Algebraic Curves

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Certificate of Examination

This is to certify that the dissertation titled "**Algebraic Curves**" submitted by **Ms. Nidhi Kaihnsa** (Reg. No. MS10020) for the partial fulfillment of BS-MS dual degree programme of the Institute, has been examined by the thesis committee duly appointed by the Institute. The committee finds the work done by the candidate satisfactory and recommends that the report be accepted.

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Dated: April 24, 2015

Declaration

The work presented in this dissertation has been carried out by me under the guidance of Dr. Alok Maharana at the Indian Institute of Science Education and Research Mohali.
This work has not been submitted in part or in full for a degree, a diploma, or a fellowship to any other university or institute. Whenever contributions of others are involved, every effort is made to indicate this clearly, with due acknowledgement of collaborative research and discussions. This thesis is a bonafide record of original work done by me and all sources listed within have been detailed in the bibliography.

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In my capacity as the supervisor of the candidate's project work, I certify that the above statements by the candidate are true to the best of my knowledge.

Dr. Alok Maharana
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Contents

Chapter 1

Introduction

In this thesis we focus on algebraic curves defined over an algebraically closed field of characteristic zero.

We begin by giving some basic definitions of terms in chapter 1 which will be used throughout. In chapter 2 and chapter 3 we define singular and normal varieties. We show that the nonsingular varieties are normal. Our main aim in these two chapters is to resolve the singularities of curves. We will show that there exists a normalization of any variety. We will conclude that normalization resolves the singularities of the curve. We then will give the construction of blowup of a surface at a point and show that an embedded curve can be resolved after finitely many blowups of the surface.

In chapter 5 and chapter 6 we discuss the notion of Weil divisors and Cartier divisors. In chapter 7 we look at the vector space of rational functions constructed with respect to a given divisor. Given a divisor we will see in chapter 8 that there is 1-1 correspondence between Cartier divisors and invertible sheaves on a projective variety, in particular a nonsingular projective curve.

After having developed the necessary machinery we will then prove the Riemann-Roch theorem for curves and look at some of its applications in chapter 10. In the next chapter given a finite morphism between two curves we look at relation between their genus. And finally, we show that any nonsingular, projective curve can be embedded in $\mathbb{P}^3$.

Chapter 2

Preliminaries

In this chapter we will give some basic definitions which will be used throughout this thesis. We will use k to denote algebraically closed field unless stated otherwise. More specifically k will denote field of complex numbers.

2.1 Sheaves

Definition 2.1. *(Presheaf) Let X be a topological space. A presheaf $\mathcal{F}$ on X consists of*

(a) for every open subset $U \subseteq X$, a set $\mathcal{F}(U)$

(b) for every inclusion $V \subseteq U$ of open subsets of X, we have a morprhism $\rho_{UV} : \mathcal{F}(U) \longrightarrow \mathcal{F}(V)$

such that

(i) $\mathcal{F}(\varnothing) = 0$

(ii) ρ_{UU} is the identity map

(iii) for $W \subseteq V \subseteq U$, $\rho_{UW} = \rho_{VW} \circ \rho_{UV}$.

Definition 2.2. *(Sheaf) A presheaf $\mathcal{F}$ on topological space X is sheaf if it satisfies following conditions*

(i) if U is an open set and $\{V_i\}$ be its open covering and if $s \in \mathcal{F}(U)$ is an element such that $s|_{V_i} = 0$ for all i, then $s = 0$.

(ii) if U is an open set and $\{V_i\}$ be its open covering and if we have $s_i \in \mathcal{F}(V_i)$ for each i such that for all i,j we have $s_i|_{V_i \cap V_j} = s_j|_{V_i \cap V_j}$, then there exists an element $s \in \mathcal{F}(U)$ such that $s|_{V_i} = s_i$ for all i.

Definition 2.3. *(Stalk) Let $\mathcal{F}$ be the presheaf on X and P be a point on X. Stalk $\mathcal{F}_P$ of $\mathcal{F}$ at P is defined to be the direct limit of sets $\mathcal{F}(U)$ for all open set containing P using the restriction map ρ.*

Definition 2.4. *(Morphism of sheaves) Let $\mathcal{F}$ and $\mathcal{G}$ be the sheaves on space X. A morphism $\psi : \mathcal{F} \longrightarrow \mathcal{G}$ consists of homomorphisms of $\psi(U) : \mathcal{F}(U) \longrightarrow \mathcal{G}(U)$ for every open set U of X such that whenever $V \subseteq U$ following diagram commutes-*

$$\begin{tikzcd}
\mathcal{G}(V) & \mathcal{F}(V) \arrow[l, "\varphi(V)"'] \\
\mathcal{G}(U) \arrow[u, "\rho_{UV}"] & \mathcal{F}(U) \arrow[l, "\varphi(U)"] \arrow[u, "\rho'_{UV}"']
\end{tikzcd}$$

where ρ and ρ' are restriction maps of $\mathcal{F}$ and $\mathcal{G}$ respectively.

Definition 2.5. *Let $\mathcal{F}$ be a presheaf. Sheafification of $\mathcal{F}$ is sheaf $\mathcal{F}^+$ with morphism $\theta : \mathcal{F} \longrightarrow \mathcal{F}^+$ such that for any given sheaf, say $\mathcal{G}$ and any morphism $\varphi : \mathcal{F} \longrightarrow \mathcal{G}$ there is a unique morphism $\psi : \mathcal{F}^+ \longrightarrow \mathcal{G}$ such that $\varphi = \psi \circ \theta$.*

To see the existence of sheafification of presheaf we construct $\mathcal{F}^+$ associated to presheaf $\mathcal{F}$ of X in the following manner. For an open set U of X let $\mathcal{F}^+$ be the set of functions s given by $U \mapsto \bigcup_{P \in U} \mathcal{F}_P$ such that for every $P \in U$

(i) we have $s(P) \in \mathcal{F}_P$

(ii) there exists $V \subseteq U$ with $P \in V$ and an element $t \in \mathcal{F}(V)$ such that for all $Q \in V$, we have germ $t_Q = s(Q)$.

Definition 2.6. *A ringed space is pair $(X, \mathcal{O}_X)$ where X is topological space and $\mathcal{O}_X$ is a sheaf of rings on X. A ringed space is called locally ringed space if for every $P \in X$ the stalk $\mathcal{O}_{X,P}$ is a local ring.*

Definition 2.7 (Morphism of Ringed Spaces). *Morphism of ringed spaces from* $(X, \mathcal{O}_X)$ *to* $(Y, \mathcal{O}_Y)$ *is given by* $(f, f^\sharp)$ *where* $f : X \longrightarrow Y$ *is a continuous map and* $f^\sharp$ *is a map of sheaves of rings on* Y *such that for any open set* $V \subseteq Y$ *we have* $f^\sharp(V) : \mathcal{O}_Y(V) \longrightarrow \mathcal{O}_X(f^{-1}(V))$.

Definition 2.8. *Let* $(X, \mathcal{O}_X)$ *be a ringed space. A sheaf* $\mathcal{F}$ *is a sheaf of* $\mathcal{O}_X$*-module if for each open set* $U \subseteq X$*,* $\mathcal{F}(U)$ *is an* $\mathcal{O}_X(U)$*-module.*

A sheaf of ideals $\mathcal{I}$ on X is a subsheaf of $\mathcal{O}_X$ such that for every open set U, $\mathcal{I}(U)$ is an ideal in $\mathcal{O}_X(U)$.

We define morphism of sheaves of $\mathcal{O}_X$-modules $\mathcal{F}$ and $\mathcal{G}$ such that map $\mathcal{F}(U) \longrightarrow \mathcal{G}(U)$ is homomorphism of $\mathcal{O}_X(U)$-module. An $\mathcal{O}_X$-module sheaf $\mathcal{F}$ is said to be free if it can be expressed as direct sum of copies of $\mathcal{O}_X$ and if $\mathcal{F}|_{U_i}$ is a free $\mathcal{O}_X|_{U_i}$-module for some covering $\{U_i\}$ of X, it is said to be locally free.

Definition 2.9. *Let* $\mathcal{F}$ *and* $\mathcal{G}$ *be* $\mathcal{O}_X$*-module sheaves. Tensor product* $\mathcal{F} \otimes \mathcal{G}$ *is defined to be the sheaf associated to presheaf* $U \mapsto \mathcal{F}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{G}(U)$.

2.2 Basic Definitions

Definition 2.10. *Let* A *be a commutative noetherian ring. The set of all prime ideals is defined to be* $\operatorname{Spec} A$.

Definition 2.11 (Krull Dimension). *In a ring* A*, the height of a prime ideal* $\mathfrak{p}$ *is the supremum of integers* n *such that there exists a chain* $\mathfrak{p}_0 \subseteq \mathfrak{p}_1 \subseteq \ldots \subseteq \mathfrak{p}_n = \mathfrak{p}$ *of distinct prime ideals. Krull dimension (or simply dimension) of* A *is the supremum of the heights of all the prime ideals.*

Definition 2.12 (Zariski Topology). *Let* $\mathfrak{a}$ *be any ideal of* A*. We define*

$$V(\mathfrak{a}) = \{\mathfrak{p} \in \operatorname{Spec} A | \mathfrak{a} \subseteq \mathfrak{p}\}.$$

Zariski topology on $\operatorname{Spec} A$ *is defined by taking subsets of the form* $V(\mathfrak{a})$ *to be the closed sets.*

Definition 2.13. *The structure sheaf* $\mathcal{O}_{SpecA}$ *is the sheaf associated to presheaf* $\operatorname{Spec} A_f \mapsto A_f$.

Definition 2.14 (Affine Scheme)**.** *An affine scheme is defined as locally ringed space* $(SpecA, \mathcal{O}_{\operatorname{Spec} A})$. *Henceforth we will denote the ringed space* $(SpecA, \mathcal{O}_{\operatorname{Spec} A})$ *by* $\operatorname{Spec} A$.

Definition 2.15 (Scheme)**.** *A locally ringed space* $(X, \mathcal{O}_X)$ *is scheme if every point has an open neighbourhood* U *such that* $(U,\mathcal{O}_X|_U)$ *is an affine scheme.*

Definition 2.16 (Affine Space)**.** *Affine n-space over k is defined as a ringed space* $\operatorname{Spec} k[x_1, \ldots, x_n]$ *and is denoted by* $\mathbb{A}^n_k$ *or simply* $\mathbb{A}^n$ *if* k *is understood.*

Definition 2.17 (Proj)**.** *Let* S *be a graded ring. Let* S_+ *be the ideal* $\oplus_{d>0} S^d$. *The set* $\operatorname{Proj} S$ *is defined to be the set of all homogeneous prime ideals* $\mathfrak{p}$, *which do not contain all of* S_+. *We define Zariski topology on* $\operatorname{Proj} S$ *by taking subsets of the form*

$$V(\mathfrak{a}) = \{\mathfrak{p} \subset \operatorname{Proj} S | \mathfrak{a} \subseteq \mathfrak{p}\}$$

where $\mathfrak{a}$ *is a homogeneous ideal of* S.

