A study of real special 2-groups using quadratic maps over fields of characteristic 2

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A thesis submitted for the partial fulfilment
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dedicated

to

my parents

Declaration

The work presented in this thesis has been carried out by me under the guidance of Dr. Amit Kulshrestha at Indian Institute of Science Education and Research Mohali.

This work has not been submitted in part or in full for a degree, a diploma, or a fellowship to any other university or institute. Whenever contributions of others are involved, every effort is made to indicate this clearly, with due acknowledgement of collaborative research and discussions. This thesis is a bonafide record of original work done by me and all sources listed within have been detailed in the bibliography.

Date:
Place:

Dilpreet Kaur

In my capacity as the supervisor of the candidate's thesis work, I certify that the above statements by the candidate are true to the best of my knowledge.

Dr. Amit Kulshrestha
(Supervisor)

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Contents

List of notations

$\mathbb{F}$	:	A field.
$\mathbb{F}_2$	:	The field containing 2-elements.
$\mathbb{C}$	:	The field of complex numbers.
$\mathbb{R}$	:	The field of real numbers.
$\mathbb{Q}$	:	The field of rational numbers.
$\mathbb{H}$	:	The quaternion algebra $\left(\frac{-1,-1}{\mathbb{Q}}\right)$.
$\lvert S\rvert$	:	Number of elements in a set S.
$\dim_{\mathbb{F}} V$	:	The dimension a vector space V over the field $\mathbb{F}$.
$\mathrm{Hom}_{\mathbb{F}}(V,\mathbb{F})$	:	The set of all $\mathbb{F}-$linear maps from V to $\mathbb{F}$.
q	:	A quadratic map.
b_q	:	Polar map associated to a quadratic map q.
$\mathrm{rad}\, b_q$	:	Radical of quadratic form q.
$[0,0]$	:	2-dimensional quadratic form $q(x,y)=xy$.
$[1,1]$	:	2-dimensional quadratic form $q(x,y)=x^2+xy+y^2$.
$\mathrm{Arf}(q)$	:	Arf Invariant of a quadratic form q.
$\mathrm{Quad}(V,W)$	:	Set of quadratic maps from V to W.
$q_1 \perp q_2$	:	Orthogonal sum of quadratic forms q_1 and q_2.
$q_1 \perp_\theta q_2$	:	A quadratic map defined by $q_1 \perp_\theta q_2(v_1,v_2)=\theta(q_1(v_1))+q_2(v_2)$.
$\mathrm{tr}(A)$	:	Trace of a matrix A.
$A \otimes B$	:	Tensor product of matrices A and B.
$\mathrm{C_n}$	:	Cyclic group of order n.
D_n	:	Dihedral group of order $2n$.
Q_n	:	Quaternion group of order $4n$.
$\mathrm{QD_n}$	:	Quasi dihedral group of order n.
$\mathrm{GL}_n(V)$	:	General linear group of $n\times n$ matrices.
$\left(\frac{\mathbb{Z}}{2\mathbb{Z}}\right)^n$	:	Elementary abelian group of order 2^n.
$V \dot{\times} W$	:	A group defined in Remark 1.3.10.
$Z(G)$	:	Center of a group G.
G'	:	Derived subgroup of a group G.
$\Phi(G)$	:	Frattini subgroup of a group G.

$\ker(\rho)$	: Kernel of a homomorphism ρ.
$\mathrm{H}^2(\mathrm{V},\mathrm{W})$	: Second cohomology group of V with coefficients in W.
c	: A normal 2-cocycle.
$s \circ c$	: Transfer of a normal 2-cocycle c by linear map s.
$\mathrm{Inf}(c)$	: Inflation of normal 2-cocycle.
$G \times H$	: Direct product of groups G and H.
$(G)^n$	: Direct product of n copies of group G.
$G \circ_\theta H$	: Central product of groups G and H with identification θ.
$G^{(n)}$	: Central product of n copies of group G.
$G \rtimes H$	: Semidirect product of groups G and H.
$\rho \times \sigma$	: Tensor product of representations ρ and σ.
$\hat{\rho}$	: Representation of quotient group induced from a representation ρ.
$\ker(\chi)$	: Kernel of a character χ.
$\mathrm{sign}\,\chi$	: Sign of a character χ.
$\nu(\chi)$	: The Schur indicator of a character χ.
$\mathbf{Z}(\chi)$	: $\{g \in G \ : \ \chi(g) \neq 0\}$ for a character χ of group G.
$s \circ q$	: Transfer of a quadratic map q by linear map s.
V_s	: $\frac{V}{\mathrm{rad}_{s\circ q}}$.
ϵ_s	: The canonical surjection from V to V_s.
q_s	: The regular quadratic form induced from $s \circ q$.
G_s	: The extraspecial 2-group associated to q_s.
ϕ_s	: The unique representation of degree at least 2 of extraspecial 2-group G_s.
χ_s	: The unique character of degree at least 2 of extraspecial 2-group G_s.
$\mathcal{C}$	: Conjugacy class of real special 2-group.
$\mathcal{S}_v$	: $\{s \in \mathrm{Hom}_{\mathbb{F}_2}(Z(G),\mathbb{F}_2) \ : \ v \in \mathrm{rad}(b_{s\circ q})\}$.
$\mathbb{F}[G]$	: $\mathbb{F}$-group algebra of a group G.
$\mathbb{C}[G]$	: Complex group algebra of a group G.
$\mathbb{Q}[G]$	: Rational group algebra of a group G.
$\mathbb{Q}[G].G'$	: Commutative part of rational group algebra of $\mathbb{Q}[G]$.
$\Delta(G,G')$	: Non-commutative part of rational group algebra of $\mathbb{Q}[G]$.
$\widehat{H}$	: An element $\frac{1}{H}\sum_{h\in H} h$ of the group algebra $\mathbb{F}[G]$ for a subgroup H of G.

$\mathbb{Q}(\chi)$	:	The field obtained by adjoining $\chi(g); g \in G$ to $\mathbb{Q}$.
$\mathrm{Gal}(\mathbb{Q}(\chi)/\mathbb{Q})$	:	The Galois group of field extension $\mathbb{Q}(\chi)$ of $\mathbb{Q}$.
$e(\chi)$	:	The primitive central idempotent of $\mathbb{C}[G]$ corresponding to χ.
$e_{\mathbb{Q}}(\chi)$	:	The primitive central idempotent of $\mathbb{Q}[G]$ corresponding to χ.
$\mathrm{tr}(e)$	:	Trace of an idempotent e.
G_{64}	:	The special 2-group associated to $q(w,x,y,z) = (z^2 + wx + wz + xy, wy)$.
G_{128}	:	The special 2-group associated to $q(w,x,y,z) = (wx + yz, wy, xy)$.
G_{256}	:	The special 2-group associated to $q(w,x,y,z,t) = (wx + wt + yz, wy, wt + xy)$.
G_{512}	:	The special 2-group associated to $q(w,x,y,z,t) = (wx + yz, wy, xy, wt)$.

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Introduction

In this thesis main objects of study are special 2-groups. A finite 2-group G is called a *special* 2-*group* if its commutator subgroup, the Frattini subgroup and the center, all three coincide and are isomorphic to an elementary abelian group. Nonabelian groups of order 8 are the simplest examples of special 2-groups. Much sophisticated examples of special 2-groups include Sylow 2-subgroups of Suzuki groups. We derive our interest in special 2-groups for the following reason.

- Special 2-groups can be described in terms of quadratic maps between vector spaces over fields of characteristic 2. This allows one to use the theory of quadratic forms over such fields to study special 2-groups. Therefore it is interesting to identify group theoretic properties which can be studied this way.

Zahinda in [Zah11] used quadratic maps to study totally orthogonal special 2-groups. In this thesis we explore it further to study various representation theoretic aspects of special 2-groups.

Study of determining the types of complex representations of groups is related to, but not equivalent to, the study of strongly real groups. For example in [Gow76], Gow mentioned that central product of dihedral group of order 8 and quaternion group of order 8 is a strongly real group with symplectic representation. In this thesis we grossly generalize this example and produce examples of strongly real special 2-groups which afford symplectic complex representations. We also tackle the question in the reverse direction and produce examples of real special 2-groups supporting orthogonal complex representations only, without being strongly real. These phenomena are not commonly exhibited by groups. That makes special 2-groups interesting.

It turns out that many representation theoretic properties of special 2-groups can be studied using the description of such groups in terms of quadratic maps. In particular, in this thesis we obtain methods to determine complex representations, characters, conjugacy classes and rational Wedderburn decomposition of real special 2-groups using quadratic maps between vector spaces over fields of characteristic 2.

Organization of the thesis

This thesis is organized into two parts. In first part we summarize results which were already known before and in the second part we describe results obtained in the thesis. There are six chapters, three chapters in each part, altogether.

In chapter 1, we discuss the basic results of theory of quadratic forms over fields of characteristic 2 and representation theory. We also discuss the connection between quadratic maps over fields of characteristic 2, special 2-groups and second cohomology groups. Since extraspecial 2-groups are building blocks of real special 2-groups, in chapter 2, we collect the results for extraspecial 2-groups such as their classification, representations and more. The chapter 3 deals with characterization of real special 2-groups and characterization of totally orthogonal special 2-groups.

In chapter 4, we give the infinite class of groups for which neither the notion of strong reality and total orthogonality implies the other. In chapter 5, we provide a method to construct the character table of real special 2-groups using only the quadratic map associated to the group. The chapter 6 concerns with the Wedderburn decomposition of rational group algebra of real special 2-groups.

Definitions

The aim of this introductory section on definitions is to briefly recall terminology that is going to be used throughout the thesis. These definitions will be explained in much detail later in the thesis chapters as and when required. A separate list of notations is provided soon after table of contents.

Let G be a finite group. A group G is said to be *real* if every element of G conjugate to its inverse. An element $g \in G$ is called *strongly real* if either g is of order 2 or equals the product of two elements of order 2. A group is called *strongly real* if all its elements are strongly real. It is evident that every strongly real group is real.

A complex representation $\rho : G \to \mathrm{GL}(U)$ is said to be *real* if its associated character is real valued. It is said to be *realizable over* $\mathbb{R}$ if the vector space U admits an $\mathbb{R}$-subspace U_0 such that $U = U_0 \otimes_{\mathbb{R}} \mathbb{C}$ and U_0 is stable under $\rho(G)$. If a complex representation is realizable over $\mathbb{R}$ then the corresponding character is real valued. A group is real if and only if all its representations are real. However, a real group may admit complex representations which are not realizable over $\mathbb{R}$. Such representations of real groups are called *symplectic*. A real representation which is realizable over $\mathbb{R}$ is called an *orthogonal* representation. A group is said to be *totally orthogonal* if its all representations are orthogonal.

Let $\mathbb{K}$ be a field and G be a group. We denote by $\mathbb{K}[G]$ the set of all formal finite linear combinations of the form $\alpha = \sum_{g \in G} \alpha_g g$ for $\alpha_g \in \mathbb{K}$ and $g \in G$. The set $\mathbb{K}[G]$, with component wise addition and with multiplication derived from the multiplication of group G forms an algebra. We call $\mathbb{K}[G]$ the *group algebra of G over* $\mathbb{K}$.

The element e of $\mathbb{K}[G]$ is called *idempotent* if $e^2 = e$. An idempotent e is called *primitive central idempotent* if it lies in the center of $\mathbb{K}[G]$ and it can not be written as $e = e' + e''$, where e' and e'' are non zero idempotents such that $e'e'' = 0$.

A theorem of Maschke states that for a finite group G and the field $\mathbb{K}$, the group algebra $\mathbb{K}[G]$ is semisimple if and only if $\mathrm{char}(\mathbb{K})$ does not divide $|G|$. The decomposition of semisimple group algebra $\mathbb{K}[G]$ as direct sum of its simple ideals is called the *Wedderburn decomposition of* $\mathbb{K}[G]$.

Let $\mathbb{F}$ be a field of characteristic 2. Let V and W be vector spaces over $\mathbb{F}$. A map $q : V \to W$ is called a *quadratic map* if $q(\alpha v) = \alpha^2 q(v)$ for all $v \in V$ and $\alpha \in \mathbb{F}$ and the map $b_q : V \times V \to W$ given by $b_q(v, w) := q(v + w) - q(v) - q(w)$ is $\mathbb{F}$-bilinear for all $v, w \in V$. The pair (V, q) is called a *quadratic space* over $\mathbb{F}$. The subspace $\mathrm{rad}(b_q) := \{w \in V : b_q(v, w) = 0 \; \forall v \in V\}$ is called the *radical* of (V, q). The quadratic space (V, q) is called *regular* if $\mathrm{rad}(b_q) = 0$.

Main results

In this section we state the main results that have been obtained in this thesis.

It is well known that the number of real irreducible representations of a group equals the number of its real conjugacy classes [[JL01], Th. 11.12]. However, a natural bijection between real irreducible representations and real conjugacy classes remains an open problem.

According to a conjecture of Tiep a finite simple group is strongly real if and only if it is totally orthogonal. There are many classes of groups which are strongly real as well as totally orthogonal and those which are neither. For example:

1. A symmetric groups S_n is strongly real as well as totally orthogonal.

2. Alternating groups A_n are real if and only if $n = 1, 2, 5, 6, 10, 14$ [Ber69, Th. 1.2]. All real classes of A_n are strongly real [Sul08, §3, corollary 3]. Moreover its all real representations are totally orthogonal [Tur92, Th. 1.1].

3. All real classes of general linear group $\mathrm{GL}_n(q)$ are strongly real [Won66, Th. 1] and all its real representations are orthogonal [Pra98, Th. 4]. Therefore in $\mathrm{GL}_n(q)$, the number of conjugacy classes of strongly real elements is same as the number of orthogonal characters.

4. The special linear group $\mathrm{SL}_2(q)$ is strongly real as well as totally orthogonal when q is even and it is neither strongly real nor totally orthogonal when q is odd [KS11, 5.3].

5. The orthogonal group $\mathrm{O}_n(q)$ is strongly real [Won66] as well as totally orthogonal [Gow85, Th. 1].

6. A real group whose Sylow 2-subgroups are abelian is strongly real as well as totally orthogonal. [Arm96, Cor. 11, Th. 12]

In thesis we exhibit an infinite class of groups, for which neither of the notions of strong reality and total orthogonality imply the other. These examples lie in the class of special 2-groups.

Let $\mathbb{F}_2$ denote the field with two elements and G be a special 2-group. Then the center $Z(G)$ and the quotient group $\frac{G}{Z(G)}$ are elementary abelian 2-groups. We can therefore consider them as vector spaces over the field $\mathbb{F}_2$. The map $q : \frac{G}{Z(G)} \to Z(G)$ given by $q(gZ(G)) = g^2$ for all $g \in G$ is a regular quadratic map [[Zah08], Th. 3.4.11]. This map is called the quadratic map associated to the special 2-group. Conversely, for a regular quadratic map $q : V \to W$ such that $\langle b_q(V \times V)\rangle = W$ there exist a special 2-group, unique up to isomorphism, such that $V = \frac{G}{Z(G)}$ and $W = Z(G)$ [[Zah11], Th. 1.4]. This group is called *special 2-group associated to the quadratic map $q : V \to W$.*

As a set special 2-group associated to the quadratic map $q : V \to W$ is same as $V \times W$ and its group operation is given in remark 1.3.10.

The theory of quadratic maps over fields of characteristic 2 plays an crucial role in checking properties of special 2-groups. The one to one correspondence between regular quadratic maps with special 2-groups is utilized to study the properties of these groups. The first result obtained in the thesis is the characterization of strongly real special 2-groups.

Theorem 1. *Let $q : V \to W$ be a regular quadratic map with $\langle b_q(V \times V)\rangle = W$ and G be the special 2-group associated to q such that $V = \frac{G}{Z(G)}$, $W = Z(G)$ and $q(xZ(G)) = x^2$. Then G is strongly real if and only if for every nonzero $v \in V$ there exists $a \in V$ with $v \neq a$ and $q(a) = q(a - v) = 0$.*

In [Zah11], Zahinda gives a characterization of totally orthogonal special 2-groups. We use this characterization (see Th. 3.2.7) and Th. 1 to get the following results:

Theorem 2. *For every $m \geq 5$ there exist special 2-groups of order 2^m which are strongly real but not totally orthogonal.*

Theorem 3. *For every $m \geq 7$ there exist special 2-groups of order 2^m which are totally orthogonal but not strongly real.*

The following lemma is useful in achieving this result.

Lemma 4. *Let G_1 and G_2 be special 2-groups such that $Z(G_1)$ is isomorphic to $Z(G_2)$. Let $q_1 : V_1 \to W_1$ and $q_2 : V_2 \to W_2$ be the regular quadratic maps associated to special 2-groups G_1 and G_2, respectively. Let $\theta : W_1 \to W_2$ be an isomorphism of $\mathbb{F}_2$-vector spaces.*

Then $q_1 \perp_\theta q_2 : V_1 \oplus V_2 \to W_2$ *defined by* $(q_1 \perp_\theta q_2)(v_1, v_2) = \theta(q_1(v_1)) + q_2(v_2)$ *is a regular quadratic map and the group associated to* $q_1 \perp_\theta q_2$ *is* $G_1 \circ_\theta G_2$*, where* $G_1 \circ_\theta G_2$ *denotes the central product of* G_1 *and* G_2 *with their centers identified through isomorphism* θ*.*

We notice that the quadratic map associated to special 2-groups gives a lot of information about the types of representations of these groups. This motivates us to explore the description of special 2-group as quadratic map to provide a method for constructing the complex character table of a real special 2-groups. Our methods to compute representations, characters and conjugacy classes can be implemented directly on the quadratic maps associated to special 2-groups. These methods are based on the understanding of representations of extraspecial 2-groups.

Let us fix some notations required to state the result. Let G be a special 2-group and $q : V \to W$ be the quadratic map associated to G. For a non-zero $s \in \mathrm{Hom}_{\mathbb{F}_2}(W, \mathbb{F}_2)$ the composition $s \circ q$ is called the *transfer of* q *by* s. If $s \circ q$ vanishes on $\mathrm{rad}(b_{s\circ q})$ then $s \circ q$ induces a regular quadratic form $q_s : V/\,\mathrm{rad}(b_{s\circ q}) \to \mathbb{F}_2$ [[Zah11] Prop. 1.5]. We denote by G_s the extraspecial 2-group associated to q_s. Following proposition is a refinement in a result of Zahinda [Zah11, Prop. 3.3].

Proposition 5. *Let* G *be a real special* 2*-group and* $q : V \to W$ *be the quadratic map associated to* G*. Then for every non-zero* $s \in \mathrm{Hom}_{\mathbb{F}_2}(W, \mathbb{F}_2)$ *there exist exactly* $|(\mathrm{rad}(b_{s\circ q}))|$ *number of inequivalent irreducible representations* φ *of degree at least* 2 *of* G *such that* $\varphi(G) = G_s$*.*

It is well known result that every extraspecial 2-group has a unique faithful representation of degree at least 2. We construct $|(\mathrm{rad}(b_{s\circ q}))|$ number of surjective homomorphism from real special 2-group G to extraspecial 2-group G_s. Then we compose these homomorphisms with unique representation of degree at least 2 of G_s to get the inequivalent irreducible representations of degree at least 2 of G, whose images are isomorphic to G_s. Therefore representations of degree at least 2 of real special 2-groups associated to a quadratic map $q : V \to W$ are indexed by $(\mathrm{Hom}_{\mathbb{F}_2}(W, \mathbb{F}_2)\backslash\{0\}) \times \mathrm{rad}(b_{s\circ q})$.

For a non-zero $s \in \mathrm{Hom}_{\mathbb{F}_2}(W, \mathbb{F}_2)$, let $\varphi_{s,i}$; $1 \leq i \leq |\,\mathrm{rad}(b_{s\circ q})|$ be inequivalent irreducible representations φ of degree at least 2 such that $\varphi(G) = G_s$. Let $\chi_{s,i}$ be the character afforded by the representation $\varphi_{s,i}$. As a set special 2-group G is same as

$V \times W$. We denote the elements of G by $(v, w) \in V \times W$. The following theorem computes the character $\chi_{s,i}$ at all elements of G.

Theorem 6. *Let G be a real special 2-group and $q : V \to W$ be the quadratic map associated to G. Let $s \in \mathrm{Hom}_{\mathbb{F}_2}(W, \mathbb{F}_2)$. Then*

1. *$\chi_{s,i}(v, w) \neq 0$ if and only if $v \in \mathrm{rad}(b_{s \circ q})$.*

2. *For $(0, w) \in G$ we have*

$$\chi_{s,i}(0, w) = \begin{cases} 2^l & \text{if } s(w) = 0 \\ -2^l & \text{if } s(w) = 1 \end{cases}$$

   where l is defined by $|G_s| = 2^{2l+1}$.

3. *Let $\{v_1, v_2, \cdots v_k\}$ be a basis of $\mathrm{rad}(b_{s \circ q})$. Then*

$$\chi_{s,i}(v_j, 0) = \begin{cases} -2^l & \text{if } A_{j,i} = 1 \\ 2^l & \text{if } A_{j,i} = 0 \end{cases}$$

   where $A_{j,i}$ denotes the coefficient of 2^j in the binary expansion $i - 1 = \sum_{j=0}^{k-1} A_{j,i} 2^j$.

4. *Let $g \in G$ be an element with $g = \prod_{j=1}^{r} (v_{i_j}, 0)(0, w)$ for $1 \leq i_1 < i_2 < \cdots < i_r \leq k$ then*

$$\chi_{s,i}(g) = \prod_{j=1}^{r} \mathrm{sign}(\chi_{s,i}(v_{i_j}, 0)).\, \mathrm{sign}(\chi_{s,i}(0, w)).2^l$$

The proof of above theorem follows from the following two facts:

1. The character $\chi_{s,i}$ of degree at least 2 of G is composition of the homomorphism from G to G_s and the unique character of degree at least 2 of extraspecial 2-group G_s.

2. The character of degree at least 2 of an extraspecial 2-group vanishes outside the center of the group.

We know that characters of a group are class functions. To complete the character table of a group, we need to know the conjugacy classes of group. It is good to have description of conjugacy classes of real special 2-groups in terms of quadratic maps. The following result gives us the same.

Theorem 7. *Let G be real special 2-group and $q : V \to W$ be quadratic map associated to G.*

1. *Let $v \in V$. If $v \notin \operatorname{rad}(b_{s \circ q})$ for all non-zero linear maps $s : W \to \mathbb{F}_2$ then $\{(v, w) : w \in W\}$ is a conjugacy class of G.*

2. *Let $v \in V$ and $\mathcal{S}_v := \{s \in \operatorname{Hom}_{\mathbb{F}_2}(Z(G), \mathbb{F}_2) : v \in \operatorname{rad}(b_{s \circ q})\}$. Then the conjugacy class of element $(v, w) \in G$ is $\{(v, w + w') : s(w') = 0$ for all $s \in \mathcal{S}_v\}$.*

We know that for two element g, h of a group G and for all irreducible characters χ of G, $\chi(g) = \chi(h)$ if and only if g and h are conjugate elements in G. We use this fact and Th. 6 to prove Th. 7.

In the thesis we also establish the utility of the quadratic map associated to a real special 2-group in determining the Wedderburn decomposition of its rational group algebra.

It is well known fact that there is one-to-one correspondence between irreducible characters of a group and primitive central idempotents of its group algebra [Yam74]. The determination of Wedderburn decomposition involves the computation of primitive central idempotents. We denote the primitive central idempotent of $\mathbb{Q}[G]$ by $e_{\mathbb{Q}}(\chi)$, where χ is an irreducible character of G. For a finite subgroup H of G, we denote $\widehat{H} := \dfrac{1}{|H|} \sum_{h \in H} h \in \mathbb{Q}[G]$.

Proposition 8. *Let G be real special 2-group.*

1. *Let χ be a one dimensional character of G. If χ is trivial character then $e_{\mathbb{Q}}(\chi) = \widehat{G}$, otherwise $e_{\mathbb{Q}}(\chi) = \widehat{\ker(\chi)} - \widehat{G}$.*

2. *Let χ be a character of degree at least 2 of G and $\mathbf{Z}(\chi) := \{g \in G : |\chi(g)| = \chi(1)\}$. Then $e_{\mathbb{Q}}(\chi) = \widehat{\ker(\chi)} - \widehat{\mathbf{Z}(\chi)}$.*

The proof of above result is very simple. It follows from the definition of primitive central idempotents using the properties of real special 2-groups and of its characters.

To determine Wedderburn decomposition of rational group algebra of real special 2-groups, we use Wedderburn decomposition of rational group algebra of extraspecial 2-groups, which is known [[VL06], Prop. 3.4]. The non-commutative part $\Delta(G, G')$ of the decomposition is given by Prop. 2.3.6. The following result gives the Wedderburn decomposition of rational group algebra $\mathbb{Q}[G]$ of real special 2-group G.

Theorem 9. *Let G be a real special 2-group and $|G| = 2^n, |Z(G)| = 2^m$. Let q be quadratic map associated to group G and $s_j : Z(G) \to \mathbb{F}_2, 1 \leq j \leq 2^m - 1$ be non-zero linear maps and q_{s_j} is regular quadratic form. Let $G_{s_j}, 1 \leq j \leq 2^m - 1$ be extraspecial 2-groups associated to quadratic forms $q_{s_j}, 1 \leq j \leq 2^m - 1$ and $|G_{s_j}| = 2^{2l_j+1}$. Then*

$$\mathbb{Q}[G] \cong 2^{n-m}\mathbb{Q} \oplus \bigoplus_{j=1}^{2^m-1} 2^{n-m-2l_i} \Delta(G_{s_j}, G_{s'_j})$$

Here $\Delta(G_{s_j}, G'_{s_j})$ denotes the non commutative part of Wedderburn decomposition of $\mathbb{Q}[G_{s_j}]$.

The crucial step in the proof of above theorem is the computation that establishes that the ideal of $\mathbb{Q}[G]$ generated by idempotent $e_{\mathbb{Q}}(\chi)$ is isomorphic to the ideal of $\mathbb{Q}[G_{s_j}]$ generated by the idempotent $e_{\mathbb{Q}}(\chi_{s_j})$. Here $e_{\mathbb{Q}}(\chi)$ and $e_{\mathbb{Q}}(\chi_{s_j})$ denote the idempotent corresponding to character χ of group G and character χ_{s_j} of degree at least 2 of group G_{s_j}.

Another interesting problem in the theory of group rings is whether the rational group algebra $\mathbb{Q}[G]$ determines up to isomorphism, the group G. In [San81], it is mentioned that the isomorphism problem was formulated by G. Higman for Integral group rings in his PhD thesis [Hig40]. The rational group algebra of extraspecial 2-groups determines the group up to isomorphism [VL06]. As consequence of Th. 9 we show that this is not the case for real special 2-groups. We explicitly show this by giving the examples of two real special 2-groups G_1 and G_2 such that $\mathbb{Q}[G_1] \cong \mathbb{Q}[G_2]$ but $G_1 \not\cong G_2$.

Part I

Notations and survey of known results

Chapter 1

Preliminaries

This chapter is divided into three sections. In §1.1 we discuss basic results of the theory of quadratic forms over fields of characteristic 2. In §1.2 we collect the definitions and results concerning basics of representation theory. In §1.3 we discuss the connection between quadratic maps over fields of characteristic 2, special 2-groups and second cohomology groups. In the end we obtain the correspondence between central products of special 2-groups and orthogonal sums of their associated quadratic maps.

1.1 Quadratic forms over fields of characteristic 2

Let $\mathbb{F}$ be a field of characteristic 2. Let V and W be finite dimensional vector spaces over $\mathbb{F}$.

Definition 1.1.1. *A map $b : V \times V \to W$ is called $\mathbb{F}$-bilinear if it satisfies the following properties:*

1. $b(\alpha v_1 + \beta v_2, w) = \alpha b(v_1, w) + \beta b(v_2, w)$ *for all* $v_1, v_2, w \in V$ *and* $\alpha, \beta \in \mathbb{F}$.

2. $b(v, \alpha w_1 + \beta w_2) = \alpha b(v, w_1) + \beta b(v, w_2)$ *for all* $v, w_1, w_2 \in V$ *and* $\alpha, \beta \in \mathbb{F}$

Definition 1.1.2. *A map $q : V \to W$ is called a quadratic map if*

1. $q(\alpha v) = \alpha^2 q(v)$ *for all* $v \in V$

2. *The map* $b_q : V \times V \to W$ *given by* $b_q(v, w) := q(v + w) - q(v) - q(w)$ *is* $\mathbb{F}$*-bilinear.*

For a quadratic map q the map b_q is called the *polar map* of q. A bilinear map b_q is called *alternating* if $b_q(v,v) = 0$ for all $v \in V$.

Definition 1.1.3. *A quadratic map $q : V \to W$ is said to be a quadratic form if $W = \mathbb{F}$. A bilinear map $b_q : V \times V \to W$ is said to be a bilinear form if $W = \mathbb{F}$. For a quadratic form $q : V \to \mathbb{F}$, the pair (V, q) is called the quadratic space over $\mathbb{F}$.*

Definition 1.1.4. *Let V be a $2n$-dimensional vector space over $\mathbb{F}$ and $b : V \times V \to \mathbb{F}$ be a bilinear form. A basis $\{e_1, f_1, \cdots, e_n, f_n\}$ of V is called symplectic if $b(e_i, e_j) = b(f_i, f_j) = 0, b(e_i, f_i) = b(f_i, e_i) = 1$ and $b(e_i, f_j) = b(f_i, e_j) = 0$ for all $i \neq j$.*

Remark 1.1.5 Every alternating bilinear form $b : V \times V \to \mathbb{F}$ has a symplectic basis (see [Sch85, p. 264, th. 8.1]).

If $B = \{e_1, e_2, \cdots, e_n\}$ is a basis of V then any $n \times n$ matrix Q satisfying $q(x) = x^t Q x$, where $x \in V$ is indeterminate column vector and x^t denotes the transpose of x, is called a *matrix of q with respect to basis B*.

Every matrix of q with respect to same basis is of the form $Q + A$, where A is an alternating matrix and Q is the unique upper triangular matrix of q with respect to basis B. If we change the basis and T is transition matrix for this change of basis then upper triangular matrix of q with respect to new basis is $T^t Q T$. The matrix of b_q is the *alternating matrix $Q + Q^t$*.

Definition 1.1.6. *Two n-dimensional quadratic spaces (V_1, q_1) and (V_2, q_2) over $\mathbb{F}$ are said to be isometric if there exists an $\mathbb{F}$-linear isomorphism $T : V_1 \to V_2$ such that $q_1(v) = q_2(T(v))$ for all $v \in V$. Isometry between two quadratic spaces is denoted by $(V_1, q_1) \simeq (V_2, q_2)$ or simply $q_1 \simeq q_2$.*

Definition 1.1.7. *A quadratic space (V, q) is called the orthogonal sum of (V_1, q_1) and (V_2, q_2) if $V = V_1 \oplus V_2$ and $q(v) = q_1(v_1) + q_2(v_2)$, where $v = (v_1, v_2) \in V$ is an arbitrary element of V. In this case we write $q = q_1 \perp q_2$.*

Conversely, let (V, q) be a quadratic space and $V_i,\ 1 \leq i \leq m$ be subspaces of V such that $V = V_1 \oplus \cdots \oplus V_m$ and $b_q(v_i, v_j) = 0$ for $v_i \in V_i, v_j \in V_j, i \neq j$. Then $q = q_1 \perp \cdots \perp q_m$ where q_i denotes the restriction of q to V_i.

Definition 1.1.8. *Let (V, q) be a quadratic space. The subspace* $\mathrm{rad}(b_q) := \{w \in V : b_q(v, w) = 0 \ \forall \ v \in V\}$ *is called the radical of (V, q). The quadratic space (V, q) is called regular if* $\mathrm{rad}(b_q) = 0$.

The following theorem is analogous to the diagonalisation of quadratic spaces over the field of characteristic different from 2.

Proposition 1.1.9 ([Pfi95], p. 13, Th. 4.3)**.** *Every quadratic space (V, q) has an orthogonal decomposition $V = U \oplus \mathrm{rad}(b_q)$ such that $(U, q|_U)$ is regular and an orthogonal sum of 2-dimensional regular quadratic spaces; whereas $(\mathrm{rad}(b_q), q|_{\mathrm{rad}(b_q)})$ is orthogonal sum of 1-dimensional quadratic spaces.*

In other words there exists a basis $\{e_i, f_i, g_j, 1 \leq i \leq r, 1 \leq j \leq s\}$ of V where $2r + s = \dim(V)$ and elements $a_i, b_i, c_j \in \mathbb{F}, 1 \leq i \leq r, 1 \leq j \leq s$ such that for all $v = \sum(x_i e_i + y_i f_i) + \sum z_j g_j$, we have

$$q(v) = \sum(a_i x_i^2 + x_i y_i + b_i y_i^2) + \sum c_j z_j^2.$$

Notation 1.1.10. *We denote the quadratic form $q(v) = \sum(a_i x_i^2 + x_i y_i + b_i y_i^2) + \sum c_j z_j^2$ by $[a_1, b_1]\perp \cdots \perp[a_r, b_r]\perp\langle c_1, \cdots, c_s\rangle$. This is called normalized form of a quadratic form q.*

A quadratic form q is *regular* if $s = 0$. If $s \neq 0$ then q is said to be *singular* whereas if $r = 0$ then q is said to be *totally singular.* If $s > 0$, then regular part of a quadratic form is generally not determined uniquely up to isometry, whereas the part $\langle c_1, ..., c_s\rangle$ is always determined uniquely up to isometry. For example $[1, 1] \oplus \langle 1\rangle \simeq [0, 0] \oplus \langle 1\rangle$ but $[1, 1] \cong [0, 0]$ holds if and only if the quadratic equation $x^2 + x + 1 = 0$ has a solution in $\mathbb{F}$.

Remark 1.1.11 It immediately follows from Prop. 1.1.9 that every regular quadratic form over a field of characteristic 2 is even dimensional. Up to isometry, there are only two regular 2-dimensional quadratic forms over the field containing 2 elements $\mathbb{F}_2$, namely $[0, 0]$ and $[1, 1]$.

A quadratic form is said to be *isotropic* if there exist $0 \neq v \in V$ such that $q(v) = 0$, otherwise it is called *anisotropic*. Quadratic form $[0, 0]$ is the only 2-dimensional regular isotropic quadratic form up to isometry. It is called the *hyperbolic plane* and denoted by

H. A quadratic space is said to be *hyperbolic space* if it is orthogonal sum of hyperbolic planes.

As all regular quadratic forms are of even dimension, the *dimension invariant* e_0 given by $q \mapsto \dim(q) \mod 2$ is trivial. In $\text{char}(\mathbb{F}) = 2$ case an invariant at the next level is the Arf invariant which is an analogue of the discriminant in $\text{char}(\mathbb{F}) \neq 2$ case. It was defined by Arf in his classical paper [Arf41]. Let $q : V \to \mathbb{F}$ be a regular $2n$-dimensional quadratic form. As b_q is an alternating form, the space (V, q) has a symplectic basis $\{e_i, f_i, 1 \leq i \leq n\}$ (see remark 1.1.5). Let $\wp(\mathbb{F}) = \{x^2 + x : x \in \mathbb{F}\}$. Since we have $x_1^2 + x_1 + x_2^2 + x_2 = (x_1 + x_2)^2 + (x_1 + x_2) \in \wp(\mathbb{F})$ and every element of $\wp(\mathbb{F})$ is its own inverse under addition. Hence the set $\wp(\mathbb{F})$ is a subgroup of $(\mathbb{F}, +)$.

Definition 1.1.12. *Let (V, q) be a quadratic space and $\{e_i, f_i, 1 \leq i \leq n\}$ be a symplectic basis of (V, q). The class of element $\sum_{i=1}^{n} q(e_i)q(f_i)$ in the quotient $\mathbb{F}/\wp(\mathbb{F})$, is called the Arf invariant of q and is denoted by $\text{Arf}(q)$. In particular, if $q = [a_1, b_1] \perp \cdots \perp [a_n, b_n]$ then $\text{Arf}(q) = a_1b_1 + \cdots + a_nb_n \in \mathbb{F}/\wp(\mathbb{F})$.*

Arf invariant is independent of the choice of symplectic basis [Sch85, p. 340]. Moreover, for two quadratic forms q_1 and q_2 we have $\text{Arf}(q_1 \perp q_2) = \text{Arf}(q_1) + \text{Arf}(q_2) \in \mathbb{F}/\wp(\mathbb{F})$.

1.2 Real representations

Let G be a finite group. Let $\mathbb{C}$ be the field of complex numbers and U be a finite dimensional vector space over $\mathbb{C}$. We denote the group of isomorphisms of U onto itself by $\text{GL}(U)$. A *complex representation* of G is a homomorphism $\rho : G \to \text{GL}(U)$.

Let U' be a subspace of U such that for all $u \in U'$, we have $\rho(g)u \in U'$ for all $g \in G$. Then we say that U' is stable under the action of G. A representation $\rho : G \to \text{GL}(U)$ is said to be irreducible if $U \neq 0$ and if U' is subspace of U, which is stable under the action of G, then either $U' = 0$ or $U' = U$.

Let tr denote the trace of a linear transformation. The map $\chi : G \to \mathbb{C}$ defined by $\chi(g) = \text{tr}(\rho(g))$ for all $g \in G$ is called the *character* afforded by the representation ρ.

