Diffusion Processes: Analysis and their Applications

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Certificate of Examination

This is to certify that the dissertation titled **Diffusion Processes: analysis and their applications** submitted by **Gaurav Aggarwal** (Reg. No. MS09051) for the partial fulfilment of BS-MS dual degree programme of the Institute, has been examined by the thesis committee duly appointed by the Institute. The committee finds the work done by the candidate satisfactory and recommends that the report be accepted.

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Declaration

The work presented in this dissertation has been carried out by me under the guidance of Dr Lingaraj Sahu at the Indian Institute of Science Education and Research Mohali.

This work has not been submitted in part or in full for a degree, a diploma, or a fellowship to any other university or institute. Whenever contributions of others are involved, every effort is made to indicate this clearly with due acknowledgements. This thesis is a bonafide record of work done by me and all sources listed within have been detailed in the bibliography.

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In my capacity as the supervisor of the candidate's project work, I certify that the above statements by the candidate are true to the best of my knowledge.

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Contents

Abstract

Differential equations are viewed as models for the trajectories of moving particles. Using differential equations to study the trajectory of a particle undergoing random motion is not straight forward. The aim of the project is to understand *diffusion processes*, which are used as models for the trajectory of particle exhibiting a random behaviour. The analysis behind defining stochastic integration and the use of Itô's formula in writing the stochastic differential equations is rigorously reproduced. The solutions of the SDEs and the sufficient conditions for their existence and uniqueness are studied, the analysis is supplemented with important examples and applications.

Chapter 1

Brownian Motion

1.1 Introduction

The term Brownian Motion was coined for the motion exhibited by a small particle suspended in a fluid, it was named after Robert Brown who observed this phenomenon in 1827. T.N. Thiele(in 1880), L. Bachelier(in 1900) and A. Einstein(in 1905) made independent efforts to model the Brownian motion. The mathematical theory of Brownian motion was given a firm foundation by Norbert Wiener in 1923, in his honour, Brownian motion is also known as Wiener process.[6]

Definition 1.1. *A Brownian motion* $\{B_t : t \geq 0\}$ *is a stochastic process defined on a probability space* $(\Omega, \mathcal{F}, \mathbb{P})$ *with*

- $B_0 = x$*;*
- *stationary, independent and normally distributed increments*
- *and almost surely continuous paths.*

If $B_0 = 0$*, the process is called a standard Brownian motion.*

We observe the meanings of some of the important terms used in the above definition. A stochastic process $X(t, \omega)$ is a real valued function $X : \mathbb{R} \times \Omega \to \mathbb{R}$. It can thus be viewed either as a collection of random variables $X(t, \cdot)$ or as a random function $X(\cdot, \omega)$. Stationary increments means that for $s < t$, the difference $B_t - B_s$ depends only on $t - s$, by independent increments we mean that $B_t - B_s$ is independent of $\{B_r : r \leq s\}$ whenever $s < t$. The distribution of $B_t - B_s$ is given by $N(0, t - s)$, also the map $t \to B_t$ is continuous [10]. The continuity of paths is crucial as it technicaly adds more information which goes beyond the finite dimensional distributions of the process. Suppose that $\{B_t : t \geq 0\}$ is a Brownian motion and T is an independent random variable which is uniformly distributed on $[0, 1]$. Then the process $\{B'_t : t \geq 0\}$ defined by

$$B'_t = \begin{cases} B_t & \text{if } t \neq T, \\ 0 & \text{if } t = T, \end{cases}$$

has the same finite dimensional distribution as a Brownian motion, but is discontinuous if $B_T \neq 0$, i.e. with probability one, and hence is not a Brownian motion.[6]

A process $\{W_t : t \geq 0\}$ with the distribution of the increments given by $N(\mu(t - s), \sigma^2(t - s))$ is called a Brownian motion with drift. Here μ is the drift coefficient and σ is the diffusion coefficient. This process can be expressed in the terms of a standard Brownian motion as follows:

$$W_t = W_0 + \mu t + \sigma B_t$$

where B_t is a standard Brownian motion. Thus for any Brownian motion, the process given by:

$$\frac{W_t - W_0 - \mu t}{\sigma}$$

represents a standard Brownian motion.

1.2 Existence of Wiener Measure

We will follow the approach as given in [10]. We first begin with a simple one dimensional random walk (beginning at origin and whose probability of moving in either direction is same) and observe that in certain limiting cases, we obtain what is known as Brownian motion. The probability function $f(t, x)$ of a simple random walk with time intervals τ and step size h satisfies the difference equation:

$$f(t + \tau, x) = f(\tau, h)f(t, x - h) + f(\tau, -h)f(t, x + h) \tag{1.1}$$

In the limits $h \to 0, \tau \to 0$ and $\dfrac{h^2}{\tau} \to 1$ we obtain that the probability density function satisfies the following equation:

$$\frac{\partial f}{\partial t} = \frac{1}{2}\frac{\partial^2 f}{\partial x^2} \tag{1.2}$$

Here we have chosen the limits for $\frac{h^2}{\tau}$ to be 1. We could have chosen any constant value for it. We can easily see that it cannot be 0 or $+\infty$ as then the process may approach $+\infty$ in a finite number of steps which cannot be true.
We have obtained the *heat equation* and we observe that the *Gaussian kernel*, $p(t,x) = \frac{1}{\sqrt{2\pi t}} \exp \frac{-x^2}{2t}$, is a solution to (1.2) under the initial value $f(0,x) = u(x)$ in the sense that $\lim_{t\to 0} f(t,x) = u(x)$. This can be interpreted as the probability that the process begins at x. It can be observed that:

$$p(t,x) \geq 0$$

$$\int p(t,x)dx = 1$$

$$\lim_{t\to 0} p(t,x) = \delta(x) = \begin{cases} 0 & \text{if } x \neq 0, \\ 1 & \text{if } x = 0, \end{cases}$$

where $\delta(x)$ is the dirac delta function, thus giving the probability that the process begins at the origin. We see that if we translate the origin to y, $p(t, x - y)$ is still the solution to the heat equation. For each $y \in \mathbb{R}^n$, $p(t, x - y)dy$ defines a probability distribution which we can interpret as giving the probability that a particle starting at y at time 0 will be in the given region in $\mathbb{R}^n$ at time t. The solution to the initial value problem can thus be expressed in the form:

$$f(t,x) = \int_{\mathbb{R}} u(y)p(t, x - y)dy \tag{1.3}$$

where $u(y)$ is bounded and continuous.

Lemma 1.1. *Let* $p(t,x) = \frac{1}{\sqrt{2\pi t}} \exp \frac{-x^2}{2t}$ *for* $t > 0$. *Then*

$$p(t,\cdot) * p(s,\cdot) = p(t+s,\cdot)$$

The proof of the above lemma can be obtained using Fourier transforms. By the virtue this lemma we can write:

$$\int_{\mathbb{R}} p(s-0, z-y)p(t-s, x-z)dz = p(t, x-y), 0 < s < t$$

$p(t-s, x-z)$ can be interpreted as the *transition probability*, i.e. the probability of moving from a space point z at time s to space point x at time t. In view of this result, we obtain that:

$$\int_{\mathbb{R}} u(y)p(t, x-y)dy = \int\int u(y)p(s, z-y)p(t-s, x-z)dzdy$$

It follows that the above holds for any finite partition $0 < t_1 < t_2 < \ldots < t_n = t$. Thus, given $0 < t_1 < t_2 < \ldots < t_n = t$ and given Borel sets $E_j \subset \mathbb{R}^n$, the probability that a path, starting at $x = 0$ at $t = 0$, lies in E_j at time t_j for each $j \in \{1, 2, \ldots n\}$ is

$$\int \ldots \int p(t_n - t_{n-1}, x_n - x_{n-1}) \ldots p(t_1, x_1)dx_n \ldots dx_1$$

Hence for each partition $(t_1, t_2, \ldots, t_n)$, we have a probability distribution function $P_{(t_1,t_2,\ldots,t_n)}$, therefore we have obtained a family of probability distribution functions corresponding to all possible n-tuples $(t_1, t_2, \ldots, t_n)$ of t. The existence of a countably additive measure characterized by the properties as above follows from the Kolmogorov's extension theorem. For a detailed proof of the theorem, refer [1].

Theorem 1. ***Kolmogorov's Extension Theorem***
If $P_{(t_1,t_2,\ldots,t_n)}$ are a system of distributions satisfying the consistency conditions given by:

$$P_{(t_1,t_2,\ldots,t_n)}(H_1 \times H_2 \times \ldots \times H_n) = P_{(t_{\pi 1},t_{\pi 2},\ldots,t_{\pi n})}(H_{\pi 1} \times H_{\pi 2} \times \ldots \times H_{\pi n}) \qquad (1.4)$$

and,

$$P_{(t_1,t_2,\ldots,t_{n-1})}(H_1 \times H_2 \times \ldots \times H_{n-1}) = P_{(t_1,t_2,\ldots,t_n)}(H_1 \times H_2 \times \ldots \times H_{n-1} \times \mathbb{R}) \quad (1.5)$$

where π is a permutation function which maps n-tuples to any one of its possible permutations. Then there exists on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ a stochastic process $\{X_t, t \in T\}$ having $P_{(t_1,t_2,\ldots,t_n)}$ as its finite dimensional distributions.

As a consequence of the above discussion and Kolmogorov's theorem, for a Brownian motion on $\Omega = C([0, \infty); \mathbb{R})$ with the Borel σ- algebra $\mathcal{F}$ and a sequence of sub-σ-algebras $\mathcal{F}_t = \sigma\{X(s), 0 < s < t\}$(this sequence is also known as filtration) we obtain a measure known as the Wiener measure which induces a product measure on $\Pi_{0<s<\infty}\mathbb{R}$. Therefore a Brownian motion with starting point x is an $\mathbb{R}$- valued stochastic process with

$X(0) = x$ and the family of distribution of the process is specified by

$$P_{(t_1,t_2,\ldots,t_n)}(A) = \int_A p(t_n - t_{n-1}, x_n - x_{n-1}) \ldots p(t_1, x_1 - x)dx_n \ldots dx_1 \tag{1.6}$$

for every Borel set A in $\mathbb{R}^n$.

1.3 Properties of Brownian Motion

We will enumerate some of the important properties of Brownian motion with brief outlines of their proofs. The arguments given are inspired from [6] [11] [3].