Definition 2.18 (Projective Space)**.** *Projective n-space over k is defined to be the locally ringed scheme* $\mathbb{P}^n_k = \operatorname{Proj} k[x_0, \ldots, x_n]$.

Definition 2.19 (Variety)**.** *A variety is a noetherian integral separated scheme of finite type over an algebraically closed field k.*

Definition 2.20 (Morphism)**.** *Let* X *and* Y *be schemes. A mapping* $\varphi : X \to Y$ *is called morphism of schemes if*

(i) φ *is continuous.*

(ii) for every open set U *of* Y *if* $f \in \Gamma(U, \mathcal{O}_Y)$, *then,* $f \circ \varphi \in \Gamma(\varphi^{-1}(U), \mathcal{O}_X)$.

Definition 2.21 (Rational Map)**.** *A rational map* $\varphi : X \to Y$ *is an equivalence class of pairs* $(U,\varphi(U))$*,where* U *is non-empty subset of* X *and* $\varphi(U)$ *is morphism from* U *to* Y. $(U,\varphi(U))$ *and* $(V,\varphi(V))$ *are equivalent if* $\varphi(U)$ *and* $\varphi(V)$ *agree on* $U \cap V$.

Chapter 3

Singularities of Algebraic Varieties

In this chapter we define the condition for a variety to be singular. We also define a normal variety and show that a nonsingular variety is a normal variety. Though a normal variety may not imply that a variety is nonsingular in general, for curves we show that a normal variety is nonsingular. We, then, define normalization of a variety and say that it exists for all the varieties.

Definition 3.1 (Nonsingular Variety)**.** *Let $Y \subseteq \mathbb{A}^n$ be an affine variety and let $f_1, f_2, \cdots, f_t \in k[x_1, x_2 \cdots, x_n]$ be the set of generators for ideal of Y. Y is nonsingular at a point $P \in Y$ if the rank of matrix $\| (\partial f_i/\partial x_j)(P) \|$ is $n - r$, where r is the dimension of Y. The affine variety Y is nonsingular if it is nonsingular at every point.*

Example 3.2 Consider a variety given by $y^2 = x^3$ in $\mathbb{A}^2$. Here, $n = 2$ and $r = 1$. The matrix is given by $[-3x^2 \quad 2y]$. At point $P = (0, 0)$ the rank of matrix is $0 < n - r$. Therefore P is singular point.

Example 3.3 Let Y be a variety given by $y^2 - x^3 + x = 0$ in $\mathbb{A}^2$. The matrix as in definition is $[-3x^2 + 1 \quad 2y]$. The rank of this matrix is 0 if $y = 0$ and $x = \pm 1/\sqrt{3}$. But this point does not lie on the curve. Since $n - r = 2 - 1 = 1$ and rank of matrix is always 1, Y is a nonsingular variety.

Example 3.4 Now, we will look at a variety given by two equations in $\mathbb{A}^3$. Let Y be given by $(x^2 - y^3, y^2 - z^3)$. Here, $n = 3$ and $r = 2$. We get the matrix as

$\begin{bmatrix} 2x & -3y^2 & 0 \\ 0 & 2y & -3z^2 \end{bmatrix}$. The rank of matrix at $P = (0,0)$ is 0. Therefore, Y is singular at P.

Definition 3.5 (Regular local ring)**.** *Let R be a noetherian local ring with maximal ideal $\mathfrak{m}$ and residue field $k = R/\mathfrak{m}$. R is a regular local ring if* $\dim \mathfrak{m}/\mathfrak{m}^2 = \dim R$.

Following theorem shows that the concept of nonsingularity is intrinsic and hence can be extended to any variety.

Theorem 3.6. *Let $Y \subseteq \mathbb{A}^n$ be an affine variety. Let $P \in Y$ be a point. Then Y is nonsingular at P if and only if the local ring $\mathcal{O}_{P,Y}$ is a regular local ring.*

Proof Let $P = (a_1, \ldots, a_n) \in \mathbb{A}^n$ and $I(P) = (x_1 - a_1, \ldots, x_n - a_n)$ be the corresponding maximal ideal in $R = k[x_1, \ldots, x_n]$. Let

$$\theta : R \to k^n$$

be a linear map given by

$$f \mapsto \langle \frac{\partial f}{\partial x_1}(P), \ldots, \frac{\partial f}{\partial x_n}(P) \rangle$$

for any $f \in R$. For all i, $\theta(x_i - a_i)$ forms a basis of k^n and $\theta(I^2(P)) = 0$. Therefore θ induces an isomorphism

$$\theta' : I(P)/I^2(P) \to k^n.$$

Consider an ideal I' of Y generated by $f_1, \ldots, f_t$. The rank of the Jacobian matrix $J = \| (\frac{\partial f_i}{\partial x_j})(P) \|$ is the dimension of $\theta(I')$ as a subspace of k^n i.e. the dimension of the subspace $(I' + I^2(P))/I^2(P)$ of $I(P)/I^2(P)$. We get the local ring $\mathcal{O}_P$ of P on Y from R by dividing by I' and localizing at maximal ideal $I(P)$. If $\mathfrak{m}$ is the maximal ideal of $\mathcal{O}_P$ we have

$$\mathfrak{m}/\mathfrak{m}^2 \simeq I(P)/(I' + I^2(P)).$$

Therefore, we have

$$\dim \mathfrak{m}/\mathfrak{m}^2 + \operatorname{rank} J = n$$

Let $\dim Y = r$. Then $\mathcal{O}_P$ is a local ring of dimension r. Now, $\mathcal{O}_P$ is regular iff $\dim_k \mathfrak{m}/\mathfrak{m}^2 = r$ i.e. $\operatorname{rank} J = n - r$, that is P is a non-singular point of Y. □

In view of the above theorem we can reformulate definition 3.1 as follows.

Definition 3.7. *Let Y be a variety. Y is nonsingular at a point $P \in Y$ if $\mathcal{O}_{P,Y}$ is a regular local ring. Y is nonsingular if it is so at every point. Y is singular if it is not nonsingular.*

Following examples illustrate the equivalence of two definitions.

Example 3.8 Let Y be the variety given by $y^2 - x^3 = 0$ in $A = k[x, y]$. We have seen that Y is singular at $P = (0, 0)$. We have $k[Y] = k[x, y]/(y^2 - x^3)$. Let $\mathcal{O}_P(Y)$ be the local ring at $P = (0, 0)$.
A function $f = g/h \in \mathcal{O}_P$ if $h(P) \neq 0$ and $f \in \mathfrak{m}_P(Y)$ if g is of the form $xk[x]+yk[x]$. So, $\mathfrak{m}_P(Y) = (x, y)$ and $\mathfrak{m}_P^2(Y) = (x^2, xy)$.
Clearly, $\dim \mathfrak{m}_P/\mathfrak{m}_P^2 = 2 \neq \dim \mathcal{O}_P = 1$.

Example 3.9 We will now consider the variety Y given by $(x^2 - y^3, y^2 - z^3)$ in $A = k[x, y, z]$. The coordinate ring of variety Y is $k[Y] = k[x, y]/(x^2 - y^3, y^2 - z^3)$. From 3.4 it is singular at $P = (0, 0)$.
A function $f = g/h \in \mathcal{O}_P$ if $h(P) \neq 0$ and $f \in \mathfrak{m}_P(Y)$ if g is of the form $xk[y] + yk[y] + zk[y] + z^2k[y] + xzk[y] + xz^2k[y]$. We have $\mathfrak{m}_P(Y) = (x, y, z)$ and $\mathfrak{m}_P^2(Y) = (y^2, z^2, xy, yz, xz)$. Now, $\dim \mathcal{O}_P = 2$ and $\dim \mathfrak{m}_P/\mathfrak{m}_P^2 = 3$.

Remark 3.10 The set of singular points is the set of points where the rank of Jacobian matrix $\parallel (\partial f_i/\partial x_j)(P) \parallel < n-r$. For a curve $r = n-1$ and hence, the singular points are where the rank of matrix is 0, i.e. common solution where $\{\partial f_i/\partial x_j = 0\}_{i,j}$. Since f_i's are polynomials, the singular points are finite and isolated.

Definition 3.11 (Normal Variety)**.** *A variety Y is said to be a normal variety at a point P if ring of regular functions $\mathcal{O}_P$ is integrally closed. Y is said to be normal if it is normal at every point.*

Example 3.12 Consider the variety X given by $y^2 - x^3 = 0$ in $A = k[x, y]$. it has a singular point at (0,0) in $\mathbf{A}^2_\mathbf{k}$. Parametrizing, one can see that $A/I(X) = k[X] = k[x, y]/(y^2 - x^3) \cong k[t^2, t^3]$. It can be easily checked that $\frac{1+t^3}{t^2}$ is not integral over $k[t^2, t^3]$.

Theorem 3.13. *A nonsingular variety is normal.*

Proof Let X be an irreducible nonsingular variety and $x \in X$ be any nonsingular point. The local ring, $\mathcal{O}_x$, at this point is a unique factorization domain. Let $\alpha \in k(X)$ be expressed as $\alpha = u/v$ such that $u, v \in \mathcal{O}_x$ have no common factor. If α is integral over $\mathcal{O}_x$ then, $\alpha^n + a_1\alpha^{n-1} + \cdots + a_n = 0$ where $a_i \in \mathcal{O}_x$ and hence, $u^n + a_1u^{n-1}v + \cdots + a_nv^n = 0$. From the equation we have $v|u^n$ but since v and u have no common factors, $\alpha \in \mathcal{O}_x$. Therefore, $\mathcal{O}_x$ is integrally closed.
Now, we have that the local rings $\mathcal{O}_x$ are integrally closed for all $x \in X$. We will prove that $k[X]$ is also integrally closed. Let $\beta \in k(X)$ be integral over $k[X]$. We have $\beta_n + b_1\beta_{n-1} + \cdots + b_n = 0$ where $b_i \in k[X]$ for all $i \in \{1, 2, \cdots, n\}$. Also, $b_i \in \mathcal{O}_x$ for every $x \in X$. Since $\mathcal{O}_x$ is integrally closed, $\beta \in \mathcal{O}_x$ for all $x \in X$. Therefore, $\beta \in \cap_{x \in X}\mathcal{O}_x$. Since $\cap_{x \in X}\mathcal{O}_x = k[X]$, $\beta \in k[X]$. $\square$

However, converse of above theorem may not always be true, i.e. a normal variety may not be nonsingular. Consider the following example.

