Classification of isometries of the complex and quaternionic hyperbolic spaces

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A thesis submitted for the partial fulfilment
of the degree of
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dedicated

to

my parents

Declaration

The work presented in this thesis has been carried out by me under the guidance of Dr. Krishnendu Gongopadhyay at Indian Institute of Science Education and Research Mohali.

This work has not been submitted in part or in full for a degree, a diploma, or a fellowship to any other university or institute. Whenever contributions of others are involved, every effort is made to indicate this clearly, with due acknowledgement of collaborative research and discussions. This thesis is a bonafide record of original work done by me and all sources listed within have been detailed in the bibliography.

Date:
Place:

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In my capacity as the supervisor of the candidate's thesis work, I certify that the above statements by the candidate are true to the best of my knowledge.

Dr. Krishnendu Gongopadhyay
(Supervisor)

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Contents

Chapter 1

Introduction

Classically, the isometries of the real hyperbolic space $\mathbf{H}^n_{\mathbb{R}}$ are classified as elliptic, parabolic and hyperbolic according to the dynamics of their fixed points. In two and three dimensional real hyperbolic geometries, this trichotomy of the isometries are classified algebraically in terms of their traces, cf. [4, Theorems 4.3.1 and 4.3.4]. There have been several attempts to generalize this algebraic classification in higher dimensional real hyperbolic geometries, for example, see [1, 9, 23, 51, 52]. Similar trichotomy based on the fixed-point dynamics of the isometries is also valid in the complex and the quaternionic hyperbolic geometries. In order to understand the geometry and dynamics of isometries in these geometries, it is natural to ask the following problem:

Problem 1. *Obtain algebraic criteria to classify isometries of the n-dimensional quaternionic and complex hyperbolic spaces?*

That is, the problem is to obtain generalization of [4, Theorems 4.3.1 and 4.3.4] in these geometries. In two-dimensional complex hyperbolic geometry similar criterion is known by the work of Goldman [18, Theorem 6.2.4], also see [17]. Cao-Gongopadhyay, Gongopadhyay [7, 24] have obtained a counterpart of Goldman's theorem in two and three dimensional quaternionic hyperbolic geometries. In chapter-3, we have generalized these results to obtain an algebraic criterion to classify isometries of the n-dimensional quaternionic hyperbolic space for any n. In chapter-4, we use the coefficients of the characteristic polynomial to give a dynamical classification of unitary matrices preserving a non-degenerate Hermitian form. As a special case, this gives us algebraic criteria to classify isometries of the n-dimensional complex hyperbolic space.

In order to illustrate and motivate the main results, let us work through the well known example of 2×2 matrices. In this case, if $A \in \mathrm{SU}(p, q)$ with $p + q = 2$ then the characteristic polynomial of A is

$$\chi_A(X) = X^2 - \tau X + 1$$

where $\tau = \mathrm{tr}(A)$, which is real. Consider the *resultant* $R(\chi_A, \chi'_A)$, which determines when $\chi_A(X)$ has a repeated root. Here the resultant is $4 - \tau^2$ and we have

(i) A is elliptic if and only if $R(\chi_A, \chi'_A) = 4 - \tau^2 > 0$.

(ii) A is parabolic (or $\pm I$) if and only if $R(\chi_A, \chi'_A) = 4 - \tau^2 = 0$.

(iii) A is loxodromic if and only if $R(\chi_A, \chi'_A) = 4 - \tau^2 < 0$.

This argument was generalised by Goldman [18, Theorem 6.2.4] to the case $p+q = 3$; see also Parker [39]. Goldman's work has been generalised in a different direction by Navarrete [38] who considers elements of $\mathrm{SL}(3, \mathbb{C})$. This is related to the theory of complex Kleinian groups, see [5]. The classification of elements of $\mathrm{SL}(2, \mathbb{R})$, $\mathrm{SL}(2, \mathbb{C})$ or $\mathrm{SU}(2, 1)$ has been useful in many contexts, see [19], [26] or [39]. Our initial motivation to this work was to provide initial tools for generalisation of these works to $\mathrm{SU}(n, 1)$ for $n \geq 3$. As we did so, we realised that it is natural to consider Hermitian forms of arbitrary signature. First, we drive the classification for $\mathrm{SU}(p, q)$, p, q arbitrary, and then, we consider the special case where $p + q = 4$. This later case gives us a foundation tool to initiate the study of the three dimensional complex hyperbolic geometry.

There is a long tradition of work to study two-generator discrete subgroups of $\mathrm{SL}(2, \mathbb{R})$ and $\mathrm{SL}(2, \mathbb{C})$ in connection with Fuchsian and Kleinian groups. The study goes back to the work of Vogt [50] and Fricke [16] who proved that a non-elementary two-generator discrete free subgroup is determined up to conjugation by the traces of the generators and their product. This result was incremental in the development of Teichmüller theory and in particular, it was used to provide Fenchel-Nielsen coordinates on the Teichmüller space. For an up to date exposition of this work see Goldman [19].

In an attempt to define analogous Fenchel-Nielsen coordinates for complex hyperbolic quasi-Fuchsian representations of surface groups, Parker and Platis [41, Theorem 7.1] proved a generalization of the result of Fricke-Vogt for two-generator discrete free subgroups of $\mathrm{SU}(2,1)$ with loxodromic generators. Parker and Platis followed an approach that uses traces of the generators and a point on the so called Koranyi-Riemann cross-ratio variety. In another approach, it follows from the work of Lawton [33], Wen [53] and Will [54, 55] that a two-generator Zariski dense subgroup of $\mathrm{SU}(2,1)$ is determined by traces of the generators and the traces of three more compositions of the generators. For a survey of these results see Parker [39]. In chapter-5, we obtain a generalization of the result of Fricke-Vogt for two-generator subgroups of $\mathrm{SU}(3,1)$ generated by loxodromic elements. This also generalizes the work of Parker and Platis [41] to some extent.

1.1 An overview of the thesis

The key idea used in [7, 24] involves an embedding of the quaternions into the matrix ring $M_2(\mathbb{C})$, and, classical analysis of nature of roots of a real cubic or biquadratic equations. In chapter-3, generalizing this approach to arbitrary dimension, we have

Theorem 1.1.1. *[28, Theorem 3.1] Let A be an element in $\mathrm{Sp}(n,1)$. Suppose $A_{\mathbb{C}}$ be the corresponding element in $\mathrm{GL}((2(n+1),\mathbb{C})$. Let $\mathcal{S}_A = \{\Delta_1, \cdots, \Delta_{n+1}\}$ be the discriminant sequence of $g_A(t)$, where $\Delta_{n+1} = \Delta$ is the usual algebraic discriminant of $g_A(t)$. Let D be the discriminant of the minimal polynomial of $A_{\mathbb{C}}$. Then the following holds.*

1. *A is regular hyperbolic if and only if $\Delta < 0$.*
2. *A is regular elliptic if and only if $\Delta > 0$.*
3. *A is semi-regular hyperbolic if and only if $\Delta = 0$ and the number of sign changes of the revised sign list of $\mathcal{S}_A$ is exactly one.*
4. *A is screw hyperbolic if and only if $\Delta = 0$ and $g_A(t)$ has a real root λ such that $|\lambda| > 2$.*
5. *A is a strictly hyperbolic if and only if $g_A(t)$ has a real root λ such that $|\lambda| > 2$ and for all $m \leq n-2$, $g_A^{(m)}(2) = 0$.*

6. *A is elliptic or parabolic if and only if* $\Delta = 0$ *and there is no sign change in the number of revised sign list of* $\mathcal{S}_A$*. Further A is parabolic if* $D = 0$*; otherwise it is elliptic. Further A is simple elliptic if the number of non-vanishing members of the revised sign list is exactly one.*

For terminology used in the above theorem, see page 12, chapter-2 and pages 21-22, chapter-3.

In chapter-4, our aim is to generalise Goldman's classification to higher values of $p + q = n$. First, we consider arbitrary n and give a general result, Theorem 1.1.2. In particular *regular* means that the eigenvalues of A are distinct. A is said to be k-loxodromic means that A has k pairs of distinct eigenvalues related by inversion in the unit circle and all other eigenvalues lie on the unit circle, so regular 0-loxodromic maps are elliptic.

Theorem 1.1.2. *[27, Theorem 3.1] Let* $A \in \mathrm{SU}(p, q)$*. Let* $R(\chi_A, \chi'_A)$ *denote the resultant of the characteristic polynomial* $\chi_A(X)$ *and its first derivative* $\chi'_A(X)$*. Then for* $m \geq 0$*, we have the following.*

(i) A is regular $2m$*-loxodromic if and only if* $R(\chi_A, \chi'_A) > 0$*.*

(ii) A is regular $(2m + 1)$*-loxodromic if and only if* $R(\chi_A, \chi'_A) < 0$*.*

(iii) A has a repeated eigenvalue if and only if $R(\chi_A, \chi'_A) = 0$*.*

An immediate corollary of Theorem 1.1.2 is a classification for $\mathrm{SU}(p, 1)$. Since $q = 1$, if A is loxodromic it must be 1-loxodromic. This simplifies the classification:

Corollary 1.1.3. *[27, Corollary 3.2] Let* $A \in \mathrm{SU}(p, 1)$*. Let* $R(\chi_A, \chi'_A)$ *denote the resultant of the characteristic polynomial* $\chi_A(X)$ *and its first derivative* $\chi'_A(X)$*. Then we have the following.*

(i) A is regular elliptic if and only if $R(\chi_A, \chi'_A) > 0$*.*

(ii) A is regular loxodromic if and only if $R(\chi_A, \chi'_A) < 0$*.*

(iii) A has a repeated eigenvalue if and only if $R(\chi_A, \chi'_A) = 0$*.*

Secondly, we give a much more detailed description in the case $p+q=4$. Here the characteristic polynomial is

$$\chi_A(X) = X^4 - \tau X^3 + \sigma X^2 - \overline{\tau} X + 1$$

where $\tau = \mathrm{tr}(A)$, which is complex, and $\sigma = \big(\mathrm{tr}^2(A) - \mathrm{tr}(A^2)\big)/2$, which is real. In this case, the locus where $R(\chi_A, \chi_A') = 0$ was studied by Poston and Stewart [46] following earlier work by Chillingworth [12]. They named this object the *holy grail*. As a subset of three dimensional space, parametrised by $(\tau, \sigma) \in \mathbb{C} \times \mathbb{R}$, the holy grail comprises a ruled surface together with four space curves, called *whiskers*. We devote some space to different ways of parametrising the holy grail and the different components of its complement. The parametrisation of the corresponding object (a deltoid) in the case of $p+q=3$ has been useful when studying complex hyperbolic representation spaces (see [22], [43] or the survey [39]). We believe that the results in this direction will be foundational to the generalisation of these theorems to higher dimensions. We prove the following

Theorem 1.1.4. *[27, Theorem 4.9] Let $A \in \mathrm{SU}(p,q)$ where $p+q=4$ and let $\tau = \mathrm{tr}(A)$ and $\sigma = \big(\mathrm{tr}^2(A) - \mathrm{tr}(A^2)\big)/2$. Let $\chi_A(X)$ be the characteristic polynomial of A and let $R(\chi_A, \chi_A')$ be the resultant of $\chi_A(X)$ and $\chi_A'(X)$. Then*

(i) A is regular 2-loxodromic if and only if $R(\chi_A, \chi_A') > 0$ and

$$\min\big\{\Re(\tau)^2 - 4\sigma + 8,\ \Im(\tau)^2 + 4\sigma + 8,\ 6 - \sigma,\ 6 + \sigma\big\} < 0.$$

(ii) A is regular 1-loxodromic if and only if $R(\chi_A, \chi_A') < 0$.

(iii) A is regular elliptic if and only if $R(\chi_A, \chi_A') > 0$ and

$$\Re(\tau)^2 - 4\sigma + 8 > 0, \quad \Im(\tau)^2 + 4\sigma + 8 > 0, \quad -6 < \sigma < 6.$$

(iv) A has a repeated eigenvalue if and only if $R(\chi_A, \chi_A') = 0$.

We also consider the automorphisms of anti de Sitter space, which may be canonically identified with $\mathrm{PSL}(2, \mathbb{R})$. This gives an identification between the automorphisms of anti de Sitter space and $\mathrm{PSL}(2,\mathbb{R})\times\mathrm{PSL}(2,\mathbb{R})$. By translating such an automorphism to $\mathrm{PSO}(2,2)$ we can use our classification to determine the dynamics. Specifically we have

Theorem 1.1.5. *[27, Theorem 5.4] Let* $(A_1, A_2) \in \mathrm{PSL}(2,\mathbb{R}) \times \mathrm{PSL}(2,\mathbb{R})$ *be an automorphism of anti de Sitter space. Then*

(i) (A_1, A_2) *is regular 2-loxodromic if at least one of* A_1 *and* A_2 *is loxodromic, and also* $\mathrm{tr}^2(A_1)$ *and* $\mathrm{tr}^2(A_2)$ *are distinct and neither of them equals* 4*.*

(ii) (A_1, A_2) *is regular elliptic if* A_1 *and* A_2 *are both elliptic and* $\mathrm{tr}^2(A_1)$ *does not equal* $\mathrm{tr}^2(A_2)$*.*

(iii) (A_1, A_2) *is not regular if* $\mathrm{tr}^2(A_1) = 4$ *or,* $\mathrm{tr}^2(A_2) = 4$ *or* $\mathrm{tr}^2(A_1) = \mathrm{tr}^2(A_2)$*.*

In chapter-5, we are motivated by the approach of Parker-Platis [41]. However for two-generator subgroups in $\mathrm{SU}(3,1)$, traces and cross-ratios of the generators are not sufficient to determine the subgroup up to conjugacy. For the determination of two-generator subgroups, one needs to look for more conjugacy invariants of the pair of generators. For this purpose, we use new invariants which are generalizations of Goldman's eta invariants. We also use Goldman's eta-invariants to derive a sufficient condition for a two-generator discrete, free loxodromic subgroup $\langle A, B\rangle$ to have an invariant $\mathbb{C}^2$-chain.

Let $\mathbb{C}^{3,1}$ be the vector space $\mathbb{C}^4$ equipped with a non-degenerate Hermitian form of signature $(3,1)$. Then $\mathbf{H}^3_{\mathbb{C}}$ is the projectivization of negative vectors in $\mathbb{C}^{3,1}$. The boundary $\partial\mathbf{H}^3_{\mathbb{C}}$ is the projectivization of null vectors. Following Goldman [18] recall that a *k-dimensional complex totally geodesic subspace* of $\mathbf{H}^3_{\mathbb{C}}$ or a *$\mathbb{C}^k$-plane* is the projectivization of a copy of $\mathbb{C}^{k,1}$ in $\mathbb{C}^{3,1}$, $k = 1, 2$. A $\mathbb{C}^1$-plane is simply called a *complex geodesic.* A *$\mathbb{C}^k$-chain* is the boundary of a $\mathbb{C}^k$-plane in $\mathbf{H}^3_{\mathbb{C}}$; a $\mathbb{C}^1$-chain is simply called a *chain.* A positive vector c is polar to a $\mathbb{C}^2$-plane C if the lift of C in $\mathbb{C}^{3,1}$ is the orthogonal complement of c. The positive vector c is polar to a $\mathbb{C}^2$-chain L if L is the boundary of a $\mathbb{C}^2$-plane C that is polar to c.

For four distinct points z_1, z_2, z_3 and z_4 in $\partial\mathbf{H}^3_{\mathbb{C}}$ the *Koranyi-Riemann cross-ratio* is defined by:

$$\mathbb{X}(z_1, z_2, z_3, z_4) = \frac{\langle \mathbf{z}_3, \mathbf{z}_1\rangle\langle \mathbf{z}_4, \mathbf{z}_2\rangle}{\langle \mathbf{z}_4, \mathbf{z}_1\rangle\langle \mathbf{z}_3, \mathbf{z}_2\rangle},$$

where $\mathbf{z_i}$ is lift of z_i in $\mathbb{C}^{3,1}$. For more details on cross ratios, see [18]. We extend the above definition to define invariants for the "generic case" that includes three null vectors and one positive vector in $\mathbb{C}^{3,1}$. For a loxodromic element A, we denote by $\mathbf{a}_A$, $\mathbf{r}_A$ the null eigenvectors of A corresponding to the fixed points and let $\mathbf{x}_A$ and $\mathbf{y}_A$ correspond to the positive eigenvectors of A.

Let A, B be two loxodromic elements in $\mathrm{SU}(3,1)$ with distinct fixed points. Then corresponding to the fixed points there are three cross-ratios $\mathbb{X}_k(A,B)$, $k = 1, 2, 3$ that determines the four points uniquely. The collection of all such cross-ratios corresponding to pair of loxodromic elements form a variety, called the *cross-ratio variety.* It follows that every point in this variety has five real degrees of freedom. For more details on cross ratios in the geometry of rank one symmetric spaces, see Platis [44]. The pair (A, B) is called *non-singular* if

(i) A and B are loxodromics without a common fixed point.

(ii) The fixed points of A and B do not lie on a common $\mathbb{C}^2$-chain.

(iii) The fixed point set of A is disjoint from at least one of the $\mathbb{C}^2$-chains polar to the positive eigenvectors of B and, the fixed point set of B is disjoint from at least one of the $\mathbb{C}^2$-chains polar to the positive eigenvectors of A.

The free subgroup $\langle A, B\rangle$ is non-singular if the generating pair is non-singular. In particular, a non-singular subgroup is Zariski-dense in $\mathrm{SU}(3,1)$. To a non-singular pair (A, B), we associate a pair of complex numbers $\alpha_i(A, B)$ and $\beta_j(A, B)$ which are given by the following:

$$\alpha_1(A,B) = \mathbb{X}(\mathbf{r}_A,\ \mathbf{a}_A,\ \mathbf{x}_B,\ \mathbf{a}_B),\quad \alpha_2(A,B) = \mathbb{X}(\mathbf{r}_A,\ \mathbf{a}_A,\ \mathbf{y}_B,\ \mathbf{a}_B).$$

$$\beta_1(A,B) = \mathbb{X}(\mathbf{r}_B,\ \mathbf{a}_B,\ \mathbf{x}_A,\ \mathbf{a}_A),\quad \beta_2(A,B) = \mathbb{X}(\mathbf{r}_B,\ \mathbf{a}_B,\ \mathbf{y}_A,\ \mathbf{a}_A).$$

We shall refer to $\alpha_1(A, B)$ or $\alpha_2(A, B)$ by *α-invariant* and, $\beta_1(A, B)$ or $\beta_2(A, B)$ by *β-invariant.* We prove the following:

Theorem 1.1.6. *Let A, B be two loxodromic elements in $\mathrm{SU}(3,1)$ such that they generate a non-singular subgroup $\langle A, B\rangle$. Then $\langle A, B\rangle$ is determined up to conjugacy by the following parameters:*

$tr(A),\ tr(B),\ \sigma(A),\ \sigma(B),\ \mathbb{X}_k(A,B),\ k=1,2,3,$ *one non-zero α-invariant and one non-zero β-invariant, where* $tr(A)=trace(A)$, $\sigma(A)=\frac{1}{2}(tr^2(A)-tr(A^2))$.

In the parameter space associated to $\langle A,B\rangle$, the parameters $\mathrm{tr}(A),\ \mathrm{tr}(B),\ \alpha$ and β are complex numbers, $\sigma(A),\ \sigma(B)$ are real numbers, $(\mathbb{X}_1,\mathbb{X}_2,\mathbb{X}_3)$ live on the cross-ratio variety, which is 5 dimensional. Thus we need a total of $(4\times2+2\times1+5)=15$ dimensional real parameters to specify $\langle A,B\rangle$ up to conjugacy.

Suppose F_2 is a free group of rank two. Let $F_2=\langle m,n\rangle$. Let us consider the $\mathrm{SU}(3,1)$-representation variety of F_2: $\mathcal{M}=Hom(F_2,\mathrm{SU}(3,1))//\mathrm{SU}(3,1)$. Let $\mathcal{R}^{lox}$ be the subset of $\mathcal{M}$ defined by

$$\mathcal{R}^{lox}=\{\rho:F_2\to\mathrm{SU}(3,1)\mid\rho(m)\text{ and }\rho(n)\text{ are loxodromic}\}.$$

For $i,j\in\{1,2\}$, let

$$\mathcal{R}^{lox}_{ij}=\{\rho\in\mathcal{R}^{lox}\mid(\rho(m),\rho(n))\text{ is non-singular and }\eta_i(\rho(m),\rho(n))\neq0\neq\nu_j(\rho(m),\rho(n))\}.$$

Let

$$\mathcal{R}^{lox}_o=\{\rho\in\mathcal{R}^{lox}\mid(\rho(m),\rho(n))\text{ is non-singular}\},\ \text{thus}$$

$$\mathcal{R}^{lox}_o=\mathcal{R}^{lox}_{11}\cup\mathcal{R}^{lox}_{12}\cup\mathcal{R}^{lox}_{21}\cup\mathcal{R}^{lox}_{22}.$$

Let $\mathcal{M}^{lox}_{ij}=\mathcal{R}^{lox}_{ij}//\mathrm{SU}(3,1)$. Then Theorem 1.1.6 classifies the representations of $\mathcal{M}^{lox}_{ij}$.

Corollary 1.1.7. *Let $\rho:F_2\to\mathrm{SU}(3,1)$ be a representation such that $\rho(m),\ \rho(n)$ are loxodromic and generates a non-singular subgroup of* $\mathrm{SU}(3,1)$. *For some $i,j\in\{1,2\}$, let $\eta_i(\rho(m),\rho(n))\neq0\neq\nu_j(\rho(m),\rho(n))$. Then there exists two non-zero complex parameters α_i and β_j such that these along with coefficients of the characteristic polynomials of $\rho(m)$, $\rho(n)$ and a point on the cross-ratio variety completely determine ρ up to conjugacy.*

The real dimension of the parameter space associated to $\mathcal{M}^{lox}_{ij}$ is 15.

Chapter 2

Preliminaries

2.1 Polynomials and their roots

Theorem 2.1.1. *Let $A \in M_n(\mathbb{C})$ has characteristic polynomial $\chi_A(x) = x^n + s_1x^{n-1} + \cdots + s_n$ and T_k =Trace of A^k. Then the coefficients s_i of $\chi_A(x)$ are given by the following recursion:*
$s_1 = -T_1,\ s_2 = -\frac{1}{2}(s_1T_1 + T_2), \cdots,\ s_n = -\frac{1}{n}(s_{n-1}T_1 + s_{n-2}T_2 + \cdots + T_n).$

Definition 2.1.2. *We define the resultant of two polynomials, $p(X) = a_rX^r + a_{r-1}X^{r-1} + \cdots + a_1X + a_0, q(X) = b_sX^s + b_{s-1}X^{s-1} + \cdots + b_1X + b_0$ as the determinant of the $(r+s) \times (r+s)$ matrix defined as follows.*

$$R(p,q) = \det \begin{pmatrix} a_r & a_{r-1} & \cdots & a_0 & 0 & 0 & \cdots & 0 \\ 0 & a_r & \cdots & a_1 & a_0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 & a_r & a_{r-1} & \cdots & a_0 \\ b_s & b_{s-1} & \cdots & b_0 & 0 & 0 & \cdots & 0 \\ 0 & b_s & \cdots & b_1 & b_0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 & b_s & b_{s-1} & \cdots & b_0 \end{pmatrix}.$$

Definition 2.1.3. *Given a polynomial $f(x) = a_0x^n + a_1x^{n-1} + \cdots + a_n$, write the first derivative of $f(x)$ as*

$$f'(x) = 0.x^n + na_0x^{n-1} + \cdots + a_{n-1}.$$

The discriminant matrix of $f(x)$ is given by $R(f, f')$ and we have,

$$\mathrm{Disc}(f) = \begin{pmatrix} a_0 & a_1 & a_2 & \cdots & a_n & & \\ 0 & na_0 & (n-1)a_1 & \cdots & a_{n-1} & & \\ 0 & a_0 & a_1 & \cdots & a_0 & a_n & \\ 0 & 0 & na_0 & \cdots & 2a_{n-2} & a_{n-1} & \\ & & & \cdots & & \cdots & \\ & & & \cdots & & \cdots & \\ & & & & a_0 & a_1 & \cdots a_n \\ & & & & 0 & na_0 & \cdots a_{n-1} \end{pmatrix}.$$

Definition 2.1.4. *For $k = 1, \cdots, n$, let $\Delta_k(f)$ or simply, Δ_k, denote the determinant of the submatrix of* Disc(f) *formed by the first $2k$ rows and first $2k$ columns. Note that* $\Delta_n = \Delta = \det(\mathrm{Disc}(f))$ *The* discriminant sequence *of $f(x)$ is defined to be the sequence* $\mathcal{S} = \{\Delta_1, \Delta_2, \cdots, \Delta_n\}$.

