Integration in Finite Terms

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Certificate of Examination

This is to certify that the dissertation titled **Integration in Finite Terms** submitted by **Nitin Serwa** (Reg. No. MS09092) for the partial fulfilment of BS-MS dual degree programme of the institute, has been examined by the thesis committee duly appointed by the institute. The committee finds the work done by the candidate satisfactory and recommends that the report be accepted.

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Dated: April 24, 2014

Declaration

The work presented in this dissertation has been carried out by me under the guidance of Dr. Varadharaj R. Srinivasan at the Indian Institute of Science Education and Research Mohali.

This work has not been submitted in part or in full for a degree, a diploma, or a fellowship to any other university or institute. Whenever contributions of others are involved, every effort is made to indicate this clearly, with due acknowledgment of collaborative research and discussions.

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In my capacity as the supervisor of the candidate's project work, I certify that the above statements by the candidate are true to the best of my knowledge.

Dr. Varadharaj R. Srinivasan
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Chapter 1

Introduction

Symbolic integration is the problem of finding a "closed form" expression for an indefinite integral. This problem attracted many mathematicians but the first substantial contribution came from Joseph Liouville (1840). In crude terms, he proved that if an algebraic function has an elementary integral then the latter is itself an algebraic function plus a sum of constant multiples of logarithms. Later, Maxwell Rosenlicht ([1], [2]) provided a purely algebraic exposition of the problem and proved this theorem of Liouville using algebraic techniques. Another serious contribution to the problem of Symbolic integration was made by Robert Risch. In his paper ([3]), building on the work of Rosenlicht, Risch produced an algorithm to determine when an indefinite integral has a finite closed form expression. In this thesis, I will elaborate the works of Rosenlicht and Risch on the theory of integration in finite terms. The rest of the thesis is arranged as follows.

In *Chapter* 2, we define the notion of differential field and differential field extensions and prove basic results that will be often used in this thesis. In *Chapter* 3, we prove the Liouville's theorem of Maxwell Rosenlicht. The Liouville criterion is also presented in Chapter 3. In particular for non zero function $f(z)$ and non-constant function $g(z)$, it states: The function fe^g has an elementary integral if and only if there is a rational function $q \in \mathbb{C}(z)$ such that $f = q' + qg'$ and then the integral is qe^g.

In *Chapter* 4, Risch algorithm is proved using previously developed theories for determining the elementary integrability of those elementary functions which can be built up (roughly speaking) using only the rational operations, exponentiation and taking logarithms.

Chapter 2

Differential Algebra

Here we define some basic terminologies from differential algebra and prove a standard result which will serve as an important tool through out this thesis.

Definition 2.1. *A differential field is a field F together with a map $d : F \to F$ which satisfies following conditions $\forall x, y \in F$:*

$$d(x + y) = d(x) + d(y),$$

$$d(xy) = xd(y) + d(x)y,$$

we call such a map d as a derivation on F

Definition 2.2. *An element c of a differential field is said to be constant if $d(c) = 0$.*

From the definition of a derivation map, we prove the following immediate consequences:

Lemma 2.1. *a) $d(1) = 0$.*

b) For all x and non zero y in F, $d(x/y) = \dfrac{(dx)y - x(dy)}{y^2}$.

c) For all x in F and n in $\mathbb{N}$, $d(x^n) = nx^{n-1}dx$.

Proof

a) $d(1) = d(1.1) = d(1).1 + 1.d(1) = 2d(1)$ which implies $d(1) = 0$

b) Let $x = zy$, on differentiation we have, $dx = d(zy) = (dz)y + z(dy)$ which implies $dz = \dfrac{dx - zdy}{y}$.

On substituting $z = x/y$, we have our desired expression

$$d\left(\frac{x}{y}\right) = \frac{(dx)y - x(dy)}{y^2}.$$

c) Through simple induction on n we get the *power rule*as, $d(x^n) = nx^{n-1}dx \forall x \in F$

Now, it is easy to see from above Lemma that the set of all constants in F is a subfield of F. In particular, if F has characteristic 0, then the constant subfield contains the rational $\mathbb{Q}$. Since, field of characteristic 0 contains $\mathbb{Q}$.

Definition 2.3. *A differential field E is a differential extension field of F, if it contains F as a subfield and for the injective homomorphism $\phi : E \to F$*

$$\phi(df) = d\phi(f),$$

for all $f \in F$.

Definition 2.4. *An isomorphism of the differential field E and F which preserves the differentiation operation is called a differential isomorphism.*

Now we will derive a standard result on algebraic extension of differential field which will be used extensively through out this document.

Proposition 2.1. *Let $(F,')$ be a differential field of characteristic zero and K an algebraic extension field of F. Then the derivation of F can be extended to a derivation on K, and this extension is unique.*

Proof Define the following maps:

$$D_0 : F[X] \to F[X],$$
$$D_0\left(\sum_{i=0}^{n} a_i X^i\right) = \sum_{i=0}^{n} a_i' X^i,$$

and

$$D_1 : F[X] \to F[X],$$
$$D_1\left(\sum_{i=0}^{n} a_i X^i\right) = \sum_{i=0}^{n} i a_i X^{i-1},$$

where $a_i \in F \ \ \forall i = 0, ..., n$.

We show that if a differential structure exist on K extending that of F, then it is unique.

For any $x \in K$ and any $A(X) \in F[X]$

$$A(x)' = (D_0 A)(x) + (D_1 A)(x)x'.$$

Now replace $A(x)$ with the minimal polynomial $f(X)$ of x. Then we have

$$f(x)' = (D_0 f)(x) + (D_1 f)(x)x'.$$

Since $f(x) = 0$, this implies

$$x' = -\frac{(D_0 f)(x)}{(D_1 f)(x)}.$$

Notice that $(D_1 f)(x)$ can never be zero because $D_1 f$ has degree one less than $f(X)$. Since, $f(X)$ is the minimal polynomial, we must have $(D_1 f)(x) \neq 0$. Hence, if a differential field structure on K exists then it is unique. Now we show that such a structure on K actually exists.

We can write $K = F(x)$ for some $x \in K$.(see appendix A.1)

Define, $D : F[X] \to F[X]$ as

$$D(A) = D_0 A + g(x) D_1 A,$$

where $g(x) \in F[X]$ to be determined later.

Since D_0 and D_1 are derivation, we observe for any $A, B \in F[X]$

$$D(A + B) = D(A) + D(B),$$

$$D(AB) = BD(A) + AD(B),$$

and for all $a \in F$, we have

$$D(a) = a'.$$

Consider the following surjective ring homomorphism

$$\phi : F[X] \to F[x]$$

$$X' \to x'$$

Let $f(x)$ be the minimal polynomial of x over F then clearly, $Ker\phi = \{p(x) \mid p(x) = 0\}$ and $f(x)$ must divide $p(x)$.Thus, we have $Ker\phi = \langle f \rangle$. By First Isomorphism theorem

$$\frac{F[X]}{\langle f \rangle} \cong F(x) = F[x].$$

Now we define the derivation D of F on $\frac{F[X]}{\langle f \rangle}$ to be $D(g(X)+\langle f \rangle) = D(g(x))+\langle f \rangle$ and once we show that D maps the Kernel of ϕ into itself then we can induce a derivation on $K = F(x)$.
Now consider $f(X)h(X) \in Ker\phi$. Then

$$D(fh) = fD(h) + hD(f).$$

Applying ϕ on both the sides, we obtain

$$\begin{aligned}\phi(D(fh)) &= \phi(fD(h)) + \phi(hD(f)) \\ &= \phi(f)\phi(Dh) + \phi(h)\phi(Df) \\ &= h(x)(Df)(x).\end{aligned}$$

since, for this new derivation $(Df)(x) = 0$, we have

$$(D_0 f)(x) + g(x)D_1 f(x) = 0.$$

Since $(D_0 f)(x) \neq 0$, we obtain

$$g(x) = \frac{-D_0 f(x)}{D_1 f(x)}$$

which is required $g(x)$, and the condition for ϕ to map $Ker\phi$ into itself. Hence, $D_0 + g(x)D_1$ maps $f(x)$ into multiple of itself, and thus maps the ideal $\langle f \rangle$ of $F[X]$ into itself. Therefore, this induces a derivation on the factor ring $F[X]/f(x)$, which is isomorphic to $F(x)$.

Chapter 3

Liouville's Theorem

In this chapter, we prove the Liouville's theorem by Maxwell Rosenlicht [1, 2] but before that, some important terminologies and results are in order which will serve as important tools in proving the theorem. Here we present two proofs for the theorem under section 3.1 and 3.2.

Definition 3.1. *If $x, y \in F$ and $x \neq 0$, we define "x" an exponential of "y" or 'y" a logarithm of "x" if $dy = \frac{dx}{x}$. And we have the following logarithmic derivative identity:*

$$\frac{d(a_1^{n_1}.a_2^{n_2}\ldots a_s^{n_s})}{a_1.a_2\ldots a_s} = n_1\frac{da_1}{a_1} + \ldots + n_s\frac{da_s}{a_s}, \tag{3.1}$$

for $a_1, \ldots, a_n$ non zero elements of F and $n_1, \ldots, n_s$ integers.

Definition 3.2. *A field $\mathbb{E}$ is an elementary extension of $\mathbb{F}$ if it is a differential field extension of $\mathbb{F}$ and there exists a finite tower of fields*

$$\mathbb{F} = \mathbb{E}_0 \subset \mathbb{E}_1 \subset \ldots \subset \mathbb{E}_{k-1} \subset \mathbb{E}_k = \mathbb{E}$$

such that each $\mathbb{E}_i = \mathbb{E}_{i-1}(\theta_i)$ where (θ_i) is a logarithmic or an exponential of an element in $\mathbb{E}_{i-1}$ or θ_i is algebraic over $\mathbb{E}_{i-1}$. Moreover, any element in $\mathbb{E}$ is called elementary function.

The following lemma is taken from [1] and an effort for detailed explanation has been made.

3.1 First Method

Lemma 3.1. *Let F be a differential field with characteristic 0, $F(t)$ be a differential extension field of F having the same subfield of constants with t transcendental over F and with either $t' \in F$ or $t'/t \in F$. If $t' \in F$, then for any polynomial in $f(t) \in F[t]$ of positive degree, $(f(t))'$ is a polynomial in $F[t]$ of the same degree as $f(t)$ or degree one less, according as the highest coefficient of $f(t)$ is not, or is, a constant. If $t'/t \in F$, then for any non zero $a \in F$ and any non-zero integer n we have $(at^n)' = ht^n$, for some non-zero $h \in F$, and furthermore, for any polynomial $f(t) \in F[t]$ of any positive degree, $f(t)'$ is a polynomial in $F[t]$ of the same degree, and is a multiple of $f(t)$ only if $f(t)$ is a monomial.*

Proof

Case I Suppose $t' \in F$ and let $f(t) = a_n t^n + a_{n-1}t^{n-1} + ... + a_0$ be any polynomial in $F[t]$ with $a_n \neq 0$. Then,

$$\begin{aligned} f(t)' &= a_n' t^n + n a_n t^{n-1} t' + a_{n-1}' t^{n-1} + ... + a_0' \\ &= a_n' t^n + (n a_n t' + a_{n-1}') t^{n-1} + ... + a_0'. \end{aligned}$$

If a_n is not a constant, then $a_n' \neq 0$ which implies $f(t)'$ has same degree as $f(t)$. If a_n is a constant and $na_n t' + a_{n-1}' = 0$, then notice $(na_n t + a_{n-1})' = na_n t' + a_{n-1}' = 0$ which implies, $na_n t + a_{n-1}$ is a constant. Since F and $F(t)$ have same subfield of constants, $na_n t + a_{n-1} \in F$, moreover $t \in F$ which is a contradiction. Hence, $na_n t' + a_{n-1}' \neq 0$ and $f(t)'$ has one degree less than $f(t)$.