Example 3.14 Let $X \subset \mathbf{A}^3$ be given by $x^2 + y^2 = z^2$. X is singular at $(0, 0, 0)$. We will prove that $k[X]$ is integrally closed in $k(X)$. $k[X]$ consists of elements $u + vz$ where $u, v \in k[x, y]$. Hence, $k[X]$ is a finite module over $k[x, y]$, and hence all elements of $k[X]$ are integral over $k[x, y]$.
If $\alpha = u + vz \in k(X)$, where $u, v \in k(x, y)$, is integral over $k[X]$ then it must also be integral over $k[x, y]$. Its minimal polynomial is

$$T^2 - 2uT + u^2 - (x^2 + y^2)v^2.$$

Since $2u \in k[x, y]$ we have $u \in k[x, y]$. Similarly, $u^2 - (x^2 + y^2)v^2 \in k[x, y]$, and hence $(x^2 + y^2)v^2 \in k[x, y]$. Now, since $x^2 + y^2 = (x + iy)(x - iy)$ is the product of two coprime irreducibles, it follows that $v \in k[x, y]$, and thus $\alpha \in k[X]$.

Theorem 3.15. *If X is a normal variety and $Y \subset X$ a codimension 1 subvariety then there exists an affine open set $X' \subset X$ with $X' \cap Y \neq 0$ such that the ideal of $Y' = X' \cap Y$ in $k[X']$ is principal.*

Proof Let $I(Y) = (v_1, \cdots, v_m)$ be the ideal corresponding to Y in $k[X]$. Let $0 \neq f \in k[X]$ such that $f \in I(Y) \subset \mathcal{O}_Y$. Then $Y \subset Z(f)$, and since both of these are codimension 1 subvarieties, Y consists of components of $Z(f)$. Suppose that $Z(f) = Y \cup Y_1$ such that $Y \not\subseteq Y_1$. Let $X_1 = X \setminus Y_1$, we have $Y \cap X_1 \neq \varnothing$ and $Y \cap X_1 = Z(f) \cap X_1$. Thus, we can assume that $Y = Z(f)$.
By the Nullstellensatz, $Y = Z(f)$ in X implies that $I(Y)^k \subset (f)$ for some $k > 0$, and

hence for some minimal k, $\mathfrak{m}_Y^k \subset (f)$ in $\mathcal{O}_Y$, where $\mathfrak{m}_Y$ is the maximal ideal of $\mathcal{O}_Y$. Then there exist $\alpha_1, \cdots, \alpha_{k-1} \in \mathfrak{m}_Y$ such that $\alpha_1 \cdots \alpha_{k-1} \notin (f)$ but $\alpha_1 \cdots \alpha_{k-1}\mathfrak{m}_Y \subset (f)$. Let $g = \alpha_1 \cdots \alpha_{k-1}$ we have $g \notin (f)$ but $g\mathfrak{m}_Y \subset (f)$.
Consider $u = f/g$. We have $u^{-1} \notin \mathcal{O}_Y$, but $u^{-1}\mathfrak{m}_Y \subset \mathcal{O}_Y$. Since $\mathcal{O}_Y$ is integrally closed, $u^{-1}\mathfrak{m}_Y \nsubseteq \mathfrak{m}_Y$. Now, $\mathfrak{m}_Y$ is the maximal ideal of $\mathcal{O}_Y$ and $u^{-1}\mathfrak{m}_Y \subseteq \mathcal{O}_Y$ but $u^{-1}\mathfrak{m}_Y \nsubseteq \mathfrak{m}_Y$ implies that $u^{-1}\mathfrak{m}_Y = \mathcal{O}_Y$. Hence, $\mathfrak{m}_Y = (u)$ is principal in $\mathcal{O}_Y$.
Since $I(Y) \subset \mathfrak{m}_Y$, we can write $v_i = uw_i$, where $w_i = c_i/d_i$, with $c_i, d_i \in k[X]$ and $d_i \notin I(Y)$. Consider

$$X' = X \setminus Z(g) \bigcup_{i=1}^{m} Z(d_i).$$

The ideal $I(Y')$ of the variety $Y' = X' \cap Y$ is the principal ideal (u) in $k[X']$. □

Theorem 3.16. *The set of singular points of a normal variety has codimension ≥ 2*

Proof Let X be the normal variety and S be collection of singular points. Suppose that S contains an irreducible component of codimension 1, say Y. Let X' and Y' be as in proof of 3.15. Let $y \in Y'$ such that y is nonsingular point of Y'. Let $u_1, \cdots, u_{n-1}$ be the local parameters of Y' at y. From 3.15 $I(Y') = (u)$ is a principal ideal of $k[X']$, and hence $k[Y'] = k[X']/(u)$ and $\mathcal{O}_{Y',y} = \mathcal{O}_{X',y}/(u)$. Consider the natural map $\mathcal{O}_{X',y} \longrightarrow \mathcal{O}_{Y',y}$, the inverse image of $\mathfrak{m}_{Y',y}$ is $\mathfrak{m}_{X',y}$. Let $v_1, \cdots, v_{n-1} \in \mathcal{O}_{X',y}$ be the arbitrary inverse images of the local parameters $u_1, \cdots, u_{n-1} \in \mathcal{O}_{Y',y}$. Then $\mathfrak{m}_{X',y} = (v_1, \cdots, v_{n-1}, u)$. This proves that $\dim \mathfrak{m}_{X',y}/\mathfrak{m}_{X',y}^2 \leq n$, and hence that y is a nonsingular point of X, which contradicts the assumption $y \in Y \subset S$. □

Remark 3.17 Since the codimension of singular varieties is greater than or equal to 2, for algebraic curves normal varieties are nonsingular.

Definition 3.18 (Normalization)**.** *A normalization of an irreducible variety X is an irreducible normal variety $\bar{X}$, together with a regular map $\psi : \bar{X} \longrightarrow X$, such that ψ is finite and birational and has a property that whenever Z is a normal variety and $\phi : Z \longrightarrow X$ is finite and birational map, there is a unique morphism $\theta : Z \longrightarrow \bar{X}$ such that $\phi = \psi \circ \theta$.*

Theorem 3.19 (Emmy Noether)**.** *If R is a finitely generated domain over a field or over the integers, and L is finite extension field of the field of fractions of R, then the integral closure of R in L is a finitely generated R-module.*

Proof *cf.* Eisenbud [Ei] [13, corollary 13.13] $\square$

Given an irreducible affine variety X following theorem shows the existence of a normalized variety $\bar{X}$.

Theorem 3.20. *An affine irreducible variety has a normalisation that is also affine.*

Proof Let X be given variety and $k[X]$ be the associated coordinate ring. Let A be the integral closure of $k[X]$ in $k(X)$. To prove the theorem we need to find an affine variety X' such that $A = k[X']$. Then X' is normal and the inclusion $k[X] \hookrightarrow k[X']$ defines a regular birational map $f : X' \longrightarrow X$.

By Noether Normalisation, there exists a subring $B \subset k[X]$ such that B is isomorphic to a polynomial ring $B \cong k[t_1, \cdots, t_r]$ and $k[X]$ is integral over B. Now, A is equal to the integral closure of B in $k(X)$ and $K = k(X)$ is a finite field extension of $k(t_1, \cdots, t_r)$, since $t_1, \cdots, t_r$ is a transcendence basis of $k(X)$. From 3.19 we have A is finite B-module.
Therefore, A is finitely generated k-algebra generated by finitely many elements say, $t_1, \cdots, t_n$. We have, $A \cong k[t_1, \cdots, t_n]/I$, where I is an ideal of the polynomial ring $k[t_1, \cdots, t_n]$. Suppose that $I = (f_1, \cdots, f_m)$. Consider the closed set $X' \subset A^n$ defined by the equations $f_1 = \cdots = f_m = 0$ we have $I(X') = I$ (by Hilbert Nullstellensatz and the fact that there are no zero divisors), from which it follows that $k[X'] \cong k[t_1, \cdots, t_n]/I \cong A$. $\square$

Proposition 3.21. *Let $A \subseteq B$ be rings and C be the integral closure of A in B. Let S be a multiplicatively closed subset of A. Then $S^{-1}C$ is the integral closure of $S^{-1}A$ in $S^{-1}B$.*

Proof Let x be an element of C and s be an element of S. Let

$$x^n + a_1 x^{n-1} + \ldots + a_n = 0$$

where $a_i \in A$.

Let x/s be an element of $S^{-1}B$. Then we obtain the equation

$$(x/s)^n + (a_1/s)(x/s)^{n-1} + \ldots + a_n/s^n = 0$$

which shows that x/s is integral over $S^{-1}A$. Hence $S^{-1}C$ is integral over $S^{-1}A$.

Conversely, let $b/s \in S^{-1}B$ be integral over $S^{-1}A$. We have an equation of the form

$$(b/s)^n + (a_1/s_1)(b/s)^{n-1} + \ldots + a_n/s_n = 0$$

with $a_i \in A$ and $s_i \in S$. Define $t = s_1 \cdots s_n$. On multiplying the above equation $(st)^n$, we get an equation of integral dependence for bt over A. Thus, we have $bt \in C$ and therefore $b/s = bt/st \in S^{-1}C$. With this the proof is complete. □

Theorem 3.22. *Normalization exists for any scheme.*

Proof (Sketch) Let X be an integral scheme covered by finite open affine schemes. Consider two such open subsets $U = \operatorname{Spec} A$ and $V = \operatorname{Spec} B$. The intersection $U \cap V$ can be covered by finitely many affine subschemes given by localisation of A and B. From 3.21 if $\tilde{A}$ is the integral closure of A, then $\tilde{A}_f$ is the integral closure of A_f. Looking at intersection of $\operatorname{Spec} A_f$ and $\operatorname{Spec} B_g$ in $\tilde{A}$ and $\tilde{B}$ we get an isomorphism and hence we can patch the normalization of local affine schemes to get global normalization. Hence for any scheme normalization exists. □

Chapter 4

Resolution of Singularities of Curves

Here we will state the general problem of the resolution of singularities of varieties and observe that on curves normalization resolves the singularities. We then explain the construction of blowup of a surface at a point and show that a curve embedded in a nonsingular surface can be resolved by finite number of sequential blowups of the surface at the singular points. In this chapter we will be using concept of divisors and canonical divisor which will be explained in the later chapters.