Definition 2.1.5. *The list $[sign(\Delta_1), \cdots, sign(\Delta_n)]$ is called the* sign list *of the discriminant sequence $\mathcal{S}$.*

Definition 2.1.6. *Given a sign list $[s_1, s_2, \cdots, s_n]$, define the* revised sign list *as follows:*

If $[s_i, s_{i+1}, \cdots, s_{i+j}]$ is a section of the given list, where

$$s_i \neq 0,\ s_{i+1} = s_{i+2} = \cdots = s_{i+j-1} = 0,\ s_{i+j} \neq 0,$$

then we replace the subsection $[s_{i+1}, \cdots, s_{i+j-1}]$ by

$$[-s_i, -s_i, s_i, s_i, -s_i, -s_i, s_i, s_i, \cdots]$$

i.e. let $e_{i+r} = (-1)^{[\frac{r+1}{2}]} s_i$ for $r = 1, 2, \cdots, j-1$. Otherwise let $e_k = s_k$. This gives us the revised sign list $[e_1, e_2, \cdots, e_n]$.

Theorem 2.1.7. `[29, Theorem 1]` *Given a polynomial $f(x)$ with real coefficients*

$$f(x) = a_0 x^n + a_{n-1} x^{n-1} + \cdots + a_n,$$

if the number of the sign changes of the revised sign list of

$$\{\Delta_1(f), \Delta_2(f), \cdots, \Delta_n(f)\}$$

is p, then the pairs of distinct conjugate imaginary roots of $f(x)$ equal p. Furthermore, if the number of non-vanishing members of the revised sign list is q, then the number of distinct real roots of $f(x)$ equals $q - 2p$.

Theorem 2.1.8. `[14, Number of Roots Theorem]` *Let $D_n = (-1)^{\frac{n(n-1)}{2}} a_0^{n-2} n^{-n} \Delta_n$. Suppose the roots of $f(x)$ are distinct. Then the number of real roots of $f(x)$ is:*

(1) if n is odd, congruent to 1 or 3 modulo 4 according as $D_n > 0$ or $D_n < 0$.

(2) if n is even, congruent to 0 or 2 modulo 4 according as D_n and the leading coefficient of $f(x)$ have the same or opposite signs.

2.2 Hyperbolic spaces and their isometry groups

Definition 2.2.1. *The division ring of quaternions is a 4-dimensional division algebra over $\mathbb{R}$ with basis $\{1, i, j, k\}$ satisfying $i^2 = -1 = j^2, ij = k = -ji$. We denote it by $\mathbb{H}$. For a quaternion $z = a + bi + cj + dk,\ a, b, c, d \in \mathbb{R}$, we define $\bar{z} = a - bi - cj - dk$ so that $\Re(z) = (z + \bar{z})/2$ and $\|z\|^2 = a^2 + b^2 + c^2 + d^2 = z\bar{z}$.*

Lemma 2.2.2. *Two quaternions z, w are conjugate iff $\Re(z) = \Re(w)$ and $\|z\| = \|w\|$.*

Lemma 2.2.3. *Let V be a right vector space over $\mathbb{H}$ and T be an invertible linear transformation of V. Then the eigenvalues of T occur in similarity classes.*

Proof. For $v \in V$, $\lambda \in \mathbb{H}^*$, suppose $Tv = v\lambda$, i.e. λ is a (right) eigenvalue of T. For $\mu \in \mathbb{H}^*$, $T(v\mu) = (v\mu)\mu^{-1}\lambda\mu$, i.e. if v is an λ-eigenvector, then $v\mu \in v\mathbb{H}$ is an $\mu^{-1}\lambda\mu$-eigenvector. This establishes the result.

Definition 2.2.4. *Let λ be a (right) eigenvalue of an invertible linear transformation of V. Then the one-dimensional right subspace of V spanned by v will be called* λ-eigenline.

Remark 2.2.5. *Each similarity class of eigenvalues contains a unique pair of complex conjugate numbers. We will denote the similarity class of an eigenvalue with its complex representative $[re^{i\theta}],\ 0 \le \theta \le \pi$.*

Theorem 2.2.6. *[57, Theorem-6.3] Every $n \times n$ quaternionic matrix A is conjugate to an $n \times n$ complex matrix A_c.*

Definition 2.2.7. *The* multiplicity *of the similarity class* $[re^{i\theta}]$ *of eigenvalues of* A *is defined to be algebraic multiplicity of an eigenvalue* $re^{i\theta}$ *of* A_c*, where* A_c *is* $n \times n$ *complex matrix conjugate to* A.

Remark 2.2.8. The eigenvalues are no more conjugacy invariants for T, but the similarity classes of eigenvalues are conjugacy invariant.

Definition 2.2.9. *Let* $\mathbb{F} = \mathbb{C}$ *or* $\mathbb{H}$ *and* V *be a (right)* n*-dimensional vector space over* $\mathbb{F}$*. Then a Hermitian form* $\langle \cdot, \cdot \rangle$ *on* V*, is a map from* $V \times V$ *to* $\mathbb{F}$ *satisfying*

1. $\langle v + z, w\rangle = \langle v, w\rangle + \langle z, w\rangle$,
2. $\langle v, z + w\rangle = \langle v, z\rangle + \langle v, w\rangle$,
3. $\langle v, z\lambda\rangle = \langle v, z\rangle\lambda$, *and*
4. $\overline{\langle v, z\rangle} = \langle z, v\rangle$ *for all* $v, z, w \in V$ *and all* $\lambda \in \mathbb{F}$.

Definition 2.2.10. *Let* $\mathbb{V} = \mathbb{F}^{n+1}$ *be the Hermitian vector space equipped with the Hermitian form of* signature (p, q) *defined by* $\langle z, w\rangle = -\bar{z}_0 w_0 - \bar{z}_1 w_1 - \cdots - \bar{z}_{p-1} w_{p-1} + \bar{z}_p w_p + \cdots + \bar{z}_{p+q-1} w_{p+q-1}$*, where* z *and* w *are the column vectors in* $\mathbb{V}$ *with entries* $z_0, \cdots, z_n$ *and* $w_0, \cdots, w_n$ *respectively. When* $\mathbb{F} = \mathbb{C}$*, the isometry group is denoted by* $\mathrm{U}(p, q)$*. An element of* $\mathrm{U}(p, q)$ *is called an unitary matrix. We often wish to consider unitary matrices with determinant equal to 1. Such matrices form the group* $\mathrm{SU}(p, q)$*. When* $\mathbb{F} = \mathbb{C}$*, or* $\mathbb{H}$ *and* $p = n,\ q = 1$*, we denote the isometry group by* $\mathrm{U}(n, 1; \mathbb{F})$*. For* $\mathbb{F} = \mathbb{C}$*, we denote it simply by* $\mathrm{U}(n, 1)$ *and for* $\mathbb{F} = \mathbb{H}$*, we denote it by* $\mathrm{Sp}(n, 1)$.

Lemma 2.2.11 (Lemma 6.2.5 of Goldman)**.** *Let* V *be a Hermitian vector space and* A *a unitary transformation of* V*. If* λ *is an eigenvalue of* A *then* $\overline{\lambda}^{-1}$ *is also an eigenvalue of* A *with the same multiplicity as* λ*. That is, the collection of eigenvalues of* A *is invariant under inversion in the unit circle.*

Definition 2.2.12. *A vector* $v \in \mathbb{V}$ *is called* time-like*, resp.* space-like*, resp.* light-like *if* $\langle v, v\rangle$ *is negative, resp. positive, resp. zero. The set of all* time-like*, resp.* space-like*, resp.* light-like *vectors is denoted by* V_-*, resp.* V_+*, rep.* V_0.

Definition 2.2.13. *A right eigenvalue λ (counted without multiplicities) of $g \in U(n,1;\mathbb{F})$ is called* negative*, resp.* positive*, resp. null if the λ-eigenline is time-like, resp. space-like, resp. light-like.*

Remark 2.2.14. *A similarity class of eigenvalues $[\lambda]$ is* negative*,* positive *or* null *according as its representative λ is negative, positive or null eigenvalue.*

Definition 2.2.15. *Consider the (right) vector space $\mathbb{V} = \mathbb{F}^{n+1}$ over $\mathbb{F}$ equipped with the Hermitian form of* signature $(n,1)$*. Let $\mathbb{P}(\mathbb{V})$ be the projective space obtained from $\mathbb{V}$, equipped with the quotient topology via the projection map $\pi : \mathbb{V} - \{0\} \to \mathbb{P}(\mathbb{V})$. The n-dimensional hyperbolic space over $\mathbb{F}$ is defined to be $\mathbf{H}^n_{\mathbb{F}} = \pi(\mathbb{V}_-)$. The boundary $\partial\mathbf{H}^n_{\mathbb{F}}$ is $\pi(\mathbb{V}_0)$.*

Definition 2.2.16. *The group $\mathrm{U}(n,1;\mathbb{F})$ acts as the isometry group of $\mathbf{H}^n_{\mathbb{F}}$. The actual group of isometries of $\mathbf{H}^n_{\mathbb{F}}$ is $\mathrm{PU}(n,1;\mathbb{F}) = \mathrm{U}(n,1;\mathbb{F})/Z(\mathrm{U}(n,1;\mathbb{F}))$, where $Z(\mathrm{U}(n,1;\mathbb{F}))$ denotes the center. When $\mathbb{F} = \mathbb{C}$, $Z(\mathrm{U}(n,1))$ is the circle group $\mathbb{S}^1 = \{\lambda I \mid |\lambda| = 1\}$, and for $\mathbb{F} = \mathbb{H}$, $Z(\mathrm{Sp}(n,1)) = \{I, -I\}$; here I denotes the identity transformation.*

Remark 2.2.17. *An isometry g of $\mathbf{H}^n_{\mathbb{F}}$ lifts to a transformation $\tilde{g}$ in $\mathrm{U}(n,1;\mathbb{F})$. For convenience, we mostly deal with the linear group $\mathrm{U}(n,1;\mathbb{F})$ rather than $\mathrm{PU}(n,1;\mathbb{F})$.*

2.2.1 Siegel domain model of the complex hyperbolic space

Let $V = \mathbb{C}^{n,1}$ be the complex vector space $\mathbb{C}^{n+1}$ equipped with the Hermitian form of signature $(n,1)$ given by

$$\langle \mathbf{z}, \mathbf{w} \rangle = \mathbf{w}^* H \mathbf{z} = z_0 \bar{w}_n + z_1 \bar{w}_1 + \cdots + z_{n-1} \bar{w}_{n-1} + z_n \bar{w}_0,$$

where $*$ denotes conjugate transpose. The matrix of the Hermitian form is given by

$$H = \begin{pmatrix} 0 & 0 & 1 \\ 0 & I_{n-1} & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

Let $\mathbb{P} : \mathbb{C}^{n,1} - \{0\} \longrightarrow \mathbb{C}P^n$ be the canonical projection onto complex projective space. The complex hyperbolic space $\mathbf{H}^n_{\mathbb{C}}$ is defined to be $\mathbb{P}V_-$. The ideal boundary $\partial\mathbf{H}^3_{\mathbb{C}}$ is

$\mathbb{P}V_0$. The canonical projection of a vector $\mathbf{z}$ is given by $z = \mathbb{P}(\mathbf{z}) = (z_0/z_n, \cdots, z_{n-1}/z_n)$. Therefore we can write $\mathbf{H}^n_{\mathbb{C}} = \mathbb{P}(V_-)$ as

$$\mathbf{H}^n_{\mathbb{C}} = \{(w_0, \cdots, w_{n-1}) \in \mathbb{C}^n \ : \ 2\Re(w_0) + |w_1|^2 + \cdots + |w_{n-1}|^2 < 0\}.$$

This gives the Siegel domain model of $\mathbf{H}^n_{\mathbb{C}}$. There are two distinguished points in V_0 which we denote by o and ∞ given by

$$\mathbf{o} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, \infty = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

Then we can write $\partial\mathbf{H}^n_{\mathbb{C}} = \mathbb{P}(V_0)$ as

$$\partial\mathbf{H}^n_{\mathbb{C}} - \infty = \{(z_0, \cdots, z_{n-1}) \in \mathbb{C}^n : 2\Re(z_0) + |z_1|^2 + \cdots + |z_{n-1}|^2 = 0\}.$$

Remark 2.2.18. *If H and H' are two $(n+1) \times (n+1)$ Hermitian matrices with the same signature $(n,1)$, then there is a $(n+1) \times (n+1)$ matrix C so that $C^*HC = H'$. Also a matrix $A \in GL(n+1, \mathbb{C})$ preserves the Hermitian form H iff $C^{-1}AC$ preserves the Hermitian form H'. In context to the Siegel domain model, the isometry group of $\mathbf{H}^n_{\mathbb{C}}$ is $C^{-1}\mathrm{U}(n,1)C$, where $\mathrm{U}(n,1)$ is same as in definition 2.2.10 and C is given by*

$$C = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 0 & I_{n-1} & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix}.$$

The projective transformation given by C is called the Cayley transform. However in this case also, we denote the isometry group by $\mathrm{U}(n,1)$.

2.3 Classification of isometries

Definition 2.3.1. *Every isometry $g \in \mathrm{U}(n,1;\mathbb{F})$ has a fixed point on the closure $\overline{\mathbf{H}^n_{\mathbb{F}}} = \mathbf{H}^n_{\mathbb{F}} \cup \partial\mathbf{H}^n_{\mathbb{F}}$. An isometry g is called* elliptic *if it has a fixed point on $\mathbf{H}^n_{\mathbb{F}}$. It is called* parabolic, *resp.* hyperbolic *if it has exactly one, resp. two fixed points on the boundary*

$\partial \mathbf{H}^n_{\mathbb{F}}$.

Theorem 2.3.2. *[11, Theorem 3.4.1]*

(a) *An elliptic element is semisimple, with eigenvalues of norm one. Two elliptic elements are conjugate if and only if they have same similarity class of negative eigenvalues (which may coincide with one of the positive classes) and same n similarity classes of positive eigenvalues (with the same multiplicities).*

(b) *A hyperbolic element has $(n-1)$ similarity classes of positive eigenvalues (which may not be different) of norm one and two similarity classes of null eigenvalues represented by $re^{i\theta}, r^{-1}e^{i\theta}$, $r > 1$, $0 \leq \theta \leq 2\pi$. Two hyperbolic elements are conjugate if and only if they have same similarity classes of eigenvalues.*

(c) *A parabolic element is not semisimple and all similarity classes of eigenvalues have norm one. it has the Jordan decomposition $g = g_s g_u = g_u g_s$, where g_s is elliptic, g_u is unipotent. Two parabolics elements are conjugate if and only if their elliptic and unipotent components are conjugate. If it is unipotent, then there are two classes chracterized by their minimal polynomials; which are $(x-1)^2$ for* vertical *and $(x-1)^3$ for* non-vertical translations.

Theorem 2.3.3 (Theorem 6.2.4 of Goldman [18])**.** *Let $A \in \mathrm{SU}(p,q)$ with $p+q=3$. A has characteristic polynomial $\chi_A(X) = X^3 - \tau X^2 + \overline{\tau} X - 1$ and resultant $R(\chi_A, \chi'_A) = -|\tau|^2 + 8\Re(\tau^3) - 18|\tau|^2 + 27$, where $\tau = \mathrm{tr}(A)$. Then*

(i) *A is regular elliptic if and only if $R(\chi_A, \chi'_A) > 0$.*

(ii) *A has a repeated eigenvalue if and only if $R(\chi_A, \chi'_A) = 0$. In this case A is either parabolic or boundary elliptic.*

(iii) *A is loxodromic if and only if $R(\chi_A, \chi'_A) < 0$.*

Moreover, if A is loxodromic or parabolic then $(p,q) = (2,1)$ or $(1,2)$.

Remark 2.3.4. *The locus where $R(\chi_A, \chi'_A) = 0$ is a classical curve called a* deltoid, *see pages 26, 27 of Kirwan [31].*

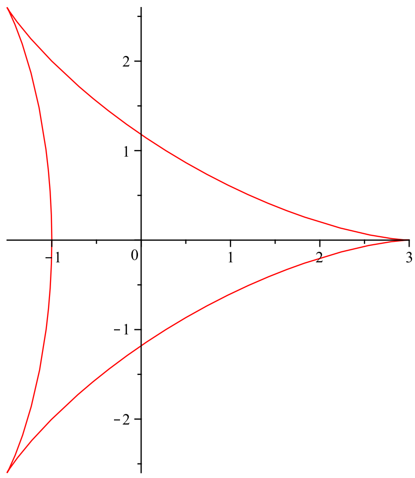

Figure 2.1: The deltoid.

2.4 Numerical invariants in complex hyperbolic geometry

All the invariants discussed in this section can be defined for $\mathrm{SU}(n,1)$. However we shall restrict to the case $n = 3$, as that is relevant to this thesis.

Definition 2.4.1. *Let $z_1,\ z_2,\ z_3$ be three distinct points of $\partial\mathbf{H}^3_{\mathbb{C}}$ with lifts $\mathbf{z}_1,\ \mathbf{z}_2$ and $\mathbf{z}_3$ respectively. Cartan's angular invariant $\mathbb{A}(z_1,\ z_2,\ z_3)$ is defined by :*

$$\mathbb{A}(z_1,\ z_2,\ z_3) = arg(-\langle \mathbf{z}_1, \mathbf{z}_2\rangle\langle \mathbf{z}_2, \mathbf{z}_3\rangle\langle \mathbf{z}_3, \mathbf{z}_1\rangle).$$

$\mathbb{A}$ is invariant under $\mathrm{SU}(3,1)$ and is independent of the chosen lifts.

Theorem 2.4.2. [18, Theorem 7.1.1] *Let $z_1,\ z_2,\ z_3$ and $z'_1,\ z'_2,\ z'_3$ be triples of distinct points of $\partial\mathbf{H}^3_{\mathbb{C}}$. Then $\mathbb{A}(z_1,\ z_2,\ z_3) = \mathbb{A}(z'_1,\ z'_2,\ z'_3)$ if and only if there exist $A \in \mathrm{SU}(3,1)$ so that $A(z_j) = z'_j$ for $j = 1,\ 2,\ 3$.*

Remark 2.4.3. *The above theorem shows that this invariant determines any triples of distinct points in $\partial\mathbf{H}^3_{\mathbb{C}}$ up to $\mathrm{SU}(3,1)$-equivalence.*

Theorem 2.4.4. [18, Theorems 7.1.3 and 7.1.4] *Let $z_1,\ z_2,\ z_3$ be three distinct points of $\partial\mathbf{H}^3_{\mathbb{C}}$ and let $\mathbb{A} = \mathbb{A}(z_1, z_2, z_3)$ be their angular invariant. Then*

1. $\mathbb{A} \in [-\frac{\pi}{2}, \frac{\pi}{2}]$.

2. $\mathbb{A} = \pm\frac{\pi}{2}$ *if and only if* z_1, z_2, z_3 *lie on the same chain.*

3. $\mathbb{A} = 0$ *if and only if* z_1, z_2, z_3 *lie on a totally real totally geodesic subspace.*

Definition 2.4.5. *Given a quadruple of distinct points* (z_1, z_2, z_3, z_4) *on* $\partial\mathbf{H}^3_{\mathbb{C}}$, *their Koranyi-Riemann cross ratio is defined by*

$$\mathbb{X}(z_1, z_2, z_3, z_4) = [z_1, z_2, z_3, z_4] = \frac{\langle \mathbf{z}_3, \mathbf{z}_1 \rangle \langle \mathbf{z}_4, \mathbf{z_2} \rangle}{\langle \mathbf{z}_4, \mathbf{z}_1 \rangle \langle \mathbf{z}_3, \mathbf{z}_2 \rangle},$$

where, for $i = 1, 2, 3, 4$, $\mathbf{z}_i$, *are lifts of* z_i. *It can be seen easily that* $\mathbb{X}$ *is independent of the chosen lifts of* z_i*'s. By choosing different ordering of the four points, we may define other cross ratios and it can be seen in [18, p.225] that there are certain symmetries that are associated with certain permutations. After taking these into account, there are only three cross-ratios that remain. Given distinct points* $z_1,\ z_2,\ z_3,\ z_4$ *in* $\partial\mathbf{H}^3_{\mathbb{C}}$, *we define :*

$$\mathbb{X}_1 = [z_1,\ z_2,\ z_3,\ z_4],\ \mathbb{X}_2 = [z_1,\ z_3,\ z_2, z_4],\ \mathbb{X}_3 = [z_2,\ z_3,\ z_1,\ z_4] \tag{2.4.1}$$

Chapter 3

Classification of Quaternionic Hyperbolic Isometries

Let $\mathbb{F}$ denote either the complex numbers $\mathbb{C}$ or the quaternions $\mathbb{H}$. Let $\mathbf{H}^n_{\mathbb{F}}$ denote the n-dimensional hyperbolic space over $\mathbb{F}$. We obtain algebraic criteria to classify the isometries of $\mathbf{H}^n_{\mathbb{F}}$.

3.1 Classification of Quaternionic hyperbolic Isometries

Let $A \in \mathrm{Sp}(n,1)$. Write $\mathbb{H} = \mathbb{C} \oplus \mathbf{j}\mathbb{C}$. Express $A = A_1 + \mathbf{j}A_2$, where $A_1, A_2 \in M_{2(n+1)}(\mathbb{C})$. This gives an embedding $A \mapsto A_{\mathbb{C}}$ of $\mathrm{Sp}(n,1)$ into $\mathrm{GL}(2(n+1), \mathbb{C})$, cf. [34, section-2], [57, section-2], where

$$A_{\mathbb{C}} = \begin{pmatrix} A_1 & -\overline{A_2} \\ A_2 & \overline{A_1} \end{pmatrix}. \tag{3.1.1}$$

Lemma 3.1.1. *The characteristic polynomial of $A_{\mathbb{C}}$ is real and self-dual i.e. it has the form*

$$\chi_{A_{\mathbb{C}}}(x) = \sum_{j=0}^{2(n+1)} a_j x^{2(n+1)-j}, \text{ where } a_j = a_{2(n+1)-j};\ a_0 = a_{2(n+1)} = 1. \tag{3.1.2}$$

where for all i, $a_i \in \mathbb{R}$.

Proof. Note that the characteristic polynomial $\chi_{A_{\mathbb{C}}}(x)$ of $A_{\mathbb{C}}$ is an invariant of the conjugacy class of A. It follows from the conjugacy class representatives in $\mathrm{Sp}(n,1)$ that $\chi_{A_{\mathbb{C}}}(x)$ is self-dual, i.e. if $\lambda \in \mathbb{C}$ is a root of $\chi_{A_{\mathbb{C}}}(x)$, so is λ^{-1}. Further if λ is an eigenvalue, then so is $\bar{\lambda}^{-1}$, cf. [18, Lemma 6.2.5, p. 205]. It follows that if λ is a root of the characteristic polynomial, so is $\bar{\lambda}$. This proves the result.

Write $\chi_{A_{\mathbb{C}}}(x) = x^{n+1} g(x + x^{-1})$, where

$$g(x + x^{-1}) = \sum_{j=0}^{n} a_j (x^{n+1-j} + x^{-(n+1-j)}) + a_{n+1}$$

Expanding the terms in the brackets, and considering $t = x + x^{-1}$ as polynomial indeterminate, we get the polynomial

$$g_A(t) = g(x + x^{-1}). \tag{3.1.3}$$

Using the Newton's identities, cf. [35, 47] the coefficients of $\chi_{A_{\mathbb{C}}}(x)$ can be expressed as a combination of several powers of $T_k = \mathrm{trace}(A_{\mathbb{C}}^k)$, $k = 1, 2, \ldots, n+1$. Hence the coefficients of $g_A(t)$ can be expressed by the numbers T_k.