A map $\psi : G \to \mathbb{C}$ is called *class function* if $\psi(g) = \psi(hgh^{-1})$ for all $g, h \in G$. The characters of a group are class functions. In fact, the set of all irreducible characters of a finite group G forms a basis of the vector space of all class functions on G ([JL01, corollary 15.4]). Thus we have the following result:

Proposition 1.2.1 ([JL01], Prop. 15.5)**.** *Let G be a finite group and $g, h \in G$. Then g is conjugate to h if and only if $\chi(g) = \chi(h)$ for all irreducible characters χ of G.*

Definition 1.2.2. *Two representations $\rho : G \to GL(U_1)$ and $\phi : G \to GL(U_2)$ are called equivalent if there exists an isomorphism $\sigma : U_1 \to U_2$ such that $\rho(g) = \sigma^{-1}(\phi(g))\sigma$ for all $g \in G$.*

The following proposition provides a criteria for equivalence of representations.

Proposition 1.2.3 ([Ser77],p. 16, Cor. 2)**.** *Let ρ_1 and ρ_2 be two representations of a group G and χ_1 and χ_2 denote the characters afforded by ρ_1 and ρ_2 respectively. The representations ρ_1 and ρ_2 are equivalent if and only if $\chi_1(g) = \chi_2(g)$ for all $g \in G$.*

To find the complete list of distinct irreducible characters of a finite group is the fundamental problem of character theory of finite groups. The following result gives an easy method to check that whether a list of distinct irreducible characters is complete.

Theorem 1.2.4 ([JL01], Th. 11.12)**.** *Let G be a finite group and $\chi_1, \chi_2, \cdots \chi_k$ be the complete list of distinct irreducible characters of G. Then $\sum_{i=1}^{k} \chi_i(1)^2 = |G|$.*

Definition 1.2.5. *A representation $\rho : G \to \mathrm{GL}(U)$ is called a real representation of a group G if $\chi(g) \in \mathbb{R}$ for all $g \in G$, where χ denotes the character afforded by the representation ρ.*

Definition 1.2.6. *Let G be a group. An element $g \in G$ is called real if there exists $h \in G$ such that $g^{-1} = hgh^{-1}$.*

Let G be a group and $g \in G$ be a real element. By definition there exists $h \in G$ such that $g^{-1} = hgh^{-1}$. Since for all $k \in G$, we have

$$khk^{-1}(kgk^{-1})khk^{-1} = k(hgh^{-1})k^{-1} = kg^{-1}k^{-1} = (kgk^{-1})^{-1}.$$

Therefore all the conjugates of a real element are also real. A conjugacy class consisting real elements is called a *real conjugacy class* .

It is well known that the number of irreducible representations of a finite group is same as the number of its conjugacy classes. The following proposition states the similar result for real representations.

Theorem 1.2.7 ([JL01], Th. 11.12)**.** *Let G be a finite group. The number of real irreducible representations of G is same as the number of real conjugacy classes of G.*

Definition 1.2.8. *A complex representation $\rho : G \to \mathrm{GL}(U)$ is said to be realizable over $\mathbb{R}$ if the vector space U admits an $\mathbb{R}$-subspace U_0 such that $U = U_0 \otimes_{\mathbb{R}} \mathbb{C}$ and U_0 is stable under $\rho(G)$.*

Using Prop. 1.2.3 it is clear that every representation that is realizable over $\mathbb{R}$ is a real representation. In fact depending on whether or not a real representation is realizable over $\mathbb{R}$, there is further classification of representations as orthogonal and symplectic.

Definition 1.2.9. *A real representation $\rho : G \to \mathrm{GL}(U)$ is called orthogonal representation if it is realizable over $\mathbb{R}$. Otherwise it is called symplectic representation.*

In our discussion, we have three types of representations, namely orthogonal, symplectic and those representations which are not real. Now we define Schur indicator of a character. The Schur indicator of a character determines the type of representation afforded by the character.

Definition 1.2.10. *Let G be a group and $\chi : G \to \mathbb{C}$ be an irreducible character of G. The Schur indicator $\nu(\chi)$ of character χ is defined by*

$$\nu(\chi) = \frac{1}{|G|} \sum_{g \in G} \chi(g^2)$$

The Schur indicator of character χ can take only three values. The values taken by Schur indicator are $0, 1$ and -1.

Proposition 1.2.11 ([JL01], Corollary 23.17)**.** *Let ρ be an irreducible representation of G and χ be the character afforded by ρ. Then*

$$\nu(\chi) = \begin{cases} 0 & \text{if } \rho \text{ is not real} \\ 1 & \text{if } \rho \text{ is orthogonal} \\ -1 & \text{if } \rho \text{ is symplectic} \end{cases}$$

Now we give the examples of representations of each type.

Example 1.2.12

1. Let $C_4 := \langle a \ : \ a^4 = 1\rangle$ be the cyclic group of order 4. Let $\rho : C_4 \to \mathbb{C}$ be the representation of C_4 defined by $\phi(a) = i$, where i denotes the primitive fourth root of unity. Let χ_1 be the character afforded by the representation ϕ. Since

   $$\nu(\chi_1) = \frac{1}{4}(\chi_1(1)+\chi_1(a^2)+\chi_1(a^4)+\chi_1(a^6)) = \frac{1}{4}(2\chi_1(1)+2\chi_1(a^2)) = \frac{1}{4}(2-2) = 0.$$

   The representation ϕ of C_4 defined above is not real representation.

2. For dihedral group $D_4 = \langle a, b \ : \ a^4 = b^2 = 1, bab^{-1} = a^{-1}\rangle$ the homomorphism $\rho : D_4 \to \mathrm{GL}(\mathbb{C}^2)$ defined by

   $$\rho(a) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \rho(b) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

   is the unique irreducible representation of degree at least 2 of D_4. Let χ_2 be the character afforded by the representation ρ. We compute

   $$\nu(\chi_2) = \frac{1}{8}\sum_{g\in D_4} \chi_2(g^2) = \frac{1}{8}((6\chi_2(1) + 2\chi_2(a^2)) = \frac{1}{4}(6 \times 2 - 2 \times 2) = 1.$$

   Therefore χ_2 is orthogonal.

3. For quaternion group $Q_2 = \langle c, d \ : \ c^4 = 1, d^2 = c^2, dcd^{-1} = c^{-1}\rangle$ the homomorphism $\sigma : Q_2 \to \mathrm{GL}(\mathbb{C}^2)$, where

   $$\sigma(c) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \sigma(d) = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

   is the unique irreducible representation of degree at least 2 of Q_2. Let χ_3 be the character afforded by the representation σ. We compute

   $$\nu(\chi_3) = \frac{1}{8}\sum_{g\in Q_2} \chi_3(g^2) = \frac{1}{8}((2\chi_3(1) + 6\chi_3(c^2)) = \frac{1}{4}(2 \times 2 - 6 \times 2) = -1.$$

   Therefore χ_3 is symplectic.

Since we need to determine representations up to equivalence, using isomorphism $\mathrm{GL}(\mathbb{C}^n) \cong \mathrm{GL}_n(\mathbb{C})$ between linear transformations and matrices that arises after fixing a basis, we may take the target group of representations as group $\mathrm{GL}_n(\mathbb{C})$. Now we define the tensor product of two representations. We begin with the definition of tensor product of two matrices.

Definition 1.2.13. *Let $A = [a_{ij}]_{n\times n}$ and B be two complex matrices. Then the tensor product of matrices A and B is defined by*

$$A \otimes B = \begin{pmatrix} a_{11}B & a_{12}B & \cdots & a_{1n}B \\ a_{21}B & a_{22}B & \cdots & a_{2n}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}B & a_{n2}B & \cdots & a_{nn}B \end{pmatrix}$$

Note that $\operatorname{tr}(A \otimes B) = \operatorname{tr}(A). \operatorname{tr}(B)$.

Definition 1.2.14. *Let G and H be two groups. Let $\rho : G \to \mathrm{GL}_n(\mathbb{C})$ and $\sigma : H \to \mathrm{GL}_m(\mathbb{C})$ be the representations of group G and H respectively. The tensor product $\rho \otimes \sigma : G \times H \to \mathrm{GL}_{nm}(\mathbb{C})$ of the representations ρ and σ is defined by $(\rho \otimes \sigma)(g, h) = \rho(g) \otimes \sigma(h)$ for all $(g, h) \in G \times H$.*

The following theorem illustrates the utility of defining the tensor product of two representations. It provides a description of representations of direct products of groups in terms of irreducible representations of direct factors.

Theorem 1.2.15 ([Gor80], Ch. 3, Th. 7.2)**.** *Let H and K be two finite groups and $G = H \times K$ be the direct product of H and K. Let $\rho : H \to \mathrm{GL}_n(\mathbb{C})$ and $\sigma : K \to \mathrm{GL}_m(\mathbb{C})$ be irreducible representations of H and K, respectively. Then the tensor product $\rho \otimes \sigma : G \to \mathrm{GL}_{nm}(\mathbb{C})$ of ρ and σ is an irreducible representation of G. Moreover, every irreducible representation of $G = H \times K$ is equivalent to a representation of the form $\rho \otimes \sigma$ for a suitable choice of ρ and σ.*

The above theorem implies that the tensor product of representations is again a representation. The following proposition helps in determining the type of tensor product of representations in terms of types of the individual components of the tensor product.

Proposition 1.2.16. *Let ρ_1 and ρ_2 be two representations of groups G and H respectively. Let χ, χ_1 and χ_2 be the characters afforded by representations $\rho_1 \otimes \rho_2$, ρ_1 and ρ_2 respectively. Then $\nu(\chi) = \nu(\chi_1).\nu(\chi_2)$.*

Proof For two matrices A and B, we have $\operatorname{tr}(A \otimes B) = \operatorname{tr}(A). \operatorname{tr}(B)$. By definition of tensor product of representations, it follows that $\chi = \chi_1.\chi_2$. Here $\chi_1.\chi_2 : G \times H \to \mathbb{C}$ is defined by $\chi_1.\chi_2(g,h) = \chi_1(g).\chi_2(h)$ for all $(g,h) \in G \times H$. Now we compute

$$\begin{aligned}
\nu(\chi) &= \nu(\chi_1.\chi_2) \\
&= \frac{1}{|G \times H|} \sum_{(g,h)\in G\times H} \chi_1.\chi_2(g,h) \\
&= \frac{1}{|G \times H|} \sum_{(g,h)\in G\times H} \chi_1(g).\chi_2(h) \\
&= \frac{1}{|G| \times |H|} \sum_{h\in H} \left(\sum_{g\in G} \chi_1(g).\chi_2(h) \right) \\
&= \left(\frac{1}{|G|} \sum_{g\in G} \chi_1(g) \right) \cdot \left(\frac{1}{|H|} \sum_{h\in H} \chi_2(h) \right) \\
&= \nu(\chi_1).\nu(\chi_2).
\end{aligned}$$

□

Corollary 1.2.17. *Let ρ_1 and ρ_2 be two representations of groups G and H respectively. The representation $\rho_1 \otimes \rho_2$ of $G \times H$ is real if and only if both the representations ρ_1 and ρ_2 are real. Moreover, the representation $\rho_1 \otimes \rho_2$ is orthogonal if and only if either both ρ_1 and ρ_2 are orthogonal or both ρ_1 and ρ_2 are symplectic. The representation $\rho_1 \otimes \rho_2$ is symplectic if and only if exactly one of ρ_1 and ρ_2 is symplectic.*

Proof The proof follows from the Prop. 1.2.11 and Prop. 1.2.16. □

Definition 1.2.18. *Let G be a group and $\rho : G \to \mathrm{GL}_n(\mathbb{C})$ a representation of G. Let N be a normal subgroup of G such that $N \subseteq \ker(\rho)$. Then ρ induce a representation $\bar{\rho} : \frac{G}{N} \to \mathrm{GL}_n(\mathbb{C})$ of $\frac{G}{N}$ defined by $\bar{\rho}(gN) = \rho(g)$ of $\frac{G}{N}$ for all $gN \in \frac{G}{N}$. It is called the representation induced by ρ.*

Proposition 1.2.19. *Let G be a group and $\rho : G \to \mathrm{GL}_n(\mathbb{C})$ be its representation. Let N be a normal subgroup of G such that $N \subseteq \ker(\rho)$. Let $\bar{\rho} : \frac{G}{N} \to \mathrm{GL}_n(\mathbb{C})$ be the representation induced by ρ. Let χ and $\bar{\chi}$ be the characters afforded by the representations ρ and $\bar{\rho}$ respectively. Then $\nu(\chi) = \nu(\bar{\chi})$.*

Proof Since $\bar{\rho}(gN) = \rho(g)$ for all $gN \in \frac{G}{N}$, we have $\bar{\chi}(gN) = \chi(g)$ for all $gN \in \frac{G}{N}$. Using definition of Schur indicator $\nu(\chi)$, we compute

$$\begin{aligned}\nu(\bar{\chi}) &= \frac{1}{|\frac{G}{N}|}\sum_{gN\in\frac{G}{N}}\bar{\chi}(gN)\\ &= \frac{|N|}{|G|}\left(\sum_{g\in G}\frac{1}{|N|}\chi(g)\right)\\ &= \frac{1}{|G|}\sum_{g\in G}\chi(g)\\ &= \nu(\chi)\end{aligned}$$

□

Corollary 1.2.20. *The type of induced representation $\bar{\rho}$ is same as the type of representation ρ.*

Definition 1.2.21. *Let G be a group. Let χ be a character of G.*

1. *We denote the subgroup $\{g \in G \ : \ \chi(g) = \chi(1)\}$ by $\ker(\chi)$.*
2. *The set $\{g \in G \ : \ |\chi(g)| = \chi(1)\}$ forms a subgroup of G. We denote this subgroup by $\mathbf{Z}(\chi)$.*

Now we state some results related to the subgroup $\mathbf{Z}(\chi)$.

Lemma 1.2.22 ([Isa69], Lemma 2.27)**.** *Let G be a group. Let ρ be a representation of G and χ be the character afforded by ρ. Then*

1. *$\mathbf{Z}(\chi) = \{g \in G \ : \ \rho(g) = \lambda I$ for some $\lambda \in \mathbb{C}\}$.*
2. *$Z\left(\frac{G}{\ker(\chi)}\right) = \frac{\mathbf{Z}(\chi)}{\ker(\chi)}$.*

Lemma 1.2.23 ([Isa69])**.** *Let G be a group and χ be a character of G. The character χ vanishes outside $\mathbf{Z}(\chi)$ if and only if $\chi(1)^2 = |G : \mathbf{Z}(\chi)|$.*

1.3 Special 2-groups

In this section $\mathbb{F}_2$ denotes the field containing two elements. Here we discuss the connection between special 2-groups and quadratic maps over the field $\mathbb{F}_2$.

For a finite group G, let $Z(G)$ and G' denote the center and derived subgroup of G, respectively.

Definition 1.3.1. *Let G be a non-trivial finite group. The intersection of maximal subgroups of G is called the Frattini subgroup of G. It is denoted by $\Phi(G)$. For the trivial group, Frattini subgroup is defined to be trivial.*

For a group G, we denote the direct product of n copies of G by $(G)^n$. A p-group G is called *elementary abelian* if G is an abelian group of exponent p. The direct product of some copies of $\frac{\mathbb{Z}}{2\mathbb{Z}}$ is *elementary abelian* 2*-group.*

Remark 1.3.2

1. For a p-group G, the Frattini subgroup $\Phi(G)$ is the smallest normal subgroup of G such that the quotient group $\frac{G}{\Phi(G)}$ is elementary abelian 2-group (see [KS04, p. 106, 5.2.8]).

2. For a finite p-group G, the Frattini subgroup $\Phi(G) = G'G^p$, where G' is derived subgroup of G and $G^p = \langle g^p \ : \ g \in G\rangle$ (see [Rob96, p.140,5.3.2]). Since every commutator is a product of squares as shown below:

$$[g,h] = g^{-1}h^{-1}gh = g^{-1}h^{-2}hgh = g^{-1}h^{-2}gg^{-2}ghgh = (g^{-1}hg)^{-2}g^{-2}(gh)^2,$$

   for $p = 2$, we have $\Phi(G) = G^2 = \langle g^2 \ : \ g \in G\rangle$.

Definition 1.3.3. *A* 2*-group G is called a special* 2*-group if $\Phi(G) = G' = Z(G) \cong \left(\frac{\mathbb{Z}}{2\mathbb{Z}}\right)^n$ for some $n \in \mathbb{N}$. Moreover, a special* 2*-group G is called extraspecial* 2*-group if $|Z(G)| = 2$.*

For special 2-group G, note that the quotient $\frac{G}{Z(G)}$ and $Z(G)$ both are elementary abelian 2-groups. Therefore we may regard them as vector spaces over $\mathbb{F}_2$.

Theorem 1.3.4 ([Zah08], Th. 3.4.11)**.** *Let G be a special* 2*-group and $q : \frac{G}{Z(G)} \to Z(G)$ be the map given by $q(xZ(G)) = x^2$ for all $x \in G$. Then q is a regular quadratic map and $b_q(xZ(G), yZ(G)) = xyx^{-1}y^{-1}$ for all $x, y \in G$.*

Proof Let $x, y \in G$. Since G is a special 2-group, $x^2, y^2 \in Z(G)$ and

$$\begin{aligned} b_q(xZ(G), yZ(G)) &= q(xyZ(G))(q(xZ(G))^{-1}(q(yZ(G))^{-1} \\ &= (xy)^2 x^{-2} y^{-2} \\ &= x^{-2} xyy^{-2} xy \\ &= x^{-1} y^{-1} xy \end{aligned}$$

Now we check that b_q is bilinear map. We recall that $Z(G) = G'$ as G is a special 2-group. Let $x, y, z \in G$.

$$\begin{aligned} b_q(xzZ(G), yZ(G)) &= (xz)^{-1} y^{-1} (xz) y \\ &= z^{-1} x^{-1} y^{-1} xzy \\ &= z^{-1} (x^{-1} y^{-1} xy) y^{-1} zy \\ &= (x^{-1} y^{-1} xy)(z^{-1} y^{-1} zy) \\ &= b_q(xZ(G), yZ(G)) b_q(zZ(G), yZ(G)) \end{aligned}$$

On similar lines, One can check that $b_q(xZ(G), yzZ(G)) = b_q(xZ(G), yZ(G)) b_q(Z(G), zZ(G))$. Therefore $q : \frac{G}{Z(G)} \to Z(G)$ is a quadratic map. Since $x \in Z(G)$ if and only if $x^{-1} y^{-1} xy = 1$ for every $y \in G$, the quadratic map $q : \frac{G}{Z(G)} \to Z(G)$ is regular. $\square$

Note that for the quadratic map q in the above theorem, the image of b_q generates $Z(G)$. This quadratic map q is called the *quadratic map associated to the special* 2*-group* G.

1.3.1 Cocycles, quadratic maps and central extensions

Let V and W denote two vector spaces over $\mathbb{F}_2$. Let $\mathrm{Quad}(V, W)$ denote the set of quadratic maps from V to W. We consider it as a group under the group operation of point wise addition of maps. In what follows, we discuss the connection between $\mathrm{Quad}(V, W)$ and $H^2(V, W)$, the second cohomology group of V with coefficients in W. We begin with the definition of normal 2-cocycle.

Definition 1.3.5. *A map $c : V \times V \to W$ is called a normal* 2*-cocycle on V with coefficients in W if for all $v, v_1, v_2, v_3 \in V$ it satisfies the following conditions:*

i. $c(v_2, v_3) - c(v_1 + v_2, v_3) + c(v_1, v_2 + v_3) - c(v_1, v_2) = 0.$

ii. $c(v, 0) = c(0, v) = 0$.

The set of normal 2-cocycles on V with coefficients in W is denoted by $Z^2(V, W)$.

The set $Z^2(V, W)$ forms an abelian group under the operation of point wise addition.

Definition 1.3.6. *Let $\lambda : V \to W$ be a linear map such that $\lambda(0) = 0$. Then the map $c_\lambda : V \times V \to W$ defined by $c_\lambda(v_1, v_2) = \lambda(v_2) - \lambda(v_1 + v_2) + \lambda(v_1)$ is a normal 2-cocycle. The normal 2-cocycles obtained this way are called normal 2-coboundaries and their collection is denoted by $B^2(V, W)$.*

The set $B^2(V, W)$ forms a subgroup of $Z^2(V, W)$.

Definition 1.3.7. *The quotient $H^2(V, W) = \frac{Z^2(V,W)}{B^2(V,W)}$ is called the second cohomology group of V with coefficients in W.*

The following proposition gives the correspondence between $H^2(V, W)$ and $\mathrm{Quad}(V, W)$.

Proposition 1.3.8 ([Zah11], Prop. 1.2)**.** *The map $\varphi : Z^2(V, W) \to \mathrm{Quad}(V, W)$ which maps $c \in Z^2(V, W)$ to the quadratic map q_c defined by $q_c(x) = c(x, x)$ induces a homomorphism between $H^2(V, W)$ and $\mathrm{Quad}(V, W)$. If the dimension of V is finite then this homomorphism is an isomorphism.*

Now we define the central extension and discuss the one-one correspondence between the elements of $H^2(V, W)$ and central extensions of W by V. We consider V and W as groups under the operation of vector space addition.

Definition 1.3.9. *A group G is called a central extension of V by W if there exist a short exact sequence of groups $1 \to W \to G \to V \to 1$ such that $W \subseteq Z(G)$.*

Two central extensions $1 \to W \overset{\alpha_1}{\to} G_1 \overset{\pi_1}{\to} V \to 1$ and $1 \to W \overset{\alpha_2}{\to} G_2 \overset{\pi_2}{\to} V \to 1$ are said to be *equivalent* if there exists an isomorphism $\vartheta : G_1 \to G_2$ such that $\pi_2 \circ \vartheta = \pi_1$ and ϑ is the identity map on W. The set of equivalent classes of central extension of W by V is in one to one correspondence with $H^2(V, W)$ [Wei94, §6.6]. We record this correspondence in the following remark for further reference.

Remarks 1.3.10

1. The central extension of W by V corresponding to a cocycle class $[c] \in H^2(V, W)$ is isomorphic to the group $V \dot{\times} W$, where the underlying set of the group $V \dot{\times} W$ is just the cartesian product $V \times W$ and its group operation is defined by

$$(v, w)(v', w') = (v + v', c(v, v') + w + w')$$

for all $v, v' \in V$ and $w, w' \in W$. The identity element this group is $(0, 0)$ and the inverse of (v, w) is $(v, c(v, v) + w)$.

2. Conversely, let $1 \to W \overset{\alpha}{\to} G \overset{\pi}{\to} V \to 1$ be a central extension. Let $s : V \to G$ be a map such that $s(0) = 1$ and $\pi \circ s$ is identity map on V. For all $v_1, v_2 \in V$, we suppose $c(v_1, v_2) = s(v_1)s(v_2)s(v_1 + v_2)^{-1}$. We compute

$$\begin{aligned} \pi(c(v_1, v_2)) &= \pi(s(v_1)s(v_2)s(v_1 + v_2)^{-1}) \\ &= \pi \circ s(v_1)\pi \circ s(v_2)\pi \circ s(v_1 + v_2)^{-1} \\ &= v_1 + v_2 - (v_1 + v_2) \\ &= 0. \end{aligned}$$

Therefore $c(v_1, v_2) \in \ker(\pi) = \mathrm{Im}(\alpha) \cong W$. From the associativity of the multiplication of G, it follows that $c : V \times V \to W$ is a normal 2-cocycle.

If the dimension of V is finite then Prop. 1.3.8 and the correspondence of elements of $H^2(V, W)$ with central extension of V by W gives a useful correspondence between $\mathrm{Quad}(V, W)$ and central extension of V by W [Zah11].

Let $q : V \to W$ be a regular quadratic map and the image of b_q generates W. The following result asserts that the group corresponding to q is a special 2-group. We reproduce the proof of this result from [Zah11].

Theorem 1.3.11 ([Zah11], Th. 1.4)**.** *Let $q : V \to W$ be a regular quadratic map. Suppose that $W = \langle b_q(V \times V)\rangle$. Then there exists a special 2-group G associated with quadratic map q such that $W = Z(G)$ and $V = \frac{G}{Z(G)}$. Such a group is unique up to isomorphism.*

Proof Recall that $\mathrm{Quad}(V, W)$ denotes the group of quadratic maps under the point wise addition. It follows from Prop. 1.3.8 that $\mathrm{Quad}(V, W)$ is isomorphic to $H^2(V, W)$ as an abelian group. Let $c \in Z^2(V, W)$ be such that class $[c]$ of c in $H^2(V, W)$ corresponds to the quadratic map $q : V \to W$. Therefore $q(v) = c(v, v)$ for all $v \in V$. Using cocycle relation (see definition 1.3.5), we compute the polar map to be $b_q(v_1, v_2) = c(v_1, v_2) + c(v_2, v_1)$.

The central extension corresponding to normal 2-cocycle $[c] \in H^2(V, W)$ is group G, where underlying set of G is $V \times W$ and the operation of G is as defined in remark 1.3.10(1).

Let $g = (v, w) \in G$ for $v \in V$ and $w \in W$. We compute

$$q(v) = c(v, v) = (0, c(v, v)) = (v + v, c(v, v) + w + w) = (v, w)(v, w) = g^2.$$

Moreover for $g_1 = (v_1, w_1), g_2 = (v_2, w_2) \in G$ for $v_1, v_2 \in V$ and $w_1, w_2 \in W$, we have

$$\begin{aligned} g_1^{-1} g_2^{-1} g_1 g_2 &= (v_1, c(v_1, v_1) + w_1)(v_2, c(v_2, v_2) + w_2)(v_1, w_1)(v_2, w_2) \\ &= (0, c(v_1, v_2) + c(v_2, v_1)) \\ &= c(v_1, v_2) + c(v_2, v_1) \\ &= b_q(v_1, v_2) \end{aligned}$$

Now we show that $W = Z(G)$. Let $g = (v, w) \in Z(G)$ for $v \in V$ and $w \in W$. For all $h \in G$, we have $g^{-1}h^{-1}gh = 1$. Hence from the above calculation, it follows that $b_q(v, v') = 0$ for all $v' \in V$. Using the hypothesis that q is a regular quadratic map, we conclude that $v = 0$. Therefore $g = (0, w) \in W$. Conversely, let $g \in W$, then $g = (0, w)$ and $b_q(0, v') = 0$ for all $v' \in V$. Therefore $g^{-1}h^{-1}gh = 1$ for all $h \in G$ and $g \in Z(G)$.

Next we claim that $W = G'$. Since for all $g_1, g_2 \in G$, $g_1^{-1} g_2^{-1} g_1 g_2$ lies in the image of b_q and the image of b_q lies in W, the derived subgroup $G' \subseteq W$ and $b_q(V \times V) \subseteq W$. Using hypothesis $W = \langle b_q(V \times V) \rangle$, we get that $W \subseteq G'$.

Finally we claim that $W = \Phi(G)$. Since W is vector space over $\mathbb{F}_2$, W is an elementary abelian 2-group. Also $q : V \to W$ is quadratic map defined by $q(v) = v^2$ for all $v \in V$. Therefore for all $g \in G$, $g^2 \in W$ and order of a non identity element of G is either 2 or 4. This implies that G is a 2-group.

Using remark 1.3.2(2), we have $\Phi(G) = \langle g^2 \;:\; g \in G \rangle$. Now from the definition of q, it follows that $\Phi(G) \subseteq W$ and $\langle q(V) \rangle = \Phi(G)$. Using $b_q(v_1, v_2) = q(v_1) - q(v_1 + v_2) + q(v_2)$, we get that $\langle b_q(V \times V) \rangle \subseteq \langle q(V) \rangle$. By hypothesis, $W = \langle b_q(V \times V) \rangle$ and it follows that $W \subseteq \langle q(V) \rangle = \Phi(G)$.

Thus G is a special 2-group. Since G is central extension of V by W, the sequence $1 \to W \overset{\alpha}{\to} G \overset{\pi}{\to} V \to 1$ is an exact sequence. Therefore $V \cong \frac{G}{\ker(\pi)} = \frac{G}{\mathrm{Im}(\alpha)} = \frac{G}{Z(G)}$. □

This group is called the *special* 2*-group associated to the quadratic map* q. From Th. 1.3.11 and Th. 1.3.4 we get a one to one correspondence between special 2-groups and regular quadratic maps $q : V \in W$ with $\langle b_q(V \times V) = W \rangle$. The quadratic map associated to a special 2-group is very useful to study the properties of these groups. For an illustration, we have the following lemma:

Lemma 1.3.12. *Let G be a special 2-group and $q : \frac{G}{Z(G)} =: V \to W := Z(G)$ be the quadratic map associated to it. The order of a non-trivial element of G is either 2 or 4. Moreover if $g = (v, w) \in G$, then $g^2 = 1$ if and only if $q(v) = 0$.*

Proof Since $q : \frac{G}{Z(G)} =: V \to W := Z(G)$ is defined by $q(gZ(G)) = g^2$ for all $gZ(G) \in \frac{G}{Z(G)}$, we have $g^2 \in Z(G)$ for all $g \in G$. This implies that $(g^2)^2 = g^4 = 1$ for all $g \in G$ as $Z(G)$ is an elementary abelian 2-group.

Let $g = (v, w) \in G$. Since $g^2 = (v, w)(v, w) = (v + v, c(v, v) + w + w) = (0, q(v))$, we get that $g^2 = 1$ if and only if $q(v) = 0$. □

1.3.2 The central product of special 2-groups

The aim of this section is to show that the orthogonal sum of quadratic maps corresponds to central product of corresponding special 2-group. We first define the notion of central product of two groups.

Definition 1.3.13. *Let G_1 and G_2 be two groups with isomorphic centers. Let $Z(G_1)$ and $Z(G_2)$ be their centers and $\theta : Z(G_1) \to Z(G_2)$ be an isomorphism of groups. Let N denote the normal subgroup $\{(x, y) \in Z(G_1) \times Z(G_2) : \theta(x)y = 1\}$ of $G_1 \times G_2$. The central product $G_1 \circ_\theta G_2$ of groups G_1 and G_2 with the identification θ is the quotient of the direct product $G_1 \times G_2$ by N.*

The following lemma relates 'orthogonal sum' of quadratic maps to central products.

Lemma 1.3.14 ([KK15], Lemma 2.5)**.** *Let G_1 and G_2 be special 2-groups such that $Z(G_1)$ is isomorphic to $Z(G_2)$. Let $q_1 : V_1 := \frac{G_1}{Z(G_1)} \to W_1 := Z(G_1)$ and $q_2 : V_2 :=$*

$\frac{G_2}{Z(G_2)} \to W_2 := Z(G_2)$ *be regular quadratic maps associated to special* 2*-groups* G_1 *and* G_2*, respectively. Let* $\theta : W_1 \to W_2$ *be an isomorphism of groups. Then* $q_1 \perp_\theta q_2 : V_1 \oplus V_2 \to W_2$ *defined by* $(q_1 \perp_\theta q_2)(v_1, v_2) = \theta(q_1(v_1)) + q_2(v_2)$ *is a regular quadratic map and the group associated to* $q_1 \perp_\theta q_2$ *is* $G_1 \circ_\theta G_2$*.*

Proof Let $q = q_1 \perp_\theta q_2$. We first show that q is a quadratic map. For $\alpha \in \mathbb{F}_2$ and $(v_1, v_2) \in V_1 \oplus V_2$, we check

$$\begin{aligned} q(\alpha(v_1, v_2)) &= q(\alpha v_1, \alpha v_2), \\ &= \theta(q_1(\alpha v_1)) + q_2(\alpha v_2), \\ &= \theta(\alpha^2 q_1(v_1)) + \alpha^2 q_2(v_2), \\ &= \alpha^2 \theta(q_1(v_1)) + \alpha^2 q_2(v_2), \\ &= \alpha^2(\theta(q_1(v_1)) + q_2(v_2)), \\ &= \alpha^2 q(v_1, v_2). \end{aligned}$$

Now we compute the polar map b_q associated to $q = q_1 \perp_\theta q_2$.

$$\begin{aligned} b_q((v_1, v_2), (v_1', v_2')) &= q((v_1, v_2) + (v_1', v_2')) - q(v_1, v_2) - q(v_1', v_2'), \\ &= q(v_1 + v_1', v_2 + v_2') - q(v_1, v_2) - q(v_1', v_2'), \\ &= \theta(q_1(v_1 + v_1')) + q_2(v_2 + v_2') - \theta(q_1(v_1)) - q_2(v_2) - \theta(q_1(v_1')) - q_2(v_2'), \\ &= \theta(q_1(v_1 + v_1') - q_1(v_1) - q_1(v_1')) + q_2(v_2 + v_2') - q_2(v_2) - q_2(v_2'), \\ &= \theta(b_{q_1}(v_1, v_1')) + b_{q_2}(v_2, v_2'), \end{aligned}$$

where $(v_1, v_2), (v_1', v_2') \in V_1 \oplus V_2$ and b_{q_1}, b_{q_2} are the polar maps associated to q_1 and q_2, respectively. Now we check that the polar map b_q is bilinear.

$$\begin{aligned} b_q((v_1, v_2) + (v_3, v_4), (v_1', v_2')) &= b_q((v_1 + v_3, v_2 + v_4), (v_1', v_2')), \\ &= \theta(b_{q_1}(v_1 + v_3, v_1')) + b_{q_2}(v_2 + v_4, v_2'), \\ &= \theta(b_{q_1}(v_1, v_1') + b_{q_1}(v_3, v_1')) + b_{q_2}(v_2, v_2') + b_{q_2}(v_4, v_2'), \\ &= \theta(b_{q_1}(v_1, v_1')) + \theta(b_{q_1}(v_3, v_1')) + b_{q_2}(v_2, v_2') + b_{q_2}(v_4, v_2'), \\ &= \theta(b_{q_1}(v_1, v_1')) + b_{q_2}(v_2, v_2') + \theta(b_{q_1}(v_3, v_1')) + b_{q_2}(v_4, v_2'), \\ &= b_q((v_1, v_2), (v_1', v_2')) + b_q((v_3, v_4), (v_1', v_2')), \end{aligned}$$

where $(v_1, v_2), (v_3, v_4), (v_1', v_2') \in V_1 \oplus V_2$. On similar lines, one can check that

$$b_q((v_1, v_2), (v_1', v_2') + (v_3', v_4')) = b_q((v_1, v_2), (v_1', v_2')) + b_q((v_1, v_2), (v_3', v_4'))$$

for $(v_1, v_2), (v_1', v_2'), (v_3', v_4') \in V_1 \oplus V_2$ and

$$b_q(\alpha(v_1, v_2), (v_1', v_2')) = \alpha b_q((v_1, v_2), (v_1', v_2')) = b_q((v_1, v_2), \alpha(v_1', v_2'))$$

for $\alpha \in \mathbb{F}_2$ and $(v_1, v_2), (v_1', v_2') \in V_1 \oplus V_2$.

Now we show that the quadratic map q is regular. Let $(v_1, v_2) \in \mathrm{rad}(b_q)$ and $(v_1', v_2') \in V_1 \oplus V_2$. Then $0 = b_q((v_1, v_2), (v_1', v_2')) = \theta(b_{q_1}(v_1, v_1')) + b_{q_2}(v_2, v_2')$. Thus $\theta(b_{q_1}(v_1, v_1')) = b_{q_2}(v_2, v_2')$ for every $(v_1', v_2') \in V_1 \oplus V_2$. In particular for $v_2' = 0$, we have $\theta(b_{q_1}(v_1, v_1')) = b_{q_2}(v_2, 0) = 0$. Since θ is an isomorphism, $b_{q_1}(v_1, v_1') = 0$ for every $v_1' \in V_1$. Thus we conclude that $v_1 = 0$ as q_1 is a regular quadratic map. That $v_2 = 0$ follows from a similar argument, and it confirms that q is regular.

