Homotopy Theory and Gottlieb Groups

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Certificate of Examination

This is to certify that the dissertation titled **" Homotopy Theory and Gottlieb Groups "** submitted by **Mr. Akash Kumar Sharma** (Reg. No. MS09012) for the partial fulfilment of BS-MS dual degree programme of the Institute, has been examined by the thesis committee duly appointed by the Institute. The committee finds the work done by the candidate satisfactory and recommends that the report be accepted.

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Dated : April 25, 2014

Declaration

The work presented in this dissertation has been carried out by me under the guidance of Dr. Mahender Singh at the Indian Institute of Science Education and Research Mohali.

This work has not been submitted in part or in full for a degree, a diploma, or a fellowship to any other university or institute. Whenever contributions of others are involved, every effort is made to indicate this clearly, with due acknowledgement of collaborative research and discussions. This thesis is a bonafide record of original work done by me and all sources listed within have been detailed in the bibliography.

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In my capacity as the supervisor of the candidate's project work, I certify that the above statements by the candidate are true to the best of my knowledge.

Dr. Mahender Singh
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Akash Kumar Sharma

Notation

$\mathbb{Z}$	The set of integers
$\mathbb{R}$	The set of real numbers
$\mathbb{C}$	The set of complex numbers
$\mathbb{R}^n$	n-dimensional Euclidean space
S^n	The unit sphere in $\mathbb{R}^{n+1}$
$\pi_n(X, x_0)$	n^{th}-homotopy group of the space X at the base point x_0
D^n	The unit disc in $\mathbb{R}^n$
X^X	Space of continuous mappings from X to X
$\mathbb{Z}_n$	The integers mod n
$G(X, x_0)$	Gottlieb group of the space X at the base point x_0
P^n	The projective space of dimension n
$\chi(X)$	Euler characteristic of the space X

Contents

Abstract

A very fundamental problem in topology is to find whether two topological spaces are homeomorphic or not. This problem cannot be solved using purely topological tools only. Algebraic topology originated to develop tools to deal this problem using algebraic methods. Fundamental group is a very basic and one of the most important invariants of a topological space. In the first chapter, some basic concepts like CW complexes, fibration and H-spaces are defined, which will be used in the later chapters. In the second chapter, homotopy groups are defined and some of their properties are discussed. Some computations are done for spheres. In the last chapter, we discuss an important subgroup of the fundamental group of a space. These groups were defined by Daniel Henry Gottlieb in 1965. Following Gottlieb [2], we discuss some properties of these groups and compute them for some nice spaces such as lens spaces, projective spaces and two dimensional manifolds.

Chapter 1

Basic Definitions

In this chapter, some basic concepts are defined. The definitions given in this chapter are based on Hatcher [1] and Aguilar-Gitler-Prieto [3].

1.1 Paths and Homotopy

In a space X, a path is a continuous map $f : I \to X$; where I is a unit interval [0,1].
Let X, Y be spaces and f, g are two continuous maps from X to Y. Then we say f is homotopic to g if there exists a continuos map $F : X \times I \to Y$ such that

$$F(x, o) = f(x)$$
$$F(x, 1) = g(x)$$

$\forall\ x \in X$.
If f and g are homotopic, then we write $f \backsimeq g$. $F(x, t)$ is called a homotopy and sometimes we denote a homotopy by simply f_t.
Similarly, two paths α and β in X are said to be said to be homotopic if there exists a continuous function $F : I \times I \to X$ such that

$$F(s, 0) = \alpha(s)$$
$$F(s, 1) = \beta(\text{s}) \ ; \qquad\qquad \forall\ s \in I.$$

A loop in a space X, based at x_0 is a path $\alpha : I \to X$ such that $\alpha(0) = \alpha(1) = x_0$.
A map f is called nullhomotopic if it homotopic to a constant map.
Let α and β are two paths in a space X such that $\alpha(1) = \beta(0)$, then we can join these two paths.

We define an operation $*$ such that

$$(\alpha * \beta)(t) = \begin{cases} \alpha(2t); & 0 \leq t \leq 1/2 \\ \beta(2t-1); & 1/2 \leq t \leq 1 \ . \end{cases} \tag{1.1}$$

If α, β are two loops in X, based at x_0 and they are homotopic, then we write $\alpha \simeq_{x_0} \beta$. $\simeq$ is an equivalence relation. The equivalence class of α is denoted by $[\alpha]$ and is also called the homotopy class of α. So all the loops based at x_0 and homotopic to α belong to $[\alpha]$.

1.2 CW complexes

By an n-cell we mean a space which is homeomorphic to an open n-disk, $\text{int}(D^n)$. We call a space is a cell if it is an n-cell for $n \geq 0$.
Note that a 0-cell is a point. We denote an n-cell by e^n.
A CW complex or cell complex is a space constructed in the following way

(a) We start with 0-cells (with discrete set X^0).
(b) Inductively, n-skeleton X^n is formed by attaching n-cells e^n_α to $(n-1)$ skeleton X^{n-1} via maps $\varphi_\alpha : S^{n-1} \to X^{n-1}$. We can view X^n as the quotient space of the disjoint union $X^{n-1} \coprod_\alpha D^n_\alpha$; x is identified with $\varphi_\alpha(x)$ $(x \sim \varphi_\alpha(x))$ for $x \in \delta D^n_\alpha$.
So as a set, $X^n = X^{n-1} \coprod_\alpha e^n_\alpha$.
(c) We can stop this process by setting $X = X^n$, or continue this inductive process indefinitively by setting $X = \bigcup X^n$.

Examples

1. S^n is an example of a CW complex. It can be constructed by attaching just two cells e_0 and e_n; where e_n is attached to e_0 via constant map $(S^n \to e_0)$.

2. An one dimensional CW complex $X = X^1$ is called a graph. It is constructed by vertices (0-cells) and edges (1-cells).

3. A torus can also be considered as a cell complex. It can be constructed by one 0-cell, two 1-cells and one 2-cell.

1.3 Fibrations

Let X, E, B be spaces and $p : E \to B$ and $f\colon X \to B$ be maps. Then lifting problem for f with respect to p is that
Is there a continuous map $f^{'} : X \to E$ such that

$$f = p \circ f^{'} \ ?$$

If there exist such a map then we say f can be lifted to E with respect to p and $f^{'}$ is a lift of f.

Definition 1.1 *A map $p : E \to B$ is said to have the homotopy lifting property if for the given maps*

$$f^{'} : X \to E \ \textit{and}\ F : X \times I \to B$$
$$\textit{such that}\ F(x, 0) = p \circ f^{'}(x)\ \textit{for}\ x \in X.$$

There exist a map $F^{'} : X \times I \to E$ such that $F'(x, 0) = f^{'}(x)$ for $x \in X$ and

$$p \circ F^{'} = F.$$

Now we move to define Fibration.

Definition 1.2 *Fibration is a map $p : E \to B$ if the map p has the homotopy lifting property with respect to every space X.*

E is called the total space of the fibration and B is called the base space of the fibration. For $b \in B$, $p^{-1}(b)$ is called the fibre over b. Sometimes we write fiber bundle as $F \to E \to B$. It can be considered as a short exact sequence of spaces.

Examples

1. Let F, B be spaces, then the projection map $B \times F \to B$ is a fibration.

2. Take E to be the Mobius band

$$E = I \times [-1, 1]/\sim;$$

where identify $(0, v) \sim (1, -v)$. Then E is a bundle over S^1 and fiber is the interval [-1, 1].

3. Klein bottle is also a bundle over the circle S^1 where fiber is also S^1.

4. Projective spaces yield very important fiber bundles. In the real case, S^n are bundles over projective space $\mathbb{R}P^n$. In the complex case, S^{2n+1} is a bundle over $\mathbb{C}P^n$ with fiber S^1. When $n = 1$, then $\mathbb{C}P^1 = S^2$ so the sequence becomes $S^1 \to S^3 \to S^2$. It is an important example which we will give us an important relation between homotopy groups of spheres.

Definition 1.3 *Let X be a space. Then $\tilde{X}$ is said to be a covering space of X if there is a continuous surjection $p : \tilde{X} \to X$ satisfying the following property*
For each point $x \in X$, there is an open neighbourhood U of x such that $p^{-1}(U)$ is a disjoint union of open sets U_j ; $j \in J$ such that $p|_{U_j} : U_j \to U$ is an homeomorphism.

p is called a covering map or a covering projection. $\tilde{X}$ is called a covering space.
$p^{-1}(x)$ is a fiber over x, and clearly it is a discrete space.
We can see that a fiber bundle but with a discrete fiber is a covering space.
Examples

1. $\mathbb{R}$ is a covering space of the circle S^1. The covering map $p : \mathbb{R} \to S^1$ can be given by

$$p(t) = (cos2\pi t, sin2\pi t).$$

2. A homeomorphism $p : \tilde{X} \to X$ is naturally a covering projection.

3. Let $q_n : S^1 \to S^1$ is a covering projection, where

$$q_n(z) = z^n.$$

4. Let $p : \tilde{X} \to X$ and $q : \tilde{X} \to X$ are covering projections. Then the product map $p \times q : \tilde{X} \times \tilde{Y} \to X \times Y$ defined by

$$(p \times q)(x, y) = (p(x), q(y))$$

is also a covering projection.