Theorem 2.3.2 motivates to a more refined classification of elements of $\mathrm{U}(n,1;\mathbb{F})$. An element is called *regular* if it has mutually disjoint classes of eigenvalues. A non-regular hyperbolic has a positive eigenvalue of multiplicity at least two. A non-regular hyperbolic isometry whose null eigenvalues are non-reals, is called *semi-regular*; it is called *screw hyperbolic* if its null eigenvalues are real numbers; it is called a *strictly hyperbolic* if it is a screw hyperbolic and all positive eigenvalues are 1. An elliptic element is called *simple elliptic* if it has only a single class of eigenvalues, i.e. it is of the form λI, $|\lambda| = 1$.

Theorem 3.1.2. *Let A be an element in $\mathrm{Sp}(n,1)$. Suppose $A_{\mathbb{C}}$ be the corresponding element in $\mathrm{GL}((2(n+1), \mathbb{C})$. Let $\mathcal{S}_A = \{\Delta_1, \cdots, \Delta_{n+1}\}$ be the discriminant sequence of $g_A(t)$, where $\Delta_{n+1} = \Delta$ is the usual algebraic discriminant of $g_A(t)$. Let D be the discriminant of the minimal polynomial of $A_{\mathbb{C}}$. Then the following holds.*

1. *A is regular hyperbolic if and only if $\Delta < 0$.*

2. *A is regular elliptic if and only if $\Delta > 0$.*

3. A is semi-regular hyperbolic if and only if $\Delta = 0$ and the number of sign changes of the revised sign list of $\mathcal{S}_A$ is exactly one.

4. A is screw hyperbolic if and only if $\Delta = 0$ and $g_A(t)$ has a real root λ such that $|\lambda| > 2$.

5. A is a strictly hyperbolic if and only if $g_A(t)$ has a real root λ such that $|\lambda| > 2$ and for all $m \leq n-2$, $g_A^{(m)}(2) = 0$.

6. A is elliptic or parabolic if and only if $\Delta = 0$ and there is no sign change in the number of revised sign list of $\mathcal{S}_A$. Further A is parabolic if $D = 0$; otherwise it is elliptic. Further A is simple elliptic if the number of non-vanishing members of the revised sign list is exactly one.

Proof. Since $\chi_{A_{\mathbb{C}}}(x)$ is a conjugacy invariant, so is $g_A(t)$. If α is a root of $\chi_{A_{\mathbb{C}}}(x)$, then $\alpha + \alpha^{-1}$ is a root of $g_A(t)$. Hence the nature of roots of $g_A(t)$ is determined by the nature of roots of $\chi_{A_{\mathbb{C}}}(x)$. It follows from the conjugacy classification in $Sp(4,1)$ that for all A in $\mathrm{Sp}(n,1)$, the number of complex conjugate roots of $g_A(t)$ can be at most 2.

For A hyperbolic, the representatives of eigenvalues of A are given by $re^{i\theta}$, $r^{-1}e^{i\theta}$, $e^{i\phi_k}$, $k = 1, \cdots, n-1$. It is easy to see from the embedding (3.1.1) that $\chi_{A_{\mathbb{C}}}$ has roots $re^{\pm i\theta}$, $r^{-1}e^{\pm i\theta}$, $e^{\pm i\phi_k}$, $k = 1, \cdots, n-1$. Thus the roots of $g_A(t)$ are given by

$$s_1 = re^{i\theta} + r^{-1}e^{-i\theta}, \; s_2 = r^{-1}e^{i\theta} + re^{-i\theta}, \; t_k = e^{i\phi_k} + e^{-i\phi_k} = 2\cos\phi_k, \tag{3.1.4}$$

for $k = 1, \cdots, n-1$. Note that if A is a screw hyperbolic, i.e. $\theta = 0$ or $m\pi$, then $g_A(t)$ has at least one real double root $r + r^{-1}$ or $-r - r^{-1}$. Note that $|r + r^{-1}| > 2$ and $|t_k| \leq 2$ for all k. . Hence if A is a screw hyperbolic, then $g_A(t)$ has exactly one double real root of absolute value > 2. If A is a strictly hyperbolic, then 1 is a root of A of multiplicity $n-1$. Hence 2 is a root of $g_A(t)$ of multiplicity $(n-1)$, and hence $g_A^{(m)}(2) = 0$ for all $m \leq n-2$.

For A elliptic or parabolic, the eigenvalues of A are represented by $e^{i\theta_i}$, $i = 1, \cdots, n+1$, (for some i, j, θ_i may be equal to θ_j). In this case, the roots of $g_A(t)$ are given by $u_1, \cdots, u_{n+1}$ where $u_i = 2\cos\theta_i$. For all i, $|u_i| \leq 2$.

If A is regular hyperbolic, then $\theta \neq m\pi$. Hence s_1 and s_2 are non-real complex conjugate numbers and $g_A(t)$ has $n-1$ real roots. Depending on n is even or odd we have the following possibilities and in each of the cases, applying Theorem 2.1.8 we see that $\Delta < 0$:

First note that $D_{n+1} = (-1)^{\frac{n(n+1)}{2}}(n+1)^{-(n+1)}\Delta_{n+1}$.

(i) For $n = 4k$, $n - 1 \equiv 3 \mod 4$: Consequently, by Theorem 2.1.8, $D_{n+1} < 0$. Since $\frac{4k(4k+1)}{2} = 2k(4k+1)$ is an even number, hence $\Delta < 0$.

(ii) For $n = 4k+1$, $n - 1 \equiv 0 \mod 4$: Since the leading coefficient of $g_A(t)$ is $1 > 0$, hence by Theorem 2.1.8, $D_{n+1} > 0$; consequently $\Delta < 0$.

(iii) For $n = 4k+2$, $n - 1 \equiv 1 \mod 4$, hence $D_{n+1} > 0$; consequently, $\Delta < 0$.

(iv) For $n = 4k+3$, $n - 1 \equiv 2 \mod 4$, hence $D_{n+1} < 0$; consequently, $\Delta < 0$.

If A is regular elliptic, then we have u_i's are mutually distinct, hence all roots of $g_A(t)$ are real and mutually distinct. Using Theorem 2.1.8 and similar arguments as above it follows that $\Delta > 0$.

If A is either of non-regular elliptic, non-regular hyperbolic or parabolic, then $g_A(t)$ has at least one root of multiplicity 2. Hence $\Delta = 0$.

Suppose $\Delta = 0$. Then $\mathcal{S}_A = \{\Delta_1, \Delta_2, \cdots, \Delta_{n+1}\}$ be the discriminant sequence of the polynomial $g_A(t)$. Let A be semi-regular hyperbolic. Then $g_A(t)$ has exactly two complex conjugate roots. Hence by Theorem 2.1.7, the number of sign changes of the revised sign list of $\mathcal{S}_A$ is exactly 1. If A is screw hyperbolic, elliptic or parabolic , then $g_A(t)$ has no complex roots, hence the number of sign changes of the revised sign list of $\mathcal{S}_A$ is zero. If A is elliptic or parabolic, then from above we have seen that all the roots have absolute value ≤ 2; A is screw hyperbolic if and only if A has a root α such that $|\alpha| > 2$.

Finally we note that if A is parabolic, it has the Jordan decomposition $A = A_s A_u$, where A_u is a vertical or non-vertical translation. Thus the minimal polynomial of $A_{\mathbb{C}}$ has a factor of the form $(x - \lambda)^m$, $m = 2$ or 3. Hence D must be zero. For A elliptic, the minimal polynomial of $A_{\mathbb{C}}$ is a product of distinct linear factors, hence $D \neq 0$. Suppose A is simple elliptic. Then all the roots of $g_A(t)$ are equal, hence it has only one real root. Hence by Theorem 2.1.7, the number of non-vanishing members of the revised sign list is exactly one.

Δ	*Type of isometry*
< 0	Regular Hyperbolic
> 0	Regular elliptic
$= 0$	Parabolic, non-regular elliptic or a non-regular hyperbolic

Table 3.1: Classification of isometries of $\mathbf{H}_{\mathbb{H}}^n$.

This proves the theorem.

Note that Theorem 3.1.2 can also be adapted to the setting of complex hyperbolic geometry. This can be done by using the embedding of $\mathrm{U}(n,1)$ into $\mathrm{GL}(2(n+1),\mathbb{R})$ given by (3.1.1) and then follow the same method as in the quaternionic case. In the action of $\mathrm{U}(n,1)$ on $\mathbf{H}_{\mathbb{C}}^n$, the regular, semi-regular and (non-strictly) screw hyperbolic isometries fall in the same class; together we call them *loxodromic*. Also the simple elliptics, i.e. the scalar matrices of the form λI, $|\lambda| = 1$, acts as the identity on $\mathbf{H}_{\mathbb{C}}^n$. We have the following.

Corollary 3.1.3. *Let A be an element in $\mathrm{U}(n,1)$. Suppose $A_{\mathbb{R}}$ be the corresponding element in $\mathrm{GL}((2(n+1),\mathbb{R})$. Let $\mathcal{S}_A = \{\Delta_1, \cdots, \Delta_{n+1}\}$ be the discriminant sequence of $g_A(t)$, where $\Delta_{n+1} = \Delta$ is the usual algebraic discriminant of $g_A(t)$. Let D be the discriminant of the minimal polynomial of $A_{\mathbb{R}}$. Then the following holds.*

1. *A is loxodromic if and only if one of the following holds:*

   (i) $\Delta < 0$.

   (ii) $\Delta = 0$ *and either the number of sign changes of the revised sign list of $\mathcal{S}_A$ is exactly one or, $g_A(t)$ has a real root λ such that $|\lambda| > 2$ and for some $m \leq n-2$,* $g_A^{(m)}(2) \neq 0$.

2. *A is regular elliptic if and only if $\Delta > 0$.*

3. *A is a strictly hyperbolic if $g_A(t)$ has a real root λ such that $|\lambda| > 2$ and for all $m \leq n-2$, $g_A^{(m)}(2) = 0$.*

4. *A is elliptic or parabolic if and only if $\Delta = 0$ and there is no sign change in the number of revised sign list of $\mathcal{S}_A$. Further A is parabolic if $D = 0$; otherwise it is elliptic.*

5. *A acts as the identity if and only if* $\Delta = 0$, $D \neq 0$ *and the number of non-vanishing members of the revised sign list is exactly one.*

3.1.1 Remarks on a complete algorithm for algebraic classification of the isometries

Theorem 3.1.2 gives us a fair classification of the isometries. However, it does not give us informations about the multiplicities of the similarity classes of eigenvalues. However, following the methods in the above proof, using the polynomial $g_A(t)$ and the algorithm in [29, p. 633], it is indeed possible to derive a complete root classification of $g_A(t)$ with multiplicities. This will give us the number of distinct eigenvalues with multiplicities. For example, if A is elliptic and $g_A(t)$ has distinct roots $2\cos\theta_1, \cdots, 2\cos\theta_k$ with multiplicities $m_1, \cdots, m_k$ respectively, then the eigenvalue classes of A are represented by $e^{i\theta_1}, \cdots, e^{i\theta_k}$ each with multiplicities $m_1, \cdots, m_k$ respectively. A limitation of this method is that it tells only the number of distinct classes of eigenvalues with multiplicities, but it does not tell us which one among the eigenvalues is time-like or space-like or light-like. To obtain this information, we need to refer to the centralizers, up to conjugacy, of the isometries. The description of the centralizers, up to conjugacy, in $\mathrm{Sp}(n, 1)$, resp. $\mathrm{U}(n, 1)$, has been obtained in [25], resp. [6]. Thus Theorem 3.1.2, the above algorithm, along with the description of the centralizers, give us a complete algorithm to determine the type of an isometry of $\mathbf{H}^n_{\mathbb{H}}$ (and also of $\mathbf{H}^n_{\mathbb{C}}$).

Chapter 4

Classification of Unitary Matrices

In this chapter, we classify the dynamical action of matrices in $\mathrm{SU}(p,q)$ using the coefficients of their characteristic polynomial. This generalises earlier work of Goldman for $\mathrm{SU}(2,1)$ and the classical result for $\mathrm{SU}(1,1)$, which is conjugate to $\mathrm{SL}(2,\mathbb{R})$. As geometrical applications, we show how this enables us to classify automorphisms of real and complex hyperbolic space and anti de Sitter space.

4.1 Classification of elements in $\mathrm{SU}(p,q)$

4.1.1 Introduction

In this section we consider matrices in $\mathrm{SU}(p,q)$ for arbitrary $n = p + q$. We discuss how to use the resultant to enumerate the different possibilities for such matrices.

4.1.2 The resultant

First recall the definition of resultant of two polynomials. Let $p(X)$ and $q(X)$ be two polynomials. Suppose that $p(X)$ has degree $r > 0$, leading coefficient a_r (so the highest order term of $p(X)$ is $a_r X^r$) and roots $\alpha_1, \ldots, \alpha_r$. Similarly, suppose that $q(X)$ has degree s, leading coefficient b_s and roots $\beta_1, \ldots, \beta_s$. Then the *resultant* of $p(X)$ and $q(X)$ is defined to be

$$R(p,q) = a_r^s b_s^r \prod_{i,j} (\alpha_i - \beta_j) = a_r^s \prod_{i=1}^{r} q(\alpha_i) = b_s^r \prod_{j=1}^{s} p(\beta_j).$$

In the case where $q(X) = p'(X)$, which is the case we are interested in, there is a simpler formula. It is easy to show that

$$p'(\alpha_j) = a_r \prod_{i \neq j} (\alpha_j - \alpha_i).$$

Hence

$$R(p, p') = a_r^{r-1} \prod_{j=1}^{r} p'(\alpha_j) = a_r^{2r-1} (-1)^{r(r-1)/2} \prod_{i<j} (\alpha_j - \alpha_i)^2.$$

4.1.3 Classification when $p + q = n$

A matrix A in $\mathrm{SU}(p, q)$ is called *k-loxodromic* if it has k pairs of eigenvalues $r_j e^{i\theta_j}$ and $r_j^{-1} e^{i\theta_j}$ with $r_j > 1$ for $j = 1, \ldots, k$, and all other eigenvalues are unit modulus complex numbers. We adopt the convention of taking $k \geq 0$ with the understanding that a 0-loxodromic means that all eigenvalues are unit modulus complex numbers. Note that in $\mathrm{SU}(p, q)$ we have $k \leq \min\{p, q\}$.

Also, A is said to be *regular* if the eigenvalues are mutually distinct, that is A has no repeated eigenvalues.

Theorem 4.1.1. *Let $A \in \mathrm{SU}(p, q)$. Let $R(\chi_A, \chi_A')$ denotes the resultant of the characteristic polynomial $\chi_A(X)$ and its first derivative $\chi_A'(X)$. Then for $m \geq 0$, we have the following.*

(i) A is regular $2m$-loxodromic if and only if $R(\chi_A, \chi_A') > 0$.

(ii) A is regular $(2m + 1)$-loxodromic if and only if $R(\chi_A, \chi_A') < 0$.

(iii) A has a repeated eigenvalue if and only if $R(\chi_A, \chi_A') = 0$.

Proof. Write $p + q = n$. Suppose A is r-loxodromic, including the case where $r = 0$ and so A is elliptic. Then A has mutually distinct eigenvalues

$$\lambda_j = e^{\ell_j + i\phi_j}, \quad \overline{\lambda}_j^{-1} = e^{-\ell_j + i\phi_j}, \quad \mu_k = e^{i\theta_k},$$

where ℓ_j is a positive real number, $j = 1, \ldots, r$, $k = 1, \ldots, s$ and $2r + s = p + q = n$. Then the squares of the differences of these eigenvalues are

$$\begin{aligned}
(\lambda_j - \overline{\lambda}_j^{-1})^2 &= e^{2i\phi_j} 4\sinh^2(\ell_j), \\
(\lambda_j - \lambda_k)^2(\overline{\lambda}_j^{-1} - \overline{\lambda}_k^{-1})^2 &= e^{2i\phi_j + 2i\phi_k}\big(2\cosh(\ell_j - \ell_k) - 2\cos(\phi_j - \phi_k)\big)^2, \\
(\lambda_j - \overline{\lambda}_k^{-1})^2(\overline{\lambda}_j^{-1} - \lambda_k)^2 &= e^{2i\phi_j + 2i\phi_k}\big(2\cosh(\ell_j + \ell_k) - 2\cos(\phi_j - \phi_k)\big)^2, \\
(\lambda_j - \mu_k)^2(\overline{\lambda}_j^{-1} - \mu_k)^2 &= e^{2i\phi_j + 2i\theta_k}\big(2\cosh(\ell_j) - 2\cos(\phi_j - \theta_k)\big)^2, \\
(\mu_j - \mu_k)^2 &= -e^{i\theta_j + i\theta_k}\big(2 - 2\cos(\theta_j - \theta_k)\big).
\end{aligned}$$

Therefore

$$\begin{aligned}
&R(\chi_A, \chi_A') \\
&= (-1)^{n(n-1)/2} \prod_j (\lambda_j - \overline{\lambda}_j^{-1})^2 \prod_{j<k} (\lambda_j - \lambda_k)^2(\overline{\lambda}_j^{-1} - \overline{\lambda}_k^{-1})^2(\lambda_j - \overline{\lambda}_k^{-1})^2(\overline{\lambda}_j^{-1} - \lambda_k)^2 \\
&\quad \cdot \prod_{j,k} (\lambda_j - \mu_k)^2(\overline{\lambda}_j^{-1} - \mu_k)^2 \prod_{j<k} (\mu_j - \mu_k)^2 \\
&= (-1)^{n(n-1)/2}(-1)^{s(s-1)/2} \prod_{j=1}^{r} e^{(n-1)2i\phi_j} \prod_{k=1}^{s} e^{(n-1)i\theta_k} \prod_j 4\sinh^2(\ell_j) \\
&\quad \cdot \prod_{j<k} \big(2\cosh(\ell_j - \ell_k) - 2\cos(\phi_j - \phi_k)\big)^2 \big(2\cosh(\ell_j + \ell_k) - 2\cos(\phi_j - \phi_k)\big)^2 \\
&\quad \cdot \prod_{j,k} \big(2\cosh(\ell_j) - 2\cos(\phi_j - \theta_k)\big)^2 \prod_{j<k} \big(2 - 2\cos(\theta_j - \theta_k)\big) \\
&= (-1)^{n(n-1)/2 + s(s-1)/2} \prod_j 4\sinh^2(\ell_j) \\
&\quad \cdot \prod_{j<k} \big(2\cosh(\ell_j - \ell_k) - 2\cos(\phi_j - \phi_k)\big)^2 \big(2\cosh(\ell_j + \ell_k) - 2\cos(\phi_j - \phi_k)\big)^2 \\
&\quad \cdot \prod_{j,k} \big(2\cosh(\ell_j) - 2\cos(\phi_j - \theta_k)\big)^2 \prod_{j<k} \big(2 - 2\cos(\theta_j - \theta_k)\big),
\end{aligned}$$

where we have used

$$\prod_{j=1}^{r} e^{(n-1)2i\phi_j} \prod_{k=1}^{s} e^{(n-1)i\theta_k} = \big(\det(A)\big)^{n-1} = 1.$$

All the product terms are real and positive provided $\ell_j > 0$ and $\theta_j \neq \theta_k$. Thus we must find the power of (-1). Since $n = 2r + s$ we have

$$n(n-1) + s(s-1) = 2n(n-1) - 4rn + 4r^2 + 2r.$$

Since $2n(n-1)$ is even, this implies $(-1)^{n(n-1)/2+s(s-1)/2} = (-1)^r$. This proves assertions (i) and (ii). Assertion (iii) follows from the definition of the resultant.

Corollary 4.1.2. *Let $A \in \mathrm{SU}(p,1)$. Let $R(\chi_A, \chi'_A)$ denotes the resultant of the characteristic polynomial $\chi_A(X)$ and its first derivative $\chi'_A(X)$. Then we have the following.*

(i) A is regular elliptic if and only if $R(\chi_A, \chi'_A) > 0$.

(ii) A is regular loxodromic if and only if $R(\chi_A, \chi'_A) < 0$.

(iii) A has a repeated eigenvalue if and only if $R(\chi_A, \chi'_A) = 0$.

4.2 Classification of matrices in $\mathrm{SU}(p,q)$ with $p+q=4$

4.2.1 Introduction

In this section we consider the case of $\mathrm{SU}(p,q)$ where $p+q=4$. In fact, up to changing the sign of the Hermitian form, there are three possible groups $\mathrm{SU}(4,0) = \mathrm{SU}(4)$, $\mathrm{SU}(3,1)$ and $\mathrm{SU}(2,2)$. Our goal will be to extend Goldman's classification of matrices in $\mathrm{SU}(2,1)$ using the resultant $R(\chi_A, \chi'_A)$ as a polynomial in $\mathrm{tr}(A)$ and $\overline{\mathrm{tr}(A)}$. In this case, the characteristic polynomial is determined by a complex and a real parameter (see [26, section 4.5]):

Lemma 4.2.1. *Let A be in $\mathrm{SU}(p,q)$, where $p+q=4$, with characteristic polynomial $\chi_A(X)$. Write $\tau = \mathrm{tr}(A)$ and $\sigma = \frac{1}{2}\big(\mathrm{tr}^2(A) - \mathrm{tr}(A^2)\big) \in \mathbb{R}$. Then*

$$\chi_A(X) = X^4 - \tau X^3 + \sigma X^2 - \overline{\tau} X + 1. \tag{4.2.1}$$

If λ_i for $i = 1, 2, 3, 4$ are the eigenvalues of A, then note that

$$\tau = \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4, \tag{4.2.2}$$

$$\sigma = \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_1\lambda_4 + \lambda_2\lambda_3 + \lambda_2\lambda_4 + \lambda_3\lambda_4. \tag{4.2.3}$$

We want conditions on σ, τ characterising when $\chi_A(X) = 0$ has repeated solutions, or equivalently when $\chi_A(X)$ and its derivative $\chi_A'(X)$ have a common root. Note that:

$$\chi_A'(X) = 4X^3 - 3\tau X^2 + 2\sigma X - \overline{\tau}. \tag{4.2.4}$$

Therefore we need to find the locus of points $(\tau, \sigma) \in \mathbb{C} \times \mathbb{R}$ where the resultant $R(\chi_A, \chi_A') = 0$. This problem was studied by Poston and Stewart [46]. Based on earlier work of Chillingworth [12], they call the locus of points where this resultant vanishes the *holy grail*; see Figure 4.1. This generalises the deltoid, which is the zero locus of the resultant for SU(2, 1).

In this section we investigate the dynamics of isometries whose parameters (τ, σ) lie on each part of the holy grail and in each component of the complement. In this section no assumption is made about the signature of H, but readers should recall that a k-loxodromic map can only occur in $\mathrm{SU}(p, q)$ when $k \leq \min\{p, q\}$.

4.2.2 Eigenvalues and parameters

Consider a unitary matrix A in $\mathrm{SU}(p, q)$ with $p + q = 4$, but at this stage we will not specify the signature of the Hermitian form. Suppose that the eigenvalues of A (that is the roots of the characteristic polynomial) are λ_1, λ_2, λ_3, λ_4. Recall from Goldman's lemma, Lemma 2.2.11, the set $\{\lambda_1, \lambda_2, \lambda_3, \lambda_4\}$ is closed under the map $\lambda \longmapsto \overline{\lambda}^{-1}$. Note that an even number of eigenvalues satisfy $|\lambda| \neq 1$ and so an even number satisfy $|\lambda| = 1$. In what follows, after rearranging them if necessary, suppose that the eigenvalues are paired up as follows.