To show that the regular quadratic map q is associated to $G_1 \circ_\theta G_2$, let $c_1 \in Z^2(V_1, W_1)$ and $c_2 \in Z^2(V_2, W_2)$ be normal 2-cocycles corresponding to quadratic maps q_1 and q_2, respectively. Let $c_\theta := \theta(c_1) \perp c_2 : (V_1 \oplus V_2) \times (V_1 \oplus V_2) \to W_2$ be the map defined by

$$c_\theta((v_1, v_2), (v_1', v_2')) = \theta(c_1(v_1, v_1')) + c_2(v_2, v_2').$$

We check that c_θ is a normal 2-cocycle on $V_1 \oplus V_2$ with coefficients in W_2. For $(v_1, v_2), (v_1', v_2'), (v_1'', v_2'') \in V_1 \oplus V_2$, we compute:

$$\begin{aligned}
&c_\theta((v_1', v_2'), (v_1'', v_2'')) - c_\theta((v_1, v_2) + (v_1', v_2'), (v_1'', v_2'')) + c_\theta((v_1, v_2), (v_1', v_2') + (v_1'', v_2'')) - c_\theta((v_1, v_2), (v_1', v_2')) \\
&= \theta(c_1(v_1', v_1'')) + c_2(v_2', v_2'') - \theta(c_1(v_1 + v_1', v_1'')) - c_2(v_2 + v_2', v_2'') + \theta(c_1(v_1, v_1' + v_1'')) + c_2(v_2, v_2' + v_2'') - \theta(c_1(v_1, v_1')) - c_2(v_2, v_2'), \\
&= \theta(c_1(v_1', v_1'') - c_1(v_1 + v_1', v_1'') + c_1(v_1, v_1' + v_1'') - c_1(v_1, v_1')) + c_2(v_2', v_2'') - c_2(v_2 + v_2', v_2'') + c_2(v_2, v_2' + v_2'') - c_2(v_2, v_2'), \\
&= \theta(0) + 0 = 0
\end{aligned}$$

Now for $(v_1, v_2) \in V_1 \oplus V_2$, we compute:

$$c_\theta((v_1, v_2), (0, 0)) = \theta(c_1(v_1, 0)) + c_2(v_2, 0) = \theta(0) + 0 = 0$$

$$c_\theta((0, 0), (v_1, v_2)) = \theta(c_1(0, v_1)) + c_2(0, v_2) = \theta(0) + 0 = 0$$

We also compute that if c_1 and c_2 are normal 2-coboundaries then c_θ also a normal 2-coboundary. Let $\lambda_1 : V_1 \to W_1$ is a map such that $\lambda_1(0) = 0$ and $c_1(v_1, v_1') = \lambda_1(v_1') - \lambda_1(v_1 + v_1') + \lambda_1(v_1)$ for $v_1, v_1' \in V_1$. Let $\lambda_2 : V_2 \to W_2$ is a map such that $\lambda_2(0) = 0$

and $c_1(v_2, v_2') = \lambda_2(v_2') - \lambda_2(v_2 + v_2') + \lambda_2(v_2)$ for $v_2, v_2' \in V_2$. Now we show that in this case c_θ is also a normal 2-coboundary. We define a map $\lambda : V_1 \oplus V_2 \to W_2$ by $\lambda(v_1, v_2) = \theta(\lambda_1(v_1)) + \lambda_2(v_2)$ for $v_1 \in V_1$ and $v_2 \in V_2$. We check that $\lambda(0,0) = \theta(\lambda_1(0)) + \lambda_2(0) = 0$. We compute:

$$\begin{aligned}
c_\theta((v_1, v_2), (v_1', v_2')) &= \theta(c_1(v_1, v_1')) + c_2(v_2, v_2') \\
&= \theta(\lambda_1(v_1') - \lambda_1(v_1 + v_1') + \lambda_1(v_1)) + \lambda_2(v_2') - \lambda_2(v_2 + v_2') + \lambda_2(v_2) \\
&= \theta(\lambda_1(v_1')) + \lambda_2(v_2') - \theta(\lambda_1(v_1 + v_1')) - \lambda_2(v_2 + v_2') + \theta(\lambda_1(v_1)) + \lambda_2(v_2) \\
&= \lambda(v_1', v_2') - \lambda((v_1, v_1',) + (v_2, v_2')) + \lambda(v_1, v_2)
\end{aligned}$$

The above calculation ensures that the association $([\theta(c_1)], [c_2]) \mapsto [c_\theta] \in H^2(V_1 \oplus V_2, W_2)$ is well-defined. Further, the normal 2-cocycle c_θ corresponds to the quadratic form q as

$$c_\theta((v_1, v_2), (v_1, v_2)) = \theta(c_1(v_1, v_1)) + c_2(v_2, v_2) = \theta(q_1(v_1)) + q_2(v_2) = q(v_1, v_2).$$

The special 2-group associated to q is $G := (V_1 \oplus V_2)\dot{\times} W_2$, with the group operation

$$((v_1, v_2), w) \cdot ((v_1', v_2'), w') = ((v_1 + v_1', v_2 + v_2'), c_\theta((v_1, v_2), (v_1', v_2')) + w + w').$$

We need to show that $G \simeq G_1 \circ_\theta G_2$. By definition $G_1 \circ_\theta G_2$ is the quotient of $G_1 \times G_2$ by N where $N := \{(x, y) \in Z(G_1) \times Z(G_2) \ : \ \theta(x) + y = 0\}$ is a normal subgroup of $G_1 \times G_2$. Define $\phi : G_1 \times G_2 \to G$ by

$$\phi((v_1, w_1), (v_2, w_2)) = ((v_1, v_2), \theta(w_1) + w_2)$$

where $(v_1, w_1) \in G_1$ and $(v_2, w_2) \in G_2$. We notice that ϕ is a group homomorphism as for $(v_1, w_1), (v_1', w_1') \in G_1$ and $(v_2, w_2), (v_2', w_2') \in G_2$ we have,

$$\begin{aligned}
&\phi(((v_1, w_1), (v_2, w_2)) \cdot ((v_1', w_1'), (v_2', w_2'))) \\
&= \phi(((v_1, w_1).(v_1', w_1')), ((v_2, w_2).(v_2', w_2'))) \\
&= \phi((v_1 + v_1', c_1(v_1, v_1') + w_1 + w_1'), (v_2 + v_2', c_2(v_2, v_2') + w_2 + w_2')) \\
&= ((v_1 + v_1', v_2 + v_2'), \theta(c_1(v_1, v_1')) + \theta(w_1) + \theta(w_1') + c_2(v_2, v_2') + w_2 + w_2') \\
&= ((v_1 + v_1', v_2 + v_2'), \theta(c_1(v_1, v_1')) + c_2(v_2, v_2') + \theta(w_1) + w_2 + \theta(w_1') + w_2') \\
&= ((v_1, v_2) + (v_1', v_2'), c_\theta((v_1, v_2), (v_1', v_2')) + \theta(w_1) + w_2 + \theta(w_1') + w_2') \\
&= ((v_1, v_2), \theta(w_1) + w_2)((v_1', v_2'), \theta(w_1') + w_2') \\
&= \phi((v_1, w_1), (v_2, w_2))\phi((v_1', w_1'), (v_2', w_2')).
\end{aligned}$$

The homomorphism ϕ is surjective because for an arbitrary $((v_1, v_2), w) \in G$ we have

$$\phi((v_1, 0), (v_2, w)) = ((v_1, v_2), w).$$

Now we compute $\ker(\phi)$.

$$\begin{aligned}
\ker(\phi) &= \{((v_1, w_1), (v_2, w_2)) \in G_1 \times G_2 : \phi((v_1, w_1), (v_2, w_2)) = ((0, 0), 0)\} \\
&= \{((v_1, w_1), (v_2, w_2)) \in G_1 \times G_2 : ((v_1, v_2), \theta(w_1) + w_2)) = ((0, 0), 0)\} \\
&= \{((v_1, w_1), (v_2, w_2)) \in G_1 \times G_2 : v_1 = 0, v_2 = 0, \theta(w_1) + w_2 = 0\} \\
&= \{((0, w_1), (0, w_2)) \in G_1 \times G_2 : \theta(w_1) + w_2 = 0\} \\
&= \{(w_1, w_2) \in Z(G_1) \times Z(G_2) : \theta(w_1) + w_2 = 0\} \\
&= N.
\end{aligned}$$

Therefore, finally we have $G \simeq \frac{G_1 \times G_2}{\ker(\phi)} = \frac{G_1 \times G_2}{N} = G_1 \circ_\theta G_2$. $\square$

Chapter 2

Extraspecial 2-groups

This chapter consists of three sections. In §2.1 we record the classification of extraspecial 2-groups using quadratic forms associated with these groups. In §2.2 and §2.3 we describe the representations of extraspecial 2-groups and Wedderburn decomposition of rational group algebra of extraspecial 2-groups, respectively. In §2.2 we also recall one dimensional representations of special 2-groups.

Recall that a special 2-group G is called extraspecial 2-group if $|Z(G)| = 2$. Let $\mathbb{F}_2$ be the field containing two elements. The classification of extraspecial 2-groups using quadratic forms over $\mathbb{F}_2$ is outlined in [Wil09]. Extraspecial 2-groups can also be classified using group theoretic methods (see [Gor80]). However the classification of extraspecial 2-groups using quadratic forms is a beautiful application of the theory of quadratic forms over fields of characteristic 2. In the following section, we use the theory of quadratic forms over $\mathbb{F}_2$ to classify extraspecial 2-groups.

2.1 Classification of extraspecial 2-groups

Let V be a vector space over $\mathbb{F}_2$ and $q : V \to \mathbb{F}_2$ be a regular quadratic form. Let G be a special 2-group associated to q (see Th. 1.3.11). One may regard $Z(G)$ as the field of two elements and therefore the group G is an extraspecial 2-group. Conversely, Let G be a extraspecial 2-group. The quadratic map associated to G given by Th. 1.3.4 is a regular quadratic form over $\mathbb{F}_2$.

Proposition 2.1.1. *The order of an extraspecial group is* 2^{2n+1} *for some* $n \in \mathbb{N}$.

Proof A regular quadratic form over a field of characteristic two is even dimensional (see remark 1.1.11). Let G be an extraspecial 2-group and $q : V \to \mathbb{F}_2$ be the quadratic form associated to it as in Th. 1.3.4. Since the quadratic form q_1 is regular, $dim_{\mathbb{F}_2} V = 2n$ for some $n \in \mathbb{N}$. From §1.3, G is in bijection with $V \times \mathbb{F}_2$. Therefore $|G| = 2^{2n} \times 2 = 2^{2n+1}$. □

From now onwards, D_4 will denote the dihedral group of order 8 and Q_2 will denote the quaternion group of order 8 . These groups can be presented as:

$$D_4 = \langle a, b \ : \ a^4 = b^2 = 1, bab^{-1} = a^{-1} \rangle$$
$$Q_2 = \langle c, d \ : \ c^4 = 1, d^2 = c^2, dcd^{-1} = c^{-1} \rangle$$

Proposition 2.1.2. *Let* V *be the two dimensional vector space over* $\mathbb{F}_2$ *and* $q : V \to \mathbb{F}_2$ *be a regular quadratic form. Then the extraspecial* 2*-group associated to* q *is either* Q_2 *and* D_4.

Proof The quadratic forms $q_1 = [0, 0]$ and $q_2 = [1, 1]$ are the only regular 2-dimensional quadratic forms over $\mathbb{F}_2$ up to isometry (see remark 1.1.11). Recall that $q_1(x, y) = xy$ and $q_2(x, y) = x^2 + xy + y^2$. We show that the dihedral group D_4 is the extraspecial 2-group associated to the quadratic form $[0, 0]$. From Th. 1.3.11 we have that $V \dot{\times} \mathbb{F}_2$ is the group associated to $q_1 : V \to \mathbb{F}_2$, where the multiplication is defined by

$$(v, \alpha)\cdot)(v', \alpha') = (v + v', c_1(v, v') + \alpha + \alpha')$$

where $v, v' \in V$, $\alpha, \alpha' \in \mathbb{F}_2$ and $c_1 \in H^2(V, \mathbb{F}_2)$ is the normal 2-cocycle such that $q_1(v) = c_1(v, v)$ for all $v \in V$.

Consider the map $\psi : D_4 \to V \dot{\times} \mathbb{F}_2$ defined by $\psi(a) = ((1, 1), 1)$ and $\psi(b) = ((1, 0), 1)$, where a and b are generating elements of D_4 as in the presentation of D_4. Now we claim that ψ is an isomorphism of groups. Clearly both D_4 and $V_1 \dot{\times} \mathbb{F}_2$ are groups of order 8. It is easy to check that the orders of $((1, 1), 1)$ and $((1, 0), 1)$ are 4 and 2, respectively.

Moreover,

$$\begin{aligned}((1,0),1)((1,1),1)^{-1} &= ((1,0),1)((1,1),c_1((1,1),(1,1))+1)\\ &= ((1,0),1)((1,1),q_1(1,1)+1)\\ &= ((1,0),1)((1,1),0)\\ &= ((1,0)+(1,1),c_1((1,0),(1,1))+1)\\ &= ((0,1),1)\\ &= ((1,1)+(1,0),c_1((1,1),(1,0))+1+1)\\ &= ((1,1),1)((1,0),1)\end{aligned}$$

Therefore ψ is an isomorphism of groups. On similar lines it can be shown that the quaternion group Q_2 is the group associated with quadratic form $q_2 = [1,1]$. In that case, an isomorphism is given by $\psi' : Q_2 \to V_2\dot{\times}\mathbb{F}_2$, where $\psi'(c) = ((1,1),1)$, $\psi'(d) = ((1,0),1)$ and c, d denote generators of group Q_2 as in the given presentation of Q_2. □

Proposition 2.1.3. *Let $q_1 = [0,0]\perp[0,0]$ and $q_2 = [1,1]\perp[1,1]$ be two quadratic forms over $\mathbb{F}_2$. Then q_1 is isometric to q_2.*

Proof We have $q_1(w,x,y,z) = wx+yz$ and $q_2(w,x,y,z) = w^2+wx+x^2+y^2+yz+z^2$. The following map is the isometry between the quadratic forms q_1 and q_2.

$$\begin{aligned} w &\mapsto x+y+z\\ x &\mapsto w+y+z\\ y &\mapsto w+x+z\\ z &\mapsto w+x+y\end{aligned}$$

□

Proposition 2.1.4. *The group $D_4 \circ D_4$ is isomorphic to $Q_2 \circ Q_2$ where $\circ$ denotes the central product of groups.*

Proof From lemma 1.3.14 and Prop. 2.1.2, the quadratic form associated to $D_4 \circ D_4$ is $[0,0] \perp [0,0]$ and that the quadratic form associated to $Q_2 \circ Q_2$ is $[1,1] \perp [1,1]$. The quadratic forms $[0,0] \perp [0,0]$ and $[1,1] \perp [1,1]$ are isometric (Prop. 2.1.3). Now the result follows from Th. 1.3.11. □

Proposition 2.1.5. *For every $n \in \mathbb{N}$, there are exactly two extraspecial 2-groups of order 2^{2n+1}, namely $D_4 \circ D_4 \circ \cdots \circ D_4$ (n copies of D_4) and $Q_8 \circ D_4 \circ \cdots \circ D_4$ ($n-1$ copies of D_4).*

Proof Let G be an extraspecial 2-group and $q : V \to \mathbb{F}_2$ be the associated regular quadratic form. Since q is regular, $\dim_{\mathbb{F}_2}(V)$ is even (see remark 1.1.11). We assume that $\dim_{\mathbb{F}_2}(V) = 2n$ for some $n \in \mathbb{N}$. From Prop.1.1.9, it follows that q is an orthogonal sum of two dimensional regular quadratic spaces over $\mathbb{F}_2$. Since $[0,0] \perp [0,0] \simeq [1,1] \perp [1,1]$ (see Prop. 2.1.3), we conclude that either $q \simeq [0,0] \perp [0,0] \perp \cdots \perp [0,0]$ or $q \simeq [1,1] \perp [0,0] \perp \cdots [0,0]$. By Prop. 2.1.2, we know that the quadratic forms $[0,0]$ and $[1,1]$ are associated with the groups D_4 and Q_2, respectively. It follows from lemma 1.3.14 that orthogonal sum of quadratic forms corresponds to the central product of extraspecial 2-groups. Therefore, we conclude that the group associated to q is either $D_4 \circ D_4 \circ \cdots \circ D_4$ (n copies of D_4) or $Q_2 \circ D_4 \circ \cdots \circ D_4$ ($n-1$ copies of D_4). This completes the classification of extraspecial 2-groups. □

Notation 2.1.6. *From now onwards, we denote the extraspecial 2-group $D_4 \circ D_4 \circ \cdots \circ D_4$ (n copies of D_4) and $Q_2 \circ D_4 \circ \cdots \circ D_4$ ($n-1$ copies of D_4) by $D^{(n)}$ and $Q_2 \circ D_4^{(n-1)}$, respectively.*

Remark 2.1.7 The Arf invariant of quadratic form $q \simeq [0,0] \perp [0,0] \perp \cdots \perp [0,0]$ associated to extraspecial 2-group $D_4^{(n)}$ is trivial. The Arf invariant of quadratic form $q \simeq [1,1] \perp [0,0] \perp \cdots \perp [0,0]$ associated to extraspecial 2-group $Q_2 \circ D_4^{(n-1)}$ is equal to 1.

2.2 Representations of extraspecial 2-groups

The method of writing one dimensional representations of extraspecial 2-groups is exactly same as that of writing one dimensional representation for special 2-group. We discuss in general, the method of writing one dimensional representation of special 2-groups.

2.2.1 One dimensional representations of special 2-groups

The representations of degree one are called one dimensional representations. Finding one dimensional representations of special 2-groups is elementary and is based on the following well-known results.

Theorem 2.2.1 ([JL01], Th. 17.11)**.** *Let G be a finite group. The one dimensional representations of G are precisely the lifts to G of the irreducible representations of $\frac{G}{G'}$. In particular, the number of distinct one dimensional representations of G equals the index of G' in G, where G' denotes the derived subgroup of G.* □

In view of Th. 2.2.1 and Th. 1.2.15 we make the following remark on the one dimensional representations of special 2-groups.

Remark 2.2.2 For a special 2-group G the quotient $\frac{G}{Z(G)}$ is isomorphic to a direct product of copies of $\frac{\mathbb{Z}}{2\mathbb{Z}}$. The group $\frac{\mathbb{Z}}{2\mathbb{Z}}$ has only one non-trivial irreducible representation. Using Th. 1.2.15 one can write all irreducible representations of $\frac{G}{Z(G)}$. Now from Th. 2.2.1 and the equality $Z(G) = G'$ for special 2-groups, one can write all one dimensional representations of special 2-groups.

The following section concerns the representations of dimension at least 2 of extraspecial 2-groups.

2.2.2 Representations of dimension at least 2 of extraspecial 2-groups

The extraspecial 2-groups are central products of copies of non-abelian groups of order 8 (see Prop. 2.1.5). The following theorem provides a method to write representations of central products of groups when the irreducible representations of individual components are known.

Theorem 2.2.3 ([Gor80],Ch. 3, Th. 7.2)**.** *Let H and K be two groups and $\zeta : Z(H) \to Z(K)$ be a group isomorphism. Let $N := \{(h,k) \in Z(H) \times Z(K) \ : \ \zeta(h)k = 1\}$ and $G = H \circ K \cong \frac{H \times K}{N}$ be the central product of groups H and K. Let $\rho : H \times K \to \mathrm{GL}(n, \mathbb{C})$ be a representation of $H \times K$. If $N \subseteq \ker(\rho)$ then ρ induce representation $\widehat{\rho}$ of G defined by $\widehat{\rho}((h,k)N) = \rho(h,k)$ for all $(h,k)N \in G$. All irreducible representations of $G = H \circ K$ are of this type. Moreover a representation $\widehat{\rho}$ is irreducible if and only if ρ is irreducible.*

□

We use the above theorem to describe representations of degree at least 2 of extraspecial 2-groups.

Remark 2.2.4

1. For the dihedral group $D_4 = \langle a, b \ : \ a^4 = b^2 = 1, bab^{-1} = a^{-1} \rangle$ the homomorphism $\rho : D_4 \to \mathrm{GL}(2, \mathbb{C})$ defined by

$$\rho(a) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \rho(b) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

is the only irreducible representation of degree at least 2.

2. For the quaternion group $Q_2 = \langle c, d \ : \ c^4 = 1, d^2 = c^2, dcd^{-1} = c^{-1} \rangle$ the homomorphism $\sigma : Q_2 \to \mathrm{GL}(2, \mathbb{C})$, where

$$\sigma(c) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \sigma(d) = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

is the only irreducible representation of degree at least 2.

Based on this remark, we have the following :

Proposition 2.2.5. *Every extraspecial* 2*-group has a unique irreducible representation of degree at least* 2*.*

Proof Let $\gamma_i : 1 \leq i \leq 4$ be one dimensional representations of the group D_4. From Th. 1.2.15 the irreducible representations of degree at least 2 of the group $D_4 \times D_4$ are $\rho \otimes \gamma_i : 1 \leq i \leq 4$ and $\rho \otimes \rho$, where ρ is as in remark 2.2.4(1). The central product $D_4 \circ D_4 = \frac{D_4 \times D_4}{N}$, where $N := \{(1, 1), (a^2, a^2)\}$ is normal subgroup of the group $D_4 \times D_4$. The subgroup N is contained in the kernel of $\rho \otimes \rho$ but not contained in the kernel of $\rho \otimes \gamma_i$ for all $1 \leq i \leq 4$. From Th. 2.2.3 it is evident that $\rho \otimes \rho$ is the only representation of degree at least 2 of $D_4 \times D_4$, which induces the representation $\widehat{\rho \otimes \rho}$ of $D_4 \circ D_4$. This generalises to the fact that the representation $\rho \otimes \widehat{\rho \otimes \cdots} \otimes \rho$ (n copies of ρ) is the only representation of degree at least 2 of $D_4^{(n)}$. Similarly the group $Q_8 \circ D_4^{(n-1)}$ has unique irreducible representation of degree at least 2, namely $\sigma \otimes \widehat{\rho \otimes \cdots} \otimes \rho$ ($n - 1$ copies of ρ). □

Remark 2.2.6 The degree of unique representation of degree at least 2 of extraspecial 2-group of order 2^{2n+1} is 2^n. This irreducible representation of degree at least 2 of extraspecial 2-group is faithful (see [Gor80, p. 208, Th. 5.5]).

Lemma 2.2.7. *Let G be an extraspecial 2-group of order 2^{2n+1} and χ be the character of its unique irreducible representation of degree at least 2. Then*

$$\chi(g) = \begin{cases} 2^n & \text{if } g \text{ is the identity element of } G \\ -2^n & \text{if } g \text{ is a non trivial element of } Z(G) \\ 0 & \text{otherwise.} \end{cases}$$

Proof The extraspecial 2-group G is isomorphic to either $D_4^{(n)}$ or $Q_2 \circ D_4^{(n-1)}$ (see Prop. 2.1.5).

Let $\varphi : G \to \mathrm{GL}(2^n, \mathbb{C})$ be the irreducible representation of degree at least 2 of $D_4^{(n)}$. Then $\varphi = \rho \otimes \widehat{\rho \otimes \cdots} \otimes \rho$ (n copies of ρ), where ρ is unique representation of degree at least 2 of D_4 as in remark 2.2.4(1).

By Th. 1.2.15, we know that $\varphi(\bar{1}) = \rho(1) \otimes \rho(1) \otimes \cdots \otimes \rho(1)$($n$ times). Let χ_φ and χ_ρ be the characters afforded by representations φ and ρ respectively. Using the fact that for two matrices A and B, $\mathrm{tr}(A \otimes B) = \mathrm{tr}(A)\,\mathrm{tr}(B)$, we have $\chi_\varphi(\bar{1}) = \chi_\rho(1) \otimes \chi_\rho(1) \otimes \cdots \otimes \chi_\rho(1)$. Since $\rho(1)$ is 2×2 identity matrix (see remark 2.2.4(1)) we get $\chi_\varphi(\bar{1}) = 2^n$. We know that order of center of $D_4^{(n)}$ is 2. If $\bar{1} \neq g \in Z(G)$ then $g = \overline{(a^2, 1, \cdots, 1)}$, where a^2 is the non-trivial element of $Z(D_4)$. We have

$$\begin{aligned} \chi_\varphi(g) &= \mathrm{tr}(\rho(a^2) \otimes \rho(1) \otimes \cdots \otimes \rho(1)) \\ &= \mathrm{tr}(\rho(a^2))\,\mathrm{tr}(\rho(1)) \cdots \mathrm{tr}(\rho(1)) \\ &= -2^n \end{aligned}$$

If $g = \overline{(g_1, g_2, \cdots, g_l)} \in G - Z(G)$, then for some $1 \leq i \leq l$, we have $g_i \in D_4 - Z(D_4)$ and $\chi_\rho(g_i) = 0$. Thus $\chi_\varphi(g) = 0$. This proves the result if $G \cong D_4 \circ D_4 \circ \cdots \circ D_4$ (l copies of D_4). On the other hand if $G \cong Q_2 \circ D_4^{(n-1)}$, then the representation of degree at least 2 of G is $\sigma \otimes \rho \widehat{\otimes \rho \otimes \cdots} \otimes \rho$ ($l-1$ copies of ρ). Here σ and ρ are unique representations of degree at least 2 of Q_2 and D_4, respectively as in remark 2.2.4. On similar lines, one can prove result in this case as well. □

The following proposition gives the type of irreducible representation of degree at least 2 of extraspecial 2-groups.

Proposition 2.2.8. *The irreducible representation of degree at least 2 of extraspecial 2-group $D_4^{(n)}$ is orthogonal, while that of $Q_8 \circ D_4^{(n-1)}$ is symplectic.*

Proof It follows from example 1.2.12(2) that the unique irreducible representation ρ of degree at least 2 of D_4 is orthogonal. The irreducible representation of degree at least 2 of $D_4 \circ D_4 \circ \cdots \circ D_4$ (l copies of D_4) is induced from the tensor product of n copies of ρ. From corollary 1.2.17 and corollary 1.2.20, it follows that irreducible representation of degree at least 2 of $D_4 \circ D_4 \circ \cdots \circ D_4$ (l copies of D_4) is orthogonal.

Example 1.2.12(3) shows that the unique irreducible representation ϕ of degree at least 2 of Q_2 is symplectic. The irreducible representation of degree at least 2 of $Q_2 \circ D_4 \circ \cdots \circ D_4$ ($l-1$ copies of D_4) is induced from the tensor product of ϕ with $l-1$ copies of representation ρ of D_4. From corollary 1.2.17 and corollary 1.2.20, it follows that the irreducible representation of degree at least 2 of $Q_2 \circ D_4 \circ \cdots \circ D_4$ (l copies of D_4) is symplectic. □

Proposition 2.2.9. *Let G be an extraspecial 2-group and $q : V \to \mathbb{F}_2$ be quadratic form associated to G. The unique irreducible representation of degree at least 2 of G is orthogonal (symplectic) if and only if* $\mathrm{Arf}(q) = 0$ *(*$\mathrm{Arf}(q) = 1$*).*

Proof It follows from the remark 2.1.7 and the Prop. 2.2.8. □

We summarize the above results regarding the type of representation of degree at least 2 of extraspecial 2-groups in following table for the further reference:

Extraspecial 2-group	**Type of representation of degree at least 2**	**Arf Invariant**
$D_4^{(n)}$	Orthogonal	0
$Q_2 \circ D_4^{(n-1)}$	Symplectic	1

Table 2.1: Arf Invariant for associated quadratic form and type of representation of degree at least 2 for extraspecial 2-groups.

2.3 Wedderburn decomposition of rational group algebra of extraspecial 2-groups

Let $\mathbb{K}$ be a field and G be a finite group. Let $\mathbb{K}[G] = \{\sum_{g\in G} \lambda_g g, \lambda_g \in \mathbb{K}\}$. We define addition and multiplication on $\mathbb{K}[G]$ as follows:

$$\sum_{g\in G} \lambda_g g + \sum_{g\in G} \mu_g g = \sum_{g\in G} (\lambda_g + \mu_g) g$$

$$\sum_{g\in G} \lambda_g g . \sum_{h\in G} \mu_h h = \sum_{g\in G, h\in G} (\lambda_h \mu_{h^{-1}g}) g$$

Under the above defined addition and multiplication, $\mathbb{K}[G]$ is $\mathbb{K}$-algebra. It is called the $\mathbb{K}$-group algebra of G .

The famous theorem of Maschke states that $\mathbb{K}[G]$ is semisimple if and only if char($\mathbb{K}$) does not divide $|G|$ (see [PMS02], Th. 3.4.7). The following theorem is another celebrated result in the theory of group rings.

Theorem 2.3.1. Wedderburn Artin Theorem *[[PMS02], Th. 3.4.9] Let G be a finite group and let $\mathbb{K}$ be a field such that* char($\mathbb{K}$) *does not divide $|G|$. Then*

1. *$\mathbb{K}[G]$ is a direct sum of a finite number of its two sided ideals A_i; $1 \le i \le r$. Also for all $1 \le i \le r$, A_i is a simple ring.*

2. *Each simple component A_i is isomorphic to $M_{n_i}(D_i)$, where D_i is a division ring containing a copy of $\mathbb{K}$ in its center. Also $\mathbb{F}G \cong \oplus_{i=1}^{r} M_{n_i}(D_i)$ as $\mathbb{K}$-algebras.*

The decomposition of group algebra $\mathbb{K}[G]$ given in Th. 2.3.1 is called the Wedderburn decomposition of $\mathbb{K}[G]$. We recall the definition of primitive central idempotent.

Definition 2.3.2. *An element e of a ring R is called idempotent if $e^2 = e$. A set of idempotents $\{e_1, e_2, \cdots, e_r\}$ is called a complete set of primitive central idempotents if*

1. *$e_i \in Z(R)$ for all $1 \le i \le r$, where $Z(R)$ denotes the center of the ring R.*

2. *$e_1 + e_2 + \cdots + e_r = 1$ and $e_i e_j = 0$ for all $1 \le i, j \le r$ and $i \ne j$.*

3. *No e_i can be written as $e_i = e' + e''$, where e' and e'' are non zero idempotents with $e'e'' = 0$.*

The decomposition of $\mathbb{K}[G] \cong A_1 \oplus A_2 \oplus \cdots \oplus A_r$ as a direct sum of simple components corresponds to a complete set primitive central idempotents $\{e_1, e_2, \cdots e_r\}$ such that $A_i \cong \mathbb{F}[G]e_i,\ 1 \leq i \leq r$.

The determination of Wedderburn decomposition of a group algebra is one of the fundamental problems in the theory of group rings. A group algebra $\mathbb{K}[G]$ is called rational group algebra if $\mathbb{K} = \mathbb{Q}$.

Notation 2.3.3. *Let G be a finite group and H be its subgroup. We denote the element $\frac{1}{|H|}\sum_{h\in H} h$ of $\mathbb{Q}[G]$ by $\widehat{H}$.*

Definition 2.3.4. *Let $\mathbb{Q}$ denote the field of rational numbers. Let G be a finite group and $\mathbb{Q}[G]$ be its rational group algebra. Then we can split the rational group algebra as.*

$$\mathbb{Q}[G] = \mathbb{Q}[G].\hat{G'} \oplus \mathbb{Q}[G].(1 - \hat{G'}).$$

The first part $\mathbb{Q}[G].\hat{G'}$ is isomorphic to $\mathbb{Q}[\frac{G}{G'}]$. It contains all the commutative ideals of $\mathbb{Q}[G]$ and is called commutative part of $\mathbb{Q}[G]$. The second part $\mathbb{Q}[G].(1 - \hat{G'})$ is called non-commutative part and is denoted by $\Delta(G, G')$.

Now we discuss the Wedderburn decomposition of extraspecial 2-groups. We first state the following well known results:

Remark 2.3.5 Let G be direct product of n copies of cyclic group C_2 of order 2. Then $\mathbb{Q}[G]$ is a direct sum of 2^n copies of $\mathbb{Q}$ (see [PMS02, p. 150, Exercise 3]). For an extraspecial 2-group G of order 2^{2n+1}, the quotient group $\frac{G}{G'}$ is an elementary abelian 2-group of order 2^{2n}. Thus we get that $\mathbb{Q}[\frac{G}{G'}] \cong \mathbb{Q} \oplus \mathbb{Q} \oplus \cdots \oplus \mathbb{Q}$ (2^{2n} copies of $\mathbb{Q}$). Therefore the commutative part of Wedderburn decomposition of extraspecial 2-group G of order 2^{2n+1} consists orthogonal sum of 2^{2n} copies of the field of rational numbers.

The following result gives us the non commutative component $\Delta(G, G')$ of Wedderburn decomposition of extraspecial 2-groups. Let $\mathbb{H}$ denote the Quaternion algebra $\left(\frac{-1,-1}{\mathbb{Q}}\right)$ over the field $\mathbb{Q}$. We recall that $\mathbb{H}$ can be described as a 4-dimensional vector space over $\mathbb{Q}[G]$ with basis $\{1, i, j, k\}$ and multiplication defined by $i^2 = j^2 = -1$ and $ij = -ji = k$.

Proposition 2.3.6 ([VL06], Prop. 3.4)**.** *Let G be an extraspecial 2-group of order $2^{2n+1}, n \geq 2$.*

1. *If* $G \cong D_4^{(n)}$ *then* $\Delta(G, G') = M_{2^n}(\mathbb{Q})$.

2. *If* $G \cong Q_2 \circ D_4^{(n-1)}$ *then* $\Delta(G, G') = M_{2^{n-1}}(\mathbb{H})$.

We conclude this chapter with a table of extraspecial 2-groups of order 2^{2n+1} with their associated quadratic forms and the non commutative component $\Delta(G, G')$ of their Wedderburn decomposition. This table will be helpful in Part II of thesis.

Extraspecial 2-group	**Quadratic form**	$\Delta(G, G')$
$D_4^{(n)}$	$[0,0] \perp [0,0] \perp \cdots \perp [0,0]$ (n copies of $[0,0]$)	$M_{2^n}(\mathbb{Q})$
$Q_2 \circ D_4^{(n-1)}$	$[1,1] \perp [0,0] \perp [0,0] \perp \cdots \perp [0,0]$ ($n-1$ copies of $[0,0]$)	$M_{2^{n-1}}(\mathbb{H})$

Table 2.2: $\Delta(G, G')$ for an extraspecial 2-group G.

Chapter 3

Totally orthogonal special 2-groups

This short chapter summarizes the results of Zahinda [Zah11]. In this chapter we mainly discuss a characterization of totally orthogonal special 2-groups based on the associated quadratic map. This is indeed a nice illustration of the utility of associating a quadratic map to special 2-groups.

3.1 Real Special 2-groups

Let G be a group. Recall from definition 1.2.6 that an element $g \in G$ is called real if there exist $h \in H$ such that $g^{-1} = hgh^{-1}$.

Definition 3.1.1. *A group is called real if its all elements are real.*

All symmetric groups, Dihedral groups and extraspecial 2-groups are examples of real groups. The following theorem gives a criterion to determine if a special 2-group is real.

Theorem 3.1.2 ([Zah11], Th. 2.1)**.** *Let G be special 2-group and $q : V \to W$ be the quadratic map associated to G. The following assertions are equivalent:*

i. The group G is real.

ii. For all $v \in V$, there exists $a \in V$ such that $q(a) = q(v - a)$.

3.2 Totally orthogonal special 2-groups

We recall from definition 1.2.9 that a representation is said to be orthogonal if it is realizable over the field $\mathbb{R}$.

Definition 3.2.1. *A group G is said to be totally orthogonal group if its all complex irreducible representations are of orthogonal type.*

To state a criterion for checking total orthogonality of a special 2-group, we need the following definition:

Definition 3.2.2. *Let V and W be vector spaces over $\mathbb{F}_2$ and $q : V \to W$ be a quadratic map. Let $s \in \mathrm{Hom}_{\mathbb{F}_2}(W, \mathbb{F}_2)$. The map $s \circ q : V \to \mathbb{F}_2$ is a quadratic form with polar form $b_{s\circ q} := s \circ b_q : V \times V \to \mathbb{F}_2$. The quadratic form $s \circ q$ is called the transfer of q by s.*

The quadratic form $s \circ q$ may not be regular. For example, consider the regular quadratic map $q : V \to W$ defined by $q(x, y, z) = (xy, xz)$ for all $(x, y, z) \in V$. We take the linear map $s : W \to \mathbb{F}_2$ defined by $s(w_1, w_2) = w_1$ for $(w_1, w_2) \in W$. Now the transfer of q by s is given by $s \circ q(x, y, z) = xy$, which is not a regular quadratic form. However, taking a suitable quotient of V may yield a regular quadratic form. This is explained in the following lemma.

Lemma 3.2.3 ([Zah11], Prop. 1.5(i))**.** *Let V and W be vector spaces over $\mathbb{F}_2$ and $q : V \to W$ be a quadratic map. Let $s \in \mathrm{Hom}_{\mathbb{F}_2}(W, \mathbb{F}_2)$ and $s \circ q$ be the transfer of q by s. If $s \circ q(\mathrm{rad}(b_{s\circ q})) = 0$ then $s \circ q$ induces a regular quadratic form q_s from $V_s := \frac{V}{\mathrm{rad}(b_{s\circ q})}$ to $\mathbb{F}_2$ defined by $q_s(\epsilon_s(v)) = s \circ q(v)$ for all $v \in V$. Here $\epsilon_s : V \to V_s$ is the canonical surjection.*

Proof Since $s \circ q(\mathrm{rad}(b_{s\circ q}) = 0$, we have $\mathrm{rad}(b_{s\circ q}) \subseteq \ker(s \circ q)$. Therefore, $s \circ q$ induces a quadratic form $q_s : V_s := \frac{V}{\mathrm{rad}(b_{s\circ q})} \to \mathbb{F}_2$ defined by $q_s(\epsilon_s(v)) = s \circ q(v)$ for all $v \in V$, where $\epsilon_s : V \to V_s$ is the canonical surjection. We first check that the quadratic form q_s is well defined.

