Dimension Subgroups and Prime Power Groups

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Certificate of Examination

This is to certify that the dissertation entitled "**Dimension Subgroups and Prime Power Groups**", submitted by **Debanjana Kundu** (Reg. No. MS10017) for the partial fulfillment of BS-MS dual degree programme of the Institute, has been examined by the thesis committee duly appointed by the Institute. The committee finds the work done by the candidate satisfactory and recommends that the report be accepted.

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Declaration

The work presented in this dissertation has been carried out by me under the guidance of Prof. I.B.S. Passi at the Indian Institute of Science Education and Research Mohali.

This work has not been submitted in part or in full for a degree, a diploma, or a fellowship to any other university or institute. Whenever contributions of others are involved, every effort is made to indicate this clearly, with due acknowledgement of collaborative research and discussions. This thesis is a bonafide record of original work done by me and all sources listed within have been detailed in the bibliography.

Debanjana Kundu

24 April 2015

In my capacity as the supervisor of the candidate's project work, I certify that the above statements by the candidate are true to the best of my knowledge.

Inder Bir Singh Passi
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Debanjana Kundu

Contents

Introduction

The main aim of my thesis is to review the major developments in the area of integral and modular dimension subgroups and study some of their applications.

One of the fundamental objects of study in group theory is the lower central series. Magnus [Mag35] was one of the first to investigate the lower central series of free groups. To recall his approach, let F be a free group ring with basis $X = \{x_i\}_{i \in I}$. Let $\mathcal{A} = \mathbb{Z}[[X_i]]$ be the ring of formal power series and $\mathcal{U}(\mathcal{A})$ the group of units of $\mathcal{A}$. Clearly, $1 + X_i$ is an invertible element with the inverse as $1 - X_i + X_i^2 - \dots$. The map $x_i \mapsto 1 + X_i$ extends to a homomorphism $\theta : F \to \mathcal{U}(\mathcal{A})$. It can be shown that θ is actually a monomorphism [MKS76, Chapter 5].

For $a \in \mathcal{A}$, let a_n be the homogeneous components of degree n so that

$$a = a_0 + a_1 + \ldots + a_n + \ldots.$$

Magnus defined dimension subgroups, $\mathcal{D}_n(F), n \geq 1$, as follows

$$\mathcal{D}_n(F) := \{f \in F \,|\, \theta(f) = 1 + \theta(f_n) + \theta(f_{n+1}) + \ldots\}. \tag{1}$$

These subgroups are normal subgroup with the property that $(F, \mathcal{D}_n(F)) \subseteq \mathcal{D}_{n+1}(F)$ for all $n \geq 1$ where for M, N subgroups of the group, G, we define (M, N) to be the subgroup generated by the commutators $(m, n) = m^{-1}n^{-1}mn$ for $m \in M$ and $n \in N$, i.e.,

$$(M, N) = \langle (m, n) = m^{-1}n^{-1}mn | m \in M \text{ and } n \in N \rangle. \tag{2}$$

Let $\mathfrak{f}$ be the augmentation ideal of $\mathbb{Z}[F]$. Define $D_n(F) = G \cap (1 + \mathfrak{f}^n)$. For free groups, it is easy to see that $\gamma_n(F) \subseteq D_n(F) \subseteq \mathcal{D}_n(F)$ for all $n \geq 1$.

The homomorphism θ can be extended to a monomorphism $\Theta : \mathbb{Z}[F] \to \mathcal{U}(\mathcal{A})$. Under this map, $\alpha \in \mathfrak{f}^n$ maps to an element where $\theta(\alpha)_i = 0$ for all $i \leq n - 1$. From the work of Grün [Gru36], Magnus [Mag37] and Witt [Wit37] it follows that the above inclusions are actually equalities i.e.,

$$\gamma_n(F) = D_n(F) = \mathcal{D}_n(F) \text{ for all } n \geq 1. \tag{3}$$

The above result gives a close relation between the lower central series and the dimension series. It was only natural to conjecture that, for any group G, the lower central series and the dimension series coincide. It was in 1972 that E. Rips [Rip72] settled this conjecture by giving a counter-example.

In the first chapter, we study integral dimension subgroups. We see that the first three terms of the integral dimension series and the lower central series of an arbitrary group coincide; however, beyond that the equality does not hold in general. We study the structure of the fourth [Tah77] and the fifth dimension subgroups [Tah81] in some detail. We also study some of the counter-examples given by Gupta [Gup90]. In the second part of this chapter we focus on dimension subgroups over fields.

In the second chapter, we study the Lie dimension subgroups, $D_{(n)}[G]$ and $D_{[n]}(G)$ for $n \geq 1$. We see that $\gamma_n(G) \subseteq D_{[n]}(G) \subseteq D_{(n)}(G) \subseteq D_n(G)$. We explore the Lie dimension subgroups in some detail to realize that more definitive results are known about them. We also discuss the identification of Lie dimension subgroups over fields as given by Passi and Sehgal [PS75].

In the last chapter, we study powerful p-groups which were introduced by Lubotzky and Mann [LM87]. These can be thought of as generalization of Abelian groups. Shalev [Sha90] introduced a double-indexed series, $\{D_{m,k}\}$, which we study in some detail. We focus on some of its properties and see how these are related to dimension subgroups [SS91]. We discuss how powerful and potent p-groups help us understand the power structure of p groups [Wil03]. Extensive work has been done in this area by A. Shalev [Sha90] [Sha91], C. Scoppola [Sco91] and others.

Chapter 1

Dimension Subgroups

In this chapter we study integral dimension subgroups, $D_{n,R}(G)$ when R is the ring $\mathbb{Z}$ of integers or a field.

1.1 Integral Dimension Subgroups

1.1.1 Preliminaries

Definition 1.1.1. Let G be any group, and R be a ring with identity. The set of elements of the form $\sum_i r_i g_i$ where $r_i \in R$ (with only finitely many r_i's non-zero) and $g_i \in G$ is called **group ring** of G with respect to R and is denoted by $R[G]$ with addition and multiplication defined as follows:

1. $\sum_{g\in G} r_g g + \sum_{g\in G} r'_g g = \sum_{g\in G}(r_g + r'_g)g$,
2. $(\sum_{g\in G} a_g g)\cdot(\sum_{h\in G} b_h h) = \sum_{g,h\in G}(a_g b_h)(gh) = \sum_{x\in G}(\sum_{gh=x} a_g b_h)(x)$.

When $R = \mathbb{Z}$ we refer to the group ring $R[G]$ as the **integral group ring** of G.

Definition 1.1.2. The map $\varepsilon : R[G] \to R$ which maps $\sum_i r_i g_i$ to $\sum_i r_i$ is called the **augmentation map**; it is a ring homomorphism. The kernel of ε is a two-sided ideal, called the **augmentation ideal**. We denote it by $\Delta_R(G)$.

When $R = \mathbb{Z}$ we drop the subscript R and abbreviate the notation to $\Delta(G)$ or $\mathfrak{g}$.

As an R-module, $\Delta_R(G)$ is generated freely by the elements $g - 1_R$, $(g \in G)$.

Definition 1.1.3. The nth **dimension subgroup** of G over the ring, R with identity is defined as

$$D_{n,R}(G) = G \cap (1 + \Delta_R^n(G)) \qquad \text{for all} n \geq 1. \tag{1.1}$$

This is a normal subgroup with the property $(G, D_{n,R}(G)) \subseteq D_{n+1,R}(G)$. We thus have the following central series

$$G = D_{1,R}(G) \supseteq D_{2,R}(G) \supseteq \ldots \supseteq D_{n,R}(G) \supseteq \ldots \tag{1.2}$$

called the **dimension series** of G over R. Again, if $R = \mathbb{Z}$ we will write $D_n(G)$ rather than $D_{n,\mathbb{Z}}(G)$.

We denote by (a, b) the commutator, $a^{-1}b^{-1}ab$, of two elements a and b of a group, G. Note that we use simple brackets in place of square brackets; this is to avoid confusion with the Lie bracket notation introduced later. We use the notation $a^b = b^{-1}ab$.

Theorem 1.1.4. *[MKS76, Theorem 5.1] For any three elements $x, y, z \in G$*

1. $(x, y) \cdot (y, x) = 1$
2. $(x, y \cdot z) = (x, y) \cdot (x, z) \cdot ((x, y), z)$
3. $(x \cdot y, z) = (x, y) \cdot ((x, z), y) \cdot (y, z)$
4. $((x, y), z^x) \cdot ((z, x), y^z) \cdot ((y, z), x^y) = 1.$

Definition 1.1.5. Define subgroups $\gamma_n(G)$ inductively as follows

$$\gamma_1(G) = G; \ \ \gamma_{n+1}(G) = (G, \gamma_n(G)) \qquad \text{for all } n \geq 1. \tag{1.3}$$

The series

$$G = \gamma_1(G) \supseteq \gamma_2(G) \supseteq \ldots \supseteq \gamma_n(G) \supseteq \ldots \tag{1.4}$$

is referred to as the **lower central series** of G.

The *lower central series* is a fundamental object of study in the theory of groups.

We note that $\gamma_n(G) \subseteq D_n(G)$ for all groups, G, and for all $n \geq 1$.

1.1.2 Dimension Property

Definition 1.1.6. Let G be a group. The quotient $D_n(G)/\gamma_n(G), n \geq 1$ is called the nth (integral) **dimension quotient** of G. The group G is said to have the **dimension property** if all the dimension quotients are trivial, i.e., $D_n(G) = \gamma_n(G)$ for all $n \geq 1$.

For a free group F, $D_n(F) = \gamma_n(F)$ for all $n \geq 1$. This is the Magnus-Grün-Witt theorem [Gup87, I.3.7], also known as the **Fundamental Theorem of Free Group Rings** which exhibits a close relationship between the lower central series and the dimension subgroups of free groups.

It is this theorem that motivated the study of dimension subgroups. It was conjectured that $D_n(G) = \gamma_n(G)$ for all G and all n. This came to be known as the **dimension subgroup conjecture**. The origin of the dimension subgroup conjecture can be traced back to Grün [Gru36] who attributes it to Magnus.

For all $n \leq 3$, it is known $D_n(G) = \gamma_n(G)$ for all groups. The proof is elementary for the case $n = 2$. G. Higman and D. Rees had shown independently that for all groups $D_3(G) = \gamma_3(G)$. For the proof, the reader may refer to [Pas79, Chapter V, Section 5]. In fact, a stronger result holds.

Theorem 1.1.7. *[Pas79, V.5.9] For every group G,*

$$\gamma_3(G) = G \cap (1 + \Delta(G)^3 + \Delta(G)\Delta(\zeta_1(G))) \tag{1.5}$$

where $\zeta_1(G)$ is the center of the group G.

Passi [Pas68a] showed that $D_4(G) = \gamma_4(G)$ for any finite p-group (p odd). A counter-example to the triviality of the fourth dimension quotient was provided by Rips [Rip72]. Subsequently counter-examples for all $n \geq 4$ were given by Narain Gupta (see [Gup90]) by constructing for each $n \geq 4$ a finite 2-group, G_n, such that the $D_n(G_n) \neq \gamma_n(G_n)$.

It was observed by G. Higman that to study the obstruction to the triviality of dimension quotients it suffices to focus on prime power groups.

Theorem 1.1.8. ***Higman's Reduction*** *[Pas79, V.4.4] If G is a group such that $D_n(G) \neq \gamma_n(G)$, then there is a sub-quotient N of G such that*

1. *N is a finite p-group of class $\leq n - 1$.*
2. *$D_n(N) \neq 1$.*

Proof Let $x \in G$ be such that $x \in D_n(G)$ and $x \notin \gamma_n(G)$. We then have an equation

$$x - 1 = \sum r(g_1 - 1)(g_2 - 1) \ldots (g_n - 1)$$

where $g_k \in G$ and $r \in \mathbb{Z}$.

Let H be the subgroup of G (finitely) generated by the elements, $g_k \in G$, which are involved in the above expansion of $x-1$. Now, $x \in D_n(H)$ but $x \notin \gamma_n(H)$.

Define a quotient group $K = H/\gamma_n(H)$. Dimension subgroups are preserved under homomorphism, hence $\overline{x} = x\gamma_n(H) \in D_n(K)$. Also, $\overline{x} \notin \gamma_n(K) = 1$.

Now, note that K is a finitely generated nilpotent group and is hence a residually prime power group [Gru57, Theorem 2.1(i)]. Thus, there exists a normal subgroup L of K such that $\overline{x} \notin L$ and $K/L = N$ is a prime power group which is a quotient of a subgroup of G (by construction) such that $\overline{x}L \in D_n(N)$ and $\overline{x}L \notin \gamma_n(N) = 1$. Hence, the theorem is proved. □

Fourth Dimension Subgroup

We next take up the discussion of the fourth dimension quotient. We will first see a result of Passi [Pas68a]. We will look at a description of $D_4(G)/\gamma_4(G)$ given by Gupta using the approach of free groups. Our main will be to understand the structure of $D_4(G)/\gamma_4(G)$ given by Tahara [Tah77].

We state without proof a result which will be required in one of the approaches to investigate the fourth dimension subgroup.

Let $1 \to N \xrightarrow{i} G \to X \to 1$ be a short exact sequence where N is an Abelian group and i is an inclusion map. We use the notation $(N, nG) = (\dots((N, \underbrace{G), G), \dots G}_{\text{n times}})$.

Theorem 1.1.9. *[Pas79, V.5.1] A homomorphism $\alpha : N \to M$, M Abelian, can be extended to a map $\varphi : G \to M$ whose linear extension to $\mathbb{Z}[G]$ vanishes on $\Delta(G)^{n+1} + \Delta(G)\Delta(N)$ iff*

1. *There exists transversal $\{w(x)\}_{x \in X}$ with $w(1) = 1$, for X in G and a map $\chi : X \to M$, $\chi(1) = 0$, such that*

$$\alpha(W(x_1, (x_2 - 1)(x_3 - 1)\dots(x_n - 1))) = \chi((x_1 - 1)(x_2 - 1)\dots(x_{n+1} - 1)),$$

   $\{x_i\}_{i=1}^{n+1} \in X$ where $W(x_1, x_2) = w(x_1, x_2)^{-1}w(x_1)w(x_2) : X \times X \to N$ is the 2-cocycle determined by w. N is a right X-module via conjugation in G. By linearity, W is extended to $\mathbb{Z}[X] \times \mathbb{Z}[X]$ and χ is extended to $\mathbb{Z}[X]$.

2. *α vanishes on (N, nG).*

Theorem 1.1.10. *If G is a p-group, $p \neq 2$, then*

$$D_4(G) = \gamma_4(G). \tag{1.6}$$

The above theorem is a straight forward corollary of a more general result.

Theorem 1.1.11. *If G is a p-group for p odd, then*

$$\gamma_4(G) = G \cap (1 + \Delta^4(G) + \Delta(G) \cdot \Delta(\zeta_1(G))) \tag{1.7}$$

where $\zeta_1(G)$ is the center of G.

Proof Theorem 1.1.8 tells us that it is sufficient to give the proof for finite $p(\neq 2)$ groups of nilpotency class 3. Let us consider the group G with $\gamma_4(G) = 1$. We only have to show that RHS $\subseteq$ LHS as the other way inclusion is obvious. Let $x(\neq 1)$ be an element in $G \cap (1 + \Delta^4(G) + \Delta(G) \cdot \Delta(\zeta_1(G)))$.

Using theorem 1.1.7 we can say that $x \in \gamma_3(G) \subseteq \zeta_1(G)$. Let T be the additive group of rationals modulo 1. There exists a homomorphism $\alpha : \zeta_1(G) \to T$ with the property that x is not mapped to 0. The crux of the proof is that the homomorphism α can be extended to a map $\varphi : G \to T$ which when extended linearly to $\mathbb{Z}[G]$ vanishes on $\Delta^4(G) + \Delta(G) \cdot \Delta(\zeta_1(G))$. Therefore, we have

$$\alpha(x) = \varphi(x) = \varphi(x - 1) = 0. \tag{1.8}$$

But this is a contradiction.

Thus our assumption must be false, and $G \cap (1 + \Delta^4(G)) + \Delta(G) \cdot \Delta(\zeta_1(G))) = 1$. $\square$

Though $D_4(G)$ may not be equal to $\gamma_4(G)$ for all groups G, we have the following result which is the best known for the fourth dimension quotient.

Theorem 1.1.12. *For every group G, $D_4(G)/\gamma_4(G)$ has exponent at most 2.*

Proof Using theorem 1.1.8, it is sufficient to consider the case of finite 2-groups of nilpotency class 3. Let G be such a group. Let x be an element in $D_4(G)$ with $x^2 \neq 1$. Since $D_3(G) = \gamma_3(G)$ for all groups, we can conclude that $x \in \gamma_3(G)$. Furthermore, there exists a homomorphism $\alpha : \gamma_3(G) \to T$ so that $2\alpha(x) = \alpha(x^2) \neq 0$. We can extend the homomorphism, 2α, to a map $\varphi : G \to T$ of degree ≤ 3 such that

$$2\alpha(x) = \varphi(x) = \varphi(x-1) = 0.$$

This results in a contradiction, meaning that our assumption is wrong. Hence $x^2 = 1$ for all $x \in D_4(G)$. This proves the theorem. □

We now look at the free group approach developed by Gupta for studying the structure of $D_4(G)/\gamma_4(G)$. This approach became useful as it later helped in coming with a large number of examples of groups without the dimension property.

Let G be any group with free presentation $\langle X|r_1, r_2, \ldots\rangle$ where X is the set of generators and r_i's are the relations. The presentation of G can be viewed as a short exact sequence

$$1 \to R \xrightarrow{i} F \xrightarrow{p} G \to 1 \tag{1.9}$$

where i is an injection and p a surjection, F is a free group on the set X and R is a normal subgroup of F. We thus have $G \simeq F/R$.

NOTATION: Let F be a free group with basis X and $\mathbb{Z}[F]$ its integral group ring. We denote the augmentation ideal $\mathbb{Z}[F](F-1)$ by $\mathfrak{f}$. For a normal subgroup R in F we have a two-sided ideal, $\mathfrak{r} = \mathbb{Z}[F](R-1)$ of $\mathbb{Z}[F]$, .

We translate the dimension subgroup problem into the language of free group rings. Let $G = F/R$, we define $D(n, R) = F \cap (1 + \mathfrak{r} + \mathfrak{f}^n)$ and call it the nth *dimension subgroup of F relative to R*. The dimension subgroup problem translates into the identification of the quotient $D(n, R)/R\gamma_n(F)$.

The following theorem of Gupta gives the structure of the fourth dimension subgroup. As will be seen subsequently, this is a translation of what was shown by Tahara [Tah77]. In this discussion G is a finitely generated metabelian group (a group with $\gamma_2(G)$ Abelian) which has the pre-Abelian presentation

$$F/R = \langle x_1, \ldots, x_m | x_1^{e_1}\xi_1, \ldots, x_m^{e_m}\xi_m, \xi_{m+1}, \ldots, F''\rangle \tag{1.10}$$

with $e_m | \ldots | e_1 \geq 0$ and for all i, $\xi_i \in F'$.

Theorem 1.1.13. *[Gup87, IV.5.1] Let $G = F/R$ with $F/RF' \simeq F/S$ where F is the free group on the set $X = \{x_1, \ldots, x_m\}$ and $S = \langle x_1^{e_1}, \ldots, x_m^{e_m}, F'\rangle$, $e_i = 2^{\alpha_i}$, and $\alpha_1 \geq \alpha_2, \geq \alpha_m \geq 1$. Then modulo $R\gamma_4(F)$, $D(4, R)$ consists of all elements $w = \prod_{1\leq i<j\leq m}(x_i^{e_i}, x_j^{e_j})^{a_{ij}}$, $a_{ij} \in \mathbb{Z}$ such that*

- $e_j \mid \binom{e_i}{2} a_{ij} \quad 1 \le i < j \le m.$
- $y_k = \prod_{i<k} x_i^{-e_i' a_{ik}} \prod_{k<j} x_j^{e_j b_{kj}} \in RF'^{e_k}\gamma_3(F),\ 1 \le k \le m$, *where we define* $b_{kj} = \frac{e_k}{e_j} a_{kj} + \frac{\binom{e_k}{2}}{e_j} a_{kj}(x_k - 1)$.

Proof To proceed with the proof we need the following theorem

Theorem 1.1.14. *[Gup87, IV.3.2] For all* $n \ge 1$, *modulo* $(F', S)\gamma_{n+2}(F)$, $D(n+2, \mathfrak{fs})$ *is generated by*

$$(x_i, x_j)^{t(x_i,e_i)a_{ij}} \quad 1 \le i < j \le m \tag{1.11}$$

where $a_{ij} = a_{ij}(x_j, \ldots, x_m) \in \mathbb{Z}[F]$ *and* $t(x_i, e_i) := 1 + x_i + \ldots + x_i^{e_i}$. *Also,* $t(x_i, e_j)a_{ij}$ *is an element of* $t(x_j, e_j)\mathbb{Z}[F] + \mathfrak{s} + \mathfrak{f}^n$. *We define* $D(n+2, \mathfrak{fs})$ *as* $F \cap (1 + \mathfrak{fs} + \mathfrak{f}^n)$.

