Subjective Probability and Inference in Thermodynamics

Harsh Katyayan (MS09057)

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Certificate of Examination

This is to certify that the dissertation titled **Subjective Probability and Inference in Thermodynamics** submitted by **Harsh Katyayan** (Reg. No. MS09057) for the partial fulfillment of BS-MS dual degree programme of the Institute, has been examined by the thesis committee duly appointed by the Institute. The committee finds the work done by the candidate satisfactory and recommends that the report be accepted.

Dr. Dipanjan Chakraborty   Dr. Abhishek Chaudhuri   Dr. R. S. Johal
(Supervisor)

Dated: April 25, 2014

Declaration

The work presented in this dissertation has been carried out by me under the guidance of Dr. Ramandeep S. Johal at the Indian Institute of Science Education and Research Mohali.

This work has not been submitted in part or in full for a degree, a diploma, or a fellowship to any other university or institute. Whenever contributions of others are involved, every effort is made to indicate this clearly, with due acknowledgement of collaborative research and discussions. This thesis is a bonafide record of original work done by me and all sources listed within have been detailed in the bibliography.

Harsh Katyayan
(Candidate)

Dated: April 25, 2014

In my capacity as the supervisor of the candidates project work, I certify that the above statements by the candidate are true to the best of my knowledge.

Dr. Ramandeep S. Johal
(Supervisor)

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Abstract

This project is a study of the notion of subjective probability and its applications in statistical physics and thermodynamics. The ideas of subjective probability are then used to make inferences on the optimality in thermodynamic processes involving maximum work extraction. This thesis starts with the definition of probability and the difference between the objective and subjective schools of thought in Chapter 1. This includes an analysis of a typical inverse probability problem which is solved using the Bayesian Statistics. In chapter 2, some of the pertinent literature related to this project are reviewed. In Chapter 3, there is an account of an application of this technique to make inferences about identical thermodynamic systems undergoing maximum work extraction process. In Chapter 4, there is an account of an attempt to extend the inference making protocol for non-identical systems. Finally, the thesis ends with the future possibilities and related exercises that can be undertaken.

Contents

Chapter 1

Introduction

Probability begins as an abstract concept in probability theory which can be understood in different ways. The way in which probability is interpreted is important as there has to be epistemological consistency in a logical theory. There are three major interpretations that have been suggested[1]:

- propensity (supported by Popper)
- degree of belief (Bayes, Laplace, Gauss, Jeffreys, de Finetti)
- relative frequency (Venn, Fisher, Neyman, von Mises)

The interpretation as *propensity* attributes the value of probability as the property of the object involved. So, a fair coin when tossed has a propensity to give heads or tails. This is entirely the property of the coin. The interpretation as *degree of belief* is different in the sense that it factors in all the information previously obtained about the event to arrive at a belief for success. The interpretation as *relative frequency* is reflective of the frequency of success when the trial is repeated a very large number of times.

The interpretation as *degree of belief* is the core of the subjective school of probability while the interpretation as *relative frequency* is the core of the objective school of probability.

1.1 Subjective and Objective Probability

The two schools have thought of themselves as correct and it becomes increasingly difficult for a physicist to choose sides. But arguments could be made to see if any one of the two interpretations is more fundamental. It turns out that the idea of degree of belief is more fundamental and without it, there would be logical difficulties in describing probabilities using the relative frequency approach.

Suppose S is the number of times there is success in T events. Then, probability is equal to the relative frequency $\frac{S}{T}$. It has been proved by many, including James Bernoulli, that

If the order of trials is unimportant, and if the *probability* of success at each trial is the same, then $\frac{S}{T} \to p$ as $T \to \infty$ with *probability* one.

In this approach, it is important that the probability of success in each trial remains unchanged. But using the relative frequency approach, defining the probability of each event would require an infinite number of trials before that event. Using the frequency interpretation strictly lands us in an infinitely recursive definition. On the other hand, the sense of the whole thing can be made if the probabilities were interpreted as degrees of belief to begin with, in which case the frequency interpretation would just update and validate the belief.

In this sense, it can be argued that the subjective notion of probability is more fundamental of the two. It gives us freedom to use the concept of probability in cases where it would be impossible to do a trial large number of times. It is a quantitative representation of the faith of a person in an outcome of a trial based on any and every previous information available. An inherent part of this approach is the process of updating one's belief as new data and information comes to light. This updating is done via *Bayesian Statistics.*

1.2 Bayes' Theorem

The statement of the Bayes' Theorem is as follows[2]:

Let $\{A_1, A_2, ...\}$ be a sequence (either finite or infinite) of disjoint events such that their union is the set of all possible outcomes of an experiment, and let B be another event, then

$$Pr(A_i|B) = \frac{Pr(B|A_i)Pr(A_i)}{\sum_j Pr(B|A_j)Pr(A_j)}$$

where $Pr(A_i)$ is the prior probability of occurrence of A_i; $Pr(A_i|B)$ is the probability of occurrence of A_i given that B has occurred and $Pr(B|A_i)$ is the probability of occurrence of B given that A_i has occurred.

1.3 Bayesian Statistics

Bayesian Statistics entails using Bayes' Theorem to solve the inverse probability problem. Let us consider the following example of a novel casino game played between Alice and Bob[3].

1.3.1 Model

A reference ball is rolled on the pool table randomly and the position where it stops is noted. Now the dealer rolls a ball and if the ball stop at the right of the reference ball, a point is won by Alice. If the ball stops to the left of where the reference ball had stopped, the point is won by Bob. Alice and Bob can't see the pool table, so they do not know where the reference ball had stopped. They are just informed about who has won a point. First one to reach six points wins the game.

1.3.2 Bet

When Alice has won five points and Bob has won 3, Alice decides she will bet Bob that she is going to win the game. Then what are the fair odds that should be offered?
The treatment of this problem seems quite simple in the sense that for Bob to win the game, he needs to win each of the next three points, or else Alice wins the game. So, if the probability that Alice wins a point is p, which will depend on the position of the reference ball and whose value is unknown to both Alice and Bob, the probability that Bob wins the game is $(1-p)^3$ and the probability that Alice wins the game is $[1-(1-p)^3]$. If the value of p was (for instance in case of a fair coin) $p = \frac{1}{2}$, then the fair odds would have been $7:1$ as $P(Bob) = \frac{1}{8}$ and $P(Alice) = \frac{7}{8}$. However, Alice does not know the value of p, and so it is not so easy for her to calculate the odds.