Case II Now suppose $t'/t \in F$. Let $a \neq 0 \in F$ and $n \neq 0 \in \mathbb{Z}$. Then,

$$\begin{aligned} (at^n)' &= a' t^n + n a t^{n-1} t' \\ &= t^n \left(a' + na \frac{t'}{t} \right) \\ &= t^n (a' + nab), \end{aligned}$$

for some $b \in F$, if $a' + nab = 0$, then $(at^n)' = 0$ which implies at^n is a constant in F which contradicts the transcendency of t over F. Therefore, $a' + nab \neq 0$ and thus, for some $h \in F$

$$(at^n)' = ht^n.$$

Clearly, $f(t)'$ has same degree as $f(t)$. We have shown that for any monomial $f(t)'$ is a multiple of $f(t)$ but that is not the case with any other $f(t) \in F(t)$. To see this, let $f(t) = a_n t^n + a_m t^m$, then after taking derivative we obtain

$$\begin{aligned} f(t)' &= a_n' t^n + n a_n t^{n-1} t' + a_m' t^m + m a_m t^{m-1} t' \\ &= \left(\frac{a_n' + nab}{a_n} \right) a_n t^n + \left(\frac{a_m' + m a_m b}{a_m} \right) a_m t^m. \end{aligned}$$

If $f(t)'$ is a multiple of $f(t)$, then

$$\frac{a_n' + n a_n b}{a_n} = \frac{a_m' + n a_m b}{a_m}$$

$$\frac{a_n'}{a_n} + n\frac{t'}{t} = \frac{a_m'}{a_m} + m\frac{t'}{t}.$$

Now observe that

$$\begin{aligned} \left(\frac{a_n t^n}{a_m t^m} \right)' &= \frac{a_m t^m (a_n t^n)' - (a_m t^m)' a_n t^n}{(a_m t_m)^2} \\ &= \frac{(a_n t^n)'}{a_m t^m} - \frac{(a_m t^m)' a_n t^n}{(a_m t^m)^2} \\ &= \frac{a_n' t^n}{a_m t^m} - \frac{n a_n t^{n-1} t'}{a_m t^m} - \left[\frac{a_m' t^m + m a_m t^{m-1} t'}{(a_m t^m)^2} \right] a_n t^n \\ &= \frac{a_n' t^n}{a_m t^m} - \frac{n a_n t^{n-1} t'}{a_m t^m} - \frac{a_m' t^m a_n t^n}{(a_m t^m)^2} - \frac{m t' a_n t^n}{a_m t^{m+1}} \\ &= \frac{a_n t^n}{a_m t^m} \left[\frac{a_n'}{a_n} + n\frac{t'}{t} - \frac{a_m'}{a_m} - m\frac{t'}{t} \right] \\ &= 0 \end{aligned}$$

arguing as earlier, it contradicts the transcendency of t over F, hence not possible.

Theorem 3.1. *(Liouville)Let F be a differential field of characteristic zero and $\alpha \in F$. If the equation $y' = \alpha$ has a solution in some elementary differential extension field of F having the same subfield of constants, then there are constants $c_1, c_2, ..., c_n \in F$*

and the elements $u_1, ...u_n, v \in F$ *such that*

$$\alpha = \sum_{i=1}^{n} c_i \frac{u_i'}{u_i} + v'. \tag{3.2}$$

Proof By assumption, we have tower of elementary differential extension fields of F

$$F \subset F(t_1) \subset \ldots \subset F(t_1, \ldots, t_n),$$

with same subfield of constants. We shall prove the theorem by induction on n.
For $n = 0$, $y' = \alpha$ has solution in F itself, i.e $\alpha = y'$ is the desired form. Now assume that this result holds for $n - 1$ and apply this to the fields $F(t_1) \subset F(t_1, \ldots, t_n)$ and we obtain

$$\alpha = \sum_{i=1}^{n} c_i \frac{u_i'}{u_i} + v', \tag{3.3}$$

but with $u_1, \ldots, u_n, v \in F(t_1)$ and $c_1, \ldots, c_n \in F$. We wish to find a similar expression for α but with all $u_1, \ldots, u_n, v \in F$, possibly with different n. For the rest of proof, we set $t_1 = t$.

Case I Suppose t is algebraic over F. So in this case we have F(t)=F[t], then there are polynomials $U_1, \ldots, U_n, V \in F[X]$ such that $U_i(t) = u_i$ and similarly $V(t) = v$ for all $i = 1, \ldots, n$. So we can rewrite 3.3 as

$$\alpha = \sum_{i=1}^{n} c_i \frac{U_i'}{U_i} + V'.$$

Let $t = \tau_1, \ldots, \tau_s$ be distinct conjugates of t over F in some suitable algebraic closure of $F(t)$ and $\sigma_1, \ldots, \sigma_s$ be $F - isomorphisms$. Notice that for all $j = 1, \ldots, s$

$$\sigma_j(\alpha) = \sum_{i=1}^{n} c_i \frac{U_i(\sigma_j(t))'}{U_i(\sigma_j(t))} + V(\sigma_j(t))'.$$

Since $\alpha \in F$, we obtain

$$\alpha = \sum_{i=1}^{n} c_i \frac{U_i(\tau_j)'}{U_i(\tau_j)} + V(\tau_j)'.$$

After taking summation over j, it follows that

$$\begin{aligned}\alpha &= \frac{1}{s}\sum_{i=1}^{n}\sum_{j=1}^{s} c_i \frac{U_i(\tau_j)'}{U_i(\tau_j)} + \frac{1}{s}\sum_{j=1}^{s} V(\tau_j)' \\ &= \frac{1}{s}\sum_{i=1}^{n} c_i \left(\frac{U_i(\tau_1)'}{U_i(\tau_1)} + \ldots + \frac{U_i(\tau_s)'}{U_i(\tau_s)}\right) + \frac{1}{s}\left(\sum_{j=1}^{s} V(\tau_j)\right)' \\ &= \frac{1}{s}\sum_{i=1}^{n} c_i \frac{(U_i(\tau_1)\ldots U_i(\tau_s))'}{U_i(\tau_1)\ldots U_i(\tau_s)} + \ldots + \frac{U_i(\tau_s)'}{U_i(\tau_s)} + \frac{1}{s}\left(\sum_{j=1}^{s} V(\tau_j)\right)'.\end{aligned}$$

Now we see that $U_i(\tau_1)\ldots U_i(\tau_s)$ and $V(\tau_1)+\ldots+V(\tau_s)$ are symmetric polynomials in $\tau_1,\ldots,\tau_s$ with coefficients in F and therefore they are fixed by each σ_j. Hence, they belong to F and we get the desired form.

A number of comments are in order before we proceed to remaining cases. We may assume that t is transcendental over F, then $F(t)$ is a Unique Factorisation Domain (UFD) and we have already

$$\alpha = \sum_{i=1}^{n} c_i \frac{u_i'}{u_i} + v', \tag{3.4}$$

with $u_1,\ldots,u_n, v \in F(t)$ and $c_1,\ldots,c_n \in F$. For any i, we can write

$$u_i = f r_1^{m_1}\ldots r_s^{m_s} q_1^{-n_1}\ldots q_p^{-n_p},$$

with $f \in F$ and monic polynomials $r_1 \ldots r_s, q_1 \ldots q_p \in F(t)$ then notice,

$$\frac{u_i(t)'}{u_i(t)} = \frac{f'}{f} + m_1\frac{r_1'}{r_1} + \ldots + m_s\frac{m_s'}{m_s} - n_1\frac{q_1'}{q_1} - \ldots - n_p\frac{q_p'}{q_p}.$$

So we can rewrite 3.4 with $u_i(t)$ either in F or a monic irreducible polynomial of $F[t]$. Also, we can always write $v(t) = \dfrac{P(t)}{Q(t)}$ with $(P,Q) = 1$ and Q be a monic polynomial. Now consider the remaining cases.

Case II Now, let t is logarithm over F. Then, for some $a \in F$ we can write $t' = \dfrac{a'}{a}$. Let $f(t)$ be a monic irreducible element of $F[t]$. From the above discussion, we can assume that $u_i(t) = f(t)$ and therefore by lemma 3.1, we know $deg(f(t)) > deg(f(t)')$. Then $f(t) \nmid f(t)'$, thus the fraction $\dfrac{u_i(t)'}{u_i(t)}$ is already in lowest form which is not possible since $\alpha \in F$. Now substitute $V = \dfrac{P}{Q}$ in 3.4

$$\alpha = \sum_{i=1}^{n} c_i \frac{u_i(t)'}{u_i(t)} + \frac{QP' - PQ'}{Q^2}.$$

On clearing common denominators

$$\prod_{i=1}^{n} u_i(t) Q^2 \alpha = Q^2 \left(\sum_{i=1}^{n} \prod_{j \neq i} c_i u_j(t) u_i(t)' \right) + \prod_{i=1}^{n} u_i(t)(QP' - PQ'),$$

$$\left(\prod_{i=1}^{n} u_i(t) \alpha - \sum_{i=1}^{n} \prod_{j \neq i} c_i u_j(t) u_i(t)' \right) Q^2 = \prod_{i=1}^{n} u_i(t)(QP' - PQ').$$

On comparing, it is easy to see that $Q^2 = \prod_{i=1}^{n} u_i(t)$. Thus, it will appear in α too, which is not possible. Therefore, $v(t) \in F[t]$ with $v' \in F$, again with the help of lemma 3.1, we obtain $v = ct + d$ with constants $c, d \in F$. Finally, we arrive at the following identity,

$$\alpha = \sum_{i=1}^{n} c_i \frac{u_i'}{u_i} + c\frac{a'}{a} + d',$$

with $v = d$ we have the desired form for α.
Finally, consider the last case.

Case III Now consider the final case, when t is exponential over F. By similar argument as in the preceding case, we can assume that $u_i(t) = f(t)$ as a monic irreducible element in $F[t]$. Lemma 3.1 implies that if $f(t)$ is a monic irreducible element of $F(t)$ other than t itself, then $f(t)' \in F[t]$ and $f(t) \nmid f(t)'$. With the same reasoning as above,we conclude that $f(t)$ cannot appear in the denominator of $v(t)$ nor can any $u_i(t) = f(t)$. Thus $v(t) = a_0 + a_1 t + \ldots + a_n t^n$ with $a_0, \ldots, a_n \in F$ and each of the quantities $\dfrac{u_i(t)'}{u_i(t)} \in F$. This implies $v(t)' \in F$, so lemma 3.1 suggests $v(t) \in F$. If each $u_i(t) \in F$, then we are done, otherwise there is only one $u_i(t)$, say $u_1(t) \notin F$. Then $u_1(t)$ and $u_2(t), \ldots, u_n(t) \in F$. So we can write

$$\alpha = \sum_{i=2}^{n} c_i \frac{u_i'}{u_i} + (v + c_1 \frac{t'}{t}).$$

Theorem 3.2. *Let g be a non-constant rational function in $\mathbb{C}(z)$. Then e^g is transcendental over $\mathbb{C}(z)$*

Proof We shall prove it by contradiction. Suppose on the contrary e^g is algebraic over $\mathbb{C}(z)$, then e^g is contained in a finite normal algebraic extension F of $\mathbb{C}(z)$. For each $\sigma \in Aut(F/\mathbb{C}(z))$,

$$g' = \frac{\sigma(e^g)'}{\sigma(e^g)}.$$

Set $e^g = t$ and on summing over all $\sigma \in Aut(F/\mathbb{C}(z))$

$$\begin{aligned}[F : \mathbb{C}(z)]g' &= \sum_{\sigma} \frac{\sigma(t)'}{\sigma(t)} \\ &= \frac{(\prod_{\sigma} \sigma(t))'}{\prod_{\sigma} \sigma(t)},\end{aligned}$$

where we have used logarithmic identity 3.1 in writing second equality. Now observe that right hand side has pole of order of one, whereas left hand side can never have pole of order one, since g is a non constant rational function.
Let f be a non zero rational function over $\mathbb{C}$ and g as above. Then we have the following theorem.