Let $v, w \in V$ such that $\epsilon_s(v) = \epsilon_s(w)$. Let $r \in \mathrm{rad}(b_{s\circ q})$ be such that $v = w + r$. We compute:

$$\begin{aligned} q_s(\epsilon_s(v)) - q_s(\epsilon_s(w)) &= s \circ q(v) - s \circ q(w) \\ &= s \circ q(w + r) - s \circ q(w) \\ &= s \circ q(w + r) - s \circ q(w) - s \circ q(r) \end{aligned}$$

$$= b_{s\circ q}(w, r)$$
$$= 0.$$

We now show that the quadratic form $q_s : V_s \to \mathbb{F}_2$ is regular. For $v, w \in V$, we compute

$$\begin{aligned} b_{q_s}(\epsilon_s(v), \epsilon_s(w)) &= q_s(\epsilon_s(v)) + q_s(\epsilon_s(w)) - q_s(\epsilon_s(v) + \epsilon_s(w)) \\ &= s(q(v)) + s(q(w)) - s(q(v+w)) \\ &= s(q(v) + q(w) - q(v+w)) \\ &= s(b_q(v, w)) \\ &= b_{s\circ q}(v, w) \end{aligned}$$

Let $\epsilon_s(v) \in \mathrm{rad}(b_{q_s})$. Then from the above computation, we conclude that $b_{s\circ q}(v, w_i) = 0$ for a set $\{w_i{}_{i=1}^r\}$ of coset representing of V in $\mathrm{rad}(b_{s\circ q})$. Let $w \in V$ be any arbitrary element. Then we write $w = w_i + w'$ for a suitable $w' \in \mathrm{rad}(b_{s\circ q})$ and $1 \leq i \leq r$. Then

$$b_{s\circ q}(v, w) = b_{s\circ q}(v, w_i) + b_{s\circ q}(v, w') = 0$$

Therefore $v \in \mathrm{rad}(b_{s\circ q})$ and $\epsilon_s(v) = 0$. Thus $\mathrm{rad}(b_{q_s})$ is the trivial subspace of V_s and the quadratic form q_s is regular. □

The following lemma is implicit in the proof of Proposition 3.3 of [Zah11]

Lemma 3.2.4. *Let G be a special 2-group and $q : V \to W$ be the quadratic map associated to group G. If G is real then for all $s \in \mathrm{Hom}_{\mathbb{F}_2}(W, \mathbb{F}_2)$, the radical $\mathrm{rad}(b_{s\circ q})$ vanishes under $s \circ q$, the transfer of q by s.*

Proof Let G be the real special 2-group associated to a quadratic map $q : V := \frac{G}{Z(G)} \to Z(G) =: W$. We recall that $q(gZ(G)) = g^2$ for all $gZ(G) \in \frac{G}{Z(G)}$ and the polar map $b_q : \frac{G}{Z(G)} \times \frac{G}{Z(G)} \to Z(G)$ of q is $b_q(g_1Z(G), g_2Z(G)) = g_1^{-1}g_2^{-1}g_1g_2$ for $g_1Z(G), g_2Z(G) \in \frac{G}{Z(G)}$ (see Th. 1.3.4). Let $s \in \mathrm{Hom}_{\mathbb{F}_2}(W, \mathbb{F}_2)$. The polar map $b_{s\circ q} := s \circ b_q : \frac{G}{Z(G)} \times \frac{G}{Z(G)} \to \mathbb{F}_2$ of the transfer $s \circ q$ turns out to be $b_{s\circ q}(g_1Z(G), g_2Z(G)) = s(g_1^{-1}g_2^{-1}g_1g_2)$ for $g_1Z(G), g_2Z(G) \in \frac{G}{Z(G)}$. We prove that quadratic form $s \circ q$ vanishes on the radical $\mathrm{rad}(b_{s\circ q})$. That will finish the proof in the view of lemma 3.2.3. Let $r \in G$ be such that $rZ(G) \in \mathrm{rad}(b_{s\circ q})$. For all $g \in G$,

$$1 = b_{s\circ q}(gZ(G), rZ(G)) = s(g^{-1}r^{-1}gr).$$

As G is real group, there exists $h \in G$ such that $r^{-1} = h^{-1}rh$. In particular for $g = h$, we have

$$1 = b_{s\circ q}(hZ(G), rZ(G)) = s(h^{-1}r^{-1}hr) = s(r^2) = s(q(rZ(G)) = s \circ q(rZ(G)).$$

Thus $s \circ q\,(\mathrm{rad}(b_{s\circ q}))$ is trivial. □

Remark 3.2.5 The above lemmas imply that for a real special 2-group G associated to a quadratic map $q : V \to W$ and $s \in \mathrm{Hom}_{\mathbb{F}_2}(W, \mathbb{F}_2)$, the quadratic form $s \circ q : V \to \mathbb{F}_2$ induces a regular quadratic form $q_s : V_s \to \mathbb{F}_2$ as defined in the statement of lemma 3.2.3.

Now through an example of special 2-group we show that the transfer of quadratic map by some linear map may not vanish on the radical if the corresponding group is not real.

Example 3.2.6 We consider the group G defined by

$$G = \langle a, b, c : a^2 = b^4 = c^4 = 1, bc = cb, aba^{-1} = bc^2, aca^{-1} = cb^2\rangle.$$

We make following observations about G.

- The center of G is $Z(G) := \langle b^2, c^2 : b^4 = c^4 = 1, bc = cb\rangle$, and the quotient by the center is $\frac{G}{Z(G)} := \langle \bar{a}, \bar{b}, \bar{c} : \bar{a}^2 = \bar{b}^2 = \bar{c}^2 = (\bar{ab})^2 = (\bar{ac})^2 = (\bar{bc})^2 = \bar{1}\rangle$. Both $Z(G)$ and $\frac{G}{Z(G)}$ are elementary abelian 2-groups.
- The group G is a special 2-group as $|G| = 32$ and $Z(G) = \Phi(G) = G' = \langle b^2, c^2 : b^4 = c^4 = 1, bc = cb\rangle$.

We identify $\frac{G}{Z(G)}$ with a 3-dimensional vector space V and $Z(G)$ with a 2-dimensional vector space W over the field $\mathbb{F}_2$. Therefore, as a set, the group G gets identified with $V \times W$. Let $\{e_1 = (1, 0, 0), e_2 = (0, 1, 0), e_3 = (0, 0, 1)\}$ be a basis of V and $\{f_1 = (1, 0), f_2 = (0, 1)\}$ be a basis of W over $\mathbb{F}_2$. The quadratic map $q : V \to W$ associated to the special 2-group G is defined by

$$q(x, y, z) = (z^2 + xy, y^2 + xz); \quad (x, y, z) = x(1, 0, 0) + y(0, 1, 0) + z(0, 0, 1) \in V.$$

We use the Th. 3.1.2 to show that group G is not real. The following table shows that for $v = (0, 1, 0)$, there does not exist $a \in V$ such that $q(a) = q(v - a)$.

a	$v-a$	$q(a)$	$q(v-a)$
$(0,0,0)$	$(0,1,0)$	$(0,0)$	$(0,1)$
$(1,0,0)$	$(1,1,0)$	$(0,0)$	$(1,1)$
$(0,1,0)$	$(0,0,0)$	$(0,1)$	$(0,0)$
$(0,0,1)$	$(0,1,1)$	$(1,0)$	$(1,1)$
$(1,1,0)$	$(1,0,0)$	$(1,1)$	$(0,0)$
$(1,0,1)$	$(1,1,1)$	$(1,1)$	$(0,0)$
$(0,1,1)$	$(0,0,1)$	$(1,1)$	$(1,0)$
$(1,1,1)$	$(1,0,1)$	$(0,0)$	$(1,1)$

Table 3.1: Special 2-group G defined in example 3.2.6 is not real.

Consider the linear map $s : W \to \mathbb{F}_2$ defined by $s(w_1, w_2) \to w_1$. The transfer $s \circ q : V \to \mathbb{F}_2$ of q by s is given by $s \circ q(x, y, z) = z^2 + xy$. We compute $\text{rad}(b_{s \circ q}) = \langle (0, 0, 1) \rangle$. Since $s \circ q(0, 0, 1) = 1$, we conclude that $s \circ q$ does not not vanish on $\text{rad}(b_{s \circ q})$.

The following theorem states a criterion to check the total orthogonality of a real special 2-group. The outline of the proof that we indicate here is due to Zahinda [Zah11]. We reproduce it here for the sake of brevity.

Theorem 3.2.7 ([Zah11], Th. 3.5)**.** *Let G be the special 2-group associated to a quadratic map $q : V \to W$. If G is real then the following are equivalent:*

i. The group G is totally orthogonal.

ii. For all non-zero $s \in \text{Hom}_{\mathbb{F}_2}(W, \mathbb{F}_2)$ the Arf invariant $\text{Arf}(q_s)$ is trivial.

Proof $(i) \Rightarrow (ii)$ Since G is real, for any non zero linear map $s : W \to \mathbb{F}_2$, the remark 3.2.5 implies that the quadratic form $s \circ q$ induces a regular quadratic form q_s as defined in lemma 3.2.3.

Let G_s be the extraspecial 2-group associated to regular quadratic form q_s (Th. 1.3.11). There is a surjective group homomorphism f_s from G to extraspecial 2-group G_s.

The extraspecial 2-group G_s has unique irreducible representation ρ_s of degree at least 2 (Prop. 2.2.6). The composition of these two maps $\rho := \rho_s \circ f_s$ is an irreducible representation of special 2-group G.

Since G is totally orthogonal special 2-group, the representation ρ of G is orthogonal. This implies the representation ρ_s of group G_s is orthogonal. Since q_s is quadratic form associated to G_s, using Prop. 2.2.9 we get $\mathrm{Arf}(q_s) = 0$.

$(ii) \Rightarrow (i)$ The outline of the proof of converse part is as follows: For any representation ρ of degree at least 2 of group G, the group $\rho(G)$ is an extraspecial 2-group. Let q_ρ be the quadratic form associated to the group $\rho(G)$ as defined in Th. 1.3.4. Note that $\rho(Z(G)$ is a cyclic group of order 2.

The restriction of ρ on $Z(G)$ is a linear map from $Z(G)$ to $\mathbb{F}_2$. We denote this by s. Since the group G is real, the transfer of quadratic map q by s induces a regular quadratic form q_s (see remark 3.2.5). Then one can show that $q_s \simeq q_\rho$.

The representation ρ induces an irreducible representation ρ' of extraspecial 2-group $\rho(G)$. The degree and type of both the representations ρ and ρ' is same. Therefore ρ' is unique irreducible representation of degree at least 2 of extraspecial 2-group $\rho(G)$. Now using the fact that $q_\rho \cong q_s$ and the hypothesis $\mathrm{Arf}(q_s) = 0$ for all non zero $s \in \mathrm{Hom}_{\mathbb{F}_2}(W, \mathbb{F}_2)$, we conclude, using Prop. 2.2.9 that ρ' is orthogonal. Therefore the representation ρ of G is orthogonal. $\square$

Part II

Results obtained in the thesis

Chapter 4

Strong reality and symplectic representations

The aim of this chapter is to exhibit an infinite class of strongly real finite groups with symplectic representations and conversely, totally orthogonal groups which are not strongly real. All examples exhibited here are special 2*-groups. It is conjecture of Tiep that such examples are not possible in the case of finite simple groups. The results of this chapter indicates that the analogue of Tiep's conjecture is false in a strong sense for special* 2*-groups.*

This chapter is divided in to two sections. In §4.1, we obtain a criterion for strong reality of special 2*-group using the associated quadratic maps. Next section is devoted to exhibit an infinite class of finite groups for which neither the notion of strong reality and total orthogonality implies the other. At the end of this chapter we tabulate the lists of all groups up to order 128, which are strongly real and afford symplectic representations and conversely, totally orthogonal groups which are not strongly real. These lists have been obtained using the computer algebra system GAP [GAP08].*

There are plenty of examples of groups which are both strongly real and totally orthogonal. The symmetric group S_n is both strongly real and totally orthogonal. The alternating group A_n is real if and only if $n = 1, 2, 5, 6, 10, 14$ [Ber69, Th. 1.2]. All real classes of A_n are strongly real [Sul08, §3, corollary 3]. Moreover its all real representations are totally orthogonal [Tur92, Th. 1.1].

All real classes of general linear group $\mathrm{GL}_n(q)$ are strongly real [Won66, Th. 1] and all its real representations are orthogonal [Pra98, Th. 4]. Therefore in $\mathrm{GL}_n(q)$, the number of conjugacy classes of strongly real elements is same as the number of orthogonal characters.

The special linear group $\mathrm{SL}_2(q)$ is strongly real as well as totally orthogonal when q is even and it is neither strongly real nor totally orthogonal when q is odd [KS11, 5.3].

The orthogonal group $\mathrm{O}_n(q)$ is strongly real [Won66] as well as totally orthogonal [Gow85, Th. 1]. A real group whose Sylow 2-subgroups are abelian is strongly real as well as totally orthogonal. [Arm96, Cor. 11, Th. 12]

This motivates us to compare the two notions for arbitrary groups.

4.1 Strongly real special 2-groups

We begin with the definition of strongly real elements. In the following definitions, G denotes a group.

Definition 4.1.1. *An element $g \in G$ is called strongly real if there exists an element $h \in G$ such that $h^2 = 1$ and $g^{-1} = hgh^{-1}$.*

An element $h \in G$ is called an *involution* if the order of h is 2. Every involution is a strongly real element, since for every involution h, the identity element $1 \in G$ satisfies $h^{-1} = 1h1^{-1}$. Let g be a strongly real element in G which is not an involution. Then by definition, there exists $h \in G$ such that $h^2 = 1$ and $g^{-1} = hgh^{-1}$. Since $gh^{-1}gh^{-1} = (gh^{-1}g)h^{-1} = h^{-2} = 1$, an strongly real element g is either a involution or a product of two involutions.

We show that if an element is strongly real then all its conjugates are strongly real. Let $g \in G$ be a strongly real element. By definition, there exists $h \in G$ be such that $h^2 = 1$ and $g^{-1} = hgh^{-1}$. Let $g' \in G$, since $h^2 = 1$, we have $(g'hg'^{-1})^2 = 1$ and

$$(g'hg'^{-1})g'gg'^{-1}(g'hg'^{-1})^{-1} = g'hgh^{-1}g'^{-1} = g'g^{-1}g'^{-1} = (g'g^{-1}g'^{-1})^{-1}.$$

Definition 4.1.2. *A group G is called strongly real if all its elements are strongly real.*

We now record a few lemmas.

Lemma 4.1.3 ([KK15], Lemma 3.5)**.** *The direct product of strongly real groups is strongly real.*

Proof Let G and G' be strongly real groups. Let $(g, g') \in G \times G'$. Then there exists $h \in G$ and $h' \in G'$ such that $h^2 = 1 = {h'}^2$, $g^{-1} = h^{-1}gh$ and ${g'}^{-1} = {h'}^{-1}g'h'$. Now $(h, h')^2 = (1, 1)$ and

$$(h, h')^{-1}(g, g')(h, h') = (h^{-1}gh, {h'}^{-1}g'h') = (g^{-1}, {g'}^{-1}) = (g, g')^{-1}.$$

Thus $G \times H$ is strongly real. □

Lemma 4.1.4 ([KK15], Lemma 3.4)**.** *The central product of two strongly real groups is a strongly real group.*

Proof The direct product of two strongly real groups is a strongly real group (see lemma 4.1.3). Central products are quotients of direct products. Thus to prove the lemma, we need to show that a quotient of a strongly real group is strongly real. Let G be a strongly real group and N be a normal subgroup of G. Let $\frac{G}{N}$ be the quotient group and $gN \in \frac{G}{N}$. Let $h \in G$ be such that $h^2 = 1$ and $g^{-1} = hgh^{-1}$. Then $hN \in \frac{G}{N}$ is such that $hNhN = h^2N = N$ and $hNgN(hN)^{-1} = hgh^{-1}N = g^{-1}N = (gN)^{-1}$. Therefore central product of two strongly real groups is strongly real. □

The following theorem gives a characterization of strongly real special 2-groups. We use the quadratic map associated to special 2-group to check the strong reality of these groups.

Theorem 4.1.5 ([KK15], Th. 3.1)**.** *Let G be the special 2-group associated to quadratic map $q : V \to W$. Then G is strongly real if and only if for every non-zero $v \in V$ there exists $a \in V$ with $v \neq a$ and $q(a) = q(a - v) = 0$.*

Proof We first suppose G to be strongly real. Let $x \in G$ and we write $xZ(G) = v \in V$. Since G is strongly real there exists $y \in G$ such that $y^2 = 1$ and $yx^{-1} = xy$. We take $a = yZ(G)$. We know that $q(a) = q(yZ(G)) = y^2 = 1$. Now we compute

$$\begin{aligned} q(a - v) &= q(yZ(G)(xZ(G))^{-1}) \\ &= q(yx^{-1}Z(G)) \\ &= (yx^{-1})^2 \\ &= yx^{-1}xy \\ &= y^2 = 1 \in G \end{aligned}$$

Thus we have $q(a-v) = q(a) = 0$.

For the converse part, we recall that $G = V \dot{\times} W$ where the group operation is defined by

$$(v, w)(v', w') = (v + v', c(v, v') + w + w')$$
$$(v, w)^{-1} = (v, c(v, v) + w).$$

where c is a normal 2-cocycle and $q(x) = c(x, x)$. Let $x = (v, w) \in V \dot{\times} W = G$. By hypothesis there exists $a \in V$ such that $q(a) = q(a-v) = 0$. We choose $y = (a - v, 0) \in G$. We first check that $y^2 = 1 \in G$, whose image in $V \dot{\times} W$ is $(0, 0)$.

$$y^2 = (a-v, 0) + (a-v, 0) = (2(a-v), c(a-v, a-v)) = (2(a-v), q(a-v)) = (0, 0)$$

Moreover

$$\begin{aligned} y^{-1}xy = (a-v, 0)(v, w)(a-v, 0) &= (2a - v, c(a-v, v) + c(a, a-v) + w) \\ &= (v, c(v, v) + c(a, a) + w) \\ &= (v, q(a) + c(v, v) + w) \\ &= (v, c(v, v) + w) \\ &= (v, w)^{-1} = x^{-1}. \end{aligned}$$

Therefore $yxy = x^{-1}$. Further since $y^2 = 1$, we conclude that $yxy^{-1} = x^{-1}$ and G is strongly real. $\square$

A quadratic map $q : V \to W$ is said to be *isotropic* if there exists a non-zero element $v \in V$ such that $q(v) = 0$. From above theorem, it is clear that the quadratic map associated to a strongly real special 2-group is always isotropic. However the converse is not true. For example, consider the special 2-group G associated to the quadratic map $q : V \to W$ defined by $q(x, y, z) = (x^2 + xy + y^2, xz)$. This quadratic map is isotropic because $q(0, 0, 1) = (0, 0)$. But we claim that the group G is not strongly real. We use the Th. 4.1.5 to establish our claim. We first find all $a \in V$ such that $q(a) = 0$. Let $a := (x, y, z) \in V$ such that $q(a) = 0$. This gives us $x^2 + xy + y^2 = xz = 0$. It implies that $x = y = 0$ and z may take any value. Hence the value of a is either $(0, 0, 0)$ or $(0, 0, 1)$. We consider $v = (1, 1, 1) \in V$. The following table confirms that there does not exist any $a \in V$ such that $q(a) = q(a - v) = 0$.

a	$v-a$	$q(v-a)$
$(0,0,0)$	$(1,1,1)$	$(1,1)$
$(0,0,1)$	$(1,1,0)$	$(1,0)$

Table 4.1: Special 2-group G associated to $q(x,y,z) = (x^2+xy+y^2, xz)$ is not strongly real.

We use Th. 4.1.5 to the study strong reality of extraspecial 2-groups. The notations are same as defined in notation 2.1.6.

Proposition 4.1.6 ([KK15], Prop. 4.1)**.** *All extraspecial 2-groups except Q_2 are strongly real.*

Proof We first show that the Dihedral group D_4 is strongly real. The quadratic form associated to Dihedral group D_4 is $[0,0]$. We recall that $[0,0]$ is 2-dimensional quadratic form $q: V \to \mathbb{F}_2$ defined by $q(x,y) = xy$. We use Th. 4.1.5 to show that D_4 is strongly real group. For each $v \in V$ we have to exhibit some $a \in V$ such that $q(a) = q(a-v) = 0$. The following table demonstrates that it is indeed possible.

v	a
$(0,0),(1,0),(0,1)$	$(0,0)$
$(1,1)$	$(1,0)$

Table 4.2: The group D_4 is strongly real.

By lemma 4.1.4, the groups $D_4^{(n)}$, $n \in \mathbb{N}$ are strongly real. Now we proof that groups $Q_2 \circ D_4^{(n-1)}$ are strongly real. We again use Th. 4.1.5. The quadratic form associated to $Q_2 \circ D_4$ is $q = [0,0] \perp [1,1]$. As a map $q: V \to \mathbb{F}_2$ is given by

$$q(w,x,y,z) = w^2 + wx + x^2 + yz$$

For each $v \in V$, the following table gives the $a \in V$ such that $q(a) = q(a-v) = 0$.

v	a
$(0,0,0,1),(0,0,1,0),(1,1,1,1),(1,0,1,1),(0,1,1,1),(0,0,0,0)$	$(0,0,0,0)$
$(0,0,1,1)$	$(0,0,0,1)$
$(1,0,0,0),(0,1,0,0),(1,1,1,0),(1,1,0,1)$	$(1,1,1,1)$
$(0,1,1,0),(0,1,0,1),(1,1,0,0)$	$(0,1,1,1)$
$(1,0,1,0),(1,0,0,1)$	$(1,0,1,1)$

Table 4.3: The group $Q_2 \circ D_4^{(n-1)}$ is strongly real.

By repeated use of lemma 4.1.4, we get that groups $Q_2 \circ D_4^{(n-1)}$ for $n \geq 2$ are strongly real.

The only left out extraspecial 2-group is Q_2, which is not strongly real. This is because there is only one involution in Q_2, which is central. □

Lemma 4.1.7. *The Quaternion group Q_2 is real.*

Proof The quadratic form associated to quaternion group Q_2 is $[1,1]$. We recall that $[1,1]$ is a 2-dimensional quadratic form $q : V \to \mathbb{F}_2$ defined by $q(x,y) = x^2 + xy + y^2$. We use Th. 3.1.2 to show that Q_2 is a real group. For each $v \in V$, the following table exhibits $a \in V$ such that $q(a) = q(a - v)$.

v	a
$(0,0),(0,1)$	$(1,0)$
$(1,0),(1,1)$	$(0,1)$

Table 4.4: The group Q_2 is real.

□

It is clear that every strongly real group is real. Therefore Prop. 4.1.6 and lemma 4.1.7 imply that all extraspecial 2-groups are real.

4.2 Examples

In this section, we give the examples of strongly real groups which are not totally orthogonal and vice-versa. The examples of such groups shall be built up on other such examples of groups of smaller orders. The following lemmas are useful in the building of infinite series of such examples.

Lemma 4.2.1 ([KK15], Lemma 3.5)**.** *The direct product of totally orthogonal groups is totally orthogonal.*

Proof Since the representations of a direct product are the tensor products of representations of the individual components [Gor80, Ch. 3, Th. 7.2] and the tensor product of two orthogonal representations is orthogonal (see corollary 1.2.17), the lemma follows. □

Lemma 4.2.2 ([KK15], Lemma 3.6)**.** *Let G and H be special 2-groups. Then*

1. *The direct product $G \times H$ is a special 2-group.*
2. *If H is totally orthogonal and if there is an isomorphism $\theta : Z(G) \to Z(H)$ between the centers of G and H then G is totally orthogonal if and only if $G \circ_\theta H$ is totally orthogonal.*

Proof

1. We know that $Z(G \times H) = Z(G) \times Z(H)$ and $(G \times H)' = G' \times H'$. Also

   $$\Phi(G\times H) = \{(g,h)^2 \,:\, (g,h) \in G\times H\} = \{(g^2,h^2) \,:\, g \in G,\ h \in H\} = \Phi(G)\times\Phi(H).$$

   Where $\Phi(G\times H)$ denotes the Frattini subgroup of $G\times H$. Since $Z(G)$ and $Z(H)$ are elementary abelian 2-groups and $Z(G\times H) = Z(G)\times Z(H)$, the center $Z(G\times H)$ is also an elementary abelian 2-group. This gives $Z(G \times H) = (G \times H)' = \Phi(G \times H)$ and these are elementary abelian 2-groups. Thus $G \times H$ is a special 2-group.

2. Let q_1 and q_2 be the quadratic maps associated to special 2-groups G and H respectively. It is easy to check that $(q_1 \perp_\theta q_2)_s = \theta(q_1)_s \perp (q_2)_s$ for a non zero linear map $s : Z(H) \to \mathbb{F}_2$. Also $\mathrm{Arf}((q_1 \perp_\theta q_2)_s) = \mathrm{Arf}(\theta(q_1)_s) + \mathrm{Arf}((q_2)_s)$. Since H is totally orthogonal, $\mathrm{Arf}((q_2)_s) = 0$ for all non zero linear maps $s : Z(H) \to \mathbb{F}_2$ (Th. 3.2.7). Therefore $\mathrm{Arf}((q_1 \perp_\theta q_2)_s) = 0$ if and only if $\mathrm{Arf}(\theta(q_2)_s) = 0$ for all non zero linear maps $s : Z(H) \to \mathbb{F}_2$. Now again using Th. 3.2.7, we have that the group G is totally orthogonal if and only if $G \circ_\theta H$ is totally orthogonal.

□

4.2.1 Totally orthogonal but not strongly real groups

This section is devoted to give examples of strongly real special 2-groups with symplectic representations. We begin with such examples in class of extraspecial 2-groups. The notations are same as defined in notation 2.1.6.

Proposition 4.2.3 ([KK15], Example 4.2)**.** *All extraspecial 2-groups $Q_2 \circ D_4^{(n-1)}$, $n \geq 2$ are examples of strongly real groups which are not totally orthogonal.*

Proof Let G be an extraspecial 2-group associated to a quadratic form q. The $\mathrm{Hom}_{\mathbb{F}_2}(\mathbb{F}_2, \mathbb{F}_2)$ consists only of one non zero element, namely the identity map. Now by Th. 3.2.7 the group G is totally orthogonal if and only if $\mathrm{Arf}(q)$ is trivial. Extraspecial 2-groups $Q_2 \circ D_4^{(n-1)}$ are not totally orthogonal because the Arf invariant of the associated quadratic form $q = [1,1] \perp [0,0] \cdots \perp [0,0]$ is not trivial. From the Prop. 4.1.6, we know that all extraspecial 2-groups expect Q_2 are strongly real. Thus all extraspecial 2-groups $Q_2 \circ D_4^{(n-1)}$, $n \geq 2$ are strongly real groups which are not totally orthogonal. □

In the view of Th. 3.2.7, we mention that for all $n \in \mathbb{N}$, extraspecial 2-groups $D_4^{(n)}$ are totally orthogonal. This is because the Arf invariant of the quadratic form $q = [0,0] \perp [0,0] \cdots \perp [0,0]$ associated to the group $D_4^{(n)}$ is trivial.

The computer algebra system GAP [GAP08] confirms that the extraspecial 2-group $Q_2 \circ D_4$ is smallest strongly real group which is not totally orthogonal. We give GAP [GAP08] code to check this in the appendix. The order of group $Q_2 \circ D_4$ is 32 and it is the only such group of order 32. The next order in which an example of strongly real group with symplectic representations is found is 64. We record strongly real special 2-group of order 64 which is not totally orthogonal group in the following example.

Example 4.2.4 ([KK15], Example 4.3) Let V (resp. W) be a vector space of dimension 4 (resp. 2) over the field $\mathbb{F}_2$. Consider the regular quadratic map $q(w,x,y,z) = (z^2 + wx + wz + xy, wy)$ from V to W. We show that the special 2-group associated to q is strongly real but not totally orthogonal. We use the Th. 4.1.5 to show that the special 2-group G is strongly real. In the following table we give $a \in V$ for every $v \in V$ such that $q(a) = q(v-a) = 0$.

v	a
$(1,0,0,0),(0,1,0,0),(0,0,1,0),(1,0,0,1),(0,1,1,1)$	$(0,0,0,0)$
$(0,0,0,1),(1,1,0,0),(1,0,1,0),(1,1,1,1)$	$(1,0,0,0)$
$(0,1,1,0),(0,1,0,1)$	$(0,0,1,0)$
$(0,0,1,1)$	$(0,1,0,0)$
$(1,1,1,0),(1,0,1,1),(1,1,0,1)$	$(1,0,0,1)$

Table 4.5: The group G_{64} is strongly real.

We know that every strongly real group is also real. We use Th. 3.2.7 to check that the special 2-group associated to q is not totally orthogonal. Let $s : W \to \mathbb{F}_2$ be the linear map given by $s(w_1, w_2) = w_1 + w_2$ for $(w_1, w_2) \in W$. The quadratic form $s \circ q : V \to \mathbb{F}_2$ given by $s \circ q(w, x, y, z) = (z^2 + wx + wz + xy + wy)$ is regular. Therefore the quadratic forms $s \circ q$ and q_s are same. The following change of variables in $s \circ q$ converts it to the form $[1, 1] \perp [0, 0]$:

$$\begin{aligned} w &\mapsto w + x + z \\ x &\mapsto x + y \\ y &\mapsto y + w \\ z &\mapsto y + z \end{aligned}$$

The Arf Invariant of $[1, 1] \perp [0, 0]$ is equal to 1. Thus we have a linear map $s : W \to \mathbb{F}_2$ for which the Arf invariant of the quadratic form q_s is not trivial. Hence by Th. 3.2.7 this group G is not totally orthogonal.

Remark 4.2.5 We have checked using GAP [GAP08] that special 2-group associated to q is only special 2-group of order 64 which is strongly real and not totally orthogonal. The GAP [GAP08] coding need to check this is given in the appendix.

Notation 4.2.6. *We denote the unique strongly real special* 2*-group of order* 64 *which is not totally orthogonal by* G_{64}*. The quadratic map associated to* G_{64} *is* $q(w, x, y, z) = (z^2 + wx + wz + xy, wy)$*.*

The groups G_{64} and $\mathcal{G} = C_2 \times (Q_2 \circ D_4)$, where C_2 is the group of order 2 are the only strongly real groups of order 64 which are not totally orthogonal. □

We now have all the ingredients to prove the following theorem.

Theorem 4.2.7 ([KK15], Th. A)**.** *For every $m \geq 5$ there exist special 2-groups of order 2^m which are strongly real but not totally orthogonal.*

Proof Let $m \in \mathbb{N}$. We first suppose that m is odd and $m = 2n + 1$. By Prop. 4.2.3 the extraspecial 2-groups $Q_2 \circ D_4^{(n-1)}$ for $n \geq 2$ are strongly real groups with symplectic representation.

Now we consider the case when m is even and the two sub cases: $m = 6 + 4n$ and $m = 8 + 4n$. First suppose that $m = 6 + 4n$. The groups $G_{64} \circ (D_4 \times D_4)^{(n)}$ are strongly real but not totally orthogonal of order $6+4n$, where G_{64} is the group as in example 4.2.4 and $(D_4 \times D_4)^{(n)}$ denotes the n-fold central product of $D_4 \times D_4$. The group D_4 is strongly real and totally orthogonal special 2-group, whereas G_{64} is strongly real special 2-group which is not totally orthogonal. Now the proof follows from lemma 4.2.2, lemma 4.1.4 and example 4.2.4.

Finally we consider the case $m = 8 + 4n$. The group $((Q_2 \circ D_4) \times D_4) \circ (D_4 \times D_4)^{(n)}$ are strongly real but not totally orthogonal of order $8 + 4n$. It follows from lemma 4.2.2, lemma 4.1.4 and Prop. 4.2.3 as the group $Q_2 \circ D_4$ is strongly real extraspecial 2-group which is not totally orthogonal. This completes the proof. □

4.2.2 Strongly real but not totally orthogonal groups

This section is devoted to construct examples of special 2-groups which are totally orthogonal but not strongly real. We construct such examples by finding a quadratic map $q : V \to W$ between vector spaces over field $\mathbb{F}_2$ such that the Arf invariant $\mathrm{Arf}(q_s)$ is trivial for all non-zero $s \in \mathrm{Hom}_{\mathbb{F}_2}(W, \mathbb{F}_2)$ and there exists a non-zero $v \in V$ for which no $a \in V$ satisfies $q(a) = q(a - v) = 0$. The Th. 4.1.5 and Th. 3.2.7 imply that the special

2-groups associated to such quadratic maps are totally orthogonal but not strongly real.

Example 4.2.8 ([KK15], Example 5.1) Let V be the 4-dimensional vector space over $\mathbb{F}_2$ and W be the 3-dimensional vector space over $\mathbb{F}_2$. We define a quadratic map $q : V \to W$ by

$$q(w, x, y, z) = (wx + yz, wy, xy); \quad (w, x, y, z) \in V \tag{4.1}$$

The polar map b_q of q is given by

$$b_q((w_1, x_1, y_1, z_1), (w_2, x_2, y_2, z_2)) = (w_1x_2 + x_1w_2 + y_1z_2 + z_1y_2, w_1y_2 + y_1w_2, x_1y_2 + y_1x_2)$$

where $(w_1, x_1, y_1, z_1), (w_2, x_2, y_2, z_2) \in V$.

We check that $\mathrm{rad}(b_q) = 0$. Let $(w, x, y, z) \in \mathrm{rad}(b_q)$. Then for all $(w_1, x_1, y_1, z_1) \in V$, we have $b_q((w, x, y, z), (w_1, x_1, y_1, z_1)) = 0$. This implies $wx_1 + xw_1 + yz_1 + zy_1 = wy_1 + yw_1 = xy_1 + yx_1 = 0$ and hence $((w, x, y, z) = 0 \in V$. Since $b_q((1, 0, 0, 0), (0, 1, 0, 0)) = (1, 0, 0)$, $b_q((1, 0, 0, 0), (0, 0, 1, 0)) = (0, 1, 0)$ and $b_q((0, 1, 0, 0), (0, 0, 1, 0)) = (0, 0, 1)$ we have $\langle b_q(V \times V)\rangle = W$. From Th. 1.3.11 there exist a unique special 2-group whose associated quadratic map is q. We denote this group by G_{128}. The order of this group is $|V| \times |W| = 128$.

To check that special 2-group associated to the quadratic map q is real, for each $v \in V$, we give $a \in V$ such the criteria of Th. 3.1.2 is satisfied.

v	a	$q(a) = q(a - v)$
$(0,0,0,0), (1,0,0,0), (0,1,0,0), (0,0,1,0),$ $(0,0,0,1), (1,0,0,1), (0,1,0,1)$	$(0,0,0,0)$	$(0,0,0)$
$(1,1,0,0), (1,1,0,1), (1,0,1,0)$	$(1,0,0,0)$	$(0,0,0)$
$(0,1,1,0), (0,0,1,1), (0,1,1,1), (1,0,1,1)$	$(0,0,1,0)$	$(0,0,0)$
$(0,0,0,1), (1,1,1,1)$	$(0,0,1,1)$	$(1,0,0)$

Table 4.6: The group G_{128} is real.

To show that G_{128} is not strongly real, we consider $v = (1, 1, 1, 1) \in V$. Then for every $a \in V$ with $q(a) = 0$ we claim that $q(v - a) \neq 0$. We first identify all $a \in V$ such that $q(a) = 0$. Let $a = (w, x, y, z) \in V$ be a vector such that $q(w, x, y, z) = 0$. This implies

$$wx + yz = wy = xy = 0$$

If $y \neq 0$ then the above condition forces $x = w = z = 0$. Thus we have $a = (0,0,1,0)$. If $y = 0$ then above condition implies that either $w = 0$ or $x = 0$. Therefore we conclude that $a \in \{(0,0,0,0),(0,1,0,0),(1,0,0,0),(0,0,0,1),(0,1,0,1),(1,0,0,1)\}$. In the following table, we compute that for every such a, $q(a-v) \neq 0$.

a	$a-v$	$q(a-v)$
$(0,0,1,0)$	$(1,1,0,1)$	$(1,0,0)$
$(0,0,0,0)$	$(1,1,1,1)$	$(0,1,1)$
$(0,1,0,0)$	$(1,0,1,1)$	$(1,1,0)$
$(1,0,0,0)$	$(0,1,1,1)$	$(1,0,1)$
$(0,0,0,1)$	$(1,1,1,0)$	$(1,1,1)$
$(0,1,0,1)$	$(1,0,1,0)$	$(0,1,0)$
$(1,0,0,1)$	$(0,1,1,0)$	$(0,0,1)$

Table 4.7: The group G_{128} is not strongly real.

The above table and the Th. 4.1.5 confirm that G_{128} is not strongly real.

Now we show that the special 2-group G_{128} associated to the quadratic map q as in equation (4.1) is totally orthogonal. Since $\dim_{\mathbb{F}_2}(W, \mathbb{F}_2) = 3$, there exist exactly 7 non-zero $\mathbb{F}_2$-linear maps from W to $\mathbb{F}_2$, which are the following

$$s_n(x,y,z) = ix + jy + kz; \qquad (x,y,z) \in W, 1 \leq n \leq 7,$$

where $n = 4i + 2j + k$ is the binary expansion of $n \in \{1,2,\cdots,7\}$. We write various transfer maps of q:

$$\begin{aligned}
s_1 \circ q(w,x,y,z) &= xy,\\
s_2 \circ q(w,x,y,z) &= wy,\\
s_3 \circ q(w,x,y,z) &= wy + xy = (w+x)y,\\
s_4 \circ q(w,x,y,z) &= wx + yz,\\
s_5 \circ q(w,x,y,z) &= wx + yz + xy = wx + (z+x)y,\\
s_6 \circ q(w,x,y,z) &= wx + yz + wy = wx + (z+w)y,\\
s_7 \circ q(w,x,y,z) &= wx + yz + wy + xy = wx + (z+w+x)y,
\end{aligned}$$

where $(w,x,y,z) \in V$. By the linear change of variables, the quadratic forms $s_1 \circ q$ and $s_3 \circ q$ are isometric to $s_2 \circ q : V \to \mathbb{F}_2$ defined by $s_2 \circ q(w,x,y,z) = wy$. Whereas the remaining quadratic forms are isometric to $s_4 \circ q : V \to \mathbb{F}_2$ defined by $s_4 \circ q(w,x,y,z) = wx + yz$ by suitable linear changes of variables.

Now $\mathrm{rad}(b_{s_2 \circ q}) = \langle (0,1,0,0),(0,0,0,1) \rangle$ and $\frac{V}{\mathrm{rad}(b_{s_2 \circ q})} = \langle (1,0,0,0),(0,0,1,0) \rangle$. Therefore $s_2 \circ q$ induces regular quadratic form $q_{s_2} : \frac{V}{\mathrm{rad}(b_{s_2 \circ q})} \to \mathbb{F}_2$ defined by $q_{s_2}(\alpha,\beta) = \alpha\beta$, where $(\alpha,\beta) \in \frac{V}{\mathrm{rad}(b_{s_2 \circ q})}$. Since the quadratic form q_{s_2} is isometric to $[0,0]$, $\mathrm{Arf}(q_{s_2}) = 0$. On the other hand, the subspace $\mathrm{rad}(b_{s_4 \circ q})$ is trivial. Therefore quadratic form $s_4 \circ q$ is regular. The quadratic form q_{s_4} is same as $s_4 \circ q$, which is $[0,0] \perp [0,0]$. Hence $\mathrm{Arf}(q_{s_4}) = 0$.