Let $w \in D(4, \mathfrak{fr}) \le D(4, \mathfrak{fs})$. Theorem 1.1.14 tells us that we can write

$$w = \prod_{1 \le i < j \le m} (x_i, x_j)^{d_{ij}} \prod_{k=1}^{m} (x_k^{e_k}, \eta_k)\xi \tag{1.12}$$

where $\xi \in \gamma_4(F)$, $\eta_k \in F'$, $d_{ij} = t(x_i, e_i)a_{ij} \equiv t(x_j, e_j)b_{ij}(\mathfrak{s} + \mathfrak{f}^2)$, $a_{ij} = a_{ij}(x_j, \ldots, x_m)$ and $b_{ij} = b_{ij}(x_1, \ldots, x_m)$ are elements of the free group ring $\mathbb{Z}[F]$.

Now, modulo $(F', S)\gamma_4(F)$, $(x_i, x_j)^{t(x_i,e_i)(x_k-1)} = 1$. This means that it is enough to prove for the case when $a_{ij} \in \mathbb{Z}$ and $b_{ij} = b'_{ij} + b''_{ij}(x_i - 1)$ with $b'_{ij}, b''_{ij} \in \mathbb{Z}$.

Now we set

$$t(x_i, e_i)a_{ij} = t(x_j, e_j)b'_{ij} + t(x_j, e_j)b''_{ij}(x_i - 1) + \sum_l \alpha_l(x_l^{e_l} - 1). \tag{1.13}$$

We write $t(x_i, e_i) = 1 + x_i + \ldots + x_i^{e_i - 1} = \frac{x_i^{e_i} - 1}{x_i - 1}$ wherever required. If we consider the augmentation of both sides, we get $b_{ij} = \frac{e_i a_{ij}}{e_j}$. Further, we use the expansion

$$(x^{e_i} - 1) = (x_i - 1 + 1)^{e_i} - 1 = e_i(x_i - 1) + \binom{e_i}{2}(x_i - 1)^2 + \ldots \tag{1.14}$$

and compare coefficients to get $e_j \mid \binom{e_i}{2} a_{ij} \quad 1 \le i < j \le m$ and $b''_{ij} = \frac{\binom{e_i}{2}}{e_j} a_{ij}$.

y_k is as defined in the statement of the theorem and $z_k := \prod (x_i, x_j)^{x_k D_{x_k}(d_{ij})}$, product over i such that $i < j$ and $i \le k$ where $D_{x_k}(u)$ is the usual partial derivative as defined for rational functions.

We can simplify each term in the expression for z_k as:

$$(x_i, x_j)^{x_k D_{x_k}(d_{ij})} = \begin{cases} (x_i, x_j)^{t(x_i,e_i)D_{x_k}(a_{ij})} & k \neq i \\ (x_i, x_j)^{t(x_j,e_j)D_{x_k}(b_{ij})} & k = i \end{cases} \tag{1.15}$$

Modulo $\mathfrak{f}^2\mathfrak{s} + \mathfrak{f}^4$, we have

$$w - 1 \equiv \sum_{k=1}^{m} (x_k - 1)(y_k z_k^{-1} \eta_k^{e_k} - 1) \tag{1.16}$$

(For detailed calculations, see [Gup87, Lemma IV.4.4].)

Therefore, we have $\sum_{k=1}^{m}(x_k - 1)(y_k z_k^{-1}\eta_k^{e_k} - 1) \equiv 0 \mod \mathfrak{f}(\mathfrak{r} + \mathfrak{f}\mathfrak{s} + \mathfrak{f}^3)$. Thus, $y_k z_k^{-1}\eta_k^{e_k} r_k - 1 \in \mathfrak{f}\mathfrak{s} + \mathfrak{f}^3 \leq \mathfrak{r} + \mathfrak{f}^3$ for some $r_k \in R$.

Furthermore, $y_k z_k^{-1}\eta_k^{e_k} r_k \in R\gamma_3(F)$ implies $y_k z_k^{-1} \in RF'^{e_k}\gamma_3(F)$.

Also, $z_k \equiv 1 \mod R\gamma_3(F)$. Thus, we get the desired result $y_k \in RF'^{e_k}\gamma_3(F)$.

Conversely, if $w = \prod_{1\leq i<j\leq m}(x_i^{e_i}, x_j)^{a_{ij}}$, $a_{ij} \in \mathbb{Z}$ satisfying the two properties mentioned above, then

$$t(x_i, e_i)a_{ij} \equiv t(x_j, e_j)b_{ij} \mod (\mathfrak{s} + \mathfrak{f}^2). \tag{1.17}$$

Theorem 1.1.14 tells us that $w \in D(4, \mathfrak{f}\mathfrak{s})$. As before, modulo $\mathfrak{f}^2\mathfrak{s} + \mathfrak{f}^4$ we have,

$$w - 1 \equiv \sum_{k=1}^{m} (x_k - 1)(y_k z_k^{-1} \eta_k^{e_k} - 1)$$

where $z_k \in R\gamma_3(F)$.
This allows us to say $(x_k - 1)(z_k^{-1} - 1) \in \mathfrak{f}\mathfrak{r} + \mathfrak{f}^4$. Also, we have from the hypothesis that $(x_k - 1)(y_k - 1) \in \mathfrak{f}\mathfrak{r} + \mathfrak{f}^4$.

Also, $(x_k - 1)(\eta_k^{e_k} - 1) \equiv (x_k^{e_k} - 1)(\eta_k - 1) \equiv 0 \mod (\mathfrak{r} + \mathfrak{f}^4)$. It is obvious that $\mathfrak{f}\mathfrak{s} + \mathfrak{f}^4 \leq \mathfrak{f}^2\mathfrak{r} + \mathfrak{f}^4 \leq \mathfrak{r} + \mathfrak{f}^4$. This allows us to conclude $w - 1 \in \mathfrak{r} + \mathfrak{f}^4$ and thus $w - 1 \in D(4, R)$. With this the proof is complete. □

Corollary 1.1.15. *If $m \leq 3$ in the above theorem, then $D(4, R) = R\gamma_4(F)$.*

This allows us to say that if G is a 2 or 3 generator group then $D_4(G) = \gamma_4(G)$.

We will now look at the structure of fourth dimension subgroup as given by Tahara [Tah77]. This approach paves the way for understanding the structure of the fifth dimension subgroup.

Recall that for an Abelian group A, the nth *symmetric power*, $Sp^n(A)$ *of* A is the quotient $A^{\otimes n}/J$, where $A^{\otimes n}$ is the nth tensor power of A and J is the subgroup of $A^{\otimes n}$ generated by all the elements $x_1 \otimes \ldots x_n - x_{\sigma(1)} \otimes \ldots x_{\sigma(n)}$ for $x_i \in A$ and $\sigma \in S_n$, the symmetric group of degree n. We denote the element $x_1 \otimes \ldots \otimes x_n + J \in Sp^n(A)$ by $x_1 \vee \ldots \vee x_n$. We define a *weight function* $w : G \to \mathbb{N} \cup \{\infty\}$ by setting

$$w(x) = \begin{cases} k & x \in \gamma_k(G) \setminus \gamma_{k+1}(G) \\ \infty & x = 1 \end{cases} \tag{1.18}$$

If $x \neq 1$, define $o^*(x)$ to be the order of the coset $\overline{x} = x\gamma_{w(x)+1}(G)$ in $\gamma_{w(x)}(G)/\gamma_{w(x)+1}(G)$. Each quotient $\gamma_i(G)/\gamma_{i+1}(G)$ is a finite Abelian group and by the Structure Theorem of finitely generated Abelian groups can be written as direct sum of $\lambda(i)$ cyclic groups. Each element $\overline{g} \in \gamma_k(G)/\gamma_{k+1}(G)$ can be written uniquely as

$$\overline{g} = e(1)\overline{x}_{k1} + \ldots + e(\lambda(k))\overline{x}_{k\lambda(k)}, \tag{1.19}$$

where $0 \leq e(i) < o^*(x_{ki})$ for all i. We can choose x_{ki} such that $o^*(x_{ki})$ divides $o^*(x_{ki+1})$.

Since our aim is to study the fourth dimension subgroup of a finite group, we can restrict ourselves to a finite nilpotent group of nilpotency class 3.

Let $d(i) = o^*(x_{1i})$ where $1 \leq i \leq s = \lambda(1)$. We have

$$x_{1i}^{d(i)} = x_{2i}^{c_{i1}} \ldots x_{2t}^{c_{it}} x_{3i} \tag{1.20}$$

with $x_{3i} \in \gamma_3(G),\ t = \lambda(2)$. This means we can write

$$\overline{x_{1i}^{d(i)}} = (c_{i1})\overline{x}_{2i} + \ldots + (c_{it})\overline{x}_{2t}. \tag{1.21}$$

Using the above notation, we have the following theorem

Theorem 1.1.16. *[Tah77, Theorem 8]* $G \cap (1 + \Delta^4(G))$ *is equal to the subgroup generated by elements*

$$\sum_{1\le i<j\le s} u_{ij}\frac{d(j)}{d(i)}(x_{1i}^{d(i)}, x_{1j})$$

for all integers u_{ij} satisfying the conditions

1. $u_{ij}\binom{d_j}{2} \equiv 0 \mod d(i)$.
2. $\sum_{1\le h<i} u_{hi}\frac{d(i)}{d(h)}c_{hk} - \sum_{i<j\le s} u_{ij}c_{jk} \equiv 0 \mod (d(i), d'(k))$

for $1 \le i \le s$ and $1 \le k \le t$ with $d(i) = o^*(x_{1i})$, $d'(k) = o^*(x_{2k})$.

Proof We know $G\cap(1+\Delta^3(G)) = \gamma_3(G)$, hence $\gamma_4(G) \subseteq G\cap(1+\Delta^4(G)) \subseteq \gamma_3(G)$. [Tah77, Theorem 7] gives us the structure of

$$\Delta^3(G)/\Delta^4(G) \simeq \{\gamma_3/\gamma_4 \oplus (\gamma_1/\gamma_2 \otimes \gamma_2/\gamma_3) \oplus Sp^3(\gamma_1/\gamma_2)\}/R \qquad (1.22)$$

where R is the submodule of

$$\gamma_3(G) \oplus (\gamma_1(G)/\gamma_2(G) \otimes \gamma_2(G)/\gamma_3(G)) \oplus Sp^3(\gamma_1(G)/\gamma_2(G)).$$

generated by

$$\frac{d(j)}{d(i)}\overline{(x_{1i}^{d(i)}, x_{1j})} \oplus \{\frac{d(j)}{d(i)}(\overline{x}_{1j}\otimes\overline{x}_{1i}^{d(i)}) - (\overline{x}_{1i}\otimes\overline{x}_{1j}^{d(j)})\} \oplus \{\binom{d_j}{2}(\overline{x}_{1i}\vee\overline{x}_{1j}\vee\overline{x}_{1j}) - \frac{d(j)}{d(i)}\binom{d_j}{2}(\overline{x}_{1i}\vee\overline{x}_{1j}\vee\overline{x}_{1j})\}$$

with $1 \le i \le j \le \lambda(1)$, $\overline{(x_{1i}^{d(i)}, x_{1j})} = (x_{1i}^{d(i)}, x_{1j})\gamma_4(G)$, $\overline{x}_{1k} = x_{1k}\gamma_2(G)$ and $\overline{x_{1k}^{d(k)}} = x_{1k}^{d(k)}\gamma_3(G)$.

Consider the homomorphism

$$\varphi_3 : \gamma_3(G) \to \Delta^3(G)/\Delta^4(G)$$

defined by

$$x \mapsto (x-1) + \Delta^4(G).$$

The kernel, $\ker(\varphi_3) = G \cap (1 + \Delta^4(G))$.

$\gamma_3(G)$ can be identified with $\gamma_3(G)\oplus\overline{0}\oplus\overline{0}$, the structure of $\Delta^3(G)/\Delta^4(G)$ above shows that $ker(\varphi_3) = R \cap \{\gamma_3(G) \oplus \overline{0} \oplus \overline{0}\}$.

Thus, $ker(\varphi_3)$ is the submodule generated by $\sum_{1\le i<j\le s} u_{ij}\frac{d(j)}{d(i)}(x_{1i}^{d(i)}, x_{1j})$ for all integers $u_{ij}(1 \le i < j \le s)$ provided the following two conditions are satisfied:

1. $\sum_{1\leq i<j\leq s} u_{ij}\{\frac{d(j)}{d(i)}(\overline{x}_{1j}\otimes \overline{x}_{1i}^{d(i)}) - (\overline{x}_{1i}\otimes \overline{x}_{1j}^{d(j)})\} = \overline{0}$

2. $\sum_{1\leq i<j\leq s} u_{ij}\{\binom{d_j}{2}(\overline{x}_{1i}\vee \overline{x}_{1j}\vee \overline{x}_{1j}) - \frac{d(j)}{d(i)}\binom{d_i}{2}(\overline{x}_{1i}\vee \overline{x}_{1j}\vee \overline{x}_{1j})\} = \overline{0}.$

We now write $\overline{x_{1i}^{d(i)}}$ as was mentioned above (1.21) and in the first condition put coefficients of $\overline{x}_{1i}\otimes \overline{x}_{2k} = 0$. Thus the first condition is equivalent to

$$\sum_{1\leq h<i} u_{hi}\frac{d(i)}{d(h)}c_{hk} - \sum_{i<j\leq s} u_{ij}c_{jk} \equiv 0 \mod (d(i), d'(k)) \ (1\leq k\leq \lambda(2)) \tag{1.23}$$

The second condition gives us the equivalent condition

$$u_{ij}\binom{d_j}{2} \equiv 0 \mod d(i). \tag{1.24}$$

This proves the theorem. □

Corollary 1.1.17. *[Tah77, Corollary 9] If G is a finite p-group with $p\neq 2$, then $G\cap(1+\Delta^4(G)) = \gamma_4(G)$, i.e., $D_4(G) = \gamma_4(G)$.*

Proof As before we may assume our group to be of class 3, i.e., $\gamma_4(G) = 1$. Since we have already assumed that p is odd, we straight away have $\binom{d_j}{2} \equiv 0 \mod d(i)$ for all $1\leq i<j\leq s$. If u_{ij} are integers satisfying condition 2 of theorem 1.1.16 then we are only left to show

$$\sum_{1\leq i<j\leq s} u_{ij}\frac{d(j)}{d(i)}(x_{1i}^{d(i)}, x_{1j}) = 0.$$

Condition 2 of the theorem implies that

$$\sum_{1\leq h<i} u_{hi}\frac{d(i)}{d(h)}(x_{1h}^{d(h)}, x_{1i}) + \sum_{1<j\leq s} u_{ij}(x_{1i}^{d(i)}, x_{1j}) = 0. \tag{1.25}$$

The conditions p odd and nilpotency class 3 imply

$$\frac{d(j)}{d(i)}(x_{1i}^{d(i)}, x_{1j}) = (x_{1i}, x_{1j}^{d(j)}) \textit{ for } 1\leq i<j\leq s. \tag{1.26}$$

These two combined give us

$$2\sum_{1\leq i<j\leq s} u_{ij}\frac{d(j)}{d(i)}(x_{1i}^{d(i)}, x_{1j}) = 0. \tag{1.27}$$

Since $p \neq 2$, we have $\sum_{1\leq i<j\leq s} u_{ij}\frac{d(j)}{d(i)}(x_{1i}^{d(i)}, x_{1j}) = 0$. □

With this the fourth dimension subgroup case is complete.

Before shifting focus to the fifth dimension subgroup case we mention an important theorem of Sjogren [Sjo79], which gives an estimate of the exponent of $D_n(G)/\gamma_n(G)$.

Theorem 1.1.18. *[Sjo79, 2.15] For every group G,*

$$D_n(G)^{c(n)} = \gamma_n(G) \text{ for all } n \geq 1 \tag{1.28}$$

where $c(1) = c(2) = 1$ *and* $c(n) = b(1)^{\binom{n-2}{1}} \dots b(n-2)^{\binom{n-2}{n-2}}$, $n \geq 3$. *For every* $k \in \mathbb{N}$, *let* $b(k) = lcm\{1, 2, \dots, k\}$.

The proof given by Sjogren is through construction of spectral sequences and analyzing its properties through combinatorial methods. Later a relatively easier proof of this theorem was given by Cliff and Hartley ([CH87]). Another proof was given by Gupta [Gup87] using the free group ring approach. Computations for small n show: $c(3) = 1$, $c(4) = 2$, $c(5) = 48$.

Corollary 1.1.19. *If G is a p-group, then $D_n(G) = \gamma_n(G)$ for $n \leq p + 1$*

Proof The proof follows from the observation $c(p + 1)$ is co-prime to p where p is a prime.

Remark 1.1.20.

1. The above corollary improves an earlier result of Moran where he had shown that for such a group G, $D_n(G) = \gamma_n(G)$ for all n less than equal to p.
2. The bound $c(n)$ for the exponent of $D_n(G)/\gamma_n(G)$ obtained by Sjogren is unlikely to be the best possible. For a metabelian group G, Gupta has shown [Gup87, Theorem IV.4.6] that $c(n)$ can be replaced by a smaller integer $2 \cdot b(1) \dots b(n-2)$ where $b(i)$'s are as defined in Sjogren's theorem.

Fifth Dimension Subgroup

In this section our aim is to study the structure of fifth dimension subgroup, $D_5(G) = G \cap \{1 + \Delta^5(G)\}$ with the help of the structure of $\Delta^3(G)/\Delta^5(G)$.

Let G be a finite group of nilpotency class 4. We set some notations which will be required throughout this section:

- $x_{1i}^{d(i)} = x_{21}^{b_{i1}} x_{22}^{b_{i2}} \cdots x_{2t}^{b_{it}} x_{31}^{c_{i1}} x_{32}^{c_{i2}} \cdots x_{3u}^{c_{iu}} y_{4i}$ where $y_{4i} \in \gamma_4(G)$ and $1 \leq i \leq s$.

- $x_{2i}^{e(i)} = x_{31}^{d_{i1}} x_{32}^{d_{i2}} \cdots x_{3u}^{d_{it}} y'_{4i}$ where $y'_{4i} \in \gamma_4(G)$ and $1 \leq i \leq t$.

- $x_{3i}^{f(i)} = x_{41}^{f_{i1}} x_{42}^{f_{i2}} \cdots x_{4v}^{f_{iv}} y_{5i}$ where $y_{5i} \in \gamma_5(G)$ and $1 \leq i \leq u$.

- $(x_{1i}^{d(i)}, x_{1j}) = x_{31}^{\alpha_1^{(ij)}} x_{32}^{\alpha_2^{(ij)}} \cdots x_{3u}^{\alpha_u^{(ij)}} x_{41}^{\beta_1^{(ij)}} x_{42}^{\beta_2^{(ij)}} \cdots x_{4v}^{\beta_v^{(ij)}} y_5^{(ij)}$ where $y_5^{(ij)} \in \gamma_5(G)$ and $1 \leq i < j \leq s$.

We have seen the statement of Sjogren's theorem in 1.1.18. This gives an upper bound on the exponent of $D_n(G)/\gamma_n(G)$. For $n = 5$ the exponent must be a divisor of 48. Tahara's approach [Tah81] could improve this result significantly and that is what we will study in this section.
The main result we wish to study is the following:

Theorem 1.1.21. *[Tah81, Corollary 6.11] If G is a finite group, then the exponent $D_5(G)/\gamma_5(G)$ is divisible by* $3!$.

This follows from theorem 1.1.22 which gives us the exact structure of $D_5(G)$.

Theorem 1.1.22. *[Tah81, Theorem 6.1] $G \cap \{1 + \Delta^5(G)\}$ is equal to the subgroup generated by the elements*

$$\sum_{1\leq i<j\leq s} u_{ij}\frac{d(j)}{d(i)}(x_{1i}^{d(i)}, x_{1j}) + \sum_{1\leq i\leq s; 1\leq k\leq t} v_{ik}(\sum_{k<l} b_{il}(x_{2l}, x_{2k})) + \sum_{1\leq i\leq j\leq k\leq s} w_{ijk}(x_{1i}^{d(i)}, x_{1j}, x_{1k}) \tag{1.29}$$

for all integers u_{ij} $(1 \leq i < j \leq s)$, v_{ik} $(1 \leq i \leq s; 1 \leq k \leq t)$, w_{ijk} $(1 \leq i \leq j \leq k \leq s)$, w'_{ijk} $1 \leq i < j \leq k \leq s$, w''_{ijk} $1 \leq i < j \leq k \leq s$ which satisfy the following conditions

1. $w_{iii} = 0$ *for* $1 \leq i \leq s$.
2. $u_{ij}\frac{d(j)}{d(i)}\binom{d(i)}{2} + w_{iij}d(i) + w''_{iij}(d(j)) = 0$ *for* $1 \leq i < j \leq s$.
3. $-u_{ij}\binom{d(j)}{2} + w_{ijj}d(i) + w'_{ijj}d(j)$ *for* $1 \leq i < j \leq s$.
4. $w_{ijk}d(i) + w'_{ijk}d(j) + w''_{ijk}d(k) = 0$ *for* $1 \leq i < j < k \leq s$.
5. $\sum_{i<h} u_{ih}b_{hk} - \sum_{h<i} u_{hi}\frac{d(i)}{d(h)}b_{hk} + v_{ik}d(i) + v'_{ik}e(k) = 0$ *for* $1 \leq i \leq s;\ 1 \leq k \leq t$.
6. $u_{ij}\frac{d(j)}{d(i)}\binom{d(i)}{3} + w_{iij}\binom{d(i)}{2} \equiv 0 \mod d(i)$ *for* $1 \leq i < j \leq s$.