Theorem 3.3. *The function fe^g has an elementary integral if and only if there is a rational function $q \in \mathbb{C}(z)$ such that*

$$f = q' + qg',$$

and then the integral is qe^g.

Proof Set $e^g = t$ for brevity and $F = \mathbb{C}(z)$. If the integral $\int ft$ is elementary then by Liouville's theorem we must have

$$ft = \sum_{i=1}^{n} c_i \frac{u_i'}{u_i} + v',$$

with $c_1, \ldots, c_n \in \mathbb{C}$ and $u_1, \ldots, u_n, v \in F(t)$. With the same argument used in Liouville's theorem, we conclude that $v \in F[t]$ and all $u_1, \ldots, u_n \in F$ with possible exception of one of $u_i(t) = t$. Since $\sum \frac{u_i(t)'}{u_i(t)} \in F$, we have $ft = v'$, hence we obtain

$v = bt$ for some $b \in F$. Writing $b = q$ we have $ft = (q' + qg')t$.
For the converse, suppose there is a $q \in F$ such that $f = q' + qg'$, then

$$\int (q' + qg')e^g = qe^g.$$

3.2 Second method

In this section, we prove the same theorem [2], but with a different approach.

Lemma 3.2. *Let F be a differential field, $F(t)$ be a differential extension field of F having the same subfield of constants with t transcendental over F and with either $t' \in F$ or $t'/t \in F$. Let $c_1, ..., c_n \in F$ be linearly independent over the rational numbers $\mathbb{Q}$ and let $u_1, ..., u_n$ be nonzero elements of $F(t), v \in F(t)$. Then if*

$$\sum_{i=1}^{n} c_i \frac{u_i'}{u_i} + v' \in F[t],$$

we have $v \in F[t]$ and in the case $t' \in F$, each $u_i \in F$ while in the case $t'/t \in F$, for each $i = 1, ..., n$ we have $u_i/t^{\nu_i} \in F$ for some integer ν_i.

Proof In a suitable finite normal algebraic extension field K of F; $u_1, ..., u_n, v$ of $F(t)$ will split into linear factors. Writing $u_i(t) = \dfrac{p(t)}{q(t)}$, we can make both polynomials monic and split linearly, and thus

$$u_i = g_i \prod_j (t - z_j)^{\mu_{ij}}.$$

Similarly by partial decomposition of v

$$v = \sum_{j,\nu} h_{\nu_j} (t - z_j)^{\nu} + f(t).$$

Above quantities hold for all $i = 1, ..., n$ with j ranging over a finite set of positive integers, ν ranges over a finite set of negative integers, each μ_{ij} is an integer and each $0 \neq g_i, z_j, h_{\nu_j} \in K$.
We work in the differential extension field $K(t)$ of $F(t)$ (see proposition 2.1). By hypothesis

$$\sum_{i=1}^{n} c_i \frac{u_i'}{u_i} + v' \in K[t].$$

On substitution

$$\sum_{i=1}^{n} c_i \frac{g_i'}{g_i} \prod_j (t-z_j)'^{\mu_{ij}} g_i \prod_j (t-z_j)^{\mu_{ij}} + \left(\sum_{j,\nu} h_{\nu_j}(t-z_j)^{\nu} + f(t)\right)' \in K[t].$$

By logarithmic derivative identity 3.1, we can write

$$f(t)' + \sum_{i=1}^{n} c_i \frac{g_i'}{g_i} + \sum_{i,j} c_i \mu_{ij} \frac{(t-z_j)'}{(t-z_j)} + \sum_{j,\nu} (h\nu_j (t-z_j)^{\nu})' \in K[t]. \tag{3.5}$$

Case I When $t' = a$ for some $a \in F$ (logarithmic case). Consider the quantity $\frac{(t-z_j)'}{t-z_j} = \frac{t'-z_j'}{t-z_j}$, substituting $t' = a$ yields,

$$\frac{t'-z_j'}{t-z_j} = \frac{a-z_j'}{t-z_j}.$$

We claim that this expression is in lowest terms i.e. $a - z_j' \neq 0$. To see this, let us assume $a - z_j' = 0$ which implies $a = z_j'$. Then for each $\sigma \in Aut(K/F)$

$$\sigma(a) = \sigma(z_j')$$
$$a = \sigma(z_j)'$$

Moreover,

$$\sum_\sigma a = \sum \sigma(z_j)'$$
$$[K:F]a = \left(\sum_\sigma \sigma(z_j)\right)'$$
$$a = \frac{(\sum_\sigma \sigma(z_j))'}{[K:F]},$$

So we can write $a = b'$ for some $b \in F$. Hence, the quantity $(t-b)' = t' - b' = a' - b' = 0$ implies $t - b$ is a constant. By assumption F and $F(t)$ have same subfield of constants. Therefore, we obtain $t \in F$ which contradicts the transcendency of t, thus $a - z_j \neq 0$.

Case II Similarly for $t' = at$ for some $a \in F$ (exponential case) we have,

$$\frac{t' - z'_j}{t - z_j} = \frac{at - z'_j}{t - z_j}.$$

We show that above expression is also in lowest terms, provided $z_j \neq 0$. Suppose it is not in the lowest terms, then $(t - z_j)$ comes as a factor of numerator and this implies $z'_j = az_j$. Given $z_j \neq 0$

$$\frac{z'_j}{z_j} = a$$

for each $\sigma \in Aut(K/F)$

$$\sum_{\sigma} \frac{(\sigma z_j)'}{\sigma z_j} = [K : F]a$$

again by logarithmic identity 3.1

$$\frac{\prod \sigma(z_j)'}{\prod \sigma(z_j)} = [K : F]a$$

Numerator and denominator are symmetric in z_j for all j. We conclude, for some $b \neq 0 \in F$

$$[K : F]a = \frac{b'}{b}$$

let $[K : F] = N$ and consider

$$\frac{(t^N)'}{t^N} = N\frac{t'}{t} = Na = \frac{b'}{b}$$

which gives

$$\begin{aligned}
\left(\frac{t^N}{b}\right)' &= \frac{b(t^N)' - t^N b'}{b^2} \\
&= \frac{(t^N)'}{b} - t^N(\frac{b'}{b^2}) \\
&= \frac{Nat^N}{b} - t^N\frac{b'}{b^2} \\
&= \frac{t^N}{b}\left(Na - \frac{b'}{b}\right) \\
&= 0
\end{aligned}$$

which implies t^N/b is a constant in F, which again contradicts the transcendency of t.
Thus, in all cases except when $z_j = 0$ and $t' = at$, the fraction $\dfrac{(t-z_j)'}{t-z_j}$ is in lowest terms. Now, consider the case when $z_j = 0$

$$\begin{aligned}\frac{(t-z_j)'}{(t-z_j)} &= \frac{t'-z_j'}{t-z_j} \\ &= \frac{t'}{t} \\ &= a \quad \in F\end{aligned}$$

Now, consider the last expression in 3.5

$$\begin{aligned}(h_{\nu_j}(t-z_j)^{\nu})' &= h_{\nu_j}'(t-z_j)^{\nu} + \nu h_{\nu_j}(t-z_j)^{\nu-1}(t'-z_j) \\ &= \frac{h_{\nu_j}'}{(t-z_j)^{-\nu}} + \nu h_{\nu_j}\frac{(t'-z_j')}{(t-z_j)^{1-\nu}}\end{aligned}$$

So what has been done above implies this expression is also in lowest terms, except in the one exceptional case when $z_j = 0$ and $t' = at$. For the exceptional case it becomes,

$$\begin{aligned}(h_{\nu_j}(t)^{\nu})' &= \nu h_{\nu_j} t^{\nu-1} t' + t^{\nu} h_{\nu_j}' \\ &= t^{\nu}(\nu h_{\nu_j} a + h_{\nu_j}')\end{aligned}$$

We claim that if $h_{\nu_j} \neq 0$, then also $\nu h_{\nu_j} a + h_{\nu_j}' \neq 0$. Let $\nu h_{\nu_j} a + h_{\nu_j}' = 0$ which implies $\frac{h_{\nu_j}'}{h_{\nu_j}} = -\nu a$ Now, on summing over each $\sigma \in Aut\,(K/F)$, we obtain

$$\begin{aligned}\sum_{\sigma}\left(\frac{\sigma h_{\nu_j}'}{\sigma h_{\nu_j}}\right) &= -N\nu a \\ \left(\frac{(\prod_{\sigma}\sigma(h_{\nu_j}'))}{(\prod_{\sigma}\sigma(h_{\nu_j}))}\right) &= -N\nu a\end{aligned}$$

and therefore, $-N\nu a = \frac{b'}{b}$ for some $b \neq 0 \in F$. Arguing similarly as in case II, we obtain $t^{-N\nu}/b \in F$.
From above discussion, we conclude that in the exceptional case, $(h_{\nu_j}(t-z_j)^{\nu})'$ has denominator $t^{-\nu}$ and in all remaining cases has $(t-z_j)^{1-\nu}$ as denominator. Therefore, we have proved that L.H.S of 3.5 would not cancel to add upto a

polynomial in t and thus $h_{\nu_j} = 0$ and $v = f(t) \in K[t]$.
Finally, we are left with

$$\sum_{i,j} c_i \mu_{ij} \frac{t' - z_j'}{t - z_j} \in K[t]$$

Then, $\sum_i c_i \mu_{ij} = 0$, since $c_1, ..., c_n$ are linearly independent over $\mathbb{R}$ and we have all $\mu_{ij} = 0$. In exceptional case, when $z_j = 0$ and $t' = at$

$$\sum_{i,j} c_i \mu_{ij} \frac{t'}{t} = \sum_{i,j} c_i \mu_{ij} a \in F$$

Thus, $u_i = bt^{\nu_i}$ for some $b \in F$ and integers ν_i.

Theorem 3.4. *Let F be a differential field of characteristic zero and $\alpha \in F$. If the equation $y' = \alpha$ has a solution in some elementary differential extension field of F having the same subfield of constants, then there are constants $c_1, c_2, ..., c_n \in F$ and the elements $u_1, ...u_n, v \in F$ such that*

$$\alpha = \sum_{i=1}^{n} c_i \frac{u_i'}{u_i} + v'.$$

Proof Let $y \in F_N$, where by assumption

$$F = F_0 \subset F_1 \subset \ldots \subset F_N,$$

and for each $i = 1, ..., N; F_i = F_{i-1}(t_i)$ where t_i is logarithm or exponential of an element of F_{i-1} or algebraic over F_{i-1}. We shall prove the theorem by induction on N.
For $N = 0$, i.e. $y \in F$

$$\alpha = y'$$

, is the required form. Now suppose result holds for $N - 1$, i.e for the chain

$$F_1 \subset F_2 \subset \ldots \subset F_N,$$

and

$$\alpha = \sum_{i=1}^{n} c_i \frac{u_i'}{u_i} + v',$$

with with $u_1, \ldots, u_n, v \in F_1$ and $c_1, \ldots, c_n \in F$. Our aim is to modify n, c_i, u_i, v in such a manner that the same expression holds for α but with all $u_i, v \in F$

First, we prove that we can always assume $c_1, \ldots, c_n$ being linearly independent over $\mathbb{Q}$. Suppose they are linearly dependent and thus, we have a relation,

$$c_n = \frac{m_1 c_1 + \ldots + m_{n-1} c_{n-1}}{m},$$

with $m_1, \ldots, m_{n-1}, m$ are integers and $m \neq 0$. We can also write

$$c_n = m_1 \frac{c_1}{m} + \ldots + m_{n-1} \frac{c_{n-1}}{m}.$$

Observe that, on replacing each u_i by u_i^m and each c_i by c_i/m, we again have the same form for α.

$$\alpha = \sum_{i=1}^{n} \frac{c_i}{m} \frac{u_i'^m}{u_i^m} + v'$$

$$\alpha = \sum_{i=1}^{n} \frac{c_i}{m} m \left(\frac{u_i'}{u_i} \right) + v' \tag{3.6}$$

$$\alpha = \sum_{i=1}^{n} c_i \frac{u_i'}{u_i} + v'$$

For $i = 1, \ldots, n-1$; set new $c_i = c_i/m$ and $u_i = u_i^m$ and now, new $c_n = m_1 c_1 + \ldots + m_{n-1} c_{n-1}$. On substitution in 3.6

$$\begin{aligned}
\alpha &= \sum_{i=1}^{n-1} c_i \frac{u_i'}{u_i} + (m_1 c_1 + \ldots + m_{n-1} c_{n-1}) \frac{u_n'}{u_n} + v' \\
&= c_1 \left(\frac{u_1'}{u_1} + m_1 \frac{u_n'}{u_n} \right) + \ldots + c_{n-1} \left(\frac{u_{n-1}'}{u_{n-1}} + m_{n-1} \frac{u_n'}{u_n} \right) + v' \\
&= c_1 \left(\frac{(u_1 u_n^{m_1})'}{u_1 u_n^{m_1}} \right) + \ldots + c_{n-1} \left(\frac{(u_{n-1} u_n^{m_{n-1}})'}{u_{n-1} u_n^{m_{n-1}}} \right) + v'
\end{aligned}$$

Hence, we have same situation as earlier, but with smaller n. Thus, we can always assume, $c_1, \ldots, c_n$ are linearly independent over $\mathbb{Q}$.
Write $F_1 = F(t)$ and we continue the induction process for different cases.