1.4 H-spaces

Let (W, w_0) be a pointed topological space. Then W is said to be an H-space if there exists a continuous map μ

$$\mu : W \times W \to W$$

such that if $e : W \to W$ is the constant map ($e(W) = w_0$) ; then this constant map e is identity map up to homotopy (or an H-identity) i. e. the composites

$$\mu \circ (e, id) : W \to W,\ \mu \circ (id, e) : W \to W$$

are homotopic to the identity map (id) of W.
A space W is called homotopy associative or H-associative if
these two composite maps

$$\mu \circ (\mu \times id), \mu \circ (id \times \mu) : W \times W \times W \to W$$

are homotpic.
H-inverse A map $j : W \to W$ determines H-inverse if both the composites

$$\mu \circ (id, j), \mu \circ (j, id) : W \to W$$

are (each) homotopic to the constant map $e : W \to W$. It means that both are nullhomotopic.
These properties are similar to the axioms of a group, but they are upto to homotopy. So we have the concept of an H-group.
H-group An H-space, which is H-associative and has a map $j : W \to W$ that determines H-inverse is called an H-group.
An H-space, or H-group is H-abelian if these two maps $\mu, \mu \circ T : W \times W \to W$ are homotopic ; where we define T as $T(x, y) = (y, x)$.
H-homomorphism Let W and W' to be H-spaces, then a continuous map
$h : W \to W'$ is said to be an H-homomorphism if the composite maps

$$h \circ \mu : W \times W \to W',\ \mu' \circ (h \times h) : W \times W \to W'$$

are homotopic.
Examples

1. Any topological group is also an H-group.

2. A loop space of a space is an H-group.

Chapter 2

Homotopy Groups and some computations

In this chapter, homotopy groups are defined and some computations are done for spheres. The definitions, theorems and related computations are based on Hatcher [1].

2.1 Fundamental groups

Definition 2.1 *The fundamental group $\pi_1(X, x_0)$ is the set of all homotopy classes of loops in X based at x_0.*

For $[\alpha]$, $[\beta] \in \pi_1(X, x_0)$, define

$$[\alpha] * [\beta] = [\alpha * \beta].$$

This operation is well defined and associative. Constant loop (maps every poit to x_0) is the identity of this group. Inverse of a loop α(t) is $\alpha^{-1}(t) = \alpha(1-t)$.
If our space X is path-connected, then it does not matter which point we are choosing as a base point. Because then

$$\pi_1(X, x_0) \text{ is isomophic to } \pi_1(X, x_1) \text{ ; for any } x_0, x_1 \in X.$$

Then we simply denote our fundamental group by $\pi_1(X)$.
The most basic and simple example of the fundamental group is the fundamental group of the circle. Computation of $\pi_1(S^1)$ is done later in this chapter. We will prove that

$\pi_1(S^1) \cong \mathbb{Z}$ and we will try to compute higher homotopy groups of spheres also.

Some properties of the fundamental groups

1. Let (X, x_0) and (Y, y_0) be two pointed spaces. Let $f : (X, x_0) \to (Y, y_0)$ be a map $(f(x_0) = y_0)$. Then there is an homomorphism induced by f

$$f_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$$

given by $f_*[\alpha] = [f \circ \alpha]$.

2. If X and Y are two path-connected spaces of the same homotopy type, then their fundamental groups are isomorphic

$$\pi_1(X) \cong \pi_1(Y).$$

3. Let (X, x_0) and (Y, y_0) be two pointed path-connected spaces. Then

$$\pi_1(X \times Y, (x_0, y_o)) \cong \pi_1(X, x_0) \times \pi_1(Y, y_0).$$

2.2 Higher homotopy groups

We have defined fundamental group in the previous section. Now we will define higher homotopy groups. Higher homotopy groups are just the higher-dimensional analogs of the fundamental group.

2.2.1 $\pi_n(X, x_0)$

The n^{th}-homotopy group for a space (X, x_0) is defined in the same manner. So $\pi_n(X, x_0)$ is the set of homotopy classes of maps (or n-dimensional loops) $f : (I^n, \delta I^n) \to (X, x_0)$; and it is required that homotopies h_t satisfy $h_t(\delta I^n) = x_0$ for all t ; where I^n is the n-dimensional cube and δI^n is the boundary of I^n.
Also for $n = 0$, this definition works. We can take $I^0 =$ a point and $\delta I^0 = \phi$. Then this definition divides our space in disjoint path connected components or path components. So $\pi_0(X, x_0)$ is the set of path components of X.

Group operation for $\pi_n(X, x_0)$ for $n \geq 2$, we define as follows

Let $f, g \in \pi_n(X, x_0)$, then

$$(f * g)(t_1, t_2, \cdots, t_n) = \begin{cases} f(2t_1, t_2, \cdots, t_n) \ ; & 0 \leq t_1 \leq 1/2 \\ g(2t_1 - 1, t_2, \cdots, t_n) \ ; & 1/2 \leq t \leq 1 \end{cases} \tag{2.1}$$

It can be easily proved that $\pi_n(X, x_0)$ is a group using the same arguments as for $\pi_1(X, x_0)$ because only the first co-ordinate is used in the group operation.
Identity element is the constant map (maps $I^n \to x_0$) and inverse is defined as

$$\bar{f}(t_1, t_2, \cdots, t_n) = f(1 - t_1, t_2, \cdots, t_n).$$

For $n \geq 2$, $\pi_n(X, x_0)$ is abelian. So sometimes we use $+$ as a group operation instead of $*$ for $\pi_n(X, x_0)$; where $n \geq 2$.
Since we can define the S^n by taking quotient of I^n with its boundary δI^n. So $S^n = I^n/\delta I^n$, we can view $\pi_n(X, x_0)$ in a different way. We can consider $\pi_n(X, x_0)$ be the set of homotopy classes of maps $(S^n, s_0) \to (X, x_0)$; where basepoint $s_0 = \delta I^n/\delta I^n$.
For path-connected space X, as same for π_1, we get isomorphic groups for different choices of the base points. Now we can simply denote it by $\pi_n(X)$.

2.2.2 Relative homotopy groups

Let X be a space, A be a subspace of X and choose a base point x_0 such that $x_0 \in A$. So we define a relative homotopy group $\pi_n(X, A, x_0)$ for a pair (X, A).
To define this, take $I^{n-1} \subset I^n$ (face of I^n) with last coordinate $t_n = 0$. Define J^{n-1} = closure of ($\delta I^n - I^{n-1}$).
Then we define $\pi_n(X, A, x_0)$ - the set of homotopy classes of maps

$$f : (I^n, \delta I^n, J^{n-1}) \to (X, A, x_0) \text{ ; for } n \geq 1.$$

By taking $A = x_0$, we get $\pi_n(X, x_0, x_0) = \pi_n(X, x_0)$. So absolute homotopy groups $\pi_n(X, x_0)$ are a special case of relative homotopy groups.
Group operation in $\pi_n(X, A, x_0)$ is the same as defined in $\pi_n(X, x_0)$. But here the last co-ordinate t_n is 0. So in general $\pi_1(X, A, x_0)$ is not a group. So for $n \geq 2$, $\pi_n(X, A, x_0)$ is a group and for $n \geq 3$, it is abelian.

Since we can view $\pi_n(X, x_0)$ as the set of homotopy classes of maps from $(S^n, s_0) \to (X, x_0)$. Similarly we can view or define $\pi_n(X, A, x_0)$ in a differrent way as the set of homotopy classes of maps from (D^n, S^{n-1}, s_0) to (X, A, x_0).

Compression criterion This criterion is the reformulation of the definition of zero (identity) element in $\pi_n(X, A, x_0)$ in a way so that it will help us to prove the exactness of the long sequence of homotopy groups. It says that
If f is a map from (D^n, S^{n-1}, s_0) to (X, A, x_0), then f is trivial (or zero) element in the group $\pi_n(X, A, x_0)$ if and only if f is homotopic (relative to S^{n-1}) to a map whose image is contained in A.
Let (X, A, x_0) and (Y, B, y_0) be two pairs of spaces; $x_0 \in A, y_0 \in B$. Let A and B are subspaces of X and Y respectively. Then a map $\varphi : (X, A, x_0) \to (Y, B, y_0)$ induces a map $\varphi_* : \pi_n(X, A, x_0) \to \pi_n(Y, B, y_0)$ and this is a homomorphism for $n \geq 2$. We define boundary maps δ. δ restrict maps $(I^n, \delta I^n, J^{n-1}) \to (X, A, x_0)$ to I^{n-1} or (D^n, S^{n-1}, s_0) to (X, A, x_0) to S^{n-1}. δ is a homomorphism when $n > 1$.

Theorem 2.2 *Let (X, A, x_0) be a space and i, j are the inclusion maps $(A, x_0) \hookrightarrow (X, x_0)$ and $(X, x_0, x_0) \hookrightarrow (X, A, x_0)$ respectively. δ is the boundary map. Then the sequence*

$$\cdots \to \pi_n(A, x_0) \xrightarrow{i_*} \pi_n(X, x_0) \xrightarrow{j_*} \pi_n(X, A, x_0) \xrightarrow{\delta} \pi_{n-1}(A, x_0) \xrightarrow{i_*} \cdots \to \pi_0(X, x_0)$$

is exact.
Here, i_* and j_* are the induced maps from the inclusion maps i and j.