As a consequence, for all $s \in \mathrm{Hom}_{\mathbb{F}_2}(W, \mathbb{F}_2)$ the Arf invariant of the quadratic form q_s is trivial and by Th. 3.2.7 the group G_{128} is totally orthogonal. $\square$

Now we give examples of totally orthogonal special 2-groups of order 2^8 and 2^9 which are not strongly real. These examples are building blocks for constructing such examples in order 2^m for every $m \geq 7$.

Example 4.2.9 ([KK15], Example 5.2) Let V and W be vector spaces over the field $\mathbb{F}_2$ with $\dim_{\mathbb{F}_2}(V) = 5$ and $\dim_{\mathbb{F}_2}(W) = 3$. We check that the special 2-group associated to the quadratic map $q : V \to W$ defined by

$$q(w,x,y,z,t) = (wx + wt + yz, wy, wt + xy); \quad (w,x,y,z,t) \in V \tag{4.2}$$

is totally orthogonal but not strongly real group of order 2^8. We show that this group is not strongly real using Th. 4.1.5. Let $a := (w,x,y,z,t) \in V$ such that $q(a) = 0$ implies that $wx + wt + yz = wy = wt + xy = 0$. If $w = 0$, this condition forces that $yz = xy = 0$. If we further suppose that $y = 0$, we get the following set of values of a.

$$\{(0,0,0,0,0),(0,1,0,0,0),(0,0,0,1,0),(0,0,0,0,1),(0,1,0,1,0),(0,0,0,1,1)(0,1,0,1,1)\}.$$

On the other hand if we consider $w = 0$ and $y \neq 0$, then we get $x = z = 0$. Thus in this case values of a are $(0,0,1,0,0)$ and $(0,0,1,0,1)$. Now we consider the case $w \neq 0$, then by condition $q(a) = 0$ we have $y = t = x = 0$ and the values of a in this case are $(1,0,0,0,0)$ and $(1,0,0,1,0)$. This completes the list of $a \in V$ with property $q(a) = 0$. Consider

$v = (1, 1, 1, 1, 1) \in V$. The calculation in the following table shows that $q(v - a) \neq 0$, for all $a \in V$ for which $q(a) = 0$.

a	$a - v$	$q(a - v)$
$(0, 0, 0, 0, 0)$	$(1, 1, 1, 1, 1)$	$(1, 1, 0)$
$(1, 0, 0, 0, 0)$	$(0, 1, 1, 1, 1)$	$(1, 0, 1)$
$(0, 1, 0, 0, 0)$	$(1, 0, 1, 1, 1)$	$(0, 1, 1)$
$(0, 0, 1, 0, 0)$	$(1, 1, 0, 1, 1)$	$(0, 0, 1)$
$(0, 0, 0, 1, 0)$	$(1, 1, 1, 0, 1)$	$(0, 1, 0)$
$(0, 0, 0, 0, 1)$	$(1, 1, 1, 1, 0)$	$(0, 1, 1)$
$(1, 0, 0, 1, 0)$	$(0, 1, 1, 0, 1)$	$(0, 0, 1)$
$(0, 1, 0, 1, 0)$	$(1, 0, 1, 0, 1)$	$(1, 1, 1)$
$(0, 1, 0, 0, 1)$	$(1, 0, 1, 1, 0)$	$(1, 1, 0)$
$(0, 0, 1, 0, 1)$	$(1, 1, 0, 1, 0)$	$(1, 0, 0)$
$(0, 0, 0, 1, 1)$	$(1, 1, 1, 0, 0)$	$(1, 1, 1)$
$(0, 1, 0, 1, 1)$	$(1, 0, 1, 0, 0)$	$(0, 1, 0)$

Table 4.8: The group G_{256} is not strongly real.

By Th. 4.1.5 we conclude that the special 2-group associated with quadratic map defined in equation (4.2) is not strongly real. Now we use Th. 3.1.2 to show that this group is real. In the following table, for every $v \in V$, we find $a \in V$ such that $q(a) = q(v - a)$.

v	a	$q(a) = q(a - v)$
$(0, 0, 0, 0, 0), (1, 0, 0, 0, 0), (0, 1, 0, 0, 0), (0, 0, 1, 0, 0),$ $(1, 0, 0, 1, 0), (0, 1, 0, 1, 0), (0, 1, 0, 0, 1), (0, 0, 0, 1, 1),$ $(0, 0, 0, 1, 0), (0, 0, 0, 0, 1), (0, 1, 0, 1, 1), (0, 0, 1, 0, 1)$	$(0, 0, 0, 0, 0)$	$(0, 0, 0)$
$(1, 1, 0, 0, 0), (1, 0, 1, 0, 0), (1, 0, 0, 0, 1), (1, 0, 0, 1, 1),$ $(1, 0, 1, 0, 1), (1, 1, 0, 0, 1), (1, 1, 0, 1, 0), (1, 1, 0, 1, 1)$	$(1, 0, 0, 0, 0)$	$(0, 0, 0)$
$(0, 1, 1, 0, 0), (0, 0, 1, 1, 0), (0, 0, 1, 1, 1), (0, 1, 1, 0, 1),$ $(0, 1, 1, 1, 0), (1, 0, 1, 1, 0), (0, 1, 1, 1, 1)$	$(0, 0, 1, 0)$	$(0, 0, 0)$
$(1, 1, 1, 0, 0), (1, 1, 1, 1, 0)$	$(0, 0, 1, 1, 0)$	$(1, 0, 0)$
$(1, 1, 1, 1, 1), (1, 1, 1, 0, 1)$	$(0, 1, 1, 1, 0)$	$(1, 0, 1)$
$(1, 0, 1, 1, 1)$	$(0, 1, 1, 0, 0)$	$(0, 0, 1)$

Table 4.9: The group G_{256} is real.

Now we explicitly make a calculation to show that special 2-group associated to the quadratic map as defined in the equation (4.2) is totally orthogonal. Since $\dim_{\mathbb{F}_2}(W, \mathbb{F}_2) = 3$, there exist exactly 7 non-zero $\mathbb{F}_2$-linear maps from W to $\mathbb{F}_2$, which are the following

$$s_n(x, y, z) = ix + jy + kz; \qquad (x, y, z) \in W, 1 \leq n \leq 7,$$

where $n = 4i + 2j + k$ is the binary expansion of $n \in \{1, 2, \cdots, 7\}$. We write various transfer maps of q:

$$\begin{aligned}
s_1 \circ q(w, x, y, z, t) &= wt + xy, \\
s_2 \circ q(w, x, y, z.t) &= wy, \\
s_3 \circ q(w, x, y, z, t) &= wy + wt + xy = wt + (w + x)y, \\
s_4 \circ q(w, x, y, z, t) &= wx + wt + yz = yz + (x + t)w, \\
s_5 \circ q(w, x, y, z, t) &= wx + wt + yz + wt + xy = yz + (w + y)x, \\
s_6 \circ q(w, x, y, z, t) &= wx + wt + yz + wy = yz + (x + t + y)w, \\
s_7 \circ q(w, x, y, z, t) &= wx + wt + yz + wy + wt + xy = wx + (z + w + x)y.
\end{aligned}$$

where $(w, x, y, z) \in V$. We first consider the quadratic form $s_2 \circ q : V \to \mathbb{F}_2$ defined by $s_2 \circ q(w, x, y, z, t) = wy$. Now we compute that $\text{rad}(b_{s_2 \circ q}) = \langle (0, 1, 0, 0, 0), (0, 0, 0, 1, 0), (0, 0, 0, 0, 1) \rangle$ and $\frac{V}{\text{rad}(b_{s_2 \circ q})} = \langle (1, 0, 0, 0, 0), (0, 0, 1, 0, 0) \rangle$. Therefore $s_2 \circ q$ induces regular quadratic form $q_{s_2} : \frac{V}{\text{rad}(b_{s_2 \circ q})} \to \mathbb{F}_2$ defined by $q_{s_2}(\alpha, \beta) = \alpha\beta$, where $(\alpha, \beta) \in \frac{V}{\text{rad}(b_{s_2 \circ q})}$. Now the quadratic form q_{s_2} is isometric to $[0, 0]$, so $\text{Arf}(q_{s_2}) = 0$.

Except $s_2 \circ q$ all the other transfer maps are isometric to $s_1 \circ q : V \to \mathbb{F}_2$ defined by $s_1 \circ q(w, x, y, z) = wt + xy$ by suitable linear changes of variables. Here $\text{rad}(b_{s_1 \circ q}) = \langle (0, 0, 0, 1, 0) \rangle$ and $\frac{V}{\text{rad}(b_{s_1 \circ q})} = \langle (1, 0, 0, 0, 0), (0, 1, 0, 0, 0), (0, 0, 1, 0, 0), (0, 0, 0, 0, 1) \rangle$. Therefore $s_1 \circ q$ induces regular quadratic form $q_{s_1} : \frac{V}{\text{rad}(b_{s_1 \circ q})} \to \mathbb{F}_2$ defined by $q_{s_1}(\alpha, \beta, \gamma, \omega) = \alpha\beta + \gamma\omega$, where $(\alpha, \beta, \gamma, \omega) \in \frac{V}{\text{rad}(b_{s_1 \circ q})}$. Now the quadratic form q_{s_1} is isometric to $[0, 0] \perp [0, 0]$, so $\text{Arf}(q_{s_1}) = 0$.

Thus we conclude that for all $s \in \text{Hom}(W, \mathbb{F}_2)$ the quadratic form q_s is isometric to either $[0, 0]$ or $[0, 0] \perp [0, 0]$, therefore $\text{Arf}(q_s) = 0$ for all $s \in \text{Hom}(W, \mathbb{F}_2)$. Thus by Th. 3.2.7, the special 2-group associated with quadratic map defined in equation (4.2) is totally orthogonal. $\square$

Example 4.2.10 ([KK15], Example 5.3) We consider two vector spaces V and W over $\mathbb{F}_2$ with $\dim_{\mathbb{F}_2}(V) = 5$ and $\dim_{\mathbb{F}_2}(W) = 4$. We check that the special 2-group associated to the quadratic map $q : V \to W$ defined by

$$q(w, x, y, z, t) = (wx + yz, wy, xy, wt); \quad (w, x, y, z, t) \in V. \tag{4.3}$$

is totally orthogonal but not strongly real group of order 2^9.

To show that the special 2-group associated to quadratic map as defined in equation (4.3) is not strongly real, we take $v = (1, 1, 1, 1, 1) \in V$. We first compute all $a := (w, x, y, z, t) \in V$ such that $q(a) = 0$. The condition $q(a) = 0$ implies that $wx + yz = wy = xy = wt = 0$. If $w = 0$, this condition forces that $yz = xy = 0$. If we further suppose that $y = 0$, we get the following set of

$$\{(0, 0, 0, 0, 0), (0, 1, 0, 0, 0), (0, 0, 0, 1, 0), (0, 0, 0, 0, 1), (0, 1, 0, 1, 0), (0, 0, 0, 1, 1)(0, 1, 0, 1, 1)\}.$$

On the other hand if we consider $w = 0$ and $y \neq 0$, then we get $x = z = 0$. Thus in this case values of a are $(0, 0, 1, 0, 0)$ and $(0, 0, 1, 0, 1)$. Now we consider the case $w \neq 0$, then by condition $q(a) = 0$ we have $y = t = x = 0$ and the values of a in this case are $(1, 0, 0, 0, 0)$ and $(1, 0, 0, 1, 0)$. This completes the list of $a \in V$ with property $q(a) = 0$.

From the following table, we have that $q(v - a) \neq 0$ for all $a \in V$ with $q(a) = 0$.

a	$a - v$	$q(a - v)$
$(0, 0, 0, 0, 0)$	$(1, 1, 1, 1, 1)$	$(0, 1, 1, 1)$
$(1, 0, 0, 0, 0)$	$(0, 1, 1, 1, 1)$	$(1, 0, 1, 0)$
$(0, 1, 0, 0, 0)$	$(1, 0, 1, 1, 1)$	$(1, 1, 0, 1)$
$(0, 0, 1, 0, 0)$	$(1, 1, 0, 1, 1)$	$(1, 0, 0, 1)$
$(0, 0, 0, 1, 0)$	$(1, 1, 1, 0, 1)$	$(1, 1, 1.1)$
$(0, 0, 0, 0, 1)$	$(1, 1, 1, 1, 0)$	$(0, 1, 1, 0)$
$(1, 0, 0, 1, 0)$	$(0, 1, 1, 0, 1)$	$(0, 0, 1, 0)$
$(0, 1, 0, 1, 0)$	$(1, 0, 1, 0, 1)$	$(0, 1, 0, 1)$
$(0, 1, 0, 0, 1)$	$(1, 0, 1, 1, 0)$	$(1, 1, 0, 0)$
$(0, 0, 1, 0, 1)$	$(1, 1, 0, 1, 0)$	$(1, 0, 0, 0)$
$(0, 0, 0, 1, 1)$	$(1, 1, 1, 0, 0)$	$(1, 1, 1, 0)$
$(0, 1, 0, 1, 1)$	$(1, 0, 1, 0, 0)$	$(0, 1, 0, 0)$

Table 4.10: The group G_{512} is not strongly real.

In view of Th. 4.1.5, the above table confirms that special 2-group associated to quadratic map as defined in equation (4.3) is not strongly real. The special 2-group associated to quadratic map as defined in equation (4.3) is real (Th. 3.1.2). The following table gives the value of $a \in V$ for all $v \in V$ such that $q(a) = q(v - a)$.

v	a	$q(a) = q(a - v)$
$(0,0,0,0,0),(1,0,0,0,0),(0,1,0,0,0),(0,0,1,0,0),$ $(1,0,0,1,0),(0,1,0,1,0),(0,1,0,0,1),(0,0,0,1,1),$ $(0,0,0,1,0),(0,0,0,0,1),(0,1,0,1,1),(0,0,1,0,1)$	$(0,0,0,0,0)$	$(0,0,0,0)$
$(1,1,0,0,0),(1,0,1,0,0),(1,0,0,0,1),(1,0,0,1,1),$ $(1,0,1,0,1),(1,1,0,0,1),(1,1,0,1,0),(1,1,0,1,1)$	$(1,0,0,0,0)$	$(0,0,0,0)$
$(0,1,1,0,0),(0,0,1,1,0),(0,0,1,1,1),(0,1,1,0,1),$ $(0,1,1,1,0),(1,0,1,1,0),(0,1,1,1,1)$	$(0,0,1,0)$	$(0,0,0,0)$
$(1,1,1,1,1),(1,1,1,1,0)$	$(1,1,0,0,0)$	$(1,0,0,0)$
$(1,0,1,1,1)$	$(1,0,0,1,0)$	$(0,0,0,0)$
$(1,1,1,0,0)$	$(0,0,1,1,0)$	$(1,0,0,0)$
$(1,1,1,0,1)$	$(0,0,1,1,1)$	$(1,0,0,0)$

Table 4.11: The group G_{512} is real.

Now we make the calculations to show that special 2-group associated to quadratic map as defined in equation (4.3) is totally orthogonal. Since $\dim_{\mathbb{F}_2}(W, \mathbb{F}_2) = 4$, there exist exactly 15 non-zero $\mathbb{F}_2$-linear maps from W to $\mathbb{F}_2$, which are the following

$$s_n(x, y, z, w) = ix + jy + kz + lw; \qquad (x, y, z, w) \in W, 1 \leq n \leq 15,$$

where $n = 8i + 4j + 2k + l$ is the binary expansion of $n \in \{1, 2, \cdots, 15\}$. We write various transfer maps of q:

$$s_1 \circ q(w, x, y, z, t) = wt,$$
$$s_2 \circ q(w, x, y, z.t) = xy,$$
$$s_3 \circ q(w, x, y, z, t) = wt + xy,$$
$$s_4 \circ q(w, x, y, z, t) = wy,$$

$$
\begin{aligned}
s_5 \circ q(w,x,y,z,t) &= wy + wt = (y+t)w, \\
s_6 \circ q(w,x,y,z,t) &= wy + xy = (w+x)y, \\
s_7 \circ q(w,x,y,z,t) &= wy + wt + xy = xy + (t+y)w, \\
s_8 \circ q(w,x,y,z,t) &= wx + yz, \\
s_9 \circ q(w,x,y,z,t) &= wx + yz + wt = yz + (x+t)w, \\
s_{10} \circ q(w,x,y,z,t) &= wx + yz + xy = yz + (w+y)x, \\
s_{11} \circ q(w,x,y,z,t) &= wx + yz + wt + xy = (x+z)y + (x+t)w, \\
s_{12} \circ q(w,x,y,z,t) &= wx + yz + wy = yz + (x+y)w, \\
s_{13} \circ q(w,x,y,z,t) &= wx + yz + wy + wt = yz + (x+y+t)w, \\
s_{14} \circ q(w,x,y,z,t) &= wx + yz + wy + xy = wx + (z+w+x)y, \\
s_{15} \circ q(w,x,y,z,t) &= wx + yz + wy + xy + wt = (x+t)w + (z+w+x)y.
\end{aligned}
$$

where $(w,x,y,z) \in V$. The quadratic forms $s_1 \circ q, s_2 \circ q, s_5 \circ q$ and $s_6 \circ q$ are isometric to the quadratic form $s_4 \circ q : V \to \mathbb{F}_2$ defined by $s_2 \circ q(w,x,y,z,t) = wy$. Now we compute $\mathrm{rad}(b_{s_4 \circ q}) = \langle (0,1,0,0,0),(0,0,0,1,0),(0,0,0,0,1)\rangle$ and $\frac{V}{\mathrm{rad}(b_{s_4 \circ q})} = \langle (1,0,0,0,0),(0,0,1,0,0)\rangle$. Therefore $s_4 \circ q$ induces a regular quadratic form $q_{s_4} : \frac{V}{\mathrm{rad}(b_{s_2 \circ q})} \to \mathbb{F}_2$ defined by $q_{s_4}(\alpha,\beta) = \alpha\beta$, where $(\alpha,\beta) \in \frac{V}{\mathrm{rad}(b_{s_4 \circ q})}$. Now the quadratic form q_{s_4} is isometric to $[0,0]$, so $\mathrm{Arf}(q_{s_4}) = 0$.

All the other transfer maps are isometric to $s_3 \circ q : V \to \mathbb{F}_2$ by suitable linear changes of variables. Here $\mathrm{rad}(b_{s_3 \circ q}) = \langle (0,0,0,1,0)\rangle$ and $s_3 \circ q$ induces regular quadratic form $q_{s_3} : \frac{V}{\mathrm{rad}(b_{s_3 \circ q})} \to \mathbb{F}_2$ defined by $q_{s_3}(\alpha,\beta,\gamma,\omega) = \alpha\beta + \gamma\omega$, where $(\alpha,\beta,\gamma,\omega) \in \frac{V}{\mathrm{rad}(b_{s_3 \circ q})}$. Now the quadratic form q_{s_3} is isometric to $[0,0] \perp [0,0]$, so $\mathrm{Arf}(q_{s_3}) = 0$.

Therefore for all $s \in \mathrm{Hom}(W, \mathbb{F}_2)$, $\mathrm{Arf}(q_s) = 0$. Now by Th. 4.1.5 and by Th. 3.2.7, the special 2-group associated to quadratic map defined in the equation (4.3) is a totally orthogonal group which is not strongly real. □

Notation 4.2.11. *We denote the special 2-groups associated to quadratic maps as defined by the equations (4.1), (4.2) and (4.3) by G_{128}, G_{256} and G_{512} respectively.*

We now have all the ingredients to prove the following theorem.

Theorem 4.2.12 ([KK15], Th. B)**.** *For every $m \geq 7$ there exist special 2-groups of order 2^m which are totally orthogonal but not strongly real.*

Proof We consider the groups $G_{128} \times D_4^n$, $G_{256} \times D_4^n$ and $G_{512} \times D_4^n$, where D_4^n denotes the n-fold direct product of D_4. It is computed that the groups G_{128}, G_{256} and G_{512} are strongly real groups which are not totally orthogonal in examples 4.2.8, 4.2.9, 4.2.10, respectively. Whereas the group D_4 is strongly real as well as totally orthogonal. Now using lemma 4.2.2, we conclude that the groups are $G_{128} \times D_4^n$, $G_{256} \times D_4^n$ and $G_{512} \times D_4^n$ are totally orthogonal special 2-groups which are not strongly real. Their orders are $2^{7+3n}, 2^{8+3n}$ and 2^{9+3n}. This completes the proof of the Theorem. □

Remark 4.2.13 We remark that the smallest totally orthogonal special 2-group which is not strongly real is of order 128. We have checked using GAP [GAP08] that the smallest totally orthogonal group which is not strongly real is of order 64. That group, though, is not a special 2-group. We give GAP [GAP08] code to check this in the appendix.

In the following table we record all totally orthogonal groups of order at most 128 which are not strongly real. Here C_n, D_n, QD_n and Q_n denote the cyclic group of order n, dihedral group of order $2n$, quasidihedral group of order n and quaternion group of order $4n$, respectively and $G \times H$, $G \rtimes H$ and $G \circ H$ denote the direct product, semidirect product and central product of groups G and H respectively. This table has been obtained using GAP [GAP08]. We give GAP [GAP08] code used to make this table in the appendix.

SmallGroups library ID	Structure description	Order	Special group
$(64, 177)$	$(C_2 \times D_8) : C_2$	64	No
$(128, 453)$	$((C_8 \times C_4) : C_2) : C_2$	128	No
$(128, 931)$	$(((C_8 \times C_2) : C_2) : C_2) : C_2$	128	No
$(128, 932)$	$((C_4 \times C_2 \times C_2) : C_4) : C_2$	128	No
$(128, 982)$	$(C_2 \times QD_{32}) : C_2$	128	No
$(128, 1345)$	$((C_2 \times C_2 \times C_2 \times D_4) : C_2$	128	Yes
$(128, 1389)$	$(C_2 \times ((C_4 \times C_4) : C_2)) : C_2$	128	Yes
$(128, 1544)$	$(C_2 \times ((C_2 \times C_2 \times C_2 \times C_2) : C_2)) : C_2$	128	Yes
$(128, 1550)$	$(C_2 \times ((C_4 \times C_4) : C_2)) : C_2$	128	Yes
$(128, 1880)$	$(C_2 \times (C_2 \times D_8) : C_2)$	128	No
$(128, 1924)$	$(C_2 \times ((C_4 \times C_2 \times C_2) : C_2)) : C_2$	128	No
$(128, 1949)$	$(C_2 \times ((C_4 \times C_4) : C_2)) : C_2$	128	No

Table 4.12: Totally orthogonal groups which are not strongly real up to order 128.

In the following table, we record all strongly real groups up to order 128 which are not totally orthogonal. Again, this table has been obtained using GAP [GAP08]. We give GAP [GAP08] code used to make this table in the appendix.

SmallGroups library ID	**Structure description**	**Order**	**Special group**
$(32, 50)$	$Q_2 \circ D_4$	32	Extraspecial
$(64, 218)$	$(C_2 \times (C_4 \times C_2) : C_2)) : C_2$	64	Yes
$(64, 265)$	$C_2 \times ((C_2 \times Q_2) : C_2)$	64	No
$(128, 1347)$	$(C_2 \times C_2 \times ((C_4 \times C_2) : C_2)) : C_2$	128	Yes
$(128, 1388)$	$(C_2 \times ((C_4 \times C_2) : C_4)) : C_2$	128	Yes
$(128, 1407)$	$(C_2 \times ((C_4 \times C_2 \times C_2) : C_2)) : C_2$	128	Yes
$(128, 2180)$	$C_2 \times ((C_2 \times (C_4 \times C_2) : C_2)) : C_2)$	128	No
$(128, 2318)$	$(C_2 \times ((C_2 \times Q_2) : C_2))) : C_2$	128	No
$(128, 2324)$	$C_2 \times (C_2 \times ((C_2 \times Q_2) : C_2)$	128	No
$(128, 2327)$	$Q_2 \circ D_4 \circ D_4$	128	Extraspecial

Table 4.13: Strongly real groups which are not totally orthogonal up to order 128.

Chapter 5

Representations of real special 2-groups

The aim of this chapter is to utilize the description of special 2-groups in terms of quadratic maps to construct the conjugacy classes, irreducible representations and complex character tables of real special 2-groups.

This chapter consists of four sections. In the first section we describe irreducible representations of real special 2-groups. In §5.2 and §5.3 we describe a method to construct the character table of real special 2-groups. These two sections deal with the characters and conjugacy classes of real special 2-groups. In §5.4, we illustrate our method of constructing character tables of real special 2-groups through examples.

Realization of abstract groups as groups of linear transformations is known as representation theory. In the theory of representations, there are number of ways of getting new representations from already known representations. In this chapter, we describe a method to write representations of real special 2-groups using the well known representations of extraspecial 2-groups. A large portion of the chapter is devoted to patch together the information of extraspecial 2-groups. This is done by converting quadratic maps associated to real special 2-groups to quadratic forms associated to extraspecial 2-groups by composing them with suitable linear maps.

Recall from example 3.2.6 that conversion of quadratic map to regular quadratic forms is indeed possible if the corresponding group is real. For this reason, we deal only with those special 2-groups which are real. A major part of this chapter concerns refining the Prop. [[Zah11], Prop. 3.3]. Before stating this proposition, we recall the notations from §3.2. Let G be a real special 2-group and $q : V \to W$ be the quadratic map associated to it. Let $s \in \mathrm{Hom}_{\mathbb{F}_2}(W, \mathbb{F}_2)$ be a non-zero linear map. The map $s \circ q$ is called the transfer of quadratic map q by s. If the group G is real, then $s \circ q$ induces a regular quadratic form from $V_s := \frac{V}{\mathrm{rad}(b_{s\circ q})}$ to $\mathbb{F}_2$ (see remark 3.2.5). We denote this form by q_s. We denote the extraspecial 2-group associated to the quadratic form q_s by G_s.

Proposition 5.0.14 ([Zah11], Prop. 3.3)**.** *Let G be a real special 2-group and $q : V := \frac{G}{Z(G)} \to Z(G) =: W$ be the quadratic map associated to G. Then*

1. *For every non-zero $s \in \mathrm{Hom}_{\mathbb{F}_2}(W, \mathbb{F}_2)$, there exists an irreducible representation φ of degree at least 2 of G such that $\varphi(G) = G_s$.*
2. *Conversely, for all irreducible representations φ of degree at least 2 of G, there exists a non-zero $s \in \mathrm{Hom}_{\mathbb{F}_2}(W, \mathbb{F}_2)$ such that $\varphi(G) = G_s$.*

5.1 Representations

This section is devoted to describe all irreducible representations of real special 2-groups. We first record a useful lemma:

Lemma 5.1.1. *Let G be a real special 2-group and $q : V \to W$ be the quadratic map associated to G. For $0 \neq s \in \mathrm{Hom}_{\mathbb{F}_2}(W, \mathbb{F}_2)$ let $q_s : V_s := \frac{V}{\mathrm{rad}(b_{s\circ q})} \to \mathbb{F}_2$ be the regular quadratic form induced from transfer map $s\circ q$. Let G_s denote the special 2-group associated to q_s. Then $V_s \simeq \frac{G_s}{Z(G_s)}$.*

Proof We recall from remark 1.3.10 that the group G_s has underlying set $V_s \times \mathbb{F}_2$ and its group operation is given by $(v, w)(v', w') = (v + v', c_s(v, v') + w + w')$, where c_s is the normal 2-cocycle corresponding to quadratic form q_s (see Prop. 1.3.8). We consider V_s as a group under addition and define $\xi : G_s \to V_s$ by $\xi(v, w) = v$. Clearly ξ is a surjection. Therefore $\frac{G_s}{\ker(\xi)} \cong V_s$. Now to prove the result, we need to show that $\ker(\xi) = Z(G_s)$. If $(v, w) \in \ker(\xi)$ then $v = 0$. For all $(v', w') \in G_s$, we have

$$(0, w)(v', w') = (0 + v', c_s(0, v') + w + w') = (v' + 0, c_s(v', 0) + w + w') = (v', w')(0, w)$$

Thus $Z(G_s)$ contains $\ker(\xi)$. Now for the reverse inclusion let $(v,w) \in Z(G_s)$. Then for all $(v',w') \in G_s$, we have

$$\begin{aligned}(v,w)(v',w') &= (v',w')(v,w)\\ \Rightarrow (v+v', c_s(v,v')+w+w') &= (v'+v, c_s(v',v)+w+w')\\ \Rightarrow c_s(v,v') &= c_s(v',v).\end{aligned}$$

By [Zah11, Prop. 1.2], we have $b_{q_s}(v,v') = c_s(v,v') - c_s(v',v)$. From the above calculation, we have $b_{q_s}(v,v') = 0$ for all $v' \in V_s$. Thus $v \in \operatorname{rad} b_{q_s}$. Since q_s is a regular quadratic form, $v = 0$ and therefore $(v,w) \in \ker(\xi)$. □

We refine of the first part of Prop. 5.0.14 by observing that for every non-zero $s \in \operatorname{Hom}_{\mathbb{F}_2}(W,\mathbb{F}_2)$, there are exactly $|\operatorname{rad}(b_{s\circ q})|$ number of inequivalent irreducible representations ϕ of degree at least 2 of G such that $\phi(G) = G_s$. Here $|\operatorname{rad}(b_{s\circ q})|$ denotes the size of the radical $\operatorname{rad}(b_{s\circ q})$.

Before stating next proposition, we record some definitions, which will be used later.

Definition 5.1.2. *1. Let $c : V\times V \to W$ be a normal 2-cocycle and $s \in \operatorname{Hom}_{\mathbb{F}_2}(W,\mathbb{F}_2)$. Then $s\circ c : V \times V \to \mathbb{F}_2$ defined by $s \circ c(v,v') = s(c(v,v'))$ for all $v,v' \in V$ is a normal 2-cocycle. It is called the transfer of c by s .*

2. Let $\epsilon_s : V \to V_s$ be the canonical surjection and $c_s : V_s \times V_s \to W$ be a normal 2-cocycle. Then $\operatorname{Inf}(c_s) : V \times V \to \mathbb{F}_2$ defined by $\operatorname{Inf}(c_s)(v,v') = c_s(\epsilon_s(v), \epsilon_s(v'))$ for $v,v' \in V$ is a normal 2-cocycle. It is called the inflation of c_s .

Proposition 5.1.3. *Let G be a real special 2-group and $q : V := \frac{G}{Z(G)} \to Z(G) =: W$ be the quadratic map associated to G. Then for every non-zero $s \in \operatorname{Hom}_{\mathbb{F}_2}(W,\mathbb{F}_2)$ there exist at least $|\operatorname{rad}(b_{s\circ q})|$ number of surjective homomorphisms from G to the extraspecial 2-group G_s.*

Proof Let $s \in \operatorname{Hom}_{\mathbb{F}_2}(W,\mathbb{F}_2)$ be a non-zero map. We have $|\operatorname{rad}(b_{s\circ q})| = 2^k$ for some $k \in \mathbb{N}$ as $\operatorname{rad}(b_{s\circ q})$ is a subspace of V. Since the order of $\operatorname{Hom}_{\mathbb{F}_2}(\operatorname{rad}(b_{s\circ q}),\mathbb{F}_2)$ is same as that of $\operatorname{rad}(b_{s\circ q})$, we have 2^k linear maps from $\operatorname{rad}(b_{s\circ q})$ to $\mathbb{F}_2$. We enumerate these linear maps as t_i; $1 \le i \le 2^k$. For rest of the proof we fix a vector space complement V' of $\operatorname{rad}(b_{s\circ q})$ in V. Thus we write $V = \operatorname{rad}(b_{s\circ q}) \oplus V'$. Define $h_i : V \to \mathbb{F}_2$ by $h_i(v) = t_i(x)$, where $v = (x, y) \in V$ with $x \in \mathrm{rad}(b_{s\circ q})$ and $y \in V'$.

Let $[c]$ denote the class of normal 2-cocycle c in $H^2(V, W)$. Using Prop. 1.3.8, we have a normal 2-cocycle c such that $\phi([c]) = q$, where ϕ is the isomorphism between $H^2(V, W)$ and $\mathrm{Quad}(V, W)$ as in Prop. 1.3.8. Let $s \in \mathrm{Hom}_{\mathbb{F}_2}(W, \mathbb{F}_2)$ be a non zero map and $s \circ c : V \times V \to \mathbb{F}_2$ be the transfer of c by s. Let c_s be a normal 2-cocycle such that $\phi([c_s]) = q_s$, where ϕ is the isomorphism between $H^2(V_s, \mathbb{F}_2)$ and $\mathrm{Quad}(V_s, \mathbb{F}_2)$ defined in Prop. 1.3.8.

Let $\mathrm{Inf}(c_s) : V \times V \to \mathbb{F}_2$ denote the inflation of c_s. Now we compute

$$\mathrm{Inf}(c_s)(v, v) = q_s(\epsilon_s(v)) = s(q(v)) = s \circ c(v, v)$$

This implies that under the isomorphism $\phi : H^2(V, \mathbb{F}_2) \to \mathrm{Quad}(V, \mathbb{F}_2)$, both $[\mathrm{Inf}(c_s)]$ and $[s \circ c]$ are preimages of the same quadratic map.

Therefore $\mathrm{Inf}(c_s)$ and $s \circ c$ are cohomologous [Zah11, Prop. 3.3] and there exists a coboundary $\lambda : V \to \mathbb{F}_2$ such that $\lambda(0) = 0$ and

$$\mathrm{Inf}(c_s)(v, v') = s \circ c(v, v') - \lambda(v + v') + \lambda(v) + \lambda(v'). \tag{5.1}$$

We have now all the ingredients to define surjective homomorphisms from G to G_s for each i ; $1 \leq i \leq 2^k$. Define $f_{s,i} : G \to G_s$ by

$$f_{s,i}(v, w) = (\epsilon_s(v), s(w) - \lambda(v) - h_i(v))$$

for $v \in V, w \in W$. It follows from the following direct computation that each $f_{s,i} : G \to G_s$ is a group homomorphism.

$$\begin{aligned} f_{s,i}((v, w)(v', w')) &= f_{s,i}(v + v', c(v, v') + w + w') \\ &= (\epsilon_s(v + v'), s(c(v, v') + w + w') - \lambda(v + v') - h_i(v + v')) \\ &= (\epsilon_s(v) + \epsilon_s(v'), c_s(\epsilon_s(v), \epsilon_s(v') + s(w) + s(w') - \lambda(v) - \lambda(v') - h_i(v) - h_i(v')) \\ &= (\epsilon_s(v), s(w) - \lambda(v) - h_i(v))(\epsilon_s(v'), s(w') - \lambda(v') - h_i(v')) \\ &= f_{s,i}((v, w))f_{s,i}((v', w')) \end{aligned}$$

for $(v, w), (v', w') \in G$.

Now we check that the homomorphisms $f_{s,i}$ for each $i; 1 \leq i \leq 2^k$ are surjective. Let $(v_s, w_s) \in G_s$ where $v_s \in V_s$ and $w_s \in \mathbb{F}_2$. The surjectivity of maps ϵ_s and s provides

$v \in V$ and $w \in W$ such that $\epsilon_s(v) = v_s$ and $s(w) = w_s$. Further, since $\lambda(v), h_i(v) \in \mathbb{F}_2$ and s is surjective, there exist $w_1, w_2 \in W$ such that $\lambda(v) = s(w_1)$ and $h_i(v) = s(w_2)$. Now we compute:

$$\begin{aligned} f_{s,i}(v, w + w_1 + w_2) &= (\epsilon_s(v), s(w + w_1 + w_2) - \lambda(v) - h_i(v)) \\ &= (\epsilon_s(v), s(w) + s(w_1) + s(w_2) - \lambda(v) - h_i(v)) \\ &= (v_s, w_s) \end{aligned}$$

This calculation confirms that $f_{s,i}$ is surjective for each $i; 1 \leq i \leq 2^k$. Further $f_{s,i}\ ; 1 \leq i \leq 2^k$ are distinct homomorphisms. If $i \neq j$, then there exists $v \in V$ such that $h_i(v) \neq h_j(v)$, this implies $f_{s,i} \neq f_{s,j}$. Hence there are at least $|\operatorname{rad}(b_{s\circ q})| = 2^k$ number of surjective homomorphisms form G to the extraspecial 2-group G_s. □

Proposition 5.1.4. *Let G be a real special 2-group and $q : V := \frac{G}{Z(G)} \to Z(G) =: W$ be the quadratic map associated to G. Then for every non-zero $s \in \operatorname{Hom}_{\mathbb{F}_2}(W, \mathbb{F}_2)$ there exist at least $|(\operatorname{rad}(b_{s\circ q}))|$ number of inequivalent irreducible representations φ of degree at least 2 of G such that $\varphi(G) = G_s$.*

Proof Let $s \in \operatorname{Hom}_{\mathbb{F}_2}(W, \mathbb{F}_2)$ be a non-zero linear map and $|\operatorname{rad}(b_{s\circ q})| = 2^k$. The group G_s is extraspecial 2-group, it has a unique irreducible faithful representation of degree at least 2 (see remark 2.2.5). Let φ_s denote the unique irreducible faithful representation of degree at least 2 of G_s. Then we define $\varphi_{s,i} := \varphi_s \circ f_{s,i}$ for $1 \leq i \leq 2^k$, where $f_{s,i}$ is as in the proof of Prop. 5.1.3. Clearly $\varphi_{s,i}$ are irreducible representations of G of the degree same as that of φ_s. Moreover we have $\varphi_{s,i}(G) = \varphi_s(f_{s,i}(G)) = \varphi_s(G_s) \cong G_s$ as the map $f_{s,i} : G \to G_s$ is surjective and φ_s is faithful.