1. Brownian motion $\{B_t : t \geq 0\}$ is a **Gaussian process**.

   Proof A process $\{X_t : t \geq 0\}$ is called Gaussian if $\forall\, 0 \leq t_1 < t_2 < \cdots < t_n$ the probability distribution of the random vector $(X_{t_1}, \ldots, X_{t_n})$ on $\mathbb{R}^n$ is normal or Gaussian. We just need to verify that for any $t_1, \ldots, t_n$, $(B_{t_1}, \ldots, B_{t_n})$ is multivariate Gaussian. From the definition of Brownian motion, it is immediate that $(B_{t_1}, B_{t_2} - B_{t_1}, \ldots, B_{t_n} - B_{t_{n-1}})$ is multivariate Gaussian. Since $(B_{t_1}, \ldots, B_{t_n})$ is a linear combination of $(B_{t_1}, B_{t_2} - B_{t_1}, \ldots, B_{t_n} - B_{t_{n-1}})$ it is multivariate Gaussian as well. From the properties of a normal distribution, it follows that the distribution of a Gaussian process is characterized by the mean $E[X_t]$ and covariance function

   $$cov(X_s, X_t) = E[X_s - E[X_s]]E[X_t - E[X_t]]$$

   Thua a Brownian motion is a Gaussian process with mean 0 and covariance $s \wedge t$, the minimum of s and t.

2. Brownian motion $\{B_t : t \geq 0\}$ is a **Markov process**. Let $s > 0$, then the process $\{B_{t+s} - B_s : t \geq 0\}$ is again a Brownian motion started in the origin and it is independent of the process $\{B_r : 0 \leq r \leq s\}$.

   Proof A process is Markov if the future development of the process depends on its present value alone, without any reference to its past history. The property of the Brownian motion that it possesses stationary and independent increments ensures that it is a Markov process.

3. **Scaling property:** If $\{X_t : t \geq 0\}$ is defined by $X_t = \dfrac{1}{\sqrt{c}} B_{ct}$ then $\{X_t : t \geq 0\}$ is also a standard Brownian motion.

   Proof Under scaling, we observe that continuity of sample paths, stationary and independent increments are preserved. We observe that $X_t - X_s$ are normally distributed.

$$X_t - X_s = \frac{1}{\sqrt{c}} B_{ct} - \frac{1}{\sqrt{c}} B_{cs} \sim \frac{1}{\sqrt{c}} N(0, c(t-s)) \sim N(0, t-s)$$

4. **Time inversion property:** If $\{X_t : t \geq 0\}$ is defined by

$$X_t = \begin{cases} 0 & \text{if } t = 0 \\ tB_{\frac{1}{t}} & \text{if } t > 0 \end{cases}$$

   then $\{X_t : t \geq 0\}$ is also a standard Brownian motion.

   Proof We check that for the above process, $E[X_t] = tE[B_{\frac{1}{t}}] = 0$ and $cov[X_t, X_s] = (ts)cov[B_{\frac{1}{t}}, B_{\frac{1}{s}}] = (ts)\frac{1}{t} = s$ for $0 < s < t$. Since this is the characteristic of a Brownian motion(which is a Gaussian process) we just need to check that the process $\{X_t, t \geq 0\}$ has almost surely continuous paths.

   Time inversion is a useful tool to relate the properties of Brownian motion in a neighbourhood of time $t = 0$ to properties at infinity.

5. **Law of Large Numbers:** Almost surely, $\lim_{t\to\infty} \frac{B_t}{t} = 0$.

   Proof Using the time inversion, we obtain that

$$\lim_{t\to\infty} \frac{B_t}{t} = \lim_{t\to\infty} X_{\frac{1}{t}} = X_0 = 0 \text{ almost surely.}$$

6. The sample paths of a Brownian motion are **nowhere differentiable**.

   Proof We define a process $\{D_t : t \geq 0\}$ as $D_t = \lim_{h\to 0} \dfrac{B_{t+h} - B_t}{h}$. This process takes the value of the slope of the paths of the Brownian motion at any time t. We observe that its distribution is given by

$$D_t \sim \lim_{h\to 0} \frac{N(0,h)}{h} \sim \lim_{h\to 0} N(0, h^{-1})$$

Thus the slope has variance that goes to infinity almost everywhere, hence the Brownian motion is nowhere differentiable.

7. The paths of Brownian motion have a non-zero finite quadratic variation, such that on interval (s,t), the **quadratic variation** is $(t-s)$

   Proof For a partition $\pi = \{s = t_1 < t_2 < \ldots < t_n = t\}$ of $[s,t]$ we can form the sum of squared increments $\sum_{i=1}^{n-1}(B_{t_{i+1}} - B_{t_i})^2$. These sums converge to a random variable $[B]_t$ in probability as mesh(π)= $max_i(t_{i+1} - t_i) \to 0$. The process obtained is called the quadratic variation.

$$\begin{aligned}
E[[B]_t] &= E[\sum_{i=1}^{n-1}(B_{t_{i+1}} - B_{t_i})^2] \\
&= \sum_{i=1}^{n-1}[E[(B_{t_{i+1}} - B_{t_i})^2]] \\
&= \sum_{i=1}^{n-1}[t_{i+1} - t_i] \\
&= t - s
\end{aligned}$$

8. The paths of Brownian motion have **infinite variation** on compact time intervals, almost surely.

   Proof The main idea that we use here is that a function that has bounded variation is differentiable almost everywhere, which is not the case with the Brownian motion. Using this contrapositive argument, we have the result.

9. **Strong Markov Property:** For every almost surely finite stopping time T, the process $\{B_{T+t} - B_T : t \geq 0\}$ is a standard Brownian motion independent of $\mathcal{F}_T^+$ where $\mathcal{F}_T^+ = \bigcap_{t>T} \mathcal{F}_t$.

   Proof Consider a sequence of stopping times $T_n = (m+1)2^{-n}$ if $m2^{-n} \leq T < (m+1)2^{-n}$. Define $B^{(k)} = \{B_t^{(k)} : t \geq 0\}$ such that $B_t^{(k)} = B_{t+\frac{k}{2^n}} - B_{\frac{k}{2^n}}$, and

$B^* = \{B^*_t : t \geq 0\}$ such that $B^*_t = B_{t+T_n} - B_{T_n}$. Now, for $E \in \mathcal{F}^+_{T_n}$ and an event $\{B^* \in A\}$, we have

$$P(\{B^* \in A\} \cap E) = \sum_{k=0}^{\infty} P(\{B^{(k)} \in A\} \cap E \cap \{T_n = k2^{-n}\})$$
$$= \sum_{k=0}^{\infty} P(\{B^{(k)} \in A\}) P(E \cap \{T_n = k2^{-n}\})$$

this follows from the independent increments property of Brownian motion. Since $B^{(k)} = \{B^{(k)}_t : t \geq 0\}$ is also a standard Brownian motion, we have that $P(\{B^{(k)} \in A\}) = P(\{B \in A\})$ is independent of k, and hence

$$\sum_{k=0}^{\infty} P(\{B^{(k)} \in A\}) P(E \cap \{T_n = k2^{-n}\}) = P(\{B \in A\}) \sum_{k=0}^{\infty} P(E \cap \{T_n = k2^{-n}\})$$
$$= P(\{B \in A\}) P(E)$$

which shows that B^* is a Brownian motion and is independent of E, and hence independent of $\mathcal{F}^+_{T_n}$.
Now as $T_n \downarrow T$ we have that $\{B_{s+T_n} - B_{T_n} : s \geq 0\}$ is a Brownian motion independent of $\mathcal{F}^+_{T_n} \supset \mathcal{F}^+_T$. Hence the increments of the process $\{B_{r+T} - B_T : r \geq 0\}$ given by

$$B_{s+t+T} - B_{t+T} = \lim_{n \to \infty} B_{s+t+T_n} - B_{t+T_n}$$

are independent and normally distributed with mean zero and variance s. The process is almost surely continuous and hence is a Brownian motion independent of $\mathcal{F}^+_T$. For a more elaborated study on stopping times and filtrations, the reader is referred to [3].

10. **Reflection Principle:** Given a standard Brownian motion $\{B_t : t \geq 0\}$, for every $a \geq 0$

$$P[\sup_{0 \leq s \leq t} B_s \geq a] = 2P[B_t \geq a] \tag{1.7}$$

Proof Define $\tau_a = \inf\{t \geq 0 : B_t = a\}$. Then by the strong Markov property it follows that $\{B_{\tau_a + t} - B_{\tau_a} : t \geq 0\}$ is a standard Brownian motion independent of

$\mathcal{F}^+_{\tau_a}$. Using this argument and that $P[B_t < 0] = \frac{1}{2}$ we have

$$\begin{aligned}
P[\sup_{0\le s\le t} B_s \ge a] &= P[\sup_{0\le s\le t} B_s \ge a, B_t \ge a] + P[\sup_{0\le s\le t} B_s \ge a, B_t < a] \\
&= P[B_t \ge a] + P[\sup_{0\le s\le t} B_s \ge a, (B_{\tau_a+t} - B_{\tau_a}) < 0] \\
&= P[B_t \ge a] + \frac{1}{2} P[\sup_{0\le s\le t} B_s \ge a] \\
&= 2P[B_t \ge a]
\end{aligned}$$

We can observe that the process $\{B^*_t : t \ge 0\}$ called the *Brownian motion reflected at* τ_a and defined by

$$B^*_t = B_t 1_{\{t\le\tau_a\}} + (2B_{\tau_a} - B_t)1_{\{t>\tau_a\}}$$

is also a standard Brownian motion.

11. **Martingale Property:** $\{B_t : t \ge 0\}$ is an $\mathcal{F}_t$- martingale. i.e. for $t \le u$

$$E[B_u|\mathcal{F}_t] = B_t$$

Proof For all $0 \le t \le u < \infty$

$$\begin{aligned}
B_u &= B_t + \int_t^u dB_s \\
E[B_u|\mathcal{F}_t] &= E[B_t + \int_t^u dB_s|\mathcal{F}_t] \\
&= E[B_t] + E[\int_t^u dB_s|\mathcal{F}_t] \\
&= B_t + 0 = B_t
\end{aligned}$$

Thus we have that a standard Brownian motion is a continuous $\mathcal{F}_t$-martingale. Some important related results are mentioned below, these can be verified following the above method.