1.3.3 Inverse Probability Problem

Alice can use the data that she has at her disposal to come to a fair odd. She has won five games to Bob's three. If she were to use this in terms of relative frequency, she could assign the probability that she wins an event as $p = \frac{5}{8}$ so that $P(Bob) = \frac{27}{512}$ and $P(Alice) = \frac{485}{512}$. In this case, the odds are about $18:1$. But even this is not a fair assessment as one can't be sure that the value of p used is indeed correct. Such a surety when using the relative frequency approach can only be achieved when there have been infinite number of trials.

This is what essentially constitutes an inverse probability problem. Because the value of p is not known, it is ideal to account for all possible values of it in the range $\{0, 1\}$. In such a situation, the probability that Bob wins the game, i.e., the next three points can be expressed as

$$P(Bob) = \int_0^1 (1-p)^3 P(p|A = 5, B = 3)dp$$

where we consider the probability that p is the correct probability given the acquired results. This inverse probability can be found using the Bayes' Theorem, hence the name Bayesian Statistics.

$$P(p|A = 5, B = 3) = \frac{P(A=5,B=3|p)P(p)}{\int_0^1 P(A=5,B=3|p)P(p)dp}$$

Now, $P(A = 5, B = 3|p) = (\frac{8!}{5!3!})p^5(1-p)^3$. The prior $P(p)$ can be assumed to be distributed uniformly over $[0, 1]$. Using this in the above equation and using the properties of $\Gamma - function$, it can be established that the correct odds for the bet would be $10 : 1$.

This example shows that when there is a lack of certainty about the model, it is more prudent to use the Bayesian Statistics for a better result. It also becomes essential that whenever there is a parameter that is itself a random variable, its inverse probability should be found out.

1.4 Prior and updating the prior

The proper protocol of going about Bayesian Statistics is to assign some probability distribution to a random variable in the problem based on the degree of belief which can be formed from the available information. This distribution is called the *prior* distribution of the random variable. As and when new information is available, it becomes important to update the probability distribution that was earlier assigned so that it now reflects that the new information that has been incorporated in our probability assignment. This updating is done via Bayes' Theorem and the distribution now obtained is called the *posterior* distribution. This posterior represents the distribution based on all the information available till that point of time. For the next cycle of updating, in case new information is obtained, the posterior now acts as the prior. For instance, in the previous example, $P(p)$ was the prior for p being the correct model. After it became known that Alice had won five points and Bob had won three points, it was necessary to update the prior and that was done using Bayes' theorem. $P(p|A = 5, B = 3)$ thus obtained is the posterior distribution which contains the information of the results. If any further results are known, then this posterior will act as a prior for further updating.

1.5 Subjective vis-$\grave{a}$-vis objective

To better understand the difference between subjective approach and objective approach, a problem was considered[4]. There are six boxes $H_0, H_1, H_2, H_3, H_4, H_5$ having five balls each. These five balls can either be white or black, and the subscript of each box denotes the number of white ball in that box. So, for example, H_2 has two white balls and three black balls. So, we know that if a ball is to be drawn randomly from a box, then the probability that a white ball is drawn is $P(white|H_i) = \frac{i}{5}$ for the box H_i.

1.5.1 The problem

A person has to take one ball out of a particular box from the six boxes and note the color of the ball. The ball is then placed back in the box. The person drawing the ball out does not know from which box he is drawing the ball out. The problem is to infer from which box is the person drawing the ball out.

1.5.2 Approach and result

In this case, it is very easy to differentiate between the subjective and the objective approach. In the frequency approach, if the probability $\{p_i\}$ associated with each $\{H_i\}$ is to be found, then the ball is taken out of the chosen box N number of times. Every time the drawn ball is white, the event is recorded as success S. The value of i for which $P(white|H_i)$ is closest to the relative frequency $\frac{S}{N}$ gives the box which has been chosen. In subjective approach, we find $P(H_i|I_k)$ where I_k includes all the information we have after the k^{th} draw. Here, that value of i for which $P(H_i|I_k)$ tends to one gives the chosen box.
In his work[4], D'Agostini produces the results of a simulated experiment for this problem (Figure 1.1). It can be seen from the table that the prior $P(H_i|I_0)$ is same for all i's before any ball is drawn. This is so because there is no information and in that case, uniform distribution is the best representation of our state of knowledge. Once a white ball is drawn, $P(H_0|I_k)$ immediately becomes 0 because the box H_0 is now ruled out with complete certainty. In contrast to this, in the objective approach, it is still possible to have $\frac{S}{N} < \frac{1}{5}$; this is in violation of the fact that we can be certain that the smallest probability of drawing a white ball even without knowing which box it is drawn from is $\frac{1}{5}$ because after the first draw it is known for sure that there is at least one white ball in the box. Also, it can be seen from the table that subjective approach gives a more confident result in fewer repetitions as compared to the objective approach.

trial k	$E^{(k)}$ (score)	H_0	H_1	H_2	H_3	H_4	H_5	$P(E_W \mid I_k)$	$f(E_W)$
		Probability of the hypotheses $P(H_j \mid I_k)$							
0	–	0.167	0.167	0.167	0.167	0.167	0.167	0.50	–
1	E_W (1,0)	0	0.067	0.133	0.200	0.267	0.333	0.73	1
2	E_B (1,1)	0	0.200	0.300	0.300	0.200	0	0.50	0.50
3	E_B (1,2)	0	0.320	0.360	0.240	0.080	0	0.42	0.33
4	E_B (1,3)	0	0.438	0.370	0.164	0.027	0	0.35	0.25
5	E_W (2,3)	0	0.246	0.415	0.277	0.062	0	0.43	0.40
10	(3,7)	0	0.438	0.468	0.092	0.002	0	0.33	0.30
20	(6,14)	0	0.458	0.522	0.020	$\approx 10^{-5}$	0	0.31	0.30
30	(7,23)	0	0.854	0.146	$\approx 10^{-4}$	$\approx 10^{-10}$	0	0.229	0.233
40	(9,31)	0	0.936	0.064	$\approx 10^{-5}$	$\approx 10^{-13}$	0	0.213	0.225
50	(9,41)	0	0.9962	0.004	$\approx 10^{-8}$	$\approx 10^{-19}$	0	0.2008	0.180
60	(11,49)	0	0.9985	0.002	$\approx 10^{-10}$	$\approx 10^{-23}$	0	0.2003	0.183
70	(11,59)	0	0.9999	$\approx 10^{-4}$	$\approx 10^{-13}$	$\approx 10^{-29}$	0	0.20002	0.157
80	(12,68)	0	1.0000	$\approx 10^{-5}$	$\approx 10^{-15}$	$\approx 10^{-34}$	0	0.200003	0.176
90	(15,75)	0	1.0000	$\approx 10^{-5}$	$\approx 10^{-16}$	$\approx 10^{-36}$	0	0.200003	0.188
100	(18,82)	0	1.0000	$\approx 10^{-5}$	$\approx 10^{-16}$	$\approx 10^{-39}$	0	0.200003	0.180

