variety X. Suppose Rf_* has a right adjoint, say g, then we would have

$$\mathrm{Hom}_{\mathrm{pt}}(Rf_*\mathcal{F}^\bullet, \mathcal{G}^\bullet) \cong \mathrm{Hom}_X(\mathcal{F}^\bullet, g\mathcal{G}^\bullet)$$

where $\mathcal{F}^\bullet \in \mathrm{D}^b(\mathrm{Coh}(X))$, and $\mathcal{G}^\bullet \in \mathrm{D}^b(\mathrm{Coh}(\mathrm{pt}))$.

Any object in $\mathrm{D}^b(\mathrm{Coh}(\mathrm{pt}))$ decomposes into direct sums of shifts of k. (Explanation due) So let $\mathcal{G}^\bullet = k$. Also for simplicity let $\mathcal{F}^\bullet$ is a complex with one coherent sheaf $\mathcal{F}$ concentrated at i, thus $\mathcal{F}^\bullet = \mathcal{F}[i]$.
Now it is easy to see that

$$\mathrm{Hom}_{\mathrm{pt}}\left(\mathbf{R}f_*\mathcal{F}[i], k\right) \cong H^i(X, \mathcal{F})^v,$$

because $f_*\mathcal{F} = \Gamma(X, \mathcal{F})$.
Now Serre duality predicts

$$H^i(X, \mathscr{F})^\vee \cong \mathrm{Ext}_X^{n-i}\left(\mathscr{F}, \omega_X\right) = \mathrm{Hom}_{\mathrm{D}^b_{\mathrm{coh}}(X)}\left(\mathscr{F}[i], \omega_X[n]\right),$$

Thus, if we set $gk = \omega_X[n]$, Serre duality yields

$$\mathrm{Hom}_{\mathrm{pt}}\left(Rf_*\mathcal{F}[i], k\right) \cong \mathrm{Hom}_X\left(\mathcal{F}[i], gk\right),$$

which is precisely the adjunction formula we want. Making $g$commute with direct sums and shifts we get a well defined $g : \mathrm{D}^b(\mathrm{coh}(\mathrm{pt})) \to \mathrm{D}^b(\mathrm{coh}(X))$ which is a right adjoint (in the derived sense) to $\mathbf{R}f_*$.

We will now generalize the above calculation to make it work for a much more general class of maps f. Our final result will be:

Theorem 6.9. *Let $f : X \to Y$ be a morphism of smooth projective schemes. Then a right adjoint to $Rf_* : D^b(\mathrm{coh}(X)) \to D^b(\mathrm{coh}(Y))$ exists, and is given by $g : D^b(\mathrm{coh}(Y)) \to D^b(\mathrm{coh}(X))$,*

$$g(-) = Lf^*\left(- \otimes \omega_Y^{-1}[-\dim Y]\right) \otimes \omega_X[\dim X].$$

Note that the previous calculation that $gk = \omega_X[\dim X]$ for the structure morphism $f : X \to \mathrm{pt}$ of a smooth scheme X agrees with this theorem.

Remark : It is crucial to emphasize at this point that it is **not** true that an equivalence of triangulated categories $F : \mathrm{D}^b(\mathrm{Coh}(X)) \to \mathrm{D}^b(\mathrm{Coh}(Y))$ between the derived categories of smooth varieties will satisfy

$$F\left(\omega_X[\dim X]\right) \cong \omega_Y[\dim Y]$$

The reason for this apparent mismatch is the fact that in general derived equivalences need not take tensor products to tensor products; thus the peculiar fact that S_X is given by tensoring with $\omega_X[\dim X]$ will not translate to F mapping ω_X to ω_Y. What is true is that F will commute with the corresponding Serre functors.

To prove the theorem we will simply use a very nice property of Serre functors. The great thing about Serre functors is that it allows one to convert from a left adjoint to a right adjoint and vice versa. Specifically, we have

Theorem 6.10. *Let $F : \mathcal{C} \to \mathcal{D}$ be a functor between the k-linear categories $\mathcal{C}, \mathcal{D}$ that admit Serre functors $S_{\mathcal{C}}, S_{\mathcal{D}}$. Assume that F has a left adjoint $G : \mathcal{D} \to \mathcal{C}, G \dashv F$. Then*

$$H = S_{\mathcal{C}} \circ G \circ S_{\mathcal{D}}^{-1} : \mathcal{D} \to \mathcal{C}$$

is a right adjoint to F.

Proof.

$$\begin{aligned}\mathrm{Hom}_{\mathcal{D}}(Fx, y) &\cong \mathrm{Hom}_{\mathcal{D}}\left(S_{\mathcal{D}}^{-1}y, Fx\right)^{\vee} \cong \mathrm{Hom}_{\mathcal{C}}\left(GS_{\mathcal{D}}^{-1}y, x\right)^{\vee} \\ &\cong \mathrm{Hom}_{\mathcal{C}}\left(x, S_{\mathcal{C}}GS_{\mathcal{D}}^{-1}y\right) = \mathrm{Hom}_{\mathcal{C}}(x, Hy)\end{aligned}$$

This immediately yields the theorem. □

6.3 The structure of $\mathbf{D}^b_{\mathbf{Coh}}(X)$

6.3.1 Coherent sheaves over X, a smooth projective curve

Purpose : In this section we will describe the structure of Coherent sheaves over a smooth Projective Curve, we will prove that any Coherent sheaf over X is a direct sum of a locally free sheaf and a sheaf supported at finitely many points. We will also prove that Every locally free sheaf over $\mathbb{P}^1$ decomposes to a direct sum of line bundles.

Definition 6.5. *For a scheme $(X, \mathcal{O}_X)$ an $\mathcal{O}_X$ module is called Quasi-coherent if for every $x \in X$, there exists an open neighborhood $U \subset X$ such that $\mathcal{F}|_U \cong \tilde{M}$, for some $\Gamma(U, \mathcal{O}_X)$ module M. We call the sheaf of modules $\mathcal{F}$ Coherent if the modules M are finitely generated. (Here $\tilde{M}$ is the $\mathcal{O}_X$ module associated to the module M)*

We give some equivalent conditions for an $\mathcal{O}_X$ module to be quasi-coherent,

Proposition 43. *Let X be a scheme and let $\mathcal{F}$ be an $\mathcal{O}_X$-module. Then the following assertions are equivalent,*
(i) The $\mathcal{O}_X$-module $\mathcal{F}$ is quasi-coherent.
(ii) For every open affine subset $U = \operatorname{Spec} A$ of X there exists an A-module M such that $\mathcal{F}_{|U} \cong \tilde{M}$.
(iii) There exists an open affine covering $(U_i)_i$ of $X, U_i = \operatorname{Spec} A_i$, and for each i an A_i-module M_i such that $\mathcal{F}_{|U_i} \cong \tilde{M}_i$ for all i.
(iv) For all $x \in X$, there exists an open neighborhood U of x and an exact sequence of $\mathcal{O}_X|_U$ modules of the form

$$\mathcal{O}_X^J|_U \to \mathcal{O}_X^J|_U \to \mathcal{F}|_U \to 0$$

where I and J are arbitrary index sets (depending on x).
(iv) For every open affine subset $U = \operatorname{Spec} A$ of X and every $f \in A$ the homomorphism

$$\Gamma(U, \mathcal{F})_f \to \Gamma(D(f), \mathcal{F})$$

is an isomorphism.

Next we define the torsion subsheaf $\mathcal{T}$ of an $\mathcal{O}_X$ module $\mathcal{F}$ as follows

$$\begin{aligned}\Gamma(U,\mathcal{T}) &= \text{the torsion submodule of } \Gamma(U,F)\\ &= \{a \in \Gamma(U,F) \mid \exists\, 0 \neq r \in \Gamma(U,\mathcal{O}_X), \text{ such that } ra = 0\}\end{aligned}$$

We will show that the Torsion sheaf is isomorphic to a finite direct sum of sky-scraper sheaves.