Theorem 2.3 *Let E and B to be spaces and $p : E \to B$ has the homotopy lifting property with respect to disks D^k for all $k \geq 0$. Let b_0 is the basepoint of B and F is the fibre of b_0 $(F = p^{-1}(b_0))$ and take $x_0 \in F$. Then the induced map $p_* : \pi_n(E, F, x_0) \to \pi_n(B, b_0)$ is an isomorphism for all $n \geq 1$. If B is path-connected, then we have a long sequence*

$$\cdots \to \pi_n(F, x_0) \to \pi_n(E, x_0) \xrightarrow{p_*} \pi_n(B, b_0) \to \pi_{n-1}(F, x_0) \to \cdots \to \pi_0(E, x_0) \to 0;$$ *which is exact.*

This type of fibration which has the homotopy lifting property with respect to all disks D^k ; $k \geq 0$, is sometimes called Serre fibration.

2.3 Some computations for spheres

2.3.1 $\pi_i(S^n)$

$\boldsymbol{\pi_1(S^1)}$

To compute this we will use path lifting property which can be stated as follows

Let $p : \tilde{X} \to X$ be a covering projection and $p(\tilde{x}_0) = x_0$. Then any path $\alpha : I \to X$ has a unique lift $\tilde{\alpha} : I \to \tilde{X}$ starting at $\tilde{x}_0$.
Since $\mathbb{R}$ is a covering of S^1, so we can apply path lifting property here. Let $\alpha : I \to S^1$ be a loop starting at $(1, 0) \in S^1$. Then there is a unique lift $\tilde{\alpha} : I \to \mathbb{R}$ such that $\tilde{\alpha}(0) = 0$. Since P maps $\tilde{\alpha}(1)$ to $(1, 0) \in S^1$, this implies that $\tilde{\alpha}(1)$ must be an integer.
We call this $\tilde{\alpha}(1)$ to be the degree of α.
If α, β are two loops in S^1 and $\alpha \backsimeq \beta$, then degree(α) = degree(β). Therefore we can talk of equivalence classes.
Now we define a map

$$\phi : \pi_1(S^1) \to \mathbb{Z}$$

by $\phi([\alpha])$ = degree(α). It can be easily proved that ϕ is an isomorphism. Therefore

$$\pi_1(S^1) \cong \mathbb{Z}.$$

Based on this we get fundamental group of torus which is

$$\pi_1(S^1 \times S^1) \cong \pi_1(S^1) \times \pi_1(S^1) \cong \mathbb{Z} \times \mathbb{Z}.$$

$\boldsymbol{\pi_i(S^1)}$

$\pi_i(S^1)$ is the trivial group for $i \geq 2$. To prove this result we will use the next theorem.

Theorem 2.4 *Let $p : (\tilde{X}, \tilde{x}_0) \to (X, x_0)$ is a covering projection. Then p induces an isomorphism $p_* : \pi_i(\tilde{X}, \tilde{x}_0) \to \pi_i(X, x_0)$ for $i \geq 2$.*

Since $\mathbb{R}$ is a covering of S^1 and $\mathbb{R}$ is a contractible space or it is homotopic to a point, so $\pi_i(\mathbb{R}) \cong 0$ for all i. Therefore $\pi_i(S^1)$ is trivial for all $i \geq 2$.

$\pi_i(S^n)$; $i < n$

Now we want to compute $\pi_i(S^n)$ for $i < n$. To define homotopy groups we take maps or n-dimensional loops $f : (I^n, \delta I^n) \to (X, x_0)$. But these maps are same as the maps of the quotient $I^n/\delta I^n = S^n$ to X and taking the basepoint $s_0 = [\delta I^n]$ to x_0.
So we can also view homotopy group $\pi_n(X, x_0)$ as homotopy classes of maps $(S^n, s_0) \to (X, x_0)$.

Definition 2.5 *A space X is called n-connected ($n \geq 0$) if for all $k \leq n$, any map from S^k to X is homotopic to a constant map or nullhomotopic.*

Theorem 2.6 *For each $n \geq 1$, the n-sphere S^n is $(n-1)$-connected.*

This theorem can be proved with the help of simplical approximation theorem. Since $\pi_i(S^n)$ is the set of homotopy classes of maps $(S^i, s_0) \to (S^n, s_0)$. By the above theorem we get that all these maps are nullhomotopic for $i < n$. This implies that

$$\pi_i(S^n) = 0$$

for $i < n$.

$\pi_i(S^3)$

As we have seen earlier $S^1 \to S^3 \to S^2$ is a fiber bundle.
Since it is a fiber so it has the homotopy lifting property, then by Theorem 2.3 we get an exact sequence of homotopy groups
$\cdots \to \pi_n(S^1) \to \pi_n(S^3) \xrightarrow{p_*} \pi_n(S^2) \to \pi_{n-1}(S^1) \to \cdots \to \pi_0(S^3) \to 0.$

From this we get an isomorphism

$$\pi_n(S^3) \cong \pi_n(S^2)$$

for $n \geq 3$ and

$$\pi_2(S^2) \cong \pi_1(S^1).$$

2.3.2 $\pi_n(S^n)$

Definition 2.7 *Let X be a space. Then we define a space SX called suspension of X. SX is the quotient space of $X \times I$ with identification $X \times \{0\}$ to one point and $X \times \{1\}$ to another point.*

Examples

1. When $X = S^n$, then $SX \cong S^{n+1}$. The suspension points are north poles and south poles of S^{n+1}.
2. In the case of n-dim. cubes I^n, it is same as spheres. When $X = I^n$, then $SX \cong I^{n+1}$.

Not only spaces but maps between them can also be suspended.
Let $f : X \to Y$ be any map, there is a induced map $Sf : SX \to SY$ given by

$$Sf(x,t) = (f(x),t).$$

But we are more interested in the suspension space of spheres.
The next theorem is called Freudenthal suspension theorem.

Theorem 2.8 *The suspension map $\pi_i(S^n) \to \pi_{i+1}(S^{n+1})$ is an isomorphism for $i < 2n-1$ and it is a surjection for $i = 2n-1$. More generally it is also true for the suspension $\pi_i(X) \to \pi_{i+1}(SX)$ if X is an (n-1)-connected CW complex.*

This theorem can be proved using Theorem 2.2. Proof of this theorem is given in Hatcher, p. 360 [1].

Corollary 2.9 *$\pi_n(S^n) \cong \mathbb{Z}$ for all $n \geq 1$.*

Proof From the Theorem 2.8, we get a suspension sequence

$$\pi_1(S^1) \to \pi_2(S^2) \to \pi_3(S^3) \to \pi_4(S^4) \to \cdots$$

The first map is surjective and all other maps are isomorphisms. We know that $\pi_1(S^1) \cong \mathbb{Z}$. From the section 2.3.4, we know that $\pi_1(S^1) \cong \pi_2(S^2)$. This implies that

$$\pi_n(S^n) \cong \mathbb{Z}.$$

Chapter 3

Gottlieb Groups

In this chapter, we define a subgroup of the fundamental group called Gottlieb group. Some properties of these groups are discussed and they are computed for many interesting spaces. The role of this group as the subgroup of the group of deck transformations of the universal covering space is discussed, leading to the computation for lens spaces and projective spaces. The relation of this group to the mapping space X^X is discussed.
The definitions, theorems and proofs given in this chapter are completely based on the fundamental paper of Gottlieb [2].

3.1 Cyclic homotopy and Gottlieb groups

To define a Gottlieb group, first we have to define cyclic homotopy, and we will always assume that our space is pathwise connected and a CW complex throughout this chapter. Let (X,x_0) be a pointed topological space. A cyclic homotopy is a homotopy $F : X \times I \to X$ which satisfy the following property

$$F(x, 0) = F(x, 1) = x.$$

In other words we can say that at t = 0 or t = 1, $F(x, t)$ is just the identity map of X.

Definition 3.1 *Let $F(x, t)$ be a cyclic homotopy and $\alpha : I \to X$ be a path.*
α is called the trace of $F(x, t)$ if

$$\alpha(t) = F(x_0, t).$$

Obviously trace is a closed path and we denote it by $\tau(F)$.

Definition 3.2 *Let (X,x_0) be a pointed topological space. The set $G(X, x_0)$ of all those loops which are trace of some cyclic homotopy forms a subgroup of the fundamental group $\pi_1(X, x_0)$ called Gottlieb group. This was first introduced by D. H. Gottlieb in his paper in 1965.* [2]

But how do we know that which types of loops are the trace of some cyclic homotopy ? The next theorem will tell us that it depends only on the homotopy classes of the loops.

Theorem 3.3 *Let α be a loop which is the trace of a cyclic homotopy and $[\alpha]$ denote the homotopy class of α. If $\beta \in [\alpha]$,then β is also the trace of some cyclic homotopy.*

Proof Let $F : X \times I \to X$ be the cyclic homotopy of which α is the trace.
Since $\beta \in [\alpha]$, we have a homotopy f_t such that $f_0(s) = \alpha(s)$ and $f_1(s) = \beta(s)$. We have already assumed that our space X is a CW complex, so we can talk of a subcomplex A of $X \times I$. Let

$$A = (X \times 0) \cup (X \times 1) \cup (x_0 \times I)$$

be the subcomplex of $X \times I$. On the subcomplex A, define a partial homotopy $h_s : A \to X$ such that

$$h_s(x, t) = x; \text{ if } t = 0 \text{ or } 1$$
$$k_s(x_0, t) = f_s(t).$$

A is a subcomplex of $X \times I$, so we can use homotopy extension property. Using homotopy extension property, we get a homotopy

$$H_s : X \times I \to X$$

such that $H_0 = F$ and $H_s \mid A = h_s$. Then the new homotopy $H_1 : X \times I \to X$ is a cyclic homotopy on X and its trace is β. This completes the proof.