Case I Suppose t is logarithmic over F. By assumption

$$\alpha = \sum_{i=1}^{n} c_i \frac{u_i'}{u_i} + v' \in F,$$

with $u_1, \ldots, u_n, v \in F(t)$. By lemma 3.2, we conclude each $u_1, \ldots, u_n \in F$ and $v \in F[t]$. We also have $v' \in F$, on writing $v = \sum_{j=1}^{m} a_j t^j$ with $a_1, \ldots, a_m \in F$,

$a_m \neq 0$ and after differentiation, we obtain

$$\begin{aligned} v' &= ma_m t^{m-1} t' + t^m a_m' + \ldots + a_0' \\ &= ma_m t^{m-1} \left(\frac{a'}{a}\right) + t^m a_m' + \ldots + a_0' \\ &= t^m a_m' + t^{m-1}\left(ma_m \frac{a'}{a} + a_{m-1}'\right) + \ldots + a_0'. \end{aligned}$$

Since $\alpha \in F$, above expression implies $a_m' = 0$ for $m > 1$ and if

$$\begin{aligned} \left(ma_m \frac{a'}{a} + a_{m-1}'\right) &= 0 \\ \left(ma_m t + a_{m-1}'\right) &= 0, \end{aligned}$$

then $ma_m t + a_{m-1}'$ is a constant in F which is not possible since t is transcendental over F. For $m = 1, a' = 0$ and hence by above discussion we have shown $v = a_0 t + d$ with a_0 and d being constants in F. Now

$$\begin{aligned} \alpha &= \sum_{i=1}^{n} c_i \frac{u_i'}{u_i} + (a_0 t + d)' \\ &= \sum_{i=1}^{n} c_i \frac{u_i'}{u_i} + a_0 \frac{a'}{a} + d' \end{aligned}$$

setting $u_{n+1} = \dfrac{a'}{a}$ and $v = d$, we have

$$\alpha = \sum_{i=1}^{n+1} c_i \frac{u_i'}{u_i} + d',$$

which is the desired form.

Case II Now, let t is exponential over F. Then lemma 3.2 suggests that $v \in F[t]$ and also, we can write $u_i = a_i t^{n_i}$ with $a_i \in F$ and integers n_i. Observe

$$\begin{aligned} \frac{u_i'}{u_i} &= \frac{a_i'\left(t^{\eta_i}\right)}{a_i\left(t^{\eta_i}\right)} + \frac{\eta_i a_i t^{\eta_i} a'}{a_i t^{\eta_i}} \\ &= \frac{a_i'}{a_i} + \eta_i a' \end{aligned}$$

Thus, we simplify α in the form

$$\alpha = \sum c_i \frac{a_i'}{a_i} + (a(\eta_1 + \ldots + \eta_n) + v)'.$$

Now, we only need to show that $v \in F$, given the fact that $v' \in F$. Now write $v = b_m t^m + \ldots + b_0$ and after differentiation , we obtain

$$\begin{aligned} v' &= b_m' t^m + m b_m t^{m-1} a t + b_{m-1} t^{m-1} + \ldots + b_0' \\ &= t^m (b_m' + m b_m a') + \ldots + b_0' \end{aligned}$$

if $m \neq 0$, then $b_m' + m b_m a' = 0$ and thus $(b_m t^m)' = 0$ which is a contradiction. Hence, $v = b_0$ and

$$\alpha = \sum c_i \frac{a_i'}{a_i} + (b_0 + a(\eta_1 + \ldots + \eta_n))',$$

is the required form.

Case III Finally, let t is algebraic over F and K be the smallest normal algebraic extension of F containing F_1. For each $\sigma \in Aut(K/F)$

$$\begin{aligned} \alpha &= \sum_{i=1}^{n} c_i \frac{\sigma(u_i)'}{\sigma(u_i)} + \sigma(v)' \\ [K:F]\alpha &= \sum_{i=1}^{n} c_i \sum_{\sigma} \frac{(\sigma u_i)'}{\sigma u_i} + \sum_{\sigma} (\sigma v)' \\ [K:F]\alpha &= \sum_{i=1}^{n} c_i \frac{(\prod_\sigma \sigma u_i)'}{\prod_\sigma \sigma u_i} + \left(\sum_{\sigma} \sigma v \right)' \\ \alpha &= \frac{1}{[K:F]} \left[\sum_{i=1}^{n} c_i \frac{(\prod_\sigma \sigma u_i)'}{\prod_\sigma \sigma u_i} + \left(\sum_{\sigma} \sigma v \right)' \right] \end{aligned}$$

and we have the required result.

Theorem 3.5. *Let $f \neq 0$ and g is a non constant rational function in $\mathbb{C}(z)$. Then the function, fe^g has an elementary integral if and only if there is a rational function $q \in \mathbb{C}(z)$ such that*

$$f = q' + qg',$$

and then the integral is qe^g.

Proof Write $e^g = t$ and $F = \mathbb{C}(z)$ for brevity. If the integral is elementary, then by Liouville's theorem.

$$ft = v_0' + \sum_{i=1}^{n} c_i \frac{u_i'}{u_i} \in F(t),$$

with $c_1, \ldots, c_n$ constants in F and $u_1, \ldots, u_n, v \in F(t)$. We may assume $c_1, \ldots, c_n$ are linearly independent over $\mathbb{Q}$. Then by lemma 3.2, $v \in F[t]$ and $u_j = a_i t^{m_i}$ where $a \in F$ and $m \in \mathbb{Z}$. Observe

$$\begin{aligned}
\frac{u_i'}{u_i} &= \frac{(a_i t^{m_i})'}{a_i t^{m_i}} \\
&= \frac{a_i' t^{m_i}}{a_i t^{m_i}} + \frac{a_i m_i t^{m_i - 1} g' t}{a_i t^{m_i}} \\
&= \frac{a_i'}{a_i} + m_i g' = r \in F,
\end{aligned}$$

and it follows that $ft = v_0' + r$. Let $v_0 = b_m t^m + \ldots + b_0 \in F$ and then after differentiation we obtain

$$\begin{aligned}
v_0' &= b_m' t^m + m b_m g' t^m + \ldots + b_0' \\
&= t^m (b_m' + m b_m g') + \ldots + b_0'
\end{aligned}$$

On comparing coefficients in $ft = u_0' + r$, for $m > 1$ we obtain

$$\begin{aligned}
b_m' + m b_m g' = 0 \\
\frac{b_m'}{b_m} = -mg' (g' \neq 0)
\end{aligned}$$

which is not possible, since RHS has no poles of order 1 while LHS has poles of order 1. This means $b_m = 0$ for $m > 1$. Hence, $v = bt + b_0$ which implies

$$ft = t(b' + bg') + b_0' + r$$

$$ft = b' + bg'$$

Converse is obvious.

3.3 Examples

In this section, we provide some important examples and also prove one of the famous result, that is, $\int e^{x^2}$ is not elementary.

Example $\int e^{z^2}$ is not elementary.
Using the notation from the previous theorem, we see $f = 1$ and $g = z^2$. If $\int e^{z^2}$ is elementary, then above theorem suggests for some $q \in \mathbb{C}(z)$

$$1 = q' + qg'$$
$$1 = q' + q(2z)$$

let $q = \dfrac{u}{v}$, $(u, v) = 1$ and v be monic, then

$$1 = \frac{u'v - v'u}{v^2} + \frac{u}{v}2z$$

$$v^2 = u'v - v'u + 2zuv$$

$$(v - 2zu - u')v = -uv',$$

it follows that $v|uv'$. Since $(u, v) = 1$, we have $v|v'$ which is not possible. Hence no such q exists and $\int e^{z^2}$ is not elementary.

Example $\int \dfrac{e^z}{z}$ is not elementary.
Similarly in this case, $f = 1/z$ and $g = z$. If $\int \dfrac{e^z}{z}$ is elementary, then there exist $q \in \mathbb{C}(z)$ such that

$$\frac{1}{z} = q' + q.$$

From earlier example we write,

$$\frac{1}{z} = \frac{u'v - v'u}{v^2} + \frac{u}{v}$$

$$v^2 = z(u, v - v'u) + uvz$$

$$zv'u = zu'v - v^2 + uvz$$

$$zuv' = v(zu' - v + uz)$$

which implies $v|z$ but $(u, v) = 1$ and thus, either $v = 1$ or $v = z$. In both cases, we get the contradiction. Therefore, $\int \frac{e^z}{z}$ is not elementary.

Chapter 4

Risch Algorithm

In this chapter, Risch algorithm will be discussed in detail. But, before we move to the integrand involving logarithms and exponential, we see how integration is dealt in field of rational function. Throughout this chapter, we work with field of rational function $K(x)$ over an arbitrary constant field with characteristic zero. Field $K(x)$ is equipped with differential structure with derivation satisfying $x' = 1$.

4.1 Integration of rational function

Hermite's method is an effective procedure to work with the integral of rational function without introducing any algebraic extensions. For $p/q \in K(x)$, it reduces the problem of integration to

$$\int \frac{p}{q} = \frac{c}{d} + \int \frac{a}{b},$$

with $c, d, a, b \in K(x), deg(a) < deg(b)$ and b is monic and square-free(see appendix B). Here, c/d is called the *rational part* of the integral and $\int a/b$ is called the *logarithmic part* of the integral. We discuss each part in different sections.
Following theorem will be used for computing the rational part of the integral. For the proof see [4].