7. $w_{ijj}\binom{d(i)}{2} + w''_{iij}\binom{d(j)}{2} \equiv 0 \mod d(i)$ *for* $1 \le i < j \le s$.

8. $-u_{ij}\binom{d(j)}{3} + w'_{ijj}\binom{d(j)}{2} \equiv 0 \mod d(i)$ *for* $1 \le i < j \le s$.

9. $w_{ijk}\binom{d(i)}{2},\ w'_{ijk}\binom{d(j)}{2},\ w''_{ijk}\binom{d(k)}{2} \equiv 0 \mod d(i)$ *for* $1 \le i < j < k \le s$.

10. $v_{ik}\binom{d(i)}{2} - \sum_{h\le i} w_{hii}b_{hk} - \sum_{i<h} w''_{iih}b_{hk} \equiv 0 \mod (d(i), e(k))$ *for* $1 \le i \le s;\ 1 \le k \le t$.

11. $\sum_{h\le i} w_{hij}b_{hk} + \sum_{i<h\le j} w'_{ihj}b_{hk} + \sum_{j<h} w''_{ijk}b_{hk} \equiv 0 \mod (d(i), e(k))$ *for* $1 \le i < j \le s;\ 1 \le k \le t$.

12. $-\sum_{h<i} u_{hi}\frac{d(i)}{d(h)}\alpha_l^{(hi)} + \sum_{i<h} u_{ih}c_{hl} - \sum_{h<i} u_{hi}\frac{d(i)}{d(h)}c_{hl} - \sum_k v'_{ik}d_{kl} - \sum_{g\le i\le h} w_{gih}\alpha_l^{(gh)} - \sum_{g\le h\le i} w_{ghi}\alpha_l^{(gh)} - \sum_{i<g\le h} w'_{igh}\alpha_l^{(gh)} \equiv 0 \mod (d(i), e(k))$ *for* $1 \le i \le s;\ 1 \le l \le u$.

13. $\sum_i v_{ik}b_{ik} \equiv 0 \mod e(k)$ *for* $1 \le k \le t$.

14. $\sum_i v_{ik}b_{il} + \sum_i v_{il}b_{ik} \equiv 0 \mod e(k)$ *for* $1 \le k < l \le t$.

To prove thoerem 1.1.21 from the above theorem the following lemmas are needed.

Lemma 1.1.23.

$$3\sum_{i<j<k} w'_{ijk}(x_{1i}, x_{1j}^{d(j)}, x_{1k}) - 3\sum_{i<j<k} w''_{ijk}(x_{1k}^{d(k)}, x_{1i}, x_{1j}) + \sum_{i<j} u_{ij}\frac{d(j)}{d(i)}\binom{d(i)}{2}(x_{1i}, x_{1j}, x_{1i})$$
$$+ \sum_{i<j} u_{ij}\binom{d(j)}{2}(x_{1i}, x_{1j}, x_{1i}) - \sum_{i<j} v_{ik}\binom{d(i)}{2}(x_{2k}, x_{1i}, x_{1i}) + \sum_{i<j} w_{iij}\binom{d(i)}{2}(x_{1j}, x_{1i}, x_{1i}, x_{1i})$$
$$- \sum_{i<j} w'_{ijj}\binom{d(j)}{2}(x_{1j}, x_{1i}, x_{1j}, x_{1i}) = 0.$$

Lemma 1.1.24.

$$2\sum_{i<j} u_{ij}(x_{1i}, x_{1j}^{d(j)}) + \sum_{i<j} u_{ij}\frac{d(j)}{d(i)}(x_{1i}^{d(i)}, x_{1j}, x_{1j}) - \sum_{i,k} v_{ij}(x_{2k}, x_{1i}^{d(i)}) = 0.$$

Lemma 1.1.25. $\sum_{i,k} v_{ik}(x_{2k}, x_{1i}^{d(i)}) + 2\sum_{i,k} v_{ik}(\sum_{k<l} b_{il}(x_{2l}, x_{2k})) = 0.$

Lemma 1.1.26. $\sum_{i<j} u_{ij}d(j)(x_{1i}, x_{1j}, x_{1i}) + \sum_{i<j} u_{ij}d(j)(x_{1i}, x_{1j}, x_{1j})$
$- \sum_{i<j} u_{ij}\frac{d(j)}{d(i)}\binom{d(i)}{2}(x_{1j}, x_{1i}, x_{1i}, x_{1j}) + \sum_{i<j} u_{ij}\binom{d(j)}{2}(x_{1i}, x_{1j}, x_{1j}, x_{1i}) = 0.$

Now we finally move on to the proof of the main result of this section (theorem 1.1.21).

Proof We know that a typical element $g \in G \cap \{1 + \Delta^5(G)\}$ is of the form written in theorem 1.1.22. We are done if we show $6g = 0$. Using relations 1-4 and 6-9 of the above theorem we can re-write

$$\begin{aligned} g &= \textstyle\sum_{i<j}(x_{1i}, x_{1j}^{d(j)}) + \sum_{i<j<k} w'_{ijk}(x_{1i}, x_{1j}^{d(j)}, x_{1k}) \\ &\quad - \textstyle\sum_{i<j<k} w''_{ijk}(x_{1k}^{d(k)}, x_{1i}, x_{1j}) + \sum_{i<j} u_{ij}\frac{d(j)}{d(i)}\binom{d(i)}{2}(x_{1i}, x_{1j}, x_{1i}) \\ &\quad + \textstyle\sum_{i,k} v_{ik}(\sum_{k<l} b_{il}(x_{2l}, x_{2k})). \end{aligned} \tag{1.30}$$

Now, we simplify the first term of the above expression using lemma 1.1.24 and the second and third term using lemma 1.1.23 and keep the last two terms as they are.

$$\begin{aligned} 6g &= -3\textstyle\sum_{i<j} u_{ij}\frac{d(j)}{d(i)}(x_{1i}^{d(i)}, x_{1j}, x_{1j}) \ + 3\sum_{i,k} v_{ik}(x_{2k}, x_{1i}^{d(i)}) \\ &\quad -2\textstyle\sum_{i<j} u_{ij}\frac{d(j)}{d(i)}\binom{d(i)}{2}(x_{1i}, x_{1j}, x_{1i}) \ - 2\sum_{i<j}\binom{d(j)}{2}(x_{1i}, x_{1j}, x_{1j}) \\ &\quad +2\textstyle\sum_{i,k} v_{ik}\binom{d(i)}{2}(x_{2k}, x_{1i}, x_{1i}) \ - 2\sum_{i<j} w_{iij}\binom{d(i)}{2}(x_{1j}, x_{1i}, x1i, x_{1i}) \\ &\quad +2\textstyle\sum_{i<j} w'_{ijj}\binom{d(j)}{2}(x_{1j}, x_{1i}, x_{1j}, x_{1j}) \ + 6\sum_{i<j} u_{ij}\frac{d(j)}{d(i)}\binom{d(i)}{2}(x_{1i}, x_{1j}, x_{1i}) \\ &\quad +6\textstyle\sum_{i,k} v_{ik}(\sum_{k<l} b_{il}(x_{2l}, x_{2k})). \end{aligned} \tag{1.31}$$

Also

$$\begin{aligned} 2\textstyle\sum_{i,k} v_{ik}\binom{d(i)}{2}(x_{2k}, x_{1i}, x_{1i}) &= 2\textstyle\sum_{i<j} w_{iij}\binom{d(i)}{2}(x_{1j}, x_{1i}, x_{1i}, x_{1j}) \\ &= 2\textstyle\sum_{i<j} w'_{ijj}\binom{d(i)}{2}(x_{1j}, x_{1i}, x_{1j}, x_{1j}) \\ &= 0 \end{aligned} \tag{1.32}$$

and

$$3\sum_{i,k} v_{ik}(x_{2k}, x_{1i}^{d(i)}) + 6\sum_{i,k} v_{ik}(\sum_{k<l} b_{il}(x_{2l}, x_{2k})) = 0. \tag{1.33}$$

Thus some of the terms cancel and the expression simplifies. We get

$$\begin{aligned} 6g &= -3\textstyle\sum_{i<j} u_{ij}\frac{d(j)}{d(i)}(x_{1i}^{d(i)}, x_{1j}, x_{1j}) \ + 4\sum_{i<j} u_{ij}\frac{d(j)}{d(i)}\binom{d(i)}{2}(x_{1i}, x_{1j}, x_{1i}) \\ &\quad -2\textstyle\sum_{i<j}\binom{d(j)}{2}(x_{1i}, x_{1j}, x_{1j}). \end{aligned} \tag{1.34}$$

Using the following

- $u_{ij}\frac{d(j)}{d(i)}\binom{d(i)}{2} \equiv 0 \mod d(i)$
- $d(i)d(j)(x_{1i}, x_{1j}, x_{1i}) = 0$
- $d(j)^2(x_{1i}, x_{1j}, x_{1j}) = 0$

we get

$$6g = -2\sum_{i<j} d(j)(x_{1i}, x_{1j}, x_{1i}) - 2\sum_{i<j} d(j)(x_{1i}, x_{1j}, x_{1j}). \tag{1.35}$$

Using

$$2\sum_{i<j} d(j)(x_{1j}, x_{1i}, x_{1i}, x_{1j}) = 2\sum_{i<j} \binom{d(j)}{2}(x_{1i}, x_{1j}, x_{1j}, x_{1i}) = 0, \tag{1.36}$$

an application of lemma 1.1.26 gives $6g = 0$. With this the proof is complete.

□

The following is an unpublished result by Shalini Gupta. I thank her for allowing me to include it in my thesis.

Proposition 1.1.27. *Let G be a metabelian 2-group with $\gamma_1(G)/\gamma_2(G) = C_1 \oplus C_2$, where C_1 and C_2 are cyclic groups. Then $D_5(G) = \gamma_5(G)$.*

Before giving the proof of the proposition, we mention the following lemma which will be used repeatedly:

Lemma 1.1.28. *[Tah81]Lemma 2.3 Let n be a non negative integer. For a group, G, let $G = \gamma_1(G) \supseteq \gamma_2(G) \ldots \supseteq \gamma_5(G) = 1$. Then*

$$\begin{array}{lrcl}
1. & (x, gg') & = & (x, g)(x, g'). \\
2. & (x, y)^n & = & (x, y^n)(x, y, y)^{-\binom{n}{2}}(x, y, y, y)^{-\binom{n}{3}}. \\
 & & = & (x^n, y)(x, y, x)^{-\binom{n}{3}}(x, y, x, x)^{-\binom{n}{3}}. \\
3. & (x, y, z)^n & = & (x^n, y, z)(y, x, x, z)^{\binom{n}{2}}. \\
 & & = & (x, y^n, z)(y, x, y, z)^{\binom{n}{2}}. \\
 & & = & (x, y, z^n)(y, x, z, z)^{\binom{n}{2}}. \\
4. & (x, y, g)(y, g, x)(g, x, y) & = & 1.
\end{array} \tag{1.37}$$

where $x, y, z \in G$ and $g, g' \in \gamma_2(G)$.

We now give the proof of proposition 1.1.27:

Proof Let G be a metabelian 2-group with $\gamma_1(G)/\gamma_2(G) = C_1 \oplus C_2$. Let the basis of $\gamma_1(G)/\gamma_2(G)$ be $\{x_{11}, x_{12}\}$ and $o(x_{1i}) = d(i)$. As per the notation in theorem 1.1.22, the fifth dimension subgroup, $D_5(G)$ (written additively) is generated by elements of the type:

$$u_{12}\frac{d(2)}{d(1)}(x_{11}^{d(1)}, x_{12}) + w_{112}(x_{11}^{d(1)}, x_{11}, x_{12}) + w_{122}(x_{11}^{d(1)}, x_{12}, x_{12}) \tag{1.38}$$

subject to the conditions:

1. $u_{12}\frac{d(2)}{d(1)}\binom{d(1)}{2} + w_{112}d(1) + w''_{112}d(2) = 0$
2. $-u_{12}\binom{d(2)}{2} + w_{122}d(1) + w'_{112}d(2) = 0$
3. $u_{12}b_{2k} + v_{1k}d(1) + v'_{1k}e(k) = 0$
4. $-u_{12}\frac{d(2)}{d(1)}b_{1k} + v_{2k}d(2) + v'_{2k}e(k) = 0$
5. $u_{12}\frac{d(2)}{d(1)}\binom{d(1)}{3} + w_{112}\binom{d(1)}{2} \equiv 0 \mod d(1)$
6. $w_{122}\binom{d(1)}{2} + w'_{112}\binom{d(2)}{2} \equiv 0 \mod d(1)$
7. $-u_{12}\binom{d(2)}{3} + w'_{122}\binom{d(2)}{2} \equiv 0 \mod d(1)$
8. $v_{1k}\binom{d(1)}{2} - w'_{112}b_{2k} \equiv 0 \mod gcd(d(1), e(k))$
9. $v_{2k}\binom{d(2)}{2} - w_{122}b_{1k} \equiv 0 \mod gcd(d(2), e(k))$
10. $\sum_{i=1}^{2} v_{ik}b_{ik} \equiv 0 \mod e(k)$
11. $\sum_{i=1}^{2} v_{ik}b_{il} + \sum_{i=1}^{2} v_{il}b_{ik} \equiv 0 \mod e(k)$ where $1 \leq k < l \leq t$
12. $w_{112}b_{1k} + w'_{122}b_{2k} \equiv 0 \mod gcd(d(1), e(k))$
13. $u_{12}c_{2l} - \sum_k v'_{1k}d_{kl} - w_{112}\alpha_l^{(12)} \equiv 0 \mod gcd(d(1), f(l))$
14. $-u_{12}\frac{d(2)}{d(1)}\alpha_l^{(12)} - u_{12}\frac{d(2)}{d(1)}c_{1l} - \sum_k v'_{2k}d_{kl} - 2w_{122}\alpha_l^{(12)} \equiv 0 \mod gcd(d(2), f(l))$

Simplifying the second term on the RHS of the generator (equation 1.38)

$$(x_{11}^{d(1)}, x_{11}, x_{12})^{w_{112}} \quad = \quad (x_{12}, x_{11}, x_{11}^{d(1)})^{-w_{112}}(x_{12}, x_{11}^{d(1)}, x_{11})^{-w_{112}}$$

Here, the first term on the RHS is trivial because G is metabelian. For the second term on the RHS, we can use lemma 1.37 to see that

$$\begin{aligned}(x_{12}, x_{11}^{d(1)}, x_{11})^{-w_{112}} &= (x_{12}, x_{11}, x_{11})^{-w_{112}d(1)}(x_{11}, x_{12}, x_{11}, x_{11})^{w_{112}\binom{d(1)}{2}} \\ &= (x_{12}, x_{11}, x_{11}^{d(1)}) \cdot 1 \\ &= 1.\end{aligned}$$

Thus the generator simplifies to

$$u_{12}\frac{d(2)}{d(1)}(x_{11}^{d(1)}, x_{12}) + w_{122}(x_{11}^{d(1)}, x_{12}, x_{12}). \tag{1.39}$$

Using 4, we have

$$\prod_k (x_{2k}, x_{12})^{-u_{12}\frac{d(2)}{d(1)}b_{1k} + v_{2k}d(2) + v'_{2k}e(k)} = 1. \tag{1.40}$$

Thus,

$$(\prod_k x_{2k}^{b_{1k}}, x_{12})^{-u_{12}\frac{d(2)}{d(1)}} \cdot \prod_k \{(x_{2k}, x_{12}^{d(2)})^{v_{2k}}(x_{2k}, x_{12}, x_{12})^{-v_{2k}\binom{d(2)}{2}}\} \cdot \prod_k (x_{2k}^{e(k)}, x_{12})^{v'_{2k}} = 1. \tag{1.41}$$

$(x_{2k}, x_{12}^{d(2)}) = 1$ because G is metabelian and hence the above equation simplifies to

$$(\prod_k x_{2k}^{b_{1k}}, x_{12})^{-u_{12}\frac{d(2)}{d(1)}} \cdot \prod_k (x_{2k}, x_{12}, x_{12})^{-v_{2k}\binom{d(2)}{2}} \cdot \prod_l (x_{3l}, x_{12})^{\sum_k v'_{2k}d_{kl}} = 1. \tag{1.42}$$

Making use of the relation 9 we have

$$(\prod_k x_{2k}^{b_{1k}}, x_{12})^{-u_{12}\frac{d(2)}{d(1)}} \cdot \prod_k (x_{2k}, x_{12}, x_{12})^{-w_{122}b_{1k}} \cdot \prod_l (x_{3l}, x_{12})^{\sum_k v'_{2k}d_{kl}} = 1. \tag{1.43}$$

We now focus on the term $\prod_l (x_{3l}, x_{12})^{\sum_k v'_{2k}d_{kl}}$.

Using the relation 14 we have

$$\prod_l (x_{3l}, x_{12})^{\sum_k v'_{2k} d_{kl}} = \prod_l (x_{3l}, x_{12})^{-u_{12}\frac{d(2)}{d(1)}\alpha_l^{(12)} - u_{12}\frac{d(2)}{d(1)}c_{1l} - 2w_{122}\alpha_l^{(12)}}$$
$$= (x_{11}^{d(1)}, x_{12}, x_{12})^{-u_{12}\frac{d(2)}{d(1)}} \cdot (\prod_l x_{3l}^{c_{il}})^{-u_{12}\frac{d(2)}{d(1)}} \cdot (x_{11}^{d(1)}, x_{12}, x_{12})^{-2w_{122}}. \tag{1.44}$$

We will now concentrate on the first and third terms on the RHS of equation 1.44.

$$(x_{11}^{d(1)}, x_{12}, x_{12})^{-u_{12}\frac{d(2)}{d(1)}} = (x_{11}, x_{12}, x_{12})^{-u_{12}d(2)} \cdot (x_{12}, x_{11}, x_{11}, x_{12})^{u_{12}\frac{d(2)}{d(1)}\binom{d(1)}{2}}$$
$$= (x_{11}, x_{12}, x_{12}^{d(2)})^{-u_{12}} (x_{12}, x_{11}, x_{12}, x_{12})^{-u_{12}\binom{d(2)}{2}} (x_{12}, x_{11}, x_{11}, x_{12})^{u_{12}\frac{d(2)}{d(1)}\binom{d(1)}{2}}. \tag{1.45}$$

These equalities follow from lemma 1.37.

The first term of equation 1.45 is identity because our group is metabelian.

mod $d(1)$ we have

$$u_{12}\frac{d(2)}{d(1)}\binom{d(1)}{2} \equiv 0 \quad \text{Thus the third term of 1.45 is also trivial.}$$
$$-u_{12}\binom{d(2)}{2} \equiv 0 \quad \text{Thus the second term of 1.45 is also trivial.}$$

Thus the first term on the RHS of equation 1.44 is trivial.

Now we look at the third term on the RHS of equation 1.44.

From 2 we observe that

$$-2w_{122}d(1) = -2u_{12}\binom{d(2)}{2} + 2w'_{122}d(2).$$

Thus we have

$$(x_{11}^{d(1)}, x_{12}, x_{12})^{-2w_{122}} = (x_{11}, x_{12}, x_{12}^{d(2)})^{(-u_{12}(d(2)-1)+2w'_{122})} \cdot (x_{12}, x_{11}, x_{12}, x_{12})^{(-u_{12}(d(2)-1)+2w'_{122})\binom{d(2)}{2}}. \tag{1.46}$$

Now again, the first term on the RHS of 1.46 is trivial.

We further have

$$u_{12}\frac{d(2)}{d(1)}\binom{d(2)}{2} \equiv 0 \quad \mod d(1).$$
$$2w'_{122}\binom{d(2)}{2} \equiv 0 \quad \mod d(2).$$

This ensures that the third term of 1.44 is also trivial.

Thus, we are left with

$$(\prod_k x_{2k}^{b_{1k}} \cdot \prod_l x_{3l}^{c1l}, x_{12})^{-u_{12}\frac{d(2)}{d(1)}} \cdot (x_{11}^{d(1)}, x_{12}, x_{12})^{-w_{122}} = 1. \tag{1.47}$$

This allows us to conclude

$$(x_{11}^{d(1)}, x_{12})^{-u_{12}\frac{d(2)}{d(1)}} \cdot (x_{11}^{d(1)}, x_{12}, x_{12})^{-w_{122}} = 1. \tag{1.48}$$

Hence $D_5(G) = 1$.

1.1.3 Groups Without Dimension Property

As mentioned earlier, the first example of a group with non-trivial dimension quotients was provided by E. Rips in his 1972 paper (see [Rip72]). He gave the example of a 2-group of nilpotency class 3 with a non-zero fourth integral dimension subgroup.