Figure 1.1: Table showing result of simulated experiment to differentiate between effectiveness of subjective approach and objective approach[4]

Chapter 2

Literature Review

As part of the project, different literature was reviewed in an attempt to identify and understand the working of Bayesian Statistics in the context of physics. Among the literature reviewed were the seminal paper by E. T. Jaynes[5] on the correlation between information theory and statistical physics and a paper by C. Caves[6] on the approach that should be taken when trying to predict future duration of a phenomenon using the information about its present age.

2.1 Information theory and statistical physics

Ever since the path breaking paper of Shannon in which he introduced the idea of information theory, its correspondence with statistical mechanics became an interesting study. In 1957, Jaynes tried to show that the information entropy and thermodynamic entropy were interchangeable and he tried to interpret the theory of statistical mechanics from a informational standpoint.

Previously, the theories were constructed based on the equations of motion supplemented by ergodicity, equal *a priori* probability, and entropy as a physical quantity was identified at the end by comparison of the resulting equations with laws of thermodynamics. Jaynes tried a new approach where the definition for entropy was taken as a starting point and the rest of the theory followed from there. While this is expected to simplify the subject mathematically and conceptually, it also makes the theory much more general in its application.

2.1.1 Maximum entropy distribution

Now that entropy is taken as the starting point in the theory, we start by finding the probability distribution that maximizes the entropy. Entropy in the sense of information theory is a measure of the lack of information about the system. Higher the informational entropy of a system, less information we have about it.

To begin with, let us assume a quantity $x \in \{x_i\}$ where $i \in \{1, 2, ..., n\}$. The corresponding probability $\{p_i\}$ are not known. The only other peice of information available is the expectation value of a function $f(x)$

$$< f(x) >= \sum_{i=1}^{n} p_i f(x_i); \tag{2.1}$$

It goes without saying that any distribution p_i we assume, it must be normalized so that

$$\sum p_i = 1. \tag{2.2}$$

This case could be considered similar to a situation where a system can have many energy levels but the distribution is not known. Only the expectation value of energy is given or known.

To find the distribution, we start by defining a function $H(p_1, p_2, ..., p_n)$ which measures the uncertainty represented by this probability distribution p_i. Because the distribution is not known, there can be great many distributions that might represent the situation. But each distribution is associated with a level of uncertainty about the system. To ensure that the distribution we have chosen is the correct distribution, the uncertainty it entails should be maximum. This way, we can be sure that no extra assumption were made which were not in accordance with the information already available. This uncertainty is quantified by $H(p_1, p_2, ..., p_n)$. The quantity H has the following properties:

1. H is a continuous function of p_i's

2. If all the p_i are equal, then the quantity $A(n) = H(\frac{1}{n}, \frac{1}{n}, ..., \frac{1}{n})$ is a monotonically increasing function of n.

3. H follows the composition law according to which, if the events $\{x_1, x_2, ..., x_n\}$ are grouped such that there are r groups ($\{w_1, w_2, ..., w_r\}$) each containing one or more x_i, then

$$H(p_1, ..., p_n) = H(w_1, ..., w_r) + w_1 H(\frac{p_1}{w_1}, ..., \frac{p_k}{w_1}) + ..., \tag{2.3}$$

where $w_1 = (p_1 + ... + p_k)$ is the group of first k of x_i's, $w_2 = (p_{k+1} + ... + p_{k+m})$ is the group of next m of x_i's and so on.

Using these three conditions, it is possible to arrive at an expression for the informational entropy proposed by Shannon, also called the Shannon entropy:

$$H(p_1, p_2, ..., p_n) = -K \sum_i p_i \ln(p_i). \tag{2.4}$$

Here K is some positive quantity so that the entropy is positive. This is just the expression for entropy which is found in statistical mechanics.

Now that the expression for entropy has been found, we need to arrive at a distribution which maximizes this entropy within the constraint that it reflect the information that is already possessed. This can be done by the method of Lagrange multipliers. Equation (2.4) has to be maximized under the constraints of (2.1) and (2.2). The Lagrange multipliers λ and μ are introduced as usual and we obtain the result

$$p_i = e^{-\lambda - \mu f(x_i)} \tag{2.5}$$

The constants λ and μ are determined by substituting this distribution in (2.1) and (2.2) and we get

$$< f(x) >= -\frac{\partial lnZ(\mu)}{\partial \mu} \tag{2.6}$$

$$\lambda = lnZ(\mu), \tag{2.7}$$

where

$$Z(\mu) = \sum_i e^{-\mu f(x_i)}, \tag{2.8}$$

is called the partition function.

In the case where the system has energy levels $E_i(\alpha_1, \alpha_2, ...)$, the above treatment results in the Boltzmann distribution

$$p_i = e^{-\lambda - \mu E_i}, \tag{2.9}$$

with

$$\lambda = \ln Z, \tag{2.10}$$

$$\mu = \frac{1}{k_b T}, \tag{2.11}$$

$$Z = \sum_i e^{-\frac{E_i}{k_b T}}. \tag{2.12}$$

The correspondence between statistical mechanics and information theory is phenomenal and in this paper, Jaynes goes on to establish that the information and thermodynamic entropy are indeed one and the same. This idea of correlation between the two streams motivated me to study about any such relation between information theory and thermodynamics.