Theorem 4.1. *Let $F[x]$ be the Euclidean domain of univariate polynomials over a field F. Let $a(x), b(x) \in F[x]$ be given non zero polynomials and let $g(x) = gcd(a(x), b(x)) \in F[x]$. Then, for any given polynomial $c(x) \in F[x]$ such that $g(x) \mid c(x)$, there exist unique polynomials $\sigma(x), \tau(x) \in F[x]$ such that*

$$\sigma(x)a(x) + \tau(x)b(x) = c(x),$$

and $deg(\sigma(x)) < deg(b(x)) - deg(g(x))$.
Moreover, if $deg(c(x)) < deg(a(x)) + deg(b(x)) - deg(g(x))$, *then* $\tau(x)$ *satisfies*

$$deg(\tau(x)) < deg(a(x)) - deg(g(x)).$$

4.1.1 Rational Part

Observe that for any $p/q \in K(x)$, we have the normalised form i.e we can write each element of $K(x)$ in the form p/q with $gcd(p,q) = 1$ and q monic. By Euclidean division, we can find $s, r \in K[x]$ such that $p = qs + r$ with $r = 0$ or $deg(r) < deg(q)$. Then

$$\int \frac{p}{q} = \int s + \int \frac{r}{q}. \tag{4.1}$$

First integral appearing on the right hand side is called *polynomial part* which is easy to compute. It also contributes to the rational part c/d. For $\int r/q$, compute the square-free factorisation of denominator q

$$q = \prod_{i=1}^{k} q_i^i,$$

where each q_i is monic and square-free with $gcd(q_i, q_j) = 1$ for $i \neq j$ and $deg(q_k) > 0$. Now, compute the partial fraction expansion of the integrand r/q using square-free factorisation of q.

$$\frac{r}{q} = \sum_{i=1}^{k} \sum_{j=1}^{i} \frac{r_{ij}}{q_i^j},$$

for $1 \leq i \leq k$ and $1 \leq j \leq i, r_{ij} \in K[x]$ and $deg(r_{ij}) < deg(q_i)$ when $deg(q_i) > 0$ and $r_{ij} = 0$ if $q_i = 1$. Then on substitution for integral $\int r/q$ (4.1), we obtain

$$\int \frac{r}{q} = \sum_{i=1}^{k} \sum_{j=1}^{i} \int \frac{r_{ij}}{q_i^j}.$$

Now consider a particular non zero integrand r_{ij}/q_i^j with $j > 1$. Since q_i is square-free, we have $gcd(q_i, q_i') = 1$. Now we can find $s, t \in K[x]$ by theorem 4.1 such that

$$sq_i + tq_i' = r_{ij},$$

where $deg(s) < deg(q_i) - 1$ and $deg(t) < deg(q_i)$. Then

$$\int \frac{r_{ij}}{q_i^j} = \int \frac{s}{q_i^{j-1}} + \int \frac{tq_i'}{q_i^j}.$$

Apply *integration by parts* on the second integral on right hand side

$$\int \frac{r_{ij}}{q_i^j} = \int \frac{s}{q_i^{j-1}} + \frac{-t}{(j-1)q_i^{j-1}} + \int \frac{t'}{(j-1)q_i^{j-1}},$$
$$= \frac{(\frac{-t}{j-1})}{q_i^{j-1}} + \int \frac{s + (\frac{t'}{j-1})}{q_i^{j-1}}.$$

After this process, we see a term appearing on the right hand side which will contribute to the rational part c/d. Also notice that the remaining integral has power one less in denominator. At any step if numerator of the integrand happens to be zero, then reduction process terminates. Otherwise, we have following two cases:

$j - 1 = 1$ Then this integral simply contributes to the logarithmic part to be considered in the next section.

$j - 1 > 1$ In this case, we can continue the reduction process until the denominators of all remaining integrands are square-free.

After the complete reduction process we obtain the rational part c/d of the integral and now we are left with only the logarithmic part. But before we proceed with the logarithmic part, an important result is in order which plays an essential role in proving the *Risch Algorithm*.

Theorem 4.2. *Let $p/q \in K(x)$ be such that $gcd(p, q) = 1, q$ monic and $deg(p) < deg(q)$. Let the rational part of the integral $\int p/q$ be c/d and a/b be the logarithmic part. Then*

$$d = gcd(q, q'),$$

and

$$b = \frac{q}{d}.$$

Furthermore, $deg(a) < deg(b)$ and $deg(c) < deg(d)$

Proof Let the square-free factorisation of q be

$$q = \prod_{i=1}^{k} q_i^i.$$

After applying Hermite's reduction, we obtain

$$\int \frac{p}{q} = \frac{c}{d} + \int \frac{a}{b},$$

where

$$b = \prod_{i=1}^{k} q_i, \qquad d = \prod_{i=2}^{k} q_i^{i-1},$$

with $deg(a) < deg(b)$ and $deg(c) < deg(d)$.
It follows from the square-free factorisation (see appendix B)

$$gcd(q, q') = \prod_{i=2}^{k} q_i^{i-1} = d,$$

and

$$\frac{q}{gcd(q, q')} = \prod_{i=1}^{k} q_i = b.$$

4.1.2 Logarithmic Part

We started with

$$\int \frac{p}{q} = \frac{c}{d} + \int \frac{a}{b}.$$

In the previous section we computed the rational part c/d. Now we see how to compute the logarithmic part $\int a/b$.
After Hermite's reduction, b is square free and let the complete factorisation of b over it's splitting field be

$$b = \prod_{i=1}^{m} (x - \beta_i),$$

where β_i are m distinct elements. Then after writing the partial fraction expansion, integral $\int a/b$ can be expressed as

$$\int \frac{a}{b} = \sum_{i=1}^{m} c_i \log(x - \beta_i).$$

At this point, the problem of integration for rational function is completely solved. But there are various practical difficulties in this method e.g. factorisation of denominator.

Fortunately complete factorisation is not needed always. To see look at the following example

Example

$$\int \frac{1}{x^3+x} = \log(x) - \frac{1}{2}\log(x-\iota) - \frac{1}{2}\log(x+\iota),$$
$$= \log(x) - \frac{1}{2}\log(x^2+1),$$

where we used product rule for logarithms in the second step. We see, to express the integral we do not require complete factorisation.
The problem of expressing an integral using minimum algebraic extensions was solved independently by Rothstein and Trager. In the following theorem, $res_x(A, B)$ is used to denote the *resultant* of the polynomials $A(x)$ and $B(x)$ with respect to the variable x (see appendix C.1). We drop x and simply write $res(A, B)$ where there is no confusion.

Theorem 4.3. *Let $K(x)$ be a differential field over some constant field K. Let $a, b \in K[x]$ be such that $gcd(a, b) = 1$ with b monic and square-free and $deg(a) < deg(b)$. Suppose that*

$$\int \frac{a}{b} = \sum_{i=1}^{n} c_i \log(v_i),$$

where c_i are distinct non zero constants and v_i are monic, square-free, pairwise relatively prime polynomials of positive degree. Then c_i are the distinct roots of the polynomial

$$R(z) = res_x(a - zb', b) \in K[z],$$

and v_i are the polynomials

$$v_i = gcd(a - c_i b', b).$$

Proof By assumption

$$\int \frac{a}{b} = \sum_{i=1}^{n} c_i \log(v_i).$$

On differentiating both sides, we obtain

$$\frac{a}{b} = \sum_{i=1}^{n} c_i \frac{v_i'}{v_i}.$$

For the rest of the proof, we write $u_i = \prod_{j \neq i}^{n} v_j$. Then

$$a \prod_{j=1}^{n} v_j = b \sum_{i=1}^{n} c_i v_i' u_i. \tag{4.2}$$

Now, we claim that

$$b = \prod_{j=1}^{n} v_j. \tag{4.3}$$

Since $gcd(a, b) = 1$, it follows that $b \mid \prod_{i=1}^{n} v_i$. Similarly for the other direction, each

$$v_j \mid b \sum_{i=1}^{n} c_i v_i' u_i,$$

$$v_j \mid b v_j' u_j.$$

Since v_j are square-free, we have $gcd(v_j, v_j') = 1$ and also $gcd(v_j, u_j) = 1$. This implies

$$\prod_{j=1}^{n} v_j \mid b.$$

where b and all v_i are all assumed to be monic, we conclude that

$$b = \prod_{j=1}^{n} v_j.$$

As a consequence, it follows that

$$a = \sum_{i=1}^{n} c_i v_i' u_i.$$

Now consider the quantity

$$\begin{aligned} a - c_j b' &= \sum_{i=1}^{n} c_i v_i' u_i - c_j \sum_{i=1}^{n} v_i' u_i, \\ &= \sum_{i=1}^{n} (c_i - c_j) v_i' u_i. \end{aligned}$$

For $i = j$, above sum vanishes while for $i \neq j, v_j \mid u_i$, which implies

$$v_j \mid (a - c_j b').$$

Also, v_j divides b, therefore we can say v_j is a common divisor of b and $(a - c_j b')$. Now we compute the $gcd(a - c_j b', v_h)$ using the fact $gcd(a + nb, b) = gcd(a, b)$

$$\begin{aligned} gcd(a - c_j b', v_h) &= \left(\sum_{i=1}^{n} (c_i - c_j) v_i' u_i, v_h \right), \\ &= gcd((c_h - c_j) v_h' u_h, v_h), \\ &= 1 \ or \ v_h \end{aligned}$$

where in the last step we used $gcd(v_h', v_h) = 1$ and $gcd(v_h, u_h) = 1$ Now notice

$$\begin{aligned} gcd(a - c_j b', b) &= gcd(a - c_j b', \prod_{i=1}^{n} v_i), \\ &= \prod_{i=1}^{n} gcd(a - c_j b', v_i), \\ &= v_j. \end{aligned}$$

This implies that $res(a - c_j b', b) = 0$, since v_j is a common factor. Thus, c_j is a root of $R(z) = res(a - zb', b)$.

Conversely, let c be any root of $R(z)$ (assume $c \in K_R$, the splitting field of $R(z)$) then $res(a - cb', b) = 0$. Therefore, we can assume

$$gcd(a - cb', b) = G, \qquad deg(G) > 0.$$

Let g be any irreducible factor of G. Since $g \mid b$, and $g \mid (a - cb')$, we see

$$g \mid \sum_{i=1}^{n} (c_i - c) v_i' u_i.$$

But $g \mid u_i$ for each $i \neq j$ (as $g \mid v_j$) and therefore

$$g \mid (c_j - c) v_j' u_j,$$

which can happen only when $c_j = c$. Therefore, c is one of the constants appearing in the integral. We have also proved, under the hypothesis $R(z)$ splits over K.

In the above theorem, if v_i are not square-free, then do the square-free factorisation

$$v_i = \prod_{j=1}^{k} v_j^j,$$

and use the replacement

$$\log(v_i) = \sum_{j=1}^{k} j \log(v_j).$$

Also, we can always assume that they are relatively prime without introducing new algebraic extensions, using

$$c_1 \log pq + c_2 \log qr = c_1 \log p + (c_1 + c_2) \log q + c_2 \log r.$$

To understand the above procedure look at the following example.

Example We wish to compute $\int \frac{1}{x^3 + x}$. From earlier notation, here $a = 1$ and $b = x^3+x$. Since b is square-free and $deg(a) < deg(b)$, the integral has only logarithmic part.
By Rothstein-Trager method, we can write

$$\int \frac{1}{x^3 + x} = \sum_{i=1}^{n} c_i \log(v_i),$$

where c_i are distinct roots of $res_x(a - cb', b)$

$$\begin{aligned} res_x(a - cb', b) &= res_x \left(1 - c(3x^2 + 1), x^3 + x\right) \\ &= res_x(1 - 3x^2c - c, x^3 + x) \\ &= res_x \left((-3c)x^2 - (1 - c), x^3 + x\right) \end{aligned}$$

$$
\begin{aligned}
&= \begin{vmatrix} -3c & 0 & 1-c & 0 & 0 \\ 0 & -3c & 0 & 1-c & 0 \\ 0 & 0 & -3c & 0 & 1-c \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \end{vmatrix} \\
&= -4c^3 + 3c + 1 \\
&= -4(c-1)(c+\frac{1}{2})^2
\end{aligned}
$$

thus, we get $c_1 = 1$ and $c_2 = -1/2$ as distinct roots of $R(z)$. Now we find the values for corresponding v_i.

$$
\begin{aligned}
v_1 &= gcd(a - c_1 b', b) \\
&= gcd(-3x^2, x^3 + x) \\
&= x, \\
v_2 &= gcd\left(\frac{3}{2}x^2 + \frac{3}{2}, x^3 + x\right) \\
&= gcd\left(\frac{3}{2}(x^2+1), x(x^2+1)\right) \\
&= x^2 + 1,
\end{aligned}
$$

which implies,

$$
\begin{aligned}
\int \frac{1}{x^3 + x} &= c_1 \log(v_1) + c_2 \log(v_2) \\
&= \log x - \frac{1}{2}\log(x^2+1)
\end{aligned}
$$

4.2 The Risch Integration Algorithm

After examining the rational function case, now we are ready to deal with the integration of transcendental elementary functions through an effective decision procedure.