- if $|\lambda_1| \neq 1$ then $\lambda_2 = \overline{\lambda}_1^{-1}$; if $|\lambda_1| = 1$ then $|\lambda_2| = 1$;
- if $|\lambda_2| \neq 1$ then $\lambda_1 = \overline{\lambda}_2^{-1}$; if $|\lambda_2| = 1$ then $|\lambda_1| = 1$;
- if $|\lambda_3| \neq 1$ then $\lambda_4 = \overline{\lambda}_3^{-1}$; if $|\lambda_3| = 1$ then $|\lambda_4| = 1$;

- if $|\lambda_4| \neq 1$ then $\lambda_3 = \overline{\lambda}_4^{-1}$; if $|\lambda_4| = 1$ then $|\lambda_3| = 1$.

With this ordering of eigenvalues, note that $|\lambda_1\lambda_2| = |\lambda_3\lambda_4| = 1$. Define $\phi \in [0, \pi)$ by $\lambda_1\lambda_2 = e^{2i\phi}$. Moreover, since the product of the eigenvalues is 1, we also have $\lambda_3\lambda_4 = e^{-2i\phi}$. The following parameters will simplify our calculations:

$$x = (\lambda_1 + \lambda_2)e^{-i\phi}, \quad y = (\lambda_3 + \lambda_4)e^{i\phi}, \quad t = 2\cos(2\phi). \tag{4.2.5}$$

The rest of this section will be devoted to investigating the properties of the change of parameters $(\tau, \sigma) \longleftrightarrow (x, y, \phi)$.

Lemma 4.2.2. *The parameters x, y and t defined by* (4.2.5) *are all real.*

Proof. Clearly t is real. In order to see that x is real, note that either $|\lambda_1| = |\lambda_2|^{-1} \neq 1$ and $\overline{\lambda}_1 = \lambda_2^{-1}$, $\overline{\lambda}_2 = \lambda_1^{-1}$ or else $|\lambda_1| = |\lambda_2| = 1$ and $\overline{\lambda}_1 = \lambda_1^{-1}$, $\overline{\lambda}_2 = \lambda_2^{-1}$. In the either case

$$\overline{x} = (\overline{\lambda}_1 + \overline{\lambda}_2)e^{i\phi} = (\lambda_1^{-1} + \lambda_2^{-1})e^{i\phi} = (\lambda_1 + \lambda_2)e^{-i\phi} = x$$

where we have used $\lambda_1\lambda_2 = e^{2i\phi}$. Thus x is real. Similarly y is real.

Lemma 4.2.3. *With τ, σ and x, y, ϕ as in* (4.2.5)*, we have*

$$\tau = xe^{i\phi} + ye^{-i\phi}, \tag{4.2.6}$$

$$\sigma = xy + 2\cos(2\phi). \tag{4.2.7}$$

Proof. From the definition of x, y and ϕ we have

$$\begin{aligned} \tau &= (\lambda_1 + \lambda_2) + (\lambda_3 + \lambda_4) = xe^{i\phi} + ye^{-i\phi}, \\ \sigma &= (\lambda_1 + \lambda_2)(\lambda_3 + \lambda_4) + \lambda_1\lambda_2 + \lambda_3\lambda_4 = xe^{i\phi}ye^{-i\phi} + e^{2i\phi} + e^{-2i\phi}. \end{aligned}$$

We now characterise when this change of variables is a local diffeomorphism.

Proposition 4.2.4. *The change of parameters $\mathbb{R}^2 \times S^1 \longrightarrow \mathbb{C} \times \mathbb{R}$ given by*

$$(x, y, e^{i\phi}) \longmapsto (\tau, \sigma) = \left(xe^{i\phi} + ye^{-i\phi},\, xy + e^{2i\phi} + e^{-2i\phi}\right)$$

is a local diffeomorphism provided

$$x^2 + y^2 - 4 - 2xy\cos(2\phi) + 4\cos^2(2\phi) \neq 0.$$

Proof. Consider the change of coordinates

$$\Re(\tau) = (x+y)\cos(\phi), \quad \Im(\tau) = (x-y)\sin(\phi), \quad \sigma = xy + e^{2i\phi} + e^{-2i\phi}.$$

Then the Jacobian is

$$\begin{aligned} J &= \det\begin{pmatrix} \cos(\phi) & \cos(\phi) & -(x+y)\sin(\phi) \\ \sin(\phi) & -\sin(\phi) & (x-y)\cos(\phi) \\ y & x & -4\sin(2\phi) \end{pmatrix} \\ &= 4\sin^2(2\phi) - (x+y)^2\sin^2(\phi) - (x-y)^2\cos^2(\phi) \\ &= -x^2 - y^2 + 4 + 2xy\cos(2\phi) - 4\cos^2(2\phi). \end{aligned}$$

Now we show the change of variables is surjective (compare Lemma 3.8 of [39]).

Proposition 4.2.5. *Given* $(\tau, \sigma) \in \mathbb{C} \times \mathbb{R}$ *then there exist* $(x, y, e^{i\phi}) \in \mathbb{R}^2 \times S^1$ *so that*

$$\Re(\tau) = (x+y)\cos(\phi), \quad \Im(\tau) = (x-y)\sin(\phi), \quad \sigma = xy + e^{2i\phi} + e^{-2i\phi}. \tag{4.2.8}$$

Proof. If there exist such x, y, $e^{i\phi}$ then, writing $t = 2\cos(2\phi)$, we have

$$|\tau|^2 = \Re(\tau)^2 + \Im(\tau)^2 = x^2 + y^2 + xyt, \tag{4.2.9}$$

$$2\Re(\tau^2) = 2\Re(\tau)^2 - 2\Im(\tau)^2 = (x^2+y^2)t + 4xy, \tag{4.2.10}$$

$$\sigma = xy + t.$$

Eliminating x and y we see that t must satisfy $q(t) = 0$ where

$$q(X) = X^3 - \sigma X^2 - 4X + \Re(\tau)^2 X + \Im(\tau)^2 X + 4\sigma - 2\Re(\tau)^2 + 2\Im(\tau)^2.$$

Evaluating at $X = \pm 2$ we see that

$$\begin{aligned} q(2) &= 8 - 4\sigma - 8 + 2\Re(\tau)^2 + 2\Im(\tau)^2 + 4\sigma - 2\Re(\tau)^2 + 2\Im(\tau)^2 = 4\Im(\tau)^2 \geq 0, \\ q(-2) &= -8 - 4\sigma + 8 - 2\Re(\tau)^2 - 2\Im(\tau)^2 + 4\sigma - 2\Re(\tau)^2 + 2\Im(\tau)^2 = -4\Re(\tau)^2 \leq 0. \end{aligned}$$

If $\Re(\tau) \neq 0$ and $\Im(\tau) \neq 0$ then, by the intermediate value theorem, we can find t with $-2 < t < 2$ so that $q(t) = 0$. Define ϕ by $2\cos(2\phi) = t$. As $\cos(2\phi) \neq \pm 1$ we have $\sin(2\phi) \neq 0$. In this case x and y are given by

$$x = \frac{\Re(\tau)\sin(\phi) + \Im(\tau)\cos(\phi)}{\sin(2\phi)}, \quad y = \frac{\Re(\tau)\sin(\phi) - \Im(\tau)\cos(\phi)}{\sin(2\phi)}.$$

If $\Im(\tau) = 0$ and $\Re(\tau) \neq 0$ then $q(2) = 0$ and

$$q_0(X) = q(X)/(X-2) = X^2 + 2X - \sigma X - 2\sigma + \Re(\tau)^2.$$

We have

$$q_0(2) = 8 - 4\sigma + \Re(\tau)^2, \quad q_0(-2) = \Re(\tau)^2 > 0.$$

If $\Re(\tau)^2 < 4\sigma - 8$ we have $q_0(2) < 0 < q_0(-2)$ and we can find t with $-2 < t < 2$ and $q_0(t) = 0$. In this case define $t = 2\cos(2\phi)$ and proceed as above. If $\Re(\tau)^2 \geq 4\sigma - 8$ then define $\phi = 0$. We must solve $\Re(\tau) = x + y$ and $\sigma = xy + 2$. A solution is

$$x = \frac{\Re(\tau) + \sqrt{\Re(\tau)^2 - 4\sigma + 8}}{2}, \quad y = \frac{\Re(\tau) - \sqrt{\Re(\tau)^2 - 4\sigma + 8}}{2}.$$

If $\Re(\tau) = 0$ and $\Im(\tau) \neq 0$ then $q(-2) = 0$. As above, if $\Im(\tau)^2 < -8 - 4\sigma$ then we can find t with $-2 < t < 2$ and $q(t) = 0$, giving a similar solution as before. If $\Im(\tau)^2 > -8 - 4\sigma$ then $\phi = \pi/2$ and

$$x = \frac{\Im(\tau) + \sqrt{\Im(\tau)^2 + 4\sigma + 8}}{2}, \quad y = \frac{\Im(\tau) - \sqrt{\Im(\tau)^2 + 4\sigma + 8}}{2}.$$

Finally, suppose $\Re(\tau) = \Im(\tau) = 0$. If $\sigma \geq 0$ then define $\phi = \pi/2$ and $x = y = \sqrt{\sigma + 2}$; if $\sigma < 0$ define $\phi = 0$ and $x = -y = \sqrt{-\sigma + 2}$.

4.2.3 The resultant

Let $\chi_A(x)$ be the characteristic polynomial of $A \in \mathrm{SU}(p, q)$ with $p + q = 4$. We have expressions for $\chi_A(x)$ and $\chi_A'(x)$ in (4.2.1) and (4.2.4). We now calculate their resultant

$R(\chi_A, \chi'_A)$ as a polynomial in τ, $\overline{\tau}$ and σ:

$$
\begin{aligned}
R(\chi_A, \chi'_A) &= \det \begin{pmatrix}
1 & -\tau & \sigma & -\overline{\tau} & 1 & 0 & 0 \\
0 & 1 & -\tau & \sigma & -\overline{\tau} & 1 & 0 \\
0 & 0 & 1 & -\tau & \sigma & -\overline{\tau} & 1 \\
4 & -3\tau & 2\sigma & -\overline{\tau} & 0 & 0 & 0 \\
0 & 4 & -3\tau & 2\sigma & -\overline{\tau} & 0 & 0 \\
0 & 0 & 4 & -3\tau & 2\sigma & -\overline{\tau} & 0 \\
0 & 0 & 0 & 4 & -3\tau & 2\sigma & -\overline{\tau}
\end{pmatrix} \\
&= 16\sigma^4 - 4\sigma^3(\tau^2 + \overline{\tau}^2) + \sigma^2|\tau|^4 - 80\sigma^2|\tau|^2 - 128\sigma^2 \\
&\quad +18\sigma(\tau^2 + \overline{\tau}^2)|\tau|^2 + 144\sigma(\tau^2 + \overline{\tau}^2) \\
&\quad -4|\tau|^6 - 27(\tau^2 + \overline{\tau}^2)^2 + 48|\tau|^4 - 192|\tau|^2 + 256 \\
&= 4\Big(\sigma^2/3 - |\tau|^2 + 4\Big)^3 - 27\Big(2\sigma^3/27 - |\tau|^2\sigma/3 - 8\sigma/3 + (\tau^2 + \overline{\tau}^2)\Big)^2.
\end{aligned}
$$

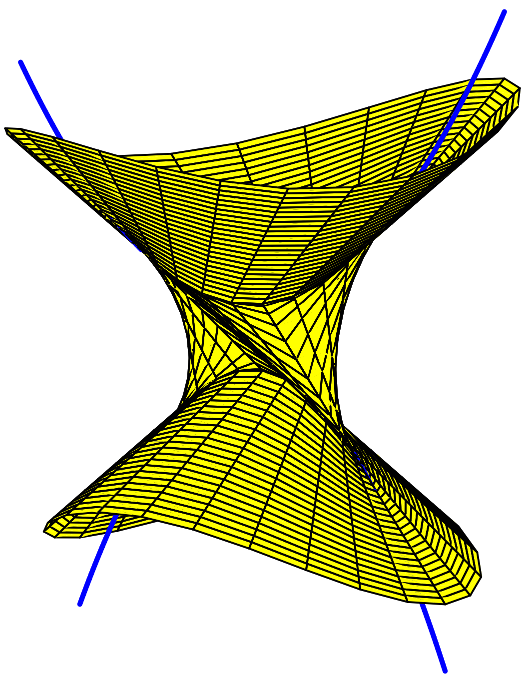

Figure 4.1: The holy grail. Here points of $\mathbb{R}^3$ have coordinates $\big(\Re(\tau), \Im(\tau), \sigma\big)$.

In [46] Poston and Stewart considered the locus of points where

$$f(z,\overline{z}) = \Re\big(\alpha z^4 + \beta z^3\overline{z} + \gamma z^2\overline{z}^2\big)$$

has repeated roots. Based on earlier work of Chillingworth [12], they call the locus of these points the *holy grail*; see Figure 4.1, which should be compared with Figures 4 and 5 of [46]. In order to see the connection between the two problems, observe that by setting $\alpha = 1$, $\beta = \tau$ and $\gamma = \sigma/2$ we have

$$f(z,\overline{z}) = \overline{z}^4 \chi_A(-z/\overline{z}).$$

When $\alpha = 1$, Poston and Stewart's equation for the holy grail, page 268 of [46], is

$$\Delta = \Big(4\gamma^2/3 - |\beta|^2 + 4\Big)^3 - 27\Big(8\gamma^3/27 - |\beta|^2\gamma/3 - 8\gamma/3 + (\beta^2 + \overline{\beta}^2)/2\Big)^2.$$

Clearly, the above substitution makes Δ agree with our expression for $R(\chi_A, \chi_A')$.

We now express $R(\chi_A, \chi_A')$ in terms of x, y and t. A consequence of this and Proposition 4.2.4 is that the change of parameters $(\tau,\sigma) \longleftrightarrow (x,y,t)$ is a local diffeomorphism when $R(\chi_A, \chi_A') \neq 0$.

Proposition 4.2.6. *In terms of the parameters x, y and t given in* (4.2.5) *the resultant is given by the following expression:*

$$R(\chi_A, \chi_A') = (x^2-4)(y^2-4)(x^2+y^2-4-xyt+t^2)^2.$$

Proof. We use equations (4.2.9), (4.2.10) and (4.2.7) substitute for τ and σ in terms of x, y and $t = 2\cos(2\phi)$. Then, expanding and simplifying, we obtain

$$\begin{aligned} R(\chi_A, \chi_A') &= 16\sigma^4 - 4\sigma^3(\tau^2+\overline{\tau}^2) + \sigma^2|\tau|^4 - 80\sigma^2|\tau|^2 \\ &\quad -128\sigma^2 + 18\sigma(\tau^2+\overline{\tau}^2)|\tau|^2 + 144\sigma(\tau^2+\overline{\tau}^2) \\ &\quad -4|\tau|^6 - 27(\tau^2+\overline{\tau}^2)^2 + 48|\tau|^4 - 192|\tau|^2 + 256 \\ &= (x^2-4)(y^2-4)(x^2+y^2-4-xyt+t^2)^2. \end{aligned}$$

We remark that there is a symmetry that arises from multiplying A by powers of i.

In several places below we will use this symmetry to avoid repetition. We note that for our geometrical applications, we will be interested in $\mathrm{PSU}(p,q) = \mathrm{SU}(p,q)/\{\pm I, \pm iI\}$ and so A is only defined up to multiplication by i.

Corollary 4.2.7. *Let x, y and t be the parameters given in* (4.2.5). *The resultant $R(\chi_A, \chi'_A)$ is preserved by the changes of variable where (x,y,t) is sent to one of*

$$\begin{array}{llll} (x,y,t), & (x,-y,-t), & (-x,y,-t), & (-x,-y,t), \\ (y,x,t), & (y,-x,-t), & (-y,x,-t), & (-y,-x,t). \end{array}$$

Moreover, this automorphism group is generated by $(\lambda_1,\lambda_2) \longleftrightarrow (\lambda_3,\lambda_4)$. and $A \longrightarrow iA$.

Proof. It is easy to see in that all the changes of variable stated above preserve the expression for $R(\chi_A, \chi'_A)$ from Proposition 4.2.6.

Now consider the effect of multiplying A by i. In the following table we give the various changes to our parameters.

A	τ	σ	ϕ	x	y	t
iA	$i\tau$	$-\sigma$	$\phi+\pi/2$	x	$-y$	$-t$
$-A$	$-\tau$	σ	$\phi+\pi$	x	y	t
$-iA$	$-i\tau$	$-\sigma$	$\phi+3\pi/2$	x	$-y$	$-t$

A further symmetry may be obtained by interchanging the pairs of eigenvalues (λ_1,λ_2) and (λ_3,λ_4). It is easy to see from (4.2.5) that this has the effect of sending (x,y,t) to (y,x,t). Repeated application of the automorphisms $A \longrightarrow iA$ and $(\lambda_1,\lambda_2) \longleftrightarrow (\lambda_3,\lambda_4)$ give all the changes of variable in the statement of the corollary.

Using Proposition 4.2.6, the condition $R(\chi_A, \chi'_A) > 0$ implies $(x^2-4)(y^2-4) > 0$. Thus, either x^2 and y^2 are both greater than 4, or they are both less than 4. In the former case A is 2-loxodromic and in the latter case it is elliptic. Thus it is useful to distinguish when $xy > 4$, $-4 < xy < 4$ and $xy < -4$. In the following lemma, we express these conditions in terms of σ and τ.

Lemma 4.2.8. *Let τ and σ be given by* (4.2.6) *and* (4.2.7). *Suppose that $R(\chi_A, \chi'_A) > 0$. Then $xy \neq \pm 4$. Furthermore:*

(i) $xy > 4$ if and only if either $\Re(\tau)^2 - 4\sigma + 8 < 0$ or $\sigma > 6$.

(ii) $xy < 4$ *if and only if both* $\Re(\tau)^2 - 4\sigma + 8 > 0$ *and* $\sigma < 6$*.*

(iii) $xy > -4$ *if and only if both* $\Im(\tau)^2 + 4\sigma + 8 > 0$ *and* $\sigma > -6$*.*

(iv) $xy < -4$ *if and only if* $\Im(\tau)^2 + 4\sigma + 8 < 0$ *or* $\sigma < -6$*.*

Note that a simple consequence of this lemma is that if $R(\chi_A, \chi_A') > 0$ then both $\min\{\Re(\tau)^2 - 4\sigma + 8,\, 6 - \sigma\}$ and $\min\{\Im(\tau)^2 + 4\sigma + 8,\, 6 + \sigma\}$ are both non-zero.

Proof. If $R(\chi_A, \chi_A') > 0$ then we have

$$0 < (x^2 - 4)(y^2 - 4) = (xy + 4)^2 - 4(x + y)^2 = (xy - 4)^2 - 4(x - y)^2.$$

Therefore $xy \neq \pm 4$. The remaining cases exhaust the other possibilities. Therefore, by process of elimination, it suffices to prove only one direction of the implications. We choose to do this from right to left.

If $\sigma > 6$ then

$$6 < \sigma = xy + 2\cos(2\phi) \leq xy + 2.$$

Therefore $xy > 4$. Similarly, if $\sigma < -6$ then $xy < -4$.

If $\Re(\tau)^2 - 4\sigma + 8 < 0$ then

$$0 > \Re(\tau)^2 - 4\sigma + 8 = (x - y)^2 \cos^2\phi + (16 - 4xy)\sin^2\phi \geq (16 - 4xy)\sin^2\phi$$

and so $xy > 4$. Similarly, if $\Im(\tau)^2 + 4\sigma + 8 > 0$ then $xy < -4$.

Now assume that $\Re(\tau)^2 - 4\sigma + 8 > 0$, $\sigma < 6$ and $R(\chi_A, \chi_A') > 0$. We note that in terms of x, y and ϕ these inequalities imply

$$0 < (x - y)^2 \cos^2\phi + (16 - 4xy)\sin^2\phi, \tag{4.2.11}$$

$$xy - 4 < 4\sin^2\phi, \tag{4.2.12}$$

$$4(x - y)^2 < (4 - xy)^2. \tag{4.2.13}$$

Using (4.2.13) to eliminate $(x - y)^2$ from (4.2.11), we see that

$$0 < 4(x - y)^2 \cos^2\phi + 16(4 - xy)\sin^2\phi < (4 - xy)\big((4 - xy)\cos^2\phi + 16\sin^2\phi\big).$$

Using (4.2.12) we see that

$$(4-xy)\cos^2\phi + 16\sin^2\phi > 4\sin^2\phi(4-\cos^2\phi) > 0.$$

Therefore $xy < 4$ as claimed.

Similarly, if $\Im(\tau)^2 + 4\sigma + 8 > 0$, $\sigma > -6$ and $R(\chi_A, \chi_A') > 0$ then $xy > -4$.

Putting this together, we have the following theorem:

Theorem 4.2.9. *Let $A \in \mathrm{SU}(p,q)$ where $p+q=4$ and let $\tau = \mathrm{tr}(A)$ and $\sigma = \big(\mathrm{tr}^2(A) - \mathrm{tr}(A^2)\big)/2$. Let $\chi_A(X)$ be the characteristic polynomial of A and let $R(\chi_A, \chi_A')$ be the resultant of $\chi_A(X)$ and $\chi_A'(X)$. Then*

(i) A is regular 2-loxodromic if and only if $R(\chi_A, \chi_A') > 0$ and

$$\min\big\{\Re(\tau)^2 - 4\sigma + 8,\ \Im(\tau)^2 + 4\sigma + 8,\ 6-\sigma,\ 6+\sigma\big\} < 0.$$

(ii) A is regular 1-loxodromic if and only if $R(\chi_A, \chi_A') < 0$.

(iii) A is regular elliptic if and only if $R(\chi_A, \chi_A') > 0$ and

$$\Re(\tau)^2 - 4\sigma + 8 > 0, \quad \Im(\tau)^2 + 4\sigma + 8 > 0, \quad -6 < \sigma < 6.$$

(iv) A has a repeated eigenvalue if and only if $R(\chi_A, \chi_A') = 0$.

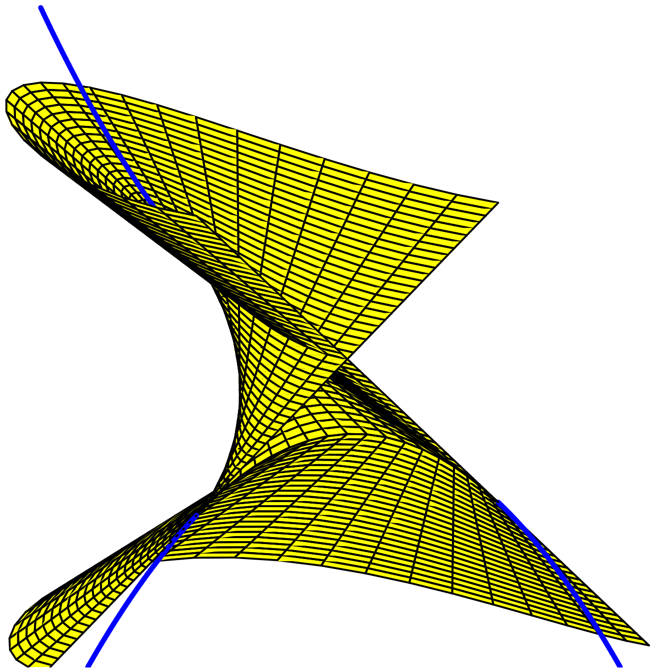

Figure 4.2: A cross section through the holy grail.

4.2.4 Parametrising the holy grail

In this section we consider the points where $R(\chi_A, \chi'_A) = 0$, called the *holy grail*. We claim that, after reordering eigenvalues, we may suppose that either $y = 2$ or else $x^2y^2 > 16$ and $x^2 + y^2 - 4 - xyt + t^2 = 0$. The former condition determines a ruled surface made up of three parts, the *upper bowl*, *central tetrahedron* and *lower bowl*, names introduced by Poston and Stewart. The latter condition determines four space curves called the *whiskers*. This is illustrated in Figure 4.1 of this chapter or in Figure 5 of Poston and Stewart [46], where the different parts are labelled.

Proposition 4.2.10. *Let x, y and t be the parameters given by* (4.2.5). *Up to applying one of the automorphisms given in Corollary 4.2.7, the condition $R(\chi_A, \chi'_A) = 0$ is equivalent to one of the following equations*

(i) $y = 2$;

(ii) $(x^2 - 4)(y^2 - 4) > 0$ *and* $x^2 + y^2 - 4 - xyt + t^2 = 0$.