2.2 Inferring future duration

In 1993, J. R. Gott[7] in his Nature article proposed a prediction mechanism for future longevity based on present age. His approach was based on Copernican Temporal Principle which states that when an observer observes a phenomenon, the observation does not happen at a special time. His *delta-t* argument went as follows:
Suppose there is a phenomenon that begins at t_0 and ends at $t_0 + T$ so that the duration is T. Now, an observation has been made in the present time t such that $t_0 < t < t_0 + T$. Then, the present age $t_p = tt_0$ and future duration $t_f = Tt_p$. If there is nothing special about the time, then t_p is a random variable uniformly distributed between 0 and T. So,

$$P(aT < t_p < bT) = b - a = f \tag{2.13}$$

From this, Gott inferred that the duration T lies between the corresponding bounds $\frac{t_p}{b}$ and $\frac{t_p}{a}$ with the same probability f. He obtained the following:

$$G(\frac{1-b}{b}t_p < t_f < \frac{1-a}{a}) = b - a = f \tag{2.14}$$

In a critical assessment of this approach, Carlton Caves[6] points out the pitfalls that should have been avoided during such a treatment as the idea of Bayesian analysis does tend to be tricky. First, Caves suggests that the quantity T is itself a random variable and thus, should have a prior probability distribution of its own. After knowing the present age upon making an observation, the prior distribution for T (or for t_f) has to be updated and the probability assignment G is incorrect. Caves further argues that in a proper Bayesian analysis, there will be a probability distribution for the start time t_0 as well, so that $\gamma(t_0)$ gives the probability $\gamma(t_0)dt_0$ that the phenomenon begins between time t_0 and $t_0 + dt_0$. Also, $p(T|t_0)dT$ gives the probability that the duration is T, given the event starts at t_0. The correct use of the temporal Copernican principle will then be to consider the phenomenon to be equally likely to begin at any time so that $\gamma(t_0)$ is constant and to ensure that the total duration is independent of the starting time.

The motivation to study this treatment was to see how even seasoned physicist could overlook details while using Subjective probability to make inferences. It is very important to keep tabs on the random variables pertinent to the problem at hand. This random variable has to be assigned prior probability and it is also important that the updating protocol must ensure that all the available data, and only the available data, are taken into consideration.

Chapter 3

Application of subjective probability in thermodynamics

The subjective notion of probability can be used to make inferences while assuming ignorance about certain information. This is in attempt to understand the relation between the information theory and thermodynamics. It can be shown that even when there is a crippling lack of information, use of subjective approach of probability makes it possible to infer outcomes of processes to a very good agreement[8].

3.1 The idea

Two identical finite reservoirs, each at an equilibrium state at their own temperatures T_+ and T_- such that $T_+ > T_-$. If these two systems undergo a reversible process, it is possible to extract work to the point till their temperatures become equal. In fact, the work that can be extracted will be the maximum extractable work from these two systems. If we consider the intermediate temperatures of the two reservoirs, that is sometime after the process is started but has not yet finished, say T_1 and T_2, it is possible to find a value of maximum work that can be extracted even if we assume that we do not know:

1. which reading (T_1 and T_2) corresponds to the hot reservoir and to the cold reservoir
2. that the process will stop once the two temperatures are equal

The only information available is that the process is reversible and the only constraint is that the extracted work should be positive. The parameters are T_1 and T_2 which are related to each other by a function as a consequence of the process being reversible. So, $T_1 = F(T_2)$. Thus, this problem is essentially in one variable, either T_1 or T_2. The condition that the work done $W \geqq 0$ gives a range over which T_1 or T_2 can vary and in this case, both $T_1 \in \{T_+, T_-\}$ and $T_2 \in \{T_+, T_-\}$. In this range, a prior probability distribution can be defined for both T_i's. For defining the two prior probability distributions, it is important to realize that the state of knowledge of an observer is the same whether the observer chooses to quantify the uncertainty in terms of parameter T_1 or T_2. So, we assign the same probabilities for those values of T_1 and T_2 which are related by the function F:

$$P(T_1)dT_1 = P(T_2)dT_2, \tag{3.1}$$

when $T_1 = F(T_2)$. Once a form for the prior has been obtained, it can be used to find the average value of W which can be expressed as a function of either T_1 or T_2. So, for instance, if $\overline{T_2}$ is the average value, then the average work extracted will be $W(\overline{T_2})$. We will see that this average for work is very close to the optimal value of work that can be extracted (the value of work that can be extracted from the beginning till the two reservoirs are at the same temperature,i.e, till the process ends) even in the far-from-equilibrium region.

3.2 The model

As already mentioned, the model consists of two identical systems (finite reservoirs) that are at final temperatures T_1 and T_2. The fundamental thermodynamic relation for the two systems can be written as $S \propto U^{\omega_1}$. Different physical systems are represented by different values of ω_1. The value of ω_1 lies between $(0, 1)$ so that the system considered has a positive heat capacity. The system considered is a black-body if $\omega_1 = \frac{3}{4}$, an ideal fermi gas if $\omega_1 = \frac{1}{2}$, an ideal bose gas if $\omega_1 = \frac{3}{5}$. The classical ideal gas can also be considered by making $\omega_1 \to 0$.
Using the basic definition that $(\frac{\partial S}{\partial U})_V = \frac{1}{T}$, it can be easily seen that

$$\Rightarrow U \propto T^{\frac{1}{1-\omega_1}} \tag{3.2}$$

$$\Rightarrow S \propto T^{\frac{\omega_1}{1-\omega_1}} \tag{3.3}$$

$$\Rightarrow S \propto T^{\omega}, \tag{3.4}$$

where $\omega = \frac{\omega_1}{1-\omega_1}$. Now because the systems are undergoing reversible process, the change in entropy is zero, or $\Delta S = 0$. This gives the relation between T_1 and T_2:

$$\Delta S = 0; \tag{3.5}$$

$$\Rightarrow T_1^{\omega} + T_2^{\omega} - T_+^{\omega} - T_-^{\omega} = 0 \tag{3.6}$$

$$\Rightarrow T_1 = (T_+^{\omega} + T_-^{\omega} - T_2^{\omega})^{\frac{1}{\omega}}. \tag{3.7}$$

Equation (2.8) gives the relation between the two intermediate temperatures and can be used to convert the expression for work into a function of only one variable. It can be noticed at this point that there is a symmetry between T_1 and T_2 in the sense that the expression for T_2 will also be of the same form as (2.8), and can be obtained just by exchanging the two T_i's. The expression for work that has been extracted is given as:

$$W = -\Delta U; \tag{3.8}$$

$$W = (T_+^{\omega_0} + T_-^{\omega_0} - T_1^{\omega_0} - T_2^{\omega_0}), \tag{3.9}$$

where $\omega_0 = \frac{1}{1-\omega_1}$.