For that we define the Support of a sheaf $\mathcal{F}$ as follows

$$\mathrm{Supp}(\mathcal{F}) = \{x \in X \mid \mathcal{F}_x \neq 0\}$$

We see that for a Coherent sheaf of modules $\mathcal{F}$, $\mathrm{Supp}(\mathcal{F})$ is a closed subset of X. Indeed, since $\mathcal{F}$ is coherent, $\mathcal{F}_x = 0$ implies that $\mathcal{F}|_U = 0$, for an open subset U containing x. Hence $\mathrm{Supp}(\mathcal{F})$ is a closed subset of X.

Now $\mathcal{T}$ is a torsion sheaf as defined above then the localization at the generic point η is zero. This shows that $\mathrm{Supp}(\mathcal{T})$ is a proper closed subset of X.

In the case when X is a curve, a proper closed subset means a collection finitely many (closed) points. This shows that any torsion sheaf $\mathcal{T}$ is supported at finitely many points, say $\{x_1, x_2, \ldots, x_n\}$. Define the sky-scraper sheaves $\mathcal{G}$ as follows

$$\Gamma(U,\mathcal{G}_i) = \begin{cases} 0 & x_i \in U \\ \mathcal{T}_{x_i} & x_i \in U \end{cases}$$

Checking on the stalk level we see that

$$\mathcal{T} \cong \bigoplus_{i=1}^{n} \mathcal{G}_i$$

From here we see that for any Coherent sheaf of modules $\mathcal{F}$ the quotient sheaf $\mathcal{S} = \mathcal{F}/\mathcal{T}$ is torsion-free.

Since X is a smooth curve the localization at a point $\mathcal{O}_{X,x}$ is a DVR, and hence a PID. This implies that $\mathcal{S}_{X,x} = \mathcal{F}_{X,x}/\mathcal{T}_{X,x}$ is a torsion-free module over $\mathcal{O}_{X,x}$, a PID. Hence $\mathcal{S}_{X,x}$ is free. Thus $\mathcal{S}$ is a locally free $\mathcal{O}_X$ module.

When X is a curve, and $\mathcal{F}$ a coherent $\mathcal{O}_X$ module, we summarize our findings in the following short exact sequence

$$0 \to \mathcal{T} \to \mathcal{F} \to \mathcal{S} \to 0$$

where $\mathcal{T}$ is a torsion sheaf of modules and $\mathcal{S}$ is locally free. For our purpose this statement is enough, but we will go one step further and show that this sequence splits.

Theorem 6.11. *Any Coherent sheaf over X is a direct sum of a locally free sheaf and a sheaf supported at finitely many points.*

Proof. (Sketch): From the previous discussion we have the following exact sequence

$$0 \to \mathcal{T} \to \mathcal{F} \xrightarrow{p} \mathcal{S} \to 0$$

where $\mathcal{T}$ is a torsion sheaf of modules and $\mathcal{S}$ is locally free. We need to show that this exact sequence splits. We apply the functor $\mathcal{H}om(\mathcal{S}, -)$ followed by the sections functor to this sequence to obtain the exact sequence

$$H^0(X, \mathcal{H}om(\mathcal{S}, \mathcal{F})) \to H^0(X, \mathcal{H}om(\mathcal{S}, \mathcal{S})) \to H^1(X, \mathcal{H}om(\mathcal{S}, \mathcal{T}))$$

The last entry vanishes since the sheaf $\mathcal{H}om(\mathcal{S}, \mathcal{T})$ is only supported at finitely many points. Hence we get that the map

$$Hom(\mathcal{S}, \mathcal{F}) \to Hom(\mathcal{S}, \mathcal{S})$$

is surjective. Hence there exists a morphism $\varphi : \mathcal{S} \to \mathcal{F}$ such that $p \circ \varphi = id|_{\mathcal{S}}$. Hence, $\mathcal{F} \cong \mathcal{S} \oplus \mathcal{T}$. □

Next we are going to proof a theorem which will be very helpful in determining the structure of $\mathrm{D}^b(\mathrm{Coh}(\mathbb{P}^1))$.

Theorem 6.12. *(Grothendieck) Every locally free sheaf (vector bundle) over $\mathcal{O}_{\mathbb{P}^1}$ decomposes into a direct sum of line bundles $\mathcal{O}_{\mathbb{P}^1}(d)$.*

Proof. See [5] □

6.3.2 The structure of $\mathbf{D}^b(\mathbf{Coh}(\mathbb{P}^1))$

Purpose : Using the structure of Coherent sheaves over $\mathbb{P}^1$, as described in the previous section we will give a proof of the fact that $\mathrm{D}^b(\mathrm{Coh}(\mathbb{P}^1)) = \langle \mathcal{O}, \mathcal{O}(-1) \rangle$.

Our aim is to show that the category $D^b_{\mathrm{Coh}}(\mathbb{P}^1)$ has a particularly simple structure, being generated by just two objects. We first make precise the idea of a generator.

Definition 6.6. *Let $\mathcal{A}$ be an arbitrary abelian category and let S be a set of objects in* $\mathrm{D}^b(\mathcal{A})$*. The category generated by S, denoted $\langle S\rangle$, is the smallest full subcategory of* $\mathrm{D}^b(\mathcal{A})$ *that contains S and is closed under shifts and distinguished triangles.*

Here, "closed under shifts" means that if $C^\bullet$ is an object of $\langle S\rangle$, then so is $C[n]^\bullet$ for any $n \in \mathbb{Z}$, and "closed under distinguished triangles" means that if

$$K^\bullet \xrightarrow{u} L^\bullet \xrightarrow{v} M^\bullet \xrightarrow{w} K[1]^\bullet$$

is distinguished in $\mathrm{D}^b(\mathcal{A})$ and two of $K^\bullet, L^\bullet, M^\bullet$ are in $\langle S\rangle$, then so is the third.

Proof of the theorem : Denote the subcategory $\langle \mathcal{O}_X, \mathcal{O}_X(-1)\rangle$ of $\mathrm{D}^b(\mathrm{Coh}(X))$ by $\mathscr{D}$. It is enough to show that every coherent sheaf $\mathcal{F}$, viewed as a single object complex in $\mathbf{D}^b_{\mathrm{Coh}}(X)$, is in $\mathscr{D}$. Indeed, an arbitrary object in $\mathbf{D}^b_{\mathrm{Coh}}(X)$ can be translated into the form

$$\cdots \longrightarrow 0 \longrightarrow \mathcal{F}_0 \xrightarrow{d^0} \mathcal{F}_1 \xrightarrow{d^1} \mathcal{F}_2 \longrightarrow \cdots \longrightarrow \mathcal{F}_k \longrightarrow 0 \longrightarrow \cdots$$

where $\mathcal{F}_0$ is in degree 0 . Then because $\mathcal{F}_0, \mathcal{F}_1$ are in $\mathscr{D}$, the map

$$\begin{tikzcd}
\dots \arrow[r] & 0 \arrow[r] \arrow[d] & \mathcal{F}_0 \arrow[r] \arrow[d, "d^0"] & 0 \arrow[r] \arrow[d] & \dots \\
\dots \arrow[r] & 0 \arrow[r] & \mathcal{F}_1 \arrow[r] & 0 \arrow[r] & \dots
\end{tikzcd}$$

is in $\mathscr{D}$, so that its cone

$$\cdots \longrightarrow 0 \longrightarrow \mathcal{F}_0 \xrightarrow{d^0} \mathcal{F}_1 \longrightarrow 0 \longrightarrow 0 \longrightarrow \cdots$$

is also an object of $\mathscr{D}$ (here, $\mathcal{F}_0$ is in degree -1). Then also $\mathcal{F}_2$ is in $\mathscr{D}$, so taking the cone of the map

$$\begin{tikzcd}
\dots \arrow[r] & 0 \arrow[r] & \mathcal{F}_0 \arrow[r, "d^0"] \arrow[d] & \mathcal{F}_1 \arrow[r] \arrow[d, "d^1"] & 0 \arrow[r] \arrow[d] & \dots \\
\dots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \mathcal{F}_2 \arrow[r] & 0 \arrow[r] & \dots
\end{tikzcd}$$

gives

$$\cdots \longrightarrow 0 \longrightarrow \mathcal{F}_0 \xrightarrow{d^0} \mathcal{F}_1 \xrightarrow{d^1} \mathcal{F}_2 \longrightarrow 0 \longrightarrow 0 \longrightarrow \cdots$$