(a) $\{{B_t}^2, t \ge 0\}$ is not an $\mathcal{F}_t$- martingale.

(b) $\{{B_t}^2 - t, t \ge 0\}$ is an $\mathcal{F}_t$- martingale.

(c) Brownian motion with drift $\{W_t, t \ge 0\}$ where $W_t = \mu t + \sigma B_t$ is not an $\mathcal{F}_t$-martingale.

(d) $\{W_t - \mu t, t \geq 0\}$ is an $\mathcal{F}_t$- martingale.

(e) $\{X_t, t \geq 0\}$, where $X_t = \exp\{\lambda B_t - \frac{1}{2}\lambda^2 t\}$ is an $\mathcal{F}_t$- martingale.

1.4 Applications and Examples

1.4.1 Geometric Brownian Motion

A stochastic process $\{S_t : t \geq 0\}$ defined such that the logarithm of the process agrees with the Brownian motion with drift is known as a geometric Brownian motion. It is widely used in the modelling of stock prices using the Black-Scholes model.[8] The process is given by the relation

$$S_t = \exp W_t \tag{1.8}$$

where,

$$W_t = W_0 + \mu t + \sigma B_t$$

Thus we can say that $\log(S_t)$ is normally distributed with mean $W_0 + \mu t$ and variance $\sigma^2 t$. We note that $S_t \geq 0 \ \forall \ t$. Using the properties of log-normal distribution we can say that:

$$E[S_t] = \exp(\mu t + W_0 + \frac{\sigma^2 t}{2})$$
$$Var[S_t] = \exp(2\mu t + 2W_0 + \sigma^2 t)((\exp \sigma^2 t) - 1)$$

1.4.2 Brownian Bridge

A stochastic process $\{S_t : t \geq 0\}$, obtained by conditioning the Brownian motion $\{B_t : t \geq 0\}$ on an interval $[0, 1]$ to the event $B_1 = 0$ is known as a Brownian bridge. [11]. The formal relation between a Brownian motion and a Brownian bridge is given by

$$S_t = B_t - t\, B_1 \tag{1.9}$$

Thus the probability distribution of the Brownian bridge is the conditional probability distribution of a Brownian motion to the given condition $B_1 = 0$. We can say that the

Brownian bridge is a Gaussian process with

$$
\begin{aligned}
E[S_t] &= E[B_t - t\,B_1] = E[B_t] = 0 \\
cov[S_s, S_t] &= cov[B_s - s\,B_1, B_t - t\,B_1] \\
&= E[(B_s - s\,B_1)(B_t - t\,B_1)] \\
&= E[B_sB_t] - tE[B_1B_s] - sE[B_1B_t] + stE[{B_1}^2] \\
&= (s \wedge t) - t(1 \wedge s) - s(1 \wedge t) + st \\
&= (s \wedge t) - st
\end{aligned}
$$

It can be verified using the mean and covariance property that

$$Y_t = (1-t)W_{\frac{t}{1-t}} \text{ for } 0 \le t \le 1; Y_1 = 0 \tag{1.10}$$

also gives a Brownian bridge.

1.4.3 Ornstein-Uhlenbeck Process

A stochastic process $\{X_t : t \ge 0\}$ given by the relation

$$X_t = e^{-t} B_{e^{2t}} \tag{1.11}$$

is known as Ornstein-Uhlenbeck process. The process is stationary, Gaussian, Markov and mean-reverting. It can be observed that X_t is standard normally distributed. i.e.

$$X_t \sim N(0,1)$$

Using the time-inversion property of Brownian motion, we can observe that the process is time reversible that is X_t and X_{-t} have the same law. [6].

1.4.4 Fractional Brownian Motion

It is an example of a stochastic process which is Gaussian, self-similar and it has stationary increments. It differs from Brownian motion as it does not possess independent increments. A gaussian stochastic process $\{X_t : t \ge 0\}$ is called a fractional Brownian motion of Hurst parameter $\alpha \in (0,1)$ if it has mean zero and covariance given by

$$cov(X_s, X_t) = \frac{1}{2}(s^{2\alpha} + t^{2\alpha} - \mid t - s \mid^{2\alpha})$$

The value of α determines the correlation between the increments of fractional Brownian motion.

$$\text{If } \alpha \begin{cases} = \frac{1}{2} & \text{increments are uncorrelated} \\ < \frac{1}{2} & \text{increments are negatively correlated} \\ > \frac{1}{2} & \text{increments are positively correlated} \end{cases}$$

Due to the self similarity of the process, it is widely used for fractal simulations.[8]

Chapter 2

Stochastic Integration

2.1 Introduction

In this chapter we first construct a stochastic integral with respect to Brownian motion. We will look at a Brownian motion $\{B_t : t \geq 0\}$ as a random function. From the previous chapter we know that this function is of infinite variation, which is why we cannot use Riemann-Stieltjes integration to define the integrals with respect to Brownian motion. Utilizing the fact that $\{B_t : t \geq 0\}$ is a random function and thus using the weaker form of limits, we can define such integration. If we consider a Riemann- Stieltjes integral:

$$\int f(t)dg(t,w) = h(w) \tag{2.1}$$

We say, that the integral makes sense if f is continuous and g is a function of bounded variation (i.e. g is the difference between two monotone functions). But we have seen that Brownian motion has infinite variation. The idea is to define an integral in the above sense for functions with infinite variation [10]. We will define the integral with respect to Brownian motion, a more general form where the integral is defined with respect to a semi-martingale exists but we will not be concerned with it in our work. To define an integral of the type:

$$\int f(t)dB(t,w) = h(w) \tag{2.2}$$

We first look at a suitable class of integrands that can be admitted so as to define this integral. We denote by $(\Omega, \mathcal{B}, \mathbb{P})$ the probability space on which our Brownian motion is defined and suppose that $\{\mathcal{F}_t : t \geq 0\}$ is a filtration to which the Brownian motion is adapted. We assume that the filtration is complete,i.e. contains all the sets of probability zero in $\mathcal{B}$. We will require the class of integrands to be progressively measurable. We define such processes as follows

Definition 2.1. *A process $\{X_t : t \geq 0\}$ is called progressively measurable if for each $t \geq 0$ the mapping $X : [0,t] \times \Omega \to \mathbb{R}$ is measurable with respect to the σ-algebra $\mathcal{B}([0,t]) \otimes \mathcal{F}_t$.*

Having defined the integrand and the integrator of the integral we now construct the integral and look forward to its interpretation.

2.2 Itô's Stochastic Integral

In the above equation, let us assume f to be a determinstic step function of the type:

$$f = \sum_{i=1}^{n} a_i \chi_{[t_i, t_{i+1})}, 0 \leq t_1 < t_2 < \ldots < t_{n+1} \tag{2.3}$$

Then following the Riemann- Stieltjes type integral we define:

$$h(w) = \int_0^\infty f(t) dB(t,w) = \sum_{i=1}^{n} a_i (B_{t_{i+1}}(w) - B_{t_i}(w)) \tag{2.4}$$

We observe that h is a random variable and its distribution is obtained as follows:

$$\begin{aligned} h(w) &= \sum_{i=1}^{n} a_i (B_{t_{i+1}}(w) - B_{t_i}(w)) \\ &\sim \sum_{i=1}^{n} a_i N(0, t_{i+1} - t_i) \\ &\sim \sum_{i=1}^{n} N\left(0, {a_i}^2 [t_{i+1} - t_i]\right) \\ &\sim N\left(0, \sum_{i=1}^{n} {a_i}^2 [t_{i+1} - t_i]\right) \end{aligned}$$

Thus,

$$\begin{aligned} E\left[h\right] &= 0; \\ E\left[h^2\right] &= \sum_{i=1}^{n} {a_i}^2 [t_{i+1} - t_i] = \|f\|_2^2 \end{aligned}$$

Now for any constant α and any two step functions f, g the following can be easily obtained:

$$\int_0^\infty (\alpha f)\, dB(t,w) = \alpha \int_0^\infty (f)\, dB(t,w);$$
$$\int_0^\infty (f+g)\, dB(t,w) = \int_0^\infty (f)\, dB(t,w) + \int_0^\infty (g)\, dB(t,w)$$

The mapping $f \to \int_0^\infty f dB$ is therefore a linear L^2 - isometry of the space S of all step functions of the type:

$$\sum_{i=1}^{n} a_i \chi_{[t_i,t_{i+1})}, 0 \leq t_1 < t_2 < \ldots < t_{n+1}$$

into the space $L^2(\Omega, \mathcal{B}, \mathbb{P})$. To extend this result uniquely to the space $L^2[0,\infty)$, we observe that the space S is dense in $L^2[0,\infty)$. Thus there is a unique linear $L^2_{\mathbb{R}}$ - isometry of $L^2[0,\infty)$ into the space $L^2(\Omega, \mathcal{B}, \mathbb{P})$. One thing that we must notice here is that, if we define this integral as a process with respect to t i.e. if we define the process $X_t = \int_0^t f(s)dB_s$, it comes out to be a martingale. We check for $u < t$:

$$E\left[X_t|\mathcal{F}_u\right] = X_u + E\left[\int_u^t f(s)dB_s|\mathcal{F}_u\right] = X_u$$

Thus we have made sense of the stochastic integral for deterministic function in $L^2[0,\infty)$. Now to extend the definition for continuous time stochastic processes we need to define the integrals of the type:

$$h(t,w) = \int_0^t Y(s,w)dB(s,w) \tag{2.5}$$

We first define simple functions Φ, extend our definition for these functions and then move towards the progressively measurable continuous functions.

Definition 2.2. *A function $\Phi : [0,\infty) \times \Omega \to \mathbb{R}$ is called simple if for a partition $0 \le s_0 < s_1 < s_2 \ldots < s_n < \ldots < \infty$ of $[0,\infty)$*

$$\Phi(s,w) = \Phi_j(w);\ for\, s \in [s_j, s_{j+1})$$

where $\Phi_j(w)$ is $\mathcal{F}_{s_j}$ measurable and bounded.