This expression for work is a function of both T_1 and T_2. Substituting (2.8) into (2.10) results in the following expression for work as a function of T_2:

$$W = T_+^{\omega_0} + T_-^{\omega_0} - (T_+^{\omega} + T_-^{\omega} - T_2^{\omega})^{\frac{1}{\omega_1}} - T_2^{\omega_0} \tag{3.10}$$

The expression for W as a function for T_1 can be obtained by simply interchanging T_1 and T_2. So, the entire analysis can be done in terms of either of the two. Here, it is done in terms of T_2. Now imposing the condition that the work extracted $W \geq 0$ the range for possible values of T_2 can be obtained and the range is $[T_-, T_+]$.
It is intuitive that the actual range of T_2 will not be $[T_-, T_+]$ because both the systems will reach the same temperature T_c which will lie in $[T_-, T_+]$ and the temperatures of the system will remain constant then on. The value of T_c is given as:

$$T_c = (\frac{T_+^{\omega} + T_-^{\omega}}{2})^{\frac{1}{\omega}}. \tag{3.11}$$

But as already mentioned, we are ignorant of that law of thermodynamics which says that the process stops once the temperatures are same. Thus in our treatment, we will assign the prior based on the entire range of T_2 which is feasible from our state of information. This is because we don't know how much the process has proceeded.

3.2.1 Assigning the prior

We are now ready to compute the prior for T_2. It is easy to see from equations (2.1) and (2.7) that the prior will be of the form

$$P(T_2) = \frac{\omega T_2^{\omega-1}}{(T_+^\omega - T_-^\omega)}. \tag{3.12}$$

Equation (2.13) gives the normalized prior distribution of T_2. The normalized prior distribution for T_1 would also have been of the same form with T_2 replaced by T_1. This is because the equations are symmetric in both T_1 and T_2.

The next step is to evaluate the expected value of T_2 which is given as

$$\overline{T_2} = \int_{T_-}^{T^+} T_2 P(T_2) dT_2. \tag{3.13}$$

If we choose $T_+ = 1$ and $\theta = \frac{T_2}{T_1}$, we get the following expression for $\overline{T_2}$:

$$\overline{T_2} = \omega_1 \frac{1-\theta^{\omega_0}}{1-\theta^\omega}. \tag{3.14}$$

3.3 Results

Once the average value of T_2 has been obtained, the expected value of the work done ($\overline{W_p}$) can easily be estimated by substituting the value of $\overline{T_2}$ from (2.15) into (2.11). The optimal value of work extracted($\overline{W_o}$) can also be found by replacing T_2 in (2.11) by T_c from equation (2.12). For the sake of comparison, the expected value of T_2 has also been found using an uniform prior (assuming no information of any sort over the range for T_2) and this value is used in eq. (2.11) to give another estimate($\overline{W_u}$).

Figure 3.1 shows the results of the numerical analysis. It plots work as a function of the ratios of initial temperature, θ; (a) ideal classical gas ($\omega_1 \to 0$), (b) ideal Fermi gas ($\omega_1 = \frac{1}{2}$), (c) degenerate Bose gas ($\omega_1 = \frac{3}{5}$), and (d) black-body radiation ($\omega_1 = \frac{3}{4}$). The dashed curve is for W_o, the thin curve is for W_p and the thick curve is for W_u.

It can be seen that the agreement between the optimal work and the work obtained using the

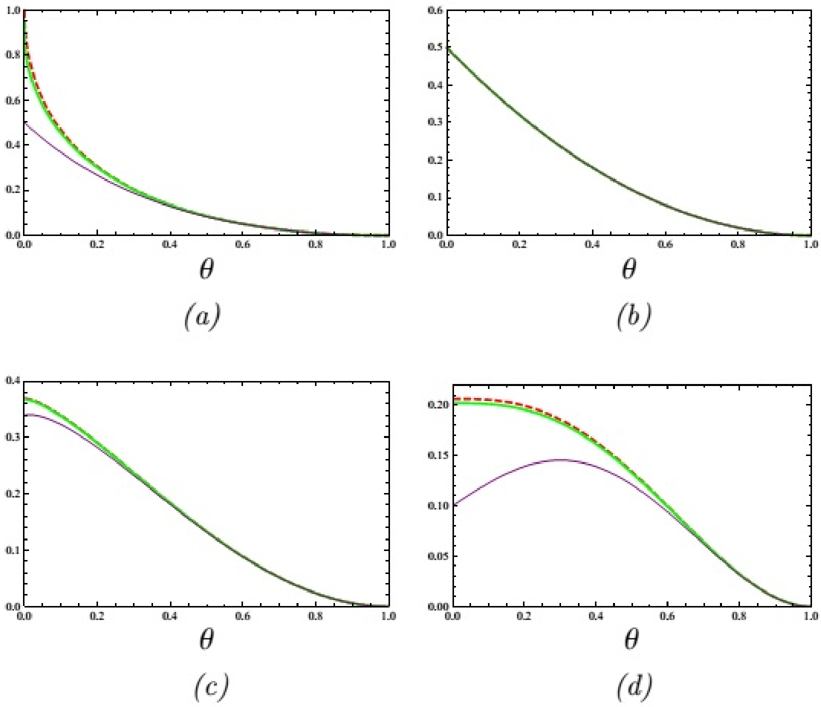


Figure 3.1: Plot showing $\overline{W}_p$, $\overline{W}_u$ and W_o as a function of ratio of bath temparature[8]

prior is remarkable and this agreement extends even in the far-from-equilibrium regime where the value of $\theta \ll 1$.
A further extension of this treatment would be to consider the finite reservoirs that are not identical. An agreement in such a case will show that the process of inference is in some way replicating the results that are obtained when there is complete information about the thermodynamic systems and processes. My further work has been to generalize this approach and I have been able to show that this agreement holds true in case of an ideal Fermi gas.