Continuing this process, we see that every object in $\mathbf{D}^b_{\mathrm{Coh}}(X)$ can be constructed, up to translation, from single object complexes; since $\mathscr{D}$ is closed under shifting, it follows that $\mathscr{D} = \mathrm{D}^b_{\mathrm{Coh}}(X)$.

We will now show that every coherent sheaf F can be constructed from $\mathcal{O}_X$ and $\mathcal{O}_X(-1)$ in a finite number of steps using short exact sequences. We know that if $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$ is a short exact sequence of sheaves, and if any two of them belong to $\mathscr{D}$, then so does the third.

Now for any arbitrary coherent sheaf $\mathcal{F}$, we have the following short exact sequence

$$0 \longrightarrow \mathcal{T} \longrightarrow \mathcal{F} \longrightarrow \mathcal{E} \longrightarrow 0$$

where $\mathcal{T}$ is torsion and $\mathcal{E}$ is locally free. If we can show that these two types of sheaves are in $\mathscr{D}$, our proof will be finished.

Sky-scraper sheaf : We will show that any sky-scraper sheaf belongs to $\mathscr{D}$. Let $\mathcal{S}_P$ be a sky-scraper sheaf supported at one point P, and D be the divisor defined by this point (since $X = \mathbb{P}^1$). Then by [4, Proposition 6.18], we have that the ideal sheaf of the closed subscheme $i : \{P\} \hookrightarrow X$ is isomorphic to $\mathcal{O}(-D) \cong \mathcal{O}(-1)$. Hence we have the following short exact sequence

$$0 \to \mathcal{O}(-1) \to \mathcal{O} \to \mathcal{S}_P \to 0$$

since $\mathcal{O}(-1), \mathcal{O}$ are in $\mathscr{D}$, we have that $\mathcal{S}_P$ is also in $\mathscr{D}$. Now since any sheaf supported at a finitely many points is actually direct sum of sky-scraper sheaves supported at a point, we have that any torsion sheaf belongs to $\mathscr{D}$.

Locally-free sheaves : We saw that in the following short exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^1}(-1) \to \mathcal{O}_{\mathbb{P}^1} \to \mathcal{F} \to 0$$

$\mathcal{F}$ is a sky-scraper sheaf, and thus tensoring with the locally free sheaf $\mathcal{O}_{\mathbb{P}^1}(1)$ we get another short exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^1} \to \mathcal{O}_{\mathbb{P}^1}(1) \to \mathcal{F}_1 \to 0$$

here also $\mathcal{F}_1$ is a sky-scraper sheaf.
From the first sequence we have that $\mathcal{F}$ is in $\mathscr{D}$, because $\mathcal{O}_{\mathbb{P}^1}(-1)$, and $\mathcal{O}_{\mathbb{P}^1}$ are in $\mathscr{D}$. And from the second equation by the similar argument we have that $\mathcal{O}_{\mathbb{P}^1}(1)$ is in $\mathscr{D}$.
Going on like this, tensoring by $\mathcal{O}_{\mathbb{P}^1}(1)$ we see that $\mathcal{O}_{\mathbb{P}^1}(n)$ is in $\mathscr{D}$, for all $n \in \mathbb{Z}$.

Now by Grothendieck's result we know that every locally free sheaf is a direct sum of line bundles $\mathcal{O}_{\mathbb{P}^1}(n)$, and thus can be placed in a short exact sequence with the other two terms as line bundles. Hence we see that any locally free sheaf is in $\mathscr{D}$. This completes our proof. $\square$

6.4 Fourier-Mukai transforms

I will now talk about Fourier-Mukai transforms and a few its applications in Algebraic Geometry. As an application I will prove that the derived categories of an abelian variety and its dual are equivalent.

We will have the following conventions: Let X and Y be smooth projective varieties over a field. We have the projections

$$p : X \times Y \to Y, \quad q : X \times Y \to X.$$

We will not write the L's and R's in front of the functors but all functors we consider are in fact derived functors.

Definition 6.7. *Let $P \in D^b(X \times Y)$, the induced Fourier-Mukai transform is the functor*

$$\begin{aligned} \Phi_P : D^b(X) &\to D^b(Y) \\ \mathcal{E}^\bullet &\mapsto p_* \left(q^* \left(\mathcal{E}^\bullet \right) \otimes P \right) \end{aligned}$$

We say P is the Fourier-Mukai kernel of Φ_P.

Remark 1. *Note that since q is flat, the derived functor q^* is just the usual pullback.*

To be less ambiguous we could write $\Phi_P^{X \to Y}$ for the Fourier-Mukai transform defined above. We also get a Fourier-Mukai transform $\Phi_P^{Y \to X} : D^b(Y) \to D^b(X)$ by reversing the roles of p and q in the definition. So one Fourier-Mukai kernel induces two Fourier-Mukai transforms. Unless we specify otherwise we take Φ_P to be the one from $D^b(X)$ to $D^b(Y)$.

Remark 2. *The Fourier-Mukai Transform is a composition of three exact (i.e. triangulated) functors and is therefore itself exact (triangulated).*

I will now give a few examples of Fourier-Mukai transforms. We will see that we have already encountered many functors which are Fourier-Mukai transforms. Before that let me mention the derived version of the projection formula. For a locally free sheaf $\mathcal{E}$ on Y and an arbitrary sheaf $\mathcal{F}$ on X, the classical projection formula gives

$$f_*(\mathcal{F} \otimes f^*\mathcal{E}) \cong f_*(\mathcal{F}) \otimes \mathcal{E},$$

where $f : X \to Y$ is a proper morphism of schemes.

Now let $f : X \to Y$ be a proper morphism of projective schemes over a field k. For any $\mathcal{F}^\bullet \in \mathrm{D}^{\mathrm{b}}(X), \mathcal{E}^\bullet \in \mathrm{D}^{\mathrm{b}}(Y)$ there exists a natural isomorphism (derived version):

$$Rf_* (\mathcal{F}^\bullet) \otimes \otimes^L \mathcal{E}^\bullet \xrightarrow{\sim} Rf_* \left(\mathcal{F}^\bullet \otimes^L Lf^* (\mathcal{E}^\bullet)\right).$$

Example 6.1. *The identity*

$$id \ : D^b(X) \to D^b(X)$$

is a Fourier-Mukai transform with kernel $\mathcal{O}_\Delta$, where Δ is the diagonal in $X \times X$. When we look at the diagonal embedding $i : X \xrightarrow{\sim} \Delta \subset X \times X$ we have $i_* \mathcal{O}_X = \mathcal{O}_\Delta$. We use this and the projection formula to get

$$\begin{aligned}
\Phi_{\mathcal{O}_\Delta} (\mathcal{E}^\bullet) &= p_* \left(q^* \mathcal{E}^\bullet \otimes i_* \mathcal{O}_X\right) \\
&= p_* \left(i_* \left(i^* q^* \mathcal{E}^\bullet \otimes \mathcal{O}_X\right)\right) \\
&= (p \circ i)_* \left((q \circ i)^* \mathcal{E}^\bullet \otimes \mathcal{O}_X\right) \\
&= \mathcal{E}^\bullet \quad (\textit{since } p \circ i = q \circ i = id).
\end{aligned}$$