Following the Riemann- Stieltjes type integral, we define

$$h(t,w) = \int_0^t \Phi(s,w)dB(s,w)$$

Lemma 2.1. *For $h(t,w)$ as defined above:*

1. *the map $\Phi \to h$ is linear.*
2. *$h(t,w)$ is an $\mathcal{F}_t$- martingale.*
3. *$E[h(t,w)] = 0$*
4. *$h^2(t,w) - \int_0^t \Phi^2 ds$ is a martingale with*

$$E[h^2(t,w)] = E[\int_0^t \Phi^2(s,w)ds] \tag{2.6}$$

Thus we obtain a similar isometry for the simple functions. It follows from the Lebesgue dominated convergence theorem that for a continuous $\mathcal{F}_t$- measurable function $Y(s,w)$ with $E\left[\int_0^t Y^2 ds\right] < \infty$, $\exists$ a sequence $\Phi_n(s,w)$ of simple functions such that

$$\lim_{n\to\infty} E[\int_0^t |\Phi_n(s,w) - Y(s,w)|^2 ds] = 0$$

All that is left is to extend the definition of the integrals from simple functions to that of $\mathcal{F}_t$- measurable functions. The following theorem takes care of this part of the definition.

Theorem 2. *Let $Y(s,w)$ be $\mathcal{F}_t$- measurable, such that*

$$E\left[\int_0^t Y^2 ds\right] < \infty$$

for each $t > 0$. Let $\{\Phi_n\}$ be a sequence of simple functions that approximate Y. Put

$$h_n(t,w) = \int_0^t \Phi_n(s,w)dB(s,w)$$

Then

1. *$lim_{n\to\infty} h_n(t,w)$ exists uniformly in probability, i.e. there exists an almost surely continuous $h(t,w)$ such that*

$$\lim_{n\to\infty} P\left(\sup_t |h_n(t,w) - h(t,w)| \geq \epsilon\right) = 0$$

   for each $\epsilon > 0$.

2. *The properties in Lemma 2.1 extend to the function $h(t,w)$.*

Proof

1. Using the property that the stochastic integral is linear for simple functions we obtain a random variable

$$(h_n - h_m)(t,w) = \int_0^t (\Phi_n - \Phi_m)(s,w)dB(s,w)$$

   Now we use the Doob's martingale inequality to get the upper bound as follows:

$$P\left(\sup_{0\leq t\leq\infty} |h_n(t,w) - h_m(t,w)| \geq \epsilon\right) \leq \frac{E[(h_n - h_m)^2(\infty,w)]}{\epsilon^2}$$

$$E[(h_n - h_m)^2(\infty,w)] = E[\int_0^\infty (\Phi_n - \Phi_m)^2 ds]$$

   Therefore,

$$\lim_{n,m\to\infty} E[(h_n - h_m)^2(\infty,w)] = 0 \tag{2.7}$$

   which gives us that $(h_n - h_m)$ is uniformly Cauchy in probability. Therefore there exists an $\mathcal{F}_t$- measurable h such that

$$\lim_{n\to\infty} P\left(\sup_{0\leq t\leq\infty} |h_n(t,w) - h(t,w)| \geq \epsilon\right) = 0, \forall\epsilon > 0$$

2. The rest of the proof can be easily constructed after observing that h now obtained is almost surely continuous.

This completes the definition of the Itô integral and we have made sense of integrating progressively measurable processes with respect to a standard Brownian motion. Although the integral has been defined, it is not straight forward to compute the integrals of the above type.[10]

2.3 Extension of the Itô integral

We have defined the integral for the class of square integrable processes that are progressively measurable i.e. for $\{X_t : t \geq 0\}$

$$\int_0^\infty X_t^2 dt < \infty$$

We now extend this to a more general class of processes that are progressively measurable and satisfy

$$P[\int_0^\infty X_t^2 dt < \infty] = 1 \tag{2.8}$$

To extend our definition to this class of processes we use localization through stopping times.

Theorem 3. *For a process $\{X_t : t \geq 0\}$ satisfying the above property and $n \in \mathbb{N}$ consider the stopping time*

$$\tau_n = \inf\{T \geq 0 : \int_0^T X_t^2 dt \geq n\}$$

Then $\lim_{n\to\infty} \tau_n = \infty$ a.s. and

$$\int_0^\infty X_t dB_t = \lim_{n\to\infty} \int_0^{\tau_n} X_t dB_t$$

exists as limit in probability.

Proof Consider an event $\Omega_N = \{\omega : \int_0^\infty X_t^2 dt < N\}$. By assumption $\bigcup_{N\geq 1} \Omega_N$ has probability one. For all $n > N$ on Ω_N we have $\tau_n = \infty$. Choosing M so large that

$$P[\bigcup_{N\geq 1}^{M} \Omega_N] \geq 1 - \epsilon$$

The random variables $\int_0^{\tau_n} X_t dB_t$ are finite for all $n \geq M$ with probability atleast $1-\epsilon$. This implies that these random variables form Cauchy sequence with respect to convergence in probability. By completeness the limit exists.

Although the limit exist in probability, it may not exist in L^2 sense. The limit process may also not have a finite expectation. The process may fail to be a martingale, the best that can be claimed is that it is locally martingale. For a more rigorous proof refer [9].

2.4 An Explicit Computation

We will explicitly solve an example $\int_0^1 B_s dB_s$ and observe the fundamental difference between the stochastic calculus and the ordinary calculus also we will see that the quadratic variation property of Brownian motion plays a central role in defining the stochastic calculus. We will follow the idea used in the proof and observe a sequence of processes whose limit is the required result. Let us consider a standard Brownian motion in interval $[0, 1]$.

$$X_s = B_s \mathbf{1}_{s \leq 1}$$

Clearly the process is adapted to the natural filtration of standard Brownian motion, also the process is square integrable

$$\int_0^\infty E[{X_s}^2]ds = \int_0^1 E[{B_s}^2]ds = \int_0^1 sds = \frac{1}{2} \leq \infty$$

Let us consider a sequence of simple functions

$$\Phi_s^{(n)} = X_{\frac{k}{2^n}} \text{ for } s \in \left[\frac{k}{2^n}, \frac{k+1}{2^n}\right)$$

Through this we observe that the sequence $\Phi_s^{(n)}$ converges in L^2 sense to X_s.

$$\begin{aligned}\int_0^\infty E[(X_s - \Phi_s^{(n)})^2]ds &= \sum_{k=0}^{2^n-1} \int_{\frac{k}{2^n}}^{\frac{k+1}{2^n}} E[(X_s - \Phi_s^{(n)})^2]ds \\ &= \sum_{k=0}^{2^n-1} \int_{\frac{k}{2^n}}^{\frac{k+1}{2^n}} E[(B_s - B_{\frac{k}{2^n}})^2]ds \\ &= \sum_{k=0}^{2^n-1} \int_{\frac{k}{2^n}}^{\frac{k+1}{2^n}} (s - \frac{k}{2^n})ds\end{aligned}$$

This tends to zero as n tends to ∞. Therefore the integral $\int_0^\infty X_s dB_s$ is the limit of the sequence of integrals $\int_0^\infty \Phi_s^{(n)} dB_s$. Using the Riemann type sums, we write this integral to be

$$\int_0^\infty \Phi_s^{(n)} dB_s = \sum_{k=0}^{2^n-1} B_{\frac{k}{2^n}}(B_{\frac{k+1}{2^n}} - B_{\frac{k}{2^n}})$$

Observe that

$$\begin{aligned} {B_1}^2 - {B_0}^2 &= \sum_{k=0}^{2^n-1} ({B_{\frac{k+1}{2^n}}}^2 - {B_{\frac{k}{2^n}}}^2) \\ &= 2\sum_{k=0}^{2^n-1} B_{\frac{k}{2^n}}(B_{\frac{k+1}{2^n}} - B_{\frac{k}{2^n}}) + \sum_{k=0}^{2^n-1} (B_{\frac{k+1}{2^n}} - B_{\frac{k}{2^n}})^2 \\ &= 2\int_0^\infty \Phi_s^{(n)} dB_s + \sum_{k=0}^{2^n-1} (B_{\frac{k+1}{2^n}} - B_{\frac{k}{2^n}})^2 \end{aligned}$$

Therefore for $n \to \infty$

$$\int_0^\infty X_s dB_s = \frac{{B_1}^2}{2} - 1 \tag{2.9}$$

Clearly the result does not match with what we would have expected it to be using ordinary calculus.

2.5 Itô's Formula

From the previous computation, we infer that the Itô integral is not an integral in the usual sense. There does not exist a fundamental theorem of calculus for these integrals, which makes it difficult to evaluate stochastic integrals. The following theorem is a chain rule for stchastic integrals and makes it possible to evaluate the integrals straight away without having to do the computation as above.

Theorem 4. *Let $u(x,t)$ be a function of $x \in \mathbb{R}$ and $t \geq 0$ that is twice continuously differentiable in x and once continuously differentiable in t, and let $\{B_t : t \geq 0\}$ be a Brownian motion. Denote by $u_t, u_x,$ andu_{xx} the first and second partial derivatives of u with respect to the variables t and x. Then*

$$u(B_t,t) - u(0,0) = \int_0^t u_x(B_s,s)dB_s + \int_0^t u_t(B_s,s)ds + \frac{1}{2}\int_0^t u_{xx}(B_s,s)ds \tag{2.10}$$

Suppose that $u(x,t)$ is such that the partial derivatives u_x, u_{xx} are bounded on $\mathbb{R} \times [0,t]$, say,

$$| u_x(B_s, s) | < C$$
$$| u_{xx}(B_s, s) | < C$$

Fix an $\omega \in \Omega_0 \subset \Omega$ such that $P(\Omega_0) = 1$ and $s \to B_s(\omega)$ is continuous for each $\omega \in \Omega_0$. Now, we partition the interval $[0,t]$ into n equal subintervals so that $0 = t_0^{(n)} < t_1^{(n)} < t_2^{(n)} < \cdots < t_n^{(n)} = t$ where $t_j^{(n)} = \frac{jt}{n}$. Thus define $\triangle_j t = t_{j+1}^{(n)} - t_j^{(n)}$ and $\triangle_j B = B_{t_{j+1}}(\omega) - B_{t_j}(\omega)$. Then

$$\begin{aligned} u(B_t(\omega), t) - u(0,0) &= \sum_{j=0}^{n-1} u(B_{t_{j+1}}(\omega), t_{j+1}) - u(B_{t_j}(\omega), t_j) \\ &= \sum_{j=0}^{n-1} \big(u(B_{t_{j+1}}(\omega), t_{j+1}) - u(B_{t_{j+1}}(\omega), t_j) + u(B_{t_{j+1}}(\omega), t_j) - u(B_{t_j}(\omega), t_j) \big) \end{aligned}$$