Proof. Using Proposition 4.2.6 we see that points on the holy grail are given by

$$0 = (x^2 - 4)(y^2 - 4)(x^2 + y^2 - 4 - xyt + t^2)^2.$$

If $(x^4 - 4)(y^2 - 4) = 0$ then either $x = \pm 2$ or $y = \pm 2$. After applying the automorphisms from Corollary 4.2.7, we see that we may take $y = 2$.

If $(x^2 - 4)(y^2 - 4) \neq 0$ then $x^2 + y^2 - 4 - xyt + t^2 = 0$. Hence

$$t = \frac{xy \pm \sqrt{(x^2 - 4)(y^2 - 4)}}{2}.$$

Since t is real, we must have $(x^2 - 4)(y^2 - 4) > 0$.

The following result is stated on page 269 of Poston and Stewart [46]. It is illustrated in the cross-section drawn in Figure 4.2.

Corollary 4.2.11. *The points on the holy grail with $y = 2$ form a ruled surface in $\mathbb{C} \times \mathbb{R}$.*

Proof. The points in $\mathbb{C} \times \mathbb{R}$ for which $y = 2$ are

$$\begin{aligned} (\tau, \sigma) &= \big(xe^{i\phi} + 2e^{-i\phi}, 2x + 2\cos(2\phi)\big) \\ &= \big(2e^{-i\phi}, 2\cos(2\phi)\big) + x\big(e^{i\phi}, 2\big). \end{aligned}$$

This is the equation of a ruled surface (see Section 3.5 of do Carmo [10], for example).

Suppose that $y = 2$. Then the three main parts of the holy grail are determined by the conditions $x > 2$, $-2 \leq x \leq 2$ and $x < -2$.

Corollary 4.2.12. *Suppose that $y = 2$. Then the parameters τ and σ are given by*

(i) If $x = 2\cosh(\ell) > 2$ then

$$\tau = 2\cosh(\ell)e^{i\phi} + 2e^{-i\phi}, \quad \sigma = 4\cosh(\ell) + 2\cos(2\phi).$$

(ii) If $x = 2\cos(\theta) \in [-2, 2]$ then

$$\tau = 2\cos(\theta)e^{i\phi} + 2e^{-i\phi}, \quad \sigma = 4\cos(\theta) + 2\cos(2\phi).$$

(iii) If $x = -2\cosh(\ell) < -2$ *then*

$$\tau = -2\cosh(\ell)e^{i\phi} + 2e^{-i\phi}, \quad \sigma = -4\cosh(\ell) + 2\cos(2\phi).$$

The parameter values of Corollary 4.2.12 exhaust the possibilities when condition (i) of Proposition 4.2.10 is satisfied. They correspond to the *upper bowl*, *central tetrahedron* and *lower bowl* respectively. We can relate these parameter values to the possible Jordan decompositions that can arise.

Proposition 4.2.13. *Suppose that* $A \in \mathrm{SU}(p, q)$ *and* $y = 2$.

(i) If $x = 2\cosh(\ell) > 2$ *or* $x = -2\cosh(\ell) < -2$ *then A is either diagonalisable or its Jordan normal form has a* 2×2 *Jordan block associated to the eigenvalue* $e^{-i\theta}$. *The latter can only happen if* $p = q = 2$.

(ii) If $x = 2\cos(\theta) \in [-2, 2]$ *then A cann have any Jordan normal form. There can be at most* $\min\{p, q\}$ *Jordan blocks of size at least* 2.

Proof. The eigenspace associated to each Jordan block of size at least 2 is spanned by a null vector. These null vectors are linearly independent. Therefore there can only be $\min\{p, q\}$ Jordan blocks of size at least 2.

In (i) the eigenvectors corresponding to the eigenvalues $e^{\pm\ell+i\phi}$ or $-e^{\pm\ell+i\phi}$ span a subspace where the restriction of H has signature $(1, 1)$. If the other eigenvalues correspond to a Jordan block of size 2, then its eigenvector is linearly independent from the above subspace. Therefore $\min\{p, q\}$ is at least 2. Since $p + q = 4$ we have $p = q = 2$.

In (ii) all eigenvalues have absolute value 1, so there is no further restriction.

In both cases, it is an easy exercise to write down matrices and Hermitian forms to demonstrate that there are no further restrictions.

We now consider what happens when condition (ii) of Proposition 4.2.10 is satisfied. Suppose that $(x^2 - 4)(y^2 - 4) > 0$ and $-4 \le xy \le 4$. Then $-2 < x < 2$ and $-2 < y < 2$. Write $x = 2\cos(\theta)$ and $y = 2\cos(\psi)$. If we also have $x^2 + y^2 - 4 - xyt + t^2 = 0$ then $t = 2\cos(2\phi) = 2\cos(\theta \pm \psi)$. In other words, $2\phi = \theta \pm \psi$ or $2\phi = -\theta \pm \psi$. There are several cases. We choose the case $2\phi = \theta + \psi$. Eliminating ψ, the eigenvalues are

$$\lambda_1 = e^{i\theta+i\phi}, \quad \lambda_2 = e^{-i\theta+i\phi}, \quad \lambda_3 = e^{-i\theta+i\phi}, \quad \lambda_4 = e^{i\theta-3i\phi}.$$

Reorder the eigenvalues by swapping λ_2 and λ_4.

$$\lambda_1' = e^{i\theta+i\phi}, \quad \lambda_2' = e^{i\theta-3i\phi}, \quad \lambda_3' = e^{-i\theta+i\phi}, \quad \lambda_4' = e^{-i\theta+i\phi}.$$

With this new parametrisation we get new parameters $e^{2i\phi'} = \lambda_1'\lambda_2' = e^{2i\theta-2i\phi}$ and

$$x' = (\lambda_1' + \lambda_2')e^{-i\phi'} = 2\cos(2\phi), \quad y' = (\lambda_3' + \lambda_4')e^{i\phi'} = 2, \quad t' = 2\cos(2\theta - 2\phi).$$

Therefore, this is a point on the central tetrahedron. The other cases are similar.

We therefore concentrate of the points with $xy > 4$ or $xy < -4$.

Lemma 4.2.14. *Suppose $x^2 + y^2 - 4 - xyt + t^2 = 0$ and $-2 \leq t \leq 2$.*

(i) If $xy > 4$ then $x = y$ and $t = 2$.

(ii) If $xy < -4$ then $x = -y$ and $t = -2$.

Proof. We have

$$0 = x^2 + y^2 - 4 - xyt + t^2 = (x-y)^2 + (2-t)(xy-4) + (2-t)^2.$$

Since $-2 \leq t \leq 2$ we see that if $xy > 4$ we must have $(x-y)^2 = (2-t)^2 = 0$. Similarly

$$0 = x^2 + y^2 - 4 - xyt + t^2 = (x+y)^2 + (2+t)(-xy-4) + (2+t)^2.$$

If $xy < -4$ then $(x+y)^2 = (2+t)^2 = 0$.

The locus of points described in Lemma 4.2.14 are the *whiskers*.

Corollary 4.2.15. *The whiskers are given by*

$$\begin{aligned} (\tau, \sigma) &= \big(\pm 2\cosh(\ell), 4\cosh^2(\ell) + 2\big), \\ (\tau, \sigma) &= \big(\pm 2i\cosh(\ell), -4\cosh^2(\ell) - 2\big) \end{aligned}$$

where $\ell > 0$ is a real parameter.

Proposition 4.2.16. *Suppose that $A \in \mathrm{SU}(p,q)$ satsfies the hypotheses of Lemma 4.2.14. Then $p = q = 2$ and A is either diagonalisable or its Jordan normal form has two blocks of size 2.*

Proof. In this case, (up to multiplying A by a power of i) the eigenvalues are e^{ℓ}, e^{ℓ}, $e^{-\ell}$, $e^{-\ell}$ where $\ell > 0$. Since there are two eigenvectors that are greater than 1, we see that $\min\{p,q\} \geq 2$. Thus $p = q = 2$.

Since each eigenvalue has multiplicity 2, the possible Jordan blocks have size 1 or 2. Using the same argument as in Lemma 2.2.11, we see that the eigenspace associated to e^{ℓ} has the same dimension as the eigenspace associated to $e^{-\ell}$. Therefore A is either diagonalisable or has two Jordan blocks of size 2. It is easy to write down matrices that show both possibilities can arise (see comment after Theorem 4.3.5).

4.2.5 When A is 2-loxodromic

In the next three sections we give a few more details about the components of the complement of the holy grail. In particular, we relate the coordinates (x, y, t) with more geometrical parameters.

Suppose that $|\lambda_1| = |\lambda_2|^{-1} > 1$ and $|\lambda_3| = |\lambda_4|^{-1} > 1$. In this case, (after possibly multiplying A by a power of i if necessary) we can write

$$\lambda_1 = e^{\ell+i\phi}, \quad \lambda_2 = e^{-\ell+i\phi}, \quad \lambda_3 = e^{m-i\phi}, \quad \lambda_4 = e^{-m-i\phi}$$

where $\ell > 0$ and $m > 0$. Hence

$$\tau = 2\cosh(\ell)e^{i\phi} + 2\cosh(m)e^{-i\phi}, \quad \sigma = 4\cosh(\ell)\cosh(m) + 2\cos(2\phi). \tag{4.2.14}$$

and $x = 2\cosh(\ell)$, $y = 2\cosh(m)$, $t = 2\cos(2\phi)$. In this case

$$\begin{aligned} &R(\chi_A, \chi'_A) \\ &\quad = \quad 256\sinh^2(\ell)\sinh^2(m)\big(\cosh(\ell+m) - \cos(2\phi)\big)^2\big(\cosh(\ell-m) - \cos(2\phi)\big)^2. \end{aligned}$$

When $\ell = m$ and $\phi = \pi/2$ then we see that $\tau = 0$ and $\sigma = 4\cosh^2(\ell) - 2 = 2\cosh(2\ell)$. Such points lie inside the top bowl of the holy grail. Therefore, by continuity, this region comprises points where $R(\chi_A, \chi_A') > 0$. The presence of the whiskers in this bowl mean these two components of the set where $R(\chi_A, \chi_A') > 0$ are not simply connected. This leads to subtleties when it comes to giving parameters. The whiskers comprise points with $\ell = m$ and $\phi = 0$ or $\phi = \pi$. We now give a characterisation in terms of σ and τ of the points where exactly one of these conditions is satisfied.

Lemma 4.2.17. *Suppose that τ and σ satisfy* (4.2.14).

(i) If $\phi = 0$ and $\ell \neq m$ then $\Im(\tau) = 0$, $\Re(\tau) > 0$ and $\Re(\tau)^2 - 4\sigma + 8 > 0$.

(ii) If $\phi = \pi$ and $\ell \neq m$ then $\Im(\tau) = 0$, $\Re(\tau) < 0$ and $\Re(\tau)^2 - 4\sigma + 8 > 0$.

(iii) If $\phi \neq 0, \pi$ and $\ell = m$ then $\Im(\tau) = 0$ and $\Re(\tau)^2 - 4\sigma + 8 < 0$.

Proof. If $\phi = 0$ and $\ell \neq m$ then

$$\tau = 2\cosh(\ell) + 2\cosh(m), \quad \sigma = 4\cosh(\ell)\cosh(m) + 2.$$

Clearly $\Im(\tau) = 0$ and $\Re(\tau) > 0$. Also

$$\Re(\tau)^2 - 4\sigma + 8 = \big(2\cosh(\ell) - 2\cosh(m)\big)^2 > 0.$$

The case where $\phi = \pi$ and $\ell \neq m$ is similar.

If $\phi \neq 0$, π and $\ell = m$ then

$$\tau = 4\cosh(\ell)\cos(\phi), \quad \sigma = 4\cosh^2(\ell) + 2\cos(2\phi).$$

Clearly $\Im(\tau) = 0$. Also,

$$\Re(\tau)^2 - 4\sigma + 8 = -16\sinh^2(\ell)\sin^2(\phi) < 0.$$

Define $\mathcal{C}$ to be the set of all $(\tau, \sigma) \in \mathbb{C} \times \mathbb{R}$ satisfying

(i) $R(\chi_A, \chi_A') > 0$,

(ii) $\min\big\{\Re(\tau)^2 - 4\sigma + 8,\ 6 - \sigma\big\} < 0$,

(iii) $\max\big\{\Re(\tau)^2 - 4\sigma + 8,\ \Im(\tau)^2\big\} > 0$.

Geometrically, conditions (i) and (ii) imply that $\mathcal{C}$ is contained "inside" or "above" the upper bowl of the holy grail. Condition (iii) means that the points with both $\Im(\tau) = 0$ and $\Re(\tau)^2 - 4\sigma + 8 \leq 0$ are not in $\mathcal{C}$. Using Lemma 4.2.17 (iii) and the description of the whiskers, we see that this excludes those points with $\ell = m$.

Proposition 4.2.18. *The map*

$$\Phi : \Big\{(\ell, m, e^{i\phi}) \in \mathbb{R}_+^2 \times S^1 \ : \ \ell > m\Big\} \longrightarrow \mathcal{C}$$

given by (4.2.14) *is a diffeomorphism.*

Proof. We have seen above that if τ and σ are given by (4.2.14) then $R(\chi_A, \chi_A') > 0$. Moreover since $xy = 4\cosh(\ell)\cosh(m) > 4$, using Lemma 4.2.8 we see that

$$\min\big\{\Re(\tau)^2 - 4\sigma + 8,\, 6 - \sigma\big\} < 0.$$

In addition,

$$\begin{aligned} \Re(\tau)^2 - 4\sigma + 8 &= 4\big(\cosh(\ell) - \cosh(m)\big)^2 - 16\Big(\big(\cosh(\ell) + \cosh(m)\big)^2 - 1\Big)\sin^2\phi, \\ \Im(\tau)^2 &= 4\big(\cosh(\ell) - \cosh(m)\big)^2 \sin^2\phi. \end{aligned}$$

Since $\ell \neq m$ either $\Im(\tau)^2 > 0$ or $\sin^2\phi = 0$. In the latter case, $\Re(\tau)^2 - 4\sigma + 8 > 0$. Therefore

$$\max\big\{\Re(\tau)^2 - 4\sigma + 8,\, \Im(\tau)^2\big\} > 0.$$

Hence the image of Φ is contained $\mathcal{C}$.

Conversely, Proposition 4.2.5 implies that given any $(\tau, \sigma) \in \mathbb{C} \times \mathbb{R}$ we can find $(x, y, e^{i\phi})$ satisfying (4.2.8). Using Lemma 4.2.8 (i) we see that if

$$R(\chi_A, \chi_A') > 0, \quad \min\big\{\Re(\tau)^2 - 4\sigma + 8,\, 6 - \sigma\big\} < 0$$

then $(x^2 - 4)(y^2 - 4) > 0$ and $xy > 4$. Thus $x > 2$ and $y > 2$. We can write $x = 2\cosh(\ell)$ and $y = 2\cosh(m)$. Using Lemma 4.2.17 (iii) we see that if

$$\max\big\{\Re(\tau)^2 - 4\sigma + 8,\, \Im(\tau)^2\big\} > 0$$

then $\ell \neq m$. Swapping the roles of x and y if necessary (as in Corollary 4.2.7) we may assume that $\ell > m$. Therefore Φ is onto.

In real coordinates

$$\begin{aligned}\Re(\tau) &= 2\big(\cosh(\ell) + \cosh(m)\big)\cos(\phi),\\ \Im(\tau) &= 2\big(\cosh(\ell) - \cosh(m)\big)\sin(\phi),\\ \sigma &= 4\cosh(\ell)\cosh(m) + 2\cos(2\phi).\end{aligned}$$

This change of variables leads to the Jacobian

$$\begin{aligned} J &= 16\sinh(\ell)\sinh(m)\det\begin{pmatrix} \cos(\phi) & \cos(\phi) & -\big(\cosh(\ell)+\cosh(m)\big)\sin(\phi) \\ \sin(\phi) & -\sin(\phi) & \big(\cosh(\ell)-\cosh(m)\big)\cos(\phi) \\ \cosh(m) & \cosh(\ell) & -\sin(2\phi) \end{pmatrix} \\ &= -16\sinh(\ell)\sinh(m)\big(\cosh(\ell+m)-\cos(2\phi)\big)\big(\cosh(\ell-m)-\cos(2\phi)\big). \end{aligned}$$

This is clearly non-zero when $\ell > m > 0$. Therefore Φ is a local diffeomorphism.

As m tends to 0 then (τ, σ) tends to the upper bowl of the holy grail; as $\ell - m$ tends to 0 then (τ, σ) tends to points where $\Im(\tau) = 0$ and $\Re(\tau)^2 - 4\sigma + 8 \leq 0$; as ℓ tends to ∞ then (τ, σ) tends to infinity. Therefore Φ is proper.

Therefore Φ is a covering map. For fixed m and very large values of ℓ we have $(\tau, \sigma) \sim (e^{\ell}e^{i\phi}, 2e^{\ell}\cosh(m))$. Hence Φ has winding number 1 for such values of ℓ and hence everywhere. Thus Φ is a global diffeomorphism.

4.2.6 Simple loxodromic case

Suppose that $|\lambda_1| = |\lambda_2|^{-1} > 1$ and $|\lambda_3| = |\lambda_4|^{-1} = 1$. In this case, (after possibly multiplying A by a power of i if necessary) we can write

$$\lambda_1 = e^{\ell+i\phi}, \quad \lambda_2 = e^{-\ell+i\phi}, \quad \lambda_3 = e^{i\psi-i\phi}, \quad \lambda_4 = e^{-i\psi-i\phi}$$

where $\ell > 0$. Then

$$\tau = 2\cosh(\ell)e^{i\phi} + 2\cos(\psi)e^{-i\phi}, \quad \sigma = 4\cosh(\ell)\cos(\psi) + 2\cos(2\phi) \tag{4.2.15}$$

and $x = 2\cosh(\ell)$, $y = 2\cos(\psi)$, $t = 2\cos(2\phi)$. In this case

$$\begin{aligned} & R(\chi_A, \chi'_A) \\ = \quad & -256\sinh^2(\ell)\sin^2(\psi)\big(\cosh(\ell) - \cos(\psi + 2\phi)\big)^2\big(\cosh(\ell) - \cos(\psi - 2\phi)\big)^2. \end{aligned}$$

When $\psi = \pi/2$ and $\phi = \pi/4$ then $\tau = \sqrt{2}\cosh(\ell)(1+i)$. Such points are outside the holy grail. Therefore by continuity, $R(\chi_A, \chi'_A) < 0$ in this region. The following proposition may be proved in a similar manner to Proposition 4.2.18 (compare Proposition 3.8 of [39]).

Proposition 4.2.19. *The map*

$$\Phi : \Big\{(\ell, \psi, e^{i\phi}) \in \mathbb{R}_+ \times (0, \pi) \times S^1\Big\} \longrightarrow \Big\{(\tau, \sigma) \in \mathbb{C} \times \mathbb{R} \ : \ R(\chi_A, \chi'_A) < 0\Big\}$$

given by (4.2.15) *is a diffeomorphism.*

We remark that, depending on the signature of the Hermitian form, Proposition 4.2.19 may still not mean that A is determined up to conjugacy by (τ, σ). Suppose that the eigenvalue λ_j corresponds to the eigenspace U_j. Since $|\lambda_1| = |\lambda_2|^{-1} > 1$, the eigenspaces U_1 and U_2 must both be null and the Hermitian form restricted to $U_1 \oplus U_2$ must have signature $(1,1)$. If the signature of the form is $(3,1)$ or $(1,3)$ then U_3 and U_4 must both be positive or negative respectively. On the other hand, if the form has signature $(2,2)$ then one of U_3 or U_4 is positive and the other is negative. This determines two conjugacy classes in this case. For example, if the form is the standard diagonal form $\mathrm{diag}(1,1,-1,-1)$ then for $\varepsilon = \pm 1$ consider the following matrices in $\mathrm{SU}(2,2)$

$$A_\varepsilon = \begin{bmatrix} \cosh(\ell)e^{i\phi} & 0 & 0 & \sinh(\ell)e^{i\phi} \\ 0 & e^{i\varepsilon\psi - i\phi} & 0 & 0 \\ 0 & 0 & e^{-i\varepsilon\psi - i\phi} & 0 \\ \sinh(\ell)e^{i\phi} & 0 & 0 & \cosh(\ell)e^{i\phi} \end{bmatrix}.$$

Both these matrices have the same values of τ and σ but yet they are not conjugate within $\mathrm{SU}(2,2)$ (even though they are conjugate in $\mathrm{SL}(4,\mathbb{C})$).

4.2.7 Regular elliptic case

Suppose that $|\lambda_1| = |\lambda_2|^{-1} = 1$ and $|\lambda_3| = |\lambda_4|^{-1} = 1$. In this case, (after possibly multiplying A by a power of i if necessary) we can write

$$\lambda_1 = e^{i\theta+i\phi}, \quad \lambda_2 = e^{-i\theta+i\phi}, \quad \lambda_3 = e^{i\psi-i\phi}, \quad \lambda_4 = e^{-i\psi-i\phi}.$$

Then

$$\tau = 2\cos(\theta)e^{i\phi} + 2\cos(\psi)e^{-i\phi}, \quad \sigma = 4\cos(\theta)\cos(\psi) + 2\cos(2\phi).$$

and $x = 2\cos(\theta)$, $y = 2\cos(\psi)$, $t = 2\cos(2\phi)$. In this case

$$\begin{aligned} R(\chi_A, \chi'_A) &= 256\sin^2(\theta)\sin^2(\psi)\sin^2\big(\phi + (\theta+\psi)/2\big)\sin^2\big(\phi - (\theta+\psi)/2\big) \\ &\quad \cdot \sin^2\big(\phi + (\theta-\psi)/2\big)\sin^2\big(\phi - (\theta-\psi)/2\big). \end{aligned}$$

When $\theta = \psi$ and $\phi = \pi/2$ then we see that $\tau = 0$ and $\sigma = 4\cos^2(\theta) - 2 = 2\cos(2\theta)$. This lies in the central tetrahedron of the holy grail. Therefore, by continuity, this region comprises points where $R(\chi_A, \chi'_A) > 0$.

4.3 Geometrical applications

4.3.1 Introduction

Our primary motivation for the classification of elements of $\mathrm{SU}(p,q)$ with $p+q=4$ was to consider $\mathrm{SU}(3,1)$, a four fold cover of $\mathrm{PSU}(3,1)$, the holomorphic isometry group of complex hyperbolic space $\mathbf{H}^3_{\mathbb{C}}$. In order to demonstrate that this classification is also of interest in the case of $\mathrm{SU}(2,2)$, we use our results in two special cases. First we show that we can embed the orientation preserving isometry group of $\mathbf{H}^1_{\mathbb{H}}$, which is isometric to $\mathbf{H}^4_{\mathbb{R}}$, into $\mathrm{PSU}(2,2)$. Secondly, we do a similar thing with automorphisms of anti de Sitter space.

4.3.2 Isometries of complex hyperbolic space $\mathbf{H}^3_{\mathbb{C}}$

The holomorphic isometry group of complex hyperbolic 3-space $\mathbf{H}^3_{\mathbb{C}}$ is the projective unitary group $\mathrm{PSU}(3,1) = \mathrm{SU}(3,1)/\{\pm I, \pm iI\}$. In this group all loxodromic maps are simple, that is they have a single pair of eigenvalues λ_1 and $\lambda_2 = \overline{\lambda_1}^{-1}$ with absolute value different from 1, as described in Section 4.2.6. The classification of elements of $\mathrm{SU}(3,1)$ via their resultant is simply the case $p=3$ of Corollary 4.1.2:

Proposition 4.3.1. *Let $A \in \mathrm{SU}(3,1)$. Let $R(\chi_A, \chi_A')$ denotes the resultant of the characteristic polynomial $\chi_A(X)$ and its first derivative $\chi_A'(X)$. Then we have the following.*

(i) A is regular elliptic if and only if $R(\chi_A, \chi_A') > 0$.