Example 6.2. *For a function $X \to Y$ we have the graph $X \to X \times Y$ where $\Gamma_f = \mathrm{id} \times f$. We have $\Gamma_{f*} \mathcal{O}_X = \mathcal{O}_{\Gamma_f}$ so similar to the identity case we get*

$$\Phi_{\mathcal{O}_{\Gamma_f}} (\mathcal{E}^\bullet) = (p \circ \Gamma_f)_* \left((q \circ \Gamma_f)^* \mathcal{E}^\bullet \otimes \mathcal{O}_X\right) = f_* \mathcal{E}^\bullet$$

We can reverse the roles of p and q to get

$$\Phi^{X \to Y}_{\mathcal{O}_{\Gamma_f}} = f_* \quad , \quad \Phi^{Y \to X}_{\mathcal{O}_{\Gamma_f}} = f^*.$$

Taking global sections can be seen as a special case of this since for $f : X \to Spec\ k$ we have $f_* = \Gamma$.

Example 6.3. *Let $L \in Pic(X)$. Then $\mathcal{E}^\bullet \mapsto \mathcal{E}^\bullet \otimes L$ defines an autoequivalence $D^b(X) \to D^b(X)$ which is isomorphis to the Fourier-Mukai transform with kernel $i_*(L)$ where $i : X \xrightarrow{\sim} \Delta \subset X \times X$. Indeed we have,*

$$\begin{aligned}
\Phi_{i_*(L)}(\mathcal{E}^\bullet) &= p_*(q^*(\mathcal{E}^\bullet) \otimes i_*(L)) \\
&= (p \circ i)_*(L \otimes (i \circ q)^*(\mathcal{E}^\bullet)) \\
&= \mathcal{E}^\bullet \otimes L.
\end{aligned}$$

In particular, the Serre functor, which is the exact equivalence

$$S_X(\cdot) = (\cdot) \otimes \omega_X[\dim X],$$

where ω_X denotes the canonical line bundle of X, is of Fourier-Mukai type.

Fourier-Mukai transforms have many nice properties. We have already seen that Fourier-Mukai transforms are exact. Now we will show that a Fourier-Mukai transform has a left and a right adjoint, in fact both the adjoints are Fourier-Mukai transforms, for which the kernels can be described explicitly.

Definition 6.8. *For any object $\mathcal{P} \in \mathrm{D^b}(X \times Y)$ we let*

$$\mathcal{P}_{\mathrm{L}} := \mathcal{P}^{\vee} \otimes p^*\omega_Y[\dim(Y)] \text{ and } \mathcal{P}_{\mathrm{R}} := \mathcal{P}^{\vee} \otimes q^*\omega_X[\dim(X)],$$

both objects in $\mathrm{D^b}(X \times Y)$. Here $\mathcal{P}^{\vee}$ is the derived dual [see [6, Page 78]].

Proposition 44. *The Fourier-Mukai transforms $\Phi_{\mathcal{P}_L}, \Phi_{\mathcal{P}_R} : D^b(Y) \to D^b(X)$ are left, respectively right adjoint to $\Phi_{\mathcal{P}}$.*

Proof. For this we need Grothendieck-Verdier duality. Let $f : X \to Y$, we define $\omega_f := \omega_X \otimes f^*\omega_Y^{\vee}$ and $\dim f := \dim X - \dim Y$.

Theorem 6.13. *(Grothendieck-Verdier duality). Let $\mathcal{F}^\bullet \in D^b(X)$ and $\mathcal{E}^\bullet \in D^b(Y)$, there is a functorial isomorphism*

$$f_*\mathscr{H}\mathrm{om}\,(\mathcal{F}^\bullet, f^*\mathcal{E}^\bullet \otimes \omega_f[\dim f]) \cong \mathscr{H}\mathrm{om}\,(f_*\mathcal{F}^\bullet, \mathcal{E}^\bullet)$$

Keep in mind that (as everywhere) the operations here are all derived functors.

For this proof we are interested in the special case where $f = q$ and we then take global sections. We get $\omega_f = \omega_{X \times Y} \otimes q^*\omega_X^{\vee} = p^*\omega_Y$ and

$$\mathrm{Hom}_{\mathrm{D}^b(X \times Y)}\left(\mathcal{F}^\bullet, q^*\mathcal{E}^\bullet \otimes p^*\omega_Y[\dim Y]\right) \cong \mathrm{Hom}_{\mathrm{D}^b(Y)}\left(q_*\mathcal{F}^\bullet, \mathcal{E}^\bullet\right).$$

We now use this and the fact that pullback and pushforward are adjoint.

$$\begin{aligned}
\mathrm{Hom}_{\mathrm{D}^b(X)}\left(\Phi_{P_L}\left(\mathcal{F}^\bullet\right),\mathcal{E}^\bullet\right) &= \mathrm{Hom}_{\mathrm{D}^b(X)}\left(q_*\left(p^*\mathcal{F}^\bullet \otimes P_L\right),\mathcal{E}^\bullet\right)\\
&= \mathrm{Hom}_{\mathrm{D}^b(X\times Y)}\left(p^*\mathcal{F}^\bullet \otimes P_L, q^*\mathcal{E}^\bullet \otimes p^*\omega_Y[\dim Y]\right)\\
&= \mathrm{Hom}_{\mathrm{D}^b(X\times Y)}\left(p^*\mathcal{F}^\bullet \otimes P^\vee, q^*\mathcal{E}^\bullet\right)\\
&= \mathrm{Hom}_{\mathrm{D}^b(X\times Y)}\left(p^*\mathcal{F}^\bullet, q^*\mathcal{E}^\bullet \otimes P\right) \quad \text{(see [6, } Page 84\text{])}\\
&= \mathrm{Hom}_{\mathrm{D}^b(Y)}\left(\mathcal{F}^\bullet, p_*\left(q^*\mathcal{E}^\bullet \otimes P\right)\right)\\
&= \mathrm{Hom}_{\mathrm{D}^b(Y)}\left(\mathcal{F}^\bullet, \Phi_{P_R}\left(\mathcal{E}^\bullet\right)\right)
\end{aligned}$$

□

Now the question arises: when does a given kernel defines a fully faithful or equivalent Fourier-Mukai transform. For this we need to be able to work with composition. Next we will show that the composition of two arbitrary Fourier–Mukai transforms is again a Fourier–Mukai transform. We will give an explicit formula for the Fourier–Mukai kernel of the composition.

Let X, Y, and Z be smooth projective varieties over k a field. Consider objects $\mathcal{P} \in \mathrm{D^b}(X \times Y)$ and $\mathcal{Q} \in \mathrm{D^b}(Y \times Z)$. Then define the object $\mathcal{R} \in \mathrm{D^b}(X \times Z)$ by the formula

$$\mathcal{R} := \pi_{XZ*}\left(\pi_{XY}^*\mathcal{P} \otimes \pi_{YZ}^*\mathcal{Q}\right),$$

where π_{XZ}, π_{XY}, and π_{YZ} are the projections from $X \times Y \times Z$ to $X \times Z, X \times Y$, respectively $Y \times Z$.