For some $\tau_j \in [t_j, t_{j+1})$ and z_j between $B_{t_j}(\omega)$ and $B_{t_{j+1}}(\omega)$, by Taylor's theorem we have

$$\begin{aligned} &= \sum_{j=0}^{n-1} u_t(B_{t_{j+1}}, \tau_j) \triangle_j t \\ &+ \sum_{j=0}^{n-1} u_x(B_{t_j}, t_j) \triangle_j B \\ &+ \frac{1}{2} \sum_{j=0}^{n-1} u_{xx}(z_j, t_j)(\triangle_j B)^2 \end{aligned}$$

We consider these three terms and observe their behaviour as $n \to \infty$. For the first term

$$u_t(B_{t_{j+1}}, \tau_j) = u_t(B_{t_{j+1}}, \tau_j) - u_t(B_{t_{j+1}}, t_{j+1}) + u_t(B_{t_{j+1}}, t_{j+1})$$

By hypothesis, we have that u_t is continuous, thus it is uniformly continuous on a compact interval. Also the map $t \to B_t$ being continuous, is bounded on a compact interval. In particular, for any $\epsilon > 0$, there is n sufficiently large such that

$$| u_t(B_{t_{j+1}}, \tau_j) - u_t(B_{t_{j+1}}, t_{j+1}) | < \epsilon \text{ whenever}$$
$$| \tau_j - t_{j+1} | < \frac{1}{n}$$

Therefore as $n \to \infty$ the sum

$$\sum_{j=0}^{n-1} u_t(B_{t_{j+1}}, \tau_j) \triangle_j t = \sum_{j=0}^{n-1} u_t(B_{t_{j+1}}, \tau_j) - u_t(B_{t_{j+1}}, t_{j+1}) \triangle_j t$$
$$+ \sum_{j=0}^{n-1} u_t(B_{t_{j+1}}, t_{j+1}) \triangle_j t$$
$$= \epsilon\, t + \int_0^t u_t(B_s, s) ds$$

Now for the second term, consider a sequence of functions given by

$$g_n(s,\omega) = \sum_{j=0}^{n-1} u_x(B_{t_j}(\omega), t_j) 1_{[t_j, t_{j+1}]}(s)$$

From the hypothesis u_x is continuous, hence uniformly continuous on a compact interval $[0,t]$ and the sequence of functions $g_n(s) \to g(s)$ uniformly on $(0,t]$, where $g(s,\omega) = u_x(B_s(\omega), s)$, therefore

$$\mid g_n(s,\omega) - g(s,\omega) \mid < 2C$$

It thus follows that

$$\int_0^t \mid g_n(s,\omega) - g(s,\omega) \mid^2 ds \le 4tC^2$$

which tends to zero as $n \to \infty$, which thus implies that

$$E\left(\int_0^t \mid g_n(s,\omega) - g(s,\omega) \mid^2 ds \right) \to 0$$

Thus using the Itô isometry we have that

$$\| \int_0^t g_n(s,\omega) dB_s - \int_0^t g(s,\omega) dB_s \|_2 \to 0 \tag{2.11}$$

That is there is a subsequence $\int_0^t g_{n_k}(s,\omega) dB_s$ converging to $\int_0^t g(s,\omega) dB_s$ almost surely. Thus we have that our second term converges to $\int_0^t g(s,\omega) dB_s$ almost surely. The third term $\frac{1}{2}\sum_{j=0}^{n-1} u_{xx}(z_j, t_j)(\triangle_j B)^2$, for any $\omega \in \Omega$ can be written as:

$$\begin{aligned}\frac{1}{2}u_{xx}(z_j, t_j)(\triangle_j B)^2 &= \frac{1}{2}u_{xx}(B_j, t_j)((\triangle_j B)^2 - \triangle_j t) \\ &+ \frac{1}{2}(u_{xx}(z_j, t_j) - u_{xx}(B_j, t_j))(\triangle_j B)^2 \\ &+ \frac{1}{2}u_{xx}(B_j, t_j) \triangle_j t\end{aligned}$$

Using the continuity and boundednes of u_{xx} and the quadratic variation property of Brownian motion, we have that the above tends to $\frac{1}{2}\int_0^t u_{xx}(B_s, s)ds$ as $n \to \infty$. Therefore combining the three results, we have proved the theorem for u_x, u_{xx} both bounded. This restriction can be removed by putting

$$y_n(x, t) = \phi_n(x)u(x, t)$$

such that $\phi_n(x) = 1$ for $|x| \leq n$ and $\phi_n(x) = 0$ for $|x| > n$. The above result holds for $y_n(x, t)$ for all n hence it holds for the limit of these functions which is continuous but not bounded. This proves the theorem and we have obtained the Itô's Formula.

2.6 Applying Itô's Formula

We shall look at some historically relevant examples to understand the use of Itô's formula. We will not be able to solve the Itô integrals directly, but using Itô's formula for relevant functions will help us in evaluating the value of that integral indirectly.

Example Let $u(x, t) = xt$. Using the Itô's formula we get:

$$\begin{aligned}u(B_t, t) - u(0, 0) &= \int_0^t s dB_s + \int_0^t B_s ds \\ tB_t &= \int_0^t s dB_s + \int_0^t B_s ds \\ \int_0^t s dB_s &= \int_0^t B_s ds - tB_t\end{aligned}$$

Example Let $u(x,t) = x^2$. Using the Itô's formula we get:

$$\begin{aligned} {B_t}^2 - 0 &= \int_0^t 2B_s dB_s + \frac{1}{2}\int_0^t 2ds \\ \frac{{B_t}^2 - t}{2} &= \int_0^t B_s dB_s \end{aligned}$$

This is the result that we have computed earlier.

Example Let $u(x,t)$ be such that it satisfies the heat equation $u_t + \frac{1}{2}u_{xx} = 0$. Also assume that u_x is properly defined. Using the Itô's formula we get:

$$\begin{aligned} u(B_t,t) - u(0,0) &= \int_0^t u_x(B_s,s)dB_s + \int_0^t u_t(B_s,s)ds + \frac{1}{2}\int_0^t u_{xx}(B_s,s)ds \\ u(B_t,t) - u(0,0) &= \int_0^t u_x(B_s,s)dB_s \end{aligned}$$

By the definition of Itô integral, the RHS is a martingale. Thus any $u(B_t,t)$ satisfying the heat equation is a martingale. Hence we can observe that $X_t = \exp\{\lambda B_t - \frac{1}{2}\lambda^2 t\}$ is a martingale.

Example *Change of Variable:* Itô's formula can also be applied to compute an integral with respect to a general process X_t which is not a standard Brownian motion. Let us consider $u(x,t)$ again, but this time the x variable is taken by a process which is not a standard Brownian motion [11]. Following the Itô formula, we have:

$$u(X_t,t) - u(0,0) = \int_0^t u_x(X_s,s)dX_s + \int_0^t u_t(X_s,s)ds + \frac{1}{2}\int_0^t u_{xx}(X_s,s)(dX_s)^2$$

Let $\{X_t : t \geq 0\}$ be any process that satisfies for some μ and σ

$$X_t - X_0 = \int_0^t \mu(X_s,s)ds + \int_0^t \sigma(X_s,s)dB_s$$

In symbolic differential form the above can be written as:

$$\begin{aligned} dX_s &= \mu\, ds + \sigma\, dB_s \\ \therefore d{X_s}^2 &= \mu^2\, ds^2 + \sigma^2\, d{B_s}^2 + 2\mu\sigma\, ds\, dB_s \end{aligned}$$

Using that $(ds)^2, (ds)(dB_s)$ are negligible and $(dB_s)^2 \sim ds$ we have $d{X_s}^2 = \sigma^2 ds$. Substituting the values of $dX_s, d{X_s}^2$ we can compute the required integral.
Let us consider $u(X_t, t) = \exp(X_t)$ where

$$X_t = X_0 + \int_0^t g dB_s - \frac{1}{2}\int_0^t g^2 ds$$

Then

$$\begin{aligned} dX_s &= -\frac{1}{2}g^2\, ds + g dB_s \\ u_t &= 0 \\ u_x &= \exp x \\ u_{xx} &= \exp x \end{aligned}$$

Therefore in the symbolic differentiation sense, we obtain that

$$\begin{aligned} du(X_s, s) &= u_s\, ds + u_x\, dX_s + \frac{1}{2}u_{xx}\, dX_s\, dX_s \\ &= \exp X_s(-\frac{1}{2}g^2\, ds + g\, dB_s) + \frac{1}{2}\exp X_s g^2\, ds \\ &= \exp X_s\, g\, dB_s \end{aligned}$$

Therefore using this change of variable and Itô's formula we can solve integrals with respect to processes that are not standard Brownian motion.

Example **Feynman-Kac Formula:** For the heat equation $u_t = \frac{1}{2}u_{xx}$ with $u(0,x) = f(x)$. The solution can be written as

$$u(t,x) = \frac{1}{\sqrt{2\pi t}}\int_{-\infty}^{\infty} f(v)\exp\frac{(v-x)^2}{2t}dv \tag{2.12}$$

The RHS can however be interpreted as the expectation $E[f(x + B_t)]$ and $x + B_t$ is a Brownian motion beginning at x, rather than at 0.