We consider functions $f \in K(x, \theta_1, \ldots, \theta_n)$ where K is the field of constants and each θ_i is transcendental logarithmic or exponential over $K(x, \theta_1, \ldots, \theta_{i-1})$. The decision procedure computes $\int f$ if it exists or prove that $\int f$ is not elementary. We view integrand f as a rational function in the extension $\theta = \theta_n$ and can write

$$f(\theta) = \frac{p(\theta)}{q(\theta)} \in F_{n-1}(\theta),$$

with $gcd(p, q) = 1$ and q monic. We discuss logarithmic and exponential extensions in different sections.

4.2.1 Logarithmic extension

First consider the case when θ is logarithmic over F_{n-1}. We proceed similarly as in Hermite's method. For simplicity we drop θ in notation and write $f(\theta) = f$ only. By Euclidean division, we can write

$$p = qs + r,$$

with $r = 0$ or $deg(r) < deg(q)$. Then

$$\int f = \int s + \int \frac{r}{q}.$$

Here we refer $\int s$ as *polynomial part* of f and $\int r/q$ as *rational part* of f. However, in this case computing polynomial part is not trivial, in fact it is harder than computing rational part.

Integration of the Rational Part

We continue with the Hermite's method. First we find the square-free factorisation of the denominator

$$q = \prod_{i=1}^{k} q_i^i,$$

where each q_i is monic, square-free and realtively prime, also $deg(q_k) > 0$. But notice, here square-free implies

$$gcd\left(q_i, \frac{d(q_i)}{d\theta}\right) = 1.$$

But, we need the stronger condition

$$gcd(q_i, q_i') = 1,$$

where $' = \frac{d}{dx}$. Fortunately, we find that this condition also holds.

Theorem 4.4. *Let F be a differential field with $x' = 1$ where $x \in F$. Let $F(\theta)$ be a differential extension field of F having the same subfield of constants where θ is transcendental logarithmic over F. Let $a \in F[\theta]$ be a monic polynomial with $deg(a) > 0$ such that*

$$gcd(a, \frac{da}{d\theta}) = 1.$$

Then

$$gcd(a, a') = 1.$$

Proof From chapter 3(see 3.1), we know $a' \in F[\theta]$. Let

$$a = \prod_{i=1}^{N} (\theta - a_i),$$

with each $a_i \in F_a$ (splitting field of a). Since a is square-free, all a_i are distinct. Then

$$a' = \sum_{i=1}^{N} \left(\frac{u'}{u} - a_i' \right) \prod_{j \neq i} (\theta - a_j),$$

for some $u \in F$. Observe if

$$\frac{u'}{u} - a_i' = 0$$

i.e $(\theta - a_i)' = 0$, which would mean $\theta - a_i$ is a constant but θ is transcendental over F. Hence, not possible.
Now, for a particular factor $\theta - a_i$, there is exactly one term in a' which is not divisible by $\theta - a_i$. This implies

$$gcd(a, a') = 1.$$

Now, we can continue with Hermite's method and after complete reduction we obtain

$$\int \frac{r}{q} = \frac{c}{d} + \int \frac{a}{b},$$

where $a, b, c, d \in F_{n-1}[\theta], deg(a) < deg(b), b$ is monic and square-free. From theorem 4.2, we have

$$d = gcd\left(q, \frac{dq}{d\theta}\right)$$
$$b = \frac{q}{d}$$

with $deg(a) < deg(b)$ and $deg(c) < deg(d)$.

Now it remains to compute the integral $\int a/b$. With the help of following theorem rational part is computed completely. We don't present the full proof as many things are similar to what we did for rational function case. We will prove only those which are non-trivial and refer to the previous section at appropriate places.

Theorem 4.5. *Let F be a field of elementary functions with constant field K. Let θ be transcendental and logarithmic over F and suppose that the transcendental elementary extension $F(\theta)$ has the same constant field K. Let $a/b \in F(\theta)$ with $a, b \in F[\theta], gcd(a, b) = 1, deg(a) < deg(b)$ and b is monic and square-free.*

i) $\int \frac{a}{b}$ *is elementary if and only if all the roots of the polynomial*

$$R(z) = res_\theta(a - zb', b),$$

are constants.

ii) If $\int \frac{a}{b}$ *is elementary, then*

$$\frac{a}{b} = \sum_{i=1}^{n} c_i \frac{v_i'}{v_i}, \qquad (4.4)$$

where c_i are the distinct roots of $R(z)$ and v_i are defined by

$$v_i = gcd(a - c_i b', b).$$

iii) Let L be the minimal algebraic extension field of F such that a/b can be expressed in the form 4.4 with constants $c_i \in L$ and $v_i \in L[\theta]$. Then $L = F(c_1, \ldots, c_m)$ where c_i are distinct roots of $R(z)$.

Proof Suppose $\int a/b$ is elementary, then by Liouville's theorem we can write

$$\frac{a}{b} = v_0' + \sum_{i=1}^{m} c_i \frac{v_i'}{v_i}, \tag{4.5}$$

with $c \in K^*$ and $v_i \in F^*(\theta)$ where K^* denotes the minimal algebraic extension of K required to express the integral and F^* denotes F with its constant field extended to K^*.
Observe that if we can prove $v_0' = 0$, then same proof will hold as in the case for rational function field $K(x)$. But, converse for part i) requires extra work.

Notice, with the help of rules for logarithms, we can always assume for $v_i (1 \leq i \leq m)$ are square-free, relatively prime polynomials in $F^*[\theta]$.
Let

$$v_0 = \frac{p}{q},$$

with $gcd(p, q) = 1$ and $deg(q) > 0$. Then

$$v_0' = \frac{p'q - q'p}{q^2},$$

which implies v_0' has a factor in its denominator which is not square-free. Since b is square-free, we conclude $deg(q) = 0$ and $v_0 \in F^*[\theta]$. But if v_0' is any non-zero polynomial, then right hand side of 4.13 will have a numerator of degree greater than or equal to degree of its denominator where on the other hand, $deg(a) < deg(b)$. Thus, we can say $v_0' = 0$ and now we can proceed similarly as in the case of $K(x)$ in earlier section.
Now, it remains to prove the converse for part i).
Suppose all the roots of $R(z)$ are constants. Let c_i be the distinct roots of $R(z)$ and define

$$v_i = gcd(a - c_i b', b).$$

Suppose for $i \neq j$

$$gcd(v_i, v_j) = w,$$

then

$$w \mid (a - c_i b'), \qquad w \mid (a - c_j b') \; and \qquad w \mid b,$$

which also implies

$$w \mid (c_i - c_j)b'.$$

Thus, $w \mid b$ and $w \mid b'$. But b is square-free and hence we conclude $w = 1$.
Since each $v_i \mid b$, it follows that

$$r \mid b.$$

where $r = \prod_{i=1}^{m} v_i$ and then $b = rs$ for some $s \in F(c_1, \ldots, c_m)[\theta]$. Suppose $deg(s) > 0$, then $res_\theta(a - zb', s)$ is a polynomial of positive degree, so let z_0 be a root of this polynomial. Then we have

$$gcd(a - z_0 b', s) \mid gcd(a - z_0 b', b) \tag{4.6}$$

since left hand side is non trivial, we conclude $res_\theta(a - zb', b) = 0$. But, then by assumption z_0 is one of c_i, say $c_1 = z_0$. We see that right hand side of 4.6 is v_1 which implies s and v_1 have a non trivial common divisor and hence $gcd(s, r) \neq 1$. But b is square-free , therefore it is not possible and hence $deg(s) = 0$. Since b and v_i are monic, we see $s = 1$ and

$$b = \prod_{i=1}^{m} v_i \tag{4.7}$$

Now define

$$\overline{a} = \sum_{i=1}^{m} c_i v_i' u_j \tag{4.8}$$

where $u_j = \prod_{j \neq i} v_j$. Since $b' = \sum_{i=1}^{m} v_i' uj$, we have for $1 \leq k \leq m$

$$\overline{a} - c_k b' = \sum_{i=1}^{m} (c_i - c_k) v_i' uj,$$

for $i = k$, above sum vanishes and it follows that

$$v_k \mid \overline{a} - c_k b'.$$

By definition

$$v_k \mid a - c_k b'.$$

From above two expressions, we conclude that

$$v_k \mid \overline{a} - a.$$

Now this holds for each k and since $gcd(v_i, v_j) = 1$ for $i \neq j$, we can say

$$b \mid \overline{a} - a \qquad\qquad (b = \prod_{i=1}^{m} v_i), \tag{4.9}$$

from 4.7 and 4.8, it follows that

$$deg(\overline{a}) < deg(b),$$

also, $deg(a) < deg(b)$. Then

$$deg(\overline{a} - a) < deg(b). \tag{4.10}$$

Finally, 4.9 and 4.10 implies $\overline{a} = a$ and from 4.7 and 4.8 it follows that

$$\frac{a}{b} = \sum_{i=1}^{m} c_i \frac{v_i'}{v_i},$$

and then clearly $\int a/b$ is elementary.

Following example is presented to understand how theorem works.

Example The logarithmic integral $\int 1/\log(x)$ is not elementary.
Suppose integral is elementary, then from above theorem it follows that

$$\frac{1}{\theta} = \sum_{i=1}^{m} c_i \frac{v_i'}{v_i},$$

where $\theta = \log(x)$ and c_i are distinct constants of $R(z)$. We find that

$$\begin{aligned} R(z) &= res_\theta(1 - z\theta', \theta) \\ &= res_\theta(1 - \frac{z}{x}, \theta) \\ &= 1 - \frac{z}{x} \in \mathbb{Q}(x)[z]. \end{aligned}$$

Since $R(z)$ has a non constant root, we conclude that the integral $\int 1/\log(x)$ is not elementary.

Integration of the Polynomial Part

Now it remains to compute the the polynomial part $\int s$. Let the integrand be

$$p = p_l\theta^l + \ldots + p_0,$$

where $p_i \in F_{n-1}$. If $\int p$ is elementary, then by Liouville's theorem, we can write

$$p = v_0' + \sum_{i=1}^{m} c_i \frac{v_i'}{v_i},$$

where $c_i \in \overline{K}$ and $v_i \in \overline{F}_{n-1}(\theta)$ for $1 \le i \le m$. Arguing as in Liouville's theorem, we can conclude that $v_0 \in \overline{F}_{n-1}[\theta]$ and $v_i \in \overline{F}_{n-1}$. From lemma 3.1, it follows that $deg(v_0) \le l+1$, so we write as

$$p_l\theta^l + \ldots + p_0 = (q_{l+1}\theta^{l+1} + \ldots + q_0)' + \sum_{i=1}^{m} c_i \frac{v_i'}{v_i},$$

where $p_i \in F_{n-1}, q_{l+1} \in \overline{K}$ and $q_i \in \overline{F}_{n-1}(0 \le i \le l)$. On differentiation, the right hand side becomes

$$\sum_{i=0}^{l} \left((i+1)q_{i+1}\theta' + q_i'\right)\theta^i + \sum_{i=1}^{m} c_i \frac{v_i'}{v_i}.$$

On comparing coefficients, we obtain

$$\begin{aligned} 0 &= q_{l+1}', \\ p_i &= (i+1)q_{i+1}\theta' + q_i' \qquad\qquad (1 \le i \le l), \\ p_0 &= q_1\theta' + q_0' + \sum_{i=1}^{m} c_i \frac{v_i'}{v_i}. \end{aligned}$$

First equation simply means

$$q_{l+1} = b_{l+1},$$

for some $b_{l+1} \in \overline{K}$. On substituting q_{l+1} in the second equation for $i = l$, we obtain

$$p_l = (l+1)p_{l+1}\theta' + q_l'.$$

On integration, we have

$$\int p_l = (l+1)b_{l+1}\theta + q_l.$$

In order to solve this equation , $\int p_l$ must satisfy following conditions:

i) $\int p_l$ is elementary.

ii) there is at most one log extension of $\overline{F}_{n-1}$ appearing in the integral.

iii) if a log extension of $\overline{F}_{n-1}$ appears in the integral, then it must be the particular one $\theta = \log u$.