Proposition 45. *The composition*

$$D^b(X) \overset{\Phi_{\mathcal{P}}}{\to} D^b(Y) \overset{\Phi_{\mathcal{Q}}}{\to} D^b(Z)$$

is isomorphic to the Fourier-Mukai transform

$$\Phi_{\mathcal{R}} : D^b(X) \to D^b(Z).$$

Proof. See [6, Proposition 5.10]. □

Theorem 6.14. *(Orlov) Let X and Y be two smooth projective varieties and let*

$$F : \mathrm{D^b}(X) \longrightarrow \mathrm{D^b}(Y)$$

be a fully faithful exact functor. If F admits right and left adjoint functors, then there exists an object $\mathcal{P} \in \mathrm{D^b}(X \times Y)$ unique up to isomorphism such that F is isomorphic to $\Phi_{\mathcal{P}}$:

$$F \simeq \Phi_{\mathcal{P}}.$$

Proof. See [10]. □

Applying this to equivalences we have,

Corollary 6.1. *Let $F : \mathrm{D^b}(X) \xrightarrow{\sim} \mathrm{D^b}(Y)$ be an equivalence between the derived categories of two smooth projective varieties. Then F is isomorphic to a Fourier-Mukai transform $\Phi_{\mathcal{P}}$ associated to a certain object $\mathcal{P} \in \mathrm{D^b}(X \times Y)$, which is unique up to isomorphism.*

Equivalence of categories $\mathbf{D}^b(A) \cong \mathbf{D}^b(\hat{A})$: Let A be an abelian variety of dimension g and $\hat{A}$ is its dual. Let $\mathcal{P} \in \mathrm{Pic}(A \times \hat{A})$ be the Poincaré bundle.

Theorem 6.15. *The natural adjunction morphism*

$$\mathrm{id}_{\mathrm{D}(A^\vee)} \to \phi_{\mathcal{P}} \circ \phi_{\mathcal{P}^{-1}[g]}$$

is an isomorphism.

$$D^b(\hat{A}) \xrightarrow{\Phi_{\mathcal{P}^{-1}[g]}} D^b(A) \xrightarrow{\Phi_{\mathcal{P}}} D^b(\hat{A})$$

Before this we need the following theorem

Theorem 6.16. *Let $A, A^\vee$ and $\mathcal{P}$ as before and write $p_2 : A \times A^\vee \to A^\vee$ for the second projection. Then we have*

$$R^n p_{2*}\mathcal{P} = \begin{cases} 0 & \text{if } n \neq g \\ k\left(e_{A^\vee}\right) & \text{if } n = g \end{cases}$$

where $k\left(e_{A^\vee}\right)$ denotes the skyscraper sheaf at $e_{A^\vee}$; and

$$\mathrm{H}^n\left(A \times A^\vee, \mathcal{P}\right) = \begin{cases} 0 & \text{if } n \neq g \\ k & \text{if } n = g \end{cases}$$

Proof. By example 7.3 it is enough to show that

$$\Phi_{\mathcal{P}^{-1}[g]} \circ \Phi_{\mathcal{P}} \cong \Phi_{\mathcal{R}} \cong \Phi_{i_* \mathcal{O}_{\hat{A}}},$$

where $i : \hat{A} \to \Delta \subset \hat{A} \times \hat{A}$. Define $d : \hat{A} \times \hat{A} \to \hat{A}$, $(a, b) \mapsto a - b$. Then we have

$$\begin{aligned}
\mathcal{R} &= p_{13*}(p_{12}^*(\mathcal{P}^{-1}[g]) \otimes p_{23}^*(\mathcal{P})) \\
&= p_{13*}(d \times id)^* \mathcal{P}[g] \\
&= d^* p_{2*} \mathcal{P}[g] \\
&= d^* \mathcal{O}_{\hat{A}} \quad \text{(by the last theorem)} \\
&= i_* \mathcal{O}_{\hat{A}}.
\end{aligned}$$

□

Part III: Motives

7 Motives

In this section, I will introduce Motives, in particular the Chow motives. After the definitions, we will calculate the Chow motive in a few elementary cases using Manin's principle. At the end we will study the motive of an abelian variety and prove the Chow-Kunneth decomposition of the motive of an abelian variety.

We fix a base field k. Let $\mathcal{V}_k$ denote the category of smooth projective k-schemes. We refer to the objects of $\mathcal{V}_k$ as *varieties*. For a morphism $\phi : X \to Y$, denote by $\Gamma_\phi \subset X \times Y$ its graph.
For a variety X and an integer d, the cycle group $Z^d(X)$ is the free abelian group generated by irreducible subvarieties of X of codimension d.
We write $A^d(X) = \mathcal{Z}^d(X) \otimes \mathbf{Q}/\sim$, where $\sim$ is the rational equivalence relation. We are aware that $A^d(X)$ enjoys the following properties:
- For a morphism $\phi : X \to Y$ there are pullback and push-forward maps $\phi^* : A^*(Y) \to A^*(X), \phi_* : A^*(X) \to A^{*+\dim Y - \dim X}(Y)$.
- There is a product structure $A^d(X) \otimes A^e(X) \to A^{d+e}(X)$ given by intersection theory (See Chapter 6, [3]).

There exists many other important *adequate equivalence relations*, but we will be concerned about only rational equivalence here. The motives arising from this choice are called the *Chow motives*.

Definition 7.1. *(I) Let $X, Y \in \mathcal{V}_k$. Define the group of correspondences of degree r, from X to Y, $Corr^r(X, Y)$ as follows. If X is purely d-dimensional, then*

$$\mathrm{Corr}^r(X, Y) = A^{d+r}(X \times Y).$$

In general, let $X = \coprod X_i$ where each X_i is a connected variety, and set

$$\mathrm{Corr}^r(X, Y) = \bigoplus_i \mathrm{Corr}^r(X_i, Y) \subset A^*(X \times Y).$$

(II) Given correspondences $f \in Corr^r(X \times Y)$ and $g \in Corr^s(Y \times Z)$, the composition of f and g is given by the formula

$$g \circ f := p_{13*}\left(p_{12}^*(f) \cdot p_{23}^*(g)\right) \in Corr^{r+s}(X \times Z),$$

where $p_{i.j}$ are the projections of $X \times Y \times Z$ onto the products of i-th and j-th factors.
(III) If $\alpha \in Corr^r(X, Y)$ is a correspondence of degree r, it induces a homomorphism of graded abelian groups

$$\alpha_* : A^*(X) \to A^{*+r}(Y)$$
$$x \mapsto (\mathrm{pr}_Y)_* \left(\alpha \cdot \mathrm{pr}_X^*(x)\right)$$

(IV) Given a correspondence $\alpha \in Corr^*(X, Y)$, let $\sigma : X \times Y \to Y \times X$ the natural isomorphism switching the two factors. We define the transpose of α as

$${}^t\alpha = \sigma^*(\alpha) \in A^*(Y \times X).$$

Fact: The composition of correspondences is associative.
Let us now give the definition of (Chow) motives right away.

Definition 7.2. *The category $\mathcal{M}_k$ of k-motives is defined as follows: an object of $\mathcal{M}_k$ is a triple (X, p, m) where X is a k-variety, m is an integer and $p = p^2 \in \mathrm{Corr}^0(X, X)$ is an idempotent. If (X, p, m) and (Y, q, n) are motives, then*

$$\mathrm{Hom}_{\mathcal{M}_k}((X, p, m), (Y, q, n)) = p\,\mathrm{Corr}^{n-m}(X, Y)q \subset \mathrm{Corr}^*(X, Y)$$

and composition is given by composition of correspondences.

Remark 3. *There is a functor $h : \mathcal{V}_k^{\mathrm{opp}} \to \mathcal{M}_k$ which on objects is given by $h(X) = (X,\ id,\ 0)$, and on morphisms $\phi : Y \to X$ by*

$$h(\phi) = [\Gamma_\phi] \in \mathrm{Corr}^0(X, Y) = \mathrm{Hom}(h(X), h(Y))$$

(usually one writes ϕ^* for $h(\phi)$).