Theorem 5. *Let $q : \mathbb{R} \to \mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}$ be bounded. Then the unique solution to the initial value problem*

$$u_t = \frac{1}{2}u_{xx} + (q)(u)$$

with $u(0,x) = f(x)$, has the unique representation

$$u(t,x) = E\left(f(x+B_t)\exp\int_0^t q(x+B_s)ds\right) \tag{2.13}$$

Proof Consider the function $f(s,y) = u(t-s,x-y)$ and apply the Itô's formula to it.

$$\begin{aligned} f_s &= -u_1 \\ f_y &= u_2 \\ f_{yy} &= u_{22} \end{aligned}$$

Therefore

$$\begin{aligned} df(s,B_s) &= -u_1(t-s,x+B_s)\,ds + \frac{1}{2}u_{22}(t-s,x+B_s)\,dB_s dB_s + u_2(t-s,x+B_s)\,dB_s \\ &= -q(x+B_s)u(t-s,x+B_s)\,ds + u_2(t-s,x+B_s)\,dB_s \end{aligned}$$

For $0 < s < t$, set

$$M_s = u(t-s,x+B_s)\exp\int_0^s q(x+B_v)\,dv$$

Therefore, by the product rule for stochastic processes and Itô's formula,

$$\begin{aligned} dM_s &= df\exp\int_0^s q(x+B_v)dv + fd(\exp\int_0^s q(x+B_v)dv) + dfd(\exp\int_0^s q(x+B_v)dv) \\ &= (-q(x+B_s)u(t-s,x+B_s)ds)(\exp\int_0^s q(x+B_v)dv) \\ &+ (u_2(t-s,x+B_s)dB_s)(\exp\int_0^s q(x+B_v)dv) \\ &+ fq(x+B_s)\exp\int_0^s q(x+B_v)dv + 0 \\ &= (u_2(t-s,x+B_s)dB_s)(\exp\int_0^s q(x+B_v)dv) \end{aligned}$$

It thus follows that

$$M_t = M_0 + \int_0^t u_2(t-s,x+B_s)\exp\int_0^s q(x+B_v)dv dB_s$$

Thus M_t is a martingale and therefore $E[M_t] = E[M_0]$. By construction, we have that $M_0 = u(t,x)$ almost surely and so $E[M_0] = u(t,x)$ and

$$\begin{aligned} E[M_t] &= E\left[u(0, x + B_t) \exp \int_0^t q(x + B_v) dv\right] \\ &= E\left[f(x + B_t) \exp \int_0^t q(x + B_v) dv\right] \end{aligned}$$

Thus the result follows.

Chapter 3

Stochastic Differential Equations

3.1 Introduction

Let $S_t = f(B_t)$ be a function of a standard Brownian motion. We wish to write a differential equation to examine the trajectory of such a function. Brownian motion being nowhere differentiable does not allow us to differentiate the equation right away. We assume the function to be nice and use the Taylor's expansion to obtain

$$S_{t+dt} - S_t = (B_{t+dt} - B_t)f'(B_t) + \frac{1}{2!}(B_{t+dt} - B_t)^2 f''(B_t) + \dots$$

Since Brownian motion have finite quadratic variation, we obtain that

$$dS_t = f'(B_t)dB_t + \frac{1}{2!}f''(B_t)dt$$

This gives us the chain rule (Itô's formula) for differentiating functions of Brownian motion. The above obtained equation is called a *stochastic differential equation*. The general form of a one-dimensional stochastic differential equation with one-dimensional driving Brownian motion is given by

$$dX_t = \mu(X_t, t)dt + \sigma(X_t, t)dB_t \tag{3.1}$$

We would define a process $\{X_t : t \geq 0\}$ as a solution to 3.1 in the sense that

$$X_t = X_0 + \int_0^t \mu(X_s, s)ds + \int_0^t \sigma(X_s, s)dB_s \tag{3.2}$$

3.2 Diffusion processes

A diffusion process $\{X_t : t \geq 0\}$ is a solution to 3.1, where μ and σ are continuous functions of t, X_t. Here μ and σ can be thought of as the infinitesimal mean and variance of the increments of the process X_t, they are the drift and diffusion coefficients of the process. Since we define diffusion processes as the solutions to 3.1, we must observe the conditions on the coefficients for these solutions to exist. Before going into the details of this, we look at some examples of diffusion processes.

1. General Brownian motion: Consider the process $W_t = W_0 + \mu t + \sigma B_t$. The SDE for this process using the Itô's formula is:

$$dW_s = \sigma dB_s + \mu ds$$

In the integral form:

$$W_t = W_0 + \sigma \int_0^t dB_s + \mu \int_0^t ds$$

2. Geometric Brownian motion: Consider the process $S_t = \exp W_t$. The SDE for this process using the Itô's formula is obtained as:

$$dS_t = (\mu + \frac{\sigma^2}{2})S_t dt + \sigma S_t dB_t$$

3. The Ornstein- Uhlenbeck process: Given a SDE,

$$dX_t = \kappa(\theta - X_t)dt + \sigma dB_t$$

where, $\kappa, \theta, \sigma > 0$ are constants. We look at the solution of the equation. The process obtained is mean-reverting i.e. there is a factor that pushes the process towards the mean. Substitute $Y_t = X_t - \theta$, we obtain:

$$dY_t = -\kappa(Y_t)dt + \sigma dB_t$$

Now, substitute $Z_t = \exp^{\kappa t} Y_t$, using the product rule we obtain:

$$\begin{aligned} dZ_t &= \kappa \exp^{\kappa t} Y_t dt + \exp^{\kappa t} dY_t \\ &= \kappa \exp^{\kappa t} Y_t dt + \exp^{\kappa t}(-\kappa(Y_t)dt + \sigma dB_t) \\ &= \sigma \exp^{\kappa t} dB_t \end{aligned}$$

Integrating both sides

$$Z_t = Z_0 + \sigma \int_0^t \exp^{\kappa s} dB_s$$

Reversing the changes of variables, we obtain the solution:

$$X_t = \theta + \exp^{-\kappa t}(X_0 - \theta) + \sigma \int_0^t \exp^{\kappa(s-t)} dB_s$$

4. The Brownian Bridge: The SDE

$$dX_t = -\frac{X_t}{1-t}dt + dB_t$$

for $t \in [0,1)$ has a unique continuous solution known as a Brownian bridge. We have seen earlier that it is a Gaussian process, therefore we check that for the solution to this equation, the expectation and covariance agrees with that of the Brownian bridge. The formulation for this process for $t \in [0,1]$ and $X_1 = 0$ is obtained as

$$X_t = \int_0^t \frac{1-t}{1-s} dB_s$$

3.3 Solutions of SDE : Definitions

The solutions of an SDE can be explicitly obtained in many cases. Itô and Skorkhod gave the theorems regarding the sufficient conditions for existence and uniqueness of the solutions to 3.1. Before we study the theorems, we define the two types of solutions that we discuss, and observe the difference between them.[3]

Let $\{B_t : t \geq 0\}$ be a standard Brownian motion on a probability space $(\Omega, \mathcal{F}, P)$ with a filtration $\{\mathcal{F}_t : t \geq 0\}$. A *strong solution* of 3.1 with initial condition $x \in \mathbb{R}$ is an adapted process with continuous paths such that for all $t \geq 0$, with probability 1

$$X_t = x + \int_0^t \mu(X_s, s)ds + \int_0^t \sigma(X_s, s)dB_s$$

There are a few key points in this definition that we observe

1. The integrals above must exist in the sense that for $t \geq 0$ and probability 1

$$\int_0^t \mid \mu(X_s, s) \mid ds < \infty, \int_0^t \sigma^2(X_s, s) ds < \infty$$

2. The solution must exist for all $t < \infty$ with probability 1.

3. The process X_t must lie on the same probability space as that of the driving Brownian motion and must be adapted to the given filtration.

A *weak solution* of 3.1 with initial condition x is a continuous stochastic process $\{X_t : t \geq 0\}$ defined on *some* probability space $(\Omega, \mathcal{F}, P)$ such that for *some* Brownian motion B_t and *some* admissible filtration, the process $\{X_t : t \geq 0\}$ is adapted and satisfies

$$X_t = x + \int_0^t \mu(X_s, s) ds + \int_0^t \sigma(X_s, s) dB_s$$

A weak solution is often denoted as a triple $(X_t, B_t, (\Omega, \mathcal{F}, P, \mathcal{F}_t))$.
Thus we observe that in the case of strong solutions we have a given probability space, Brownian motion and its filtration whereas for a weak solution, the probability space, Brownian motion and filtration are obtained as a part of a solution. We explicitly look at the theorems for the solutions in the next part.

3.4 Existence and uniqueness of Solutions

3.4.1 Strong solutions

The major outline for this section is borrowed from [4].

Theorem 6. *Assume that $\mu : \mathbb{R} \to \mathbb{R}$ and $\sigma : \mathbb{R} \to \mathbb{R}_+$ are uniformly Lipschitz, that is, $\exists$ a constant $C < \infty$ such that for all $x, y \in \mathbb{R}$,*

$$\begin{aligned} \mid \mu(x) - \mu(y) \mid &\leq C \mid x - y \mid \textit{ and} \\ \mid \sigma(x) - \sigma(y) \mid &\leq C \mid x - y \mid \end{aligned} \tag{3.3}$$

Then 3.1 has strong solutions, in particular, for any standard Brownian motion B_t, any admissible filtration $\{\mathcal{F}_t : t \geq 0\}$, and any initial value $x \in \mathbb{R}$ there exists a unique adapted process $\{X_t : t \geq 0\}$ with continuous paths such that, a.s.

$$X_t = x + \int_0^t \mu(X_s, s) ds + \int_0^t \sigma(X_s, s) dB_s$$

The proof will make use of the Gronwall's inequalities.

Lemma 3.1 (Gronwall). *Let $y(t)$ be a non-negative function such that for some $T < \infty$ there exists constants $A, B \geq 0$ such that $\forall\, 0 \leq t \leq T$*

$$y(t) \leq A + B \int_0^t y(s)ds < \infty$$

Then $\forall\, 0 \leq t \leq T$

$$y(t) \leq Ae^{Bt} \tag{3.4}$$

Proof Assuming $C = \int_0^T y(s)ds < \infty$ and $T < \infty$, it follows that $y(t) \leq D = A + BC$ on the interval $[0, T]$. Iterating this equation we obtain that after k iterations:

$$y(t) \leq A + ABt + \frac{AB^2t^2}{2!} + \ldots + \frac{AB^kt^k}{k!} + I_k$$

where I_k is a $(k+1)$-fold integral given by

$$I_k = B^k \int_0^t \int_0^s \cdots \int_0^r (A + B \int_0^q y(p)dp) \ldots dqdrds$$

Since in the interval $[0, T]$, $y(t) \leq D$ the series is bounded above for all $t \leq T$. Therefore $I_k \leq \dfrac{B^k D t^{(k+1)}}{(k+1)!}$ and thus converges to zero uniformly for $t \leq T$ as $k \to \infty$. Hence the result follows.