Let $\chi_{s,i}$ and $\chi_{s,j}$ be the characters of representations $\varphi_{s,i}$ and $\varphi_{s,j}$, respectively. Now using Prop.1.2.3, we show that $\varphi_{s,i}$ and $\varphi_{s,j}$ are equivalent if and only if $i = j$. For this if $i \neq j$, we need to find $g \in G$ such that $\chi_{s,i}(g) \neq \chi_{s,j}(g)$.

Let χ_s be the character afforded by the representation φ_s of group G_s. From the definition of representations $\varphi_{s,i}$ and $\varphi_{s,j}$, it follows that $\chi_{s,i} = \chi_s \circ f_{s,i}$ and $\chi_{s,j} = \chi_s \circ f_{s,j}$.

If $|(\operatorname{rad}(b_{s\circ q}))| = 1$ then there is nothing to prove. If $|(\operatorname{rad}(b_{s\circ q}))| > 1$, then for $i \neq j$ there exists $(v, 0) \in G$ where $v \in \operatorname{rad}(b_{s\circ q})$ such that $h_i(v) \neq h_j(v)$. For the homomorphism

$f_{s,i}$ as defined in Prop. 5.1.3, we have

$$f_{s,i}(v,0) = (0, -\lambda(v) - h_i(v)), \qquad f_{s,j}(v,0) = (0, -\lambda(v) - h_j(v))$$

It implies that one of $f_{s,i}(v,0)$ and $f_{s,j}(v,0)$ is the identity element of the group G_s, while the other is the non-trivial element of $Z(G_s)$. Without loss of generality we assume that

$$f_{s,i}(v,0) = (0,0) \in Z(G_s), \quad f_{s,j}(v,0) = (0,1) \in Z(G_s).$$

Let the order of G_s be 2^{2m+1}. Then by lemma 2.2.7

$$\chi_{s,i}(v,0) = \chi_s \circ f_{s,i}(v,0) = \chi_s(0,0) = 2^m$$
$$\chi_{s,j}(v,0) = \chi_s \circ f_{s,j}(v,0) = \chi_s(0,1) = -2^m$$

This proves that $\varphi_{s,i}$ and $\varphi_{s,j}$ are inequivalent if $i \neq j$. □

The following theorem implies that all irreducible representations of degree at least 2 of real special 2-groups are of the form $\varphi_{s,i}$ for suitable $s \in \mathrm{Hom}_{\mathbb{F}_2}(W, \mathbb{F}_2)$ and $1 \leq i \leq |(\mathrm{rad}(b_{s\circ q}))|$.

Theorem 5.1.5. *Let G be a real special 2-group and $q : V := \frac{G}{Z(G)} \to Z(G) =: W$ be the quadratic map associated to G. Then $\{\varphi_{s,i} : s \in \mathrm{Hom}_{\mathbb{F}_2}(W, \mathbb{F}_2), 1 \leq i \leq 2^k\}$ is the complete list of irreducible representations of degree at least 2 of G; where $\varphi_{s,i}$ are as in Prop. 5.1.4 and 2^k is the size of the radical $\mathrm{rad}(b_{s\circ q})$.*

Proof Let $|G| = 2^n$ and $|Z(G)| = 2^m$. Since the order of $\mathrm{Hom}_{\mathbb{F}_2}(Z(G), \mathbb{F}_2)$ and $Z(G)$ are same, the number of non-zero linear maps from $Z(G)$ to $\mathbb{F}_2$ is $|Z(G)| - 1 = 2^m - 1$. We denote these linear maps by $s_1, s_2, \cdots, s_{2^m-1}$.

First we use Prop. 1.2.3 to prove that $\varphi_{s_p,i}$ and $\varphi_{s_q,j}$ are inequivalent representations of G if either $p \neq q$ or $i \neq j$. If $p = q$ then it follows from the proof of Prop. 5.1.4 that $\varphi_{s_p,i}$ is not equivalent to $\varphi_{s_q,j}$ if $i \neq j$.

Suppose $p \neq q$. From Prop.5.1.4, it follows that $\varphi_{s_p,i}(G) \cong G_{s_p}$ and $\varphi_{s_q,j}(G) \cong G_{s_q}$. If $G_{s_p} \ncong G_{s_q}$ then $\varphi_{s_p,i}$ is not equivalent to $\varphi_{s_q,j}$.

Suppose that $G_{s_p} \cong G_{s_q}$ and $|G_{s_p}| = |G_{s_q}| = 2^{2l+1}$. Since s_p and s_q are distinct linear maps, there exist $w \in W$ such that $s_p(w) \neq s_q(w)$. Then $f_{s_p,i}(0,w) = (0, s_p(w))$ and $f_{s_q,j}(0,w) = (0, s_q(w))$. Therefore one of $f_{s_p,i}(0,w)$ and $f_{s_q,j}(0,w)$ is the identity element of group $G_{s_p} \cong G_{s_q}$ while other is non identity element of the center of group. Let $\chi_s, \chi_{s_p,i}$

and $\chi_{s_q,j}$ be the characters afforded by representations $\varphi_s, \varphi_{s_p,i}$ and $\varphi_{s_q,j}$, respectively. By the description of $\varphi_{s_p,i}$ and $\varphi_{s_q,j}$ as in the proof of Prop. 5.1.4, it follows that

$$\chi_{s_p,i}(0,w) = \chi_s(f_{s_p,i}(0,w)) \qquad \chi_{s_q,j}(0,w) = \chi_s(f_{s_q,j}(0,w))$$

without loss of generality, using lemma 2.2.7 we assume that $\chi_{s_p,i}(0,w) = 2^l$ and $\chi_{s_q,j}(0,w) = -2^l$. Now from Prop. 1.2.3, it follows that the representations $\varphi_{s_p,i}$ and $\varphi_{s_q,j}$ are inequivalent if either $p \neq q$ or $i \neq j$.

We know that sum of squares of degrees of all the inequivalent irreducible representations of a finite groups is equal to the order of the group (Th. 1.2.4). We now prove the result by showing that squares of degrees of representations $\varphi_{s,i}$ and one dimensional representations of group G adds up to 2^n, which is the order of group G.

Let $|G_{s_j}| = 2^{2l_j+1}$ for some non-zero linear map $s_j \in \mathrm{Hom}_{\mathbb{F}_2}(W, \mathbb{F}_2)$. Also we have $|Z(G_{s_j})| = 2$ as G_{s_j} are extraspecial 2-groups. By lemma 5.1.1, it follows that

$$\frac{|\frac{G}{Z(G)}|}{|\frac{G_{s_j}}{Z(G_{s_j})}|} = |\operatorname{rad} b_{s_j \circ q}|$$

Therefore $|\operatorname{rad} b_{s_j \circ q}| = \frac{2^{n-m}}{2^{2l_j}}$. By Prop. 5.1.4, there are at least $|\operatorname{rad} b_{s_j \circ q}|$ number of irreducible inequivalent representations φ of degree at least 2 of G such that $\varphi(G) \cong G_{s_j}$. The degree of these representations is same as the degree of the faithful representation φ_{s_j} of G_{s_j}. It follows from remark 2.2.6 that the degree of the faithful representation of degree at least 2 of extraspecial 2-groups of order 2^{2l_j+1} is 2^{l_j}. Using the Th. 2.2.1 and the fact that $G' = Z(G)$, we have that apart from the representations of degree at least 2, G has $|\frac{G}{Z(G)}|$ number of one dimensional representations. Now we compute the sum of squares of degrees of irreducible representations of G.

$$|\frac{G}{Z(G)}|.1^2 + \sum_{j=1}^{2^m-1} |\operatorname{rad}(b_{s_j \circ q})|.(2^{l_j})^2 = |\frac{G}{Z(G)}|.1^2 + \sum_{j=1}^{2^m-1} \frac{|\frac{G}{Z(G)}|}{|\frac{G_{s_j}}{Z(G_{s_j})}|}.(2^{l_j})^2$$

$$
\begin{aligned}
&= 2^{n-m} + \sum_{j=1}^{2^m-1} \frac{2^{n-m}}{2^{2l_j}}.(2^{l_j})^2 \\
&= 2^{n-m} + \sum_{j=1}^{2^m-1} 2^{n-m} \\
&= 2^{n-m} + (2^m - 1)2^{n-m} \\
&= 2^n \\
&= |G|.
\end{aligned}
$$

Therefore G can not afford representations of degree at least 2 except $\varphi_{s,i}$ and $\{\varphi_{s,i} : s \in \mathrm{Hom}_{\mathbb{F}_2}(W, \mathbb{F}_2)\}$ where $1 \leq i \leq 2^k$ is the complete list of irreducible representations of degree at least 2 of G. □

We now record a lemma for further reference with all the notations same as above.

Lemma 5.1.6. *Let $\chi_{s,i}$ be a character of degree at least 2 of real special 2-group G then $\frac{G}{\ker(\chi_{s,i})} \cong G_s$.*

Proof Let $\phi_{s,i}$ be the representation afforded by the character $\chi_{s,i}$ then from Prop. 5.1.4 we have $\phi_{s,i}(G) \cong G_s$. Now $\phi_{s,i}$ is a group homomorphism, so $\phi_{s,i}(G) \cong \frac{G}{\ker(\phi_{s,i})}$. Also we know that $\ker(\phi_{s,i}) = \ker(\chi_{s,i})$. Therefore $\frac{G}{\ker(\chi_{s,i})} \cong G_s$. □

5.2 Characters

The aim of this section is to provide a method to write the characters of degree at least 2 of real special 2-groups. With the notations of Prop. 5.1.4, we have the following:

Proposition 5.2.1. *Let G be a real special 2-group. Let $\chi_{s,i}$ be the character of the representation $\varphi_{s,i}$, as described in the proof of Prop. 5.1.4. Let the order of G_s be 2^{2l+1}. Then*

$$
\chi_{s,i}(g) = \begin{cases} 2^l & \text{if } f_{s,i}(g) = 1 \\ -2^l & \text{if } f_{s,i}(g) \text{ is non-trivial element of } Z(G_s) \\ 0 & \text{otherwise} \end{cases}
$$

Proof Let φ_s be the irreducible representation of degree at least 2 of the extraspecial 2-group G_s and χ_s be the character afforded by φ_s. It follows from lemma 2.2.7 that for $g \in G_s$, we have

$$\chi_s(g) = \begin{cases} 2^l & \text{if } g = 1 \\ -2^l & \text{if } g \text{ is the non-trivial element of } Z(G_s) \\ 0 & \text{otherwise} \end{cases}$$

From Prop. 5.1.4, we know that $\varphi_{s,i} = \varphi_s \circ f_{s,i}$. Thus we have $\chi_{s,i} = \chi_s \circ f_{s,i}$. Now the result follows by using lemma 2.2.7 and the fact that $\chi_{s,i}(g) = \chi_s(f_{s,i}(g))$ for all $g \in G$.

□

Lemma 5.2.2. *Let $\varphi_{s,i}$ be the irreducible representation of degree at least 2 of real special 2-group G. Let $\operatorname{diag}(1, 1, \cdots, 1)$ denote the identity matrix of order 2^l and $\operatorname{diag}(-1, -1, \cdots, -1)$ denote the diagonal matrix of order 2^l with diagonal entries equal to -1. Then for all $g \in G$:*

$$\varphi_{s,i}(g) = \begin{cases} \operatorname{diag}(1, 1, \cdots, 1) & \text{if } \chi_{s,i}(g) = 2^l \\ \operatorname{diag}(-1, -1, \cdots, -1) & \text{if } \chi_{s,i}(g) = -2^l \end{cases}$$

Proof It follows from Prop. 5.2.1 that if $\chi_{s,i}(g) = 2^l$ then $f_{s,i}(g)$ is the identity element of extraspecial 2-group G_s. Let φ_s be the unique irreducible representation of degree at least 2 of G_s. Form Prop. 5.1.4, we know that $\varphi_{s,i} = \varphi_s \circ f_{s,i}$. Thus if $f_{s,i}(g) = 1$ then $\varphi_{s,i}(g) = \varphi_s(f_{s,i}(g)) = \varphi_s(1) = \operatorname{diag}(1, 1, \cdots, 1)$.

On the other hand if $\chi_{s,i}(g) = -2^l$ then $f_{s,i}(g)$ is the non identity element of $Z(G_s)$ (Prop. 5.2.1). The representation φ_s of group G_s is either equivalent to $\rho \otimes \widehat{\rho \otimes \cdots} \otimes \rho(l$ copies of $\rho)$ or $\sigma \otimes \rho \widehat{\otimes \rho \otimes} \cdots \otimes \rho(l-1$ copies of $\rho)$ depending on whether $G_s \cong D_4^{(l-1)}$ or $G_s \cong Q_2 \circ D_4^{(l)}$ (see remark 2.2.5). Here ρ and σ are irreducible representations of degree at least 2 of group D_4 and Q_2, respectively.

Suppose $G_s \cong D_4^{(l)}$ and $g = \overline{(a^2, 1, 1, \cdots, 1)}$ be the non identity element of $Z(G_s)$, where a^2 denotes the non identity element of $Z(D_4)$. We know that $\rho(a^2) = \operatorname{diag}(-1, -1)$

(see remark 2.2.4(1)). We compute

$$\begin{aligned}\varphi_s(g) &= \varphi_s(\overline{(a^2,1,1,\cdots,1)}) \\ &= \rho\otimes\widehat{\rho\otimes\cdots}\otimes\rho(\overline{(a^2,1,1,\cdots,1)}) \\ &= \rho(a^2)\otimes\rho(1)\otimes\cdots\otimes\rho(1) \\ &= \mathrm{diag}(-1,-1,\cdots,-1)\end{aligned}$$

Now the result follows from the fact $\varphi_{s,i} = \varphi_s \circ f_{s,i}$. On similar lines, one can prove the result for the case $G_s \cong Q_2 \circ D_4^{(l-1)}$. ☐

For a character χ of a group G, we recall that $\mathbf{Z}(\chi) = \{g \in G \ : \ |\chi(g)| = \chi(1)\}$.

Definition 5.2.3. *For the character $\chi_{s,i}$ of real special 2-group G and for $g \in \mathbf{Z}(\chi_{s,i})$, we define*

$$\mathrm{sign}(\chi_{s,i}(g)) = \begin{cases} -1 & \text{if } \chi_{s,i}(g) \text{ is negative} \\ 1 & \text{if } \chi_{s,i}(g) \text{ is positive}\end{cases}$$

Theorem 5.2.4. *Let G be a real special 2-group and $q : V \to W$ be the quadratic map associated to G. Let $s \in \mathrm{Hom}_{\mathbb{F}_2}(W, \mathbb{F}_2)$. Then*

1. *$\chi_{s,i}(v, w) \neq 0$ if and only if $v \in \mathrm{rad}(b_{s\circ q})$.*
2. *For $(0, w) \in G$ we have*

$$\chi_{s,i}(0, w) = \begin{cases} 2^l & \text{if } s(w) = 0 \\ -2^l & \text{if } s(w) = 1\end{cases}$$

   where l is defined by $|G_s| = 2^{2l+1}$.

3. *Let $\{v_1, v_2, \cdots v_k\}$ be an ordered basis of $\mathrm{rad}(b_{s\circ q})$. Then*

$$\chi_{s,i}(v_j, 0) = \begin{cases} -2^l & \text{if } A_{j,i} = 1 \\ 2^l & \text{if } A_{j,i} = 0\end{cases}$$

   where $A_{j,i}$ denotes the coefficient of 2^j in the binary expansion $i - 1 = \sum_{j=0}^{k-1} A_{j,i}2^j$.

4. *Let $g \in G$ be an element with $g = \prod_{j=1}^{r}(v_{i_j}, 0)(0, w)$ for $1 \leq i_1 < i_2 < \cdots < i_r \leq k$ then*

$$\chi_{s,i}(g) = \prod_{j=1}^{r} \mathrm{sign}(\chi_{s,i}(v_{i_j}, 0)).\, \mathrm{sign}(\chi_{s,i}(0, w)).2^l$$

Proof

1. Let χ_s is the irreducible character of degree at least 2 of the extraspecial 2-group G_s and $f_{s,i}$ is homomorphism defined in Prop. 5.1.3. From the Prop. 5.1.4, we have $\chi_{s,i} = \chi_s \circ f_{s,i}$. Let $(v, w) \in G$, then

$$\begin{aligned}\chi_{s,i}(v, w) &= \chi_s(f_{s,i}(v, w)) \\ &= \chi_s(\epsilon_s(v), s(w) - \lambda(v) - h_i(v))\end{aligned}$$

By lemma 2.2.7, it follows for $g \in G_s$, we have $\chi_s(g) \neq 0$ if and only if $g \notin Z(G_s)$. Now $(x, y) \in Z(G_s)$ if and only if $x = 0$. Thus $\chi_{s,i}(v, w) = 0$ if and only if $\epsilon_s(v) = 0$. Here $\epsilon_s : V \to V_s$ is the canonical surjection. Therefore $\epsilon_s(v) = 0$ precisely when $v \in \mathrm{rad}(b_{s\circ q})$. Hence the result follows.

2. Let $(0, w) \in Z(G)$ then

$$\begin{aligned}\chi_{s,i}(0, w) &= \chi_s(f_{s,i}(0, w)) \\ &= \chi_s(\epsilon_s(0), s(w) - \lambda(0) - h_i(0)) \\ &= \chi_s(0, s(w))\end{aligned}$$

Let l be defined by $|G_s| = 2^{2l+1}$. From lemma 2.2.7 and the above calculation, we have

$$\chi_{s,i}(0, w) = \begin{cases} 2^l & \text{if } s(w) = 0 \\ -2^l & \text{if } s(w) = 1 \end{cases}$$

3. Let $\{v_1, v_2, \cdots v_k\}$ be a basis of $\mathrm{rad}(b_{s\circ q})$. Let $A_{j,i}$ be defined by the binary expansion

$$i - 1 = A_{0,i}2^0 + A_{1,i}2^1 + \cdots + A_{k-1,i}2^{k-1},$$

where $1 \leq i \leq 2^k$. Consider a map $\lambda : V \to \mathbb{F}_2$ as in equation 5.1 in the proof of Prop. 5.1.3. We define a map $\theta : \mathrm{rad}(b_{s\circ q}) \to \mathbb{F}_2$ by

$$\theta\left(\sum_{j=1}^{r} v_{i_j}\right) = \sum_{j=1}^{r} \lambda(v_{i_j}) \text{ for } 1 \leq i_1 < i_2 < \cdots < i_r \leq k$$

Notice that the map θ is nothing but the linear extension of $v \mapsto \lambda(v)$ to $\mathrm{rad}(b_{s\circ q})$. Thus $\theta \in \mathrm{Hom}_{\mathbb{F}_2}(\mathrm{rad}(b_{s\circ q}), \mathbb{F}_2)$. We recall from the proof of Prop. 5.1.3 the description of maps $h_i : V \to \mathbb{F}_2$; $1 \leq i \leq 2^k$. By definition, $h_i(v_j) = t_i(v_j)$ as

$v_j \in \mathrm{rad}(b_{s\circ q})$. Let $t'_i \in \mathrm{Hom}_{\mathbb{F}_2}(\mathrm{rad}(b_{s\circ q}), \mathbb{F}_2)$ be defined by $t'_i(v_j) = A_{j,i}$. Since both $\{t_i : 1 \leq i \leq 2^k\}$ and $\{\theta - t'_i : 1 \leq i \leq 2^k\}$ denote the same set, namely the set of all the linear maps from $\mathrm{rad}(b_{s\circ q})$ to $\mathbb{F}_2$, by a suitable permutation we may assume that $t_i = \theta - t'_i$. Thus we have

$$\begin{aligned}\chi_{s,i}(v_j, 0) &= \chi_s(f_{s,i}(v_j, 0)) \\ &= \chi_s(\epsilon_s(v_j), s(0) - \lambda(v_j) - h_i(v_j)) \\ &= \chi_s(0, -\lambda(v_j) - t_i(v_j)) \\ &= \chi_s(0, \lambda(v_j) - \theta(v_j) + t'_i(v_j)) \\ &= \chi_s(0, \lambda(v_j) - \lambda(v_j) + t'_i(v_j)) \\ &= \chi_s(0, t'_i(v_j)) \\ &= \chi_s(0, A_{j,i})\end{aligned}$$

Therefore

$$\chi_{s,i}(v_j, 0) = \begin{cases} \chi_s(0,1) & \text{if } A_{j,i} = 1 \\ \chi_s(0,0) & \text{if } A_{j,i} = 0 \end{cases} = \begin{cases} -2^l & \text{if } A_{j,i} = 1 \\ 2^l & \text{if } A_{j,i} = 0 \end{cases}$$

4. Let $\{v_1, v_2, \cdots, v_k\}$ be a basis of $\mathrm{rad}(b_{s\circ q})$. Take $g = \prod_{j=1}^{r}(v_{i_j}, 0)(0, w) \in G$; where $1 \leq i_1 < i_2 < \cdots < i_r \leq k$. Since $\varphi_{s,i}$ is a group homomorphism, we have

$$\begin{aligned}\varphi_{s,i}(g) &= \varphi_{s,i}\left(\prod_{j=1}^{r}(v_{i_j}, 0)(0, w)\right) \\ &= \prod_{j=1}^{r}\varphi_{s,i}(v_{i_j}, 0)\varphi_{s,i}(0, w)\end{aligned}$$

Now it follows from from Th. 5.2.4(2) and Th. 5.2.4(3) that the value of $\chi_{s,i}(v_j, 0)$ and $\chi_{s,i}(0, w)$ is either 2^l or -2^l. Using Lemma 5.2.2, we get that $\varphi_{s,i}(v_j, 0)$ and $\varphi_{s,i}(0, w)$ is equal to $\mathrm{diag}(1, 1, \cdots, 1)$ or $\mathrm{diag}(-1, -1, \cdots, -1)$. Now using above calculation and definition 5.2.3, we have

$$\chi_{s,i}\left(\prod_{j=1}^{r}(v_{i_j}, 0)(0, w)\right) = \prod_{j=1}^{r}\mathrm{sign}(\chi_{s,i}(v_{i_j}, 0)).\,\mathrm{sign}(\chi_{s,i}(o, w)).2^l.$$

Hence the result follows. ∎

The following table summarizes the Th. 5.2.4. Notations of table are same as Th. 5.2.4. Let $|G| = 2^n$, $|Z(G)| = 2^m$ and $|G_s| = 2^{2l+1}$. We recall from the proof of Th. 5.1.5 that in this case $|\operatorname{rad}(b_{s\circ q}| = 2^{n-m-2l}$. We fix an ordered basis $\{v_1, v_2, \cdots, v_{2k}\}$ of $\operatorname{rad}(b_{s\circ q})$.

Type of element	$\chi_{s,i}(v,w)$	**Number of elements**
$\{(v,w) : v \neq 0, v \notin \operatorname{rad}(b_{s\circ q}\}$	0	$2^n - 2^{n-2l}$
$\{(0,w) : s(w) = 0\}$	2^l	
$\{(0,w) : s(w) = 1\}$	-2^l	2^m
$\{(v_j,0) : A_{j,i} = 1\}$	-2^l	
$\{(v_j,0) : A_{j,i} = 0\}$	2^l	$(n-m-2l).2^m$
$\{(v_j,w) : 0 \neq (0,w) \in Z(G)\}$	$\operatorname{sign}(\chi_{s,i}(v_j,0)).\operatorname{sign}(\chi_{s,i}(0,w)).2^l$	
$\prod_{j=1}^{r}(v_{i_j},0)(0,w)$ where $1 \leq i_1 < i_2 < \cdots < i_r \leq k$ and $j \geq 2$	$\prod_{j=1}^{r}\operatorname{sign}(\chi_{s,i}(v_{i_j},0)).\operatorname{sign}(\chi_{s,i}(0,w)).2^l$	$(2^{n-m-2l} - (n-m-2l)-1).2^m$

Table 5.1: Character values for characters of degree at least 2 of real special 2-groups.

Corollary 5.2.5. *Every irreducible character χ of degree at least 2 of real special 2-groups vanishes outside $\mathbf{Z}(\chi)$.*

Proof Let χ be the character afforded by the representation ϕ. From lemma 1.2.22, it follows that $\mathbf{Z}(\chi) = \{g \in G : \phi(g) = \lambda I$ for some $\lambda \in \mathbb{C}\}$. Thus we have $\mathbf{Z}(\chi) = \{g \in G : \chi(g) = \lambda\chi(1)$ for some $\lambda \in \mathbb{C}\}$. Now the result follows from Th. 5.2.4. ☐

5.3 Conjugacy classes

To write the character table of a group, we need to know its conjugacy classes. In this section, we form the conjugacy classes of real special 2-groups using the quadratic map associated to it. To distinguish two conjugacy classes, we use the well known fact that two elements g, h of a group G are conjugate if and only if $\chi(g) = \chi(h)$ for all irreducible characters χ of G (Prop. 1.2.1). The aim of this section is to prove the following theorem:

Theorem 5.3.1. *Let G be real special 2-group and $q : \frac{G}{Z(G)} =: V \to W := Z(G)$ be the quadratic map associated to G. Let $v \in V$.*

1. *If* $v \notin \mathrm{rad}(b_{s\circ q})$ *for all non-zero linear maps* $s : W \to \mathbb{F}_2$ *then* $\{(v, w) : w \in W\}$ *is a conjugacy class of* G.

2. *Consider the set* $\mathcal{S}_v := \{s \in \mathrm{Hom}_{\mathbb{F}_2}(Z(G), \mathbb{F}_2) : v \in \mathrm{rad}(b_{s\circ q})\}$. *Then the conjugacy class of element* $(v, w) \in G$ *is* $\{(v, w + w') : s(w') = 0$ *for all* $s \in \mathcal{S}_v\}$.

Proof We know that the elements of $W = Z(G)$ form conjugacy classes containing only one element. We claim that for $v_1 \neq v_2 \in V$ and $v_1 \neq 0,\ v_2 \neq 0$ in V, the elements of the set $\{(v_1, w) : w \in W\}$ are not conjugate to any element of set $\{(v_2, w) : w \in Z(G)\}$. Since $Z(G) = G'$, the one dimensional characters of G are the lifts of irreducible characters of $\frac{G}{Z(G)} = V$ (Th. 2.2.1). Therefore for all one dimensional characters χ of G, $\chi(v, w) = \chi(v, 0)$ for all $w \in W = Z(G)$. It follows from Prop. 1.2.1 and the fact that V is a abelian group that for $v_1 \neq 0,\ v_2 \neq 0$ in V, there exist a character $\bar{\chi}$ of V such that $\bar{\chi}(v_1) \neq \bar{\chi}(v_2)$. We denote the lift of the character $\bar{\chi}$ by χ. Then for all $w \in W$,

$$\chi(v_1, w) = \chi(v_1, 0) = \bar{\chi}(v_1) \neq \bar{\chi}(v_2) = \chi(v_2, 0) = \chi(v_2, w)$$

Again using Prop. 1.2.1 we conclude that the elements of the set $\{(v_1, w) : w \in W\}$ are not conjugate to any element of set $\{(v_2, w) : w \in Z(G)\}$. Therefore the sets $\{(v, w) : w \in W\}$ indexed by non-zero $v \in V$ are mutually disjoint. This way, we divide the non-central elements of G into $|\frac{G}{Z(G)}| - 1$ number of sets each containing $|Z(G)|$ elements.

We now prove the part 1 of the statement: Let $v \notin \mathrm{rad}(b_{s\circ q})$ for all non-zero linear maps $s : W \to \mathbb{F}_2$. From above discussion, we know that all the one dimensional characters takes same value on all the elements of the set $\{(v, w) : w \in W\}$. Now suppose that χ is character of degree at least 2. Since $v \notin \mathrm{rad}(b_{s\circ q})$, by Th. 5.2.4(1) $\chi(v, w) = 0$ for all the elements of set $\{(v, w) : w \in Z(G)\}$. Thus in this case all the characters of group G takes same values on the elements of set $\{(v, w) : w \in W\}$. Now it follows from the Prop. 1.2.1 that $\{(v, w) : w \in W\}$ is a conjugacy class of G.

We now prove the part 2 of the statement: Let $w_1 \in Z(G)$ be such that $s(w_1) = 1$ for some non-zero $s \in \mathcal{S}_v$. We know that $\chi_{s,i} = \chi_s \circ f_{s,i}$, where χ_s is unique irreducible character of degree at least 2 of extraspecial 2-group G_s and $f_{s,i}$ is the homomorphism from G to G_s as given in Prop. 5.1.3.

$$\chi_{s,i}(v, w) = \chi_s(\epsilon_s(v), s(w) - \lambda(v) - h_i(v)) = \chi_s(0, s(w) - \lambda(v) - h_i(v))$$

$$\chi_{s,i}(v, w + w_1) = \chi_s(\epsilon_s(v), s(w + w_1) - \lambda(v) - h_i(v)) = \chi_s(0, s(w) + 1 - \lambda(v) - h_i(v))$$

It is clear that one of $\chi_{s,i}(v, w)$ and $\chi_{s,i}(v, w + w_1)$ is $\chi_s(0, 0) = 2^l$, while the other one is $\chi_s(0, 1) = -2^l$, where 2^l is the degree of the character $\chi_{s,i}$. Thus $\chi_{s,i}(v, w_1) \neq \chi_{s,i}(v, w_1 + w)$ and it follows from Prop. 1.2.1 that (v, w) and $(v, w + w_1)$ are not conjugates.

Let $v \in \text{rad}(b_{s\circ q})$ and $w' \in W$ be such that $s(w') = 0$ for all non-zero linear maps in $\mathcal{S}_v$. We know that for one dimensional characters χ we have $\chi((v, w)) = \chi((v, w + w'))$. Let $\chi_{s,i}$ be a character of degree at least 2 such that $v \notin \text{rad}(b_{s\circ q})$, then $\chi_{s,i}((v, w)) = \chi_{s,i}((v, w + w')) = 0$.

Finally we consider the characters $\chi_{s,i}$ of degree at least 2 such that $v \in \text{rad}(b_{s\circ q})$. Then as earlier

$$\chi_{s,i}(v, w) = \chi_s(0, s(w) - \lambda(v) - h_i(v))$$

$$\chi_{s,i}(v, w + w') = \chi_s(0, s(w) + s(w') - \lambda(v) - h_i(v)) = \chi_s(0, s(w) - \lambda(v) - h_i(v))$$

Thus again in this case $\chi_{s,i}((v, w)) = \chi_{s,i}((v, w + w'))$. By Prop. 1.2.1, (v, w) is conjugate to $(v, w + w')$. □

We summarize the types of conjugacy classes of special 2-group G in the following table. The notations in the table are same as that of Th. 5.3.1.

Type of element	**Conjugacy class**
$v \notin \text{rad}(b_{s\circ q})$ for all $0 \neq s \in \text{Hom}(Z(G), \mathbb{F}_2)$	$\{(v, w) : w \in Z(G)\}$
$v \in \text{rad}(b_{s\circ q})$ for all $s \in \mathcal{S}_v$	$\{(v, w + w') : s(w') = 0 \ \forall \ s \in \mathcal{S}_v\}$

Table 5.2: Conjugacy classes of real special 2-groups.

5.4 Examples

In this concluding section we demonstrate through examples that the results proved in earlier sections can be used to construct the character table of a real special 2-groups.

Example 5.4.1 The first example that we consider is of the group G defined by

$$G = \langle a, b, c, d, f : a^2 = b^2 = (ab)^2 = d, c^2 = (ac)^2 = f, d^2 = f^2 = (bc)^2 = (df)^2 = 1\rangle.$$

We make following observations about G.

- The center of G is $Z(G) := \langle d, f : d^2 = f^2 = (df)^2 = 1\rangle$, and the quotient by the center is $\frac{G}{Z(G)} := \langle \bar{a}, \bar{b}, \bar{c} : \bar{a}^2 = \bar{b}^2 = \bar{c}^2 = (\bar{ab})^2 = (\bar{ac})^2 = (\bar{bc})^2 = \bar{1}\rangle$. Both $Z(G)$ and $\frac{G}{Z(G)}$ are elementary abelian 2-groups.
- The group G is a special 2-group as $|G| = 32$ and $Z(G) = \Phi(G) = G' = \langle d, f : d^2 = f^2 = (df)^2 = 1\rangle$.

We identify $\frac{G}{Z(G)}$ with a 3-dimensional vector space V and $Z(G)$ with a 2-dimensional vector space W over the field $\mathbb{F}_2$. Therefore, as a set, the group G gets identified with $V \times W$. Let $\{e_1 = (1,0,0), e_2 = (0,1,0), e_3 = (0,0,1)\}$ be a basis of V and $\{f_1 = (1,0), f_2 = (0,1)\}$ be a basis of W over $\mathbb{F}_2$. The quadratic map $q : V \to W$ associated to the special 2-group G is defined by

$$q(x, y, z) = (x^2 + xy + y^2, z^2 + xz); \quad (x, y, z) = x(1,0,0) + y(0,1,0) + z(0,0,1) \in V.$$

We claim that the group G is real. We use Th. 3.1.2 to justify this claim. Let $v \in V$. We find $a \in V$ such that $q(a) = q(v - a)$ to show that G is indeed real. The following table explicitly exhibits such $a \in V$ for a given $v \in V$.

v	a	$q(a) = q(v - a)$
$(0,0,0,), (0,1,0)$ $(0,0,1), (0,1,1)$	$(1,0,0)$	$(1,0)$
$(1,0,0,), (1,1,0)$ $(1,0,1), (1,1,1)$	$(0,1,0)$	$(1,0)$

Table 5.3: The group G defined in the example 5.4.1 is real.

It follows therefore that G is real.

Since $\dim_{\mathbb{F}_2}(W, \mathbb{F}_2) = 2$, there are exactly three non zero linear maps $s : W \to \mathbb{F}_2$. In the following table we compute the radical $\mathrm{rad}(b_{s_i \circ q})$ for each non-zero linear map $s_i : W \to \mathbb{F}_2$.

Linear map (s)	$s \circ q$	$b_{s \circ q}$	$\text{rad}(b_{s\circ q})$	$\lvert\text{rad}(b_{s\circ q})\rvert$
$s_1(w_1, w_2) = w_1$	$q(x,y,z) =$ $x^2 + xy + y^2$	$b_{s_1 \circ q}((x,y,z),(x',y',z'))$ $= xy' + x'y$	$\langle e_3 \rangle$	2
$s_2(w_1, w_2) = w_2$	$q(x,y,z) =$ $z^2 + xz$	$b_{s_2 \circ q}((x,y,z),(x',y',z'))$ $= xz' + x'z$	$\langle e_2 \rangle$	2
$s_3(w_1, w_2) = w_1 + w_2$	$q(x,y,z) =$ $x^2 + x(y+z)$ $+(y+z)^2$	$b_{s_3 \circ q}((x,y,z),(x',y',z'))$ $= x(y'+z') + x'(y+z)$	$\langle e_2 + e_3 \rangle$	2

Table 5.4: Calculation of $\text{rad}(b_{s\circ q})$ for all non zero $s \in \text{Hom}_{\mathbb{F}_2}(W, \mathbb{F}_2)$ and q associated to group G defined in example 5.4.1.

We compute the conjugacy classes of G using Th. 5.3.1. Since $e_3 \in \text{rad}(b_{s_1 \circ q})$, the set $\{(e_3, w) : w \in W\}$ splits into two conjugacy classes $\{(e_3, 0), (e_3, f_2)\}$ and $\{(e_3, f_1), (e_3, f_1 + f_2)\}$. Similarly for $e_2 \in \text{rad}(b_{s_2 \circ q})$ and $e_2 + e_3 \in \text{rad}(b_{s_3 \circ q})$, the sets $\{(e_2, w) \mid w \in W\}$ and $\{(e_2 + e_3, w) \mid w \in W\}$ also split in to more than one conjugacy classes. Whereas the remaining elements of V do not belong to any $\text{rad}(b_{s\circ q})$ for all non zero linear map $s : W \to \mathbb{F}_2$. Therefore the set $\{(v, w) : w \in W\}$ for $v \in V$ and $v \notin \{e_3, e_2, e_2 + e_3\}$ forms one conjugacy class. We write all the conjugacy classes of G in the following table:

$\mathcal{C}_1 = \{(0,0)\}$	$\mathcal{C}_2 = \{(0, f_1)\}$
$\mathcal{C}_3 = \{(0, f_2)\}$	$\mathcal{C}_4 = \{(0, f_1 + f_2)\}$
$\mathcal{C}_5 = \{(e_1, 0), (e_1, f_1), (e_1, f_2), (e_1, f_1 + f_2)\}$	$\mathcal{C}_6 = \{(e_2, 0), (e_2, f_1)\}$
$\mathcal{C}_7 = \{(e_2, f_2), (e_2, f_1 + f_2)\}$	$\mathcal{C}_8 = \{(e_3, 0), (e_3, f_2)\}$
$\mathcal{C}_9 = \{(e_3, f_1), (e_3, f_1 + f_2)\}$	$\mathcal{C}_{10} = \{(e_1,0)(e_2,0), (e_1,0)(e_2,0)(0,f_1),$ $(e_1,0)(e_2,0)(0,f_2), (e_1,0)(e_2,0)(0,f_1+f_2)\}$
$\mathcal{C}_{11} = \{(e_1,0)(e_3,0), (e_1,0)(e_3,0)(0,f_1),$ $(e_1,0)(e_3,0)(0,f_2), (e_1,0)(e_3,0)(0,f_1+f_2)\}$	$\mathcal{C}_{12} = \{(e_2,0)(e_3,0), (e_2,0)(e_3,0)(0,f_1+f_2)\}$
$\mathcal{C}_{13} = \{(e_2,0)(e_3,0)(0,f_1), (e_2,0)(e_3,0)(0,f_2)\}$	$\mathcal{C}_{14} = \{(e_1,0)(e_2,0)(e_3,0),$ $(e_1,0)(e_2,0)(e_3,0)(0,f_1),$ $(e_1,0)(e_2,0)(e_3,0)(0,f_2),$ $(e_1,0)(e_2,0)(e_3,0)(0,f_1+f_2)\}$

Table 5.5: Conjugacy classes of group G defined in example 5.4.1.