Sketch of Grothendieck's construction: Let me give a sketch of the usual definition of motives found in the literature. The construction of the category of motives $\mathcal{M}_k$ with respect to an adequate equivalence relation, say rational equivalence, proceeds in several steps

$$\mathcal{V}_k^{\text{opp}} \to \text{Corr}^0(k) \to \mathcal{M}_k^{\text{eff}} \to \mathcal{M}_k$$

where, Step I: We have our category of smooth projective schemes.
Step II: $\text{Corr}^0(k)$ has the same objects as $\mathcal{V}_k$, but the morphisms are the degree zero correspondences and the composition is the composition of correspondences. This generalizes the morphism between varieties. $\text{Corr}^0(k)$ is an additive category.
Step III: (1) $\mathcal{M}_k^{\text{eff}}$ is called the category of *effective motives*. The objects are pairs (X, p), where as in our definition $X \in \mathcal{V}_k$ and $p = p^2 \in \text{Corr}^0(X, X)$ is an idempotent. Here we have the morphisms:

$$\text{Hom}_{\mathcal{M}_k^{\text{eff}}}((X, p), (Y, q)) = q \circ \text{Corr}^0_{\sim}(X, Y) \circ p.$$

Composition comes from composition of correspondences. The category $\mathcal{M}_k^{\text{eff}}$ is the *pseudo-abelian completion* of the category $\mathcal{V}_k$.
Step IV (Final step): We have our category of motives with a fully faithful embedding $\mathcal{M}_k^{\text{eff}} \to \mathcal{M}_k$, $(X, p) \mapsto (X, p, 0)$.

The previous construction shows that every variety can be seen as a motive. Intuitively, a motive $(X, p, 0)$ should be seen as a "piece" of X that is "responsible" for a certain part of the geometric and (or) arithmetic properties of X. To be precise, $(X, p, 0)$ is a direct summand of $h(X)$.

7.1 Examples and Properties

(I) The unit motive or motive of a point

$$\mathbf{1} = (\text{Spec}\, k, id, 0) = h(\text{point}).$$

The Lefschetz motive

$$\mathbb{L} = (\text{Spec}\, k, id, -1).$$

(**Tensor product of motives**) The tensor product in $\mathcal{M}_k$ is defined as follows

$$(X, p, m) \otimes (Y, q, n) = (X \times Y, p \otimes q, m + n),$$

and on morphisms

$$p_1f_1q_1 \otimes p_2f_2q_2 = (p_1 \otimes p_2)\,(f_1 \otimes f_2)\,(q_1 \otimes q_2) \in \mathrm{Corr}^*\,(X_1 \times X_2, Y_1 \times Y_2)$$
$$\text{if } p_if_iq_i : (X,p,m) \to (Y,q,n)\,.$$

First notice that

$$\mathrm{Hom}_{\mathcal{M}_k}(h(X),h(X)) = \mathrm{Hom}_{\mathcal{M}_k}((X,id,0),(X,id,0)) = \mathrm{Corr}^0(X,X).$$

Now suppose $p \in \mathrm{End}h(X) = \mathrm{Corr}^0(X,X)$ is a projector, then $ph(X) = p(X,id,0) = (X,p,0)$. Hence we get the following

$$\begin{aligned}(X,p,m) &= (X,p,0) \otimes (\mathrm{Spec}\,k, id, m)\\ &= ph(X) \otimes \mathbb{L}^{-m} \subset h(X) \otimes \mathbb{L}^{-m}.\end{aligned}$$

Here $\mathbb{L}^{-m} = \mathbb{L}^{\otimes -m}$. The diagonal $\Delta : X \to X \times X$ defines a product structure on $h(X)$. First notice that $h(X) \otimes h(X) = (X,id,0) \otimes (X,id,0) = (X \times X, id, 0) = h(X \times X)$, and $\Delta^* = [\Delta] \in \mathrm{Corr}^0(X \times X, X)$. This gives us the product structure,

$$m_X : h(X) \otimes h(X) \xrightarrow{\Delta^*} h(X).$$

(II) **(Transpose)** Let $\phi : Y \to X$, and X, Y are purely d and e-dimensional, respectively. Then the transpose $[{}^t\Gamma_\phi] \in A^d(Y \times X) = A^{d-e+e}(Y \times X) = \mathrm{Corr}^{d-e}(Y,X)$ is a correspondence of degree $d-e$ from Y to X.
We get more,

$$\begin{aligned}[{}^t\Gamma_\phi] &\in \mathrm{Corr}^{d-e}(Y \times X)\\ &= id \circ \mathrm{Corr}^{d-e}(Y,X) \circ id\\ &= \mathrm{Hom}_{\mathcal{M}_k}((Y,id,0),(X,d-e,0))\\ &= \mathrm{Hom}_{\mathcal{M}_k}(h(Y), h(X) \otimes \mathbb{L}^{e-d}).\end{aligned}$$

Hence this defines a morphism

$$\phi_* : h(Y) \to h(X) \otimes \mathbb{L}^{e-d}.$$

Suppose that $d = e$ and that ϕ is generically finite, of degree r. Then the composite $\phi_* \circ \phi^* \in \operatorname{End} h(X)$ is multiplication by r. The following rough diagram may help understand this (remember $d = e$),

$$h(X) \xrightarrow{\phi_*} h(Y) \xrightarrow{\phi^*} h(X)$$

$$\begin{tikzcd}
& \{(\phi(y), y, X)\} \cap \{(X, y, \phi(y))\} = (\phi(y), y, \phi(y))) & \\
& X \times Y \times X \arrow[dl] \arrow[dr, "p_{23}"] \arrow[dd, "p_{13}"'] \arrow[ddd, bend left] & \\
X \times Y \arrow[uur, bend left, "p_{12}^*([\Gamma_\phi])"] & & Y \times X \arrow[uul, bend right, "p_{23}^*([{}^t\Gamma_\phi])"'] \\
& X \times X & \\
& (\phi(y), \phi(y)) &
\end{tikzcd}$$

Hence we have the following,

$$\phi_* \circ \phi^* = p_{13*}\left(p_{12}^*\left[\Gamma_\phi\right] \cdot p_{23}^*\left[{}^t\Gamma_\phi\right]\right) = p_{13*}(\phi, \mathrm{id}, \phi)_*[Y] = r\left[\Delta_X\right]$$

where $\Delta_X \subset X \times X$ is the diagonal.

(III) Now suppose X is a smooth projective variety of dimension d over an algebraically closed field k. Let $x \in X(k)$, then define the following,

$$p_0(X) = \{x\} \times X, \quad p_{2d}(X) = X \times \{x\}.$$

$p_0(X), p_{2d}(X) \in \operatorname{Corr}^0(X)$ are projectors which are orthogonal to each other, i.e. $p_0 {}^\circ p_{2d} = p_{2d} {}^\circ p_0 = 0$. This then defines the two motives

$$h^0(X) = (X, p_0(X), 0)\,, \quad h^{2d}(X) = (X, p_{2d}(X), 0)\,.$$

We then have the following important isomorphisms

$$h^0(X) \simeq (X, \{x\} \times X, 0) \simeq \mathbf{1}$$
$$h^{2d}(X) \simeq (X, X \times \{x\}, 0) \simeq \mathbb{L}^d.$$

(Direct sum of motives) Let $M = (X, p, m)$ and $N = (Y, q, m)$. Then one can define a motive $M \oplus N$. Let us only give the definition in case $m = n$:

$$M \oplus N = (X \sqcup Y, p \oplus q, m),$$

here $\sqcup$ denotes the disjoint union.