Lemma 3.2. *Let $y_n(t)$ be a sequence of non-negative functions with $y_0(t) \equiv C$ (constant) such that for some constant B, $\forall\, t \leq T$ and $n \in \mathbb{Z}^+$*

$$y_{n+1}(t) \leq B \int_0^t y_n(s)ds < \infty$$

Then $\forall\, t \leq T$

$$y_n(t) \leq \frac{CB^nT^n}{n!} \tag{3.5}$$

Proof The result is again obtained by iterating the inequality

$$\begin{aligned} y_{n+1}(t) &\leq B\int_0^t y_n(s)ds \\ &\leq B\int_0^t (B\int_0^s y_n(r)dr)ds \\ &\leq B\int_0^t (B\int_0^s (B\int_0^r y_n(q)dq)dr)ds \\ &\leq \dots \\ &\leq CB^{(n+1)}I_{(n+1)} \leq CB^{(n+1)}\frac{T^{(n+1)}}{(n+1)!} \end{aligned} \tag{3.6}$$

This holds for all $t \leq T$ and $n \in \mathbb{Z}^+$

Proof [Theorem6] To sketch the proof of the theorem, we first for inspiration consider the case where σ is a constant, then we will extend the results to the general case. The solution of the equation is of the form

$$X_t = x + \int_0^t \mu(X_s, s)ds + \int_0^t \sigma dW_s$$

Using the hypothesis that μ is uniformly Lipschitz, we look at the uniqueness and existence of the solution of the above form. Let us assume that for some initial value x there are two solutions

$$\begin{aligned} X_t &= x + \int_0^t \mu(X_s, s)ds + \int_0^t \sigma dW_s \\ Y_t &= x + \int_0^t \mu(Y_s, s)ds + \int_0^t \sigma dW_s \end{aligned}$$

Then the difference of the two solutions satisfies

$$Y_t - X_t = \int_0^t (\mu(Y_s, s) - \mu(X_s, s))ds,$$

and since μ is uniformly Lipschitz, there exists a constant $B < \infty$, such that $\forall\, t < \infty$

$$\mid Y_t - X_t \mid \leq B\int_0^t \mid Y_s - X_s \mid ds$$

3.4 above implies that $Y_t - X_t = 0$. Thus, for an initial value x, the SDE can have atmost one solution. To check the existence of the solution, we define a sequence of adapted processes $X_n(t)$ with initial value x by

$$X_0(t) = x$$
$$X_{n+1}(t) = x + \int_0^t \mu(X_n(s), s)ds + \sigma W(t).$$

Since μ is continuous, it follows that X_{n+1} have continuous paths. We will show that the sequence converges uniformly on compact intervals, and the limit process X_t satisfies 3.1. We again look at the difference of two successive processes and using the uniform Lipschitz property of μ there exists a constant B such that

$$X_{n+1}(t) - X_n(t) = \int_0^t (\mu(X_n(s), s) - \mu(X_{n-1}(s), s))ds$$
$$\mid X_{n+1}(t) - X_n(t) \mid \leq B \int_0^t \mid X_n(s) - X_{n-1}(s) \mid ds$$

3.5 implies that for $t \leq T < \infty$ and some $n, m > 0$

$$\mid X_{n+1}(t) - X_n(t) \mid \leq \frac{xB^nT^n}{n!}$$
$$\mid X_{m+n}(t) - X_n \mid = \sum_{k=n}^{n+m-1} \mid X_{k+1}(t) - X_k(t) \mid$$
$$\leq (m-1)\frac{xB^nT^n}{n!}$$

Thus the processes $X_n(t)$ converge uniformly on compact time intervals, therefore the limit process X_t has continuous path. Using dominated convergence theorem and continuity of μ we can show that the limit process solves the SDE. Thus we have obtained that a solution of this form exists.

Now for a general case when σ is not a constant function, the similar approach will not lead to vanishing of the Itô integral. Instead we will have to use the Gronwall inequality for the second moments and observe the convergence in L^2 sense. To prove the existence, we again define a sequence of processes as follows:

$$X_0(t) = x$$
$$X_{n+1}(t) = x + \int_0^t \mu(X_n(s), s)ds + \int_0^t \sigma(X_n(s), s)dW_s.$$

The processes are well defined and have continuous paths, this follows through induction. We would show that these converge uniformly on the compact intervals and the limit process is a solution to 3.1. The first two terms of the sequence $X_0(t) = x$ and $X_1(t) = x + \mu(x)t + \sigma(x)W_t$ are uniformly bounded in L^2 for t in any bounded interval $[0, T]$, thus for each $T < \infty$ $\exists$ $C_T < \infty$ such that $\forall\, t \leq T$

$$E[(X_1(t) - X_0(t))^2] \leq C_T \tag{3.7}$$

By hypothesis, we have that μ and σ are uniformly Lipschitz, hence $\exists$ a suitable constant $B < \infty$ such that $\forall$ $t \geq 0$

$$\mid \mu(X_n(t)) - \mu(X_{n-1}(t)) \mid \leq B \mid X_n(t) - X_{n-1}(t) \mid,$$
$$\mid \sigma(X_n(t)) - \sigma(X_{n-1}(t)) \mid \leq B \mid X_n(t) - X_{n-1}(t) \mid$$

Thus for the convergence of processes $X_n(t)$ in L^2 sense,

$$E[(X_{n+1}(t) - X_n(t))^2]$$

$$\leq E\left[\left(\int_0^t (\mu(X_n(s)) - \mu(X_{n-1}(s)))ds + \int_0^t (\sigma(X_n(s)) - \sigma(X_{n-1}(s)))dW_s\right)^2\right]$$
$$\leq 2E\left[\left(\int_0^t (\mu(X_n(s)) - \mu(X_{n-1}(s)))ds\right)^2\right] + 2E\left[\left(\int_0^t (\sigma(X_n(s)) - \sigma(X_{n-1}(s)))dW_s\right)^2\right]$$
$$\leq 2B^2E\left[\left(\int_0^t \mid X_n(s) - X_{n-1}(s) \mid ds\right)^2\right] + 2B^2E\left[\left(\int_0^t \mid X_n(s) - X_{n-1}(s) \mid dW_s\right)^2\right]$$
$$\leq 2B^2E\left[\left(t\int_0^t \mid X_n(s) - X_{n-1}(s) \mid^2 ds\right)\right] + 2B^2\int_0^t E\left[\mid X_n(s) - X_{n-1}(s) \mid^2\right] ds$$
$$\leq 2B^2(T+1)\int_0^t E\left[\mid X_n(s) - X_{n-1}(s) \mid^2\right] ds \;\forall\; t \leq T$$

Now consider a sequence of functions $\alpha_n(t) = E\left[\mid X_{n+1}(t) - X_n(t) \mid^2\right]$. From 3.7 we have that $\alpha_0(t) \leq C_T$, thus from 3.5 we have that $\forall\, t \leq T$

$$\alpha_n(t) = E\left[\mid X_{n+1}(t) - X_n(t) \mid^2\right] \leq \frac{C_T[2B^2(T+1)]^n T^n}{n!}$$

Therefore for each $t \leq T$ the sequence of random variables $X_n(t)$ converges uniformly in L^2. It remains to show that the limit process satisfies 3.1. Since $X_n(t) \to X_t$ in L^2, the

Lipschitz property of μ and σ implies that in L^2 sense and for $t \leq T$

$$\int_0^t \sigma(X_n(s), s)dW_s \to \int_0^t \sigma(X_s, s)dW_s$$
$$\int_0^t \mu(X_n(s), s)ds \to \int_0^t \mu(X_s, s)ds$$

Thus $X_t = x + \int_0^t \mu(X_s, s)ds + \int_0^t \sigma(X_s, s)dW_s$ is the solution of the SDE. X_t being the uniform limit has continuous paths.
We need to show that the solution obtained is unique. We assume that for initial condition x there are two continuous solutions given by:

$$X_t = x + \int_0^t \mu(X_s, s)ds + \int_0^t \sigma(X_s, s)dW_s$$
$$Y_t = x + \int_0^t \mu(Y_s, s)ds + \int_0^t \sigma(Y_s, s)dW_s$$

Then the difference of these solutions satisfy

$$Y_t - X_t = \int_0^t (\mu(Y_s, s) - \mu(X_s, s))ds + \int_0^t (\sigma(Y_s, s) - \sigma(X_s, s))dW_s$$

Although the second integral may not be bounded pathwise, but since σ is Lipschitz its second moment can be bounded

$$E\left[\int_0^t (\sigma(Y_s) - \sigma(X_s))dW_s\right]^2 \leq B^2 \int_0^t E(Y_s - X_s)^2 ds$$

It is not obvious that $E(Y_s - X_s)^2$ is always meaningful, but still using second moments we obtain

$$E\left[Y_t - X_t\right]^2 \leq (2B^2 + 2B^2T) \int_0^t E[Y_s - X_s]^2 ds \tag{3.8}$$

Had $E[Y_s - X_s]^2$ been defined almost surely, using 3.4, we would have obtained that $E[Y_s - X_s]^2 \equiv 0$, hence the uniqueness. To circumvent this, we use localization and define stopping times as follows

$$\tau_A = \inf\{t : X_t^2 + Y_t^2 \geq A\}$$

We have that $\tau_A \to \infty$ as $A \to \infty$, this follows because X_t, Y_t are continuous for all t, thus bounded almost surely on compact time intervals. Hence with probability one,

$t \wedge \tau_A = t$ for all sufficietly large values of A. Thus in a compact interval [0,T] and for stopping time $\tau = \tau_A$

$$E\left[Y_{t\wedge\tau} - X_{t\wedge\tau}\right]^2 \leq (2B^2 + 2B^2T)\int_0^t E[Y_{s\wedge\tau} - X_{s\wedge\tau}]^2 ds$$

Thus we prevent our process from exploding in the compact interval. Now using 3.4 we have

$$E\left[Y_{t\wedge\tau} - X_{t\wedge\tau}\right]^2 = 0$$

It thus follows that $X_t = Y_t$ almost surely. Hence the uniqueness is obtained.

3.4.2 Weak solutions

We will first look at two concepts of uniqueness for weak solutions of the SDEs. The *pathwise uniqueness* is the generalisation of the strong uniqueness concept whereas the *uniqueness in law* is a weaker sense of uniqueness. Pathwise uniqueness implies uniqueness in law.

Definition 3.1. *Pathwise uniqueness for the solution of the SDE holds whenever two weak solutions* $(X_t, B_t, (\Omega, \mathcal{F}, P, \mathcal{F}_t))$, $(\tilde{X}_t, B_t, (\Omega, \mathcal{F}, P, \tilde{\mathcal{F}}_t))$ *with* $P(X_0 = \tilde{X}_0) = 1$ *satisfy*

$$P\left(\forall t \geq 0 : X_t = \tilde{X}_t)\right) = 1$$

that is with respect to different filtrations on the same probability space and for same Brownian motion the path of the solution is almost surely the same.