We have already defined Gottlieb group $G(X, x_0)$ as the set of all homotopy classes of loops $[\alpha]$ of $\pi_1(X, x_0)$; where α is the trace some cyclic homotopy. Now let's prove that it is a subgroup of the fundamenatal group.

Theorem 3.4 $G(X, x_0)$ *forms a subgroup of the fundamental group* $\pi_1(X, x_0)$ *.*

Proof Let $[\alpha]$ and $[\beta]$ be the elements of $G(X, x_0)$. Let f_t and g_t be the cyclic homotopies. α and β are the trace of f_t and g_t respectively. We have to prove that $\alpha * \beta$ is also the trace of some cyclic homotopy. We define a new homotopy h_t

$$h_t(x) = \begin{cases} f_{2t}(x); & 0 \leq t \leq 1/2. \\ g_{2t-1}(x); & 1/2 \leq t \leq 1 \ . \end{cases} \tag{3.1}$$

So $\alpha * \beta$ is the trace of the cyclic homotopy h_t. Therefore $[\alpha * \beta] = [\alpha] * [\beta] \in G(x, x_0)$. Inverse of an element is also belongs to $G(X, x_0)$. α^{-1} is the trace of the cyclic homotopy f_{1-t}. Since $[\alpha]^{-1} = [\alpha^{-1}]$, so $[\alpha]^{-1} \in G(X, x_0)$. Identity element also belogs to $G(X, x_0)$ because it is the trace of the cyclic homotopy $h_t(x) = x$. This completes the proof.

The fundamental group of a path-connected space has the property that it does not depend on the choice of the base point. Gottlieb group also has the same property. The next proves that $G(X, x_0)$ has this property.

Theorem 3.5 *Let x_0 and x_1 be two points in the space X and σ be a path connecting x_0 to x_1, say $\sigma(0) = x_0$ and $\sigma(1) = x_1$. Let $\sigma_* : \pi_1(X, x_1) \cong \pi_1(X, x_0)$ be the induced isomorphism. Then this induced isomophism is also an isomorphism between $G(X, x_1)$ and $G(X, x_0)$*

$$\sigma_* : G(X, x_1) \cong G(X, x_0).$$

Proof Since we know that σ_* is 1-1, so we just have to show that $\sigma_*(G(X, x_1)) \subseteq G(X, x_0)$. Let $[\alpha] \in G(X, x_1)$, so there is a cyclic homotopy $F : X \times I \to X$ of which α is the trace. Here we again use the homotopy extension property which says that there exists a homotopy $H : X \times I \to X$ such that $H(x, 0) = x$ and $H(x_0, t) = \sigma(t)$.
Define a new homotopy $J : X \times I \to X$ by

$$J(x, t) = \begin{cases} H(x, 3t); & 0 \leq t \leq 1/3. \\ F(H(x, 1), 3t - 1); & 1/3 \leq t \leq 2/3 \\ H(x, 3(1 - t)); & 2/3 \leq t \leq 1. \end{cases} \tag{3.2}$$

Observe that J is a cyclic homotopy and trace of J with respect to x_0 is $\sigma * \alpha * \sigma^{-1}$. Therefore $\sigma_*[\alpha] = [\sigma * \alpha * \sigma^{-1}] \in G(X, x_0)$. This completes the proof.

Since $G(X, x_0)$ is independent of the choice of the base point, so from now on we may denote it by simply $G(X)$ to avoid any confusion.

3.2 $P(X, x_0)$ and some computations

We first establish some notations. If (X, x_0) and (Y, y_0) are two spaces, then we will always take the point (x_0, y_0) as the base point for the space $X \times Y$. In this case, by space X we mean $X \times y_0$ and by Y we mean $x_0 \times Y$ and $X \vee Y = (X \times y_0) \cup (x_0 \times Y)$.

Remark 3.6 Let $\sigma : (S^1, s_0) \to (X, x_0)$ or a closed loop in X at x_0. Then $[\sigma]$ is an element of $G(X, x_0)$ if and only if we can extend the map $f : X \vee S^1 \to X$ to $X \times S^1$; where

$$f(x) = x \; ; \; x \in X \text{ and } f(s) = \sigma(s) \; ; \text{ if } s \in S^1.$$

The elements of fundamental group $\pi_1(X, x_0)$ acts as a group of automorphisms on $\pi_n(X, x_0)$.

Definition 3.7 *The set of all those elements of $\pi_1(X, x_0)$ which acts trivially on all $\pi_n(X, x_0)$ also forms a subgroup of $\pi_1(X, x_0)$. We denote this subgroup by $P(X, x_0)$.*

Remark 3.8 An element $[\alpha]$ of the fundamental group $\pi_1(X, x_0)$ acts trivially on $\pi_n(X, x_0)$ if and only if there is an extesion $H : S^n \times S^1 \to X$ for every map $h : S^n \to X$ such that $H|S^1 = \alpha$.

Theorem 3.9

$$G(X, x_0) \subseteq P(X, x_0).$$

Proof Let $[\sigma]$ be an element of $G(X, x_0)$. Then there exist a map $f : X \times S^1 \to X$ (by Remark 3.6) such that

$$f(x) = x \; ; \; x \in X \text{ and } f(s) = \sigma(s) \; ; \text{ if } s \in S^1.$$

Let $\alpha : (S^n, r_0) \to (X, x_0)$ be an n^{th}-loop. Then define a map $H : S^n \times S^1 \to X$ as $H(r, s) = f(\alpha(r), s)$ for $r \in S^n$ and $s \in S^1$.
Because $H(r, s_0) = f(\alpha(r), s_0) = f(r)$ and $H(r_0, s) = f(x_0, s) = \sigma(s)$, this implies that

$$H|S^1 = \sigma.$$

So $[\sigma]$ satisfies the condition in the Remark 3.8, therefore

$$G(X, x_0) \subseteq P(X, x_0).$$

This completes the proof.

Also $G(X, x_0) \subseteq Z(\pi_1(X, x_0))$. Because the subgroup of the fundamental group $\pi_1(X, x_0)$ which acts trivially on itself is precisely the center $Z(\pi_1(X, x_0))$ of $\pi_1(X, x_0)$. Therefore, $P(X, x_0) \subseteq Z(\pi_1(X, x_0))$. This implies

$$G(X) \subseteq Z(\pi_1(X)).$$

Corollary 3.10 *If X is any 1-dimensional polyhedron and it is not homotopic to circle, then $G(X) = 1$.*

Corollary 3.11 *Let P^n to be the projective n-space. Then for $n > 1$,*

$$G(P^{2n}) = 1.$$

Proof Since we know that P^{2n} is not a 2n-simple space i.e. action of $\pi_1(P^{2n})$ on $\pi_{2n}(P^{2n})$ is not trivial. Since $\pi_1(P^{2n}) \cong Z_2$, This implies that action of the generator α of $\pi_1(P^{2n})$ on $\pi_{2n}(P^{2n})$ is not trivial.
Therefore $\alpha \notin P(P^{2n}, x_0)$ implies that subgroup $P(X, x_0)$ is trivial. Therefore $G(P^{2n})$ is also trival.

Corollary 3.12 *Any closed two-dimensional manifold M except for torus and Klein bottle has the trivial Gottlieb group.*

Proof If $M = P^2$, then by previous corollary we have $G(M) = 1$. Otherwise center of $\pi_1(M)$ is trivial. Since $G(M) \subseteq Z(M)$, so $G(M) = 1$.

Theorem 3.13 *If a space X is also an H-space, then its Gottlieb group is precisely the fundametal group itself*

$$G(X) = \pi_1(X).$$

Proof We have defined an H-space earlier. So an H-space (X, x_0) has an element x_0 and a continuous multiplication μ such that the maps

$$\mu \circ (e, id) : X \to X, \mu \circ (id, e) : X \to X$$

are homotopic to the identity map (id) on X. Since we have already assumed that our space is a CW complex and since $X \vee X$ is a subcomplex of $X \times X$, so we can apply the homotopy extension property here. So there exists a continuous multiplication $*$, such that x_0 becomes a multiplicative identity with $*$. Let α be a loop at x_0 in X. Then we define a cyclic homotopy as

$$H(x, t) = \alpha(t) * x.$$

Trace of this homotopy $H(x, t)$ is

$$H(x_0, t) = \alpha(t) * x_0 = \alpha(t).$$

So $\alpha(t)$ is the trace of the cyclic homotopy $H(x, t)$. α is arbitarily chosen, therefore $G(X) = \pi_1(X)$. This completes the proof.

Therefore to compute Gottlieb group of a circle S^1 and a torus T, we can use the above theorem. From this we conclude that

$$G(S^1) = \pi_1(S^1) \ ; \ G(T) = \pi_1(T)$$

because they are H-spaces.
Furthermore, if a space X is a topological group, then $G(X) = \pi_1(X)$.

3.3 Basic properties of Gottlieb groups

Fundametal groups have a property that for any continuous map $f : (X, x_0) \to (Y, y_0)$ there is a induced homomorphism

$$f_* : \pi_1(X, x_0) \to \pi_1(Y, y_0).$$

But Gottlieb groups do not have this important property. It is not necessary that $f_*(G(X))$ is contained in $G(Y)$.