As an application, for any smooth projective variety X of dimension d, the correspondence $p^+(X) := \Delta_X - p_0(X) - p_{2d}(X)$ is a projector (since p_0 and p_{2d} are orthogonal) and if we put $h^+(X) := (X, p^+(X), 0)$ there is a direct sum decomposition

$$h(X) = h^0(X) \oplus h^+(X) \oplus h^{2d}(X).$$

(Motive of the projective line) Applying the above decomposition to $\mathbb{P}^1$ we can see that

$$h(\mathbb{P}^1) = h^0(\mathbb{P}^1) \oplus h^+(\mathbb{P}^1) \oplus h^{2d}(\mathbb{P}^1).$$

But now, $\Delta_{\mathbb{P}^1} \sim \mathbb{P}^1 \times \{x\} + \{x\} \times \mathbb{P}^1 \sim p_0(\mathbb{P}^1) + p_{2d}(\mathbb{P}^1) \implies h(\mathbb{P}^1) = h^0(\mathbb{P}^1) \oplus h^{2d}(\mathbb{P}^1) \cong \mathbf{1} \oplus \mathbb{L}$.

Similarly, we have $\Delta_{\mathbb{P}^n} \sim \sum_{i=0}^n \mathbb{P}^i \times \mathbb{P}^{n-i}$. The $\mathbb{P}^i \times \mathbb{P}^{n-i}$ are orthogonal idempotents so

$$h\left(\mathbb{P}^n\right) = \oplus_{i=0}^n \left(\mathbb{P}^n, \mathbb{P}^i \times \mathbb{P}^{n-i}\right).$$

In fact $(\mathbb{P}^n, \mathbb{P}^i \times \mathbb{P}^{n-i}) \cong \mathbb{L}^{\otimes i}$ so

$$\mathfrak{h}\left(\mathbb{P}^n\right) \cong \oplus_{i=0}^n \mathbb{L}^i$$

7.2 Motives of Curves

Suppose C is a smooth projective curve over an algebraically closed field k. Then as above let $p_1(C) = \Delta_C - p_0(C) - p_2(C)$. Then we have the decomposition,

$$h(C) = h^0(C) \oplus h^1(C) \oplus h^2(C),$$

where $h^i(C) = (X, p_i(C), 0)$.

In the last part of this section we will show that for an abelian variety A of dimension g, there exists a unique decomposition in $\mathcal{M}_k$,

$$h(A) = \bigoplus_{i=0}^{2g} h^i(A).$$

As we are already aware of the structure of $h^0(C)$ and $h^2(C)$, we will investigate $h^1(C)$. We will see that this motive is closely related to the Jacobian variety, $J(C)$ of C. The following important theorem fulfills the purpose:

Theorem 7.1. *Let C, C' are smooth projective curves, then*

$$\mathrm{Hom}_{\mathcal{M}_k}(h^1(C), h^1(C')) \cong \mathrm{Hom}_{AV}(J(C), J(C')).$$

Before the proof we need a definition and a few properties. See

Definition 7.3. *(Degenerate divisors) Let X, Y are smooth projective varieties over an k. The subgroup of degenerate divisors on $X \times Y$ is the subgroup generated by divisors D such that $pr_X(D) \neq X$ or $pr_Y(D) \neq Y$. The subgroup of classes of degenerate divisors is denoted by $A^1_{\equiv}(X \times Y)$.*

Fact: A divisor D is degenerate if and only if $D = D_1 \times Y + X \times D_2$, where $D_1 \in \operatorname{div} X, D_2 \in \operatorname{div} Y$.

Theorem 7.2. *(Weil) For any two smooth projective curves C, C' we have*

$$\mathrm{Hom}_{AV}(J(C), J(C')) \cong A^1(C \times C')/A^1_{\equiv}(C \times C'),$$

where $J(C), J(C')$ are the Jacobians of C, C' respectively.

Proof. (Theorem 8.1) First we will work on motives with integer coefficients, denoted by $\mathcal{M}_k^{\mathbb{Z}}$. By the definition of a motive, we get the following homomorphism,

$$A^1(C \times C') \to Hom_{\mathcal{M}_k^{\mathbb{Z}}}(A^1(C), A^1(C')),$$

with the kernel $A^1_{\equiv}(C \times C')$. Therefore we get the isomorphism

$$A^1(C \times C')/A^1 \equiv (C \times C') \cong Hom_{\mathcal{M}_k^{\mathbb{Z}}}(A^1(C), A^1(C')).$$

Consider the full subcategory $\mathcal{M}''_{\mathbb{Z}}$ of $\mathcal{M}_k^{\mathbb{Z}}$ of motives isomorphic to $A^1_{\mathbb{Z}}(C)$ for some smooth projective curve C, and let $F : \mathcal{M}''_Z \to \{\text{category of Jacobians of curves}\}$ be the functor defined by $F(A^1_Z(C)) = J(C)$. Since F is clearly surjective, and by what we have just proved, fully faithful, F is an equivalence of categories. Passing to rational coefficients in the correspondences and taking $\mathrm{Hom}(J(C), J(C')) \otimes \mathbb{Q}$ we have an equivalence

$$F_Q : \mathcal{M}''_{\mathbb{Q}} \longrightarrow \{\text{category of Jacobians of curves}\} \otimes \mathbb{Q}.$$

$\square$

7.3 Manin's identity principle

As a consequence of the definition of motives, we can interpret the Chow groups of any variety in terms of motives. Namely, we have the following isomorphisms:

$$A^i(X) \cong \mathrm{Corr}^i(\mathrm{Spec}(k), X) \cong \mathrm{Hom}_{\mathcal{M}_k}(\mathbf{1}, h(X) \otimes \mathbb{L}^{-i}).$$

For any cycle $\xi \in A^i(X)$ we will still denote the morphism $\mathbf{1} \to h(X)\mathbb{L}^{-i}$ as ξ and, if X is equi-dimensional, we will denote as ${}^t\xi : h(X) \otimes \mathbb{L}^{\dim(X)-i} \to \mathbf{1}$ its transpose. Taking inspiration from this point of view, we define the i-th Chow group of a motive as

$$A^i(M) := \mathrm{Hom}_{\mathcal{M}(k)}(\mathbf{1}, M \otimes \mathbb{L}^{-i})$$

which extends A^* to a $\mathbb{Z}$-graded, $\mathbb{Q}$-linear tensor functor $A^* : \mathcal{M}_k \longrightarrow \mathrm{sVec}\mathbb{Q}$ The following result is a specialization of the Yoneda Lemma to our framework.

Lemma 7.3. *The functor*

$$\omega : \mathcal{M}_k \longrightarrow \mathrm{Fun}(\ \mathit{SmProj}\ /k,\ \ \mathit{Set}\ )$$

is fully faithful where ω_M is defined for any $M \in \mathcal{M}_k$ as:

$$\begin{aligned} \omega_M : \mathrm{SmProj}\,/k &\longrightarrow\ \ \mathit{Set} \\ Y &\longrightarrow \bigoplus_r \mathrm{Hom}_{\mathcal{M}_k}(h(Y), M \otimes \mathbb{L}^{-r}) \end{aligned}$$

while ω_f acts by composition with f.