If all the conditions are satisfied, then

$$\int p_l = c_l\theta + d_l,$$

for some $c_l \in \overline{K}$ and $d_l \in \overline{F}_{n-1}$, which implies

$$b_{l+1} = \frac{c_l}{l+1}, q_l = d_l + b_l,$$

where $b_l \in \overline{K}$ is an arbitrary constant of integration. Now by recursion, for $1 \leq i \leq l$ we find that

$$\int p_i = c_i\theta + d_i,$$

$$b_{i+1} = \frac{c_i}{i+1}, q_i = d_i + b_i.$$

Thus, our final equation becomes

$$p_0 = (d_1 + b_1)\theta' + q_0' + \sum_{i=1}^{m} c_i \frac{v_i'}{v_i}.$$

On rearranging and applying integration, we obtain

$$\int (p_0 - d_1\theta') = b_1\theta + q_0 + \sum_{i=1}^{m} c_i \log v_i.$$

This time we require only one condition that $\int (p_0 - d_1\theta')$ is elementary. So, let

$$\int (p_0 - d_1\theta') = d_0,$$

this implies b_1 (possibly zero) is the coefficient in d_0 of θ and

$$q_0 + \sum_{i=1}^{m} c_i \log v_i = d_0 - b_1\theta.$$

Thus, after the whole process, we obtain

$$\int p = b_{l+1}\theta^{l+1} + \ldots + q_1\theta + d_0 - b_1\theta.$$

Now, we present an example in order to grasp the computation mentioned above. In general, to integrate $\log(x)$, we simply use $by - parts$ method, but in the following example we use the above theory.

Example Consider the integral $\int \log(x)$.
Write $\theta = \log(x)$, then we are required to compute $\int \theta$. If the integral is elementary, then from the above discussion, it follows that

$$\int \theta = b_2\theta^2 + q_1\theta + \overline{q}_0,$$

where $\overline{q_0} = d_0 - b_1\theta$. After differentiation and comparing coefficients, we obtain

$$\begin{aligned} b_2' &= 0, \\ 1 &= 2b_2\theta' + q_1', \\ 0 &= q_1\theta' + \overline{q}_0'. \end{aligned}$$

From first equation, we see b_2 is some undetermined constant in the closure of rational $\overline{\mathbb{Q}}$. From integration of second equation, it follows that

$$b_2 = 0, q_1 = x + b_1.$$

Finally on substitution in the third equation, we obtain,

$$-x\theta' = b_1\theta' + \overline{q}_0'.$$

Using $\theta' = \dfrac{1}{x}$ and taking integration yields

$$-x = b_1\theta + \overline{q}_0,$$

which implies $b_1 = 0$ and $\overline{q}_0 = -x$. Notice, we ignored the constant of integration in the final step and hence our desired result is

$$\int \log(x) = x\log(x) - x$$

4.2.2 Exponential extension

We use the same notation as in previous sections but θ being exponential ($\theta'/\theta = u'$) this time. Similarly, here for any $f \in F_{n-1}(\theta)$, after euclidean division we have

$$f = s + \frac{r}{q}, \tag{4.11}$$

where $r, q, s \in F_{n-1}[\theta]$.
If we proceed with the Hermite's process, then we encounter following problem in this case. For square-free factorisation of the denominator $q = \prod_{i=1}^{k} q_i^i$

$$gcd(q_i, \frac{dq_i}{d\theta}) = 1 \text{ does not imply } gcd(q_i, q_i') = 1,$$

where $' = d/dx$. We check this by substituting $q_i = \theta$. For this, $gcd(\theta, 1) = 1$ whereas, $gcd(\theta, \theta u') = \theta$. Thus, in order to avoid this problem, we try to modify the decomposition 4.11. Write $q = \theta^l \overline{q}$ where $\theta \nmid \overline{q}$. Then, $gcd(\theta^l, \overline{q}) = 1$ and now we can apply theorem 4.1 to find $\overline{r}, w \in F_{n-1}[\theta]$ such that

$$\overline{r}\theta^l + w\overline{q} = r,$$

where $deg(\overline{r}) \leq \overline{q}$ and $deg(w) \leq l$. Then, equation 4.11 becomes

$$\begin{aligned} f &= s + \frac{w}{\theta^l} + \frac{\overline{r}}{\overline{q}} \\ &= \overline{s} + \frac{\overline{r}}{\overline{q}}, \end{aligned}$$

which is the required decomposition, where $\overline{s} = s + \theta^{-l}w$ is the new "polynomial" and $\int \overline{r}/\overline{q}$ is the new rational part.

Integration of the Rational Part

Theorem 4.6. *Let F be a differential field with $x' = 1$ where $x \in F$. Let $F(\theta)$ be a differential extension field of F having the same subfield of constants where θ is transcendental and exponential over F. Let $a \in F[\theta]$ be a monic polynomial with $deg(a) > 0$ and $\theta \nmid a$ such that*

$$gcd\left(a, \frac{da}{d\theta}\right) = 1.$$

Then, $gcd(a, a') = 1$.

Proof From chapter 3 (see 3.1), we know $a' \in F[\theta]$. Let

$$a = \prod_{i=1}^{N} (\theta - a_i),$$

with each $a_i \in F_a$ (splitting field of a). Since a is square-free, all a_i are distinct. Then

$$a' = \sum_{i=1}^{N} (u'\theta - a_i') \prod_{j \neq i} (\theta - a_j),$$

where $u \in F$. Now, for a particular factor $\theta - a_i$ there is exactly one term in a' which might be divisible by $\theta - a_i$ i.e it may happen

$$(\theta - a_i) \mid (u'\theta - a_i'),$$

this implies for some $p \in F[\theta]$, we have

$$u'\theta - a_i' = p(\theta - a_i),$$
$$u'\theta - a_i' = p\theta - pa_i.$$

On comparing, we get $p = u'$ which means $u'a_i = a_i'$. Since a_i is non zero, we can consider the following expression

$$\begin{aligned} \left(\frac{\theta}{a_i}\right)' &= \frac{a_i\theta' - a_i\theta}{a_i^2}, \\ &= \frac{a_i(u'\theta) - (u'a_i)\theta}{a_i^2}, \\ &= 0. \end{aligned}$$

So we infer, θ/a_i is a constant in $F_a(\theta)$ which contradicts the transcendency of θ. Thus, we get our desired result

$$gcd(a, a') = 1$$

.

Now, we can continue with the Hermite's reduction process to obtain

$$\int \frac{\overline{r}}{\overline{q}} = \frac{c}{d} + \int \frac{a}{b},$$

where $a, b, c, d \in F_{n-1}[\theta], deg(a) < deg(b)$ and b is a monic, square-free polynomial with $\theta \nmid b$. We use following theorem, in order to compute $\int a/b$.

Theorem 4.7. *Let F be a field of elementary functions with constant field K. Let θ be transcendental and logarithmic over F and suppose that the transcendental elementary extension $F(\theta)$ has the same constant field K. Let $a/b \in F(\theta)$ with $a, b \in F[\theta], gcd(a, b) = 1, deg(a) < deg(b)$ and b is monic and square-free with $\theta \nmid b$.*

i) $\int \frac{a}{b}$ *is elementary if and only if all the roots of the polynomial*

$$R(z) = res_\theta(a - zb', b),$$

are constants.

ii) If $\int \frac{a}{b}$ *is elementary, then*

$$\frac{a}{b} = g' + \sum_{i=1}^{n} c_i \frac{v_i'}{v_i}, \tag{4.12}$$

where c_i are the distinct roots of $R(z)$ and v_i are defined by

$$v_i = gcd(a - c_i b', b)$$

and $g \in F(c_1, \ldots, c_m)$ is defined by

$$g = -\left(\sum_{i=1}^{m} c_i deg(v_i)\right) u'.$$

iii) Let L be the minimal algebraic extension field of F such that a/b can be expressed in the form 4.12 *with constants $c_i \in L$ and $v_i \in L[\theta]$. Then, $L = F(c_1, \ldots, c_m)$ where c_i are distinct roots of $R(z)$.*

Proof Notice, as we did in logarithmic case, we only need to prove part i). Suppose $\int a/b$ is elementary, then by liouville's theorem we can write

$$\frac{a}{b} = v_0' + \sum_{i=1}^{m} c_i \frac{v_i'}{v_i}, \tag{4.13}$$

with $c \in K^*$ and $v_i \in F^*(\theta)$, where K^* denotes the minimal algebraic extension of K required to express the integral and F^* denotes F with its constant field extended to K^*. With the help of theorem 3.4 (see case II for detailed explanation), we can simplify it to

$$\frac{a}{b} = h_0' + \sum_{i=1}^{m} c_i \frac{v_i'}{v_i}, \tag{4.14}$$

where $h_0 \in F^*$. By similar argument presented in theorem 4.3, we can conclude that

$$b = \prod_{j=1}^{m} v_j.$$

Notice in equation 4.14, the left hand side a/b is a proper rational function. But this time $deg(v_i') = deg(v_i)$, therefore we need to arrange some terms accordingly. In order to do this, write h_0 in the form

$$h_0 = g + h,$$

for some $h \in F^*$ and g as defined above in the statement of the theorem. On substitution, we obtain

$$\frac{a}{b} = h' + \sum_{i=1}^{m} c_i \left(\frac{v_i'}{v_i} - deg(v_i)u' \right).$$

Here, second term on the right hand side is a proper rational function. On comparing, we see $h' = 0$ and thus we have proved the "if" part.

For the converse, Suppose all the roots of $R(z)$ are constants. Let c_i be the distinct roots of $R(z)$ and define

$$v_i = gcd(a - c_i b', b).$$

Suppose, for $i \neq j$

$$gcd(v_i, v_j) = w.$$

With the same reasoning as in theorem 4.5, we have

$$b = \prod_{i=1}^{m} v_i,$$

where v_i are monic and $gcd(v_i, v_j) = 1$ for $i \neq j$. Now define,

$$\overline{a} = g'b + \sum_{i=1}^{m} c_i v_i' u_j,$$

where $u_j = \prod_{j \neq i} v_j$. Since $b' = \sum_{i=1}^{m} v_i' uj$, we have for $1 \leq k \leq m$

$$\overline{a} - c_k b' = g'b + \sum_{i=1}^{m} (c_i - c_k) v_i' uj,$$

it follows that

$$v_k \mid \overline{a} - c_k b'.$$

By definition, $v_k \mid a - c_k b'$, consequently $v_k \mid \overline{a} - a$. Now, we are following the same procedure as in theorem 4.5. After routine process, we finally obtain

$$\frac{a}{b} = g' + \sum_{i=1}^{m} c_i \frac{v_i'}{v_i},$$

and then clearly $\int a/b$ is elementary.