(ii) A is regular loxodromic if and only if $R(\chi_A, \chi_A') < 0$.

(iii) A has a repeated eigenvalue if and only if $R(\chi_A, \chi_A') = 0$.

Furthermore, using Proposition 4.2.13 we can say slightly more about the case when A has a repeated eigenvalue.

Proposition 4.3.2. *Suppose that $A \in \mathrm{SU}(3,1)$ has a repeated eigenvalue. If A is diagonalisable, then it is either elliptic or loxodromic (and both possibilities arise). Otherwise it is parabolic, and the possible minimal polynomials of A are:*

(i) $m(x) = (x - e^{-i\phi})^2(x - e^{i\theta + i\phi})(x - e^{-i\theta+i\phi})$ where $\theta \neq 0,\ \pi,\ \pm 2\phi$ (mod 2π);

(ii) $m(x) = (x - e^{-i\phi})^2(x - e^{i\phi})$ where $\phi \neq 0,\ \pi$ (mod 2π);

(iii) $m(x) = (x - e^{-i\phi})^2(x - e^{3i\phi})$ where $\phi \neq 0,\ \pi/2,\ \pi,\ 3\pi/2$ (mod 2π);

(iv) $m(x) = (x - e^{-i\phi})^3(x - e^{3i\phi})$ where $\phi \neq 0,\ \pi/2,\ \pi,\ 3\pi/2$ (mod 2π);

(v) $m(x) = (x - e^{-ik\pi/2})^2$ for $k = 0,\ 1,\ 2,\ 3$;

(vi) $m(x) = (x - e^{-ik\pi/2})^3$ for $k = 0,\ 1,\ 2,\ 3$.

For a detailed classification of elements of $\mathrm{SU}(3,1)$ with repeated eigenvalues see [24]. With respect to the Hermitian form

$$H = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \tag{4.3.1}$$

we can find representatives of cases (i) to (vi) with one of the following two forms:

$$A_1 = \begin{pmatrix} e^{-i\phi} & 0 & 0 & ie^{-i\phi} \\ 0 & e^{i\theta+i\phi} & 0 & 0 \\ 0 & 0 & e^{-i\theta+i\phi} & 0 \\ 0 & 0 & 0 & e^{-i\phi} \end{pmatrix}, \quad A_2 = \begin{pmatrix} e^{-i\phi} & 0 & -2e^{-i\phi} & -2e^{-i\phi} \\ 0 & e^{3i\phi} & 0 & 0 \\ 0 & 0 & e^{-i\phi} & 2e^{-i\phi} \\ 0 & 0 & 0 & e^{-i\phi} \end{pmatrix}.$$

In (i) we have A_1; in (ii) we have A_1 with $\theta = 0$; in (iii) we have A_1 with $\theta = 2\phi$; in (iv) we have A_2; in (v) we have A_1 with $\theta = 0$ and $\phi = k\pi/2$; in (vi) we have A_2 with $\phi = k\pi/2$.

Our goal in remainder of this section is to relate our parameters for loxodromic maps in $\mathrm{SU}(3,1)$ with the geometry of their action on $\mathbf{H}^3_{\mathbb{C}}$. This generalises the work in Parker [39] where the geometry of loxodromic maps in $\mathrm{SU}(2,1)$ was considered.

We now recall the notation of Section 4.2.6. Suppose that $A \in \mathrm{SU}(3,1)$ has eigenvalues

$$\lambda_1 = e^{\ell+i\phi}, \quad \lambda_2 = e^{-\ell+i\phi}, \quad \lambda_3 = e^{i\psi-i\phi}, \quad \lambda_4 = e^{-i\psi-i\phi}. \tag{4.3.2}$$

The eigenspaces V_1 and V_2 in $\mathbb{C}^{3,1}$ corresponding to λ_1 and λ_2 are both null. After projectivisation, they correspond to fixed points q_1 and q_2 of A on $\partial\mathbf{H}^3_{\mathbb{C}}$. Also, $V_1 \oplus V_2$ is indefinite. Its projectivisation is a complex line, whose intersection L with $\mathbf{H}^3_{\mathbb{C}}$is a copy of the Poincaré disc model of the hyperbolic plane, called the *complex axis* of A. The (Poincaré) geodesic in L with endpoints q_1 and q_2 is called the *axis* of A and is denoted $\alpha(A)$. The eigenspaces V_3 and V_4 in $\mathbb{C}^{3,1}$ corresponding to λ_3 and λ_4 are each positive. They are orthogonal to $V_1 \oplus V_2$, whose projectivisation intersects $\mathbf{H}^3_{\mathbb{C}}$ in L.

Proposition 4.3.3. *Let A in* $\mathrm{SU}(3,1)$ *be a loxodromic map with axis α and complex axis L. Let ℓ, ϕ and ψ be the parameters associated to A given by* (4.3.2)*. Then A translates a Bergman distance 2ℓ along α and rotates the complex lines orthogonal to L by angles $-2\phi + \psi$ and $-2\phi - \psi$.*

Proof. We use the diagonal Hermitian form $\langle , \rangle$ given by $H = \mathrm{diag}(1, 1, 1, -1)$ and we follow the ideas of Parker [39, Proposition 3.10]. In this case we may represent points z in $\mathbf{H}^3_{\mathbb{C}}$ by $(z_1, z_2, z_3) \in \mathbb{C}^3$ with $|z_1|^2 + |z_2|^2 + |z_3|^2 < 1$. If the eigenvalues of A are given by (4.3.2) then, up to conjugacy, we may suppose

$$A = \begin{pmatrix} \cosh(\ell)e^{i\phi} & 0 & 0 & \sinh(\ell)e^{i\phi} \\ 0 & e^{i\psi - i\phi} & 0 & 0 \\ 0 & 0 & e^{-i\psi - i\phi} & 0 \\ \sinh(\ell)e^{i\phi} & 0 & 0 & \cosh(\ell)e^{i\phi} \end{pmatrix}.$$

Thus A fixes $(\pm 1,\, 0,\, 0)$ on $\partial \mathbf{H}^3_{\mathbb{C}}$. The action of A on $\mathbf{H}^3_{\mathbb{C}}$ is given by

$$A \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ 1 \end{bmatrix} \sim \begin{bmatrix} \big(\cosh(\ell)z_1 + \sinh(\ell)\big)/\big(\sinh(\ell)z_1 + \cosh(\ell)\big) \\ e^{i\psi - 2i\phi} z_2/\big(\sinh(\ell)z_1 + \cosh(\ell)\big) \\ e^{-i\psi - 2i\phi} z_3/\big(\sinh(\ell)z_1 + \cosh(\ell)\big) \\ 1 \end{bmatrix}$$

where $\sim$ stands for projective equality. The axis of A is the geodesic α joining the fixed points and the complex axis of A is the unique complex line containing α. They are given by

$$\alpha = \Big\{(x,0,0) \in \mathbf{H}^3_{\mathbb{C}} \ : \ -1 < x < 1\Big\}, \quad L = \Big\{(z,0,0) \in \mathbf{H}^3_{\mathbb{C}} \ : \ |z| < 1\Big\}.$$

Suppose that $p = (x,0,0)$ is a point of the axis α of A. Let $\mathbf{p}$ denote the lift of p to $\mathbb{C}^4$ given by $\mathbf{p} = (x,0,0,1)^t$. Then the translation length of A along α is $\rho\big(A(p),p\big)$. We have

$$\cosh\Big(\rho\big(A(p),p\big)/2\Big) = \left|\frac{\langle A\mathbf{p},\mathbf{p}\rangle}{\langle \mathbf{p},\mathbf{p}\rangle}\right| = \left|\frac{\cosh(\ell)(x^2-1)}{x^2-1}\right| = \cosh(\ell).$$

This implies $\rho\big(A(p),p\big) = 2\ell$ as claimed.

The tangent vectors to $\mathbf{H}^3_{\mathbb{C}}$ spanning the complex lines orthogonal to L are given by ξ and η:

$$\xi = (0,1,0,0)^2, \quad \eta = (0,0,1,0)^t.$$

Clearly the (projective) action of A sends ξ in $T_p(\mathbf{H}^2_{\mathbb{C}})$ to $e^{i\psi-2i\phi}\xi$ in $T_{A(p)}(\mathbf{H}^2_{\mathbb{C}})$ and η to $e^{-i\psi-2i\phi}\eta$. The rest of the result follows.

4.3.3 Isometries of $\mathbf{H}^1_{\mathbb{H}} = \mathbf{H}^4_{\mathbb{R}}$

Quaternionic hyperbolic 1-space $\mathbf{H}^1_{\mathbb{H}}$ may be identified with hyperbolic 4-space $\mathbf{H}^4_{\mathbb{R}}$. The isometries of quaternionic hyperbolic 1-space are contained in the projective symplectic group $\mathrm{PSp}(1,1) = \mathrm{Sp}(1,1)/(\pm I)$. The group $\mathrm{Sp}(1,1)$ is the group of 2×2 quaternionic matrices preserving a quaternionic Hermitian form of signature $(1,1)$; see Parker [40] for example. There is a canonical way to identify a quaternion with a 2×2 complex matrix and therefore to identify a 2×2 quaternionic matrix with a 4×4 complex matrix; see Gongopadhyay [23] for example. When we do this, the quaternionic Hermitian form of signature $(1,1)$ becomes a complex Hermitian form of signature $(2,2)$. The upshot of this construction is that it is possible to embed (the double cover of) the group of orientation preserving isometries of hyperbolic 4-space into $\mathrm{SU}(2,2)$. In this section we show how the classification given in the previous sections relate to the well known classification of four dimensional hyperbolic isometries. Our construction follows Gongopadhyay [23], where arbitrary invertible 2×2 quaternionic matrices were considered. See also Parker and Short [45] for an alternative method of classifying quaternionic Möbius transformations.

Let $A_{\mathbb{H}}$ be a 2×2 matrix of quaternions acting on a column vector $\mathbf{z}_{\mathbb{H}}$ of quaternions as

$$A_{\mathbb{H}}\mathbf{z}_{\mathbb{H}} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} z \\ w \end{pmatrix} = \begin{pmatrix} az+bw \\ cz+dw \end{pmatrix}.$$

If A is in $\mathrm{Sp}(1,1)$ then $|a|=|d|$, $|b|=|c|$, $|a|^2-|c|^2=1$, $\overline{a}b=\overline{c}d$ and $a\overline{c}=b\overline{d}$; see Lemma 1.1 of [9] or Proposition 6.3.1 of [40] for example. If a is a quaternion we can write it as $a=a_1+ja_2$ where $a_1, a_2\in\mathbb{C}$. Then a corresponds to the following matrix in $\mathbb{C}^2$:

$$\begin{pmatrix} a_1 & -\overline{a}_2 \\ a_2 & \overline{a}_1 \end{pmatrix}.$$

It is not hard to show that this identification is a group homomorphism from $\mathbb{H}$ with quaternionic multiplication to $\mathrm{M}(2,\mathbb{C})$ with matrix multiplication.

Using this identification, the matrix $A_{\mathbb{H}}$ corresponds to a 4×4 complex matrix A given by:

$$A = \begin{pmatrix} a_1 & -\overline{a}_2 & b_1 & -\overline{b}_2 \\ a_2 & \overline{a}_1 & b_2 & \overline{b}_1 \\ c_1 & -\overline{c}_2 & d_1 & -\overline{d}_2 \\ c_2 & \overline{c}_1 & d_2 & \overline{d}_1 \end{pmatrix}.$$

Likewise $\mathbf{z}_{\mathbb{H}}$ corresponds to 4×2 matrix and we only consider its first column, which is a vector $\mathbf{z}$ in $\mathbb{C}^4$. The action of $A_{\mathbb{H}}$ on $\mathbf{z}_{\mathbb{H}}$ induces the standard action of A on $\mathbf{z}\in\mathbb{C}^4$ by matrix multiplication. Using this identification, we see that if $A_{\mathbb{H}}$ is in $\mathrm{Sp}(1,1)$ then

$A \in \mathrm{SU}(2,2)$.

Suppose that $\lambda_{\mathbb{H}} \in \mathbb{H}$ is a right eigenvalue for $A_{\mathbb{H}}$. This means that there is a quaternionic vector $\mathbf{v}$ so that $A_{\mathbb{H}}\mathbf{v} = \mathbf{v}\lambda_{\mathbb{H}}$. It is always possible to find a unit quaternion μ so that $\lambda = \mu^{-1}\lambda_{\mathbb{H}}\mu$ is in $\mathbb{C}$; see Parker and Short [45] or Gongopadhyay [23] for example. (That is, writing $\lambda = \lambda_1 + j\lambda_2$ with $\lambda_1, \lambda_2 \in \mathbb{C}$ gives $\lambda_2 = 0$.) In this case

$$A_{\mathbb{H}}(\mathbf{v}\mu) = \mathbf{v}\lambda_{\mathbb{H}}\mu = (\mathbf{v}\mu)\lambda.$$

Hence $\lambda \in \mathbb{C}$ is also a right eigenvalue of $A_{\mathbb{H}}$. (In the language of quaternions, right eigenvalues of quaternionic matrices are defined up to similarity.) It is easy to show that λ is also an eigenvalue of A. Since we can also find $\nu \in \mathbb{H}$ so that $\overline{\lambda} = \nu^{-1}\lambda_{\mathbb{H}}\nu$, a similar argument shows that $\overline{\lambda}$ is also an eigenvalue of A. Hence, if $|\lambda| \neq 1$, using Goldman's lemma, Lemma 2.2.11, the eigenvalues of A are

$$\lambda, \quad \overline{\lambda}, \quad \lambda^{-1}, \quad \overline{\lambda}^{-1}.$$

If $|\lambda| = 1$ then this is true of all eigenvalues and they are

$$e^{i\theta}, \quad e^{-i\theta}, \quad e^{i\psi}, \quad e^{-i\psi}.$$

This implies that τ is real (which could have been seen by inspection) and so the characteristic polynomial $\chi_A(X)$ of A has real coefficients. Hence the coefficients of X and X^3 in $\chi_A(X)$ are the same. This rules out case (i) of [23] Theorem 1.1; see also Corollary 6.2 of Parker and Short [45]. Putting $\tau \in \mathbb{R}$ in the expression for $R(\chi_A, \chi_A')$ in terms of σ and τ in Section 4.2.3 gives.

$$\begin{aligned} R(\chi_A, \chi_A') &= \big(\sigma^2 + 4\sigma + 4 - 4\tau^2\big)\big(\tau^2 - 4\sigma + 8\big)^2 \\ &= \big(\sigma + 2 - 2\tau\big)\big(\sigma + 2 + 2\tau\big)\big(\tau^2 - 4\sigma + 8\big)^2. \end{aligned}$$

We can now state our classification theorem, which should be compared to Theorem 1.1 of Gongopadhyay [23].

Proposition 4.3.4. *Let $A \in \mathrm{SU}(2,2)$ correspond to a map in $\mathrm{Sp}(1,1)$. Then A has* *characteristic polynomial*

$$\chi_A(X) = X^4 - \tau X^3 + \sigma X^2 - \tau X + 1$$

where $\mathrm{tr}(A) = \tau \in \mathbb{R}$ *and* $\sigma \in \mathbb{R}$. *Moreover*

(i) A is regular 2-loxodromic if and only if $\tau^2 - 4\sigma + 8 < 0$.

(ii) A is regular elliptic if and only if $\tau^2 - 4\sigma + 8 > 0$ *and* $(\sigma + 2)^2 \neq 4\tau^2$.

(iii) A has a repeated eigenvalue if and only if $\tau^2 - 4\sigma + 8 = 0$ *or* $(\sigma + 2)^2 = 4\tau^2$.

We note that the connection between our notation and that of Gongopadhyay is that $c_1 = c_3 = \tau^2/4$ and $c_2 = \sigma$. The main difference between our result and Theorem 1.1 of Gongopadhyay [23] is that his result does not involve $(\sigma + 2)^2 - 4\tau^2$. We now explain this. Using our expression for the eigenvalues of A, we see that when $|\lambda| \neq 1$ then

$$\big(\sigma + 2 - 2\tau\big)\big(\sigma + 2 + 2\tau\big) = |\lambda + \lambda^{-1} - 2|^2|\lambda + \lambda^{-1} + 2|^2 > 0.$$

Otherwise $\tau = 2\cos(\theta) + 2\cos(\psi)$ and $\sigma = 4\cos(\theta)\cos(\psi) + 2$ and

$$\big(\sigma + 2 - 2\tau\big)\big(\sigma + 2 + 2\tau\big) = 16\big(1 - \cos(\theta)\big)\big(1 - \cos(\psi)\big)\big(1 + \cos(\theta)\big)\big(1 + \cos(\psi)\big) \geq 0.$$

Hence $\big(\sigma + 2 - 2\tau\big)\big(\sigma + 2 + 2\tau\big) = 0$ if and only if $e^{i\theta} = \pm 1$ or $e^{i\psi} = \pm 1$. If both of these are true then $\tau^2 - 4\sigma + 8 = 0$. Otherwise, the eigenvalues of A are

$$e^{i\theta}, \quad e^{-i\theta}, \quad \pm 1, \quad \pm 1.$$

where $e^{i\theta} \neq \pm 1$. In this case $\tau^2 - 4\sigma + 8 = 4(1 \mp \cos\theta)^2 > 0$. Furthermore, the repeated eigenvalue $\lambda = \pm 1$ corresponds to the same quaternionic eigenvector $\lambda_{\mathbb{H}} = \pm 1$. Thus there is a two dimensional complex eigenspace associated to this eigenvector and A is elliptic.

4.3.4 Automorphisms of anti de Sitter space

There is a canonical identification between $\mathbb{R}^4$ and $\mathrm{M}(2, \mathbb{R})$, the collection of 2×2 real matrices. Under this identification, the determinant map $\det : \mathrm{M}(2, \mathbb{R}) \longrightarrow \mathbb{R}$ corresponds to a quadratic form of signature $(2,2)$ on $\mathbb{R}^4$. *Anti de Sitter space* is the projectivisation of the positive vectors with respect to this quadratic form. It may be canonically identified with $\mathrm{PSL}(2,\mathbb{R})$ by considering the section where this quadratic form takes the value $+1$; see Section 7 of Mess [36] or Section 2 of Goldman [20]. The automorphism group of anti de Sitter space with its Lorentz structure is $\mathrm{PSL}(2,\mathbb{R}) \times \mathrm{PSL}(2,\mathbb{R})$. Using the identification of anti de Sitter space with $\mathbb{R}^4$ gives an isomorphism between $\mathrm{PSL}(2,\mathbb{R}) \times \mathrm{PSL}(2,\mathbb{R})$ and $\mathrm{PSO}_0(2,2) = \mathrm{SO}_0(2,2)/(\pm I)$, where $\mathrm{SO}_0(2,2)$ is the identity component of $\mathrm{SO}(2,2)$; again see Mess [36] or Goldman [20].

Let us make this explicit. Identify $\mathbb{R}^4$ and $\mathrm{M}(2,\mathbb{R})$ by the map:

$$F : \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \longmapsto X = \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix}.$$

The determinant map $\det(X)$ corresponds to the quadratic form $Q(\mathbf{x}) = x_1x_4 - x_2x_3$. This is associated to the symmetric matrix H of signature $(2,2)$ where

$$H = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

Let A_1, $A_2 \in \mathrm{SL}(2,\mathbb{R})$. Then the pair (A_1, A_2) acts on $\mathrm{SL}(2,\mathbb{R})$ and this action corresponds to $A \in \mathrm{SO}(2,2)$ as follows:

$$F(A\mathbf{x}) = A_1F(\mathbf{x})A_2^{-1}.$$

(Note we invert the matrix on the right so that the map from $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$ to $\mathrm{SO}(2,2)$ is a homomorphism.) If

$$A_1 = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}.$$

Then it is easy to see that

$$A = \begin{pmatrix} a_1d_2 & -a_1c_2 & b_1d_2 & -b_1c_2 \\ -a_1b_2 & a_1a_2 & -b_1b_2 & b_1a_2 \\ c_1d_2 & -c_1c_2 & d_1d_2 & -d_1c_2 \\ -c_1b_2 & c_1a_2 & -d_1b_2 & d_1a_2 \end{pmatrix}.$$

Clearly $\tau = \mathrm{tr}(A) = (a_1 + d_1)(a_2 + d_2) = \mathrm{tr}(A_1)\mathrm{tr}(A_2)$. It is not hard to see that

$$\begin{aligned} \sigma &= \frac{1}{2}\big(\mathrm{tr}^2(A) - \mathrm{tr}(A^2)\big) \\ &= \frac{1}{2}\big(\mathrm{tr}^2(A_1)\mathrm{tr}^2(A_2) - \mathrm{tr}(A_1^2)\mathrm{tr}(A_2^2)\big) \\ &= \frac{1}{2}\Big(\mathrm{tr}^2(A_1)\mathrm{tr}^2(A_2) - \big(\mathrm{tr}^2(A_1) - 2\big)\big(\mathrm{tr}^2(A_2) - 2\big)\Big) \\ &= \mathrm{tr}^2(A_1) + \mathrm{tr}^2(A_2) - 2. \end{aligned}$$

Theorem 4.3.5. *Let $(A_1, A_2) \in \mathrm{PSL}(2,\mathbb{R}) \times \mathrm{PSL}(2,\mathbb{R})$ be an automorphism of anti de Sitter space. Then*

(i) (A_1, A_2) is regular 2-loxodromic if either A_1 or A_2 is loxodromic and also $4 \neq \mathrm{tr}^2(A_1) \neq \mathrm{tr}^2(A_2) \neq 4$.

(ii) (A_1, A_2) is regular elliptic if A_1 and A_2 are both elliptic and $\mathrm{tr}^2(A_1) \neq \mathrm{tr}^2(A_2)$.

(iii) (A_1, A_2) is not regular if $\mathrm{tr}^2(A_1) = 4$ or $\mathrm{tr}^2(A_2) = 4$ or $\mathrm{tr}^2(A_1) = \mathrm{tr}^2(A_2)$.

Proof. Consider the parameters x, y and t defined in (4.2.5). Since $\mathrm{tr}(A)$ is real, we have $t = 2$, that is $\phi = 0$ or $\phi = \pi$. Moreover

$$\begin{aligned} (x+y)^2 &= |\tau|^2 = \mathrm{tr}^2(A_1)\mathrm{tr}^2(A_2), \\ xy + 2 &= \sigma = \mathrm{tr}^2(A_1) + \mathrm{tr}^2(A_2) - 2. \end{aligned}$$

A consequence of this is that

$$\begin{aligned} (x^2-4)(y^2-4) &= (xy)^2 - 4(x^2+y^2) + 16 = \big(\mathrm{tr}^2(A_1) - \mathrm{tr}^2(A_2)\big)^2, \\ x^2 + y^2 - 4 - xyt + t^2 &= (x+y)^2 - 4xy = \big(\mathrm{tr}^2(A_1) - 4\big)\big(\mathrm{tr}^2(A_2) - 4\big). \end{aligned}$$

Therefore, using the identity from Proposition 4.2.6, we have

$$\begin{aligned} R(\chi_A, \chi'_A) &= (x^2-4)(y^2-4)(x^2+y^2-4-xyt+t^2)^2 \\ &= \big(\mathrm{tr}^2(A_1)-\mathrm{tr}^2(A_2)\big)^2\big(\mathrm{tr}^2(A_1)-4\big)^2\big(\mathrm{tr}^2(A_2)-4\big)^2. \end{aligned}$$

Then A has a repeated eigenvalue if and only if one of the following conditions hold:

$$\mathrm{tr}(A_2) = \pm\mathrm{tr}(A_1), \quad \mathrm{tr}(A_1) = \pm 2, \quad \mathrm{tr}(A_2) = \pm 2.$$

Otherwise A is 2-loxodromic or elliptic. Furthermore, we have

$$\begin{aligned} \Re(\tau)^2 - 4\sigma + 8 &= \big(\mathrm{tr}^2(A_1)-4\big)\big(\mathrm{tr}^2(A_2)-4\big), \\ \Im(\tau)^2 + 4\sigma + 8 &= 4\mathrm{tr}^2(A_1) + 4\mathrm{tr}^2(A_2), \\ 6 - \sigma &= 8 - \mathrm{tr}^2(A_1) - \mathrm{tr}^2(A_2). \end{aligned}$$

Then using Theorem 4.2.9 we see (A_1, A_2) is elliptic if and only if A_1 and A_2 are both elliptic with $\mathrm{tr}^2(A_1) \neq \mathrm{tr}^2(A_2)$.