The general problem of the resolution of singularities of variety X is to find a nonsingular variety X' along with a proper birational morphism $\phi : X' \longrightarrow X$.

For some curve Y, from 3.16, normalisation gives the nonsingular curve Y' and a birational map $\psi : Y' \longrightarrow Y$ and hence resolves the singularities on curves. Normalisation, however, may not resolve singularities in higher dimension.

Theorem 4.1. *Let C be a nonsingular curve and P be a point in C. Let Y be a projective variety and $\varphi : C - P \longrightarrow Y$ be a morphism. Then there exists a unique morphism $\bar{\varphi} : C \longrightarrow Y$ extending φ.*

Proof Embed Y as a closed subset of $\mathbb{P}^n$ for some n. It will be sufficient to show that φ extends to a morphism of C into $\mathbb{P}^n$, because if it does, the image is necessarily contained in Y since Y is closed in $\mathbb{P}^n$.

Let $\mathbb{P}^n$ have the homogeneous coordinates $x_0, \cdots, x_n$ and U be the open set such that $x_i \neq 0 \ \forall \ i$. If $\varphi(C - P) \cap U = \varnothing$ then $\varphi(C - P) \subseteq \mathbb{P}^n - U$. Since $\mathbb{P}^n - U$ is the union of hyperplanes given by $x_i = 0$ and $\varphi(C-P)$ is irreducible, $\varphi(C-P) \subset H_i$ for some i. Now, $H_i \cong \mathbb{P}^{n-1}$. Hence, by using induction on n we can assume that $\varphi(C-P) \cap U \neq \varnothing$.

Let x_i/x_j be regular functions of U $\forall$ i, j. Let f_{ij} be the regular function on C obtained by pulling back x_i/x_j by φ. $f_{ij} \in k(C)$ where $k(C)$ is the function field of C. Let ν be the valuation of $k(C)$ with respect to valuation ring R_P. Let $r_i = \nu(f_{i0})$ for $i = 0, 1, \cdots, n$. Since $x_i/x_j = (x_i/x_0)(x_0/x_j)$ we have $\nu(f_{ij}) = r_i - r_j$. Let k be such that $r_k = \min\{r_0, \cdots, r_n\}$. Then, $\nu(f_{ik}) \geq 0$ for all i. Hence, $f_{0k}, \cdots, f_{nk} \in R_P$.

Now, we define $\bar{\varphi}$ as $\bar{\varphi}(P) = (f_{0k}(P), \cdots, f_{nk}(P))$ and $\bar{\varphi}(Q) = \varphi(Q)$ for $Q \neq P$. To show that $\bar{\varphi}$ is a morphism it is enough to show that a regular function in the neighbourhood of $\bar{\varphi}(P)$ pulls back to a regular function on C. Consider $U_k \subseteq \mathbb{P}^n$ be open set given by $x_k \neq 0$. Then $\bar{\varphi}(P) \in U_k$ since $f_{kk}(P) = 1$. Since U_k is affine with coordinate ring $k[x_0/x_k, \cdots, x_n/x_k]$, these functions pull back to $f_{ok}, \cdots, f_{nk}$ which are regular at P by construction.

To prove uniqueness let us assume there exists another morphism $\psi : C \longrightarrow \mathbb{P}^n$ such that $\psi|_{C-P} = \bar{\varphi}|_{C-P}$. Consider

$$\psi \times \bar{\varphi} : C \longrightarrow \mathbb{P}^n \times \mathbb{P}^n.$$

Let

$$\Delta = \{R \times R | R \in \mathbb{P}^n\}$$

be the diagonal subset of $\mathbb{P}^n \times \mathbb{P}^n$. By hypothesis $\psi \times \bar{\varphi}(C-P) \subset \Delta$. Since $(C-P)$ is dense in C and Δ is closed subset of $\mathbb{P}^n \times \mathbb{P}^n$, we have $\psi \times \bar{\varphi}(C) \subset \Delta$. Hence the morphism is unique. $\square$

4.1 Blow-up

Now, we will construct the blowing up of $\mathbb{A}^n$ at the point $O = (0, \ldots, 0)$.

We have a quasi-projective variety $\mathbb{A}^n \times \mathbb{P}^{n-1}$. Let $x_1, \ldots, x_n$ be the affine coordinates of $\mathbb{A}^n$ and $y_1, \ldots, y_n$ be the homogeneous coordinates of $\mathbb{P}^{n-1}$. The polynomials in $x_i y_j$ are the closed subsets of $\mathbb{A}^n \times \mathbb{P}^{n-1}$.

Definition 4.2. *The blowing-up of $\mathbb{A}^n$ at the point $O = (0, \cdots, 0)$ is the closed subset X of $\mathbb{A}^n \times \mathbb{P}^{n-1}$ given by the equations*

$$\{x_i y_j = x_j y_i | i, j = 1, \ldots, n\}.$$

Let $\varphi : X \to \mathbb{A}^n$ be the natural morphism obtained by restricting the projection map of $\mathbb{A}^n \times \mathbb{P}^{n-1}$.

We now look at some of its properties

(i) Let $O \neq P \in \mathbb{A}^n$, then $\varphi^{-1}(P)$ is a single point. Moreover, φ gives an isomorphism of $X - \varphi^{-1}(O)$ onto $\mathbb{A}^n - O$. We can see this if we let $P = (a_1, \ldots, a_n)$ where not all $a_i = 0$. For each j, if $P \times (y_1, \ldots, y_n) \in \varphi^{-1}(P)$, we have $y_j = (a_j/a_i)y_i$. We see that $(y_1, \ldots, y_n)$ is uniquely determined as a point in $\mathbb{P}^{n-1}$. If we put $y_i = a_i$, we have $\varphi^{-1}(P)$ to be a single point. We can also define an inverse morphism ψ, by setting $\psi(P) = (a_1, \ldots, a_n) \times (a_1, \ldots, a_n)$ for a point $P \in \mathbb{A}^n - O$. This gives us the desired isomorphism.

(ii) We have the isomorphism $\varphi^{-1}(O) \simeq \mathbb{P}^{n-1}$. This can be seen when we observe that $\varphi^{-1}(O)$ consists of all points $O \times Q$ (where $Q \in \mathbb{P}^{n-1}$) subject to no restriction.

(iii) Let L be a line through O in $\mathbb{A}^n$. We can give a parametric equation $x_i = a_i t$ for all i where a_i's are not all 0 and $t \in \mathbb{A}^1$. Consider the line $L' = \varphi^{-1}(L - O)$ in $X - \varphi^{-1}(O)$. This is given by the parametric equation $x_i = a_i t$ and $y_i = a_i t$ with $t \in \mathbb{A}^1 - 0$. Since y_i's are homogeneous coordinates in $\mathbb{P}^{n-1}$, we can write the equation of L' as $x_i = a_i t, y_i = a_i$ for $t \in \mathbb{A}^1 - O$. These equations will now make sense even for $t = 0$. This gives the closure $\overline{L'}$ of L' in X. We observe that $\overline{L'}$ meets $\varphi^{-1}(O)$ at $Q = (a_1, \ldots, a_n) \in \mathbb{P}^{n-1}$. On sending L to Q we have a 1-1 correspondence between lines in $\mathbb{A}^n$ passing through O and points of $\varphi^{-1}(O)$.

(iv) X is the union of $X - \varphi^{-1}(O)$ and $\varphi^{-1}(O)$. We have seen that $X - \varphi^{-1}(O)$ is isomorphic to $\mathbb{A}^n - O$ which is irreducible. Since every point of $\varphi^{-1}(O)$ is in $\overline{L'}$, $X - \varphi^{-1}(O)$ is dense. Therefore, X is irreducible.

Definition 4.3. *Let Y be a closed subvariety of $\mathbb{A}^n$ passing through O. The blowing up of Y at O is defined as $\tilde{Y} = (\varphi^{-1}(Y - O))^-$, where φ is the blowing up of $\mathbb{A}^n$ at O as above. The morphism obtained by restriction of $\varphi : X \to \mathbb{A}^n$ to $\tilde{Y}$ is also denoted by $\varphi : \tilde{Y} \to Y$. To blow up at any other point $P \in \mathbb{A}^n$ we can make a linear change in coordinates.*

Since φ induces an isomorphism from $\tilde{Y} - \varphi^{-1}(O)$ to $Y - O$, it is birational morphism of $\tilde{Y}$ to Y.

Definition 4.4. *The inverse image of O is called the exceptional curve. We will denote it with E.*

Definition 4.5. *(Monoidal Transformation) Let X be a surface. Then blowing up a single point P on X is called the monoidal transformation of X.*

Let X be a nonsingular surface. Let Y be a curve in X. The problem of embedded resolution is to find a birational morphism $\phi : X' \longrightarrow X$ such that X' is a nonsingular surface and irreducible components of $\phi^{-1}(Y)$ are nonsingular and at any point P of $\phi^{-1}(Y)$ the local equations of the irreducible components forms the regular system of parameters at P. Then we say that $\phi^{-1}(Y)$ is a divisor with *normal crossings* in X'.

To achieve this we proceed by blowing up the surface X at a point x where Y is singular. Let $\phi_1 : X_1 \longrightarrow X$ be the morphism obtained by blowing up X at x. Let Y_1 be the closure of $\phi_1^{-1}(Y - x)$. Y_1 is said to be the *strict transform* of Y on X_1. We repeat the process at singular points of Y_1, if any.

By theorem 4.11 we prove that embedded curve can be resolved after finite number of monoidal transformations.

Definition 4.6. *Let C be an effective Cartier divisor on the surface X, and let f be a local equation for C at the point P. Then, we define the multiplicity of C at P to be the largest integer r such that $f \in \mathfrak{m}_P^r$, where $\mathfrak{m}_P \subset \mathcal{O}_{P,X}$ is the maximal ideal.*

Theorem 4.7. *Let C be an effective divisor. Let P be a point of multiplicity r on C and $\pi : \tilde{X} \to X$ be the monoidal transformation at P. Then,*

$$\pi^*(C) = \tilde{C} + rE$$

where E is the exceptional curve.