Theorem 1.1.29. *[Rip72] Let G be a group with generators $a_0, a_1, a_2, a_3, b_1, b_2, b_3, c$ and relations*

$$b_1^{64} = b_2^{16} = b_3^4 = c^{256} = 1,$$

$$(b_2, b_1) = (b_3, b_1) = (b_3, b_2) = (c, b_1) = (c, b_2) = (c, b_3) = 1,$$

$$a_0^{64} = b_1^{32}, a_1^{64} = b_2^{-4}b_3^{-2}, a_2^{16} = b_1^4 b_3^{-1}, a_3^4 = b_1^2 b_2,$$

$$(a_1, a_0) = b_1c^2, (a_2, a_0) = b_2c^8, (a_3, a_0) = b_3c^{32},$$

$$(a_2, a_1) = c, (a_3, a_1) = c^2, (a_3, a_2) = c^4,$$

$$(b_1, a_1) = c^4, (b_2, a_2) = c^{16}, (b_3, a_3) = c^{64},$$

$$(b_i, a_j) = 1 \textit{ for } i \neq j, (c, a_i) = 1 \textit{ for } i = 0, 1, 2, 3.$$

Calculations show that $\gamma_4(G) = \{1\}$ and the non-trivial element

$$(a_1, a_2)^{128}(a_1, a_3)^{64}(a_2, a_3)^{32} = c^{128} \in D_4(G).$$

We now give Gupta's construction of the 2-group G_n where $n \geq 4$ (see [Gup90]) such that the n^{th} dimension subgroup is not equal to the n^{th} term of the lower central series.

Theorem 1.1.30. *Let n be fixed integer greater than 4, and consider the group $\langle r, a, b, c\rangle$. Set $x_0 = y_0 = z_0 = r$ and define $x_i = [x_{i-1}, a]$, $y_i = (y_{i-1}, b)$, $z_i = (z_{i-1}, c)$*

iteratively for all i. Consider the group generated by $\{r, a, b, c\}$ *subject to the conditions*

$$r^{2^{2n-1}} = 1,\ a^{2^{n+2}} = y_{n-3}^4 z_{n-3}^2,\ b^{2^n} = x_{n-3}^{-4} z_{n-3},\ c^{2^{n-2}} = x_{n-3}^{-2} y_{n-3}^{-1},$$

$$z_{n-2} = y_{n-2}^4,\ y_{n-2} = x_{n-2}^4,$$

$$x_{n-1} = 1,\ y_{n-1} = 1, z_{n-1} = 1,$$

$$(a, b, g) = (b, c, g) = (a, c, g) = 1 \textit{ for all } g$$

$$(x_i, b) = (x_i, c) = (y_i, a) = (y_i, c) = (z_i, a) = (z_i, b) = 1 \textit{ where } i \geq 1,$$

$$(x_i, x_j) = (x_i, y_j) = (x_i, z_j) = (y_i, y_j) = (y_i, z_j) = (z_i, z_j) = 1 \textit{ for all } i, j \geq 0.$$

Then, $D_n(G) \neq \gamma_n(G)$.

G is in fact a metabelian group of nilpotency class $n - 1$. The non trivial element $(a, b)^{2^{2n-1}}(a, c)^{2^{2n-2}}(b, c)^{2^{2n-3}}$ is in $D_n(G)$.

1.2 Dimension Subgroups Over Fields

In this section we shift our focus to dimension subgroups over a field k. The study of dimension subgroups over fields of characteristic p were introduced by Jennings in the 1940's.

1.2.1 N-Series and Filtration of the Augmentation Ideal

We recall a few definitions.

Definition 1.2.1. A series $G = H_1 \supseteq H_2 \supseteq \ldots \supseteq H_i \supseteq \ldots$ of subgroups of a group G is called an **N-series** if $(H_i, H_j) \subseteq H_{i+j}$ for all $i, j \geq 1$.

Definition 1.2.2. An N-series $\{H_i\}_{i\geq 1}$ is called **restricted N-series relative to prime** p if $x \in H_i$ implies that $x^p \in H_{ip}$ for all $i \geq 1$.

The most familiar example of an N-series is the lower central series $\{\gamma_i(G)\}_{i\in\mathbb{Z}}$. This series has the additional property that for any N-series $\{H_i\}$ in G, $\gamma_i(G) \subseteq H_i$ for all i.

Remark 1.2.3. The associative powers of augmentation ideal, $\Delta_R(G)$, have the following properties

$$\Delta_R^i(G) \cdot \Delta_R^j(G) \subseteq \Delta_R^{i+j}(G) \text{ for all } i, j \geq 1. \tag{1.49}$$

Lemma 1.2.4. *[Pas79, III.1.3] If G is a group and R a ring with identity, then $\{D_{n,R}\}_{n\geq 1}$ is an N-series. If characteristic of R is a prime p, then the series is a restricted N-series relative to p.*

Definition 1.2.5. A decreasing series

$$\Delta_R(G) = A_1 \supseteq A_2 \supseteq \ldots \supseteq A_n \supseteq \ldots \tag{1.50}$$

of two-sided ideals of $R[G]$ is called a **filtration of the augmentation ideal** $\Delta_R(G)$.

Every N-series $\{H_i\}$ of a group, G, induces a *weight function*, $w : G \to \mathbb{N} \cup \infty$:

$$w(x) = \begin{cases} k & if \quad x \in H_k \setminus H_{k+1} \\ \infty & if \qquad x \in \cap_i H_i \end{cases} \tag{1.51}$$

A natural way in which the N-series arise is from filtrations of the augmentation ideals. Let R be a ring with identity and G a group with the N-series $\{H_i\}$. For $n \geq 1$ we can define A_n to be the R-submodule of $R[G]$ spanned by all the products $(g_1 - 1)(g_2 - 1)\ldots(g_s - 1)$ with $\sum_{i=1}^{s} w(g_i) \geq n$ where w is the weight function defined as above.

Then we have $A_iA_j \subseteq A_{i+j}$ for all $i, j \geq 1$ with $A_1 = \Delta_R(G)$. This gives the filtration

$$\Delta_R(G) = A_1 \supseteq A_2 \supseteq \ldots \supseteq A_n \supseteq \ldots \tag{1.52}$$

called the *canonical filtration* of $R[G]$ induced by the N-series $\{H_i\}$.

Remark 1.2.6. The canonical filtration of $R[G]$ induced by the lower central series, $\{\gamma_n(G)\}_{n\geq 1}$ is the filtration given by the powers of the augmentation ideal, i.e.,

$$A_n = \Delta_R^n(G) \qquad \text{for all } n \geq 1.$$

An interesting problem is to investigate the conditions under which the N-series determined by the canonical filtration of the augmentation ideal is $\{H_i\}$ itself, i.e., $H_i = G \cap (1 + A_i)$. The dimension subgroup problem corresponds to the case when $\{H_i\}$ is the lower central series of the group, G.

1.2.2 The Main Results

We begin this section by stating a theorem of Parmenter, Passi and Sehgal [PPS73] which shows that the dimension series $\{D_{n,R}(G)\}_{n\geq 1}$ depends only on the characteristic of the ring R.

Theorem 1.2.7. *[PPS73, 5.1] Let G be any group and R any arbitrary ring with identity.*

1. *If characteristic of R is 0, then*

$$D_{n,R}(G) = \prod_{p\in\sigma(R)} \{\tau_p(G\ mod D_{n,\mathbb{Z}}(G)) \cap D_{n,\mathbb{Z}/p^e\mathbb{Z}}(G)\} \tag{1.53}$$

where $\sigma(R) = \{p|p$ is a prime, $p^nR = p^{n+1}R$ for some $n \geq 0\}$. For $p \in \sigma(R)$, p^e is the smallest power of p for which $p^eR = p^{e+1}R$. $\tau_p(G \mod D_{n,\mathbb{Z}}(G))$ is the p-torsion subgroup of $G \mod D_{n,\mathbb{Z}}(G)$. When $\sigma(R)$ is empty the right hand side of the above equation is interpreted as $D_{n,\mathbb{Z}}(G)$.

2. *If characteristic of R is $r > 0$, then for $n \geq 1$,*

$$D_{n,R}(G) = D_{n,\mathbb{Z}/r\mathbb{Z}}(G) = \bigcap_i D_{n,\mathbb{Z}/p_i^{e_i}\mathbb{Z}}(G) \tag{1.54}$$

where $r = p_i^{e_i}$ is the prime factorization of r.

In particular, when G is a group and k a field, then for every integer $n \geq 1$ we have

$$D_{n,k}(G) = \begin{cases} D_{n,\mathbb{Q}}(G) & \text{if characteristic of } k \text{ is zero} \\ D_{n,\mathbb{Z}/p\mathbb{Z}}(G) & \text{if characteristic of } k \text{ is } p > 0 \end{cases} \tag{1.55}$$

NOTATION: Let H be a subset of a group G. We define

$$\sqrt{H} = \{x \in G | x^m \in H \text{ for some positive integer } m\}. \tag{1.56}$$

It is a result of Jennings that

Theorem 1.2.8. *For all $n \geq 1$,*

$$D_{n,\mathbb{Q}}(G) = \sqrt{\gamma_n(G)}$$

The proof of the above theorem indicated here uses the approach adopted by Passi [?]. The proof makes use of the following lemmas:

Lemma 1.2.9. *[Pas79, IV.1.3]* $G = \sqrt{\gamma_1(G)} \supseteq \sqrt{\gamma_2(G)} \supseteq \ldots \supseteq \sqrt{\gamma_n(G)} \supseteq \ldots$ *is an* N*-series in* G*.*

Proof We observe that $\gamma_n(G) \subseteq \sqrt{\gamma_n(G)}$. Periodic elements of nilpotent groups form a subgroup and hence $\sqrt{\gamma_n(G)}$ is a subgroup of G. It is trivially normal. Let x, y be elements of G and r, s be positive integers such that $x^r \in \gamma_m(G)$ and $y^s \in \gamma_n(G)$. To prove the lemma it needs to be shown that $(x, y) \in \sqrt{\gamma_{n+m}(G)}$. We may assume $\sqrt{\gamma_{n+m}(G)}$ is 1. If the commutator, $(x, y) \neq 1$ then there exists an integer greater than 1 such that $(x, y) \in Z_i \setminus Z_{i-1}$ where $\{Z_i\}_{i\geq 0}$ is the upper central series. Since it is assumed $\sqrt{\gamma_{n+m}(G)} = 1$, G is a torsion-free nilpotent group. Furthermore,

$$1 = (x^r, y^s) \equiv (x, y)^{rs} (\mod Z_{i-1}). \tag{1.57}$$

This means that $(x, y)Z_{i-1}$ is a non-identity torsion element of G/Z_{i-1}. This is not possible because we have seen earlier that G is torsion-free and nilpotent. Thus, $(x, y) = 1$ and $\{\sqrt{\gamma_{n+m}(G)}\}_{n\geq 1}$ is an N-series. □

Lemma 1.2.10. *[Pas79, IV.1.4] The canonical filtration* $\{A_n\}_{n\geq 1}$ *of* $\Delta_{\mathbb{Q}}(G)$ *induced by the* N*-series* $\{\sqrt{\gamma_n(G)}\}_{n\geq 1}$ *is the* $\Delta_{\mathbb{Q}}(G)$*-adic filtration i.e.,* $A_n = \Delta^n_{\mathbb{Q}}(G)$ *for all* $n \geq 1$*.*

Proof Let w be the weight function on G by the N-series $\{\sqrt{\gamma_n(G)}\}_{n\geq 1}$. By definition, we have A_n is the $\mathbb{Q}$-subspace of $\mathbb{Q}[G]$ spanned by products $(g_1 - 1)(g_2 - 1)\ldots(g_s - 1)$ with $s \geq 1$ and $\sum_i w_{g_i} \geq n$. Let $g \in \sqrt{\gamma_n(G)}$ and m be a positive integer such that $g^m \in \gamma_n(G)$. Then, $g^m - 1 \in \Delta^n_{\mathbb{Q}}(G)$. We have the equation

$$g^m - 1 = m(g - 1) + \binom{m}{2}(g - 1)^2 + \ldots + (g - 1)^m. \tag{1.58}$$

This gives $g - 1 \in \Delta^n_{\mathbb{Q}}(G)$. Thus, $\sqrt{\gamma_n(G)} \subseteq D_{n,\mathbb{Q}}(G)$ and $A_n \subseteq \Delta^n_{\mathbb{Q}}(G)$ for all positive integers, n. For all $n \geq 1$, we have $\Delta^n_{\mathbb{Q}}(G) \subseteq A_n$. This gives $A_n = \Delta^n_{\mathbb{Q}}(G)$ for all $n \geq 1$. □

Moving on to the proof of the theorem

Proof Fix a positive integer n. We go modulo $\sqrt{\gamma_n(G)}$ (when necessary) and assume that $\sqrt{\gamma_n(G)} = 1$. We observe that each quotient, $\sqrt{\gamma_i(G)}/\sqrt{\gamma_{i+1}(G)}$, is torsion-free. Hence in view of the following theorem of Jennings,

Theorem 1.2.11. *Let $G = H_1 \supseteq H_2 \supseteq \ldots \supseteq H_c \supseteq H_{c+1} = 1$ be a finite N-series with each successive quotient torsion-free. Let R be a ring with identity of characteristic 0. Then for all positive integers i,*

$$H_i = G \cap (1 + A_i)$$

where $\{A_i\}$ is the canonical filtration of $R[G]$ induced by $\{H_i\}$.

we have

$$G \cap (1 + \Delta^n_{\mathbb{Q}}(G)) = G \cap (1 + A_n) = \sqrt{\gamma_n(G)}. \tag{1.59}$$

□

Corollary 1.2.12. *If G is a group with $\gamma_i(G)/\gamma_{i+1}(G)$ torsion-free for all positive integers i, then $D_{i,\mathbb{Z}}(G) = \gamma_i(G)$ for all i. In particular, when F is a free group $D_{i,\mathbb{Z}}(F) = \gamma_i(F)$.*

The proof follows when we observe that $\sqrt{\gamma_i(G)} = \gamma_i(G)$ for all positive integers i when $\gamma_i(G)/\gamma_{i+1}(G)$ are all torsion-free. Also, $\gamma_2(G) \subseteq D_{i,\mathbb{Z}}(G) \subseteq D_{i,\mathbb{Q}}(G)$.

Remark 1.2.13. When F is a free group, we get the fundamental theorem of free group rings of Magnus Grün and Witt.

We now wish to calculate the dimension series $\{D_{n,\mathbb{Z}/p\mathbb{Z}}(G)\}_{n\geq 1}$ where G is any group and p is a prime. Lemma 1.2.4 tells us that $\{D_{n,\mathbb{Z}/p\mathbb{Z}}(G)\}_{n\geq 1}$ is a restricted N-series relative to p. It is thus a central series with the property $x \in D_{i,\mathbb{Z}/p\mathbb{Z}}(G)$ implies $x^p \in D_{ip,\mathbb{Z}/p\mathbb{Z}}(G)$ for all $i \geq 1$.

This means that $\{D_{n,\mathbb{Z}/p\mathbb{Z}}(G)\}_{n\geq 1}$ must contain the *Brauer Jennings Zassenhaus M*-series $\{M_{n,p}(G)\}_{n\geq 1}$, the minimal central series with the property

$$x \in M_{n,p}(G) \Rightarrow x^p \in M_{np,p}(G) \text{ for all } n \geq 1. \tag{1.60}$$

For any group G and a prime p, the series $\{M_{n,p}(G)\}_{n\geq 1}$ is defined inductively as

$$M_{1,p}(G) = G$$

and

$$M_{n,p}(G) = (G, M_{n-1,p}(G))M^p_{\lceil \frac{n}{p} \rceil,p}(G) \textit{ for } n \geq 2.$$

$\{M_{n,p}(G)\}_{n\geq 1}$ is a central series and hence contains the lower central series of G, $\{\gamma_n(G)\}_{n\geq 1}$. Thus, when $x \in \gamma_i(G)$ and $ip^j \geq n$ then $x^{p^j} \in M_{n,p}(G)$.

This means the M-series contains the series $\{G_{n,p}\}_{n\geq 1}$ of normal subgroups defined by setting

$$G_{n,p} = \prod_{ip^j \geq n} \gamma_i(G)^{p^j} \textit{ for } n \geq 1 \tag{1.61}$$

Theorem 1.2.14. *[Pas79, IV.1.9] For every group G, prime p and all positive integers n,*

$$G_{n,p} = M_{n,p}(G) = D_{n,\mathbb{Z}/p\mathbb{Z}}(G). \tag{1.62}$$

We know that the series $\{G_{n,p}\}_{n\geq 1}$ is a restricted N-series with respect to the prime, p ([Pas79, IV.1.22]).

The proof of the above theorem makes use of the following lemma:

Lemma 1.2.15. *The canonical filtration $\{A_n\}_{n\geq 1}$ of $\Delta_{\mathbb{Z}/p\mathbb{Z}}(G)$ induced by the N-series $\{G_{n,p}\}_{n\geq 1}$ is the $\Delta_{\mathbb{Z}/p\mathbb{Z}}(G)$-adic filtration i.e., $A_n = \Delta^n_{\mathbb{Z}/p\mathbb{Z}}(G)$ for all $n \geq 1$.*

Proof The canonical filtration of a group ring, $R[G]$, induced by the lower central series, $\{\gamma_i(G)\}$, is the filtration given by the powers of the augmentation ideal. In particular, when $R = \mathbb{Z}/p\mathbb{Z}$ (as in this case) we have $A_n \supseteq \Delta^n_{\mathbb{Z}/p\mathbb{Z}}(G)$ for all $n \geq 1$. We thus have to show $G_{n,p} \subseteq 1 + \Delta^n_{\mathbb{Z}/p\mathbb{Z}}(G)$ for all $n \geq 1$. However, from our discussion preceding theorem 1.2.14 ensures that $G_{n,p} \subseteq D_{n,\mathbb{Z}/p\mathbb{Z}}(G)$ for all $n \geq 1$. This proves the lemma. □

We now sketch a proof of the main theorem (theorem 1.2.14).

Proof Let n be a positive integer. We know that $G_{n,p} \subseteq D_{n,\mathbb{Z}/p\mathbb{Z}}(G)$. To prove the equality, we must show that $D_{n,\mathbb{Z}/p\mathbb{Z}}(G) = 1$ when $G_{n,p} = 1$.

Without loss of generality we may assume that G is finitely generated. Furthermore, since we assume that $G_{n,p} = 1$, it means G is finite.

Lemma 1.2.15 tells us that the canonical filtration of $\Delta_{\mathbb{Z}/p\mathbb{Z}}(G)$ induced by $\{G_{i,p}\}_{i\geq 1}$ is the $\Delta_{\mathbb{Z}/p\mathbb{Z}}(G)$-adic filtration of $\Delta_{\mathbb{Z}/p\mathbb{Z}}(G)$.

We have the following result of Lazard [Laz54]:

Proposition 1.2.16. *[Pas79, III.1.7] Let $G = H_1 \supseteq \ldots \supseteq H_c \supseteq H_{c+1} = 1$ be a finite restricted N-series relative to a prime p. Let A_i be the canonical filtration of $\Delta_R(G)$ where R is a ring of characteristic p with identity. Then $H_i = G \cap (1 + A_i)$ for all i.*

The above result yields

$$D_{n,\mathbb{Z}/p\mathbb{Z}}(G) = G \cap (1 + \Delta^n_{\mathbb{Z}/p\mathbb{Z}}(G)) = G_{n,p} = 1. \tag{1.63}$$

This completes our proof. □

Corollary 1.2.17. *Let p be a prime number and let G be a group with $G^p = 1$. Then $D_{i,\mathbb{Z}}(G) = \gamma_i(G)$ for all positive integers i.*

Proof From definition, we have $G_{i,p} = \gamma_i(G)$ when $G^p = 1$. The result follows from

$$\gamma_i(G) \subseteq D_{i,\mathbb{Z}}(G) \subseteq D_{i,\mathbb{Z}/p\mathbb{Z}}(G) = G_{i,p}. \tag{1.64}$$

□

Chapter 2

Lie Dimension Subgroups

Closely related to the dimension subgroups are the Lie dimension subgroups. In this chapter we study integral Lie dimension subgroups, $D_{(n),R}(G)$ when R is the ring $\mathbb{Z}$ of integers or a field. In contrast to the integral dimension subgroups, we will see that many more definite results are known about Lie dimension subgroups.

2.1 Integral Lie Dimension Subgroups

2.1.1 Preliminaries

Definition 2.1.1. A **Lie ring**, L, is an Abelian group with an operation $[\cdot,\cdot]$ that has the following properties:

1. Bilinearity

$$[x+y,z]=[x,y]+[y,z]; \qquad [z,x+y]=[z,x]+[z,y] \tag{2.1}$$

   for all $x,y,z\in L$.

2. Jacobi identity

$$[x,[y,z]]+[y,[z,x]]+[z,[x,y]]=0 \tag{2.2}$$

   for all $x,y,z\in L$.