Integration of the Polynomial Part

Here, we have "extended polynomial" rather than a simple polynomial as in earlier cases i.e $\overline{p} = \sum_{j=-k}^{l} p_j \theta^j$. If $\int \overline{p}$ is elementary, then by Liouville's theorem, we can write

$$\overline{p} = v_0' + \sum_{i=1}^{m} c_i \frac{v_i'}{v_i},$$

where $c_i \in \overline{K}$ and $v_i \in \overline{F}_{n-1}(\theta)$. From the discussion in Liouville's theorem and lemma 3.1, we can conclude that

$$\overline{p} = \left(\sum_{j=-k}^{l} q_j \theta^j \right)' + \sum_{i=1}^{m} c_i \frac{v_i'}{v_i}, \tag{4.15}$$

where $q_j \in \overline{F}_{n-1}, c_i \in \overline{K}$ and $v_i \in \overline{F}_{n-1}$. Notice that

$$(q_j \theta^j)' = (q_j' + ju' q_j)\theta^j, -k \leq j \leq l.$$

On equating coefficients in equation 4.15, we obtain following system of differential equations

$$p_j = q_j' + ju' q_j,$$

$$p_0 = \overline{q}_0',$$

where $\overline{q}_0 = q_0 + \sum i = 1^m c_i \log v_i, p_j \in \overline{F}_{n-1}, q_j \in \overline{F}_{n-1}$ and $\overline{q}_0 \in \overline{F}_{n-1}(\log v_1, \dots, \log v_m)$. From second equation, we have $\overline{q}_0 = \int p_0$. If $\int p_0$ is not elementary, then so is $\int \overline{p}$. Otherwise, we have our required $\overline{q}_0$. Now, for each $j \neq 0$, we solve a particular differential equation (Risch differential equation) of the form

$$y' + fy = g,$$

where f, g are given functions in F_{n-1} and we are required to determine a solution $y \in \overline{F}_{n-1}$. If any of the Risch differential equation fails to have a solution, then we conclude that $\int \overline{p}$ is not elementary. Otherwise, we have found

$$\int \overline{p} = \sum_{j \neq 0} q_j \theta^j + \overline{q}_0.$$

Example Consider the integral $\int x^x$. Write

$$x^x = e^{x \log(x)},$$

and then, we have

$$\int \theta_2 = q_1 \theta_2,$$

with $\theta_1 = \log(x)$, $\theta_2 = e^{x\theta_1}$ and $q_1 \in \mathbb{Q}(x, \theta_1)$. On differentiation, we obtain

$$\theta_2 = q_1' \theta_2 + q_1(\theta_1 + 1)\theta_2.$$

On comparing, we see

$$1 = q_1' + (\theta_1 + 1)q_1.$$

Moreover,

$$1 = q_1' + q_1, 0 = q_1$$

which has no solution. Therefore, we infer $\int x^x$ is not elementary.

Example We want to compute the integral of

$$f = \frac{-e^x - x + ln(x)x + ln(x)xe^x}{x(e^x + x)^2}.$$

The elementary field we obtain is $\mathbb{Q}(x)(\theta_1, \theta_2)$ with $\theta_1 = e^x, \theta_2 = ln(x)$, and so the integrand becomes

$$\frac{-\theta_1 - x + \theta_2 x + \theta_2 x \theta_1}{x(\theta_1 + x)^2} = -\frac{1}{x(\theta_1 + x)} + \theta_2 \frac{1 + \theta_1}{(\theta_1 + x)^2}.$$

We set $A_0 = -\frac{1}{x(\theta_1 + x)}$ and $A_1 = \frac{1 + \theta_1}{(\theta_1 + x)^2}$ and now we wish to compute

$$\int A_0 + A_1 \theta_2 = B_0 + B_1 \theta_2 + B_2 \theta_2^2.$$

After differentiation we obtain following equations:

$$\begin{aligned} 0 &= B_2' \\ A_1 &= B_1' + 2B_2\theta_2' = B_1' + 2B_2\frac{1}{x} \\ A_0 &= B_0' + B_1\theta_2' \end{aligned}$$

Thus B_2 is a constant. Integrating the second equation gives us that

$$\int A_1 dx - 2B_2\theta_2 = B_1 - b_1,$$

where b_1 is a constant. From our notation, we obtain

$$\int A_1 = \int \frac{1 + \theta_1}{(\theta_1 + x)^2} = \int \frac{1 + e^x}{(e^x + x)^2} = -\frac{1}{e^x + x} = -\frac{1}{\theta_1 + x}.$$

As no θ_2 term is involved we find that $B_2 = 0$ and we set $B_1 = \overline{B}_1 + b_1$ with $\overline{B}_1 = -\frac{1}{\theta_1 + x}$, and b_1 is an unknown constant.
Now we integrate the third equation. We get:

$$\int A_0 - B_1\theta_2' dx = \int A_0 - \overline{B}_1\theta_2' dx - b_1\theta_2 = B_0 + c.$$

(where we ignore the constant of integration c). Substituting values, we get

$$A_0 - \overline{B}_1\theta_2' = -\frac{1}{x(\theta_1 + x)} - \left(-\frac{1}{(\theta_1 + x)}\right)\frac{1}{x} = 0.$$

This shows that $b_1 = 0$ and also $B_0 = 0$. The integral thus is

$$\int f = \theta_2 B_1 = -\theta_2 \frac{1}{(\theta_1 + x)} = -ln(x) \frac{1}{e^x + x}.$$

Appendix A

Simple Extension

Here, we will prove a basic result on finite separable field extensions that was used in Chapter 2.

Theorem A.1. *Every finite separable extension is simple.*

Proof We prove the result when field k is infinite. Let K be an algebraic extension of k of degree n. Suppose $K = k(\alpha_1, \alpha_2, ..., \alpha_n)$ and let ϕ_i be the minimal polynomial of α_i over k, then $f = \prod_{i=1}^{n} \phi_i \in k[X] = K[X]$. Let N be the splitting field of $f(x)$ over k, then N will be a normal extension of k containing K as a subfield.
We know K over k is separable, therefore there exists n distinct k-isomorphism $\sigma_1, \sigma_2, ..., \sigma_n$ of K into N. Let $V_{i,j} = \{x \in K \mid \sigma_i(x) = \sigma_j(x)\}$ for $i \neq j$. $V_{i,j}$ is a proper subspace of k-vector space of K. For a vector space V over an infinite field K, we know that V cannot be the union of finite number of proper subspaces. Therefore, $\bigcup V_{i,j} \neq K$, hence there exists an element $\alpha \in K$ such that $\sigma_i(\alpha) \neq \sigma_j(\alpha)$ for all $i, j; i \neq j$. Since, K over k is algebraic extension, there exists a monic irreducible polynomial $f(x) \in k[x]$ such that $f(\alpha) = 0$. Say, $f(x) = x^m + a_{m-1}x^{m-1} + ... + a_0$ where $a_0, a_1, ..., a_{m-1} \in k$. We have,

$$f(\alpha) = 0$$
$$\alpha^m + a_{m-1}\alpha^{m-1} + ... + a_0 = 0$$

Apply σ_i on both the sides

$$\sigma_i(\alpha^m) + a_{m-1}\sigma_i(\alpha^{m-1}) + ... + \sigma_i(a_0) = 0$$
$$\sigma_i(\alpha)^m + a_{m-1}\sigma_i(\alpha)^{m-1} + ... + \sigma_i(a_0) = 0$$

$\sigma_i(\alpha)$ also satisfies $f(x)$ for all; hence, $f(x)$ has n distinct roots and therefore $k(\alpha) = K$.

Appendix B

Square Free Factorisation

We observed that polynomial factorisation plays a crucial role in symbolic integration. However, factoring a polynomial is not trivial and requires a lot of work. In this section, an algorithm for square-free factorisation is presented. This process reduces the factorisation problem to one of factoring those polynomials which are known to have no repeated factors.

Definition B.1. *Let $p(x)$ be a polynomial in $F(x)$ where F is any unique factorisation domain. Then $p(x)$ is square-free if it has no repeated roots.*

Definition B.2. *A square-free factorisation of p is a factorisation $p = \prod_{i=0}^{k}(p_i)^i$ such that all p_i are square-free and $gcd(p_i, p_j) = 1$ for $i \neq j$.*

Theorem B.1. *A polynomial p in a differential field $F(x)$ is square-free if and only if $gcd(p, p') = 1$.*

Proof If p is square-free, with irreducible factorisation

$$p = p_1 p_2 \dots p_k$$

Since p is square-free, we notice all p_i are distinct, on differentiation

$$p' = p_1' \prod_{i=2}^{k} + \dots + \prod_{i=1}^{k-1} p_k'$$

Now suppose on the contrary $gcd(p, p') \neq 1$, then any non trivial divisor of p and p' must be a multiple of some p_i. Without loss of generality we can assume p_1 is such a p_i. Then p_1 must divide p_1' which is not possible, since $deg(p_1) > deg(p_1')$. Thus,

$gcd(p, p') = 1$. For the converse, assume p has some repeated factor, so we can write $p = q^n h$ for $n > 1$ with some q and $h \in F(x)$. Then $p' = nq^{n-1}q'h + q^n h$, so q^{n-1} is the common factor for p and p' which contradicts our hypothesis.

For a polynomial p, it is difficult to compute the full factorisation in to irreducibles, However, we will present that it is not so difficult to compute the square-free factorisation.

Let $p \in F(x)$ with square-free factorisation $p = p_1^1 p_2^2 \dots p_k^k$, now we will present a method to compute all p_i. We have,

$$p' = \sum_{i=1}^{k} \left(i p_i^{i-1} p_i' \prod_{i \neq j} p_j^j \right)$$

Let

$$a_1 = gcd(p, p') = p_2 p_3^2 \dots p_k^{k-1}$$

Divide p by a_1

$$b_1 = \frac{p}{a_1} = p_1 p_2 \dots p_k$$

find

$$c_1 = gcd(a_1, b_1) = p_2 p_3 \dots p_k$$

this implies

$$p_1 = \frac{b_1}{c_1}$$

In similar manner, define

$$a_2 = \frac{a_1}{c_1}$$

$$b_2 = c_1$$

Now, we can iterate

$$c_i = gcd(a_i, b_i)$$

$$a_{i+1} = \frac{a_i}{c_i}$$

$$b_{i+1} = c_i$$

Thus, the square-free factors p_i are given by $p_i = \dfrac{b_i}{c_i}$.

Appendix C

Resultant

In this section, we provide essential background knowledge required for computations carried out in Chapter 4. We define the *resultatnt* of polynomials and state few important theorems that have been used earlier in this thesis. Throughout this section, R is an UFD.

Definition C.1. *Let* $A(x) = \sum_{i=0}^{m} a_i x^i$ *and* $B(x) = \sum_{i=0}^{n} b_i x^i$ *be non-zero polynomials in* $R[x]$*. Then the* ***Sylvester matrix*** *of* A *and* B *is the* $m+n$ *by* $m+n$ *matrix*

$$\begin{pmatrix} a_m & a_{m-1} & \dots & a_1 & a_0 & & & \\ & a_m & a_{m-1} & \dots & a_1 & a_0 & & \\ & & & \dots & \dots & \dots & & \\ & & & & a_m & & \dots & a_0 \\ b_n & \dots & \dots & \dots & b_1 & b_0 & & \\ & b_n & \dots & \dots & \dots & b_1 & b_0 & \\ & & \dots & \dots & \dots & \dots & & \\ & & & b_n & \dots & \dots & \dots & b_0 \end{pmatrix}$$

where the first n *rows consists of coefficients of* $A(x)$ *and the remaining* m *rows consists of coefficients of* $B(x)$ *while the missing entries are zero. The determinant of the Sylvester matrix is defined as the* ***resultant*** *of* $A(x)$ *and* $B(x)$ *and is denoted by* $res(A, B)$*.*

Theorem C.1. *Let* $f(x) = a_m \prod_{i=1}^{m}(x - \alpha_i)$ *and* $g(x) = b_n \prod_{i=1}^{n}(x - \beta_i)$ *be polynomials over an integral domain* D *with indeterminates* α_i, β_i*. Then*

$$res(f, g) = a_m^n b_n^m \prod_{i=1}^{m} \prod_{j=1}^{n} (\alpha_i - \beta_j)$$

We omit the proof here. The interested reader may consult [4] for the proof and more elaborate reading on the resultant. The following corollary is an immediate consequence of the theorem $C.1$

Corollary C.1. *(Sylvester's Criterion) Let* $A(x)$ *and* $B(x) \in R[x]$*. Then* $A(x)$ *and* $B(x)$ *have a non trivial common factor if and only of* $res(A, B) = 0$*.*

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