Proof Let f be the local equation of C on X. Since multiplicity of P on C is r, f can be written as

$$f = f_r(x, y) + g$$

where, f_r is a non-zero homogeneous polynomial of degree r and $g \in \mathfrak{m}^{r+1}$.
In the blowup, consider the open affine subset defined by $y = ux$. We can write

$$\pi^* f = x^r(f(1, u) + xh)$$

x is the local equation of exceptional curve E and occurs with multiplicity exactly r. $\square$

Definition 4.8 (Intersection Multiplicity)**.** *Let $D_1, \dots, D_n$ be effective divisors on a nonsingular variety X at a point $x \in X$ with local equations $f_1, \dots, f_n$ in some neighbourhood of x. Then $\dim_k(\mathcal{O}_x/(f_1, \dots, f_n))$ is the intersection multiplicity or local intersection number of $D_1, \dots, D_n$ at x.*

Proposition 4.9. *Let $\pi : \bar{X} \longrightarrow X$ be monoidal transformation of nonsingular projective surface X. Then,*

(i) if $C, D \in \operatorname{Pic} X$, $\pi^*(C).\pi^*(D) = C.D$

(ii) if $C \in \operatorname{Pic} X$, then $\pi^*(C).E = 0$

(iii) $E^2 = -1$.

Proof Let $S \subset X$ be the set of singular points and $T = \pi^{-1}(S)$ be its set theoretic inverse image. So we have an isomorphism $\bar{X} \setminus T \simeq X \setminus S$ defined by π.
If $\operatorname{Supp} C \cap S = \operatorname{Supp} D \cap S = \varnothing$ we have $\pi^*(C).\pi^*(D) = C.D$ due to isomorphism. Else we can find C' and D' such that $C' \sim C$ and $D' \sim D$ and $\operatorname{Supp} C' \cap S = \operatorname{Supp} D' \cap S = \varnothing$. Since $C.D = C'.D'$ and $C'.D' = \pi^*(C').\pi^*(D')$ due to isomorphism and $\pi^*(C') \sim \pi^*(C)$, we have the desired equality.

If $\operatorname{Supp} C \cap S = \varnothing$ then $\pi^*(C) = \tilde{C}$. So we have $\pi^*(C).E = 0$. If $\operatorname{Supp} C \cap S \neq \varnothing$ we can find C' such that $C' \sim C$ and $\operatorname{Supp} C' \cap S = \varnothing$.

Consider the curve C with local equation y. Then by 4.7 we have $\pi^*(C) = \tilde{C} + E$. Since x is the local equation of E we have $\tilde{C}.E = 1$. Now, since $\pi^*(C).E = 0$, we have $(\tilde{C} + E).E = 0$ which implies $\tilde{C}.E + E^2 = 0$. Since $\tilde{C}.E = 1$, we get $E^2 = -1$. $\square$

Theorem 4.10. *Let C be an irreducible curve in the nonsingular surface X. Then there exists a finite sequence of monoidal transformations $X_n \to X_{n-1} \to \cdots \to X_0 = X$ such that the strict transform C_n of C on X_n is nonsingular.*

Proof If C is nonsingular there is nothing to prove. Let C be singular at some point P with multiplicity $r \geq 2$. Let $X_1 \to X$ be the monoidal transformation at P and let C_1 be the strict transform of C. Then, by 10.16 we have $g(C_1) < g(C)$. If C_1 is nonsingular we have the required monoidal transformation. If not, we pick a singular point on C_1 and proceed as before.

Since the arithmetic genus of an irreducible curve is non-negative and we have $g(C_i) < g(C_{i-1})$, the sequence terminates after finite transformations. $\square$

Theorem 4.11 (Embedded Resolution of curves in surfaces)**.** *Let Y be any curve in the surface X.Then there exists a finite sequence of monoidal transformations*

$$X' = X_n \to X_{n-1} \to \cdots \to X_0 = X$$

such that if $f : X' \to X$ is their composition, then the strict transform of Y on X' is smooth. Moreover, after possibly finitely many more blow-ups the total transform can be made a divisor with simple normal crossing.

Proof Let $\pi : \bar{X} \longrightarrow X$ be the monoidal tranformation at P and let multiplicity of Y at P be r. The divisor $\pi^{-1}(Y) = \bar{Y} + E = \pi^*(Y) - (r-1)E$ and by 10.16

$$g(\pi^{-1}(Y)) = g(Y) - \frac{1}{2}(r-1)(r-2)$$

If multiplicity of Y at $P \geq 3$ then, $g(\pi^{-1}(Y)) \leq g(Y)$. So there can only be finitely many steps. If multiplicity is 2 i.e. we have $Y.E = 2$, there are three possibilities.

(i) $\bar{Y}$ meets E transversally in two distinct points we have achieved the resolution.

(ii) $\bar{Y}$ and E meets at one point with multiplicity 2, we can blowup the point again to get a triple point and then we can blow it up to get the desired resolution.

(iii) $\bar{Y}$ has singular point of multiplicity 2 and E passes through the point. So, multiplicity of singular point on $\bar{Y}+E$ is 3. So, we blowing it up will make $g(Y)$ drop. $\square$

Chapter 5

Weil Divisors

Let X be a noetherian, integral, separated scheme which is regular in codimension one.

Definition 5.1 (Prime Divisor). *A prime divisor on X is a closed integral subscheme Y of codimension one.*

Let $\operatorname{Div} X$ be the free abelian group generated by the prime divisors.

$$\operatorname{Div} X = \{D | D = \Sigma n_i Y_i\}$$

where n_i are integers and Y_i are the prime divisors.

Definition 5.2 (Weil Divisor). *A Weil Divisor by definition is an element of the group* $\operatorname{Div} X$.

Definition 5.3 (Degree of Divisor). *The degree of a divisor $D = \Sigma n_i Y_i$ is given by*

$$\deg D = \Sigma n_i \deg Y_i.$$

From the definition we can observe that $\deg(D_1 + D_2) = \deg(D_1) + \deg(D_2)$.

Definition 5.4 (Effective Divisor). *A divisor $D = \Sigma n_i Y_i$ is an effective divisor if all* $n_i \geq 0$.

Definition 5.5 (Support of Divisor). *The support of a divisor $D = \Sigma n_i Y_i$ is the union of prime divisors Y_i and is denoted by* $\operatorname{Supp} D$.

If Y is an irreducible subvariety of codimension one, the local ring is discrete valuation ring. Let v_Y be the valuation of Y.

Definition 5.6 (Zeroes and Poles)**.** *Let $f \in K^*$. If $v_Y(f)$ is positive at a point, we say f has a zero at that point along Y of order $v_Y(f)$. If $v_Y(f)$ is negative at a point f is said to have a pole that point along Y of order $-v_Y(f)$.*

Definition 5.7 (Principal Divisor)**.** *Let $f \in K^*$ and X be as defined above, then divisor of f is given by*

$$(f) = \Sigma v_Y(f)Y$$

Any divisor equal to the divisor of a function is called principal divisor.

Proposition 5.8. *Let A be a noetherian domain. If A is unique factorization domain then divisors on $X = \operatorname{Spec} A$ are principal divisors.*

Proof Since A is unique factorization domain, every prime ideal of A of height 1 is principal. Consider prime divisor $Y \subset X = \operatorname{Spec} A$. The divisor Y corresponds to a prime ideal $\mathfrak{p}$ of height 1 generated by f, say. We have $Y = (f)$. □

Definition 5.9 (Linear Equivalence)**.** *Two divisors are said to be linearly equivalent if their difference is a principal divisor.*

Definition 5.10 (Class Group)**.** *The group $\operatorname{Div} X$ modulo subgroup of principal divisors is the divisor class group of X, denoted by $\operatorname{Cl} X$.*

Proposition 5.11. *Let X be projective space $\mathbb{P}^n_k$ and D be any divisor of degree d, then $D \sim dH$ where H is the hyperspace represented by $x_0 = 0$. Moreover, we have an isomorphism $\deg : \operatorname{Cl} X \longrightarrow \mathbb{Z}$.*

Proof Let D be a divisor of degree d. We express it as difference of two effective divisors D_1 and D_2 of degrees, say, d_1 and d_2 respectively. By 5.8 D_1 and D_2 are principal divisors. So, let $D_1 = (g_1)$ and $D_2 = (g_2)$.
Now, consider a rational function $f = g_1/x_0^d g_2$.

$$D - (f) = D_1 - D_2 - (f) = dH$$

Hence, $D \sim dH$.
Since degree of hypersurface H is 1, we have $deg : \mathrm{Cl}\, X \longrightarrow \mathbf{Z}$ is an isomorphism. □

Corollary 5.12. $\mathrm{Cl}(\mathbb{P}^1) \cong \mathbb{Z}$.

This isomorphism given by deg map is generated by a point.

Corollary 5.13. $\mathrm{Cl}(\mathbb{P}^2) \cong \mathbb{Z}$.

This isomorphism is generated by a line in $\mathbb{P}^2$.

Proposition 5.14. *Let Z be an irreducible proper closed subset of codimension 1 of X and let U=X-Z. There is an exact sequence*

$$\mathbb{Z} \longrightarrow \mathrm{Cl}\, X \longrightarrow \mathrm{Cl}\, U \longrightarrow 0.$$

Proof If $f \in K^*$ and $(f) = \Sigma n_i Y_i$, then considering f as a rational function on U we have $(f)_U = \Sigma n_i (Y_i \cap U)$. So we have a homomorphism $\mathrm{Cl}\, X \longrightarrow \mathrm{Cl}\, U$. Since every principal divisor on U is the restriction of its closure in X, the map is surjective. The kernel of the map is a divisor whose support is in Z. Since Z is irreducible, kernel is just the subgroup of $\mathrm{Cl}\, X$ generated by $1.Z$. □

Example 5.15 Let $A = k[x, y, z]/(xy - z^2)$ and $X = SpecA$. Let Y be given by $y = z = 0$. Since Y is a prime divisor, we have

$$\mathbb{Z} \longrightarrow \mathrm{Cl}\, X \longrightarrow \mathrm{Cl}(X - Y) \longrightarrow 0.$$

Since $y = 0 \Rightarrow z^2$ and z generates the maximal ideal of the local ring at the generic point of Y, the divisor of y is $2.Y$. Hence $X - Y = \mathrm{Spec}\, A_y$. Now $A_y = k[x, y, y^{-1}, z]/(xy - z^2) \cong k[y, y^{-1}, z]$. Since this is unique factorisation domain, $\mathrm{Cl}(X - Y) = 0$. Therefore $\mathrm{Cl}\, X$ is generated by Y and 2.Y=0. Let $\mathfrak{m} = (x, y, z)$ be a maximal ideal. Therefore $\mathfrak{m}/\mathfrak{m}^2$ is a vector space of dimension 3 over k. Let $\mathfrak{p} = (y, z) \subseteq \mathfrak{m}$ and image of $\mathfrak{p}$ in $\mathfrak{m}/\mathfrak{m}^2$ is $\bar{y}$ and $\bar{z}$ and hence $\mathfrak{p}$ is not a principal ideal. Therefore Y is not a principal divisor. So we have $\mathrm{Cl}\, X = \mathbb{Z}/2\mathbb{Z}$.