3. For all $x\in L$

$$[x,x]=0. \tag{2.3}$$

From the last condition we have

$$\begin{aligned} 0 &= [x+y, x+y] \\ &= [x,x]+[x,y]+[y,x]+[y,y] \\ &= [x,y]+[y,x]. \end{aligned} \tag{2.4}$$

Thus $[x, y] = -[y, x]$.

Let $\mathbb{Z}[G]$ be the integral group ring and let $\Delta(G)$ be its augmentation ideal.

Definition 2.1.2. The **Lie powers** $\Delta^{(n)}(G)$, $n \geq 1$, of $\Delta(G)$ are defined inductively as follows:

$$\Delta^{(1)}(G) = \Delta(G) \tag{2.5}$$

and

$$\Delta^{(n)}(G) = [\Delta^{(n-1)}, \Delta(G)]\mathbb{Z}[G] = Ideal_{\mathbb{Z}[G]}\{[x,y] | x \in \Delta^{(n-1)}(G), y \in \Delta(G)\} \tag{2.6}$$

where $[x, y] = xy - yx$ is the Lie product.

$\Delta^{(n)}(G)$ is a two sided ideal. We can see this from the following calculations. Let $\alpha \in \Delta^{(n-1)}(G)$, $\beta \in \Delta(G)$ and $g \in \mathbb{Z}[G]$ then we have

$$\begin{aligned} g \cdot (\alpha\beta - \beta\alpha) &= (g\alpha g^{-1} g\beta g^{-1} - g\beta g^{-1} g\alpha g^{-1}) \cdot g \\ &= (\alpha^g \beta^g - \beta^g \alpha^g) \cdot g \end{aligned}$$

where we use the notation α^g for conjugation by g.

Remark 2.1.3. Even when A and B are ideals of $\mathbb{Z}[G]$, it must be noted that the Lie bracket $[A, B] = \langle [a, b] | a \in A \text{ and } b \in B \rangle$ need not be an ideal.

We have the decreasing series

$$\Delta^{(1)}(G) \supseteq \Delta^{(2)}(G) \supseteq \ldots \Delta^{(k)}(G) \supseteq \ldots. \tag{2.7}$$

Definition 2.1.4. The n^{th} **Lie dimension subgroup** of G is defined as

$$D_{(n)}(G) = G \cap (1 + \Delta^{(n)}(G)) \qquad n \geq 1. \tag{2.8}$$

Clearly we have

$$G = D_{(1)}(G) \supseteq D_{(2)}(G) \supseteq \dots D_{(k)}(G) \supseteq \dots. \tag{2.9}$$

Let $\alpha \in D_{(i)}(G)$ and $\beta \in G$. Then we have

$$\begin{aligned}(\alpha, \beta) - 1 &= \alpha^{-1}\beta^{-1}(\alpha\beta - \beta\alpha) \\ &= \alpha^{-1}\beta^{-1}((\alpha - 1)(\beta - 1) - (\beta - 1)(\alpha - 1)) \\ &= \alpha^{-1}\beta^{-1}[\alpha - 1, \beta - 1].\end{aligned}$$

By definition, $\alpha^{-1}\beta^{-1}[\alpha-1, \beta-1] \in \Delta^{(i+1)}(G)$ since $\alpha - 1 \in \Delta^{(i)}(G)$ and $\beta - 1 \in \Delta(G)$. Thus, $(\alpha, \beta) \in D_{(i+1)}(G)$ and this means that $(D_{(i)}(G), G) \subseteq D_{(i+1)}(G)$. Thus, we see that the series (equation 2.9) is a central series.

Hence, for all $n \geq 1$ and all groups, G, we have the chain

$$\gamma_n(G) \subseteq D_{(n)}(G) \subseteq D_n(G). \tag{2.10}$$

Lemma 2.1.5. *Let G be any group and m, n be positive integers then*

1. *[Pas79, Proposition 1.7(ii)]*$[\Delta^{(m)}(G), \Delta^{(n)}(G)] \subseteq \Delta^{(m+n)}(G)$.
2. *[PS75, Prop 2.2]* $\Delta^{(m)}(G) \cdot \Delta^{(n)}(G) \subseteq \Delta^{(m+n-1)}(G)$.

Proof These statements are proven by induction. Both the statements hold true (trivially) for $m = 1$ and all $n \geq 1$. Suppose that both hold true for some $m \geq 1$ and all $n \geq 1$. We have the following identity in $\mathbb{Z}[G]$

$$[xy, z] = x[y, z] + [x, z]y. \tag{2.11}$$

Thus we have

$$\begin{aligned}[\Delta^{(m+1)}(G), \Delta^{(n)}(G)] &= [[\Delta(G), \Delta^{(m)}(G)]\mathbb{Z}[G], \Delta^{(n)}(G)] \\ &\subseteq [\Delta(G), \Delta^{(m)}(G)][\mathbb{Z}[G], \Delta^{(n)}(G)] + [[\Delta(G), \Delta^{(m)}(G)], \Delta^{(n)}(G)]\mathbb{Z}[G] \\ &\subseteq [\Delta(G), \Delta^{(m)}(G)]\Delta^{(n+1)}(G) + [[\Delta(G), \Delta^{(m)}(G)], \Delta^{(n)}(G)]\mathbb{Z}[G]\end{aligned}$$

Using equation 2.11 and the induction hypothesis we can conclude that

$$[\Delta(G), \Delta^{(m)}(G)]\Delta^{(n+1)}(G) \subseteq \Delta^{(m+n+1)}(G). \tag{2.12}$$

Using the Jacobi identity (equation 2.2) and induction hypothesis we can show that

$$[[\Delta(G), \Delta^{(m)}(G)], \Delta^{(n)}(G)]\mathbb{Z}[G] \subseteq \Delta^{(m+n+1)}(G). \tag{2.13}$$

Therefore we have

$$[\Delta^{(m+1)}(G), \Delta^{(n)}(G)] \subseteq \Delta^{(m+n+1)}(G) \tag{2.14}$$

which proves the first part of the lemma.

For proving the second part, we again make use of equation 2.11 and the induction hypothesis to conclude

$$[\Delta(G), \Delta^{(m)}(G)]\Delta^{(n)}(G) \subseteq \Delta^{(m+n)}(G). \tag{2.15}$$

Thus giving us for all $n \geq 1$,

$$\Delta^{(m+1)}(G) \cdot \Delta^{(n)}(G) \subseteq \Delta^{(n+m)}(G). \tag{2.16}$$

□

Let $\alpha \in D_{(m)}(G)$ and $\beta \in D_{(n)}(G)$. We have

$$(\alpha, \beta) - 1 = \alpha^{-1}\beta^{-1}[\alpha - 1, \beta - 1]. \tag{2.17}$$

(This calculation has been carried out; see preceding text)

Since here we have, $\alpha - 1 \in \Delta^{(m)}(G)$ and $\beta - 1 \in \Delta^{(n)}(G)$, from lemma 2.1.5 we can say that $[\alpha - 1, \beta - 1] \in \Delta^{(m+n)}$. Therefore, $(\alpha, \beta) \in D_{(m+n)}(G)$. This means,

$$(D_{(m)}(G), D_{(n)}(G)) \subseteq D_{(m+n)}(G). \tag{2.18}$$

Therefore, the series 2.9 is an N-series.

Remark 2.1.6. The above definitions and results hold for any ring, R in general.

Sandling gave an explicit formula for $\Delta^{(n)}(G)$ for all n and all G in his 1972 paper, [San72a]. This is the main result discussed in this section.

Theorem 2.1.7. *Let G be any group. Then*

$$\Delta^{(n)}(G) = \Delta(\gamma_n(G))\mathbb{Z}[G] + \sum \prod_j \Delta(\gamma_{n_j}(G))\mathbb{Z}[G].$$

where the sum is over all n_j *such that* $n \geq n_j > 1$ *with* $\sum_j (n_j - 1) = n - 1$.

We make an important observation:

For all $n \geq 1$, we know that $\gamma_n(G)$ is a normal subgroup of G. Consider the map

$$\varphi : G \to G/\gamma_n(G)$$

This map induces a natural epimorphism

$$\tilde{\varphi} : \mathbb{Z}[G] \twoheadrightarrow \mathbb{Z}[G/\gamma_n(G)].$$

Let $\Delta(G, \gamma_n(G))$ denote the kernel of $\tilde{\varphi}$. The kernel is a two-sided ideal generated by $\Delta(\gamma_n(G))$, i.e.

$$\Delta(G, \gamma_n(G)) = \Delta(\gamma_n(G))\mathbb{Z}[G] = \mathbb{Z}[G]\Delta(\gamma_n(G)). \tag{2.19}$$

We now present a proof of theorem 2.1.7.

Proof First we show that RHS $\subseteq$ LHS. We consider the first term on the RHS, $\Delta(\gamma_n(G))\mathbb{Z}[G] = \Delta(G, \gamma_n(G))$. We will proceed by induction on n. $\Delta^{(1)}(G) = \Delta(G)$ and hence for $n = 1$, the first term of the RHS is contained in the LHS. Now, let $g \in G$ and $x \in \gamma_{n-1}(G)$ (induction hypothesis). Let $\Delta(G, \gamma_{n-1}(G)) \subseteq \Delta^{(n-1)}(G)$ where $n \geq 2$. We have seen in our calculations that

$$(g, x) - 1 = g^{-1}x^{-1}[g - 1, x - 1] \in \Delta^{(n)}(G)$$

where $x - 1 \in \Delta^{(n-1)}(G)$ by induction hypothesis.
$\Delta(G, \gamma_n(G))$ is generated by elements of the type $(g, x) - 1$, thus proving that the first term of the RHS is contained in the LHS. Using the second statement of lemma 2.1.5, we can easily see that the second term on the RHS is also contained in the LHS.

To show that LHS $\subseteq$ RHS we will again proceed by induction. For $n = 2$, the statement holds as $\Delta^{(2)}(G) = [\Delta(G), \Delta((G)]\mathbb{Z}[G]$ which is contained in $\Delta^{(2)}(G)\mathbb{Z}[G]$. Suppose $n > 2$ and

$$\Delta^{(n-1)}(G) = \Delta(\gamma_{n-1}(G))\mathbb{Z}[G] + \sum \prod_j \Delta(\gamma_{n_j}(G))\mathbb{Z}[G]$$

where the sum is over all n_j and $n - 1 \geq n_j > 1$ with $\sum_j (n_j - 1) = n - 2$.

Let $g, h \in G$ and $x \in \gamma_{n-1}(G)$. Now

$$\begin{aligned} [(x-1)g, h-1] &= (x-1)[g, h-1] + [x-1, h-1]g \\ &= (x-1)gh((g,h)-1) + hx((x,h)-1)g \end{aligned} \tag{2.20}$$

where the first equality follows from equation 2.11.

The first term on the RHS of equation 2.20 is an element of $\Delta(G, \gamma_{n-1}(G)) \cdot \Delta(G, \gamma_2(G))$. The second term is an element of $\Delta(G, \gamma_n(G))$. Therefore

$$[\Delta(G, \gamma_{n-1}(G)), \Delta(G)] \subseteq \Delta(G, \gamma_n(G)) + \Delta(G, \gamma_{n-1}(G)) \cdot \Delta(G, \gamma_2(G)). \tag{2.21}$$

We will be done if we can show that

$$[\prod_j \Delta(\gamma_{n_j}(G))\mathbb{Z}[G], \Delta(G)] \subseteq \Delta(G, \gamma_n(G)) + \sum \prod_i \Delta(G, \gamma_{m_i}(G)) \tag{2.22}$$

where the sum is over all m_i such that $n \geq m_i > 1$ and $\sum_i (m_i - 1) = n - 1$. Also, $n - 1 \geq n_j > 1$ and $\sum_j (n_j - 1) = n - 2$.

Using equation 2.11, calculations like those done above allow us to conclude that

$$[\prod_j \Delta(\gamma_{n_j}(G))\mathbb{Z}[G], \Delta(G)] \subseteq \prod_j \Delta(\gamma_{n_j}(G)) \cdot \Delta(\gamma_2(G)) \cdot \mathbb{Z}[G] + [\prod_j \Delta(\gamma_{n_j}(G)), \Delta(G)]\mathbb{Z}[G]. \tag{2.23}$$

The first term on the RHS of the equation 2.23 is of the required type. Furthermore, repeatedly using equation 2.11 and observing that

$$[\Delta(\gamma_i(G)), \Delta(G)] \subseteq \Delta(G, \gamma_{i+1}(G)) \tag{2.24}$$

shows that the second term is also of the desired type. With this the proof is complete. □

Another notion related to Lie dimension subgroups is the restricted Lie powers of $\Delta(G)$.

Definition 2.1.8. The **restricted Lie powers** of $\Delta_(G)$ are defined inductively by

$$\Delta^{[1]}(G) = \Delta(G) \tag{2.25}$$

and

$$\Delta^{[n]}(G) = [\Delta^{[n-1]}, \Delta(G)]. \tag{2.26}$$

Definition 2.1.9. The **restricted Lie dimension subgroup** for all $n \geq 1$ is defined as

$$D_{[n]}(G) = G \cap (1 + \Delta^{[n]}(G)\mathbb{Z}[G]). \tag{2.27}$$

In [GL83], Gupta and Levin showed that $\gamma_n(G) \subseteq D_{[n]}(G)$. For an arbitrary group G, by induction we can get the following chain

$$\gamma_n(G) \subseteq D_{[n]}(G) \subseteq D_{(n)}(G) \subseteq D_n(G) \text{ for all } n \geq 1. \tag{2.28}$$

2.1.2 Lie Dimension Subgroup Problem

An example of a group without dimension property was given by Rips. When the above chain 2.28 was observed it was natural to question whether $D_{(n)(G)} = \gamma_n(G)$ for all n and all G. Sandling [San72a] showed that the Lie dimension subgroup property holds for all groups when $n \leq 6$. Hurley and Sehgal [HS91] showed that $D_{(n)}(G) \neq \gamma_n(G)$ in general for $n \geq 9$. They constructed a 2-group G such that $D_{(n)}(G) \neq \gamma_n(G)$. The case of $n = 7$ and 8 was settled by Gupta and Tahara [GT93] in the affirmative. As a consequence of equation 2.28, the analogous restricted Lie dimension subgroup property holds for $n \leq 8$. Hurley and Sehgal [HS91] showed that the $D_{[n]}(G) \neq \gamma_n(G)$ in general for $n \geq 14$. The gap is resolved negatively by Gupta and Srivastava [GS91] where they show that $\gamma_n(G) \neq D_{[n]}(G)$ for all $9 \leq n \leq 13$.

The Case $n \leq 6$

We will now see that for all groups, G, the **Lie dimension property** holds, i.e., $D_{(n)}(G) = \gamma_n(G)$. Since $D_n(G) = \gamma_n(G)$ for all groups when $n \leq 3$, $D_{(n)}(G) = \gamma_n(G)$ for all groups when $n \leq 3$. Here we will give independent proofs of $D_n(G) = \gamma_n(G)$ for all groups when $n \leq 6$ using Sandling's formula.

For $n = 1$, there is nothing to prove.

For $n = 2$, we have to show that $D_{(2)}(G) = \gamma_2(G)$. Now, $\gamma_2(G) \subseteq D_{(2)}(G)$ holds always so we only have to show the other way inclusion. Let $g \in D_{(2)}(G)$, therefore

$g-1 \in \Delta^{(2)}(G)$. From theorem 2.1.7 we know that $\Delta^{(2)}(G) = \Delta(\gamma_2(G))\mathbb{Z}[G]$. We can write $g - 1 = \sum_i n_i g_i [f_i - 1, h_i - 1]$.

Consider the map

$$\varphi : G \to G/\gamma_2(G). \tag{2.29}$$

This map then extends to

$$\bar{\varphi} : \mathbb{Z}[G] \to \mathbb{Z}[G/\gamma_2(G)]. \tag{2.30}$$

Now since $G/\gamma_2(G)$ is Abelian we have

$$\begin{aligned} \bar{\varphi}(g-1) \;=\; \varphi(g) - 1 \;&=\; \textstyle\sum_i n_i \varphi(g_i)[\varphi(f_i) - 1, \varphi(h_i) - 1] \\ &=\; 0. \end{aligned} \tag{2.31}$$

Therefore $g \cdot \gamma_2(G) = \gamma_2(G)$ i.e., $g \in \gamma_2(G)$.

For $n = 3$, we must show that $D_{(3)}(G) = \gamma_3(G)$.

Using theorem 2.1.7 we have

$$\Delta^{(3)}(G) = \mathbb{Z}[G]\Delta(\gamma_3(G)) + \mathbb{Z}[G]\Delta(\gamma_2(G))^2. \tag{2.32}$$

Let $g \in D_{(3)}(G)$ then $g - 1 \in \Delta^{(3)}(G)$.

We consider the map

$$\theta : G \to \gamma_2(G). \tag{2.33}$$

This map extends to

$$\bar{\theta} : \mathbb{Z}[G] \to \mathbb{Z}[\gamma_2(G)]. \tag{2.34}$$

Therefore

$$\bar{\theta}(g-1) \in \mathbb{Z}[\gamma_2(G)]\Delta(\gamma_3(G)) + \mathbb{Z}[\gamma_2(G)]\Delta(\gamma_2(G))^2. \tag{2.35}$$

We know that $\gamma_2(G)/\gamma_3(G)$ is Abelian and hence

$$D_2(\gamma_2(G)/\gamma_3(G)) = \gamma_2(\gamma_2(G)/\gamma_3(G)) = 1.$$

Now, we consider the map

$$\varphi : \gamma_2(G) \to \gamma_2(G)/\gamma_3(G) \tag{2.36}$$

which has a linear extension to

$$\bar{\varphi} : \mathbb{Z}[\gamma_2(G)] \to \mathbb{Z}[\gamma_2(G)/\gamma_3(G)]. \tag{2.37}$$

Equation 2.35 under the map $\bar{\varphi}$ yields

$$g \cdot \gamma_3(G) = \gamma_3(G) \Rightarrow g \in \gamma_3(G).$$

For $n = 4$, our aim is to show that $D_{(4)}(G) = \gamma_4(G)$. It is enough to show that RHS contains the LHS. We consider an element g in $D_{(4)}(G)$. Thus, $g - 1$ is an element of $\Delta^{(4)}(G)$. Theorem 2.1.7 gives us

$$\Delta^{(4)}(G) = \mathbb{Z}[G]\Delta(\gamma_4(G)) + \mathbb{Z}[G]\Delta(\gamma_2(G))^3 + \mathbb{Z}[G]\Delta(\gamma_3(G))\Delta(\gamma_2(G)) + \mathbb{Z}[G]\Delta(\gamma_2(G))\Delta(\gamma_3(G)). \tag{2.38}$$

We consider the map

$$\theta : G \to \gamma_2(G). \tag{2.39}$$

This map extends to

$$\bar{\theta} : \mathbb{Z}[G] \to \mathbb{Z}[\gamma_2(G)]. \tag{2.40}$$

Therefore

$$\bar{\theta}(g - 1) \in \mathbb{Z}[\gamma_2(G)]\Delta(\gamma_4(G)) + \mathbb{Z}[\gamma_2(G)]\Delta(\gamma_2(G))^2. \tag{2.41}$$

We know that $\gamma_2(G)/\gamma_4(G)$ is Abelian and hence

$$D_2(\gamma_2(G)/\gamma_4(G)) = \gamma_2(\gamma_2(G)/\gamma_4(G)) = 1.$$

Now, we consider the map

$$\varphi : \gamma_2(G) \to \gamma_2(G)/\gamma_4(G) \tag{2.42}$$

which has a linear extension to

$$\bar{\varphi} : \mathbb{Z}[\gamma_2(G)] \to \mathbb{Z}[\gamma_2(G)/\gamma_4(G)]. \tag{2.43}$$

Equation 2.41 under the map $\bar{\varphi}$ yields

$$g \cdot \gamma_4(G) = \gamma_4(G) \Rightarrow g \in \gamma_4(G).$$

For $n = 5$, our aim is to show that $D_{(5)}(G) = \gamma_5(G)$. Again, it is enough to show that RHS contains the LHS, since the other way inclusion is trivial. We consider an element $g \in D_{(5)}(G)$. Just as we have seen earlier, $g-1$ is an element of $\Delta^{(5)}(G)$. Using theorem 2.1.7 we get

$$\begin{aligned}\Delta^{(5)}(G) &= \mathbb{Z}[G]\Delta(\gamma_5(G)) + \mathbb{Z}[G]\Delta(\gamma_4(G))\Delta(\gamma_2(G)) + \mathbb{Z}[G]\Delta(\gamma_2(G))\Delta(\gamma_4(G)) \\ &+ \mathbb{Z}[G]\Delta(\gamma_3(G))^2 + \mathbb{Z}[G]\Delta(\gamma_2(G))^4 + Z[G]\Delta(\gamma_3(G))\Delta(\gamma_2(G))^2 \\ &+ \mathbb{Z}[G]\Delta(\gamma_2(G))^2\Delta(\gamma_3(G)) + \mathbb{Z}[G]\Delta(\gamma_2(G))\Delta(\gamma_3(G))\Delta(\gamma_2(G)).\end{aligned} \tag{2.44}$$