Example 3.14 Let X to be the circle (S^1, s_0) and Y to be the figure eight. Let $f : (S^1, s_0) \to (Y, y_0)$ be the embedding of S^1 onto one of the loops of figure eight (Y, y_0). Take α to be a generator of $\pi_1(S^1, s_0)$ and $f_*(\alpha)$ is not the identity element 1 of $\pi_1(Y, y_0)$. Since we know that $G(Y, y_0) = 1$, this implies that $f_*(\alpha) \notin G(Y, y_0)$.
Since we know that $G(S^1, s_0) = \pi_1(S^1, s_0)$, therefore $f_*G(S^1, s_0) \nsubseteq G(Y)$.

But there are some properties which tell us about the relations between Gottlieb groups of two spaces, for that we have the following theorems.

Theorem 3.15 *Let r to be a retraction from (X, x_0) to (Y, y_0). Then*

$$r_*(G(X, x_0)) \subseteq G(Y, y_0).$$

Proof Let i to be the inclusion map from Y to X. y_0 is the base point of Y, take $i(y_0)$ be the base point of X. Let $[\beta] \in G(X, i(y_0))$. Then by Remark 3.6, we have a map $F : X \times S^1 \to X$ such that

$$F|X = 1_X \text{ and } F|S^1 = \beta.$$

Now define a new map $J : Y \times S^1 \to Y$ as

$$J(y, s) = r \circ F(i(y), s); \text{ where } y \in Y ands \in S^1.$$

Now we calculate $J(y, s_0)$ and $J(y_0, s)$. So

$$J(y, s_0) = r \circ F(i(y), s_0) = r(i(y)) = y$$
$$\text{and } J(y_0, s) = r \circ F(i(y_0), s) = r \circ \beta(s).$$

This implies that $[r \circ \beta]$ belongs to $G(y, y_0)$. Since we know that $r_*[\beta] = [r \circ \beta]$, so this implies that

$$r_*(G(X, i(y_0))) \subseteq G(Y, y_0).$$

Now we have to prove it for any arbitary base point of space X. So let any point $x_0 \in X$ and $r(x_0) = y_0$. Let γ be a path from $i(y_0)$ to x_0. Then there is an isomorphism $\gamma_* : \pi_1(X, x_0) \cong \pi_1(X, i(y_0)$ induced by γ as follows

$$\gamma_*[\beta] = [\gamma * \beta * \gamma^{-1}]$$

Let $[\beta] \in G(X, x_0)$. So by Theorem 3.3, $\gamma_*[\beta] \in G(X, i(y_0))$ and $r_*(\gamma_*[\beta]) \in G(X, y_0)$. But

$$r_*(\gamma_*[\beta]) = r_*[\gamma * \beta * \gamma^{-1}] = [r \circ \gamma * r \circ \beta * r \circ \gamma^{-1}].$$

Since r is a retraction, so $r \circ \gamma$ and $r \circ \gamma^{-1}$ are closed paths in the space Y. So we have

$$r_*(\gamma_*[\beta]) = [r \circ \gamma] * [r \circ \beta] * [r \circ \gamma^{-1}]$$

$$= [r \circ \gamma] * [r \circ \beta] * [r \circ \gamma]^{-1}.$$

Since $G(Y, y_0) \subseteq Z(\pi_1(Y, y_0))$ and $r_*(\gamma_*[\beta]) \in G(Y, y_0)$, we can multiply $r_*(\gamma_*[\beta])$ by $[r \circ \gamma]$ and $[r \circ \gamma]^{-1}$ as

$$r_*(\gamma_*[\beta]) = [r \circ \gamma]^{-1} * r_*(\gamma_*[\beta]) * [r \circ \gamma]$$

$$= [r \circ \gamma]^{-1} * ([r \circ \gamma] * [r \circ \beta] * [r \circ \gamma]^{-1}) * [r \circ \gamma]$$
$$= [r \circ \beta].$$

So finally, we get $r_*(\gamma_*[\beta]) = [r \circ \beta]$. Therefore $r_*[\beta] \in G(Y, y_0)$. This completes the proof.

Now we move to the next property of Gottlieb groups. The next theorem tells us that if two spaces are homotopic, then there is an isomorphism between their Gottlieb groups.

Theorem 3.16 *Let (X, x_0) and (Y, y_0) be two homotopy equivalent spaces. Let $f : (X, x_0) \to (Y, y_0)$ be a homotopy equivalence. Then f induces an isomorphism*

$$f_* : G(X, x_0) \cong G(Y, y_0).$$

Proof Since $f : (X, x_0) \to (Y, y_0)$ is a homotopy equivalence. It implies that there is a map $g : (Y, y_0) \to (X, x_0)$ such that $f \circ g \simeq 1_Y$ and $g \circ f \simeq 1_X$. Let $F : Y \times I \to Y$ be a homotopy such that

$$F(y, 0) = f \circ g(y) \textit{ and } F(y, 1) = y.$$

Now let $[\alpha] \in G(X, x_0)$. Since X and Y are homotopy equivalent so we know that $f_* : \pi_1(X, x_0) \cong \pi_1(Y, y_0)$ is an isomorphism. So we know that $f_*[\alpha] \in \pi_1(Y, y_0)$. Therefore we just need to show that $f_*[\alpha] \in G(Y, y_0)$. Let $H : X \times I \to X$ be that cyclic homotopy whose trace is α. Define a homotopy

$$J : Y \times I \to Y$$

such that $J(y,t) = f \circ H(g(y),t)$.
Then $J(y,0) = f \circ g(y) = J(y,1)$ and $J(y_0,t) = f(H(x_0,t)) = f(\alpha(t)) = f \circ \alpha(t)$ for all $t \in I$.
Define a new homotopy $K : Y \times I \to Y$ such that

$$K(y,t) = F(y, 1-3t);\ for\ 0 \le t \le 1/3$$

$$K(y,t) = J(y, 3t-1);\ for\ 1/3 \le t \le 2/3$$

$$K(y,t) = F(y, 3t-2);\ for\ 2/3 \le t \le 1.$$

Now from $K(y,0) = F(y,1) = y$ and $K(y,1) = F(y,1) = y$, we get that K is a cyclic homotopy. Let $\sigma : I \to Y$ be a path in Y such that $\sigma(t) = F(y_0,t)$. Now we $\sigma(0) = F(y_0,0) = y_0$ and $\sigma(1) = F(y_0,1) = y_0$, so σ is a closed path.
The trace $\tau(K)$ of K is

$$\tau(K) = K(y_0,t) = (\sigma^{-1} * (f \circ \alpha) * \sigma)(t).$$

Therefore $[\tau] = [\sigma]^{-1} * [f \circ \alpha] * [\sigma] \in G(Y, y_0)$. Since $G(Y) \subseteq Z(\pi_1(Y))$, so we can always multiply $[\tau]$ with $[\sigma]$ and $[\sigma]^{-1}$ on both side as we did in the proof of the previous theorem to get the desired result. Therefore $[f \circ \alpha] \in G(Y, y_0)$. This completese the proof.

The next theorem states another property that $G(X)$ share with the fundamental group.

Theorem 3.17 *Let (X, x_0) and (Y, y_0) be two spaces. Then*

$$G(X \times Y, (x_0, y_0)) \cong G(X, x_0) \oplus G(Y, y_0).$$

Proof Let $Z = X \times Y$ and $z_0 = (x_0, y_0)$. Let p and q are the projections of Z onto X and Y respectiely and let p_* and q_* are the induced homomorphisms. We know that fundamental group has a property that there is an isomorphism

$$f : \pi_1(Z, z_0) \to \pi_1(X, x_0) \oplus \pi_1(Y, y_0)$$

given by

$$f([\alpha]) = p_*([\alpha]) \oplus q_*([\alpha]).$$

Since projections are retractions and by applying Theorem 3.10, we get that $f(G(Z)) \subseteq G(X, x_0) \oplus G(Y, y_0)$. Let $[\alpha] \in G(X, x_0)$ and $[\beta] \in G(Y, y_0)$. Now let j and k be maps such

that j injects $X \to X \times y_0$ and k injects $Y \to x_0 \times Y$.
Then $f^{-1}([\alpha] \oplus [\beta]) = [(j \circ \alpha) * (k \circ \beta)]$. Since $[\alpha] \in G(X, x_0)$ and $[\beta] \in G(Y, y_0)$, so there are cyclic homotopies, $F(x,t)$ with trace α and $H(x,t)$ with trace β.
Define a new homotopy $J : X \times Y \times I \to X \times Y$ as

$$J(x, y, t) = (F(x, 2t), y); \;\; for \;\; 0 \leq t \leq 1/2$$

$$J(x, y, t) = (x, H(y, 2 - 2t)); \;\; for \;\; 1/2 \leq t \leq 1.$$

$J(x, y, 0) = (F(x, 0), y) = (x, y)$ and $J(x, y, 1) = (x, H(y, 0)) = (x, y)$, so J is a cyclic homotopy and it can be easily checked that its trace is $(j \circ \alpha) * (k \circ \beta)$. Therefore
$h^{-1}([\alpha] \oplus [\beta]) \in G(Z, z_0)$, thus $f^{-1}(G(X) \oplus G(Y)) \subseteq G(Z)$. Therefore
$f(G(Z)) \supseteq G(X, x_0) \oplus G(Y, y_0)$. This completes the proof.