Chapter 4

Inference of optimality in thermodynamic processes involving non-identical systems

It has been shown[8] the optimal work done can be inferred in the case where two identical finite reservoirs are undergoing maximum work extraction process. Extending this would require assuming non-identical reservoirs to begin with. The reservoirs can differ in volume, number of particles, nature of particles (say monoatomic ideal gas and diatomic ideal gas), etc. But the restriction that we will still adhere to is that the fundamental thermodynamic relation for both the systems will be of the form $S = kU^{\omega_1}$ where ω_1, as earlier, represents the different types of systems that can be used.

4.1 The idea revisited

Two non-identical finite reservoirs, each at an equilibrium state at their own temperatures T_+ and T_- such that $T_+ > T_-$. If these two systems undergo a reversible process, it is possible to extract work to the point till their temperatures become equal. If we consider the intermediate temperatures of the two reservoirs, that is sometime after the process is started but has not yet finished, say T_1 and T_2. But unlike in the previous case where we did not require to assign the value T_1 and T_2 to the hot or the cold reservoir, it is now essential to identify T_1 with the hot reservoir and T_2 with the cold reservoir. This is because of the fact that the two reservoirs are now non-identical and the set of equations which will follow will not be symmetric with respect to T_1 and T_2. But we still consider that we are unaware of the fact that the process will stop once the temperatures become the same and we still do not know the exact values of the parameters. The only information available is that the process is reversible and the only constraint is that the extracted work should be positive. The parameters T_1 and T_2 are once again related to each other by a function as a consequence of the process being reversible. So, $T_1 = F(T_2)$. Thus, this problem is essentially in one variable, either T_1 or T_2.

The condition that the work done $W \geqq 0$ gives a range over which T_1 or T_2 can vary. But once again, unlike the previous case where both $T_1 \in \{T_+, T_-\}$ and $T_2 \in \{T_+, T_-\}$, the two parameters will have different range now and in their respective ranges, a prior probability distribution can be defined for both T_i's. While it is still true that the two prior probability distributions will be of the same form, but because the ranges are now different, so the normalization constant will now be different for the two.

$$P(T_1)dT_1 = P(T_2)dT_2, \tag{4.1}$$

$$P(T_1) = \frac{f(T_1)}{\int_{T_{1,min}}^{T_{1,max}} f(T_1)dT_1}, \tag{4.2}$$

$$P(T_2) = \frac{f(T_2)}{\int_{T_{2,min}}^{T_{2,max}} f(T_2)dT_2}. \tag{4.3}$$

Once a form for the prior has been obtained, it can be used to find the average value of W which can be expressed as a function of either T_1 or T_2. But unlike the previous case, now the two expressions will not ideally be the same. Let's start with the math.

4.2 New expressions for non-identical systems

Now there are two non-identical reservoirs with the hot reservoir at T_1 and the cold reservoir at T_2. The two fundamental thermodynamic relation for the two systems can be written as $S = kU^{\omega_1}$. Different systems are represented by different values of ω_1. The value of ω_1 lies between $(0, 1)$ so that the system considered has a positive heat capacity. The system considered is a black-body if $\omega_1 = \frac{3}{4}$, is an ideal fermi gas if $\omega_1 = \frac{1}{2}$, is an ideal bose gas if $\omega_1 = \frac{3}{5}$. The classical ideal gas can also be considered by making $\omega_1 \to 0$.

Using the basic definition that $(\frac{\partial S}{\partial U})_V = \frac{1}{T}$, it can be easily seen that

$$\Rightarrow U = (\omega_1 kT)^{\frac{1}{1-\omega_1}}, \tag{4.4}$$

$$\Rightarrow S = k^{\frac{\omega_1}{1-\omega_1}} (\omega_1 T)^{\frac{\omega_1}{1-\omega_1}}, \tag{4.5}$$

$$\Rightarrow S = k^{1+\omega}(\omega_1 T)^{\omega}. \tag{4.6}$$

where $\omega = \frac{\omega_1}{1-\omega_1}$.

So, for the two systems, $S_1 = k_1^{1+\omega}(\omega_1 T_1)^{\omega}$ and $S_2 = k_2^{1+\omega}(\omega_1 T_2)^{\omega}$. Similarly, $U_1 = (\omega_1 k_1 T_1)^{\frac{1}{1-\omega_1}}$ and $U_2 = (\omega_1 k_2 T_2)^{\frac{1}{1-\omega_1}}$.

Now because the systems are undergoing reversible process, the change in entropy is zero, or $\Delta S = 0$. This gives the relation between T_1 and T_2:

$$k_1^{1+\omega}(\omega_1 T_1)^{\omega} - k_1^{1+\omega}(\omega_1 T_+)^{\omega} + k_2^{1+\omega}(\omega_1 T_2)^{\omega} - k_2^{1+\omega}(\omega_1 T_-)^{\omega} = 0 \tag{4.7}$$

$$k_1^{1+\omega}(T_1^{\omega} - T_+^{\omega}) + k_2^{1+\omega}(T_2^{\omega} - T_-^{\omega}) = 0 \tag{4.8}$$