Definition 3.2. *Uniqueness in law for the solution of SDE holds whenever two weak solutions* $(X_t, B_t, (\Omega, \mathcal{F}, P, \mathcal{F}_t))$, $(\tilde{X}_t, \tilde{B}_t, (\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P}, \tilde{\mathcal{F}}_t))$ *with the same initial distribution have the same law, that is* $\forall\, n \in \mathbb{N}$, $t_1, t_2, \ldots t_n > 0$ *and Borel sets* $B_1, B_2, \ldots B_n$

$$P[X_{t_1} \in B_1, X_{t_2} \in B_2, \ldots X_{t_n} \in B_n] = \tilde{P}[\tilde{X}_{t_1} \in B_1, \tilde{X}_{t_2} \in B_2, \ldots \tilde{X}_{t_n} \in B_n]$$

We will first state the Girsanov's theorem and then using it we shall prove the existence of weak solutions. Consider for a process $\{X_t : t \geq 0\}$

$$Y_t = \exp(\int_0^t X_s dB_s - \frac{1}{2}\int_0^t (X_s)^2 ds)$$

The processes of this form are called exponential martingales, for suitable conditions on X_t, the process Y_t is a martingale, in particular

$$E[Y_t] = E[Y_0] = 1$$

Theorem 7 (Girsanov)**.** *Let* $\{X_t : t \geq 0\}$ *be a process defined on* $(\Omega, \mathcal{F}, P)$ *and* $\{B_t : t \geq 0\}$ *be a standard Brownian motion on this space. Consider a process*

$$Y_t = \exp(\int_0^t X_s dB_s - \frac{1}{2}\int_0^t (X_s)^2 ds)$$

such that for $T \in [0, \infty]$*,*

$$E[Y_T] = 1$$

. On the measurable space $(\Omega, \mathcal{F})$*, define a measure*

$$\tilde{P}(d\omega) = Y_T(\omega)P(d\omega)$$

Then $(\Omega, \mathcal{F}, \tilde{P})$ *is a probability space and* $\{\tilde{B}_t : t \leq T\}$ *is a Brownian motion on* $(\Omega, \mathcal{F}, \tilde{P})$ *where*

$$\tilde{B}_t = \int_0^t X_s ds + B_t$$

.

Proof That $(\Omega, \mathcal{F}, \tilde{P})$ is a probability space follows from

$$\tilde{P}(d\omega) = Y_T(\omega)P(d\omega)$$
$$\tilde{P}(\Omega) = \int_\Omega Y_T(\omega)P(d\omega) = E[Y_T] = 1$$

To observe that $\{\tilde{B}_t : t \leq T\}$ is a Brownian motion on $(\Omega, \mathcal{F}, \tilde{P})$, we use the Leѷy's characterisation of Brownian motion, which says that a martingale starting at 0 and quadratic variation t is a Brownian motion. Defining a process $Z_t = Y_t\tilde{B}_t$. Using the definition of the measure $\tilde{P}$ and that Z_t is a martingale, for $s < t$

$$\tilde{E}[\tilde{B}_t|\mathcal{F}_s] = \frac{E[Y_t\tilde{B}_t|\mathcal{F}_s]}{E[Y_t|\mathcal{F}_s]} = \frac{Z_s}{Y_s} = \tilde{B}_s \tag{3.9}$$

and the quadratic variation

$$\begin{aligned}\tilde{E}[\sum_{i=1}^{n-1}(\tilde{B}_{t_{i+1}}-\tilde{B}_{t_i})^2] &= E[\sum_{i=1}^{n-1}(\int_0^{t_{i+1}} X_s ds - \int_0^{t_i} X_s ds + B_{t_{i+1}} - B_{t_i})^2] \\ &= \sum_{i=1}^{n-1}\left(E[\int_{t_i}^{t_{i+1}} X_s ds]^2 + E[B_{t_{i+1}} - B_{t_i}]^2 + 2E[(\int_{t_i}^{t_{i+1}} X_s ds)(B_{t_{i+1}} - B_{t_i})]\right) \\ &= t\end{aligned}$$

Thus we have obtained a probability space $(\Omega, \mathcal{F}, \tilde{P})$ and the Brownian motion $\{\tilde{B}_t : t \leq T\}$ on it.

Now we can articulate the proof for the existence of weak solution of a stochastic differential equation.

Theorem 8. *Consider the SDE*

$$\begin{aligned}dX_t &= \mu(X_t, t)dt + \sigma(X_t, t)dB_t,\ 0 \leq t \leq T \\ X_0 &= x\end{aligned}$$

If the coefficients μ and σ are measurable then there exists a weak solution.

Proof Assume on a probability space $(\Omega, \mathcal{F}, P)$ a process $\{S_t : t \geq 0\}$ such that for some $T > 0$

$$dS_t = \alpha(S_t, t)dt + \sigma(S_t, t)dB_t,\ 0 \leq t \leq T$$

Further assume a measurable function

$$\beta(x, t) := -\frac{\mu(x, t) - \alpha(x, t)}{\sigma(x, t)}$$

Then using the exponential martingale

$$Z_t = \exp\left(\int_0^t \beta(S_t, t)dB_t - \frac{1}{2}\int_0^t \beta(S_t, t)^2 ds\right)$$

and the Girsanov's theorem, there exists a probability measure such that

$$\tilde{P}(d\omega) = Z_T P(d\omega)$$

and a Brownian motion

$$\tilde{B}_t = B_t + \int_0^t \beta(S_s, s)ds$$

Therefore,

$$
\begin{aligned}
d\tilde{B}_t &= dB_t + \beta(S_t,t)dt \\
d\tilde{B}_t &= dB_t - \frac{\mu(x,t)-\alpha(x,t)}{\sigma(S_t,t)}dt \\
\sigma(S_t,t)\, d\tilde{B}_t &= \sigma(S_t,t)dB_t + \alpha(S_t,t)dt - \mu(S_t,t)dt \\
\sigma(S_t,t)d\tilde{B}_t &= dS_t - \mu(S_t,t)dt
\end{aligned}
$$

We obtain that $(S_t, \tilde{B}_t, (\Omega, \mathcal{F}, \tilde{P}))$ is the required weak solution of the SDE.

3.4.3 Examples of Existence of strong and weak solutions

Some examples from [9] are reproduced here:

1. **Deterministic SDE:** The following SDE has *no solutions*.

$$
\begin{aligned}
X_0 &= 0 \\
dX_t &= -sgn(X_t)dt
\end{aligned}
\tag{3.10}
$$

Proof Let Z_t be a solution. Then

$$Z_t = -\int_0^t sgn(Z_s)ds$$

For some $t > 0$ assume that $Z_t = a > 0$ and define $v = \inf\{t \geq 0 : Z_t = a\}$ and $u = \sup\{t \leq v : Z_t = 0\}$. We therefore have

$$a = Z_v - Z_u = -(v-u)$$

a contradiction as it implies $Z_t \leq 0$. Similarly a contradiction $Z_t \geq 0$ is obtained for $a < 0$. Thus, $Z_t \equiv 0$, but it is not a solution of the given SDE.

2. The SDE

$$
\begin{aligned}
dX_t &= -\frac{1}{2X_t} I(X_t \neq 0)dt + dB_t \\
X_0 &= 0
\end{aligned}
\tag{3.11}
$$

has *no solutions*.

Proof Suppose (Z, B) is a solution (weak or strong). Then

$$dZ_t = -\frac{1}{2Z_t} I(Z_t \neq 0)dt + dB_t$$
$$d{Z_t}^2 = d{B_t}^2$$

Using Itô's formula, it follows that

$$\begin{aligned} Z_t^2 &= -\int_0^t \frac{2Z_s}{2Z_s} I(Z_s \neq 0)ds + \int_0^t 2Z_s dB_s + \int_0^t ds \\ &= \int_0^t (1 - I(Z_s \neq 0))ds + \int_0^t 2Z_s dB_s \\ &= \int_0^t I(Z_s = 0)ds + \int_0^t 2Z_s dB_s \end{aligned}$$

$\int_0^t I(Z_s = 0)ds$ is the local time of the process at 0 i.e. the total time spent by the process Z_t at 0, this is 0. Hence the process Z_t^2 is a positive local martingale and thus a supermartingale

$$E[Z_t^2] \leq Z_0^2 = 0$$

This implies that $Z_t^2 = 0$ a.s. But $Z_t = 0$ is not a solution of the SDE, therefore it has no solutions.

3. **Two-sided Tanaka's equation:** This has no strong solution but has a weak solution.

$$\begin{aligned} dX_t &= sgn(X_t)dB_t \\ X_0 &= 0 \end{aligned} \tag{3.12}$$

Proof Clearly $sgn(X_t)dX_t = sgn^2(X_t)dB_t$ therefore $B_t = \int_0^t sgn(X_s)dX_s$. Using the Tanaka's formula for local times

$$|X_t| = \int_0^t sgn(X_s)dX_s + L_t^0(X)$$

Where $L_t^0(X)$ is the total time spent by the process at zero. Therefore we have

$$B_t = |X_t| - L_t^0(X)$$

We observe taht both the processes X_t and $|X_t|$ spend equal time at zero, thus $L_t^0(X) = L_t^0(|X|)$. Hence $\mathcal{F}_t^B \subset \mathcal{F}_t^{|X|}$. For X_t to be a strong solution $\mathcal{F}_t^X \subset \mathcal{F}_t^B$ but this would imply that $\mathcal{F}_t^X \subset \mathcal{F}_t^{|X|}$ which cannot be true. Hence this has a weak solution but no strong solution.

4. **One-sided Tanaka's equation:** This has a strong solution.

$$\begin{aligned} dX_t &= I(X_t > 0)dB_t \\ X_0 &= a \end{aligned} \tag{3.13}$$

Proof Consider the stopping time $v = \inf\{t \geq 0 : a + B_t \leq 0\}$ and set $X_t = a + B_{t\wedge v}$. Using the Itô's formula we observe that

$$X_t - X_0 = \int_0^t I(a + B_{t\wedge v} > 0)dB_s = \int_0^t I(X_s > 0)dB_s$$

Moreover it is adapted to the filtration generated by $B_t \cup \{a\}$. Hence it is a strong solution.

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