3.4 Gottlieb groups of aspherical spaces

We have proved that $G(X) \subseteq P(X)$. Here two important questions arise : (1) When the equality of two groups occur ? (2) Is there any space for which the containment of two groups is strict ? These questions are not answered completely. But we have something important in the next theorem.
We give some definitions first. Let $[\alpha] \in \pi_1(X, x_0)$, define a map

$$f_\alpha : X \vee S^1 \to X$$

such that $f_\alpha | S^1 = \alpha$ and $f_\alpha | X = 1_X$. Let $X^{(n)}$ be the n-skeleton of X and

$$f_{\alpha^{n+1}} : (X \vee S^1) \cup (X^{(n)} \times S^1) \to X$$

be an extension of f_α. We say $[\alpha]$ is $(n+1)$-extensible if $f_{\alpha^{n+1}}$ exists.
Let $G^{(n)}(X, x_0)$ is the set of all $(n+1)$-extensible $[\alpha]$ and it is a subgroup of $\pi_1(X, x_0)$. Then we get a decreasing sequence of groups as

$$G^{(1)}(X) \supseteq G^{(2)}(X) \supseteq .... \supseteq G(X).$$

Now let $P^{(n)}(X, x_0)$ be the subgroup of $\pi_1(X, x_0)$ consists of all those $[\alpha]$ which trivially operate on $\pi_i(X, x_0)$ for $i \leq n$. Then we get another decending sequence of groups as

$$P^{(1)}(X) \supseteq P^{(2)}(X) \supseteq .... \supseteq P(X).$$

Theorem 3.18
$$G^{(1)}(X, x_0) = P^{(1)}(X, x_0) = Z(\pi_1(X, x_0)).$$

Proof We know that the subgroup of $\pi_1(X, x_0)$ which acts trivially on $\pi_1(X, x_0)$ itself is exactly the center of $\pi_1(X, x_0)$. So we don't have to show that $P^{(1)}(X, x_0) = Z(\pi_1(X, x_0))$. To show $G^{(1)}(X, x_0) = Z(\pi_1(X, x_0))$, we have to show that $f_\alpha : X \vee S^1 \to S^1$ is 2-extensible over $X \times S^1$ if and only if $[\alpha]$ belongs to $Z(\pi_1(X, x_0))$. It can be shown in the following way

Let K be CW-complex and L be a connected subcomplex of K. Let $l_0 \in L$ and Y be a path-connected space. Let f be a map from (L, l_0) to (Y, y_0) and i be the inclusion map from L to K and

$$f_* : \pi_1(L, l_0) \to \pi_1(Y, y_0), \quad i_* : \pi_1(L, l_0) \to \pi_1(K, l_0)$$

be the induced homomorphisms. Then f is 2-extensible if and only if there exists a homomorphism $g : \pi_1(K, l_0) \to \pi_1(Y, y_0)$ such that $f_* = gi_*$.

In our case, let $L = X \vee S^1, K = X \times S^1, Y = X$ and $f = f_\alpha$. Then $\pi_1(L)$ is the free product of $\pi_1(X)$ and $\pi_1(S^1)$ and $\pi_1(K) \cong \pi_1(X) \oplus \pi_1(S^1)$.

Now let $a \in \pi_1(X), b \in \pi_1(S^1)$ and z be the generator of the $\pi_1(S^1)$. Then $i_*(a * b) = a \oplus b$ and $f_*(a * z) = a \cdot [\alpha]$.

Now suppose g exists and $f_* = gi_*$. Since $f_*(z) = [\alpha]$ and $i_*(z) = 1 \oplus z$, so $[\alpha] = g(1 \oplus z)$ and also $a = f_*(a) = gi_*(a) = g(a \oplus 1)$.

Therefore if g exists, then g satisfy the equation $g(a \oplus z) = a \cdot [\alpha]$ for all $a \in \pi_1(X)$. Now $f_*(a * z) = a \cdot [\alpha]$ and $f_*(z * a) = [\alpha] \cdot a$.

But $gi_*(a * z) = a \cdot [\alpha] = gi_*(z * a)$ and $f_* = gi_*$.

Therefore $a \cdot [\alpha] = [\alpha] \cdot a$ for all $a \in \pi_1(X)$. Therefore $[\alpha]$ must belong to $Z(\pi_1(X, x_0))$. This completes the proof.

Definition 3.19 *A space X is said to be aspherical if $\pi_n(X) = 0$ for $n > 1$.*

Corollary 3.20 *If a space is aspherical, then its Gottlieb group is exactly the center of its fundamental group.*

Proof Let X be a space and it is aspherical. Then $X \times S^1$ is also aspherical. Therefore any 2-extensible map $f_\alpha : X \vee S^1 \to X$ must be extensible over $X \times S^1$.

From the previous corollary, we get a nice result about Klein bottle. If K is a Kleinbottle, then $G(K) = Z(\pi_1(K))$.

3.5 Gottlieb groups and deck tranformations

As we have already assumed in the section 3.1 that our space X is always path-connected CW-complex. Therefore there always exists a universal cover $\tilde{X}$ for X. We will denote our covering projection by $p : (\tilde{X}, \tilde{x}_0) \to (X, x_0)$.
$D(X)$ will denote the group of Deck tranformations acting on $\tilde{X}$ and v will denote the natural isomorphism between $\pi_1(X, x_0)$ and $D(X)$. Therefore there is a correspondence between $G(X, x_0)$ and a subgroup of $D(X)$ under v. So the subgroup $vG(X)$ is naturally defined in the $D(X)$.

Theorem 3.21 *$G(X, x_0)$ is isomorphic to the subgroup of those elements of $D(X)$ which are homotopic to the identity map $1_{\tilde{X}}$ by fiber preserving properties.*

Proof Suppose $[\alpha] \in \pi_1(X, x_0)$ and $f : \tilde{X} \to \tilde{X}$ is a deck transformation generated by $[\alpha]$. That means that image of any path $\omega : \tilde{x}_0 \to f(\tilde{x}_0)$ will be a closed path $p \circ \omega \in [\alpha]$.
Let $[\alpha] \in G(X, x_0)$. Then by definition there is a cyclic homotopy $h_t : X \to X$ with trace α. Identity map $1_{\tilde{X}} : \tilde{X} \to \tilde{X}$ covers the map $1_X \circ p : \tilde{X} \to X$.
Since the map $h_t \circ p : \tilde{X} \to X$ is a homotopy of the map $1_X \circ p$, then by homotopy lifting property there exist a homotopy $\tilde{h}_t : \tilde{X} \to \tilde{X}$ which is lift of $h_t \circ p$ such that $h_t \circ p = p \circ \tilde{h}_t$.
Since h_t is a cyclic homotopy, $h_1 = 1_X$ so $p = p \circ \tilde{h}_1$. Therefore h_1 must be a deck transformation of $\tilde{X}$. Now $\tilde{h}_1 = f$, since the path $\tilde{\tau}(t) = \tilde{h}_t(\tilde{x}_0)$ starting from $\tilde{x}_0$ to $\tilde{h}_1(\tilde{x}_0)$ is a lift of α.
Thus $\tilde{h}_t$ is the desired fiber preserving homotopy from $1_{\tilde{X}}$ to f.
Conversely, suppose $\tilde{h}_t$ is a fiber preserving homotopy so that $\tilde{h}_0 = 1_{\tilde{X}}$ and $\tilde{h}_1 = f$, then there is a cyclic homotopy $h_t : X \to X$ so that $h_t \circ p = p \circ \tilde{h}_t$. It is clear that $h_t : X \to X$ is a cyclic homotopy and trace of h_t is $\tau(t) = h_t(x_0) \in [\alpha]$. This completes the proof.

For covering spaces, there is a very nice condition for the homotopies to be fiber preserving.

Theorem 3.22 *Let $D(X)$ be the group of Deck tranformations. Then the homotopy $\tilde{h}_t : \tilde{X} \to \tilde{X}$ is fiber preserving if and only if $k \circ \tilde{h}_t = \tilde{h}_t \circ k$ for every $k \in D(X)$.*

Proof Let $\tilde{h}_t$ be a fiber preserving homotopy. Suppose k be any deck transformation and $x \in \tilde{X}$ be any point. So x and $k(x)$ are in the same fiber. Since both $\tilde{h}_t(x)$ and $\tilde{h}_t(k(x))$ belong to the same fiber, so there exists a $l \in D(X)$ such that $l \circ \tilde{h}_t(x) = \tilde{h}_t \circ k(x)$. For a sufficiently small $\epsilon > 0$, we have $l \circ \tilde{h}_{t-\epsilon}(x) = \tilde{h}_{t-\epsilon} \circ k(x)$. In the set of t's, greatest lower bound must occur at $t = 0$ satisfying $l \circ \tilde{h}_t(x) = \tilde{h}_t \circ k(x)$. So by the continuity, we get $l \circ \tilde{h}_0(x) = \tilde{h}_0 \circ k(x)$. But we have $\tilde{h}_0 = 1_{\tilde{X}}$, so $l(x) = k(x)$. This implies that $l = k$. Therefore $k \circ \tilde{h}_t = \tilde{h}_t \circ k$ for every $k \in D(X)$.
Conversely, let $k \circ \tilde{h}_t = \tilde{h}_t \circ k$ for every $k \in D(X)$. Let $x, y \in \tilde{X}$, both are in the same fiber of p. Suppose $k(x) = y$. Since $\tilde{h}_t = k^{-1} \circ \tilde{h}_t \circ k$, so we get $\tilde{h}_t(x) = k^{-1} \circ \tilde{h}_t(y)$. Thus $\tilde{h}_t(x)$ and $\tilde{h}_t(y)$ are in the same fiber. Therefore $\tilde{h}_t$ is fiber preserving. This completes the proof.