The expression for work extracted W can be written as:

$$W = -\Delta U; \tag{4.9}$$

$$W = (\omega_1^{\omega_0})((k_1 T_+)^{\omega_0} + (k_2 T_-)^{\omega_0} - (k_1 T_1)^{\omega_0} - (k_2 T_2)^{\omega_0}) \tag{4.10}$$

where $\omega_0 = \frac{1}{1-\omega_1}$.
Here we once again make a simplifying choice of $T_+ = 1$, $\frac{T_-}{T_+} = \theta$ and $\frac{k_1}{k_2} = \gamma$. Then, we can write the following:

$$T_1 = (1 + \gamma^{\omega+1}\theta^{\omega} - \gamma^{\omega+1}T_2^{\omega})^{\frac{1}{\omega}}, \tag{4.11}$$

$$T_2 = (\gamma^{-(\omega+1)} + \theta^{\omega} - \gamma^{-(\omega+1)}T_1^{\omega})^{\frac{1}{\omega}}, \tag{4.12}$$

$$W = (\omega_1 k_1)^{\omega_0}(1 + (\gamma\theta)^{\omega_0} - T_1^{\omega_0} - (\gamma T_2)^{\omega_0}), \tag{4.13}$$

$$W_1(T_1) = (\omega_1 k_1)^{\omega_0}[1 + (\gamma\theta)^{\omega_0} - T_1^{\omega_0} - \gamma^{\omega_0}(\gamma^{-\omega_0} + \theta^{\omega} - \gamma^{-\omega_0}T_1^{\omega})^{\frac{1}{\omega_1}}], \tag{4.14}$$

$$W_2(T_2) = (\omega_1 k_1)^{\omega_0}[1 + (\gamma\theta)^{\omega_0} - (1 + \gamma^{\omega_0}\theta^{\omega} - \gamma^{\omega_0}T_2^{\omega})^{\frac{1}{\omega_1}} - (\gamma T_2)^{\omega_0}]. \tag{4.15}$$

So, we have obtained the expression for work that can be extracted from the process in terms of both the temperatures. It can be seen that the two expressions, W_1 and W_2 are not same and so, the range for T_1 and T_2 they result in when we impose the condition that $W \geq 0$ are different. Plotting these expressions for different values of γ shows that the two parameters have different ranges.
As can be seen, when $\gamma = 1$, T_1 and T_2 both lie in the same range $[\theta, 1]$. But when $\gamma \neq 1$, range for T_1 and T_2 is different. T_1 goes from some $T_{1,min}$ to 1 and T_2 goes from θ to some $T_{2,max}$. If these limits $T_{1,min}$ and $T_{2,max}$ are known, we can find the prior and thus, the expected value of

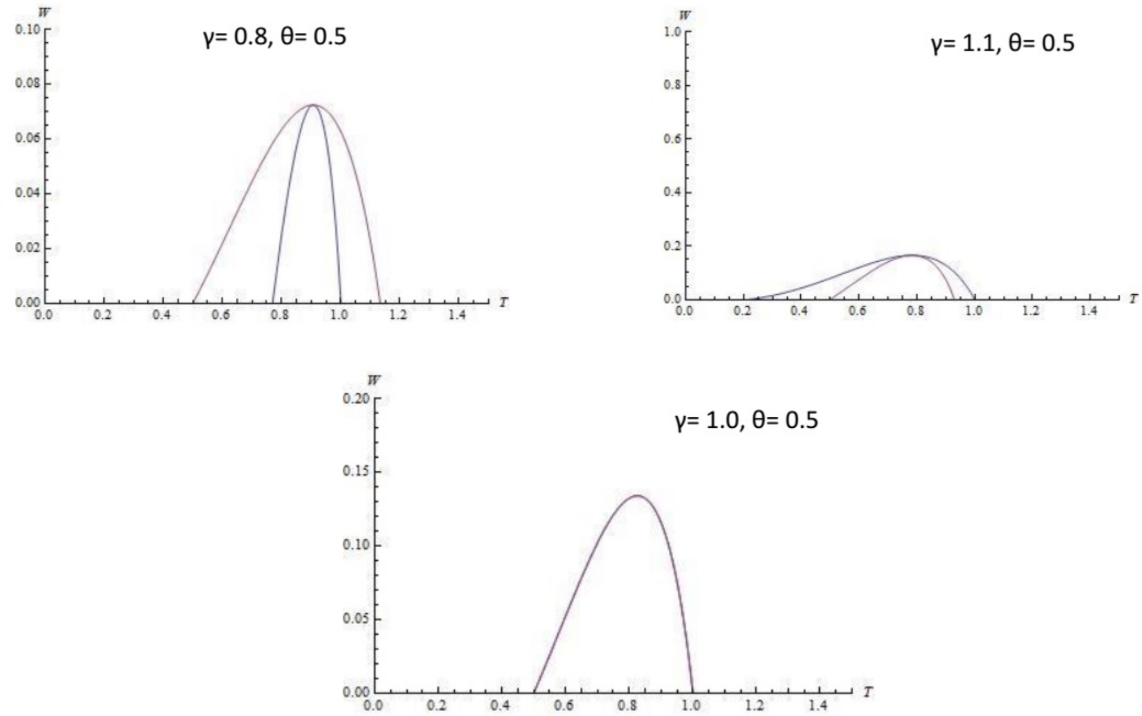


Figure 4.1: Range for T_1 and T_2

work extracted in the process.

For an arbitrary value of ω_1, equations (3.16) and (3.17) have to be solved numerically to obtain the values of $T_{1,min}$ and $T_{2,max}$. These equations have to be solved by running iterations over small intervals of θ. But the case of ideal Fermi gas, for which $\omega_1 = \frac{1}{2}$ can be done analytically.

4.3 Assigning priors for the Ideal Fermi Gas system

When we assign the value for $\omega_1 = \frac{1}{2}$, we get the following as the limits for T_1 and T_2:

$$T_1 \in \left[\frac{1-\gamma^2+2\gamma^2\theta}{1+\gamma^2}, 1\right] \tag{4.16}$$

$$T_2 \in \left[\theta, \frac{2-\theta+\gamma^2\theta}{1+\gamma^2}\right] \tag{4.17}$$

Using these limits, the priors for T_1 and T_2 can be evaluated. The normalized prior turns out to be uniform priors (these do not depend on T_1 or T_2):

$$P(T_1) = \frac{1+\gamma^2}{2\gamma^2(1-\theta)}, \tag{4.18}$$

$$P(T_2) = \frac{1+\gamma^2}{2(1-\theta)}, \tag{4.19}$$

So, we see that the two priors are not equal. This is a consequence of the two systems being non-identical. Putting $\gamma = 1$ in all the equations will replicate the treatment discussed in chapter 2.