Now, for each non-zero linear map $s : W \to \mathbb{F}_2$ we compute the regular quadratic forms q_s up to isometry and determine the extraspecial 2-groups G_s associated to these quadratic forms using remark 2.1.5.

Linear map (s)	$s \circ q$	q_s	G_s	$\lvert G_s \rvert$	Characters	Degree
$s(w_1, w_2) = w_1$	$q(x, y, z) = x^2 + xy + y^2$	$[1, 1]$	Q_2	8	$\chi_{s_1,1}, \chi_{s_1,2}$	2
$s(w_1, w_2) = w_2$	$q(x, y, z) = z^2 + xz$	$[0, 0]$	D_4	8	$\chi_{s_2,1}, \chi_{s_2,2}$	2
$s(w_1, w_2) = w_1 + w_2$	$q(x, y, z) =$ $x^2 + x(y + z) + (y + z)^2$	$[1, 1]$	Q_2	8	$\chi_{s_3,1}, \chi_{s_3,2}$	2

Table 5.6: Characters of degree at least 2 of group G defined in example 5.4.1.

For each non-zero linear map $s : W \to \mathbb{F}_2$ we compute $\lvert \operatorname{rad}(b_{s \circ q}) \rvert$ number of irreducible characters $\chi_{s,j}$ of degree at least 2 using Th. 5.2.4. The one dimensional characters of group G are determined using the remark 2.2.2.

This summarizes to the following character table of G.

G	$\mathcal{C}_1$	$\mathcal{C}_2$	$\mathcal{C}_3$	$\mathcal{C}_4$	$\mathcal{C}_5$	$\mathcal{C}_6$	$\mathcal{C}_7$	$\mathcal{C}_8$	$\mathcal{C}_9$	$\mathcal{C}_{10}$	$\mathcal{C}_{11}$	$\mathcal{C}_{12}$	$\mathcal{C}_{13}$	$\mathcal{C}_{14}$
χ_1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
χ_2	1	1	1	1	1	1	1	-1	-1	1	-1	-1	-1	-1
χ_3	1	1	1	1	1	-1	-1	1	1	-1	1	-1	-1	-1
χ_4	1	1	1	1	1	-1	-1	-1	-1	-1	-1	1	1	1
χ_5	1	1	1	1	-1	1	1	1	1	-1	-1	1	1	-1
χ_6	1	1	1	1	-1	1	1	-1	-1	-1	1	-1	-1	1
χ_7	1	1	1	1	-1	-1	-1	1	1	1	-1	-1	-1	1
χ_8	1	1	1	1	-1	-1	-1	-1	-1	1	1	1	1	-1
$\chi_{s_1,1}$	2	-2	2	-2	0	0	0	2	-2	0	0	0	0	0
$\chi_{s_1,2}$	2	-2	2	-2	0	0	0	-2	2	0	0	0	0	0
$\chi_{s_2,1}$	2	2	-2	-2	0	2	-2	0	0	0	0	0	0	0
$\chi_{s_2,2}$	2	2	-2	-2	0	-2	2	0	0	0	0	0	0	0
$\chi_{s_3,1}$	2	-2	-2	2	0	0	0	0	0	0	0	2	-2	0
$\chi_{s_3,2}$	2	-2	-2	2	0	0	0	0	0	0	0	-2	2	0

Table 5.7: character table of group G defined in example 5.4.1.

Example 5.4.2 The next example that we consider is of the group $G = \langle a, b, c, d : a^4 = b^4 = c^4 = d^2 = 1, c^2 = a^2, aca = bcb = dcd = c, bab = dadb^2 = a, dbd = b\rangle$.

We make following observations about G.

- The center of G is $Z(G) := \langle a^2, b^2 : a^4 = b^4 = 1, bab = a\rangle$, and the quotient by the center is $\frac{G}{Z(G)} := \langle \bar{a}, \bar{b}, \bar{c}, \bar{d} : \bar{a}^2 = \bar{b}^2 = \bar{c}^2 = \bar{d}^2 = (\bar{ab})^2 = (\bar{ac})^2 = (\bar{ad})^2 = (\bar{bc})^2 = (\bar{bd})^2 = (\bar{cd})^2 = \bar{1}\rangle$. Both $Z(G)$ and $\frac{G}{Z(G)}$ are elementary abelian 2-groups.
- The group G is a special 2-group as $|G| = 64$ and $Z(G) = \Phi(G) = G' = \langle a^2, b^2 : a^4 = b^4 = 1, bab = a\rangle$.

We identify $\frac{G}{Z(G)}$ with a 4-dimensional vector space V and $Z(G)$ with a 2-dimensional vector space W over the field $\mathbb{F}_2$. Therefore, as a set, the group G gets identified with $V \times W$. Let $\{e_1 = (1, 0, 0, 0), e_2 = (0, 1, 0, 0), e_3 = (0, 0, 1, 0), e_4 = (0, 0, 0, 1)\}$ be a basis of V and $\{f_1 = (1, 0), f_2 = (0, 1)\}$ be a basis of W over $\mathbb{F}_2$. The quadratic map $q : V \to W$ associated to the special 2-group G is defined by

$$q(x, y, z, w) = (y^2 + xy + yz + xw, x^2 + z^2 + xz);$$

where $(x, y, z, w) = x(1, 0, 0, 0) + y(0, 1, 0, 0) + z(0, 0, 1, 0) + w(0, 0, 0, 1) \in V$.

We claim that the group G is real. We use Th. 3.1.2 to justify this claim. Let $v \in V$. We find $a \in V$ such that $q(a) = q(v - a)$ to show that G is a real group. The following table explicitly exhibits such $a \in V$ for a given $v \in V$.

v	a	$q(a) = q(v - a)$
$(0, 0, 0, 0), (0, 1, 0, 0), (1, 0, 1, 1), (1, 1, 1, 0)$ $(0, 0, 1, 0), (1, 0, 1, 0), (0, 1, 1, 1), (1, 1, 1, 1)$	$(1, 0, 0, 0)$	$(0, 1)$
$(1, 0, 0, 0), (1, 1, 0, 1)$ $(1, 1, 0, 1), (0, 0, 0, 1)$	$(0, 0, 1, 0)$	$(0, 1)$
$(0, 1, 1, 0,), (0, 0, 1, 1)$	$(1, 1, 0, 0)$	$(0, 1)$
$(1, 0, 0, 1), (1, 1, 0, 0)$	$(0, 1, 1, 0)$	$(0, 1)$

Table 5.8: The group G defined in the example 5.4.2 is real.

It follows therefore that G is real.

Since $\dim_{\mathbb{F}_2}(W, \mathbb{F}_2) = 2$, there are exactly three non zero linear maps $s : W \to \mathbb{F}_2$. In the following table we compute the radical $\mathrm{rad}(b_{s_i \circ q})$ for each non-zero linear map $s_i : W \to \mathbb{F}_2$.

s	$s \circ q$	$b_{s\circ q}$	$\mathrm{rad}(b_{s\circ q})$
$s_1(w_1, w_2)$ $= w_1$	$q(x,y,z,w) =$ $y^2 + xy + yz + xw$	$b_{s_1\circ(q)}((x,y,z,w),(x',y',z',w'))$ $= y(x'+z') + y'(x+z)$ $+xw' + x'w$	$\langle 1\rangle$
$s_2(w_1, w_2)$ $= w_2$	$q(x,y,z,w) =$ $x^2 + z^2 + xz$	$b_{s_2\circ(q)}((x,y,z,w),(x',y',z',w'))$ $= xz' + x'z$	$\langle e_2, e_4\rangle$
$s_3(w_1, w_2) =$ $w_1 + w_2$	$q(x,y,z,w) =$ $x(y+w) + z^2+$ $z(x+y) + (x+y)^2$	$b_{s_3\circ(q)}((x,y,z,w),(x',y',z',w'))$ $= x(y'+w') + x'(y+w)$ $+z(x'+y') + z'(x+y)$	$\langle 1\rangle$

Table 5.9: Calculation of $\mathrm{rad}(b_{s\circ q})$ for all non zero $s \in \mathrm{Hom}_{\mathbb{F}_2(W,\mathbb{F}_2)}$ and q associated to group G defined in example 5.4.2.

We compute the conjugacy classes of G using Th.5.3.1. Except the elements in the set $\{e_2, e_4, e_2 + e_4\}$, no other non trivial element of V belongs to any $\mathrm{rad}(b_{s\circ q})$. Therefore the sets $\{(v, w) \ : \ w \in W\}$ for $v \in V$ and $v \notin \{e_2, e_4, e_2 + e_4\}$ form a conjugacy classes in G. Since $\{e_2, e_4, e_2 + e_4\} \subseteq \mathrm{rad}(b_{s_2\circ q})$, the sets $\{(v, w) \ : \ w \in W\}$ for $v \in \{e_2, e_4, e_2 + e_4\}$ split into two conjugacy classes, namely $\{(v, 0), (v, f_1)\}$ and $\{(v, f_2), (v, f_1 + f_2)\}$. We give the conjugacy classes of G is the following table:

$\mathcal{C}_1 = \{(0,0)\}$	$\mathcal{C}_2 = \{(0, f_1)\}$
$\mathcal{C}_3 = \{(0, f_2)\}$	$\mathcal{C}_4 = \{(0, f_1 + f_2)\}$
$\mathcal{C}_5 = \{(e_1, 0), (e_1, f_1), (e_1, f_2), (e_1, f_1 + f_2)\}$	$\mathcal{C}_6 = \{(e_2, 0), (e_2, f_1)\}$
$\mathcal{C}_7 = \{(e_2, f_2), (e_2, f_1 + f_2)\}$	$\mathcal{C}_8 = \{(e_3, 0), (e_3, f_1)\}, \{(e_3, f_2), (e_3, f_1 + f_2)\}$
$\mathcal{C}_9 = \{(e_4, 0), (e_4, f_1)\}$	$\mathcal{C}_{10} = \{(e_4, f_2), (e_4, f_1 + f_2)\}$
$\mathcal{C}_{11} = \{(e_1,0)(e_2,0), (e_1,0)(e_2,0)(0,f_1),$ $(e_1,0)(e_2,0)(0,f_2), (e_1,0)(e_2,0)(0,f_1+f_2)\}$	$\mathcal{C}_{12} = \{(e_1,0)(e_3,0), (e_1,0)(e_3,0)(0,f_1),$ $(e_1,0)(e_3,0)(0,f_2), (e_1,0)(e_3,0)(0,f_1+f_2)\}$
$\mathcal{C}_{13} = \{(e_1,0)(e_4,0), (e_1,0)(e_4,0)(0,f_1),$ $(e_1,0)(e_4,0)(0,f_2), (e_1,0)(e_4,0)(0,f_1+f_2)\}$	$\mathcal{C}_{14} = \{(e_2,0)(e_3,0), (e_2,0)(e_3,0)(0,f_1),$ $(e_2,0)(e_3,0)(0,f_2), (e_2,0)(e_3,0)(0,f_1+f_2)\}$
$\mathcal{C}_{15} = \{(e_2,0)(e_4,0), (e_2,0)(e_4,0)(0,f_1)\}$	$\mathcal{C}_{16} = \{(e_2,0)(e_4,0)(0,f_2), (e_2,0)(e_4,0)(0,f_1+f_2)\}$

$\mathcal{C}_{17} = \{(e_3,0)(e_4,0), (e_3,0)(e_4,0)(0,f_1),$ $(e_3,0)(e_4,0)(0,f_2), (e_3,0)(e_4,0)(0,f_1+f_2)\}$	$\mathcal{C}_{18} = \{(e_1,0)(e_2,0)(e_3,0),$ $(e_1,0)(e_2,0)(e_3,0)(0,f_1),$ $(e_1,0)(e_2,0)(e_3,0)(0,f_2),$ $(e_1,0)(e_2,0)(e_3,0)(0,f_1+f_2)\}$
$\mathcal{C}_{19} = \{(e_1,0)(e_2,0)(e_4,0),$ $(e_1,0)(e_2,0)(e_4,0)(0,f_1),$ $(e_1,0)(e_2,0)(e_4,0)(0,f_2),$ $(e_1,0)(e_2,0)(e_4,0)(0,f_1+f_2)\}$	$\mathcal{C}_{20} = \{(e_1,0)(e_3,0)(e_4,0),$ $(e_1,0)(e_3,0)(e_4,0)(0,f_1),$ $(e_1,0)(e_3,0)(e_4,0)(0,f_2),$ $(e_1,0)(e_3,0)(e_4,0)(0,f_1+f_2)\}$
$\mathcal{C}_{21} = \{(e_2,0)(e_3,0)(e_4,0),$ $(e_2,0)(e_3,0)(e_4,0)(0,f_1),$ $(e_2,0)(e_3,0)(e_4,0)(0,f_2),$ $(e_2,0)(e_3,0)(e_4,0)(0,f_1+f_2)\}$	$\mathcal{C}_{22} = \{(e_1,0)(e_2,0)(e_3,0)(e_4,0),$ $(e_1,0)(e_2,0)(e_3,0)(e_4,0)(0,f_1),$ $(e_1,0)(e_2,0)(e_3,0)(e_4,0)(0,f_2),$ $(e_1,0)(e_2,0)(e_3,0)(e_4,0)(0,f_1+f_2)\}$

Table 5.10: Conjugacy classes of group G defined in example 5.4.2.

Now, for each non-zero linear map $s : W \to \mathbb{F}_2$ we compute the regular quadratic forms q_s up to isometry and determine the extraspecial 2-groups G_s associated to these quadratic forms using remark 2.1.5.

Linear map (s)	$s \circ q$	q_s	G_s	$\lvert G_s\rvert$	Characters	Degree
$s(w_1,w_2)$ $= w_1$	$q(x,y,z) =$ $y(y+x+z)+xw$	$[0,0] \perp [0,0]$	$D_4 \circ D_4$	32	$\chi_{s_1,1}$	4
$s(w_1,w_2)$ $= w_2$	$q(x,y,z) =$ x^2+z^2+xz	$[1,1]$	Q_2	8	$\chi_{s_2,1}, \chi_{s_2,2},$ $\chi_{s_2,3}, \chi_{s_2,4}$	2
$s(w_1,w_2)$ $= w_1+w_2$	$q(x,y,z) =$ $x(y+w)+z^2+$ $z(x+y)+(x+y)^2$	$[0,0] \perp [1,1]$	$D_4 \circ Q_2$	32	$\chi_{s_3,1}$	4

Table 5.11: Characters of degree at least 2 of group G defined in example 5.4.2.

For each non-zero linear map $s_i : W \to \mathbb{F}_2$ we compute $|\operatorname{rad}(b_{s_i \circ q})|$ number of irreducible characters $\chi_{s,j}$ of degree at least 2 using Th.5.2.4. The one dimensional characters of G are determined using remark 2.2.2.

The following is the character table of group G defined in example 5.4.2.

G	$\mathcal{C}_1$	$\mathcal{C}_2$	$\mathcal{C}_3$	$\mathcal{C}_4$	$\mathcal{C}_5$	$\mathcal{C}_6$	$\mathcal{C}_7$	$\mathcal{C}_8$	$\mathcal{C}_9$	$\mathcal{C}_{10}$	$\mathcal{C}_{11}$	$\mathcal{C}_{12}$	$\mathcal{C}_{13}$	$\mathcal{C}_{14}$	$\mathcal{C}_{15}$	$\mathcal{C}_{16}$	$\mathcal{C}_{17}$	$\mathcal{C}_{18}$	$\mathcal{C}_{19}$	$\mathcal{C}_{20}$	$\mathcal{C}_{21}$	$\mathcal{C}_{22}$
χ_1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
χ_2	1	1	1	1	1	1	1	1	−1	−1	1	1	−1	1	−1	−1	−1	1	−1	−1	−1	−1
χ_3	1	1	1	1	1	1	1	−1	1	1	1	−1	1	−1	1	1	−1	−1	1	−1	−1	−1
χ_4	1	1	1	1	1	1	1	−1	−1	−1	1	−1	−1	−1	−1	−1	1	−1	−1	1	1	1
χ_5	1	1	1	1	1	−1	−1	1	1	1	−1	1	1	−1	−1	−1	1	−1	−1	1	−1	−1
χ_6	1	1	1	1	1	−1	−1	1	−1	−1	−1	1	−1	−1	1	1	−1	−1	1	−1	1	1
χ_7	1	1	1	1	1	−1	−1	−1	1	1	−1	−1	1	1	−1	−1	−1	1	−1	−1	1	1
χ_8	1	1	1	1	1	−1	−1	−1	−1	−1	−1	−1	−1	1	1	1	1	1	1	1	−1	−1
χ_9	1	1	1	1	−1	1	1	1	1	1	−1	−1	−1	1	1	1	1	−1	−1	−1	1	−1
χ_{10}	1	1	1	1	−1	1	1	1	−1	−1	−1	−1	1	1	−1	−1	−1	−1	1	1	−1	1
χ_{11}	1	1	1	1	−1	1	1	−1	1	1	−1	1	−1	−1	1	1	−1	1	−1	1	−1	1
χ_{12}	1	1	1	1	−1	1	1	−1	−1	−1	−1	1	1	−1	−1	−1	1	1	1	−1	1	−1
χ_{13}	1	1	1	1	−1	−1	−1	1	1	1	1	−1	−1	−1	−1	−1	1	1	1	−1	−1	1
χ_{14}	1	1	1	1	−1	−1	−1	1	−1	−1	1	−1	1	−1	1	1	−1	1	−1	1	1	−1
χ_{15}	1	1	1	1	−1	−1	−1	−1	1	1	1	1	−1	1	−1	−1	−1	−1	1	1	1	−1
χ_{16}	1	1	1	1	−1	−1	−1	−1	−1	−1	1	1	1	1	1	1	1	−1	−1	−1	−1	1
$\chi_{s_1,1}$	4	−4	4	−4	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$\chi_{s_2,1}$	2	2	−2	−2	0	2	−2	0	2	−2	0	0	0	0	2	−2	0	0	0	0	0	0
$\chi_{s_2,2}$	2	2	−2	−2	0	2	−2	0	−2	2	0	0	0	0	−2	2	0	0	0	0	0	0
$\chi_{s_2,3}$	2	2	−2	−2	0	−2	2	0	2	−2	0	0	0	0	−2	2	0	0	0	0	0	0
$\chi_{s_2,4}$	2	2	−2	−2	0	−2	2	0	−2	2	0	0	0	0	2	−2	0	0	0	0	0	0
$\chi_{s_3,1}$	4	−4	−4	4	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

Chapter 6

Wedderburn decomposition of rational group algebra of real special 2-groups

The aim of this chapter is to describe a method to write the Wedderburn decomposition of rational group algebra of real special 2-groups. We make use of quadratic maps associated to these groups to obtain this decomposition.

This chapter is divided into three sections. The first section is devoted to the computation of primitive central idempotents of rational group algebra of real special 2-groups. In §6.2 we obtain the Wedderburn decomposition of rational group algebra $\mathbb{Q}[G]$ of a real special 2-group G using the associated quadratic map. In §6.2.1 we illustrate our method of determining the Wedderburn decomposition through examples. In §6.3 we show that rational group algebra of real special 2-groups does not determine the group. This is in contrast with the case of extraspecial 2-groups [VL06].

Let G be a finite group and $\mathbb{Q}[G]$ be the rational group algebra of G. The problem of determining a complete set of primitive central idempotents and the Wedderburn decomposition of $\mathbb{Q}[G]$ is one of the fundamental problems of group rings. Apart from various classical methods of finding explicit expressions of primitive central idempotents, there are character free methods or methods without involving computation in extensions of the rationals such as using subgroup structure as in [JLP03] and [BP12].

In this chapter, for a real special 2-group G, we demonstrate a new approach to determine the Wedderburn decomposition of $\mathbb{Q}[G]$. We determine the Wedderburn decomposition of $\mathbb{Q}[G]$ using the theory of quadratic forms over fields of characteristic 2.

6.1 Primitive central idempotents

We begin by recalling the theorem of Maschke for a field $\mathbb{K}$ and group G, the group algebra $\mathbb{K}[G]$ is semisimple if and only if $\text{char}(\mathbb{K})$ does not divide $|G|$. Let $\text{char}(\mathbb{K})$ does not divide $|G|$, the decomposition of $\mathbb{K}[G]$ as the direct sum of simple ideals is called the Wedderburn decomposition of $\mathbb{K}[G]$. An element e of $\mathbb{K}[G]$ is called *idempotent* if $e^2 = e$. Moreover an idempotent e of $\mathbb{K}[G]$ is called *primitive central* if it lies in center of $\mathbb{K}[G]$ and can not be written as $e = e' + e''$, where e' and e'' are non zero idempotents such that $e'e'' = 0$. We say that the set $\{e_1, e_2, \cdots, e_s\}$ of primitive central idempotents is *complete* if $e_1 + e_2 + \cdots + e_s = 1$. and $e_ie_j = 0$ for all $1 \leq i, j \leq s$ and $i \neq j$. It is well known that the decomposition $\mathbb{K}[G] \cong A_1 \oplus A_2 \oplus \cdots \oplus A_s$ as a direct sum of simple components corresponds to a complete set of primitive central idempotents $\{e_1, e_2, \cdots e_s\}$ such that $A_i \cong \mathbb{K}[G]e_i$, $1 \leq i \leq s$. Thus the problem of determining Wedderburn decomposition involves the computation of primitive central idempotents. This section is devoted to compute a complete set of primitive central idempotents of a real special 2-group.

For a finite group G, the set of primitive central idempotents of complex algebra $\mathbb{C}[G]$ corresponds to set of its irreducible complex characters of G. Let χ denote an irreducible character of group G. The primitive central idempotent of complex algebra $\mathbb{C}[G]$ of group G corresponding to χ is given by

$$e(\chi) = \frac{\chi(1)}{|G|} \sum_{g \in G} \chi(g) g^{-1}.$$

We now describe the primitive central idempotents for $\mathbb{Q}[G]$. Let χ be a complex character of G. Let $\mathbb{Q}(\chi)$ denote the field obtained by adjoining all character values $\chi(g); g \in G$ to $\mathbb{Q}$. To obtain $e_{\mathbb{Q}}(\chi)$, the primitive central idempotent of $\mathbb{Q}[G]$ corresponding to χ, we add idempotents $e(\sigma \circ \chi)$ with $\sigma \in \text{Gal}(\mathbb{Q}(\chi)/\mathbb{Q})$. Thus it is given by $e_{\mathbb{Q}}(\chi) = \sum_{\sigma \in \text{Gal}(\mathbb{Q}(\chi)/\mathbb{Q})} e(\sigma \circ \chi)$ [Yam74].

Remark 6.1.1 For a real special 2-group G, it follows from remark 2.2.2 and Th. 5.2.4 that for all irreducible characters χ of G, $\chi(g) \in \mathbb{Z}$ for all $g \in G$. Thus in case of real special 2-groups, $\mathrm{Gal}(\mathbb{Q}(\chi)/\mathbb{Q})$ is trivial for all irreducible characters and $e_{\mathbb{Q}}(\chi) = e(\chi) = \frac{\chi(1)}{|G|}\sum_{g\in G}\chi(g)g^{-1} = \frac{\chi(1)}{|G|}\sum_{g\in G}\chi(g)g$.

We will make use of this remark in the computation of primitive central idempotents of rational algebra $\mathbb{Q}[G]$ for real special 2-group G. We begin with computation of primitive central idempotents corresponding to one dimensional characters. Recall that for a finite subgroup H of G, $\widehat{H} = \frac{1}{|H|}\sum_{h\in H} h$ denotes an element of the group algebra $\mathbb{Q}[G]$.

Lemma 6.1.2. *Let χ be a one dimensional character of a real special 2-group G. If χ is a trivial character then $e_{\mathbb{Q}}(\chi) = \widehat{G}$, otherwise $e_{\mathbb{Q}}(\chi) = \widehat{\ker(\chi)} - \widehat{G}$.*

Proof Let χ be a one dimensional character of real special 2-group. If $\ker(\chi) = G$ then $\chi(g) = 1$ for all $g \in G$. In this case $e_{\mathbb{Q}}(\chi) = \frac{\chi(1)}{|G|}\sum_{g\in G}\chi(g)g^{-1} = \frac{1}{|G|}\sum_{g\in G} g = \widehat{G}$.

Now we consider the case $\ker(\chi) \neq G$. The one dimensional characters of G are the lifts of irreducible characters of $\frac{G}{G'}$. In case of special 2-groups, $\frac{G}{G'}$ is an elementary abelian 2-group so $\bar{\chi}(\frac{G}{G'}) = \{\pm 1\}$ for all characters $\bar{\chi}$ of $\frac{G}{G'}$. Since $\chi : G \to \mathbb{C}$ is defined by $\chi(g) = \bar{\chi}(gG')$, we have $\chi(G) = \{\pm 1\}$. Further $\chi : G \to \{\pm 1\}$ is a group homomorphism, therefore $\frac{G}{\ker(\chi)} \cong \{\pm 1\}$. Hence $|\frac{G}{\ker(\chi)}| = 2$. Also $\chi(g) = 1$ for all $g \in \ker(\chi)$ and $\chi(g) = -1$ for all $g \in G - \ker(\chi)$. Since the group G is real, $\chi(g) = \chi(g^{-1})$. Thus we have

$$\begin{aligned}
e_{\mathbb{Q}}(\chi) &= \frac{\chi(1)}{|G|}\sum_{g\in G}\chi(g)g^{-1} \\
&= \frac{1}{|G|}\left(\sum_{g\in\ker(\chi)} g - \sum_{g\in G-\ker(\chi)} g\right) \\
&= \frac{1}{|G|}\left(2\sum_{g\in\ker(\chi)} g - \sum_{g\in G} g\right) \\
&= \frac{2}{|G|}\sum_{g\in\ker(\chi)} g - \frac{1}{|G|}\sum_{g\in G} g \\
&= \widehat{\ker(\chi)} - \widehat{G}.
\end{aligned}$$

□

Before computing the primitive central idempotents corresponding to characters of degree at least 2, we recall the notations of chapter 5. Let G be a real special 2-group and $q : V \to W$ be the quadratic map associated to it. Let $s \in \mathrm{Hom}_{\mathbb{F}_2}(W, \mathbb{F}_2)$ be non zero linear maps and $s \circ q$ be the transfer of q by s. Then $s \circ q$ induce regular quadratic forms q_s from $V_s := \frac{V}{\mathrm{rad}(b_{s\circ q})}$ to $\mathbb{F}_2$ (see remark 3.2.5). We denote the extraspecial 2-groups associated to the quadratic forms q_s by G_s. From Prop. 5.1.4 and Th. 5.1.5, it is clear that for every non-zero $s \in \mathrm{Hom}_{\mathbb{F}_2}(W, \mathbb{F}_2)$, G has exactly $|(\mathrm{rad}(b_{s\circ q}))|$ number of inequivalent irreducible representations φ of degree at least 2 of G such that $\varphi(G) = G_s$. We denote the characters associated to these representations by $\chi_{s,i} \ : \ 1 \leq i \leq |(\mathrm{rad}(b_{s\circ q}))|$.

We recall from definition 1.2.21 that $\mathbf{Z}(\chi) = \{g \in G \mid \chi(g) \neq 0\}$.

Proposition 6.1.3. *Let χ be a character of degree at least 2 of real special 2-group G then $e_{\mathbb{Q}}(\chi) = \widehat{\ker(\chi)} - \widehat{\mathbf{Z}(\chi)}$.*

Proof It follows from corollary 5.2.5 that χ vanishes outside $\mathbf{Z}(\chi)$. Therefore $\chi(1)^2 = |G : \mathbf{Z}(\chi)|$ using lemma 1.2.23. Let $\chi = \chi_{s,i}$ for non zero linear map $s : Z(G) \to \mathbb{F}_2$ and $1 \leq i \leq |(\mathrm{rad}(b_{s\circ q}))|$. From lemma 5.1.6, we get that $\frac{G}{\ker(\chi)} \cong G_s$, thus $Z(\frac{G}{\ker(\chi)}) \cong Z(G_s)$. Also lemma 1.2.22(2) gives that $Z(\frac{G}{\ker(\chi)}) = \frac{\mathbf{Z}(\chi)}{\ker(\chi)}$. Therefore $|\frac{\mathbf{Z}(\chi)}{\ker(\chi)}| = |Z(G_s)| = 2$ as G_s is an extraspecial 2-groups.

By definition of $\ker(\chi)$ we have that $\chi(g) = \chi(1)$ for all $g \in \ker(\chi)$. From Th. 5.2.4, it follows that $\chi(g) = -\chi(1)$ for all $\mathbf{Z}(\chi) - \ker(\chi)$. Since G is real, $\chi(g) = \chi(g^{-1})$ for all $g \in G$. Now we compute the primitive central idempotent $e_{\mathbb{Q}}(\chi)$ using remark 6.1.1.

$$\begin{aligned} e_{\mathbb{Q}}(\chi) &= \frac{\chi(1)}{|G|} \sum_{g \in G} \chi(g) g \\ &= \frac{\chi(1)}{|G|} \left(\sum_{g \in \ker(\chi)} \chi(1) g - \sum_{g \in \mathbf{Z}(\chi) - \ker(\chi)} \chi(1) g \right) \\ &= \frac{\chi(1)^2}{|G|} \left(\sum_{g \in \ker(\chi)} g - \sum_{g \in \mathbf{Z}(\chi) - \ker(\chi)} g \right) \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{|\mathbf{Z}(\chi)|}\left(\sum_{g\in\ker(\chi)} g - \sum_{g\in\mathbf{Z}(\chi)-\ker(\chi)} g\right)\\
&= \frac{1}{|\mathbf{Z}(\chi)|}\left(2\sum_{g\in\ker(\chi)} g - \sum_{g\in\mathbf{Z}(\chi)} g\right)\\
&= \frac{2}{|\mathbf{Z}(\chi)|}\sum_{g\in\ker(\chi)} g - \frac{1}{|\mathbf{Z}(\chi)|}\sum_{g\in\mathbf{Z}(\chi)} g\\
&= \widehat{\ker(\chi)} - \widehat{\mathbf{Z}(\chi)}.
\end{aligned}$$

□

Corollary 6.1.4. *Let χ be the character of degree at least 2 of an extraspecial 2-group G then $e_{\mathbb{Q}}(\chi) = 1 - \widehat{Z(G)}$.*

Proof Let G be extraspecial 2-group. The unique character χ of degree at least 2 of G is faithful (see remark 2.2.6). From lemma 2.2.7, it follows that $\mathbf{Z}(\chi) = Z(G)$. Using Prop. 6.1.3, we have $e_{\mathbb{Q}}(\chi) = 1 - \widehat{Z(G)}$. □

The primitive central idempotents of extraspecial 2-groups given by the above corollary are also computed in [BP12, 3.1].

6.2 Wedderburn decomposition

In this section we determine the Wedderburn decomposition of rational group algebra $\mathbb{Q}[G]$ of a real special 2-group G. Let G be a group and $\frac{G}{H}$ be a quotient of G. Let $\bar{\phi} : G \to \frac{G}{H}$ be the canonical map defined by $\bar{\phi}(g) = gH$ for all $g \in G$. Let $\mathbb{Q}[G]$ and $\mathbb{Q}[\frac{G}{H}]$ be rational group algebras of groups G and $\frac{G}{H}$, respectively. Then $\phi : \mathbb{Q}[G] \to \mathbb{Q}[\frac{G}{H}]$ is the linear expansion of the map $\bar{\phi}$. The algebra homomorphism $\phi : \mathbb{Q}[G] \to \mathbb{Q}[\frac{G}{H}]$ is defined by $\phi(\sum \alpha_g g) = \sum \alpha_g \phi(g) = \sum \alpha_g gH$ for $g \in G$ and $\alpha_g \in \mathbb{Q}$.

Lemma 6.2.1. *Let $\phi : \mathbb{Q}[G] \to \mathbb{Q}[\frac{G}{H}]$ be the canonical map and K be a normal subgroup of G containing H then $\phi(\widehat{K}) = \widehat{K/H}$.*

Proof Let K be a normal subgroup of G such that $H \subset K$ then

$$\phi(\widehat{K}) = \phi\left(\frac{1}{|K|}\sum_{k\in K} k\right) = \frac{1}{|K|}\sum_{k\in K} kH = \frac{|H|}{|K|}\sum_{kH\in \frac{K}{H}} kH = \widehat{K/H}.$$

□

Definition 6.2.2. *Let e denote an idempotent in rational group algebra $\mathbb{Q}[G]$. The trace $\operatorname{tr}(e)$ of e is defined to be the coefficient of 1 in e.*

Remark 6.2.3 The $\mathbb{Q}$-vector space dimension of ideal $\mathbb{Q}[G]e$ is $|G|.\operatorname{tr}(e)$. To see this let $e = \sum_{g\in G} a_g g$ be a central idempotent and $P : \mathbb{Q}[G] \to \mathbb{Q}[G]$ be the map defined by $P(\sum_{g\in G} b_g g) = \left(\sum_{g\in G} b_g g\right) e$ for all $\sum_{g\in G} b_g g \in \mathbb{Q}[G]$. Let $\operatorname{tr}(P)$ denote the trace of linear map P. Since $\mathbb{Q}[G] = \operatorname{Im}(P) \oplus \ker(P)$ we have $\dim(\operatorname{Im}(P)) = \dim(\mathbb{Q}[G]e) = \operatorname{tr}(P)$. We take G as a basis of $\mathbb{Q}[G]$, we compute $\operatorname{tr}(P) = |G|.a_1 = |G|.\operatorname{tr}(e)$. Therefore $\dim(\mathbb{Q}[G]e) = |G|.\operatorname{tr}(e)$.

Proposition 6.2.4. *Let G be a real special 2-group and $\chi_{s,i}$ be the character of degree at least 2 of G as described in section §5.2. Then $\mathbb{Q}[G].e_{\mathbb{Q}}(\chi_{s,i}) \cong \mathbb{Q}[G_s](1 - \widehat{Z(G_s)})$.*

Proof Let $\phi : \mathbb{Q}[G] \to \mathbb{Q}[\frac{G}{\ker(\chi_{s,i})}]$ be the canonical map. Let $e_{\mathbb{Q}}(\chi_{s,i})$ be the primitive central idempotent corresponding to the character $\chi_{s,i}$ of degree at least 2 of group G. Then from Prop. 6.1.3 we have $e_{\mathbb{Q}}(\chi_{s,i}) = \widehat{\ker(\chi_{s,i})} - \widehat{\mathbf{Z}(\chi_{s,i})}$. Now using lemma 6.2.1 and the fact that $\frac{\mathbf{Z}(\chi)}{\ker(\chi)} \cong Z(\frac{G}{\ker(\chi)})$ (see lemma 1.2.22) we compute

$$\phi(e_{\mathbb{Q}}(\chi_{s,i})) = \phi(\widehat{\ker(\chi_{s,i})} - \widehat{\mathbf{Z}(\chi_{s,i})}) = \widehat{1} - \widehat{\mathbf{Z}(\chi_{s,i})/\ker(\chi_{s,i})} = \widehat{1} - \widehat{Z(\frac{G}{\ker(\chi_{s,i})})}.$$

Now using ϕ, we define a map $\phi' : \mathbb{Q}[G].e_{\mathbb{Q}}(\chi_{s,i}) \to \mathbb{Q}[\frac{G}{\ker(\chi_{s,i})}].(\widehat{1} - \widehat{Z(\frac{G}{\ker(\chi_{s,i})})})$ by

$$\phi'(\sum_{g\in G} \alpha_g g.e_{\mathbb{Q}}(\chi_{s,i})) = \sum_{g\in G} \alpha_g \phi(g).(\widehat{1} - \widehat{Z(\frac{G}{\ker(\chi_{s,i})})}).$$

Since the map ϕ is surjective, the map ϕ' is also a surjective map. Now we show that vector space dimension of ideal $\mathbb{Q}[G].e_{\mathbb{Q}}(\chi_{s,i})$ and $\mathbb{Q}[\frac{G}{\ker(\chi_{s,i})}].(\widehat{1} - \widehat{Z(\frac{G}{\ker(\chi_{s,i})})})$ over $\mathbb{Q}$ are same.

We compute the dimension of ideals using remark 6.2.3, we first calculate

$$\mathrm{tr}(e_{\mathbb{Q}}(\chi_{s,i})) = \frac{1}{|\ker(\chi_{s,i})|} - \frac{1}{|\mathbf{Z}(\chi_{s,i})|} = \frac{1}{|\ker(\chi_{s,i})|} - \frac{1}{2|\ker(\chi_{s,i})|} = \frac{1}{2|\ker(\chi_{s,i})|}.$$

Hence dimension of $\mathbb{Q}[G].e_{\mathbb{Q}}(\chi_{s,i})$ is equal to $\frac{|G|}{2|\ker(\chi_{s,i})|}$. Now by lemma 5.1.6, the group $\frac{G}{\ker(\chi_{s,i})} \cong G_s$. This gives that $|Z(\frac{G}{\ker(\chi_{s,i})})| = 2$. Hence the dimension of $\mathbb{Q}[\frac{G}{\ker(\chi_{s,i})}](1 - \widehat{Z(\frac{G}{\ker(\chi_{s,i})})})$ is $\frac{|G|}{2|\ker(\chi_{s,i})|}$. Now using lemma 5.1.6, we get $\mathbb{Q}[G].e_{\mathbb{Q}}(\chi_{s,i}) \cong \mathbb{Q}[G_s](1 - \widehat{Z(G_s)})$. □

Recall from definition 2.3.4 that $\Delta(G, G')$ denotes the non-commutative part $\mathbb{Q}[G].(1 - \widehat{G'})$ of the group algebra $\mathbb{Q}[G]$ of G.