Chapter 6

Cartier Divisors

Let X be a scheme and $U = \operatorname{Spec} A$ be an open affine subset. Let S be the set of non-zero divisors. The localization of A by the multiplicative set S is the total quotient ring $K(U)$.
For each open set U, $S(U)^{-1}\Gamma(U, \mathcal{O}_X)$ forms a presheaf. Then sheaf of total quotient rings of $\mathcal{O}$ is the associated sheaf of rings to the presheaf and is denoted by $\mathcal{K}$.
Let $\mathcal{K}^*$ be the sheaf of inverible elemnts in the sheaf of rings $\mathcal{K}$.

Definition 6.1 (Cartier Divisor)**.** *Cartier divisor on X is given by an open cover $\{U_i\}$ of X, and for each i an element $f_i \in \Gamma(U_i, \mathcal{K}^*)$, such that for each $i, j, f_i/f_j \in \Gamma(U_i \cap U_j, \mathcal{O}^*)$.*
We represent cartier divisor by $\{U_i, f_i\}$.

Definition 6.2 (Principal Cartier Divisor)**.** *A Cartier divisor is said to be principal if it is in the image of the natural map*

$$\Gamma(X, \mathcal{K}^*) \longrightarrow \Gamma(X, \mathcal{K}^*/\mathcal{O}^*)$$

Two Cartier divisors are linearly equivalent if their difference is principal.

Theorem 6.3. *For an integral, noetherian, separated locally factorial scheme X the group $\operatorname{Div} X$ of Weil divisors is isomorphic to the group of Cartier divisors $\Gamma(X, \mathcal{K}^*/\mathcal{O}^*)$.*

Proof Let $\{U_i, f_i\}$ be the given cartier divisor. Let Y be any prime divisor. We take its coefficient to be $v_Y(f_i)$ whenever $Y \cap U_i \neq \varnothing$. On $U_i \cap U_j$ we have f_i/f_j is invertible and hence $v_Y(f_i/f_j) = 0$, that is, $v_Y(f_i) = v_Y(f_j)$. We can associate divisor

$D = \Sigma v_Y(f_i)Y$ to the given Cartier divisor.

Conversely, Let D be a Weil divisor on X. Let $x \in X$ be any point. The divisor D induces a Weil divisor D_x on $\operatorname{Spec}\mathcal{O}_x$. Since $\mathcal{O}_x$ is unique factorization domain, D_x is a principal divisor. Let $D_x = (f_x)$. Now,the divisor (f_x) and D on X differ at some prime divisors not passing through X. There is an open neighborhood U_x of x such that D and (f_x) are same on the neighborhood. We can cover X with such open sets and the respective functions on those open sets give the Cartier divisor on X. $\square$

Corollary 6.4. *The notions of Weil and Cartier divisors coincide for nonsingular varieties, in particular nonsingular projective varieties viz. curves.*

Chapter 7

Linear Systems

Let X be a nonsingular projective variety and $K = k(X)$ be the function field of X. Given a divisor D we construct a vector space of rational functions on X with zeroes and poles having orders no worse than the corresponding coefficient in the divisor D.

Proposition 7.1. *Let $0 \neq f \in K$. Then, the following are equivalent*

(i) $div(f) \geq 0$

(ii) $f \in k$

(iii) $div(f) = 0$.

Proof If $div(f) \geq 0$, $f \in \mathcal{O}_P(X)$ for all $P \in X$. If $f(P_0) = \lambda_0$ for some P_0, then $div(f - \lambda_0) \geq 0$ and $deg(div(f - \lambda_0)) > 0$, a contradiction, unless $f - \lambda_0 = 0$, i.e., $f \in k$. □

Remark 7.2

(i) The proposition shows that a meromorphic function can not just have zeroes or just poles.

(ii) If $D \sim D'$ (linearly equivalent), then $degD = degD'$.

(iii) If $D \sim D'$ and $D_1 \sim D_1'$, then $D + D_1 \sim D' + D_1'$.

Definition 7.3. *Let X be a curve and $D = \Sigma n_P P$ be a divisor of X. For a given divisor D we define*

$$L(D) = \{f \,|ord_P(f) \geq -n_P \,\forall\, P \in X\}.$$

$L(D)$ forms a vector space over k.

Definition 7.4 (Complete linear system)**.** *A complete linear system denoted by $|D|$ is a set of all effective divisors linearly equivalent to a given divisor D. Any subvector space of $L(D)$ is called a linear system.*

Proposition 7.5. *(i) If $D \leq D_0$, then $L(D) \subset L(D_0)$, and*

$$\dim_k(L(D_0)/L(D)) \leq \deg(D_0 - D).$$

(ii) *$L(0) = k$; $L(D) = 0$ if $\deg(D) < 0$.*

(iii) *$L(D)$ is finite dimensional for all D. If $\deg(D) \geq 0$, then* $\dim_k L(D) \leq \deg(D) + 1$.

(iv) *If $D \sim D_0$, then $\dim_k L(D) = \dim_k L(D_0)$.*

Proof

(i) The proposition can be reduced to showing $\dim_k(L(D+P)/L(D)) \leq 1$ Let t be a uniformizing parameter in $\mathcal{O}_P(X)$, and $r = nP$ be the coefficient of P in D. Consider $\phi : L(D+P) \longrightarrow k$ given by $\phi(f) = (t^{r+1}f)(P)$. Since $\ker(\phi) = L(D)$, we have $\bar{\phi} : L(D+P)/L(D) \longrightarrow k$ is an injective map. Hence $\dim_k(L(D+P)/L(D)) \leq 1$.

(ii) This follows immediately from proposition 7.1 and remark 7.2.

(iii) Let $\deg D = n \geq 0$. For some $P \in X$ consider $D' = D - (n+1)P$. Since $\deg D' < 0$, we get $L(D') = 0$. From (1), we have

$$\dim_k(L(D)/L(D')) \leq n + 1 = \deg(D) + 1.$$

(iv) If $D = D'$ there is nothing to prove. Let $D = D' + div(g)$ where $g \neq 0$ Consider the map $\psi : L(D) \longrightarrow L(D')$ given by $\psi(f) = fg$. The map ψ is injective because $fg = 0$ implies $f = 0$ as $g \neq 0$.

To prove ψ is surjective consider $f' \in L(D')$.

$$(f'/g) + D = (f') - (g) + D = (f') + D' \geq 0.$$

Hence ψ is an isomorphism. $\square$

Chapter 8

Invertible sheaves and Divisors

Let X be a ringed space. Given a cartier divisor D on X we define invertible sheaf $\mathcal{L}(D)$ associated with the divisor and show the 1-1 correspondence between the cartier divisors and the invertible sheaves on X.

Definition 8.1. *An invertible sheaf on X is a locally free $\mathcal{O}_X$-module of rank 1.*

We recall that tensor product of two $\mathcal{O}_X$-modules $\mathcal{L}$ and $\mathcal{M}$ be the sheaf associated to the presheaf $U \mapsto \mathcal{L}(U) \otimes_{\mathcal{O}_{X(U)}} \mathcal{M}(U)$.

Proposition 8.2. *If $\mathcal{L}$ and $\mathcal{M}$ are invertible sheaves on a ringed space X, then, $\mathcal{L} \otimes \mathcal{M}$ is also an invertible sheaf. Also, if $\mathcal{L}$ is any invertible sheaf on X, then there exists an invertible sheaf $\mathcal{L}^{-1}$ on X such that $\mathcal{L} \otimes \mathcal{L}^{-1} \cong \mathcal{O}_X$.*

Proof Since $\mathcal{L}$ and $\mathcal{M}$ are both locally free $\mathcal{O}_X$-module of rank 1 and $\mathcal{O}_X \otimes \mathcal{O}_X \cong \mathcal{O}_X$, we have $\mathcal{L} \otimes \mathcal{M}$ is an invertible sheaf.

Let $\mathcal{L}$ be any invertible sheaf, then $\mathcal{L} \otimes Hom(\mathcal{L}, \mathcal{O}_X) \cong Hom(\mathcal{L}, \mathcal{L}) = \mathcal{O}_X$. □

Definition 8.3 (Picard Group)**.** *The group of isomorphism classes of invertible sheaves under the $\otimes$ operation is defined to be the Picard group of X,* $\operatorname{Pic} X$.

Let D be a Cartier divisor on X represented by $\{U_i, f_i\}$. Let $\mathcal{L}(D)$ be subsheaf of sheaf of total quotient rings $\mathcal{K}$ such that $\mathcal{L}(D)$ is the sub-$\mathcal{O}_X$-module of $\mathcal{K}$ generated by f_i^{-1} on U_i. Then $\mathcal{L}(D)$ is the sheaf associated with the divisor D.

Proposition 8.4. *Let D be a cartier divisor and $\mathcal{L}(D)$ be the associated invertible sheaf. Then,* *(i) the map $D \longrightarrow \mathcal{L}(D)$ gives the 1-1 correspondence between Cartier divisors on X and invertible subsheaves of $\mathcal{K}$.*

(ii) $\mathcal{L}(D_1 - D_2) \cong \mathcal{L}(D_1) \otimes \mathcal{L}(D_2)^{-1}$.

(iii) $D_1 \sim D_2$ if and only if $\mathcal{L}(D_1) \cong \mathcal{L}(D_2)$.

Proof

(i) Since $f_i \in \Gamma(U_i, \mathcal{K}^*)$, the map $\mathcal{O}_{U_i} \mapsto \mathcal{L}(D)|_{U_i}$ given by $1 \mapsto f_i^{-1}$ is an isomorphism and hence, $\mathcal{L}(D)$ is an invertible sheaf. Given $\mathcal{L}(D)$ and its embedding in $\mathcal{K}$ one can find the divisor D by taking f_i to be inverse of local generator of $\mathcal{L}(D)$ on U_i.

(ii) Let D_1 and D_2 be locally defined by f_i and g_i respectively. The invertible sheaf $\mathcal{L}(D_1 - D_2)$ is locally generated by $f_i^{-1} g_i$. Hence $\mathcal{L}(D_1 - D_2) = \mathcal{L}(D_1).\mathcal{L}(D_2)^{-1} \cong \mathcal{L}(D_1) \otimes \mathcal{L}(D_2)^{-1}$.