Using these priors, we are in a situation to find the average values for T_1 and T_2:

$$\overline{T_1} = \frac{1+\gamma^2\theta}{1+\gamma^2}, \tag{4.20}$$

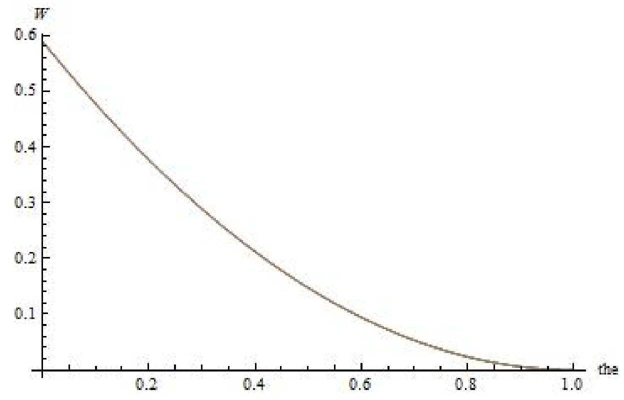
$$\overline{T_2} = \frac{1+\gamma^2\theta}{1+\gamma^2}. \tag{4.21}$$

So we see that the average value for T_1 and T_2 turn out to be the same. These values can be used to find the expected value for work extracted in the entire process. Using the value $\overline{T_1}$ and $\overline{T_2}$, we can find $\overline{W_1}$ $\overline{W_2}$ respectively. These can be compared with the optimal value of the work extracted, which can be found by replacing $\overline{T_1}$ or $\overline{T_2}$ by T_c whose value is obtained by putting $T_1 = T_2 = T_c$ in eq. (3.10).

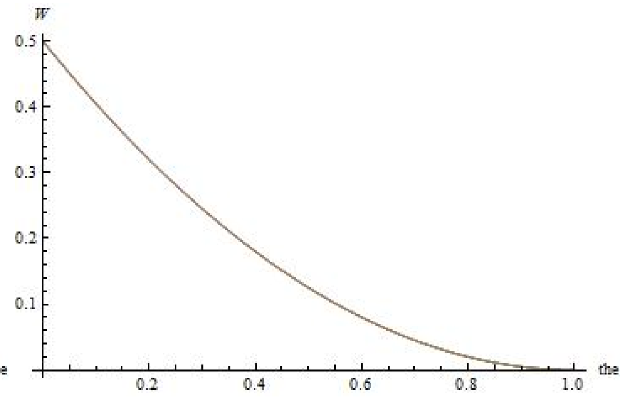
$$T_c = \frac{1+\gamma^2\theta}{1+\gamma^2} \tag{4.22}$$

So, T_c turns out to be same as $\overline{T_1}$ and$\overline{T_2}$. Plotting $\overline{W}_1$, $\overline{W}_2$ and W_o as a function of θ shows that the inference made using the prior is once again in excellent agreement with the optimal value of work that can be extracted.

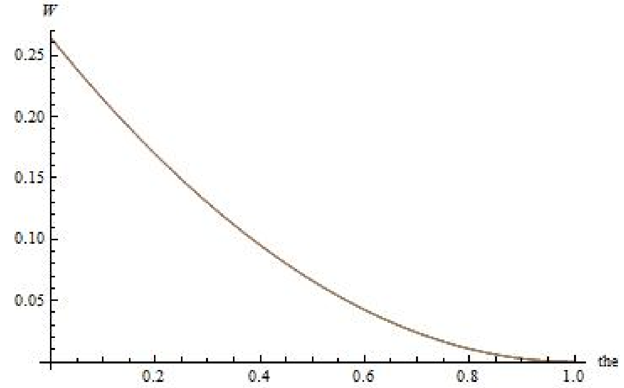
As expected, the three curves are in complete agreement, and for $\gamma = 1$, the plot is the same as the one in chapter 2 for identical systems.



(a) for $\gamma = 1.2$



(b) for $\gamma = 1.0$



(c) for $\gamma = .6$

Figure 4.2: Plot showing $\overline{W}_1$, $\overline{W}_2$ and W_o as a function of θ for different values of γ.

Chapter 5

Future direction

We have seen that in the case of absence of complete knowledge about a thermodynamic process, it is possible to invoke the ideas of subjective probability which can help us infer the outcomes of a process. These inferences are in a very good agreement with the values we would have got had the complete information been given. We see that in case where the two finite reservoirs are identical as well as in the case where the two finite reservoirs of ideal Fermi gas are non-identical, the inferred value of maximum work extracted was in very close agreement with the actual maximum value of the work that could have been extracted in the entire process.

The next step is to check if this correspondence holds for other systems such as ideal classical gas, degenerate Bose gas, black-body radiation etc. We already have the recipe for extending this work to other type of systems and these can be solved numerically using any standard mathematical software such as Mathematica. It will also be interesting to check for this agreement if any one of the hot or cold reservoir becomes infinitely large. This will represent the limiting cases for $\gamma \to 0$ and $\gamma \to \infty$. In terms of making new inferences, we could include the inference of efficiency which will give greater insight.

The entire work urges one to examine the relationship between thermodynamics and information more deeply.

Bibliography

[1] Prosper, Harrison B. "Bayesian analysis." arXiv preprint hep-ph/0006356 (2000).

[2] Raeside, D. E. "A Physicist's Introduction to Bayesian Statistics." American Journal of Physics 40.5 (2005): 688-694.

[3] Eddy, Sean R. "What is Bayesian statistics?." Nature biotechnology 22.9 (2004): 1177-1178.

[4] DAgostini, Giulio. "Teaching statistics in the physics curriculum: Unifying and clarifying role of subjective probability." American Journal of Physics 67.12 (1999): 1260-1268.

[5] Jaynes, Edwin T. "Information theory and statistical mechanics." Physical review 106.4 (1957): 620.

[6] Caves, Carlton M. "Predicting future duration from present age: A critical assessment." Contemporary Physics 41.3 (2000): 143-153.

[7] Gott, J. Richard. "Implications of the Copernican principle for our future prospects." Nature 363.6427 (1993): 315-319.

[8] Aneja, Preety, and Ramandeep S. Johal. "Prior information and inference of optimality in thermodynamic processes." Journal of Physics A: Mathematical and Theoretical 46.36 (2013): 365002.

[9] Johal, Ramandeep S. "Universal efficiency at optimal work with Bayesian statistics." Physical Review E 82.6 (2010): 061113.