Theorem 6.2.5. *Let G be a real special 2-group and q be the associated quadratic map. Let $s : Z(G) \to \mathbb{F}_2$ be a non-zero linear map and q_s is be the regular quadratic form as defined in the statement of lemma 3.2.3. Let G_s be extraspecial 2-group associated to q_s. Then $\Delta(G_s, G_s')$ appears $|(\mathrm{rad}(b_{s\circ q})|$ many times in the Wedderburn decomposition of $\mathbb{Q}[G]$.*

Proof It follows from Prop. 5.1.4 and Th.5.1.5 that for every non-zero $s \in \mathrm{Hom}_{\mathbb{F}_2}(Z(G), \mathbb{F}_2)$, there exist exactly $|(\mathrm{rad}(b_{s\circ q}))|$ number of inequivalent irreducible representations ϕ of degree at least 2 of G such that $\phi(G) = G_s$. Let $\chi_{s,i}$; where $1 \leq i \leq |(\mathrm{rad}(b_{s\circ q}))|$ be irreducible character associated to these representations and $e_{\mathbb{Q}}(\chi_{s.i})$; where $1 \leq i \leq |(\mathrm{rad}(b_{s\circ q}))|$ be primitive central idempotents corresponding to these characters. Now from Prop. 6.2.4, the left ideal generated by each of these idempotents is isomorphic to $\mathbb{Q}[G_s](1 - \widehat{Z(G_s)})$. Since G_s is extraspecial 2-group, we have $Z(G_s) \cong G_s'$. This gives that $\mathbb{Q}[G_s](1 - \widehat{Z(G_s)}) \cong \mathbb{Q}[G_s](1 - \widehat{G_s'}) = \Delta(G_s, G_s')$. □

Theorem 6.2.6. *Let G be a real special 2-group and $|G| = 2^n, |Z(G)| = 2^m$. Let q be the quadratic map associated to G and $s_j : Z(G) \to \mathbb{F}_2, 1 \leq j \leq 2^m - 1$ be non-zero linear maps. Let q_{s_j} be regular quadratic forms as defined in the statement of lemma 3.2.3. Let $G_{s_j}, 1 \leq j \leq 2^m - 1$ be extraspecial 2-groups associated to quadratic forms $q_{s_j}, 1 \leq j \leq 2^m - 1$ and $|G_{s_j}| = 2^{2l_j+1}$. Then*

$$\mathbb{Q}[G] \cong 2^{n-m}\mathbb{Q} \oplus \bigoplus_{j=1}^{2^m-1} 2^{n-m-2l_j}\Delta(G_{s_j}, G_{s_j}')$$

Proof Since G is a special 2-group, $G' = Z(G)$ and the quotient group $\frac{G}{G'}$ is an elementary abelian 2-group. Using remark 2.3.5, the rational group algebra $\mathbb{Q}[\frac{G}{G'}]$ is a direct sum of $|\frac{G}{G'}|$ copies of $\mathbb{Q}$. Now $|G| = 2^n$ and $|G'| = |Z(G)| = 2^m$. Therefore $\mathbb{Q}[G]$ contains direct sum of 2^{n-m} copies of $\mathbb{Q}$.

The group associated to the quadratic form $q_{s_j} : \frac{\frac{G}{Z(G)}}{\mathrm{rad}(b_{s_j \circ q})} \to \mathbb{F}_2$ is the extraspecial 2-group G_{s_j}. Using lemma 5.1.1, we have $\frac{\frac{G}{Z(G)}}{\mathrm{rad}(b_{s_j \circ q})} \cong \frac{G_{s_j}}{Z(G_{s_j})}$. Therefore $|\mathrm{rad}(b_{s_j \circ q})| = \frac{|\frac{G}{Z(G)}|}{|\frac{G_{s_j}}{Z(G_{s_j})}|}$. Since G_{s_j} is an extraspecial 2-group, $|\mathrm{rad}(b_{s \circ q})| = 2^{n-m-2l_j}$. From Th.6.2.5, it follows that $\bigoplus_{j=1}^{2^m-1} 2^{n-m-2l_j}\Delta(G_{s_j}, G'_{s_j})$ is direct summand in the Wedderburn decomposition of $\mathbb{Q}[G]$. Adding the 2^{n-m} copies of the field of rational numbers corresponding to linear part, we get that $2^{n-m}\mathbb{Q} \oplus \bigoplus_{j=1}^{2^m-1} 2^{n-m-2l_j}\Delta(G_{s_j}, G'_{s_j})$ is a direct summand in the Wedderburn decomposition of $\mathbb{Q}[G]$. Now the proof follows from dimension count over field of rational numbers $\mathbb{Q}$. The dimension of $\mathbb{Q}[G]$ over $\mathbb{Q}$ is $|G| = 2^n$. Since $|G_{s_j}| = 2^{2l_j+1}$, the dimension of $\Delta(G_{s_j}, G'_{s_j})$ is 2^{2l_j} (see Prop.2.3.6. Hence the dimension of $2^{n-m}\mathbb{Q} \oplus \bigoplus_{j=1}^{2^m-1} 2^{n-m-2l_j}\Delta(G_{s_j}, G'_{s_j})$ is $2^{n-m} \times 1 + 2^m - 1 \times 2^{n-m-2l_j} \times 2^{2l_j} = 2^n$. $\square$

Note that G_{s_j} are extraspecial 2-groups and the non-commutative part $\Delta(G_{s_j}, G'_{s_j})$ of the Wedderburn decomposition of their rational group algebra has been described by Prop. 2.3.6 and summarized in table 2.2 in §2.3.

6.2.1 Examples

In this section, we explicitly compute the Wedderburn decomposition of two real special 2-groups using the method developed in this chapter.

1. Consider the group $G = \langle a, b, c, d, f : a^2 = b^2 = (ab)^2 = d, c^2 = (ac)^2 = f, d^2 = f^2 = (bc)^2 = (df)^2 = 1\rangle$. We know from example 5.4.1 that the group G is real special 2-group of order 32. The quadratic map $q : \frac{G}{Z(G)} \to Z(G)$ associated to G is given by

$$q(x, y, z) = (x^2 + xy + y^2, z^2 + xz); \quad (x, y, z) = x(1, 0, 0) + y(0, 1, 0) + z(0, 0, 1) \in V.$$

The extraspecial 2-groups G_s corresponding to the regular quadratic forms q_s induced from the transfers $s \circ q$ of q by s for all non-zero linear maps $s : W \to \mathbb{F}_2$ are summarized in the following table. The non-commutative part $\Delta(G_s, G_s')$ of Wedderburn decomposition of $\mathbb{Q}[G_s]$ is given in table 2.2. The detailed calculation is done in example 5.4.1.

Linear map (s)	q_s	G_s	$\lvert G_s \rvert$	$\Delta(G_s, G_s')$
$s_1(w_1, w_2) = w_1$	$[1,1]$	Q_2	2^3	$\mathbb{H}$
$s_1(w_1, w_2) = w_2$	$[0,0]$	D_4	2^3	$M_2(\mathbb{Q})$
$s_1(w_1, w_2) = w_1 + w_2$	$[1,1]$	Q_2	2^3	$\mathbb{H}$

Table 6.1: Computation of non-commutative part of Wedderburn decomposition of $\mathbb{Q}[G]$ for group G defined in example 5.4.1.

Now using Th. 6.2.6, the Wedderburn decomposition of rational group algebra $\mathbb{Q}[G]$ is

$$\mathbb{Q}[G] \cong 8\mathbb{Q} \oplus 2M_2(\mathbb{Q}) \oplus 4\mathbb{H}.$$

2. For the next example, we consider the group $G = \langle a, b, c, d : a^4 = b^4 = c^4 = d^2 = 1, c^2 = a^2, aca = bcb = dcd = c, bab = dadb^2 = a, dbd = b \rangle$. From example 5.4.2, it follows that the group G is real special 2-group of order 64. The map $q : \frac{G}{Z(G)} \to Z(G)$ defined by $q(x, y, z, w) = (y^2 + xy + yz + xw, x^2 + z^2 + xz);\quad (x, y, z, w) = x(1,0,0,0) + y(0,1,0,0) + z(0,0,1,0) + w(0,0,0,1) \in V$ is the quadratic map associated to group G.

   The extraspecial 2-groups G_s corresponding to the regular quadratic forms q_s induced from the transfers $s \circ q$ of q by s for all non-zero linear maps $s : W \to \mathbb{F}_2$ are summarized in the following table. The non-commutative part $\Delta(G_s, G_s')$ of Wedderburn decomposition of $\mathbb{Q}[G_s]$ is given in table 2.2. For a detailed calculation, see example 5.4.2.

Linear map (s)	q_s	G_s	$\|G_s\|$	$\Delta(G_s, G'_s)$
$s_1(w_1, w_2) = w_1$	$[0,0] \perp [0,0]$	$D_4 \circ D_4$	2^5	$M_4(\mathbb{Q})$
$s_1(w_1, w_2) = w_2$	$[1,1]$	Q_2	2^3	$\mathbb{H}$
$s_1(w_1, w_2) = w_1 + w_2$	$[1,1] \perp [0,0]$	$Q_2 \circ D_4$	2^5	$M_2(\mathbb{H})$

Table 6.2: Computation of non-commutative part of Wedderburn decomposition of $\mathbb{Q}[G]$ for group G defined in example 5.4.2.

Now using Th. 6.2.6, the Wedderburn decomposition of rational group algebra of group G is

$$\mathbb{Q}[G] \cong 16\mathbb{Q} \oplus M_4(\mathbb{Q}) \oplus 4\mathbb{H} \oplus M_2(\mathbb{H}).$$

6.3 Isomorphism Problem

An interesting problem in the theory of group rings is the Isomorphism problem. In [San81], it is mentioned that the isomorphism problem was formulated by G. Higman for Integral group rings in his PhD thesis of [Hig40]. The isomorphism problem asks whether a group algebra determines the group. It means whether the ring isomorphism $R[G] \cong R[H]$ implies that $G \cong H$ for some ring R, for some groups G and H. It is shown in [VL06, Corollary 3.5] that for extraspecial 2-groups, $\mathbb{Q}[G] \cong \mathbb{Q}[H]$ if and only if $G \cong H$. In this section, we show that this is not the case for special 2-groups.

We exhibit two examples to show that the isomorphism $\mathbb{Q}[G] \cong \mathbb{Q}[H]$ does not imply that the isomorphism of G and H, where G and H are real special 2-groups.

Example 6.3.1 $G_1 := \langle a, b, c, d, f \ : \ a^2 = b^2 = c^4 = d^2 = f^2 = dcd^{-1}c = a^{-1}b^{-1}ab = b^{-1}c^{-1}bc = a^{-1}d^{-1}ad = a^{-1}c^{-1}ac = 1, faf = ac^2, fbf = b, fcf = c^{-1}, fdf = bd\rangle$.

The order of group G_1 is 2^6. The center of G_1 is $Z(G_1) := \langle b, c^2 \ : \ b^2 = c^4 = 1, bc^2 = c^2b\rangle$ and $|Z(G)| = 2^2$. The group G_1 is a special 2-group. We identify $\frac{G_1}{Z(G_1)}$ with a 4-dimensional vector space V_1 and $Z(G_1)$ with a 2-dimensional vector space W_1 over the field $\mathbb{F}_2$. Therefore, as a set, the group G_1 gets identified with $V_1 \times W_1$. Let $\{e_1 = (1,0,0,0), e_2 = (0,1,0,0), e_3 = (0,0,1,0), e_4 = (0,0,0,1)\}$ be a basis of V_1 and $\{f_1 = (1,0), f_2 = (0,1)\}$ be a basis of W_1 over $\mathbb{F}_2$. The quadratic map $q_1 : V_1 \to W_1$

associated to G_1 is given by $q_1(x,y,z,w) = (y^2 + xw + yz + xy, xz)$. For every $v \in V_1$, the following table gives an element $a \in V_1$ such that $q_1(v-a) = q_1(a)$. Thus Th. 3.1.2 confirms that G_1 is a real group.

v	a
$(0,0,0,0),(0,1,0,0),(1,0,1,1),(1,1,1,0),(1,1,1,1)$	$(1,0,0,0)$
$(0,0,1,0),(1,0,0,1),(0,1,1,0),(0,1,1,1),(1,1,0,1)$	$(0,0,0,1)$
$(0,0,0,1),(1,0,1,0),(0,1,0,1),(0,0,1,1)$	$(0,0,1,0)$
$(1,1,0,0)$	$(1,0,0,1)$
$(1,0,0,0)$	$(0,1,0,1)$

Table 6.3: The group G_1 defined in example 6.3.1 is real.

Since $\dim_{\mathbb{F}_2}(W_1,\mathbb{F}_2) = 2$, there are exactly three non zero linear maps $s : W_1 \to \mathbb{F}_2$. In the following table we compute the radical $\mathrm{rad}(b_{s_i \circ q)})$ for these linear maps.

s	$s \circ q$	$b_{s\circ q}$	$\mathrm{rad}(b_{s\circ q})$
$s_1(w_1,w_2)$ $= w_1$	$q(x,y,z,w) =$ $y^2 + xy + yz + xw$	$b_{s_1\circ(q)}((x,y,z,w),(x',y',z',w'))$ $= y(x'+z') + y'(x+z)$ $+xw' + x'w$	$\langle 1\rangle$
$s_2(w_1,w_2)$ $= w_2$	$q(x,y,z,w) =$ xz	$b_{s_2\circ(q)}((x,y,z,w),(x',y',z',w'))$ $= xz' + x'z$	$\langle e_2, e_4\rangle$
$s_3(w_1,w_2) =$ $w_1 + w_2$	$q(x,y,z) =$ $y(y+z)+$ $x(w+y+z)$	$b_{s_3\circ(q)}((x,y,z,w),(x',y',z',w'))$ $= y(y'+z') + y'(y+z)$ $+x(w'+y'+z') + x'(w+y+z)$	$\langle 1\rangle$

Table 6.4: Calculation of $\mathrm{rad}(b_{s\circ q})$ for all non zero $s \in \mathrm{Hom}_{\mathbb{F}_2}(W,\mathbb{F}_2)$ and q associated to group G_1 defined in example 6.3.1.

The extraspecial 2-groups G_s corresponding to the regular quadratic forms q_s induced from the transfers $s \circ q_1$ of q_1 by s for all non-zero linear maps $s : W_1 \to \mathbb{F}_2$ are summarized in the following table. The non-commutative part $\Delta(G_s, G'_s)$ of Wedderburn decomposition of $\mathbb{Q}[G_s]$ is given in table 2.2.

Linear map (s)	q_s	G_s	$\lvert G_s\rvert$	$\Delta(G_s, G_s')$
$s_1(w_1, w_2) = w_1$	$[0,0] \perp [0,0]$	$D_4 \circ D_4$	2^5	$M_4(\mathbb{Q})$
$s_1(w_1, w_2) = w_2$	$[0,0]$	D_4	2^3	$M_2(\mathbb{Q})$
$s_1(w_1, w_2) = w_1 + w_2$	$[0,0] \perp [0,0]$	$D_4 \circ D_4$	2^5	$M_4(\mathbb{Q})$

Table 6.5: Computation of non-commutative part of Wedderburn decomposition of $\mathbb{Q}[G_1]$ for group G_1 defined in example 6.3.1.

Now using Th. 6.2.6, the Wedderburn decomposition of rational group algebra $\mathbb{Q}[G_1]$ is

$$\mathbb{Q}[G_1] \cong 16\mathbb{Q} \oplus 2M_4(\mathbb{Q}) \oplus 4M_2(\mathbb{Q}).$$

Example 6.3.2 $G_2 := \langle a, b, c, d, f \; : \; a^2 = b^2 = c^4 = d^2 = f^2 = dcd^{-1}c = a^{-1}b^{-1}ab = b^{-1}c^{-1}bc = a^{-1}d^{-1}ad = a^{-1}c^{-1}ac = 1, faf = ac^2, fbf = b, fcf = bc^{-1}, fdf = c^2d\rangle$.

The order of group G_2 is 2^6. The center of G_2 is $Z(G_2) := \langle b, c^2 \; : \; b^2 = C^4 = 1, bc^2 = c^2b\rangle$ and $|Z(G_2)| = 2^2$. The group G_2 is special 2-group. We identify $\frac{G_2}{Z(G_2)}$ with a 4-dimensional vector space V_2 and $Z(G_2)$ with a 2-dimensional vector space W_2 over the field $\mathbb{F}_2$. Therefore, as a set, the group G_2 gets identified with $V_2 \times W_2$. Let $\{e_1 = (1,0,0,0), e_2 = (0,1,0,0), e_3 = (0,0,1,0), e_4 = (0,0,0,1)\}$ be a basis of V_2 and $\{f_1 = (1,0), f_2 = (0,1)\}$ be a basis of W_2 over $\mathbb{F}_2$. The quadratic map $q_2 : V_2 \to W_2$ associated to G_2 is given by $q(x,y,z,w) = (xw + yz + xy, xz)$. For every $v \in V_2$, the following table gives an element $a \in V_2$ such that $q_2(v - a) = q_2(a)$. Thus Th. 3.1.2 confirms that G_2 is a real group.

v	a
$(0,1,0,0), (1,1,0,0), (0,1,0,1), (0,0,1,0), (1,0,0,1), (0,0,1,1)$	$(0,0,0,1)$
$(0,0,0,1), (0,1,1,0), (1,0,1,0), (0,1,1,1), (1,1,1,1)$	$(0,0,1,0)$
$(0,0,0,0), (1,0,1,1), (1,1,0,1)$	$(1,0,0,0)$
$(1,0,0,0), (1,1,1,0)$	$(1,1,0,1)$

Table 6.6: The group G_2 defined in example 6.3.2 is real.

Here $\dim_{\mathbb{F}_2}(W_2, \mathbb{F}_2) = 2$. Therefore there are exactly three non zero linear maps

$s : W_2 \to \mathbb{F}_2$. In the following table we compute the radical $\text{rad}(b_{s_i \circ q)})$ for these linear maps.

s	$s \circ q$	$b_{s \circ q}$	$\text{rad}(b_{s \circ q})$
$s_1(w_1, w_2)$ $= w_1$	$q(x, y, z, w) =$ $xy + yz + xw$	$b_{s_1 \circ (q)}((x, y, z, w), (x', y', z', w'))$ $= y(x' + z') + y'(x + z)$ $+xw' + x'w$	$\langle 1 \rangle$
$s_2(w_1, w_2)$ $= w_2$	$q(x, y, z, w) =$ xz	$b_{s_2 \circ (q)}((x, y, z, w), (x', y', z', w'))$ $= xz' + x'z$	$\langle e_2, e_4 \rangle$
$s_3(w_1, w_2) =$ $w_1 + w_2$	$q(x, y, z, w) =$ $x(y + w) + z^2 +$ $z(x + y) + (x + y)^2$	$b_{s_3 \circ (q)}((x, y, z, w), (x', y', z', w'))$ $= yz' + z'y$ $+x(w' + y' + z') + x'(w + y + z)$	$\langle 1 \rangle$

Table 6.7: Calculation of $\text{rad}(b_{s \circ q})$ for all non zero $s \in \text{Hom}_{\mathbb{F}_2}(W, \mathbb{F}_2)$ and q associated to group G_2 defined in example 6.3.2.

The extraspecial 2-groups G_s corresponding to the regular quadratic forms q_s induced from the transfers $s \circ q_2$ of q_2 by s for all non-zero linear maps $s : W_2 \to \mathbb{F}_2$ are summarized in the following table. The non-commutative part $\Delta(G_s, G_s')$ of Wedderburn decomposition of $\mathbb{Q}[G_s]$ is given in table 2.2.

Linear map (s)	q_s	G_s	$\lvert G_s \rvert$	$\Delta(G_s, G_s')$
$s_1(w_1, w_2) = w_1$	$[0, 0] \perp [0, 0]$	$D_4 \circ D_4$	2^5	$M_4(\mathbb{Q})$
$s_1(w_1, w_2) = w_2$	$[0, 0]$	D_4	2^3	$M_2(\mathbb{Q})$
$s_1(w_1, w_2) = w_1 + w_2$	$[0, 0] \perp [0, 0]$	$D_4 \circ D_4$	2^5	$M_4(\mathbb{Q})$

Table 6.8: Computation of non-commutative part of Wedderburn decomposition of $\mathbb{Q}[G_2]$ for group G_2 defined in example 6.3.2.

Now using Th. 6.2.6, the Wedderburn decomposition of rational group algebra is

$$\mathbb{Q}[G_2] \cong 16\mathbb{Q} \oplus 2M_4(\mathbb{Q}) \oplus 4M_2(\mathbb{Q}).$$

Proposition 6.3.3. *The groups G_1 defined in example 6.3.1 and G_2 defined in example 6.3.2 are not isomorphic.*

Proof We prove that the groups G_1 and G_2 are not isomorphic by showing that the group G_1 has 11 conjugacy classes with elements of order 2, while the group G_2 has 13 conjugacy classes with elements of order 2. We compute the conjugacy classes of these groups by using Th. 5.3.1. From lemma 1.3.12, we recall that the order of a non trivial element g of special 2-group is 2 if and only if q vanishes on $gZ(G)$, where q is quadratic map associated to group G. If q does not vanish $gZ(G)$ then order of g is 4. We use this to compute the order of elements of groups G_1 and G_2. For a conjugacy class $\mathcal{C}$, let $\circ(\mathcal{C})$ denotes the order of its elements. With the notations same as in example 6.3.1, we compute conjugacy classes of G_1 and order of its element in the following table:

Conjugacy class	$\circ(\mathcal{C})$	**Conjugacy class**	$\circ(\mathcal{C})$
$\mathcal{C}_1 = \{(0,0)\}$	2	$\mathcal{C}_2 = \{(0,f_1)\}$	2
$\mathcal{C}_3 = \{(0,f_2)\}$	2	$\mathcal{C}_4 = \{(0,f_1+f_2)\}$	2
$\mathcal{C}_5 = \{(e_1,0),(e_1,f_1),(e_1,f_2),(e_1,f_1+f_2)\}$	2	$\mathcal{C}_6 = \{(e_2,0),(e_2,f_1)\}$	4
$\mathcal{C}_7 = \{(e_2,f_2),(e_2,f_1+f_2)\}$	4	$\mathcal{C}_8 = \{(e_3,0),(e_3,f_2),(e_3,f_1),(e_3,f_1+f_2)\}$	2
$\mathcal{C}_9 = \{(e_4,0),(e_4,f_1)\}$	2	$\mathcal{C}_{10} = \{(e_4,f_2),(e_4,f_1+f_2)\}$	2
$\mathcal{C}_{11} = \{(e_1,0)(e_2,0),(e_1,0)(e_2,0)(0,f_1),$ $(e_1,0)(e_2,0)(0,f_2),(e_1,0)(e_2,0)(0,f_1+f_2)\}$	2	$\mathcal{C}_{12} = \{(e_1,0)(e_3,0),(e_1,0)(e_3,0)(0,f_1),$ $(e_1,0)(e_3,0)(0,f_2),(e_1,0)(e_3,0)(0,f_1+f_2)\}$	4
$\mathcal{C}_{13} = \{(e_1,0)(e_4,0),(e_1,0)(e_4,0)(0,f_1),$ $(e_1,0)(e_4,0)(0,f_2),(e_1,0)(e_4,0)(0,f_1+f_2)\}$	4	$\mathcal{C}_{14} = \{(e_2,0)(e_3,0),(e_2,0)(e_3,0)(0,f_1),$ $(e_2,0)(e_3,0)(0,f_2),(e_2,0)(e_3,0)(0,f_1+f_2)\}$	2
$\mathcal{C}_{15} = \{(e_2,0)(e_4,0),(e_2,0)(e_4,0)(0,f_1)\}$	4	$\mathcal{C}_{16} = \{(e_2,0)(e_4,0)(0,f_2),(e_2,0)(e_4,0)(0,f_1+f_2)\}$	4
$\mathcal{C}_{15} = \{(e_3,0)(e_4,0),(e_3,0)(e_4,0)(0,f_1),$ $(e_3,0)(e_4,0)(0,f_2),(e_3,0)(e_4,0)(0,f_1+f_2)\}$	2	$\mathcal{C}_{16} = \{(e_1,0)(e_2,0)(e_3,0),$ $(e_1,0)(e_2,0)(e_3,0)(0,f_1),$ $(e_1,0)(e_2,0)(e_3,0)(0,f_2),$ $(e_1,0)(e_2,0)(e_3,0)(0,f_1+f_2)\}$	4
$\mathcal{C}_{16} = \{(e_1,0)(e_2,0)(e_4,0),$ $(e_1,0)(e_2,0)(e_4,0)(0,f_1),$ $(e_1,0)(e_2,0)(e_4,0)(0,f_2),$ $(e_1,0)(e_2,0)(e_4,0)(0,f_1+f_2)\}$	4	$\mathcal{C}_{16} = \{(e_1,0)(e_3,0)(e_4,0),$ $(e_1,0)(e_3,0)(e_4,0)(0,f_1),$ $(e_1,0)(e_3,0)(e_4,0)(0,f_2),$ $(e_1,0)(e_3,0)(e_4,0)(0,f_1+f_2)\}$	4
$\mathcal{C}_{16} = \{(e_2,0)(e_3,0)(e_4,0),$ $(e_2,0)(e_3,0)(e_4,0)(0,f_1),$ $(e_2,0)(e_3,0)(e_4,0)(0,f_2),$ $(e_2,0)(e_3,0)(e_4,0)(0,f_1+f_2)\}$	2	$\mathcal{C}_{16} = \{(e_1,0)(e_2,0)(e_3,0)(e_4,0),$ $(e_1,0)(e_2,0)(e_3,0)(e_4,0)(0,f_1),$ $(e_1,0)(e_2,0)(e_3,0)(e_4,0)(0,f_2),$ $(e_1,0)(e_2,0)(e_3,0)(e_4,0)(0,f_1+f_2)\}$	4

Table 6.9: Conjugacy classes and order of their elements of group G_1.

With the notations same as in example 6.3.2, we compute conjugacy classes of G_2 and order of its element in the following table:

Conjugacy class	$\circ(\mathcal{C})$	**Conjugacy class**	$\circ(\mathcal{C})$
$\mathcal{C}_1 = \{(0,0)\}$	2	$\mathcal{C}_2 = \{(0,f_1)\}$	2
$\mathcal{C}_3 = \{(0,f_2)\}$	2	$\mathcal{C}_4 = \{(0,f_1+f_2)\}$	2
$\mathcal{C}_5 = \{(e_1,0),(e_1,f_1),(e_1,f_2),(e_1,f_1+f_2)\}$	2	$\mathcal{C}_6 = \{(e_2,0),(e_2,f_1)\}$	2
$\mathcal{C}_7 = \{(e_2,f_2),(e_2,f_1+f_2)\}$	2	$\mathcal{C}_8 = \{(e_3,0),(e_3,f_2),(e_3,f_1),(e_3,f_1+f_2)\}$	2
$\mathcal{C}_9 = \{(e_4,0),(e_4,f_1)\}$	2	$\mathcal{C}_{10} = \{(e_4,f_2),(e_4,f_1+f_2)\}$	2
$\mathcal{C}_{11} = \{(e_1,0)(e_2,0),(e_1,0)(e_2,0)(0,f_1),$ $(e_1,0)(e_2,0)(0,f_2),(e_1,0)(e_2,0)(0,f_1+f_2)\}$	4	$\mathcal{C}_{12} = \{(e_1,0)(e_3,0),(e_1,0)(e_3,0)(0,f_1),$ $(e_1,0)(e_3,0)(0,f_2),(e_1,0)(e_3,0)(0,f_1+f_2)\}$	4
$\mathcal{C}_{13} = \{(e_1,0)(e_4,0),(e_1,0)(e_4,0)(0,f_1),$ $(e_1,0)(e_4,0)(0,f_2),(e_1,0)(e_4,0)(0,f_1+f_2)\}$	4	$\mathcal{C}_{14} = \{(e_2,0)(e_3,0),(e_2,0)(e_3,0)(0,f_1),$ $(e_2,0)(e_3,0)(0,f_2),(e_2,0)(e_3,0)(0,f_1+f_2)\}$	4
$\mathcal{C}_{15} = \{(e_2,0)(e_4,0),(e_2,0)(e_4,0)(0,f_1)\}$	2	$\mathcal{C}_{16} = \{(e_2,0)(e_4,0)(0,f_2),(e_2,0)(e_4,0)(0,f_1+f_2)\}$	2
$\mathcal{C}_{17} = \{(e_3,0)(e_4,0),(e_3,0)(e_4,0)(0,f_1),$ $(e_3,0)(e_4,0)(0,f_2),(e_3,0)(e_4,0)(0,f_1+f_2)\}$	2	$\mathcal{C}_{18} = \{(e_1,0)(e_2,0)(e_3,0),$ $(e_1,0)(e_2,0)(e_3,0)(0,f_1),$ $(e_1,0)(e_2,0)(e_3,0)(0,f_2),$ $(e_1,0)(e_2,0)(e_3,0)(0,f_1+f_2)\}$	4
$\mathcal{C}_{19} = \{(e_1,0)(e_2,0)(e_4,0),$ $(e_1,0)(e_2,0)(e_4,0)(0,f_1),$ $(e_1,0)(e_2,0)(e_4,0)(0,f_2),$ $(e_1,0)(e_2,0)(e_4,0)(0,f_1+f_2)\}$	2	$\mathcal{C}_{20} = \{(e_1,0)(e_3,0)(e_4,0),$ $(e_1,0)(e_3,0)(e_4,0)(0,f_1),$ $(e_1,0)(e_3,0)(e_4,0)(0,f_2),$ $(e_1,0)(e_3,0)(e_4,0)(0,f_1+f_2)\}$	4
$\mathcal{C}_{21} = \{(e_2,0)(e_3,0)(e_4,0),$ $(e_2,0)(e_3,0)(e_4,0)(0,f_1),$ $(e_2,0)(e_3,0)(e_4,0)(0,f_2),$ $(e_2,0)(e_3,0)(e_4,0)(0,f_1+f_2)\}$	4	$\mathcal{C}_{22} = \{(e_1,0)(e_2,0)(e_3,0)(e_4,0),$ $(e_1,0)(e_2,0)(e_3,0)(e_4,0)(0,f_1),$ $(e_1,0)(e_2,0)(e_3,0)(e_4,0)(0,f_2),$ $(e_1,0)(e_2,0)(e_3,0)(e_4,0)(0,f_1+f_2)\}$	4

Table 6.10: Conjugacy classes and order of their elements of group G_2.

□

We conclude the chapter with the following remark:

Remark 6.3.4 The computation in example 6.3.1 and example 6.3.2 shows that $\mathbb{Q}[G_1] \cong \mathbb{Q}[G_2]$ and by Prop.6.3.3 groups G_1 and G_2 are not isomorphic.

This establishes that rational group algebras of real special 2-groups do not determine the group.

Part III

Appendix

Appendix A

GAP Codes

We record our GAP code to check whether a group is special 2-group, extraspecial 2-group, real, strongly real or totally orthogonal.

Program 1

```
#Checks if G is a real group.

G;
"real";

output := 0;

N := NrConjugacyClasses(G);
T := CharacterTable(G);
R := RealClasses(T);

if N = Size(R) then
    output := "real";
fi;
```

Program 2

```
#Checks if G is a special 2-group or extraspecial 2-group.
G;
"2-Group";
"special";
"extraspecial";
output := 0;
if IsPGroup(G) = True and Order(G) mod(2) = 0 then
    output := "2-Group";
fi;
if output = "2-Group" then
    D := DerivedSubgroup(G);
    C := Centre(G);
    F := FrattiniSubgroup(G);
    l := Elements(C);
    j := [];
    for g in l do
if g*g = l[1] then
    Add( j, g );
fi;
    od;
    H := Group( j );

    if D = C and C = F and F = H then
output := "special";
    fi;
fi;
if
    output = "special" and Size(C) = 2 then
    output := "extraspecial";
fi;
```

Program 3

```
#Checks if G is a strongly real group.

G;
"stronglyreal";
output := 0;
ccsize := NrConjugacyClasses(G);
involutions := [];
tested := [];

Add(tested, ConjugacyClass(G,Identity(G)));
for g in G do
    if Order(g) = 2 then
        Add(involutions, g);
    fi;
od;
Add(involutions, Identity(G));

for g in involutions do
   for h in involutions do
if ((Size(tested) = ccsize) = false) then
Add(tested, ConjugacyClass(G, g*h));
Add(tested, ConjugacyClass(G, h*g));
        fi;
tested := Set(tested);
   od;
od;

if Size(tested) = ccsize then
    output := "stronglyreal";
fi;
```

Program 4

```
#Check if G is a totally orthogonal group.

G;
"real";
"totallyorthogonal";

output := 0;
output1 := 0;

T := CharacterTable(G);
N := NrConjugacyClasses(T);
R := RealClasses(T);
I := Indicator(T,2);

if N = Size(R) then
    output1 := "real";
fi;

if output1 = "real" and -1 in I then
    output := 0;
    else
output := "totallyorthogonal";
fi;
```

In the following, we provide our GAP code used to make table 4.12 and table 4.13. This GAP code returns a list of totally orthogonal groups that are not strongly real up to order 128 and a list of strongly real groups with symplectic representations up to order 128.

```
# Returns a list of real groups up to order 128.

real := [];
for i in [1..128] do
    m := Size(AllSmallGroups(i));
    for j in [1..m] do
G := SmallGroup(i,j);
```

Run Program 1

```
if output = "real" then
    Add(real, G);
fi;
    od;
od;

#Returns a list of strongly real groups up to order 128.

stronglyreal := [];
for G in real do
```

Run Program 3

```
    if output = "stronglyreal" then
Add(stronglyreal,G);
    fi;
od;

#Returns a list of totally orthogonal groups up to order 128.

totallyorthogonal := [];
for G in real do
```

Run Program 4

```
      if output = "totallyorthogonal" then
Add(totallyorthogonal,G);
    fi;
od;

#Returns a list of totally orthogonal groups up to order 128
 which are not strongly real.

k := [];
TNS := [];
final1 := [];
for G in totallyorthogonal do
    if G in stronglyreal then
Add(k,G);
else
    Add(TNS,G);
    fi;
od;

for G in TNS do
    I:=IdSmallGroup(G);
    D:=StructureDescription(G);
```

Run Program 1

```
    Add(final1, I);
    Add(final1, D);
    Add(final1, output);
od;

#Returns a list of strongly real groups up to order 128
 which are not totally orthogonal.

l := [];
SNT := [];
```

```
final2 := [];
for G in stronglyreal do
    if G in totallyorthogonal then
Add(l,G);
else
    Add(SNT,G);
    fi;
od;

for G in SNT do
    I:=IdSmallGroup(G);
    D:=StructureDescription(G);
```

Run Program 1

```
    Add(final2, I);
    Add(final2,D);
    Add(final2, output);
od;
```

Now we discuss the output of above program. The following list contains totally orthogonal groups which are not strongly real up to order 128. It is clear from the list that the smallest totally orthogonal group which is not strongly real is of order 64 whereas smallest such special 2-group is of order 128 as mentioned in remark 4.2.13.

```
gap> final1;
 [ [ 64, 177 ], "(C2 x D16) : C2", "2-Group",
   [ 128, 453 ], "(C2 x D16) : C2", "2-Group",
   [ 128, 931 ], "(((C8 : C2) : C2) : C2) : C2", "2-Group",
   [ 128, 932 ], "((C4 x C2 x C2) : C4) : C2", "2-Group",
   [ 128, 982 ], "((C4 x C2 x C2) : C4) : C2", "2-Group",
   [ 128, 1345 ], "(C2 x C2 x C2 x D8) : C2", "special",
   [ 128, 1389 ], "(C2 x ((C4 x C4) : C2)) : C2", "special",
   [ 128, 1544 ], "(C2 x ((C2 x C2 x C2 x C2) : C2)) : C2", "special",
   [ 128, 1550 ], "(C2 x ((C4 x C4) : C2)) : C2", "special",
   [ 128, 1880 ], "C2 x ((C2 x D16) : C2)", "2-Group",
```

```
   [ 128, 1924 ], "(C2 x ((C4 x C2 x C2) : C2)) : C2", "2-Group",
   [ 128, 1949 ], "(C2 x ((C4 x C4) : C2)) : C2", "2-Group" ]
```

The following list contains strongly real groups which are not totally orthogonal up to order 128. It is clear from the list that the smallest strongly real group which is not totally orthogonal is of order 32 which is isomorphic to extraspecial 2-group $Q_2 \circ D_4$. As mentioned in remark 4.2.5 there is only one special 2-group of order 64 which is strongly real but not totally orthogonal.

```
gap> final2;
[ [ 32, 50 ], "(C2 x Q8) : C2", "extraspecial",
  [ 64, 218 ], "(C2 x ((C4 x C2) : C2)) : C2", "special",
  [ 64, 265 ], "(C2 x ((C4 x C2) : C2)) : C2", "2-Group",
  [ 128, 1347 ], "(C2 x C2 x ((C4 x C2) : C2)) : C2", "special",
  [ 128, 1388 ], "(C2 x ((C4 x C2) : C4)) : C2", "special",
  [ 128, 1407 ], "(C2 x ((C4 x C2 x C2) : C2)) : C2", "special",
  [ 128, 2180 ], "C2 x ((C2 x ((C4 x C2) : C2)) : C2)", "2-Group",
  [ 128, 2318 ], "(C2 x ((C2 x Q8) : C2)) : C2", "2-Group",
  [ 128, 2324 ], "C2 x C2 x ((C2 x Q8) : C2)", "2-Group",
  [ 128, 2327 ], "(C2 x ((C2 x Q8) : C2)) : C2", "extraspecial" ]
```

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