Corollary 3.23 *Let A be the subgroup of $D(X)$ consisting of those elements of $D(X)$ which are homotopic to identity via a homotopy which commutes with every element of $D(X)$. Then A is isomorphic to $G(X, x_0)$.*

Let $p, q \in \mathbb{Z}$ and they are relatively prime. Let $L(p, q)$ be a three dimensional lens space. Then $\pi_1(L(p, q))$ is isomorphic to the cyclic group of order p.

Theorem 3.24

$$G(L(p, q)) \cong \pi_1(L(p, q)).$$

Proof Let S^3 given by the complex coordinates (Z_0, Z_1) and $Z_0\bar{Z}_0 + Z_1\bar{Z}_1 = 1$. Let $f : S^3 \to S^3$ be a map given by

$$f(Z_0, Z_1) = (Z_0 e^{2\pi i/p}, Z_1 e^{2\pi q i/p}).$$

So f is the generator of Z_p; cyclic group of rotations. Every element of this group Z_p is fixed point free. Then S^3/Z_p is the lens space $L(p, q)$.
Now let $h_t : S^3 \to S^3$ be a homotopy given by

$$h_t(Z_0, Z_1) = (Z_0 e^{2\pi t i/p}, Z_1 e^{2\pi q i t/p}).$$

So h_0 is the identity map 1_{S^3} and $h_1 = f$. Also $f \circ h_t = h_t \circ f$, therefore h_t commutes with all Z_p. So f belongs to $vG(L(p, q))$; where v is the natural isomorphism between $\pi_1(X)$ and $D(X)$. Therefore $G(L(p, q)) \cong Z_p$.

Theorem 3.25 *Let P^n denote the n-dimensional real projective space. Then*

$$G(P^{2n+1}) = \pi_1(P^{2n+1}) \cong Z_2.$$

Proof We can define sphere S^{2n+1} in the complex plane using (n + 1) tuples $(Z_0, Z_1, ..., Z_n)$ satisfying the condition $Z_0\bar{Z}_0 +....\ Z_n\bar{Z}_n = 1$. We get P^{2n+1} by identifying antipodal points. Let $k \in D(X)$ and $k(Z_0, Z_1, ..., Z_n) = (-Z_0, -Z_1, ..., -Z_n)$. Then define a homotopy $f_t : S^{2n+1} \to S^{2n+1}$ where

$$f_t(Z_0, ..., Z_n) = (Z_0 e^{\pi ti}, ..., Z_n e^{\pi ti}).$$

Then by f_t we get that f_0 is the identity and $f_1 = k$. We also have $k \circ f_t = f_t \circ k$ i.e. f_t commutes with every deck transformation. So we can apply Corollary 3.23 here. Therefore $G(P^{2n+1}) = \pi_1(P^{2n+1}) \cong Z_2$.

Let $H(X)$ to be the set all those elements of $D(X)$ which are in $Z(D(X))$ and homotopic to the identity; where $Z(D(X))$ is the center of $D(X)$.
$H(X)$ is a subgroup of $D(X)$.

Theorem 3.26

$$G(X, x_0) \subseteq H(X) \subseteq P(X, x_0).$$

Proof The first part $G(X, x_0) \subseteq H(X)$ is obvious using Corollary 3.23.
Let $k \in D(X)$ and $f_t : \tilde{X} \to \tilde{X}$ is a homotopy such that f_0 is the identity and $f_1 = k$. Let $\tilde{\alpha} : I \to \tilde{X}$ given by $\tilde{\alpha}(t) = f_t(x_0)$. Let $\alpha = p \circ \tilde{\alpha}$. Then k is the corresponding deck transformation to $[\alpha]$ under v. Suppose α operates on $[\sigma] \in \pi_n(X, x_0); n > 1$. Define a map $h : S^n \vee S^1 \to X$; where $h|S^n = \sigma$ and $h|S^1 = \alpha$. Then α operates trivially on σ if and only if h can be extended to a map $h' : S^n \times S^1 \to X$.
We have to define h'. By the property of the universal covering space we get a map $g : S^n \to \tilde{X}$ so that $p \circ g = h|S^n$ for $n > 1$. Consider S^1 as a unit interval i.e 0 is identified with 1, then $h'(s, t) = (p \circ f_t \circ g, s)$. Since $h'(s, 0) = p(g(s)) = p \circ k(g(s)) = p \circ f_1 \circ g(s) = h'(s, 1)$. h' is well-defined and it is an extension of h. Therefore we have shown that $H(X) \subseteq$ subgroup of $\pi_1(X, x_0)$ given by those elements which operates trivially on $\pi_n(X, x_0); n > 1$. Since we know $H(X)$ is contained in $Z(\pi_1(X, x_0))$, $\alpha \in H(X)$ operates trivially on $\pi_1(X, x_0)$. Therefore $H(X) \subseteq P(X, x_0)$. This completes the proof.

For a space X whose universal covering space $\tilde{X}$ is compact and odd-dimensional sphere, we have $H(X) = P(X) = Z(\pi_1(X)$.

Theorem 3.27 *Let X be a contractible CW complex. Let G be a discrete group of homeomorphisms of X onto itself and G acts freely on X. If f is an element of $Z(G)$, then there* *exists a homotopy h_t such that $h_0 = 1_X$ and $h_1 = f$ and h_t commutes with all elements $g \in G$.*

Proof Let X/G be the space obtained by identifying the orbits under G. Then X can be seen as the universal covering space of X/G. G can be seen as the deck transformations of the covering and thus as the fundamental group of X/G. Since X/G is aspherical, this implies that $G(X/G) \cong Z(G)$. Therefore $Z(G)$ consists of all those elements of G which are homotopic to 1_X by a homotopy h_t and h_t commutes with all elements of G.

3.6 Gottlieb groups of mapping spaces

In this section, we will compute Gottlieb group of the space of all continuous mappings from a space to itself.
Let X^X be the space of continuous mappings from from X to X having compact-open topology. Compact-open topology is generated by a subbasis which is formed by the sets

$$S(C,U) = \{f \mid f \in X^X, f(C) \subset U\} \ ;$$

where C is a compact subspace and U is an open subset of X.
Let Ω denote the path connected component of X^X having the identity 1_X.
Let $\rho : X^X \to X$ be the evaluation $\rho(f) = f(x_0)$. We want ρ to be continuous, so we will assume that the space X is locally compact throughout this section.

Remark 3.28 Let $(X^X)^{S^n})$ and $X^{X \times S^n}$ be spaces of maps, then there is a natural homeomorphism $\varphi : (X^X)^{S^n}) \to X^{X \times S^n}$ given by $\varphi(f)(x,s) = (f(s))(x)$; where $x \in X, s \in S^n$ *and* $f \in X^X$. Also $f \cong g$ if and only if $\varphi(f) \cong \varphi(g)$.

Theorem 3.29

$$\rho_* \pi_1(X^X, 1_X) \cong G(X, x_0).$$

Proof By Remark 3.28, we get that the loop $\alpha : S^1 \to X^X$ corresponds to the cyclic homotopy $\varphi(\alpha) : X \times S^1 \to X$. Now $\rho \circ \alpha : S^1 \to X$ is equal to $\varphi(\alpha)|S^1$ for $\rho(\alpha)(x_0, s) = \alpha(s)(x_0) = \rho(\alpha(s)) = \rho \circ \alpha(s)$.
So every closed loop $\alpha \in \Omega \subseteq X^X$ is a cyclic homotopy of X and trace of α equals $\rho \circ \alpha$. Conversely every cyclic homotopy of X is a closed path α in Ω such that $\rho \circ \alpha$ is equal to the trace of the cyclic homotopy. This completes the proof.

Theorem 3.30 *Let X be a locally finite, aspherical and path-connected simplical polyhedron. Then*

$$\rho_* : \pi_1(X^X, 1_X) \cong Z(\pi_1(X, x_0))$$

and for $n > 1; \pi_n(X^X, 1_X) = 0$.

Proof To prove this theorem we will use the following lemmas.

Lemma 3.31 *For $n > 1$,*

$$\pi_n(X^X, 1_X) = 0.$$

Proof Let $f : X \times S^n \to X$ be a map such that $f(X, s_0) = x; \forall x \in X$ and s_0 is the base point of S^n. Define $g : X \times S^n \to X$ such that $g(x, s) = x$. g is just the projection of $X \times S^1$ onto X. If we can show that $f \cong g$, then by Remark 3.28, we get $\varphi^{-1}(f) \cong \varphi^{-1}(g)$. Since $\varphi^{-1}(g) : S^1 \to X^X$ is the constant map onto 1_X, so we are done.
Since X is aspherical, then $f \cong g$ if and only if $f_* : \pi_1(X \times S^n, x_0 \times s_0) \to \pi_1(X, x_0)$ and $g_* : \pi_1(X \times S^n, x_0 \times s_0) \to \pi_1(X, x_0)$ are equivalent, i. e.

$$f_*(\alpha) = \zeta^{-1} \cdot g_*(\alpha) \cdot \zeta$$

$\forall\ \alpha \in \pi_1(X \times S^n)$ and some $\zeta \in \pi_1(X)$.
So now $\pi_1(X \times S^n, x_0 \times s_0) \cong \pi_1(X, x_0) \oplus \pi_1(S^n, s_0) \cong \pi_1(X, x_0)$. Both f_* and g_* act like the identity and thus $f_* = g_*$. Therefore $f \cong g$. This completes the proof.