Proof. The functor $M \longmapsto \mathrm{Hom}_{\mathcal{M}_k}(-, M)$ is fully faithful by Yoneda Lemma. Notice that, for every $M, N \in \mathcal{M}_k$

$$\mathrm{Hom}_{\mathcal{M}_k}(N, M) = \mathrm{Hom}_{\mathcal{M}_k}\left(\mathbf{1}, M \otimes N^\vee\right) = A^0\left(M \otimes N^\vee\right),$$

where for purely d-dimensional X, and $M = (X, p, m)$, we define $M^\vee = (X, {}^tp, d - m)$. We know that any motive N is a direct factor of $h(Y) \otimes \mathbb{L}^n$, for some $Y \in S\mathrm{mProj}/k$ and $n \in \mathbb{Z}$ and that $A^0(M \otimes h(Y) \otimes \mathbb{L}^{-n})$ is isomorphic to $\mathrm{Hom}_{\mathcal{M}_k}(h(Y), M \otimes \mathbb{L}^{-r})$ for some r. This proves the claim. $\square$

From this lemma we can deduce the following proposition,

Proposition 46. *(Manin's Identity Principle). (1) Let $f : M \to N$ be a morphism of motives, f is an isomorphism if and only if $\omega_f(Y) : \omega_M(Y) \to \omega_N(Y)$ is so for every $Y \in \mathrm{SmProj}/k$.*

(2) Let $f, g : M \to N$ be morphisms, $f = g$ if and only if $\omega_f(Y) = \omega_g(Y)$ for any $Y \in \mathrm{SmProj}/k$.

(3) Given morphisms $M_1 \xrightarrow{i_1} M \xrightarrow{p_2} M_2$ in $\mathcal{M}_k$, there exist $p_1 : M \to M_1$ and $i_2 : M_2 \to M$ such that M is a direct sum of M_1 and M_2 via (i_1, i_2, p_1, p_2), if and only if the sequence $0 \longrightarrow \mathrm{Hom}\left(h(Y), M_1 \otimes \mathbb{L}^*\right) \xrightarrow{\omega_{i_1}(Y)} \mathrm{Hom}(h(Y), M \otimes \mathbb{L}^*) \xrightarrow{\omega_{p_2}(Y)} \mathrm{Hom}\left(h(Y), M_2 \otimes \mathbb{L}^*\right) \longrightarrow 0.$ is exact for any $Y \in \mathrm{SmProj}/k$.

Example 7.1. *Let us calculate the motive of $\mathbb{P}^n$ using Manin's principle. We will have to borrow a result from intersection theory,*

$$A^*\left(X \times \mathbb{P}^n\right) \simeq A^*(X)[t]/\left(t^{n+1}\right)$$

where $t \in A^1\left(\mathbb{P}^n\right)$ is the class of a hyperplane in $\mathbb{P}^n$, see [9, Theorem 2.1]. Manin's principle yields an isomorphism of motives

$$h\left(\mathbb{P}^n\right) = \bigoplus_{s=0}^{n} \mathbf{1} \otimes \mathbb{L}^s = \bigoplus_{s=0}^{n} \mathbb{L}^s.$$

In fact, for any $Y \in$ SmProj/ k equi-dimensional of dimension d we have

$$\mathrm{Hom}\left(h(Y), h\left(\mathbb{P}^n\right)(r)\right) = A^{d+r}\left(Y \times \mathbb{P}^n\right) = \bigoplus_{s=0}^{n} A^{d+r-s}(Y) = \mathrm{Hom}\left(h(Y), \bigoplus_{s=0}^{n} \mathbb{L}^{s-r}\right).$$

7.4 Motive of an Abelian variety

In this section we will consider an abelian variety X over k of dimension g. For an integer n, write $[\times n] : X \to X$ for multiplication by n. Also let $\mu : X \times X \to X$ be the group law, $\varepsilon \in X(k)$ the identity element and $\sigma : X \to X$ multiplication by -1.

As we have already seen using the projectors $p_0(X), p_{2d}(X), p_+(X) = \Delta_X - p_0(X) - p_{2d}(X)$ there always exist decomposition of a space X of the form

$$h(X) = h^0(X) \oplus h^+(X) \oplus h^{2d}(X).$$

We will now see that there exists an even finer decomposition of the motive of an abelian variety,

Theorem 7.4. *There is a unique decomposition in* $\mathcal{M}_k$

$$h(X) = \bigoplus_{i=0}^{2g} h^i(X)^{\text{can}}$$

which is stable under $[\times n]^*$*, and such that* $[\times n]^*|_{h^i(X)}$ *is multiplication by the scalar* n^i*, for every* $n \in \mathbf{Z}$*.*

We first introduce some simple notations. If $i \in \mathbf{Z}$ then write $A^*(X)^{(i)}$ for the subspace comprising all $c \in A^*(X)$ such

$$[\times n]^*(c) = n^i c \quad \text{for every } n \in \mathbf{Z}.$$

Likewise if X' is a second abelian variety, of dimension g', write $A^*\left(X \times X'\right)^{(i,j)}$ for the set of all $c \in A^*\left(X \times X'\right)$ such that

$$([\times m] \times [\times n])^*(c) = m^i n^j c \quad \text{for all } m, n \in \mathbf{Z}.$$

Proposition 47. $c \in A^*\left(X \times X'\right)^{(i,j)}$ *if and only if, for every* $n \in \mathbf{Z}$*, one has identities of correspondences*

$$[\times n]^*_{X' \circ C} = n^j c \text{ and } c \circ [\times n]^*_X = n^{2g-i} c.$$

Therefore if $c \in A^*\left(X \times X'\right)^{(i,j)}$ and $d \in A^*\left(X' \times X''\right)^{(r,s)}$ one has $d \circ c = 0$ unless $j = 2g' - r$.

We say $\xi \in A^*(X)$ is symmetric if $\sigma^*\xi = \xi$. For such ξ define $\lambda = \mu^*\xi - pr_1^*\xi - pr_2^*\xi \in A^1(X \times X)$.

Proof. Let $\xi \in A^1(X)$ be a symmetric ample line bundle. If $0 \leq i \leq 2g$ define

$$f_i = \sum_{\max(0,i-g) \leq j \leq i/2} \frac{1}{j!(g-i+j)!(i-2j)!} pr_1^*\xi^j \cdot pr_2^*\xi^j \cdot \lambda^{i-2j} \in A^i(X \times X)^{(i,i)},$$

$$q_i = \sum_{\max(0,i-g) \leq j \leq i/2} \frac{1}{j!(g-i+j)!(i-2j)!} pr_1^*\xi^{g-i+j} \cdot pr_2^*\xi^j \cdot \lambda^{i-2j} \in A^g(X \times X)^{(2g-i,i)}.$$

In particular $q_i \circ q_{i'} = 0$ if $i \neq i'$. Now

$$\begin{aligned}
\sum_{i=0}^{2g} q_i &= \sum_{\substack{0 \le i \le 2g \\ \max(0, i-g) \le j \le i/2}} \frac{1}{j!(g-i+j)!(i-2j)!} pr_1^* \xi^{g-i+j} \cdot pr_2^* \xi^j \cdot \lambda^{i-2j} \\
&= \frac{1}{g!} \left(pr_1^* \xi + pr_2^* \xi + \lambda \right)^g \\
&= \frac{1}{g!} \mu^* \xi^g = d \mu^* [\varepsilon] = d \left[\Gamma_\sigma \right]
\end{aligned}$$

where the last equality follows from the following fact: if $\eta \in A^1(X)$ is symmetric, then

$$\xi^g = g! d[\varepsilon] \in A^g(X).$$

Now define $p_i^{\text{can}} = \frac{1}{d} \sigma^* \circ q_i = \frac{(-1)^i}{d} q_i \in A^g(X \times X)^{(2g-i,i)}$. Then we have,

$$\sum p_i^{\text{can}} = 1 \quad \text{and} \quad \sum p_i^{\text{can}} \circ p_i^{\text{can}} = \left(\sum p_i^{\text{can}} \right)^2 = 1$$

This forces $p_i^{\text{can}} \circ p_i^{\text{can}} = p_i^{\text{can}}$, and then $h^i(X)^{\text{can}} = (X, p_i^{\text{can}}, 0)$ satisfies the properties given in the statement. $\square$

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