We consider the map

$$\theta : G \to \gamma_4(G). \tag{2.45}$$

This map extends to

$$\bar{\theta} : \mathbb{Z}[G] \to \mathbb{Z}[\gamma_4(G)]. \tag{2.46}$$

Therefore

$$\bar{\theta}(g-1) \in \mathbb{Z}[\gamma_4(G)]\Delta(\gamma_5(G)) + \mathbb{Z}[\gamma_4(G)]\Delta(\gamma_4(G))^2. \tag{2.47}$$

We know that $\gamma_4(G)/\gamma_5(G)$ is Abelian and hence

$$D_2(\gamma_4(G)/\gamma_5(G)) = \gamma_2(\gamma_4(G)/\gamma_5(G)) = 1.$$

Now, we consider the map

$$\varphi : \gamma_4(G) \to \gamma_4(G)/\gamma_5(G) \tag{2.48}$$

which has a linear extension to

$$\bar{\varphi} : \mathbb{Z}[\gamma_4(G)] \to \mathbb{Z}[\gamma_4(G)/\gamma_5(G)]. \tag{2.49}$$

Equation 2.47 under the map $\bar{\varphi}$ yields

$$g \cdot \gamma_5(G) = \gamma_5(G) \Rightarrow g \in \gamma_5(G).$$

For $n = 6$, our aim is to show that $D_{(6)}(G) = \gamma_6(G)$. Here also we will show that RHS contains the LHS, since the other way inclusion is trivial. We consider an element $g \in D_{(6)}(G)$. Just as we have seen earlier, $g - 1$ is an element of $\Delta^{(6)}(G)$. Using theorem 2.1.7 we have

$$\begin{aligned}
\Delta^{(6)}(G) &= \mathbb{Z}[G]\Delta(\gamma_6(G)) + \mathbb{Z}[G]\Delta(\gamma_5(G))\Delta(\gamma_2(G)) + \mathbb{Z}[G]\Delta(\gamma_2(G))\Delta(\gamma_5(G)) \\
&+ \mathbb{Z}[G]\Delta(\gamma_3(G))^2\Delta(\gamma_2(G)) + \mathbb{Z}[G]\Delta(\gamma_2(G))\Delta(\gamma_3(G))^2 + Z[G]\Delta(\gamma_3(G))\Delta(\gamma_2(G))\Delta(\gamma_3(G)) \\
&+ \mathbb{Z}[G]\Delta(\gamma_2(G))^5 + \mathbb{Z}[G]\Delta(\gamma_4(G))\Delta(\gamma_3(G)) + \mathbb{Z}[G]\Delta(\gamma_4(G))\Delta(\gamma_3(G)) \\
&+ \mathbb{Z}[G]\Delta(\gamma_2(G))^2\Delta(\gamma_4(G)) + \mathbb{Z}[G]\Delta(\gamma_4(G))\Delta(\gamma_2(G))^2 + Z[G]\Delta(\gamma_2(G))\Delta(\gamma_4(G))\Delta(\gamma_2(G)).
\end{aligned} \tag{2.50}$$

We consider the map

$$\theta : G \to \gamma_3(G). \tag{2.51}$$

This map extends to

$$\bar{\theta} : \mathbb{Z}[G] \to \mathbb{Z}[\gamma_3(G)]. \tag{2.52}$$

Therefore

$$\bar{\theta}(g - 1) \in \mathbb{Z}[\gamma_3(G)]\Delta(\gamma_6(G)) + \mathbb{Z}[\gamma_3(G)]\Delta(\gamma_3(G))^2. \tag{2.53}$$

We know that $\gamma_3(G)/\gamma_6(G)$ is Abelian and hence

$$D_2(\gamma_3(G)/\gamma_6(G)) = \gamma_2(\gamma_3(G)/\gamma_6(G)) = 0.$$

Now, we consider the map

$$\varphi : \gamma_3(G) \to \gamma_3(G)/\gamma_6(G) \tag{2.54}$$

which has a linear extension to

$$\bar{\varphi} : \mathbb{Z}[\gamma_3(G)] \to \mathbb{Z}[\gamma_3(G)/\gamma_6(G)]. \tag{2.55}$$

Equation 2.53 under the map $\bar{\varphi}$ yields

$$g \cdot \gamma_6(G) = \gamma_6(G) \Rightarrow g \in \gamma_6(G).$$

Thus we have shown that $D_{(n)}(G) = \gamma_n(G)$ for all $n \leq 6$.

The Case $n = 7$ and 8

Gupta and Tahara [GT93] proved that all groups have the Lie dimension property for $n = 7$ and 8. This was done by translating the problem into the language of free group rings. It suffices if we can prove it for finite nilpotent groups. Let G be such a group given by the free presentation

$$1 \to R \to F \to G \to 1 \tag{2.56}$$

where F is a free group on the set $X = \{x_1, \ldots x_m\}$ and R is a normal subgroup of F.

The Lie dimension subgroup problem in this language translates to identifying the quotient $F \cap (1 + \mathfrak{r} + \mathfrak{f}^{(n)})/R\gamma_n(G)$.

The Case $n \geq 9$

For this section we will refer to [HS91].

In this section, we study the example of a group with non-trivial Lie dimension property provided by Hurley and Sehgal. They showed $D_{(n)}(G) \neq \gamma_n(G)$ in general when $n \geq 9$ by giving an example which is a modification of the Gupta group.

Let F be a free group on a set of generators X. Let n is a positive integer greater than or equal to 9. We define

$$t = \begin{cases} \frac{(n-3)}{2} & \text{if} \quad n \text{ is odd} \\ \frac{(n-4)}{2} & \text{if} \quad n \text{ is even} \end{cases} \tag{2.57}$$

Let us assume $a,\ b,\ c$ are distinct simple basic commutators of weight t, which are arranged as $a > b > c$. Furthermore, r is another basic commutator. It is of weight 2 when n is odd and of weight 3 when n is even. By convention, we let $r < c$ when r and c have the same weight. We define $p = n - t - 1$. Let there be at least one symbol in each $a,\ b,\ c,\ r$ not in others.

The construction of such a group is carried out in a series of steps.

1. Set $R_1 = \gamma_n(F)$.

2. Define H as the normal closure of $\{(a,b,x),(b,c,x),(a,c,x)|x \in F\}$ in F, i.e., $H = \langle (a,b,x),(b,c,x),(a,c,x)|x \in F\rangle^F$.

3. Set $R_2 = \gamma_n(F) \cdot H$.

4. Define $R_3 = \langle (a^{64},r),(b^{16},r),(c^4,r)\rangle^F \cdot R_2$.

5. $R_4 = \langle (c,r,c)^{-1}(b,r,b)^4, (b,r,b)^{-1}(a,r,a)^4, (a,b)^{-16}(a,r,a)^4, (b,c)^{-4}(a,r,a)^4, (c,a)^{-4}(a,r,a)^2\rangle^F \times R_3$.

6. Set $R = \langle a^{-64}(b,r)^4(c,r)^2, b^{-16}(a,r)^{-4}(c,r), c^{-4}(a,r)^{-2}(b,r)^{-1}\rangle^F \cdot R_4$.

7. Finally set $G = F/R$.

This completes the construction of G.

The following lemmas are needed for the proof.

Lemma 2.1.10. *[HS91, Lemma 3] In F/R_3 we have*

1. $(a,r,a)^{64} = 1$; $(b,r,b)^{16} = 1$; $(c,r,c)^4 = 1$.

2. $(a,r)^{64} = (a,r,a)^{32}$; $(b,r)^{16} = (b,r,b)^8$; $(c,r)^4 = (c,r,c)^2$.

3. $(a,r)^{128} = (b,r)^{32} = (c,r)^8 = 1$.

Extensive calculations done in [HS91, section 3] prove that the order of (a,b) in G is 256. To disprove the conjecture, it can be shown that there exists $g \in F$ such that modulo R, $g \equiv (a,b)^{128}$ and that $g - 1 \equiv 0 \mod (\mathfrak{f}^{(n)} + \mathfrak{r})$. Since $(a,b)^{128} \not\equiv 1 \mod R$ therefore the image of g in G (say $\bar{g}$) is not equal to 1. Therefore, $\bar{g} \in \gamma_n(G)$ where $\bar{g} \neq 1$. Hence $D_{(n)}(G) \neq \gamma_n(G)$.

Consider the element $g = (a,b)^{128}(a,c)^{64}(b,c)^{32} \in F$. Using relators 5,

$$\begin{aligned} (a,c)^4 &\equiv (a,r,a)^2 \\ \Rightarrow (a,c)^{64} &\equiv (a,r,a)^{32}, \\ (b,c)^{32} &\equiv (a,r,a)^{32}. \end{aligned}$$

Therefore, $(a,c)^{64}(b,c)^{32} \equiv (a,r,a)^{64} = 1$. Hence, $g \equiv (a,b)^{128} \mod R$.

Now, we move towards showing that $g - 1 \in \mathfrak{f}^{(n)} + \mathfrak{r}$. For this purpose we will work modulo $\mathfrak{f}^{(n)} + \mathfrak{r}$.

We first observe that $(a,b)^{128}, (a,c)^{64}, (b,c)^{32} \in \gamma_{n-4}(F)$. Using the known identity

$$xy - 1 = (x-1) + (y-1) + (x-1)(y-1), \tag{2.58}$$

we can conclude that

$$g - 1 \equiv ((a,b)^{128} - 1) + ((a,c)^{64} - 1) + ((b,c)^{32} - 1). \tag{2.59}$$

Consider the first term, $((a,b)^{128} - 1) \equiv 128((a,b) - 1)$. We also have the identity

$$(a,b) - 1 = (a^{-1}b^{-1} - 1)\{(a-1)(b-1) - (b-1)(a-1)\} + \{(a-1)(b-1) - (b-1)(a-1)\}. \tag{2.60}$$

We observe that $b^{32} - 1 \equiv 0$ and hence $128(a^{-1}b^{-1} - 1)\{(a-1)(b-1) - (b-1)(a-1)\} \equiv 0$. Therefore we have the equivalence,

1. $(a,b)^{128} - 1 \equiv 128\{(a-1)(b-1) - (b-1)(a-1)\}$.

Similarly, we have

2. $(a,c)^{64} - 1 \equiv 64\{(a-1)(c-1) - (c-1)(a-1)\}$.
3. $(b,c)^{32} - 1 \equiv 32\{(b-1)(c-1) - (c-1)(b-1)\}$.

Equation 2.59 therefore becomes

$$\begin{aligned} g - 1 \quad \equiv \quad & \{128(a-1)(b-1) - (b-1)(a-1)\} + \{64(a-1)(c-1) - (c-1)(a-1)\} \\ & + \{32(b-1)(c-1) - (c-1)(b-1)\}. \end{aligned} \tag{2.61}$$

For the final step the following two lemmas are needed:

Lemma 2.1.11. *Modulo* $\mathfrak{f}^{(n)} + \mathfrak{r}$,

1. $64(c-1)(a-1) \equiv (c-1)(a^{64} - 1)$.
2. $64(a-1)(c-1) \equiv (a-1)(c^{64} - 1)$.
3. $128(b-1)(a-1) \equiv (b-1)(a^{128} - 1)$.
4. $128(a-1)(b-1) \equiv (a-1)(b^{128} - 1)$.

5. $32(b-1)(c-1) \equiv (b-1)(c^{32}-1)$.

6. $64(c-1)(b-1) \equiv (c-1)(b^{32}-1)$.

Proof The first part is proven, the others follow in the same way

$$\begin{aligned} 64(c-1)(a-1) &\equiv (c-1)((a^{64}-1)+\lambda_1 \cdot 32(a-1)^2+\lambda_2 \cdot 16(a-1)^3) \qquad \lambda_i \in \mathbb{Z}. \\ &\equiv (c-1)(a^{64}-1)+\lambda_1 \cdot 32(c-1)(a-1)^2+\lambda_2 \cdot 16(c-1)(a-1)^3. \\ &\equiv (c-1)(a^{64}-1). \end{aligned} \tag{2.62}$$

The final equivalence follows from the observation that modulo $\mathfrak{f}^{(p)}+\mathfrak{r}$,

$$\begin{aligned} c^8-1 &\equiv 8(c-1)+\binom{8}{2}(c-1)^2 \\ &\equiv 8(c-1)+7(c^4-1)(c-1) \end{aligned} \tag{2.63}$$

where $p = n-t-1$ as before.

Since $c^4 \in \gamma_p(F)$, we have $7(c^4-1)(c-1)$ and c^8-1 in $\mathfrak{f}^{(p)}$. Hence $8(c-1) \in \mathfrak{f}^{(p)}+\mathfrak{r}$.

□

Lemma 2.1.12. *1.* $b^{128}c^{64} \in \gamma_p^{64}(F) \cdot \gamma_{n-1}(F) \cdot R$.

2. $a^{128}c^{-32} \in \gamma_p^{16}(F) \cdot \gamma_{n-1}(F) \cdot R$.

3. $a^{64}b^{32} \in \gamma_p^{4}(F) \cdot \gamma_{n-1}(F) \cdot R$.

Proof The proof of the first part is given; the proof of the other two parts will follow in a similar fashion.

$$\begin{aligned} b^{128}c^{64} &= b^{16\cdot 8}c^{4\cdot 16} \\ &\equiv ((a,r)^{-4}(c,r))^8 \cdot ((a,r)^{-2}(b,r)^{-1})^{16} && \text{mod } R \text{ using } 6 \\ &\equiv (a,r)^{-32} \cdot (c,r)^8 \cdot (a,r)^{-32} \cdot (b,r)^{-16} && \text{mod } R \\ &\equiv (a,r)^{-64} \cdot (b,r)^{-16} \cdot (c,r)^8 && \text{mod } R \\ &\equiv 1 && \text{mod } (\gamma_{n-1}(F) \cdot R) \text{ using 2.1.10.} \end{aligned} \tag{2.64}$$

□

Using lemma 2.1.11 and relators 6 we see that 2.61 transforms to

$$\begin{aligned} g-1 &\equiv (a-1)(b^{128}-1)-(b-1)(a^{128}-1)+(a-1)(c^{64}-1)-(c-1)(a^{64}-1) \\ &\quad +(b-1)(c^{32}-1)-(c-1)(b^{32}-1) \\ &\equiv (a-1)(b^{128}c^{64}-1)-(b-1)(a^{128}c^{-32}-1)-(c-1)(a^{64}b^{32}-1) \\ &\equiv (a-1)(d_a^{64}-1)-(b-1)(d_b^{16}-1)-(c-1)(d_c^{4}-1). \end{aligned} \tag{2.65}$$

With $d_a, d_b, d_c \in \gamma_p(F)$ and using lemma 2.1.12 we have

$$\begin{aligned} g-1 &\equiv 64(a-1)(d_a-1)-16(b-1)(d_b-1)-4(c-1)(d_c-1) \\ &\equiv (a^{64}-1)(d_a-1)-(b^{16}-1)(d_b-1)-(c^{4}-1)(d_c-1) \\ &\equiv 0 \text{ using 6.} \end{aligned} \tag{2.66}$$

With this the proof of $D_{(n)}(G) \neq \gamma_n(G)$ when $n \geq 9$ is complete.

2.2 Lie Dimension Subgroups Over Fields

Analogous to the study of dimension subgroups over fields, the Lie dimension subgroups over fields have also been studied. We start this section by mentioning a theorem of Parmenter, Passi and Sehgal [PPS73] which gives us an explicit formula for Lie dimension subgroups over arbitrary rings of coefficients. What is important to note is that just like the dimension series $\{D_{n,R}(G)\}_{n\geq 1}$ depends only on the characteristic of R, the Lie dimension series $\{D_{(n),R}(G)\}_{n\geq 1}$ also depends only on the characteristic of R.

Theorem 2.2.1. *[PPS73, 6.1] Let G be any group and R be any arbitrary ring.*

1. *If characteristic of R is 0, then*

$$D_{(n),R}(G) = \prod_{p\in\sigma(R)} \gamma_2(G) \cap \{\tau_p(G \; mod D_{(n),\mathbb{Z}}(G)) \cap D_{(n),\mathbb{Z}/p^e\mathbb{Z}}(G)\} \tag{2.67}$$

where $\sigma(R) = \{p | p \text{ is a prime, } p^nR = p^{n+1}R \text{ for some } n \geq 0\}$. For $p \in \sigma(R)$, p^e is the smallest power of p for which $p^eR = p^{e+1}R$. $\tau_p(G \mod D_{(n),\mathbb{Z}}(G))$ is the p-torsion subgroup of $\mathbb{Z}[G] \mod D_{(n),\mathbb{Z}}(G)$. When $\sigma(R)$ is empty then the right hand side is interpreted as $D_{(n),\mathbb{Z}}(G)$.

2. If characteristic of R is $r > 0$, then for $n \geq 1$,

$$D_{(n),R}(G) = D_{(n),\mathbb{Z}/r\mathbb{Z}}(G) = \bigcap_i D_{(n),\mathbb{Z}/p_i^{e_i}\mathbb{Z}}(G) \tag{2.68}$$

where $r = p_i^{e_i}$ is the prime factorization of r.

In particular, when G is a group and k a field, for every integer $n \geq 1$ we have

$$D_{(n),k}(G) = \begin{cases} D_{(n),\mathbb{Q}}(G) & \textit{if} \quad \textit{characteristic of } k \textit{ is zero} \\ D_{(n),\mathbb{Z}/p\mathbb{Z}}(G) & \textit{if} \quad \textit{characteristic of } k \textit{ is } p > 0 \end{cases} \tag{2.69}$$

Passi and Sehgal found an explicit formula for the case characteristic of the field is 0.

Theorem 2.2.2. *[PS75] For all $n \geq 2$, we have*

$$D_{(n),\mathbb{Q}}G = \sqrt{\gamma_n(G)} \cap \gamma_2(G)$$

Proof Claim: $D_{(2),\mathbb{Q}}(G) = \gamma_2(G)$.

Justification: Let x, y be elements of the group, G. We have seen that

$$\begin{aligned} (x,y) - 1 &= x^{-1}y^{-1}\{(x-1)(y-1) - (y-1)(x-1)\} \\ &\in \Delta_R^{(2)}(G). \end{aligned} \tag{2.70}$$

Therefore, $\gamma_2(G)$ is contained in $D_{(2),\mathbb{Q}}(G)$. For the reverse inclusion, we can assume that the group is Abelian. This would mean that $\Delta_R^{(2)}(G) = 0$ and hence $D_{(2),\mathbb{Q}}(G) = 1$. With this we have proven the claim.

We have from equation 2.28, $D_{(n),\mathbb{Q}}(G) \subseteq D_{n,\mathbb{Q}}(G)$. Using theorem 1.2.8, we have

$$D_{(n),\mathbb{Q}}(G) \subseteq \sqrt{\gamma_n(G)} \cap \gamma_2(G). \tag{2.71}$$

To prove the other way inclusion we consider an element $x \in \sqrt{\gamma_n(G)} \cap \gamma_2(G)$ for some $n \geq 2$. By definition, there exists a positive integer m such that x^m is an element of $\gamma_n(G)$. We consider the equation

$$x^m - 1 = m(x-1) + \binom{m}{2}(x-1)^2 + \ldots + (x-1)^m. \tag{2.72}$$

Let $x-1 \in \Delta_{\mathbb{Q}}^{(j)}(G) \setminus \Delta_{\mathbb{Q}}^{(j+1)}(G)$ where $2 \leq j < n$. Using lemma 2.1.5, the above equation 2.72 tells us

$$x^m - 1 \in \Delta_{\mathbb{Q}}^{(s)}(G) \qquad \text{where } s = min\{n, 2j-1\}.$$

However, since $s \geq j+1$, we arrive at a contradiction. Hence $x-1 \in \Delta_{\mathbb{Q}}^{(n)}(G)$ i.e., $x \in D_{(n),\mathbb{Q}}(G)$. With this the proof is complete. □

Let G be a group and p be any prime. Analogous to the M-series we had earlier, we define a series $\{M_{(n),p}(G)\}_{n\geq 1}$ as follows:

$$M_{(1),p}(G) = G, \qquad M_{(2),p}(G) = \gamma_2(G)$$

and

$$M_{(n),p}(G) = (G, M_{(n-1),p}(G))M^p_{\lceil \frac{n+p-1}{p} \rceil,p}(G) \textit{ for } n \geq 3.$$

This series has the property

$$M^p_{(i),p}(G) \subseteq M_{(ip-p+1),p}(G) \text{ for all } i \geq 1. \tag{2.73}$$

We define a series $\{G_{(n),p}\}_{n\geq 1}$ by setting

$$G_{(n),p} = \prod_{(i-1)p^j \geq n} \gamma_i(G)^{p^j}. \tag{2.74}$$

We observe that $G_{(1),p} = \gamma_2(G)$.

Passi and Sehgal [PS75, Section 4] showed that the above defined series is a restricted N-series of $\gamma_2(G)$. Our aim now is to study the structure of the Lie dimension subgroups $D_{(n),\mathbb{Z}/p\mathbb{Z}}(G)$ for all $n \geq 1$.