Lemma 3.32

$$\rho_*(\pi_1(X^X, 1_X)) = Z(\pi_1(X, x_0)).$$

Proof Since X is aspherical, we have $Z(\pi_1(X, x_0)) = G(X, x_0)$. Therefore by the Theorem 3.29, we get $\rho_*(\pi_1(X^X, 1_X)) = Z(\pi_1(X, x_0))$.

Lemma 3.33 *Let $\Omega_0 \subseteq X^X$ denote the space of all those maps f such that $f(x_0) = x_0$. Then $\pi_1(\Omega_0, 1_X) = 0$.*

Proof Let $g : X \times S^1 \to X$ such that $g(x, s) = x$. Let $f : X \times S^1 \to X$ be any arbitary map such that $f(x_0, s) = x_0$ for all $s \in S^1$. We have to show that there exist a homotopy $k_t : X \times S^1 \to X$ such that $k_0 = f$ and $k_1 = g$ and $k_t(x_0, s) = x_0$ for all $s \in S^1$ and $t \in I$. Then $\varphi^{-1}(k_t)$ will be a homotopy connecting $\varphi^{-1}(f) \in \Omega_0$ and $\varphi^{-1}(g)$ where $\varphi^{-1}(g)$ is the constant map from S^1 to 1_X. Since $\varphi^{-1}(k_t) \in \Omega_0$ for each $t \in I$, so the lemma will be proved.

We can view S^1 as I where 0 and 1 are identified in I. Therefore we can consider f and g as maps from $X \times I$ to X.

Let

$$A = (X \times 0 \times I) \cup (X \times 1 \times I) \cup (x_0 \times I \times I) \cup (X \times I \times 0) \cup (X \times I \times 1).$$

Define $K^{(1)} : A \to X$ such that

$$K^{(1)}(x, s, 0) = f(x, s)$$

$$K^{(1)}(x, s, 1) = g(x, s)$$

$$K^{(1)}(x_0, s, t) = x_0$$

$$K^{(1)}(x, 0, t) = K^{(1)}(x, 1, t) = x.$$

We want to extend $K^{(1)}$ to a map $K : X \times I \times I \to X$. Because then $K(x, s, t) = k_t(x, s)$ which is the desired homotopy.

Let $X^{(n)}$ denote the n-skeleton of X. Let $L = X \times I \times I$. Consider I to be decomposed into $\{0\}$, $\{1\}$ and (0, 1). Then

$$L^{(1)} \subseteq A \text{ and } L^{(2)} \subseteq X^{(0)} \times I \times I \cup A.$$

Then we shall extend $K^{(1)} : A \to X$ to $K^{(2)} : L^{(2)} \to X$ by the following procedure. Let $x_i \in X^{(0)}$. Then

$$S_{i^1} = (x_i \times I \times 0) \cup (x_i \times 1 \times I) \cup (x_i \times I \times 1) \cup (x_i \times 0 \times I)$$

forms a circle. Since $S_{i^1} \in A, K^{(1)}|S_{i^1} : S_{i^1} \to X$. Observe that $K^{(1)}|S_{i^1}$ is null homotopic, so it can be extended to $K_i^{(2)} : X_i \times I \times I$. Define $K^{(2)} : L^{(2)} \to X$ by

$$K^{(2)}(y) = \begin{cases} K^{(1)}(y) \ ; & \text{if } y \in A \\ K_i^{(2)}(y) \ ; & \text{if } y \in x_i \times I \times I. \end{cases} \tag{3.3}$$

Since X is aspherical, we can extend $K^{(2)} : L^{(2)} \to X$ to $K : X \times I \times I \to X$. Since $K(x, 0, t) = K(x, 1, t)$, K can be considered as a map from $X \times S^1 \times I$ to X. Now we can define $k_t(x, s) = K(x, s, t)$ and see that $k_0 = f$ and $k_1 = g$ and $k_t(x_0, s) = x_0$. This completes the proof.

Lemma 3.34

$$\rho_* : (\pi_1(X^X, 1_X)) \cong Z(\pi_1(X, x_0)).$$

Proof Consider the homotopy sequence

$$\pi_1(\Omega_0) \xrightarrow{i_*} \pi_1(X^X) \xrightarrow{\rho_*} \pi_1(X).$$

By previous lemma, we get $\pi_1(\Omega_0) = 0$, so ρ_* must be one-one. But $\rho_*(\pi_1(X^X, 1_X)) = Z(\pi_1(X, x_0))$, so $\rho_* : (\pi_1(X^X, 1_X)) \cong Z(\pi_1(X, x_0))$. This completes the proof.
Lemmas 3.31 through 3.34 prove the Theorem 3.30.

Corollary 3.35 *If X is a path-connected, aspherical, locally finite simplical polyhedron. Then Ω, the path connected component of X^X containing the identity 1_X, is contractible.*

Corollary 3.36 *If X is a locally finite, aspherical, pathwise connected simplical polyhedron. Then $\rho : \Omega \to X$ is a homotopy equivalence if and only if $\pi_1(X, x_0)$ is abelian.*

3.7 Euler characteristic and Gottlieb groups

Theorem 3.37 *Suppose X has the same homotopy type as a compact and connected polyhedron and suppose that the Euler characteristic $\chi(X) \neq 0$, then $G(X)$ is trivial.*

Proof This proof is an application of Nielson-Wecken theory of fixed point classes. Here we have summarized the pertinent facts needed for the proof. These facts are proved in Wecken [5].
By Theorem 3.15, we can assume that X is a compact, connected polyhedron. Let $\tilde{X}$ be the universal covering of X. We regard $\pi_1(X)$ as the group of deck transformations on X. Let $f : X \to X$ is a map. Consider the set of all lifts of f to maps $\tilde{f} : \tilde{X} \to \tilde{X}$. We define an equivalence relation among these lifts as follows: $\tilde{f} \equiv \tilde{f}_1$ if and only if $\tilde{f}_1 = \gamma^{-1} \circ \tilde{f} \circ \gamma$ for some $\gamma \in \pi_1(X)$.
Let $[\tilde{f}]$ denote the equivalence class of $\tilde{f}$. The set of fixed points of $\tilde{f}$ project down, by the covering map p, onto a subset of fixed points of $\tilde{f}$. The fixed points of any $\tilde{f}_1$ in the same equivalence class as $\tilde{f}$ also project down to the same subset of fixed points of f. If $\tilde{f}_1$ is not equivalent to f, then the fixed points of $\tilde{f}_1$ project down to a subset of fixed points of f disjoint from those of $\tilde{f}$. This procedure partitions the fixed points of f into disjoint subsets, called fixed point classes. Thus each fixed point class is uniquely associated with an equivalence class of lifts of f. We can also have lifts, $\tilde{f}$, of f with no fixed points, and so the equivalence class of $\tilde{f}$ corresponds to a void class of fixed points.
If $h_t : f \simeq g$ for $g : X \to X$, then h_t defines a 1-1 correspondence between the lifts of f

and those of g preserving equivalence classes. Hence there is a 1-1 correspondence between fixed point classes.
With each fixed point class $[\tilde{f}]$, it is possible to assign a number v such that $v = 0$ if $[\tilde{f}]$ is empty and such that v is preserved under homotopy. That is if $[\tilde{f}]$ corresponds to $[\tilde{g}]$ under a homotopy from f to g, then v for $[\tilde{g}]$ is equal to the v for $[\tilde{f}]$. Finally the sum of all the v's equals $\mathbf{L}_f$, the Lefschetz number.
Suppose that $f = 1_X$. Then every $v = 0$ except possibly for v_1, the number associated with the fixed point class given by the identity $\tilde{1} : \tilde{X} \to \tilde{X}$. This follows since every other lift of 1_X has no fixed point. Also we know that $\mathbf{L}_f = \chi(X)$ when $f = l_X$. Assume that $\chi(X) \neq 0$. Then $v_1 = \chi(X) \neq 0$.

Let $\alpha \in G(X)$. Then there is a cyclic homotopy $h_t : X \to X$ which can be lifted to a homotopy $\tilde{h_t} : \tilde{1} \backsimeq \alpha$ where we regard α as a deck transformation. So $[\tilde{1}]$ corresponds to $[\alpha]$. But $\alpha : \tilde{X} \to \tilde{X}$ has no fixed points, unless $\alpha = \tilde{1}$. Since $v \neq 0$ for $[\alpha]$, the associated fixed point class must be nonempty so $\alpha = \tilde{1}$. Therefore $\alpha = 1 \in \pi_1(X)$. Hence $G(X) = 1$. [2]

Using this theorem, we get a number of interesting corollaries.

Corollary 3.38 *Let X be the homotopy type of a compact connected polyhedron. If $\chi(X) \neq 0$ and X is an H-space, then $\pi_1(X) = 1$.*

Proof By Theorem 3.13, we get that $G(X) = \pi_1(X)$. Since $G(X) = 1$, by the previous theorem, we get $\pi_1(X) = 1$.

Corollary 3.39 *Let X be the same homotopy type as a compact, connected polyhedron. If X is aspherical and $\chi(X) \neq 0$, then $Z(\pi_1(X)) = 1$.*

Proof By Corollary 3.19, we get $G(X) = Z(\pi_1(X))$. Thus $Z(\pi_1(X)) = 1$.

Corollary 3.40 *If X is any 2-dimensional manifold except for torus, projective space and the Klein Bottle. Then $Z(\pi_1(X))$ is trivial.*

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