Note that taking A_1 to be loxodromic and A_2 to be parabolic gives an example of a matrix in $\mathrm{SU}(2,2)$ lying on one of the whiskers and whose Jordan normal form has two blocks of size 2; see Proposition 4.2.16.

Chapter 5

Classification of Pair of Loxodromic Elements

In this chapter, we determine, up to conjugation, non-singular pair of loxodromics elements in $\mathrm{SU}(3,1)$.

Following the notation of Section 2.2.1, we consider the Siegel domain model of $\mathbf{H}^3_{\mathbb{C}}$. The holomorphic isometry group of $\mathbf{H}^3_{\mathbb{C}}$ is the projective unitary group $\mathrm{PSU}(3,1) = \mathrm{SU}(3,1)/\{\pm I, \pm iI\}$.

5.1 Loxodromic isometries

Let $A \in \mathrm{SU}(3,1)$ represents a loxodromic isometry. Then A has eigenvalues of the form $re^{i\theta},\ r^{-1}e^{i\theta},\ e^{i\phi},\ e^{-i(2\theta+\phi)}$. We can assume $\theta,\ \phi \in (-\pi, \pi]$ and $\theta \leq \phi$. Then $(r,\ \theta, \phi) \in S$, where S is the region defined by:

$$S = \{(r,\ \theta,\ \phi) \in \mathbb{R}^3 : r > 1,\ \theta,\ \phi \in (-\pi,\ \pi],\ \theta \geq \phi\}.$$

Let $a_A \in \partial\mathbf{H}^3_{\mathbb{C}}$ be the attractive fixed point of A. then any lift $\mathbf{a}_A$ of a_A to V_0 is an eigenvector of A and corresponding eigenvalue is $re^{i\theta}$. Similarly if $r_A \in \partial\mathbf{H}^3_{\mathbb{C}}$ is the repelling fixed point of A, then any lift $\mathbf{r_A}$ of r_A to V_0 is an eigenvector of A with eigenvalue $r^{-1}e^{i\theta}$.

For $(r, \theta, \phi) \in S$, define $E(r, \theta, \phi)$ as

$$E(r,\theta,\phi) = \begin{pmatrix} re^{i\theta} & & & \\ & e^{i\phi} & & \\ & & e^{-i(2\theta+\phi)} & \\ & & & r^{-1}e^{i\theta} \end{pmatrix}. \tag{5.1.1}$$

It is easy to see that $E = E(r, \theta, \phi) \in \mathrm{SU}(3,1)$ represent a loxodromic map with attractive fixed point $a_E = \infty$ and repelling fixed point $r_E = o$.

Let $\mathbf{x}_A$, $\mathbf{y}_A$ be the eigenvectors corresponding to the eigenvalues $e^{i\phi}$, $e^{-i(2\theta+\phi)}$ respectively, scaled so that $\langle \mathbf{x}_A, \mathbf{x}_A \rangle = 1 = \langle \mathbf{y}_A, \mathbf{y}_A \rangle$. Let $C_A = \begin{bmatrix} \mathbf{a}_A & \mathbf{x}_A & \mathbf{y}_A & \mathbf{r}_A \end{bmatrix}$ be the 4×4 matrix, where the lifts $\mathbf{a_A}$ and $\mathbf{r_A}$ are chosen so that C_A has determinant 1. Then $C_A \in \mathrm{SU}(3,1)$ and $A = C_A E_A(r, \theta, \phi) C_A^{-1}$, where $E_A(r, \theta, \phi)$ is given by (5.1.1).

Lemma 5.1.1. *Let $A \in \mathrm{SU}(3,1)$. Then A has characteristic polynomial*

$$\chi_A(X) = X^4 - \tau_A X^3 + \sigma_A X^2 - \bar{\tau}_A X + 1,$$

where $\tau_A = tr(A)$ and $\sigma_A = \frac{1}{2}(tr^2(A) - tr(A^2))$. Moreover σ_A is real.

For a proof see [26]. We also denote σ_A by $\sigma(A)$ in the sequel.

Proposition 5.1.2. *Two loxodromic elements in $\mathrm{SU}(3,1)$ are conjugate if and only if they have the same eigenvalues.*

For a proof see [11]. An immediate consequence of Lemma 5.1.1 and Proposition 5.1.2 is:

Corollary 5.1.3. *Two loxodromic elements A and A' in $\mathrm{SU}(3,1)$ are conjugate if and only if $\tau_A = \tau_{A'}$ and $\sigma_A = \sigma_{A'}$.*

Lemma 5.1.4. *Let $A = [Ae_1, Ae_2, Ae_3, Ae_4] \in \mathrm{SU}(3,1)$, then the vector Ae_3 is uniquely determined by the vectors Ae_1, Ae_2 and Ae_4.*

Proof. Let W be the subspace spanned by Ae_1, Ae_2, Ae_4. Let $W^\perp$ be the orthogonal complement of W in $\mathbb{C}^{3,1}$. Observe that since $A \in \mathrm{SU}(3,1)$, $W \cap W^\perp = \{0\}$ and $W^\perp \neq \{0\}$ is an one dimensional subspace of $\mathbb{C}^4$. Let $W^\perp = \langle w \rangle$ for some $w \in \mathbb{C}^4$. Then

$Ae_3 \in W^{\perp}$ implies that $Ae_3 = \lambda w$ for some $\lambda \in \mathbb{C}$. Further the condition $\det(A) = 1$ determines λ uniquely and the assertion follows.

Corollary 5.1.5. *Let* $A = [Ae_1,\ Ae_2,\ Ae_3,\ Ae_4]$, $B = [Be_1,\ Be_2,\ Be_3,\ Be_4] \in \mathrm{SU}(3,1)$ *and* $C \in \mathrm{SU}(3,1)$ *be such that* $CAe_i = Be_i$ *for* $i = 1,\ 2,\ 4$*, then* $CAe_3 = Be_3$.

From Lemma 5.1.4 and Corollary 5.1.3 we have the following.

Corollary 5.1.6. *Let* A *and* A' *are two loxodromic elements in* $\mathrm{SU}(3,1)$ *such that* $\tau_A = \tau_{A'}$, $\sigma_A = \sigma_{A'}$, $a_A = a_{A'}$, $r_A = r_{A'}$ *and* $x_A = x_{A'}$*, then* $A = A'$.

5.2 The cross-ratios

Parker and Platis [41], also see Falbel [16], have shown that the triples of cross-ratios of an ordered quadruple of points in $\partial\mathbf{H}^2_{\mathbb{C}}$ satisfy two real equations. If the ordered triples of points belongs to $\partial\mathbf{H}^3_{\mathbb{C}}$, the corresponding cross-ratios satisfy only one real equation and one real inequality as shown in the following proposition.

Proposition 5.2.1. *Let* $z_1,\ z_2,\ z_3,\ z_4$ *be four distinct points in* $\partial\mathbf{H}^3_{\mathbb{C}}$*. Let* $\mathbb{X}_1,\ \mathbb{X}_2,\ \mathbb{X}_3$ *be defined by 2.4.1, then*

$$|\mathbb{X}_2| = |\mathbb{X}_1||\mathbb{X}_3|. \tag{5.2.1}$$

$$2|\mathbb{X}_1|^2\Re(\mathbb{X}_3) \geq |\mathbb{X}_1|^2 + |\mathbb{X}_2|^2 + 1 - 2\Re(\mathbb{X}_1 + \mathbb{X}_2). \tag{5.2.2}$$

Further, equality holds in (5.2.2) if and only if either of the following holds.

(i) $z_1,\ z_2,\ z_4$ *lie on the same complex line.*

(ii) $z_1,\ z_3,\ z_4$ *lie on the same complex line.*

(iii) $z_1,\ z_2,\ z_3,\ z_4$ *lie on the same complex line.*

Proof. Since $\mathrm{SU}(3,1)$ acts doubly transitively on $\partial\mathbf{H}^3_{\mathbb{C}}$, we may suppose $z_2 = \infty$ and $z_3 = \mathbf{o}$. Let $\mathbf{z}_1, \mathbf{z}_4$ be lifts of z_1 and z_4 chosen so that $\langle \mathbf{z}_1, \mathbf{z}_4 \rangle = 1$. We write them in coordinates as

$$\mathbf{z}_1 = \begin{bmatrix} \xi_1 \\ \eta_1 \\ \alpha_1 \\ \delta_1 \end{bmatrix}, \ \mathbf{z}_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \ \mathbf{z}_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \ \mathbf{z}_4 = \begin{bmatrix} \xi_4 \\ \nu_2 \\ \zeta_4 \\ \delta_4 \end{bmatrix}.$$

Then we have

$$0 = \langle \mathbf{z}_1, \ \mathbf{z}_1 \rangle = \xi_1 \bar{\delta}_1 + \bar{\xi}_1 \delta_1 + |\eta_1|^2 + |\alpha_1|^2 \tag{5.2.3}$$

$$1 = \langle \mathbf{z}_4, \ \mathbf{z}_1 \rangle = \xi_4 \bar{\delta}_1 + \bar{\xi}_4 \delta_1 + \nu_2 \bar{\eta}_1 + \zeta_4 \bar{\alpha}_1 \tag{5.2.4}$$

$$0 = \langle \mathbf{z}_4, \ \mathbf{z}_4 \rangle = \xi_4 \bar{\delta}_4 + \bar{\xi}_4 \delta_4 + |\nu_2|^2 + |\zeta_4|^2 \tag{5.2.5}$$

From the definitions of the cross-ratios we have:

$$\mathbb{X}_1 = [\mathbf{z}_1, \ \mathbf{z}_2, \ \mathbf{z}_3, \ \mathbf{z}_4] = \frac{\langle \mathbf{z}_3, \ \mathbf{z}_1 \rangle \langle \mathbf{z}_4, \ \mathbf{z}_2 \rangle}{\langle \mathbf{z}_4, \ \mathbf{z}_1 \rangle \langle \mathbf{z}_3, \ \mathbf{z}_2 \rangle} = \bar{\xi}_1 \delta_4$$

$$\mathbb{X}_2 = [\mathbf{z}_1, \ \mathbf{z}_3, \ \mathbf{z}_2, \ \mathbf{z}_4] = \frac{\langle \mathbf{z}_2, \ \mathbf{z}_1 \rangle \langle \mathbf{z}_4, \ \mathbf{z}_3 \rangle}{\langle \mathbf{z}_4, \ \mathbf{z}_1 \rangle \langle \mathbf{z}_2, \ \mathbf{z}_3 \rangle} = \xi_4 \bar{\delta}_1$$

$$\mathbb{X}_3 = [\mathbf{z}_2, \ \mathbf{z}_3, \ \mathbf{z}_1, \ \mathbf{z}_4] = \frac{\langle \mathbf{z}_1, \ \mathbf{z}_2 \rangle \langle \mathbf{z}_4, \ \mathbf{z}_3 \rangle}{\langle \mathbf{z}_4, \ \mathbf{z}_2 \rangle \langle \mathbf{z}_1, \ \mathbf{z}_3 \rangle} = \frac{\xi_4 \delta_1}{\xi_1 \delta_4}$$

We immediately see that

$$|\mathbb{X}_3| = \frac{|\mathbb{X}_2|}{|\mathbb{X}_1|}.$$

Using eqs. 5.2.3 – 5.2.5, we have:

$$\begin{aligned}
|\mathbb{X}_1|^2|\mathbb{X}_3-1|^2 &= |\xi_4\delta_1-\xi_1\delta_4|^2\\
&= |\xi_4\delta_1|^2+|\xi_1\delta_4|^2-\xi_4\delta_1\bar{\xi}_1\bar{\delta}_4-\bar{\xi}_4\bar{\delta}_1\xi_1\delta_4\\
&= |\bar{\xi}_1\delta_4|^2+|\xi_4\bar{\delta}_1|^2+\xi_4\bar{\delta}_4(\xi_1\bar{\delta}_1+|\eta_1|^2+|\alpha_1|^2)+\bar{\xi}_4\delta_4(\bar{\xi}_1\delta_1+|\eta_1|^2+|\alpha_1|^2)\\
&= |\bar{\xi}_1\delta_4+\xi_4\bar{\delta}_1|^2-(|\nu_2|^2+|\zeta_4|^2)(|\eta_1|^2+|\alpha_1|^2)\\
&= |\bar{\xi}_1\delta_4+\xi_4\bar{\delta}_1|^2-|\nu_2\bar{\alpha}_1+\zeta_4\bar{\alpha}_1|^2-|\nu_2\alpha_1-\eta_1\zeta_4|^2\\
&= |\mathbb{X}_1+\mathbb{X}_2|^2+|1-\mathbb{X}_1-\mathbb{X}_2|^2-|\nu_2\alpha_1-\eta_1\zeta_4|^2
\end{aligned}$$

This implies

$$|\mathbb{X}_1|^2|\mathbb{X}_3-1|^2-|\mathbb{X}_1+\mathbb{X}_2|^2+|1-\mathbb{X}_1-\mathbb{X}_2|^2=-|\nu_2\alpha_1-\eta_1\zeta_4|^2\le 0$$

Rearranging this gives the inequality we want. Further the above inequality is an equality if and only if

$$\nu_2\alpha_1-\eta_1\zeta_4=0,\ i.e.\ \frac{\nu_2}{\eta_1}=\frac{\zeta_4}{\alpha_1}.$$

This means either of the conditions (i), (ii) , (iii) given in the statement. This proves the proposition.

Platis [44] has proved a generalization of the above proposition for arbitrary rank 1 symmetric spaces of non-compact type and has applied it to derive Ptolemaean inequality on the boundary of a rank 1 symmetric space of non-compact type. Since we have restricted ourselves only to three dimensional complex hyperbolic geometry, our proof above is much simpler.

Corollary 5.2.2. *Let* $\mathbb{X}_1,\ \mathbb{X}_2,\ \mathbb{X}_3$ *be defined by 2.4.1, then* $2\Re(\mathbb{X}_1+\mathbb{X}_2)\ge 1$.

Proof. Since $\Re(\mathbb{X}_3)\le|\mathbb{X}_3|$,

$$\begin{aligned}
2\Re(\mathbb{X}_1+\mathbb{X}_2)-1 &\ge |\mathbb{X}_1|^2+|\mathbb{X}_2|^2-2|\mathbb{X}_1|^2\Re(\mathbb{X}_3)\\
&\ge |\mathbb{X}_1|^2+|\mathbb{X}_2|^2-2|\mathbb{X}_1|^2|\mathbb{X}_3|\\
&= |\mathbb{X}_1|^2+|\mathbb{X}_2|^2-2|\mathbb{X}_1||\mathbb{X}_2|\\
&= (|\mathbb{X}_1|-|\mathbb{X}_2|)^2\ge 0
\end{aligned}$$

In particular $2\Re(\mathbb{X}_1 + \mathbb{X}_2) \geq 1$

Proposition 5.2.3. *Let $z_1, \cdots, z_4$ be distinct points of $\partial \mathbf{H}^3_{\mathbb{C}}$ and let $\mathbb{X}_1,\ \mathbb{X}_2,\ \mathbb{X}_3$ denote the cross-ratios defined by 2.4.1. Suppose $\mathbb{X}_1$, $\mathbb{X}_2$ and $\mathbb{X}_3$ are non-real complex numbers. Let $\mathbb{A}_1 = \mathbb{A}(z_4,\ z_3,\ z_2)$ and $\mathbb{A}_2 = \mathbb{A}(z_3,\ z_2, z_1)$. Then*

1. $\mathbb{A}_1 + \mathbb{A}_2 = arg(\overline{\mathbb{X}}_1 \mathbb{X}_2)$.
2. $\mathbb{A}_1 - \mathbb{A}_2 = arg(\mathbb{X}_3)$.

Note that the above proposition is not true if $\mathbb{X}_i$'s are real numbers. Cuhna-Gusevskii [13, p.279] have given a counter-example to the above proposition when $\mathbb{X}_i$'s are real numbers. However, when all the cross-ratios are non-real complex numbers, the argument as in the proof of [41, Proposition 5.8] goes through and we have the above proposition. An explanation that the proof of [41, Proposition 5.8] does not carry over to the real cross ratio case is that the principal argument of complex numbers is a well-defined function from $\mathbb{C} - \{0\}$ to the semi-open interval $(-\pi, \pi]$. On the other hand, $\mathbb{A}_1 \pm \mathbb{A}_2$ are well-defined functions from distinct triple points on $\partial \mathbf{H}^n_{\mathbb{C}}$ onto the closed interval $[-\pi, \pi]$. So, the principal argument can not be identified with $\mathbb{A}_1 \pm \mathbb{A}_2$, especially on the boundary points of the intervals and those cases correspond when the cross ratios are real numbers.

Proposition 5.2.4. *Let $z_1,\ z_2,\ z_3,\ z_4$ be distinct points of $\partial \mathbf{H}^3_{\mathbb{C}}$ with non-real cross ratios $\mathbb{X}_1,\ \mathbb{X}_2,\ \mathbb{X}_3$. Let $z_1',\ z_2',\ z_3',\ z_4'$ be another set of distinct points of $\partial \mathbf{H}^3_{\mathbb{C}}$ with corresponding cross ratios $\mathbb{X}_1',\ \mathbb{X}_2',\ \mathbb{X}_3'$. If $\mathbb{X}_i' = \mathbb{X}_i$ for $i = 1,\ 2,\ 3$, then there exist $A \in \mathrm{SU}(3,1)$ such that $A(z_j) = z_j'$ for $j = 1,\ 2,\ 3,\ 4$.*

Proof. Since $\mathrm{SU}(3,1)$ acts doubly transitively on $\partial \mathbf{H}^3_{\mathbb{C}}$, wlog we may assume $z_2 = z_2' = \infty,\ z_3 = z_3' = o$. We write the lifts of other points as

$$\mathbf{z}_1 = \begin{bmatrix} \xi_1 \\ \eta_1 \\ \alpha_1 \\ \delta_1 \end{bmatrix},\ \mathbf{z}_4 = \begin{bmatrix} \xi_4 \\ \nu_2 \\ \zeta_4 \\ \delta_4 \end{bmatrix}.\ \mathbf{z}_1' = \begin{bmatrix} \xi_1' \\ \eta_1' \\ \alpha_1' \\ \delta_1' \end{bmatrix},\ \mathbf{z}_4' = \begin{bmatrix} \xi_4' \\ \nu_2' \\ \zeta_4' \\ \delta_4' \end{bmatrix}.$$

We may suppose that lifts of these points are chosen so that $\langle \mathbf{z}_4,\ \mathbf{z}_1 \rangle = \langle \mathbf{z}_4',\ \mathbf{z}_1' \rangle$, i.e

$$\bar{\xi}_1 \delta_4 + \xi_4 \bar{\delta}_1 + \nu_2 \bar{\eta}_1 + \zeta_4 \bar{\alpha}_1 = \bar{\xi}_1' \delta_4' + \xi_4' \bar{\delta}_1' + \nu_2' \bar{\eta}_1' + \zeta_4' \bar{\alpha}_1'.$$

Then our condition on the cross-ratios gives :

$$\bar{\xi}_1\delta_4 = \bar{\xi}'_1\delta'_4,$$

$$\xi_4\bar{\delta}_1 = \xi'_4\bar{\delta}'_1,$$

$$\frac{\xi_4\delta_1}{\xi_1\delta_4} = \frac{\xi'_4\delta'_1}{\xi'_1\delta'_4}.$$

Hence we also have

$$\nu_2\bar{\eta}_1 + \zeta_4\bar{\alpha}_1 = \nu'_2\bar{\eta}'_1 + \zeta'_4\alpha'_1 \tag{5.2.6}$$

Let us denote the angular invariants of the points by $\mathbb{A}_1 = \mathbb{A}(z_4,\ z_3,\ z_2)$, $\mathbb{A}_2 = \mathbb{A}(z_3,\ z_2,\ z_1)$, $\mathbb{A}'_1 = \mathbb{A}(z'_4,\ z'_3,\ z'_2)$, $\mathbb{A}'_2 = \mathbb{A}(z'_3,\ z'_2,\ z'_1)$. Using Proposition 5.2.3, we see that $\mathbb{A}_1 + \mathbb{A}_2 = \mathbb{A}'_1 + \mathbb{A}'_2$ and $\mathbb{A}_1 - \mathbb{A}_2 = \mathbb{A}'_1 - \mathbb{A}'_2$. Hence $\mathbb{A}_1 = \mathbb{A}'_1$ and $\mathbb{A}_2 = \mathbb{A}'_2$. From Theorem 2.4.2, we see that there exist $A_1,\ A_2 \in \mathrm{SU}(3,1)$ such that $A_1(z_2) = z'_2, A_1(z_3) = z'_3,\ A_1(z_4) = z'_4$ and $A_2(z_1) = z'_1,\ A_2(z_2) = z'_2,\ A_2(z_3) = z'_3$. Because A_1 fixes $z_2 = \infty$ and $z_3 = 0$, it is of form

$$\begin{pmatrix} \lambda & & \\ & U_1 & \\ & & \bar{\lambda}^{-1} \end{pmatrix}$$

where $|\lambda| \neq 1$ and $U_1 \in \mathrm{U}(2)$. Hence we have $\lambda\xi_4 = \xi'_4,\ \bar{\lambda}^{-1}\delta_4 = \delta'_4$ and $U_1 \begin{bmatrix} \nu_2 \\ \zeta_4 \end{bmatrix} = \begin{bmatrix} \nu'_2 \\ \zeta'_4 \end{bmatrix}$. Therefore

$$\begin{aligned} \xi'_1 &= \frac{\bar{\delta}_4}{\bar{\delta}'_4}\xi_1 \\ &= \lambda\xi_1 \\ \delta'_1 &= \delta_1\frac{\bar{\xi}_4}{\bar{\xi}'_4}\delta_1 \\ &= \bar{\lambda}^{-1}\delta_1. \end{aligned}$$

Hence A_2 is of form

$$\begin{pmatrix} \lambda & & \\ & U_2 & \\ & & \bar{\lambda}^{-1} \end{pmatrix}$$

where, $U_2 \in \mathrm{U}(2)$ so that

$$U_2 \begin{bmatrix} \eta_1 \\ \alpha_1 \end{bmatrix} = \begin{bmatrix} \eta_1' \\ \alpha_1' \end{bmatrix}.$$

It is enough to prove that there exist $U \in \mathrm{U}(2)$ such that

$$U \begin{bmatrix} \nu_2 \\ \zeta_4 \end{bmatrix} = \begin{bmatrix} \nu_2' \\ \zeta_4' \end{bmatrix} \text{ and } U \begin{bmatrix} \eta_1 \\ \alpha_1 \end{bmatrix} = \begin{bmatrix} \eta_1' \\ \alpha_1' \end{bmatrix}.$$

Let us denote by $\mathbf{y_1} = \begin{bmatrix} \eta_1 \\ \alpha_1 \end{bmatrix}$, $\mathbf{y_4} = \begin{bmatrix} \nu_2 \\ \zeta_4 \end{bmatrix}$, $\mathbf{y_1}' = \begin{bmatrix} \eta_1' \\ \alpha_1' \end{bmatrix}$, $\mathbf{y_4}' = \begin{bmatrix} \nu_2' \\ \zeta_4' \end{bmatrix}$.
From (5.2.6), we have

$$\ll \mathbf{y_4},\ \mathbf{y_1} \gg = \ll \mathbf{y_4}',\ \mathbf{y_1}' \gg \tag{5.2.7}$$

where $\ll .,. \gg$ is the standard positive-definite Hermitian form on $\mathbb{C}^2$. Also we have $U_1\mathbf{y_4} = \mathbf{y_4}'$ and $U_2\mathbf{y_1} = \mathbf{y_1}'$. Then $U_1,\ U_2 \in \mathrm{U}(2)$ implies

$$\ll \mathbf{y_4}, \mathbf{y_4} \gg = \ll \mathbf{y_4}',\ \mathbf{y_4}' \gg \tag{5.2.8}$$

$$\ll \mathbf{y_1}, \mathbf{y_1} \gg = \ll \mathbf{y_1}',\ \mathbf{y_1}' \gg \tag{5.2.9}$$

Suppose $\mathbf{y_1}$ and $\mathbf{y_4}$ are linearly independent over $\mathbb{C}$ and so forms a basis of $\mathbb{C}^2$. Let U be the 2×2 matrix so that $U\mathbf{y_1} = \mathbf{y_1}'$ and $U\mathbf{y_4} = \mathbf{y_4}'$. Then from (5.2.7) – (5.2.9) it follows that U preserves the Hermitian form $\ll .,. \gg$ on $\mathbb{C}^2$, so $U \in \mathrm{U}(2)$ and we are done.