Theorem 2.2.3. *[PS75, Theorem 4.10] For every group G, prime p and for all $n \geq 1$,*

$$G_{(n),p} = M_{(n+1),p}(G) = D_{(n+1),\mathbb{Z}/p\mathbb{Z}}(G).$$

Proof The proof this theorem is given in two steps. First, we will show that

$$G_{(n),p} \subseteq M_{(n+1),p}(G) \subseteq D_{(n+1),\mathbb{Z}/p\mathbb{Z}}(G). \tag{2.75}$$

From the way we have defined the series $\{M_{(n),p}(G)\}_{n\geq 1}$, it is a central series in G. Therefore, $\gamma_n(G) \subseteq M_{(n),p(G)}$ for all positive integers n. From equation 2.73 we have

$$\gamma_i(G)^{p^j} \subseteq M_{((i-1)^{p^j}+1),p}(G). \tag{2.76}$$

If we impose the condition $ip^j \geq n + p^j$, i.e., $(i-1)p^j + 1 \geq n+1$, the we have,

$$\gamma_i(G)^{p^j} \subseteq M_{(n+1),p}(G). \tag{2.77}$$

This gives us the first inclusion of equation 2.75.

The second inclusion can be shown by induction on n. The inclusion holds for $n = 1$ since

$$M_{(2),p}(G) = \gamma_2(G) = D_{(2),\mathbb{Z}/p\mathbb{Z}}(G). \tag{2.78}$$

Let $m \geq 2$ and let us assume that the inclusion holds for all $n < m$. By definition

$$M_{(m+1),p}(G) = (G, M_{(m),p}(G))M^p_{\lceil \frac{m+p}{p} \rceil,p}(G). \tag{2.79}$$

By induction hypothesis

$$M_{(m),p}(G) \subseteq D_{(m),\mathbb{Z}/p\mathbb{Z}}(G). \tag{2.80}$$

We have seen that the Lie dimension series is an N-series. Therefore

$$(G, M_{(m),p}(G)) \subseteq (G, D_{(m),\mathbb{Z}/p\mathbb{Z}}(G)) \subseteq D_{(m+1),\mathbb{Z}/p\mathbb{Z}}(G). \tag{2.81}$$

Let $s = \lceil \frac{m+p}{p} \rceil$ and x be an element of $M_{(s),p}(G)$. By induction

$$x \in D_{(s),\mathbb{Z}/p\mathbb{Z}}(G) \Rightarrow x - 1 \in \Delta^{(s)}_{\mathbb{Z}/p\mathbb{Z}}(G).$$

Using lemma 2.1.5 we have

$$x^p - 1 = (x-1)^p \in (\Delta^{(s)}_{\mathbb{Z}/p\mathbb{Z}}(G))^p \subseteq \Delta^{(ps-p+1)}_{\mathbb{Z}/p\mathbb{Z}}(G). \tag{2.82}$$

From the way we have set s, we have $ps - p + 1 \geq m + 1$. Hence

$$x^p - 1 \in \Delta^{(m+1)}_{\mathbb{Z}/p\mathbb{Z}}(G) \Rightarrow x^p \in D_{(m+1),\mathbb{Z}/p\mathbb{Z}}(G). \tag{2.83}$$

Hence, $M_{(m+1),p}(G) \subseteq D_{(m+1),\mathbb{Z}/p\mathbb{Z}}(G)$. With this the first step of our proof is complete.

We will now show

$$G_{(n),p} \supseteq M_{(n+1),p}(G) \supseteq D_{(n+1),\mathbb{Z}/p\mathbb{Z}}(G) \tag{2.84}$$

by going modulo $G_{(n),p}$ and assuming (when necessary) $G_{(n),p} = 1$. The theorem is proven if we can show that $D_{(n+1),\mathbb{Z}/p\mathbb{Z}}(G) = 1$. We may assume that G is finitely generated. We already have that G is nilpotent. Hence we can conclude $\gamma_2(G)$ is a finitely generated nilpotent group with exponent a power of p and hence has order a power of p.

To complete the proof of this theorem the following lemma is needed:

Lemma 2.2.4. *Let R be a commutative ring with identity and G be a group. Let characteristic of R be a prime, p. The canonical filtration $\{A_n\}_{n\geq 1}$ of $R(\gamma_2(G))$ defined by $\{G_{(n),p}\}_{n\geq 1}$ is given by*

$$A_n = R(\gamma_2(G)) \cap \Delta_R^{(n+1)}(G) \textit{ for all } n \geq 1.$$

As mentioned before, $\{G_{(i),p}\}_{i\geq 1}$ is a restricted N-series of $\gamma_2(G)$. The above lemma 2.2.4 suggests that the canonical filtration of $\{A_n\}_{n\geq 1}$ of $\mathbb{Z}/p\mathbb{Z}(\gamma_2(G))$ is

$$A_i = (\mathbb{Z}/p\mathbb{Z})(\gamma_2(G)) \cap \Delta_{\mathbb{Z}/p\mathbb{Z}}^{(i+1)}(G) \text{ for all } i \geq 1. \tag{2.85}$$

Lazard ([Laz54]) had shown

Theorem 2.2.5. *Let $G = H_1 \supseteq H_2 \supseteq \ldots \supseteq H_c \supseteq H_{c+1} = 1$ be a finite restricted N-series relative to a prime p. Let R be a ring with identity of characteristic p and $\{A_i\}$ be the canonical filtration of $\Delta_R(G)$. Then*

$$H_i = G \cap (1 + A_i) \textit{ for all } i \geq 1.$$

This gives

$$\begin{aligned} 1 = G_{(n),p} &= \gamma_2(G) \cap (1 + A_n) \\ &= \gamma_2(G) \cap (1 + \Delta_{\mathbb{Z}/p\mathbb{Z}}^{(n+1)}(G)) \\ &= D_{(n+1),\mathbb{Z}/p\mathbb{Z}}(G). \end{aligned}$$

With this the proof is complete. □

Chapter 3

Prime Power Groups

Though dimension subgroups were primarily thought to be tools used to study ring theoretic aspects of group algebras, it is now understood that they have some pure group theoretic applications as well. Various problems in modular p-groups are solved through extensive study of dimension subgroups.

In the earlier chapters we focussed on the identifying the dimension subgroups and Lie dimension subgroups. Higman's reduction, theorem 1.1.8, suggested that the study of dimension subgroups is primarily a study of finite p-groups. In this chapter the focus will be on applying the theory of dimension subgroups to the power structure of p groups ([SS91]). In particular we will use the properties developed by Shalev [Sha90]. Then we will shift focus to the powerful and potent groups and see how the two theories come together ([Wil03]).

NOTATION: For all $i \geq 0$, define $\mho_i(G) = \langle x^{p^i} | x \in G \rangle = G^{p^i}$ which is the subgroup generated by the p^ith powers. For all $i \geq 0$, we define $\mho_{(i)}(G)$ inductively:

$$\mho_{(0)}(G) = G$$

and

$$\mho_{(i)}(G) = \mho_1(\mho_{(i-1)}(G)) \qquad \text{for all } i \geq 1.$$

We have the obvious inclusion:

$$\mho_{(i)}(G) \supseteq \mho_i(G) \supseteq \{x^{p^i} | x \in G\}. \tag{3.1}$$

3.1 The Series $\{D_{m,k}\}$

From theorem 1.2.14 we have $D_{m,\mathbb{Z}/p\mathbb{Z}}(G) = \prod_{jp^i \geq m} \gamma_j(G)^{p^i}$ (where $i \geq 0$) for positive integer m and a prime p. As per the notation introduced above we can rewrite this as

$$D_{m,\mathbb{Z}/p\mathbb{Z}}(G) = \prod_{jp^i \geq m} \mho_i(\gamma_j(G)). \tag{3.2}$$

This is Lazard's explicit expression. For ease of writing, we drop $\mathbb{Z}/p\mathbb{Z}$ if there is no ambiguity about the field (of characteristic p) over which we are considering our dimension subgroups.

We will now introduce the double-indexed series $\{D_{m,k}\}$ described by Shalev [Sha90].

Definition 3.1.1. For integers $m \geq 1$ and $k \geq 0$, we define

$$D_{m,k}(G) = \prod_{jp^i \geq m} \mho_i(\gamma_{j+k}(G)). \tag{3.3}$$

We define non-negative integers $d_{m,k}$ by

$$p^{d_{m,k}} = (D_{m,k} : D_{m+1,k}). \tag{3.4}$$

As can be easily seen, we get Lazard's formula (equation 3.2) when we put $k = 0$. For ease of notation when $k = 0$, we will drop the second index. For a prime p and positive integer m, we denote $\nu_p(m)$ as the maximal integer ν such that p^ν divides m. We will use the notation, $(m)_{p'} = m/p^\nu$ where $\nu = \nu_p(m)$.

We will need the following definition.

Definition 3.1.2. For positive integer m and non-negative integers k and ν we define

1. $D^{\leq \nu}_{m,k} = \prod_{jp^i \geq m} \mho_i(\gamma_{j+k})$ where the product is over $i \leq \nu$.
2. $D^{> \nu}_{m,k} = \prod_{jp^i \geq m} \mho_i(\gamma_{j+k})$ where the product is over $i > \nu$.

Passi and Sehgal found an explicit formula for $D_{(m)}(G)$ (theorem 2.2.3). Using the notation introduced above it can be formulated as

$$D_{(m+1)}(G) = D_{m,1}(G) \qquad \text{for all } m \geq 1. \tag{3.5}$$

We will now summarize some of the basic properties of the series $\{D_{m,k}\}$. For the proofs of these properties, one may refer to [SS91, Section 2] and [Sha90, Section 1].

Theorem 3.1.3. *1. $D_{m,k+1}(G) = (G, D_{m,k}(G))$ for all $m \geq 1, k \geq 0$.*

2. $d_{m,k} = 0$ implies $d_{m,h} = 0$ for all $h > k$.

Lemma 3.1.4. *Set $\nu_p(m) = \nu$. Then*

1. $D_{m+1}(G) = (G, D_m(G)) \cdot D_m^{>\nu}(G)$.
2. $d_m(G) = 0$ *iff* $D_m(G) = D_m^{>\nu}(G)$.

Theorem 3.1.5. *Let $d_{m,k} = 0$ and $n \geq m$. Then*

1. $D_{n,k} = \mho_1(D_{\lceil \frac{n}{p} \rceil,k})$.
2. $D_{n,k}^{\leq\nu} \leq (G, D_{n,k})$ *where* $\nu = \nu_p(m)$.
3. $D_{n,k} = D_{n,k}^{>\nu}$ *where* $\nu = \nu_p(m)$.
4. $(n)_{p'} \geq (m)_{p'}$ *implies* $d_{n,k} = 0$.

Theorem 3.1.6. *Let $(m)_{p'} < p$ holds and $d_m(G) = 0$. Let $H = \langle \gamma_k, x \rangle$ be a subgroup of G where $k > 1$ and $x \in G$. Then $d_m(H) = 0$.*

The following theorem of P. Hall ([Hal32]) is of much importance:

Theorem 3.1.7. *For any group G, elements x, y in G, prime p and positive integer k the following hold:*

1. $(xy)^{p^k} \equiv x^{p^k} y^{p^k} \mod \gamma_2(\langle x, y \rangle)^{p^k} \cdot \prod_{i=1}^{k} \gamma_{p^i}(\langle x, y \rangle)^{p^{k-i}}$.
2. $(x^{p^k}, y) \equiv (x, y)^{p^k} \mod \gamma_2(\langle x, (x, y) \rangle)^{p^k} \cdot \prod_{i=1}^{k} \gamma_{p^i}(\langle x, (x, y) \rangle)^{p^{k-i}}$.

Theorem 3.1.8. *[Sha90, 1.12] The map $\varphi : D_m(G)/D_{m+1}(G) \to D_{pm}(G)/D_{pm+1}$ given by $x \mapsto x^p$ is well defined. If $d_l = 0$ for some $m < l < pm$ where $\nu_p(l) \geq \nu_p(m)$, then φ is an epimorphism. In particular, $d_{pm} \leq d_m$.*

Before mentioning some further properties we need to introduce another notation. Let M and N be subgroups of a group G. We inductively define

$$(M, N; 0) = M$$

and

$$(M, N; l + 1) = ((M, N; l), N).$$

Let M and N be normal subgroups of G. Then the above theorem (3.1.7) allows us to conclude that

$$(M^p, N) \subseteq (M, N)^p(N, M; p) \tag{3.6}$$

and

$$(MN)^p \subseteq M^p N^p (M, N, MN; p - 2) \tag{3.7}$$

Furthermore, the Three Subgroup Lemma allows us to conclude that for normal subgroups M and N of G

$$(M, \gamma_k(N)) \subseteq (M, N; k). \tag{3.8}$$

There are many results on commutators of dimension subgroups which are known. In the next few theorems we will summarize those which are of importance to us.

For a finite p-group G we have

$$(D_k(G), G; l) = \prod_{ip^j \geq k} \mho_{p^j}(\gamma_{i+l}(G)). \tag{3.9}$$

Theorem 3.1.9. *Let G be a finite p-group. Let $a \geq b \geq 1$ be integers. Then $(D_a(G), D_b(G)) \subseteq \gamma_{a+b}(G) D_{a+pb}(G)$.*

A slightly more robust result is the following:

Theorem 3.1.10. *Let G be a finite p-group. Let a, b and l are positive integers. Then*

1. $(\gamma_b(G), D_a(G); l) \subseteq \gamma_{la+b}(G)D_{((l-1)p+1)a+pb}(G)$.

2. $\gamma_l(D_a(G)) \subseteq \gamma_{la}(G)D_{(p(l-1)+1)a}(G)$.

3. *if* $a \geq b$, *then* $(D_a(G), D_b)G)) \subseteq \gamma_{a+b}(G)\gamma_{\lceil \frac{a}{p} \rceil + b}(G)^p D_{a+p^2b}(G)D_{pa+pb}(G)$.

Another lemma which will be needed is the following

Lemma 3.1.11. *Let* G *be a finite* p*-group and* $\gamma_m \subseteq D_{p^h}(G)$ *where* $m < p^h$. *Then* $D_n(G) = D_{\lceil \frac{n}{p} \rceil}(G)^p$ *if* $n \geq m+1$. *If* $\gamma_{p^k}(G) \subseteq D_{p^{k+1}}(G)$ *then* $D_{p^k}(G) = D_{p^{k-1}}(G)^p$.

3.2 Powerful p-Groups

The notion of powerful groups was introduced by Lubotzky and Mann in their 1987 paper ([LM87]). Powerful groups can be thought of as a generalization of Abelian groups.

Definition 3.2.1. Let G be a finite p group where p is an odd prime. We say G is **powerful** if $(G, G) \subseteq G^p$. A normal subgroup, N, is said to be **powerfully embedded** in G if $(N, G) \subseteq N^p$.

As is evident from the definition, G is powerful if and only if it is powerfully embedded in itself. If N is powerfully embedded in G, then N is powerful. If $N \subseteq H \subseteq G$ and H/N is cyclic, then

$$H^p \supseteq N^p \supseteq (N, H) = (H, H). \tag{3.10}$$

For p-groups with $p \neq 2$, there is a related notion of being potent.

Definition 3.2.2. A p-group G (where $p \neq 2$) is called **potent** if $\gamma_{p-1}(G) \subseteq G^p$.

Since for all 2-groups, $(G, G) \subseteq G^2$, we need a change in the definition for the case $p = 2$.

Definition 3.2.3. Let G be a finite 2-group. We say G is **powerful** if $(G, G) \subseteq G^4$. Like before, a normal subgroup, N, is said to be **powerfully embedded** in G if $(N, G) \subseteq N^4$.

Next, we have an important result of powerful groups.

Proposition 3.2.4. *[DdMS93, 2.6] If* G *is a powerful* p*-group then every element of* $\mho_1(G)$ *is a pth power in* G.

Proof The proof is by induction on the order of the group G. We begin with some observations

1. For powerful p-groups G, we have

$$\mho_1(G)(G,G) = \mho_1(G) = \Phi(G). \tag{3.11}$$

   where $\Phi(G)$ is the Frattini subgroup.

Define $P_i(G)$ inductively as follows:

$$P_1(G) = G$$

and

$$P_{i+1}(G) = \mho_1(P_i(G))(P_i(G), G) \text{ for all } i \geq 1.$$

2. We can extend the above observation (by induction), to a general statement that for each i, $P_i(G)$ is powerfully embedded in G and

$$P_{i+1}(G) = \mho_1(P_i(G)) = \Phi(P_i(G)). \tag{3.12}$$

3. The map $x \mapsto x^p$ induces a surjective homomorphism

$$G/P_2(G) \to P_2(G)/P_3(G). \tag{3.13}$$

Now, let $g \in \mho_1(G)$, the above observations allow us to conclude that there exist $x \in G$ and $y \in P_3(G)$ such that $g = x^p y$.

Define $H = \langle \mho_1(G), x\rangle$. Since, $\mho_1(G)$ is normal subgroup of H we have the obvious equality $(H,H) = (N,H)$. Furthermore, we know that $\mho_1(G)$ is powerfully embedded in G and hence $(H,H) \subseteq (\mho_1(G))^p \subseteq H^p$ (respectively for the $p = 2$ case). Therefore H is also powerful.

We can see that $g \in \mho_1(H)$ because $y \in P_3(G) = \mho_1(P_2(G))$.

Two cases can arise. Firstly, when $H \neq G$, induction hypothesis gives that g is a pth power in H and hence a pth power in G. Secondly, when $H = G$, we have $G = \langle \mho_1(G), x\rangle = \Phi(G)\langle x\rangle = \langle x\rangle$ where $\Phi(G)$ is the Frattini subgroup of G. The result follows trivially in this case. $\square$

Example 3.2.5. Some examples of powerful p-groups are mentioned below.

1. Let G be a **regular** p-group. For all $x, y \in G$, there exists $c \in \mho_1(H')$ where $H = \langle x, y \rangle$ such that $x^p y^p = (xy)^p c$. It is a straightforward observation that $\mho_1(\mho_1(G)) = \{x^p | x \in \mho_1(G)\} = \{x^{p^2} | x \in G\} = \mho_2(G)$. Also, it can be shown that $(\mho_1(G), \mho_1(G)) = \mho_2(G')$ ([Hup67, III.10]). This clearly gives us that $\mho_1(G)$ is powerful. ([LM87])

2. Consider a regular metabelian p-group, G, with $p \neq 2$ and exponent at most p. If $N \lhd G$ and is contained in the Frattini subgroup of G, then N is powerful. This can be seen if we consider a counterexample N such that $N^p = 1$ and N' is a normal subgroup of order p. For a p-group, the Frattini subgroup $\Phi(G) = G'G^p$. Therefore, we have $(N, \Phi(G)) = (N, G'G^p) = (N, G')(N, G^p)$. The regularity condition ensures that $(N, G^p) = (N, G)^p = 1$ (follows from $(N, G) \subseteq N$ and G is metabelian). We also have

$$\begin{aligned} (N, G') &\leq (G'G^p, G') = (G', G')(G^p, G') \\ &= (G, G')^p = \gamma_3(G)^p = 1. \end{aligned} \tag{3.14}$$

   Thus, $(N, \Phi(G)) = 1$ and hence $N' = 1$. This is a contradiction. ([Kin73, Corollary 4])

3. Consider the p-group G. Let N be a 2-generator normal subgroup of G contained in the Frattini subgroup. Then N is powerful. ([Kin73, Theorem 7])

4. The central product $D_8 \circ C_8$ is powerful, where D_8 is the dihedral group of order 8 and C_8 is the cyclic group of order 8. ([HL03])

We will see some ways as to how powerful p-groups can be viewed as generalizations of Abelian groups.

Let G be a powerful p-group (p odd). By definition of powerful groups, $\gamma_2(G) \subseteq \mho_1(G)$. Thus, from Lazard's formula, equation 3.2, we have $D_2(G) = D_3(G) = \mho_1(G)$, i.e., $d_2 = 0$. We can apply theorem 3.1.5 (part 4) to conclude that $d_m \neq 0$ only when $m = p^i$ for some i. If we let $l = 2p^{i-1}$ in theorem 3.1.8, we can say that $d_{p^{i-1}} \geq d_{p^i}$ for all $i \geq 1$. Applying lemma 3.1.4 with $m = 2p^{i-1}$ gives $D_{2p^{i-1}}(G) = D_{2p^{i-1}}^{>(i-1)}(G) = \mho_i(G)$. $D_m(G)$ is fixed for $p^{i-1} < m \leq p^i$, therefore for such an m, $D_m(G) = \mho_i(G)$. The **dimension class** of a group G, $dc(G) = \sup\{m | D_m(G) \neq 1\}$. Here, dimension class is p^{e-1} where e is the exponent of the group. Thus, powerful p-groups behave just like Abelian groups with respect to dimension subgroups in characteristic p.

The **Loewy series**, $\{c_i\}$, corresponds to the filtration $\{\Delta^i\}$ of $k[G]$ where k is a field of characteristic p and G is a finite p-group. We define $c_i = \dim_k \Delta^i(G)/\Delta^{i+1}(G)$. Let f be the generating function, then $f(x) = \sum_i c_i \cdot x^i$. Jennings [Jen41] had shown that f is symmetric polynomial of degree $t(G) - 1$ where $t(G)$ is the nilpotency index of the group. Huppert raised a question whether f is **unimodal** , i.e., whether $c_{i-1} \leq c_i$ for all positive integers $i \leq \frac{t(G)-1}{2}$. Manz and Staszewski [MS86] showed that Loewy series is unimodal for regular powerful p-groups, in particular for Abelian groups. In [Sha90, Section 4], Shalev drops the regularity condition when $p \neq 2$.