(iii) Using (2), if $D_1 - D_2$ is principal divisor, then $\mathcal{L}(D_1) \cong \mathcal{L}(D_2)$. Hence it is sufficient to prove that $D = D_1 - D_2$ is principal iff $\mathcal{L}(D) \cong \mathcal{O}_X$. Let D be a principal divisor generated by $f \in \Gamma(X, \mathcal{K}^*)$. Then $\mathcal{L}(D)$ is globally generated by f^{-1} and the map $1 \mapsto f^{-1}$ gives the isomorphism $\mathcal{O}_X \cong \mathcal{L}(D)$. Conversely, if $\mathcal{O}_X \cong \mathcal{L}(D)$ inverse of image of 1 will define D as the principal divisor. $\square$

Let D be an effective cartier divisor on X represented by $\{(U_i, f_i)\}$. Then $\mathcal{L}(-D)$ is subsheaf of $\mathcal{O}_X$ locally generated by f_i.
So $\mathcal{L}(-D)$ is the ideal sheaf of associated locally principal closed subscheme. This gives an exact sequence

$$0 \longrightarrow \mathcal{L}(-D) \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_D \longrightarrow 0.$$

Tensoring the above exact sequence with $\mathcal{O}_C$ we get

$$0 \longrightarrow \mathcal{L}(-D) \otimes \mathcal{O}_C \longrightarrow \mathcal{O}_C \longrightarrow \mathcal{O}_{C \cap D} \longrightarrow 0.$$

Thus $\mathcal{L}(D) \otimes \mathcal{O}_C$ gives the invertible sheaf on C corresponding to the divisor $C \cap D$. Hence $\deg_C(\mathcal{L}(D) \otimes \mathcal{O}_C)$ gives the transversal intersection number $C \cdot D$.

8.1 Differentials

Let R be ring containing field k with quotient field K and M be an R-module.

Definition 8.5 (Derivation)**.** *A k-linear map $D : R \longrightarrow M$ is a derivation over k if $D(xy) = xD(y) + yD(x)$.*

Let R and K are as defined above and M is a vector space over K and D be any derivation $D : R \longrightarrow M$. Consider $z \in K$, we can write $z = x/y$ for $x, y \in R$. We can extend the derivation D to $\bar{D} : K \longrightarrow M$ as $D(x) = y\bar{D}(z) + zD(y)$ since, $x = yz$. We have $\bar{D}(z) = y^{-1}(D(x) - zD(y))$.

Let F be a free R-module with R as basis (basis elements denoted as $[x]$) and N generated by

(i) $\{[x + y] - [x] - [y] \mid x, y \in R\}$

(ii) $\{[\lambda x] - \lambda[x] \mid x \in R, \lambda \in k\}$

(iii) $\{[xy] - x[y] - y[x] \mid x, y \in R\}$.

be submodule of F.

Let $\Omega_k(R) = F/N$ be the quotient module. Let dx be the image of $[x]$ in F/N and let map $d : R \longrightarrow \Omega_k(R)$ take x to dx.
Following lemma will show that any derivation $D : R \longrightarrow M$ factors through d. $\Omega_k(R)$ is an R-module called module of differentials and d is a derivation.

Lemma 8.6. *For any R-module M, and any derivation $D : R \longrightarrow M$, there is a unique homomorphism of R-module $\phi : \Omega_k(R) \longrightarrow M$ such that $D(x) = \phi(dx)$ for all $x \in R$.*

Proof We define $\phi' : F \longrightarrow M$ as

$$\phi'(\Sigma x_i[y_i]) = \Sigma x_i D(y_i).$$

Clearly, $\phi'(N) = 0$. So, we have induced map

$$\phi : \Omega_k(R) \longrightarrow M$$

such that $D(x) = \phi(dx)$. □

$\Omega_k(R)$ is called module of differentials of R over k. More generally we define module of differentials for B, an A-algebra, where A is a commutative ring (with identity)as follows.

Definition 8.7 (Module of Relative Differentials)**.** *Module of relative differential forms of B over A is B-module $\Omega_{B/A}$ together with an A-derivation $d : B \longrightarrow \Omega_{B/A}$ and satisfying the following property : for any B module M, and for any A-derivation $D : B \longrightarrow M$ there exists a unique B-module homomorphism $\phi : \Omega_{B/A} \longrightarrow M$ such that $D = \phi \circ d$.*

Proposition 8.8. *Let B be an A-algebra. Let $f : B \otimes_A B \longrightarrow B$ be the diagonal homomorphism defined by $f(b \otimes b') = bb'$. Let I be the $ker(f)$. Then, I/I^2 inherits the structure of B-module. Let $d : B \longrightarrow I/I^2$ be a map given by $db = 1 \otimes b - b \otimes 1$. Then, $(I/I^2, d)$ is module of differentials for B over A.*

Proof *cf.* Matsumura [Ma] [9, Section 25] □

Theorem 8.9 (First Exact Sequence)**.** *Let $A \longrightarrow B \longrightarrow C$ be rings and their homomorphisms. Then there is an exact sequence of C-modules*

$$\Omega_{B/A} \otimes C \longrightarrow \Omega_{C/A} \longrightarrow \Omega_{C/B} \longrightarrow 0.$$

Proof *cf.* Matsumura [Ma] [9, Theorem 25.1] □

Let $f : X \longrightarrow Y$ be morphism of schemes. Let $\Delta : X \longrightarrow X \times_Y X$ be diagonal morphism i.e., Δ when composed with projection maps $p_1, p_2 : X \times_Y X \longrightarrow X$ gives the identity map $X \mapsto X$.

Definition 8.10. *Let $\mathcal{I}$ be the sheaf of ideals of $\Delta(X)$. The sheaf of relative differentials of X over Y is defined to be the sheaf $\Omega_{X/Y} = \Delta^*(\mathcal{I}/\mathcal{I}^2)$ on X.*

Now, we try to observe that the definition of sheaf of differentials is compatible with the definition of module of differential as defined above (at least in affine case). Let $U = SpecA$ be an open affine subset of Y and $V = SpecB$ be an open affine subset of X such that $f(V) \subseteq U$. Then $V \times_U V$ is an open affine subset of $X \times_Y X$ isomorphic to $Spec(B \otimes_A B)$. $\Delta(X) \cap (V \times_U V)$ is the closed subscheme defined by the kernel of the diagonal homomorphism $B \otimes_A B \to B$. Therefore $\mathcal{I}/\mathcal{I}^2$ is the sheaf associated to the module I/I^2. $\Omega_{V/U} \simeq (\Omega_{B/A})^{\sim}$.

Further observe that we can cover X and Y with open affine subsets and glue the corresponding sheaves $(\Omega_{B/A})^{\sim}$. The derivations $d : B \to \Omega_{B/A}$ glue together to give a map $d : \mathcal{O}_X \to \Omega_{X/Y}$ of sheaves of Abelian groups on X, which is a derivation of the local rings at each point.

Theorem 8.11. *Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be morphisms of schemes. Then there is an exact sequence of sheaves on X,*

$$f^*\Omega_{Y/Z} \to \Omega_{X/Z} \to \Omega_{X/Y} \to 0.$$

Proof *cf.* Hartshorne [Ha] [II, Proposition 8.11] □

Definition 8.12 (Canonical Sheaf)**.** *The canonical sheaf of X is defined as $\omega_X = \bigwedge^n \Omega_{X/k}$, the n-th exterior power of the sheaf of differentials, where n is dimension of X. It is an invertible sheaf on X.*

Theorem 8.13. *Let Y be a nonsingular variety of codimension 1 in a nonsingular variety X. Let $\mathcal{L}$ be the associated invertible sheaf on X. Then $\omega_Y \cong \omega_X \otimes \mathcal{L} \otimes \mathcal{O}_Y$.*

Proof *cf.* Hartshorne [Ha] [II,Proposition 8.20] □

Chapter 9

Cohomology

This chapter is mainly used as a tool to prove Riemann-Roch theorem. Many theorems are stated without proof which we will refer to in the later sections.

Definition 9.1. *A complex $A^{\cdot}$ in an Abelian category $\mathfrak{A}$ is a collection of objects A_i, where i is an integer and morphisms $d^i : A^i \to A^{i+1}$ such that the composition of any two consecutive morphisms is 0 (for all i).*

Definition 9.2. *The i-th cohomology object $h^i(A^{\cdot})$ of the complex $A^{\cdot}$ is $ker(d^i)/im(d^{i-1})$.*

Let X be a topological space and $\mathcal{F}$ be a sheaf of abelian groups on X. Let $\mathfrak{A} = \{\mathfrak{A}_i\}_{i\in I}$ be an open cover of X where the indexing set is well ordered.

A complex $C^{\cdot}(\mathfrak{A}, \mathcal{F})$ is defined as follows. For each $p \geq 0$, let

$$C^p(\mathfrak{A}, \mathcal{F}) = \prod_{i_0<\ldots<i_p} \mathcal{F}(U_{i_0,\ldots,i_p}).$$

Thus, an element $\alpha \in C^p(\mathfrak{A}, \mathcal{F})$ is determined by giving an element

$$\alpha_{i_0,\ldots,i_p} \in \mathcal{F}(U_{i_0,\ldots,i_p}),$$

for each $(p+1)$-tuple $i_0 < \ldots < i_p$ of elements of I.

The coboundary map $d : C^p \to C^{p+1}$ is given by

$$(d\alpha)_{i_0,\ldots,i_{p+1}} = \sum_{k=0}^{p+1} (-1)^k \alpha_{i_0,\ldots,\hat{i}_k,\ldots,i_{p+1}|U_{i_0,\ldots,i_{p+1}}}$$

where the notation $\hat{i}_k$ means omit i_k.

Remark 9.3 The composition map $d^2 = 0$. Therefore, we have a chain complex of abelian groups.

Definition 9.4. *The p-th Čech cohomology group of $\mathcal{F}$ with respect to the covering $\mathfrak{A}$, is defined as*

$$\check{H}^p(\mathfrak{A}, \mathcal{F}) = h^p(C^{\cdot}(\mathfrak{A}, \mathcal{F})).$$

In the following example we will compute the cohomology groups $\check{H}^n(\mathbb{P}^1, \mathcal{O})$.

Example 9.5 Let $\mathfrak{A}$ be the covering of $X = \mathbb{P}^1$ by two open sets $U = \mathbb{A}^1$ with affine coordinate x and $V = \mathbb{A}^1$ with affine coordinate $y = 1/x$. Then cochain groups are
$C^0 = \mathcal{O}(U) \times \mathcal{O}(V) = (f, g)$ such that $f \in k[x]$ and $g = k[y]$
$C^1 = \mathcal{O}(U \cap V) = k[x, 1/x]$ and $C^n = \varnothing$ for $n \geq 2$.

Let d be the coboundary map. Then $d(f, g) = f - g$. So $\ker d$ is set of points (a, a) where $a \in k$. Therefore $\check{H}^0(\mathbb{P}^1, \mathcal{O}) = k$.
Since im $d = k[x, 1/x]$, we have $\check{H}^1(\mathbb{P}^1, \mathcal{O}) = 0$.