Now consider the case when $\mathbf{y_1}$ and $\mathbf{y_4}$ are linearly dependent over $\mathbb{C}$ i.e. $\mathbf{y_4} = \mu\mathbf{y_1}$ for some $\mu \in \mathbb{C}$. Then since the form $\ll .,. \gg$ is positive definite and using 5.2.7–5.2.9, this is true if and only if

$$\begin{aligned} &\ll \mathbf{y_4} - \mu\mathbf{y_1}, \mathbf{y_4} - \mu\mathbf{y_1} \gg = \mathbf{0} \\ \Leftrightarrow \quad &\ll \mathbf{y_4}' - \mu\mathbf{y_1}', \mathbf{y_4}' - \mu\mathbf{y_1}' \gg = \mathbf{0} \\ \Leftrightarrow \quad &\mathbf{y_4}' = \mu\mathbf{y_1}' \end{aligned}$$

Therefore either of U_1 and U_2 works and this completes the proof.

When cross ratios are all real

Suppose all the three cross-ratios are real. Then (5.2.1) implies $\mathbb{X}_3 = \pm\mathbb{X}_2/\mathbb{X}_1$. The following result can be proved along the same line as in the proof of [41, Proposition 5.12].

Lemma 5.2.5. *Suppose* $\mathbb{X}_1$, $\mathbb{X}_2$ *and* $\mathbb{X}_3$ *are all real.*

1. *If* $\mathbb{X}_3 = -\mathbb{X}_2/\mathbb{X}_1$, *then the points* z_j *all lie on a chain.*

2. *If* $\mathbb{X}_3 = \mathbb{X}_2/\mathbb{X}_1$, *then the points* z_j *all lie in a totally real Lagrangian subspace.*

The following result follows from [18, p.225].

Lemma 5.2.6. *Suppose* z_1, z_2, z_3 *and* z_4 *all lie on the same chain. Then* $\mathbb{X}_1$, $\mathbb{X}_2$ *and* $\mathbb{X}_3$ *are each real.*

Lemma 5.2.7. *If* z_1, z_2, z_3, z_4 *are contained in the same totally real totally geodesic subspace, then* $\mathbb{X}_1$, $\mathbb{X}_2$. $\mathbb{X}_3$ *are real numbers.*

Proof. Let ι be the anti-holomorphic involution fixing the totally real totally geodesic subspace. Then for $i = 1, 2, 3$, applying ι we get $\mathbb{X}_i = \overline{\mathbb{X}}_i$. Hence all the cross-ratios are real.

Summarizing the above lemmas we have the following.

Proposition 5.2.8. *Let* z_1, z_2, z_3, z_4 *are distinct points on* $\partial\mathbf{H}^3_{\mathbb{C}}$. *Then the cross ratios* $\mathbb{X}_1$, $\mathbb{X}_2$ *and* $\mathbb{X}_3$ *are real numbers if and only if* z_1, z_2, z_3, z_4 *all lie on the same chain or the same totally real totally geodesic subspace.*

5.3 A sufficient condition for irreducibility

Let A, B be loxodromic elements in $\mathrm{SU}(3,1)$ and following the notation of Section 5.1, let

$$C_A = \begin{bmatrix} \mathbf{a}_A & \mathbf{x}_A & \mathbf{y}_A & \mathbf{r}_A \end{bmatrix}, \; C_B = \begin{bmatrix} \mathbf{a}_B & \mathbf{x}_B & \mathbf{y}_B & \mathbf{r}_B \end{bmatrix}$$

be the eigen matrices associated with A and B respectively. The Koranyi-Riemann cross-ratios of A and B are defined by

$$\mathbb{X}_1(A,\ B) = [a_B,\ a_A,\ r_A,\ r_B] = \frac{\langle \mathbf{r}_A, \mathbf{a}_B \rangle \langle \mathbf{r}_B, \mathbf{a}_A \rangle}{\langle \mathbf{r}_B, \mathbf{a}_B \rangle \langle \mathbf{r}_A, \mathbf{a}_A \rangle} \tag{5.3.1}$$

$$\mathbb{X}_2(A,\ B) = [a_B,\ r_A,\ a_A,\ r_B] = \frac{\langle \mathbf{a}_A, \mathbf{a}_B \rangle \langle \mathbf{r}_B, \mathbf{r}_A \rangle}{\langle \mathbf{r}_B, \mathbf{a}_B \rangle \langle \mathbf{a}_A, \mathbf{r}_A \rangle} \tag{5.3.2}$$

$$\mathbb{X}_3(A,\ B) = [a_A,\ r_A,\ a_B,\ r_B] = \frac{\langle \mathbf{a}_B, \mathbf{a}_A \rangle \langle \mathbf{r}_B, \mathbf{r}_A \rangle}{\langle \mathbf{r}_B, \mathbf{a}_A \rangle \langle \mathbf{a}_B, \mathbf{r}_A \rangle} \tag{5.3.3}$$

In [18] , Goldman defines η-invariant for a triple of points with two points on $\partial \mathbf{H}^3_{\mathbb{C}}$ and one point on $\mathbb{P}(V_+)$. Following Goldman's definition, we define η-invariants associated to A and B as follows

$$\eta_1(A, B) = \eta(a_A, r_A; x_B) = \frac{\langle \mathbf{a}_A, \mathbf{x}_B \rangle \langle \mathbf{x}_B, \mathbf{r}_A \rangle}{\langle \mathbf{a}_A, \mathbf{r}_A \rangle \langle \mathbf{x}_B, \mathbf{x}_B \rangle}$$

$$\eta_2(A, B) = \eta(a_A, r_A; y_B) = \frac{\langle \mathbf{a}_A, \mathbf{y}_B \rangle \langle \mathbf{y}_B, \mathbf{r}_A \rangle}{\langle \mathbf{a}_A, \mathbf{r}_A \rangle \langle \mathbf{y}_B, \mathbf{y}_B \rangle}$$

$$\nu_1(A, B) = \eta(a_B, r_B; x_A) = \frac{\langle \mathbf{a}_B, \mathbf{x}_A \rangle \langle \mathbf{x}_A, \mathbf{r}_B \rangle}{\langle \mathbf{a}_B, \mathbf{r}_B \rangle \langle \mathbf{x}_A, \mathbf{x}_A \rangle}$$

$$\nu_2(A, B) = \eta(a_B, r_B; y_A) = \frac{\langle \mathbf{a}_B, \mathbf{y}_A \rangle \langle \mathbf{y}_A, \mathbf{r}_B \rangle}{\langle \mathbf{a}_B, \mathbf{r}_B \rangle \langle \mathbf{y}_A, \mathbf{y}_A \rangle}$$

We define

$$\zeta_o(A, B) = [y_A,\ x_A,\ x_B,\ y_B] = \frac{\langle \mathbf{x}_B, \mathbf{y}_A \rangle \langle \mathbf{y}_B, \mathbf{x}_A \rangle}{\langle \mathbf{x}_B, \mathbf{x}_A \rangle \langle \mathbf{y}_B, \mathbf{y}_A \rangle}$$

It is clear from the definition that the $\mathbb{X}_i$'s, η_j's and ζ_o are conjugacy invariants for the two generator subgroup $\langle A, B \rangle$ of $\mathrm{SU}(3,1)$ and their values are independent of the chosen lifts of eigenvectors.

Theorem 5.3.1. *Let $\langle A, B \rangle$ be a discrete, free subgroup of $\mathrm{SU}(3,1)$ that is generated by two loxodromic elements A and B. Then $\langle A, B \rangle$ preserves a $\mathbb{C}^2$-plane if and only if one of the following holds.*

(i) $\zeta_o = 0$ and, either $\eta_1(A, B) = 0 = \nu_1(A, B)$ or $\eta_2(A, B) = 0 = \nu_2(A, B)$.

(ii) $\zeta_o = \infty$ and, either $\eta_1(A, B) = 0 = \nu_2(A, B)$ or $\eta_2(A, B) = 0 = \nu_1(A, B)$.

Proof. Note that a two dimensional totally geodesic subspace of $\mathbf{H}^3_{\mathbb{C}}$ corresponds to a copy of $\mathbb{C}^{2,1}$.

The condition is necessary. Suppose $\langle A, B\rangle$ preserve a copy of $\mathbb{C}^{2,1}$. Observe that $\langle A, B\rangle$ preserve a copy of $\mathbb{C}^{2,1}$ if and only if A and B have a common space-like eigenvector. Thus, either of the following cases arises:

(a) $x_A = x_B$

(b) $y_A = y_B$

(c) $y_A = x_B$

(d) $x_A = y_B$

The result follows from the definition of $\eta_i(A, B)$'s and $\zeta_o(A, B)$.

The condition is sufficient. Suppose $\zeta_o = 0$. We discuss the case (i) i.e. let

$$\eta_1(A, B) = 0 = \nu_1(A, B) = 0 = \zeta_o(A, B).$$

We claim that $x_A = x_B$. We have

$$\langle \mathbf{a}_A, \mathbf{x}_B\rangle\langle \mathbf{x}_B, \mathbf{r}_A\rangle = 0$$

$$\langle \mathbf{a}_B, \mathbf{x}_A\rangle\langle \mathbf{x}_A, \mathbf{r}_B\rangle = 0$$

$$\langle \mathbf{x}_B, \mathbf{y}_A\rangle\langle \mathbf{y}_B, \mathbf{x}_A\rangle = 0$$

Different subcases arises, it is enough to consider the following subcase

$$\langle \mathbf{a}_A, \mathbf{x}_B\rangle = 0,\ \langle \mathbf{a}_B, \mathbf{x}_A\rangle = 0,\ \langle \mathbf{y}_B, \mathbf{x}_A\rangle = 0. \tag{5.3.4}$$

Since, $\{\mathbf{a}_B,\ \mathbf{x}_B,\ \mathbf{y}_B,\ \mathbf{r}_B\}$ is a basis for $\mathbb{C}^{3,1}$, hence there exists scalars $\mu_1,\ \mu_2,\ \mu_3,\ \mu_4$ such that

$$\mathbf{x}_A = \mu_1\mathbf{a}_B + \mu_2\mathbf{x}_B + \mu_3\mathbf{y}_B + \mu_4\mathbf{r}_B.$$

The conditions $\langle \mathbf{a}_B, \mathbf{x}_A\rangle = 0 = \langle \mathbf{y}_B, \mathbf{x}_A\rangle$ implies $\mu_3 = 0 = \mu_4$. Hence

$$\mathbf{x}_A = \mu_1\mathbf{a}_B + \mu_2\mathbf{x}_B.$$

This implies

$$0 = \langle \mathbf{x}_A, \mathbf{a}_A \rangle = \mu_1 \langle \mathbf{a}_B, \mathbf{a}_A \rangle + \mu_2 \langle \mathbf{x}_B, \mathbf{a}_A \rangle.$$

Using (5.3.4) we have $\mu_1 \langle \mathbf{a}_B, \mathbf{a}_A \rangle = 0$. Since $\langle \mathbf{a}_B, \mathbf{a}_A \rangle \neq 0$, we have $\mu_1 = 0$. Hence $\mathbf{x}_A = \mu_2 \mathbf{x}_B$ i.e. $x_B = x_A$, proving the result for the case (i). The argument in the other cases are similar.

Note that if $\zeta_o = \infty$, then $1/\zeta_o = 0$ and similar arguments work in these cases also.

The subgroup $\langle A, B \rangle$ of $\mathrm{SU}(3,1)$ is called *irreducible* or *Zariski-dense* if it does not preserves a totally geodesic subspace of $\mathbf{H}^3_{\mathbb{C}}$. Using the above theorem and the results on cross ratios, it is possible to derive many conditions for irreducibility of $\langle A, B \rangle$. As a special case we have the following.

Corollary 5.3.2. *Let A and B be two loxodromic elements in* $\mathrm{SU}(3,1)$ *such that $\langle A, B \rangle$ is non-singular. Then $\langle A, B \rangle$ is irreducible.*

5.4 Classification of non-singular pair of loxodromics

In this section we follow the notations from Section 5.1. First we shall show that for a non-singular pair (A, B) one can always get a well-defined α-invariant and a well-defined β-invariant.

5.4.1 α and β-invariants are well-defined

Let A and B be two loxodromics such that they form a non-singular pair. Without loss of generality, we can assume A is a diagonal matrix, that is $C_A = [\mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3}, \mathbf{e_4}]$, where $\{\mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3}, \mathbf{e_4}\}$ is the standard orthonormal basis of $\mathbb{C}^{3,1}$. Let $B = C_B E(\lambda, \psi) C_B^{-1}$, where $C_B = [\mathbf{a}_B, \mathbf{x}_B, \mathbf{y}_B, \mathbf{r}_B]$. Let

$$\mathbf{a}_B = \begin{bmatrix} a \\ e \\ j \\ n \end{bmatrix}, \ \mathbf{x}_B = \begin{bmatrix} b \\ f \\ k \\ s \end{bmatrix}, \ \mathbf{y}_B = \begin{bmatrix} c \\ g \\ l \\ p \end{bmatrix}, \ \mathbf{r}_B = \begin{bmatrix} d \\ h \\ m \\ q \end{bmatrix}.$$

Now we see that

$$\alpha_1(A, B) = \frac{nb}{as}, \quad \alpha_2(A, B) = \frac{nc}{ap},$$

$$\beta_1(A,B) = \frac{\bar{n}\bar{h}}{\bar{q}\bar{e}}, \quad \beta_2(A,B) = \frac{\bar{n}\bar{m}}{\bar{q}\bar{j}}.$$

Since $\mathbf{a}_B$ and $\mathbf{r}_B$ are negative vectors, we must have a, n and q non-zeros. Now note that

$$\langle \mathbf{a}_A, \mathbf{x}_B \rangle = b, \ \langle \mathbf{r}_A, \mathbf{x}_B \rangle = s, \tag{5.4.1}$$

$$\langle \mathbf{a}_A, \mathbf{y}_B \rangle = c, \ \langle \mathbf{r}_A, \mathbf{y}_B \rangle = p, \tag{5.4.2}$$

$$\langle \mathbf{a}_B, \mathbf{x}_A \rangle = e, \ \langle \mathbf{r}_B, \mathbf{x}_A \rangle = h, \tag{5.4.3}$$

$$\langle \mathbf{a}_B, \mathbf{y}_A \rangle = j, \ \langle \mathbf{r}_B, \mathbf{y}_A \rangle = m. \tag{5.4.4}$$

It follows from condition (iii) of the definition of non-singularity that neither of $\mathbf{a}_A$ and $\mathbf{r}_A$ belong to at least one of the $\mathbb{C}^2$-chains $\mathbf{x}_B^{\perp}$ and $\mathbf{y}_B^{\perp}$ and also, neither of $\mathbf{a}_B$ and $\mathbf{r}_B$ belong to one of the $\mathbb{C}^2$-chains $\mathbf{x}_A^{\perp}$ and $\mathbf{y}_A^{\perp}$. Thus, at least one of the equations (5.4.1) and (5.4.2) must have entirely non-zero solution. Similarly, the solution of one of the equations (5.4.3) and (5.4.3) will also be entirely non-zero. Thus at least one α-invariant and one β-invariant are always well-defined complex numbers for a non-singular pair of loxodromics.

It can further be seen from the definition of Goldman's eta invariants that the well-definedness of α-invariant and β-invariant can be stated equivalently by saying that for some $i, j \in \{1, 2\}$, $\eta_i(A,B) \neq 0$ and $\nu_j(A,B) \neq 0$.

Lemma 5.4.1. *Let A, B, A', B' be loxodromic elements in $\mathrm{SU}(3,1)$. Let $\langle A, B \rangle$ be a non-singular subgroup in $\mathrm{SU}(3,1)$ such that for some $i, j \in \{1,2\}$, $\eta_i(A,B) \neq 0$ and $\nu_j(A,B) \neq 0$. Suppose $\alpha_i(A,B) = \alpha_i(A',B')$, $\beta_j(A,B) = \beta_j(A',B')$ and, for $k = 1,2,3$, $\mathbb{X}_k(A,\ B) = \mathbb{X}_k(A',\ B')$. Then there exist $C \in \mathrm{SU}(3,1)$ such that $C(a_A) = a_{A'}, C(x_A) = x_{A'},\ C(y_A) = y_{A'},\ C(r_A) = r_{A'}$ and $C(a_B) = a_{B'}, C(x_B) = x_{B'},\ C(y_B) = y_{B'},\ C(r_B) = r_{B'}$.*

Proof. We shall prove the lemma assuming that $(i,j) = (1,1)$. The rest of the cases are similar.

Since $\mathbb{X}_i(A,\ B) = \mathbb{X}_i(A',\ B')$, $i = 1,2,3$, by Proposition 5.2.4 it follows that there exist $C \in \mathrm{SU}(3,1)$ such that $a_{A'} = C(a_A)$, $r_{A'} = C(r_A)$, $a_{B'} = C(a_B)$ and $r_{B'} = C(r_B)$.

Since $\alpha_1(A',\ B') = \alpha_1(A,\ B)$, we have

$$\begin{aligned}\frac{\langle \mathbf{x}_B, \mathbf{r}_A\rangle\langle \mathbf{a}_B, \mathbf{a}_A\rangle}{\langle \mathbf{a}_B, \mathbf{r}_A\rangle\langle \mathbf{x}_B, \mathbf{a}_A\rangle} &= \frac{\langle \mathbf{x}'_B, \mathbf{r}'_A\rangle\langle \mathbf{a}'_B, \mathbf{a}'_A\rangle}{\langle \mathbf{a}'_B, \mathbf{r}'_A\rangle\langle \mathbf{x}'_B, \mathbf{a}'_A\rangle} \\ &= \frac{\langle C^{-1}(\mathbf{x}'_B), \mathbf{r}_A\rangle\langle \mathbf{a}_B, \mathbf{a}_A\rangle}{\langle \mathbf{a}_B, \mathbf{r}_A\rangle\langle C^{-1}(\mathbf{x}'_B), \mathbf{a}_A\rangle} \\ \implies \frac{\langle \mathbf{x}_B, \mathbf{r}_A\rangle}{\langle C^{-1}(\mathbf{x}'_B), \mathbf{r}_A\rangle} &= \frac{\langle \mathbf{x}_B, \mathbf{a}_A\rangle}{\langle C^{-1}(\mathbf{x}'_B), \mathbf{a}_A\rangle}\end{aligned}$$

Let

$$\lambda = \frac{\langle \mathbf{x}_B, \mathbf{r}_A\rangle}{\langle C^{-1}(\mathbf{x}'_B), \mathbf{r}_A\rangle} = \frac{\langle \mathbf{x}_B, \mathbf{a}_A\rangle}{\langle C^{-1}(\mathbf{x}'_B), \mathbf{a}_A\rangle}$$

This implies

$$\langle \mathbf{x}_B - \lambda C^{-1}(\mathbf{x}'_B), \mathbf{r}_A\rangle = 0. \tag{5.4.5}$$

$$\langle \mathbf{x}_B - \lambda C^{-1}(\mathbf{x}'_B), \mathbf{a}_A\rangle = 0. \tag{5.4.6}$$

On the other hand, note that

$$\langle \mathbf{x}_B - \lambda C^{-1}(\mathbf{x}'_B), \mathbf{r}_B\rangle = \langle \mathbf{x}_B, \mathbf{r}_B\rangle - \overline{\lambda}\langle C^{-1}(\mathbf{x}'_B) - \mathbf{r}_B\rangle = 0 - \overline{\lambda}\langle \mathbf{x}'_B, \mathbf{r}'_B\rangle = 0. \tag{5.4.7}$$

Similarly,

$$\langle \mathbf{x}_B - \lambda C^{-1}(\mathbf{x}'_B), \mathbf{a}_B\rangle = 0. \tag{5.4.8}$$

Let L_A and L_B denote the two-dimensional time-like subspaces of $\mathbb{C}^{3,1}$ that represent the complex axes of A and B respectively. Thus $\{\mathbf{a}_A, \mathbf{r}_A\}$ and $\{\mathbf{a}_B, \mathbf{r}_B\}$ are the respective bases of L_A and L_B.

It follows from (5.4.5) – (5.4.8) that $v = \mathbf{x}_B - \lambda C^{-1}(\mathbf{x}'_B)$ is orthogonal to to both L_A and L_B. We must have $\langle v, v\rangle > 0$. Thus v is polar to the $\mathbb{C}^2$-chain (copy of $\mathbf{H}^2_{\mathbb{C}}$) that is represented by $\mathrm{V} = v^{\perp}$. Since $\mathbb{C}^{3,1} = \mathrm{V} \oplus \mathbb{C}v$, hence L_A and L_B must be subsets in V. Thus the fixed points of A and B belongs to boundary of the $\mathbb{C}^2$-chain $\mathbb{P}(\mathrm{V})$. This is a contradiction to the non-singularity of (A, B). Hence we must have $v = 0$, that is $C(\mathbf{x}_B) = \lambda \mathbf{x}'_B$. Thus, $C(x_B) = x'_B$. Consequently, $C(y_B) = y'_B$.

Similarly $\beta_1(A, B) = \beta_1(A', B')$ implies $C(x_A) = x'_A$ and hence $C(y_A) = y'_A$. This proves the lemma.

Theorem 5.4.2. *Let A, B be two loxodromic elements in* $\mathrm{SU}(3, 1)$ *such that they gen-*

erate a non-singular subgroup $\langle A, B\rangle$. Then $\langle A, B\rangle$ is determined up to conjugacy by the following parameters:

$tr(A),\ tr(B),\ \sigma(A),\ \sigma(B),\ \mathbb{X}_k(A,B),\ k=1,2,3,$ *one non-zero* α*-invariant and one non-zero* β*-invariant, where* $tr(A) = trace(A)$, $\sigma(A) = \frac{1}{2}(tr^2(A) - tr(A^2))$.

Proof. Suppose that $A,\ B,\ A',\ B'$ are loxodromic elements such that

$$tr(A) = tr(A'),\ tr(B) = tr(B'),\ \sigma(A) = \sigma(A'),\ \sigma(B) = \sigma(B');$$

$\alpha_i(A,B) = \alpha_i(A',B'),\ \beta_j(A,B) = \beta_j(A',B')$ and for $k = 1,2,3,\ \mathbb{X}_k(A,\ B) = \mathbb{X}_k(A',\ B')$.

Following the notation in Section 5.1, $A = C_A E_A C_A^{-1},\ B = C_B E_B C_B^{-1}$ and similarly for A' and B'. Since the cross-ratios are equal, by Lemma 5.4.1 it follows that there exist $C \in \mathrm{SU}(3,1)$ such that $C(a_A) = a_{A'}, C(x_A) = x_{A'},\ C(y_A) = y_{A'},\ C(r_A) = r_{A'}$ and $C(a_B) = a_{B'}, C(x_B) = x_{B'},\ C(y_B) = y_{B'},\ C(r_B) = r_{B'}$. Therefore CAC^{-1} and A' have same eigenvectors. Since $tr(A') = tr(CAC^{-1}),\ \sigma(A') = \sigma(CAC^{-1})$, by Corollary 5.1.3 and Proposition 5.1.2, we must have $CAC^{-1} = A'$. Similarly, $B' = CBC^{-1}$. Thus $\langle A',\ B'\rangle = \langle CAC^{-1},\ CBC^{-1}\rangle = C\langle A,\ B\rangle C^{-1}$ as claimed.

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