Remark 3.2.6. Powerful p-groups have played a crucial role in understanding analytic pro-p groups [DdMS93].

3.3 Power Structure of Finite p-Groups

In this section we will understand how dimension subgroups in characteristic p are used to study the power structure of finite p-groups. Our focus will be on the results proven by Scopolla and Shalev [SS91] and the generalization of one of their theorems by Wilson [Wil03].

We had seen the chain, equation 3.1, earlier in section 3.2. It is a natural question to ask when do all the three notions coincide. For regular p-groups, the three notions are equivalent. This is a straight forward consequence of the following theorem of P. Hall.

Theorem 3.3.1. *[Hal32, Theorem 4.21] Let G be a regular p-group. The product of the p^k-th powers of two or more elements of G is itself a p^k-th power of some element of G.*

It has been shown by Lubotzky and Mann [LM87] that a similar phenomenon occurs in case of powerful p-groups as well. One can deduce them from the following theorems:

Proposition 3.3.2. *[LM87, Theorem 1.3 and 4.1.3] Let G be a powerful p-group. Then $\mho_i(\mho_j(G)) = \mho_{i+j}(G)$.*

and

Proposition 3.3.3. *[LM87, Theorem 1.7 and 4.1.7] Let G be a powerful p-group. Then each element of $\mho_i(G)$ can be written as x^{p^i} for some $x \in G$.*

In [Wil03], Wilson states without proof a similar statement for potent groups.

Proposition 3.3.4. *Let G be a potent p-group, $\mho_{(k)}(G) = G^{p^k} = \{x^{p^k} | x \in G\}$.*

The above mentioned phenomena can be generalized in view of the following theorem.

Theorem 3.3.5. *Let $m = ap^{\alpha-1}$ with $a < p$ and $\alpha \geq 1$. Let G be a p-group with $d_m = 0$. Then for all $i \geq \alpha$ we have $\mho_{(i)}(G) = \mho_i(G) = \{x^{p^i} | x \in G\}$.*

We have a straightforward corollary:

Corollary 3.3.6. *Let G be any group of order p^n. For any positive integer $i \geq \frac{n-1}{p-1}$, $\mho_{(i)}(G) = \mho_i(G) = \{x^{p^i} | x \in G\}$.*

The corollary follows from simple counting that $D_m(G) = D_{m+1}(G)$ for some m of the form $ap^{\alpha-1}$ when a is a positive integer less than p and $\alpha \leq \lceil \frac{n-1}{p-1} \rceil$.

We will now mention a proof of theorem 3.3.5.

Proof The proof of this theorem proceeds by induction on the order of the group G. We observe that $\{D_m(G)\}$ is an N_p-series. Thus, $\mho_{(i)}(G) \subseteq D_{p^i}(G) = \mho_i(G)$. The equality follows straight from Lazard's formula, equation 3.2. We will be done if we can show that every element of $D_{p^i}(G) = \mho_i(G)$ is a p^ith power of an element of G, for all $i \geq \alpha$. We may assume $\alpha = i$ (otherwise replace m by $m' = mp^{i-\alpha}$).

The Hall's collection formula ensures

$$(xy)^{p^i} \equiv x^{p^i} y^{p^i} \mod \mho_i(\gamma_2(G)) \cdot \prod_{r=1}^{i} \mho_{i-r}(\gamma_{p^r}(G)). \tag{3.15}$$

It is a straight forward observation from definition 3.1.2 that

$$D_{p^i}^{\leq(i-1)}(G) = \prod_{r=1}^{i} \mho_{i-r}(\gamma_{p^r}(G)). \tag{3.16}$$

Theorem 3.1.5 (part 2) allows us to say

$$D_{p^i}^{\leq(i-1)}(G) \subseteq (G, D_{p^i}(G)) = (G, \mho_i(G)). \tag{3.17}$$

From definition, we have

$$D_{m,1}(G) = \prod_{jp^i \geq ap^{i-1}} \mho_i(\gamma_{j+1}(G)) = \prod_{jp^i \geq ap^{i-1}; i > i-1} \mho_i(\gamma_{j+1}(G)) = D_{m,1}^{>(i-1)}(G) = \mho_i(\gamma_2(G)). \tag{3.18}$$

Combining the above equations gives us

$$(xy)^{p^i} \equiv x^{p^i} y^{p^i} \mod \mho_i(\gamma_2(G)). \tag{3.19}$$

For every element $g \in \mho_i(G)$, we can say there must exist $x \in G$ such that

$$g \in x^{p^i} \cdot \mho_i(\gamma_2(G)) \subseteq \mho_i(N) \subseteq D_{p^i}(N) \text{ where } N = \langle \gamma_2(G), x \rangle \tag{3.20}$$

N is a proper subgroup of G. The conditions of theorem 3.1.6 are satisfied and we have $d_m(N) = 0$. We can apply the induction hypothesis on N. Every element of $D_{p^i}(N)$ can be written as a p^ith power. Therefore, $g = x^{p^i}$ for some $x \in N$.

With this the proof is complete. □

In [Man76], Mann shows in section 1 that for a group G of nilpotency class c, and $k = \lfloor \frac{c-1}{p-1} \rfloor$, we have $\mho_{i+k} \subseteq \{x^{p^i} | x \in G\}$. This result can be improved as has been done in the following theorem.

Theorem 3.3.7. *Let G be a p group ($p \neq 2$) of nilpotency class c, and let k be the minimal integer such that $c < (p-1)p^k$. Then $\mho_{(i+k)} \subseteq \{x^{p^i} | x \in G\}$.*

There is a slight generalization of this theorem discussed in section 3 of [Wil03].

Proposition 3.3.8. *Let G be a finite p-group ($p \neq 2$). If there exists k such that $\gamma_{(p-1)p^k}(G) \subseteq D_{p^{k+1}}(G)$ then $D_{p^{k+l}}(G) \subseteq \{x^{p^l} | x \in G\}$.*

Proof In view of the following lemma, $D_{p^{k+l}}(G)$ is potent.

Lemma 3.3.9. *Let G be a finite p-group with p odd. If $\gamma_m(G) \subseteq D_{p^h}(G)$ where $m < p^h$ then $D_i(G)$ is potent if the following two conditions hold:*

- *$i \geq m/(p-1)$*
- *$i \geq (m - p^{h-1})/(p-2)$.*

Repeated application of lemma 3.1.11 gives $D_{p^{k+l}(G)} = \mho_{(l)}(D_{p^k}(G))$. In view of proposition 3.3.4, $\mho_{(l)}(D_{p^k}(G)) = \{x^{p^l} | x \in D_{p^k}(G)\}$. Thus the proposition follows. □

Remark 3.3.10. It is evident that theorem 3.3.7 can now be attained as a corollary of the above theorem 3.3.8 once we observe that $\mho_{(i)}(G) \subseteq D_{p^i}(G)$ for all i. In fact, slightly more can be said. Under the hypothesis that the nilpotency class of group G is c and k is the minimal integer such that -$c < (p-1)p^k$, we have $\mho_{(k+l)}(G) \subseteq D_{p^{k+l}}(G) \subseteq \{x^{p^l} | x \in G\}$.

Below is a proof of theorem 3.3.7

Proof It is given that the nilpotency class is $c < (p-1)p^k$, thus, $\gamma_{(p-1)p^k}(G) = 1$. Also, we observe that $(p-1)p^k + 1$ is relatively prime to p and $\gamma_{(p-1)p^k+1}(G) = 1$. From Lazard's formula, equation 3.2, we have $D_{(p-1)p^k+1}(G) = D_{(p-1)p^k+2}(G)$. By definition, $d_{(p-1)p^k+1} = 0$, and this allows us to conclude (using theorem 3.1.5 part 1) that

$$D_{p^{k+1}}(G) = \mho_1(D_{p^k}(G)). \tag{3.21}$$

We formulate Lemma 1.5 of [Sco91], for the particular case here. If p is an odd prime, and $\gamma_{(p-1)p^k}(G) = 1$, then

$$\gamma_{p-1}(D_{p^k}(G)) \subseteq D_{p^{k+1}}(G). \tag{3.22}$$

Therefore, $\gamma_{p-1}(D_{p^k}(G)) \subseteq \mho_1(D_{p^k}(G))$. Using Lazard's formula, 3.2, we have

$$D_{p-1}(D_{p^k}(G)) = \mho_1(D_{p^k}(G)) = D_p(D_{p^k}(G)). \tag{3.23}$$

Therefore, $d_{p-1}(D_{p^k}(G)) = 0$. We can hence apply theorem 3.3.5. We can say that

$$\mho_{(i)}(D_{p^k}(G)) = \{g^{p^i} | g \in D_{p^k}(G)\} \text{ for all } i > 0. \tag{3.24}$$

We had observed that $\mho_{(k)}(G) \subseteq D_{p^k}(G)$. Therefore, $\mho_{(i+k)}(G) \subseteq \mho_{(i)}(D_{p^k}(G))$.

Thus,

$$\mho_{(i+k)}(G) \subseteq \{x^{p^i} | x \in G\} \textit{ for all } i > 0. \tag{3.25}$$

This completes the proof. □

Remark 3.3.11. The constant k is the best possible for most values of c. In any case, it can not be reduced by more than 1.

This can be seen in view of the following theorem:

Theorem 3.3.12. *For all integers $k \geq 1$, there exists a p-group ($p \neq 2$), G, of class $c = p^{k+1}$ such that $\mho_{1+k} \nsubseteq \{x^p | x \in G\}$. In particular, let F be the free group on two generators, then $G = F/D_{p^{k+1}+1}(F)$ has the required property.*

Proof Let F be a free group on two generators, x and y. Define $G = F/D_{p^{k+1}+1}(F)$. G is a finite p-group and its nilpotency class is p^{k+1}.

Let $z = x^{p^{k+1}} y^{p^{k+1}} \in \mho_{1+k}(G)$. We have seen that $\mho_i(G) \subseteq D_{p^i}(G)$ for all i. Hence, $z \in D_{p^{k+1}}(G)$.

The proof of the theorem is by contradiction. Let $z = a^p$ for some element a in G. This means that $a^p \in D_{p^{k+1}}(G) \setminus D_{p^{k+1}+1}(G)$. Then by a result of Scoppola ([Sco91, Lemma 1.10]), we conclude that $a \in D_{p^k}(G) \setminus D_{p^{k+1}}(G)$.

Now, we had seen in theorem 3.1.8 that the map $x \mapsto x^p$ induces a well defined map

$$\varphi : D_{p^k}(G)/D_{p^k+1}(G) \to D_{p^{k+1}}(G)/D_{p^{k+1}+1}(G). \tag{3.26}$$

We also have the canonical projection

$$\pi : D_{p^{k+1}}(G)/D_{p^{k+1}+1}(G) \to D_{p^{k+1}}(G)/\gamma_{p^{k+1}}(G)D_{p^{k+1}+1}(G). \tag{3.27}$$

The composition map $\mu_{p^k} = \pi \circ \varphi$ is a homomorphism. In fact, it can be shown that μ_{p^k} is an isomorphism ([Sco91, Lemma 1.11]).

Let us define $\bar{G} = G/\gamma_{p^{k+1}}(G)$.

The map $g \mapsto g^p$ induces the map

$$\psi : D_{p^k}(\bar{G})/D_{p^k+1}(\bar{G}) \to D_{p^{k+1}}(\bar{G}) \tag{3.28}$$

which can be identified with μ_{p^k}.

We have

$$\mu_{p^k}(\bar{x}^{p^k}\bar{y}^{p^k}D_{p^k+1}(\bar{G})) = \bar{x}^{p^{k+1}}\bar{y}^{p^{k+1}} = \bar{z} = \mu_{p^k}(\bar{a}D_{p^k+1}(\bar{G})). \tag{3.29}$$

Since $\bar{a} \equiv \bar{x}^{p^k} \bar{y}^{p^k} \mod D_{p^k+1}(\bar{G})$ and μ_{p^k} is an injective map this means $a \equiv x^{p^k} y^{p^k} \mod D_{p^k+1}(G)$. Without loss of generality we may assume that $a = x^{p^k} y^{p^k}$.

Using Hall's collection formula, we have $a^p = x^{p^{k+1}} y^{p^{k+1}} R$ where R is the product of basic commutators of weight p in x^{p^k} and y^{p^k}. Calculations show $(y^{p^k}, x^{p^k}, y^{p^k}; p-2)$ is the only basic commutator of weight 1 in x^{p^k} that appears in P. Also, the exponent of this commutator component is -1 ([Sco91, Lemma 2.3]).

We have a standard result (see [Zas40] and [Laz54]) that allows us to conclude

$$(y^{p^k}, x^{p^k}, y^{p^k}; p-2) = ((y, x), y; p^{k+1} - 1). \tag{3.30}$$

Using the method adopted by Meier-Wunderli [Mei52] we have

$$a^p = x^{p^{k+1}} y^{p^{k+1}} ((y, x), y; p^{k+1} - 1)^{-1} P \tag{3.31}$$

where P is a product of some basic commutators of total weight p^{k+1} in x and y whose partial weight in x is at least $2p^k$.

This expression of a^p is in terms of a basis as defined by Scoppola ([Sco91, Lemma 1.11]) and is unique. Thus, $a^p \neq x^{p^{k+1}} y^{p^{k+1}}$. With this our proof is complete. □

The theorem discussed above (theorem 3.3.7) holds only for the case when p is an odd prime. The following theorem of Wilson ([Wil03, Theorem 4.1]) strengthens theorem 3.3.8 further and also deals with the case of $p = 2$.

Theorem 3.3.13. *Let G be a finite p group with $\gamma_{p^k}(G) \subseteq D_{p^{k+1}}(G)$ for some k. Then $D_{p^{k+l-1}}(G) \subseteq \{x^{p^l} | x \in G\}$ for positive integers l.*

The proof of the theorem is both lengthy and involved. Here, we will only indicate a sketch of the proof.

Proof The proof is divided into three parts

Case 1: $k = 1$

In this case, as per the hypothesis of the theorem, we have a finite p-group, G, with $\gamma_p(G) \subseteq D_{p^2}(G)$.

It can be observed that $D_{p^2}(G) = \gamma_{p^2}(G) \cdot \gamma_p^p(G) \cdot \mho_2(G)$. Therefore the hypothesis $\gamma_p(G) \subseteq D_{p^2}(G)$ is equivalent to assuming $\gamma_p(G) \subseteq \mho_2(G)$.

Similarly, it can be seen that $D_p(G) = \gamma_p(G) \cdot \mho_1(G)$. This means that by our assumption we have $D_p(G) = \mho_1(G) = \{x^p | x \in G\}$.

If $p = 2$, then it is easy to observe that the group, G, is powerful and using standard results we have that $\mho_1(G) = G^2$ is also powerful in this case.

In view of the following lemma we see that even when p is odd, $D_p(G)$ is powerful.

Lemma 3.3.14. *Let G be a finite p-group with p odd. If $\gamma_m(G) \subseteq D_{p^h}(G)$ where $m < p^h$ then $D_i(G)$ is powerful if the following two conditions hold:*

- $i \geq m/2$
- $i \geq (m - p^{h-1})$.

Using lemma 3.1.11 repeatedly we have $D_{p^l} = \mho_{(l-1)}(D_p(G))$. Since we have already seen that $D_p(G)$ is powerful we can conclude that every element of $D_{p^l}(G)$ is a p^{l-1}-th power of an element of $D_p(G)$. Thus it is a p^l-th power of an element of G.

The proof of the first case is thus complete.

Case 2: $p = 2$ and $k = 2$

The hypothesis in this case is that G is a finite p group with $\gamma_4(G) \subseteq D_8(G)$.

Use of lemma 3.1.11 gives that $D_4(G) = D_2^2(G)$ and $D_8(G) = D_4^2(G)$. It can be shown that $D_4(G)$ is powerful and as a matter of fact, so is $D_3(G)$.

We consider two elements in $D_2(G)$, say x and y. $(xy)^2 = x^2y^2(y, x)^y$ can be shown to be the square of an element of $D_2(G)$. Since x and y were arbitrary elements of $D_2(G)$, we can conclude that $D_4(G) = D_2^2(G)$ are squares of elements of $D_2(G)$. $\mho_{(l-1)}(D_4(G)) = D_{2^{l+1}}(G)$ by the use of lemma 3.1.11. Since, $D_4(G)$ is powerful, elements of $\mho_{(l-1)}(D_4(G))$ is a 2^{l-1}-th power of an element of $D_4(G)$. Thus, every element of $D_{2^{l+1}}(G)$ is a 2^l-th power of an element of $D_2(G)$.

This case is also done.

Case 3: p odd with $k > 1$ and $p = 2$ with $k > 2$

Using lemma 3.3.14 we can say that $D_{p^k}(G)$ is powerful when p is odd.

Even when $p = 2$, it can be shown with some (non trivial) computation that $D_{2^k}(G)$ is powerful.

For all p, use of lemma 3.1.11 yields $D_{p^k}(G) = D^p_{p^{k-1}}(G)$ and $D_{p^{k+1}}(G) = D^p_{p^k}(G)$.

We consider two elements x, y in $D_{p^{k-1}}(G)$. It can be shown that $x^p y^p$ is an element in H^p where H is a potent subgroup of $D_{p^{k-1}}(G)$.

Thus, $x^p y^p$ is a p-th power of an element of $D_{p^{k-1}}(G)$ and every element of $D^p_{p^{k-1}}(G)$ is a p-th power of an element of $D_{p^{k-1}}(G)$. Since $D_{p^k}(G) = D^p_{p^{k-1}}(G)$, elements of $D_{p^k}(G)$ are p-th powers. This yields the result for $l = 1$.

When we use the lemma 3.1.11 repeatedly we get $D_{p^{k+l-1}}(G) = \mho_{(l-1)}(D_{p^k}(G))$. Since we have already seen that $D_{p^k}(G)$ is powerful, an element of $\mho_{(l-1)}(D_{p^k}(G))$ is a p^{l-1}-th power of an element of $D_{p^k}(G)$ and therefore a p^l-th power of an element of $D_{p^{k-1}}(G)$.

Thus the theorem is proved. $\square$

Closing Remarks

In this chapter, I will highlight some of the open questions and also point out what I wish to do in the future.

Open Problems

We saw in chapter 1 that $D_n(G) = \gamma_n(G)$ for all $n \leq 3$. A natural question then arises what is the structure of $D_n(G)/\gamma_n(G)$ in general? Tahara provided an answer to the question for $n = 4$ and $n = 5$. The structure of higher dimension subgroups are still unknown and remain as open problems. Tahara raised a problem in [Tah81] whether the exponent $D_n(G)/\gamma_n(G)$ is divisible by (n-2)! for all $n \geq 2$?

We saw in corollary 1.1.15 that for 2 or 3 generator groups $D_4(G) = \gamma_4(G)$. A natural question arises, what can we say for $D_5(G)$?

Some problems were raised in [MP09]. If G is a nilpotent group of class 3, is $D_5(G)$ always trivial? For an arbitrary group, G, is $(D_5(G), G, G) = D_7(G)$?

Future Plans

In two semesters I could only read very little portion of the literature that is already available. I would want to read in greater depth and work on the open problems in the future.

Dimension subgroups have turned out to be useful in several interesting problem which at first seemed unrelated. To cite a few examples: the isomorphism problem for local group algebras ([PS72] and [Roh87]), Frobenius-Wielandt complements ([Sco91]), generators of ideals in local group algebras ([Sha90]) and the Lie-structure of local group algebras ([Sha91]). Dimension subgroups give rise to restricted Lie algebras, which have also played a small role in the study of the restricted Burnside problem.

Myriad features of dimension subgroups in characteristic p were derived by Shalev in [Sha90]. This theory of dimension subgroups has been used to generalize Koshitani's theorem on nilpotency index of the augmentation ideal of the group ring, $k[G]$, where k is a field of characteristic $p(\neq 2)$ and G is a finite p-group. Further, it can be used to generalize some properties of the Loewy series $\{c_i\}$ of $k[G]$ where c_i is defined as $\dim(\Delta^i(G)/\Delta^{i+1}(G))$. In particular, he also derived a connection between the m-th dimension subgroup $D_m(G)$ and the $(m+1)$-st Lie dimension subgroup $D_{(m+1)}(G)$. I wish to understand these applications in greater detail.

There are some applications of dimension subgroups in topology as well, I would be interested in exploring them. In [Mas06], Massuyeau shows that Goussarov-Habiro conjecture (in topology) is a variation of the classical problem in algebra, namely the *dimension subgroup problem*. He then uses purely algebraic methods to prove an analogue of the conjecture for finite-type invariants in a fixed field.

The thesis of Smyth [Smy10] links the study of powerful p-groups and pro-p groups with Galois groups. I would be interested in looking at such a link as this will integrate what I have studied over the past years with what I have worked on for my thesis.

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