Definition 9.6 (Flasque sheaf)**.** *Let X be a topological space. A sheaf $\mathcal{F}$ is flasque on X if for every inclusion $V \subseteq U$ of open sets we have $\mathcal{F}(U) \mapsto \mathcal{F}(V)$ is surjective.*

Example 9.7 (Skyscraper sheaf) Let X be a topological space and A be an abelian group. Let $P \in X$. We define skyscraper sheaf $i_P(A)$ as $i_P(A)(U) = A$ if $P \in U$, else 0.

Proposition 9.8. *If $\mathcal{F}$ is a flasque sheaf on a topological space X, then $H^i(X, \mathcal{F}) = 0$ for all $i \geq 1$.*

Proof *cf.* Hartshorne [Ha] [III, Proposition 2.5] □

Proposition 9.9. *Let $\mathfrak{A}$ be an open covering of X. Let $\mathcal{F}$ be a flasque sheaf of abelian groups on X. Then for all $p > 0$ we have $\check{H}^p(\mathfrak{A}, \mathcal{F}) = 0$.*

Proof *cf.* Hartshorne [Ha] [III, Proposition 4.3] □

Theorem 9.10 (Serre Duality)**.** *Let X be a smooth projective space of dimension n over k. Then for any locally free sheaf $\mathcal{F}$ on X we have*

$$H^i(X, \mathcal{F}) \cong H^{n-i}(X, \omega_X \otimes \mathcal{F}^{\sim})$$

for all $0 \leq i \leq n$.

Proof *cf.* Hartshorne [Ha] [III, Corollary 7.7] □

Chapter 10

Riemann-Roch Theorem

In this chapter we will prove the Riemann-Roch theorem for curves. This theorem helps compute the dimension of space of meromorphic functions having zeroes and poles of certain order at given points. This is given by $l(D)$, where D is the divisor with the support as points of zeros and poles and the coefficients as their respective orders. A curve or surface will mean nonsingular, projective curve or surface through the rest of thesis unless stated otherwise.

Definition 10.1 (Geometric Genus)**.** *For X both projective and nonsingular curve, we define geometric genus of X to be $p_g = \dim_k H^1(X, \mathcal{O}_X)$.*

In this section we will refer to geometric genus as genus and denote it by g.

Theorem 10.2 (Riemann-Roch Theorem)**.** *Let D be a divisor on curve X of genus g. Then,*

$$l(D) - l(K_X - D) = \deg D - g + 1$$

where $l(D) = \dim_k H^0(X, \mathcal{L}(D))$ and K_X is the canonical divisor.

Proof The divisor $K_X - D$ corresponds to $(\omega_X \otimes \mathcal{L}(D)\check{})$. Since by Serre Duality theorem $H^0(X, \omega_X \otimes \mathcal{L}(D)\check{})$ is dual to $H^1(X, \mathcal{L}(D)$, their dimensions are same. We prove the theorem by induction on divisor.
Taking $D = 0$ we have

$$\begin{aligned}\dim_k H^0(X, \mathcal{L}(D)) - \dim_k H^1(X, \mathcal{L}(D)) &= \dim_k H^0(X, \mathcal{O}_X) - \dim_k H^1(X, \mathcal{O}_X)\\ &= 1 - g\\ &= 0 + 1 - g\end{aligned}$$

$$= \deg D + 1 - g$$

Now, let D be any non-zero divisor and P be any point on curve X. If the theorem is true for divisor D, we prove that it is true for divisor $D + P$.

$$0 \longrightarrow \mathcal{L}(D) \longrightarrow \mathcal{L}(D+P) \longrightarrow k(P) \longrightarrow 0$$

is an exact sequence where $k(P)$ is the skyscraper sheaf at P. (In 9.7 we have observed that skyscraper sheaf is a flasque sheaf and hence from 9.8 we have $H^1(X, k(P)) = 0$). This induces a long exact sequence

$$0 \to H^0(X, \mathcal{L}(D)) \to H^0(X, \mathcal{L}(D+P)) \to k \to H^1(X, \mathcal{L}(D)) \longrightarrow H^1(X, \mathcal{L}(D+P)) \to 0.$$

Hence we have

$$\dim H^0(X, \mathcal{L}(D)) - \dim H^0(X, \mathcal{L}(D+P)) + 1 - \dim H^1(X, \mathcal{L}(D)) + \dim H^1(X, \mathcal{L}(D+P)) = 0$$

$$\begin{aligned} &\dim H^0(X, \mathcal{L}(D)) - \dim H^1(X, \mathcal{L}(D)) + 1 \\ &\qquad = \dim H^0(X, \mathcal{L}(D+P) - \dim H^1(X, \mathcal{L}(D+P)) \\ &\deg D + 1 - g + 1 = \dim H^0(X, \mathcal{L}(D+P) - \dim H^1(X, \mathcal{L}(D+P)) \end{aligned}$$

Since $\deg(D+P) = \deg D + 1$, we have the desired result for divisor $D+P$ as

$$\dim H^0(X, \mathcal{L}(D+P)) - \dim H^1(X, \mathcal{L}(D+P)) = \deg(D+P) + 1 - g.$$

□

Following are few corollaries of the Riemann-Roch theorem.

Corollary 10.3. $l(div(\omega_X)) = g$.

Proof Let $D = 0$, we have

$$l(0) - l(K_X) = 0 - g + 1$$

which gives $l(K_X) = g$. □

Corollary 10.4. *On a curve X of genus g, canonical divisor K_X has degree 2g-2.*

Proof Let $D = K_X$, we get

$$l(K_X) - l(0) = \deg K_X + 1 - g.$$

Since $l(0) = 1$ and $l(K_X) = g$, we have the desired result. $\square$

Corollary 10.5. *If D is a divisor such that $\deg D > 2g-2$ then $l(D) = \deg D - g + 1$.*

Proof Let D be any divisor. If $l(D) \neq 0$ then $|D|$ is nonempty. Therefore D is linearly equivalent to some effective divisor, say D'. Since $\deg D = \deg D'$, we have $\deg D \geq 0$.

Now, if $\deg D > 2g - 2$, we have $\deg(K_X - D) < 0$. Therefore, $l(K_X - D) = 0$. Hence $l(D) = \deg D - g + 1$. $\square$

Corollary 10.6. *Let $D' = K_X - D$. Then $l(D) - \frac{1}{2}\deg(D) = l(D') - \frac{1}{2}\deg(D')$.*

This is a duality result without reference to the genus of the curve.

Proof By Riemann-Roch theorem we have

$$l(D) - l(K_X - D) = \deg D - g + 1.$$

Putting $D = K_X - D$ we get

$$l(K_X - D) - l(D) = \deg K_X - D - g + 1.$$

Taking difference of the two equations we get

$$l(D) - \frac{1}{2}\deg(D) = l(K_X - D) - \frac{1}{2}\deg(K_X - D).$$

$\square$

Example 10.7 Let X be a curve and $P \in X$ be a point. We can use Riemann-Roch theorem to see the existence of a function $f \in K(X)$ which is regular everywhere except at finitely many points. It would be enough to observe this for a single point. Consider divisor nP where $n \gg 0$ we have $l(K_X - nP) = 0$ and $l(nP) = n + 1 - g > 0$ for $n \gg 0$.

Corollary 10.8 (Riemann Inequality)**.** *For divisor D, $l(D) \geq \deg D + 1 - g$*

Proof $l(K_X - D) \geq 0$. $\square$

Every divisor D on curve X defines a map $\psi_D : X \longrightarrow \mathbb{P}^n$ given by $(f_0 : f_1 : \ldots : f_n)$ where $n = l(D) - 1$ and f_i generates vector space $L(D)$.

Corollary 10.9. *The map ψ_D is an embedding if* $\deg D \geq 2g + 1$.

Proof Let D be a divisor of $\deg D \geq 2g + 1$ and P and Q be points on X. So we have $l(K_X - D) = l(K_X - D + P + Q) = 0$. Now by Riemann-Roch theorem $l(D - P - Q) = g$ and $l(D) = g + 2$. Hence we have $\dim |D - P - Q| = \dim |D| - 2$. This implies that there exists a divisor $D' \sim D - P$ such that $Q \notin \operatorname{Supp} D'$. Therefore $\psi : X \longrightarrow \mathbb{P}^n$ is injective.

Since X is smooth $\dim T_P(X) = 1$. Let $D_1 \sim D$. If P has multiplicity 1 in D_1 then $\dim T_P(D_1) = 0$. Now, since $\dim |D - 2P| = \dim |D| - 2$, there exists a divisor D_2 equivalent to $(D - P)$ such that $P \notin \operatorname{Supp} D_2$. Let $D_1 \sim D_2 + P$. Then $\psi : X \longrightarrow \mathbb{P}^n$ maps the tangent spaces isomorphically. $\square$

Proposition 10.10. *A smooth proper algebraic curve X is isomorphic to $\mathbb{P}^1$ if and only if genus g of X is 0.*

Proof We have seen in 9.5 that $\check{H}^1(\mathbb{P}^1, \mathcal{O}) = 0$. So genus of $\mathbb{P}^1$ is 0. Now let X be a smooth proper algebraic curve with genus $g = 0$. Let P and Q be two distinct points on X. Consider the divisor $D = P - Q$. Since $\deg(K_X - D) = 2g - 2 - 0 = -2$, we have $l(K_X - D) = 0$. Hence $l(D) = \deg D + 1 - g = 1$. Since $\deg D = 0$, we get $D \sim 0$ which implies that $P \sim Q$. which implies we have a rational function $f \in K(X)$ such that $(f) = P - Q$. Consider the morphism $\psi : X \longrightarrow \mathbb{P}^1$ determined by f such that $(f) = \psi^*(\{0\} - \{\infty\})$. Now, since $\psi^*(0) = P$, ψ is a morphism of degree 1. Since X is a nonsingular proper curve, X is isomorphic to $\mathbb{P}^1$. $\square$

Theorem 10.11 (Luroth's Theorem)**.** *Any subfield $F \neq k$ of $k(x)$ is also of pure transcendence degree 1 .*

Proof Note that char $(k) = 0$, hence all field extensions are separable. Since F is subfield of $k(x)$, transcendence degree of F is either 0 or 1. Since k is algebraically closed and $F \neq k$, transcendence degree of F is 1. Now, since $F \subset k(x)$ and both are transcendence degree 1, $k(x)$ is algebraic over F.
Let F be function field of some curve X. Since $k(x)$ is function field of $\mathbb{P}^1$ we have a

morphism $f : \mathbb{P}^1 \longrightarrow X$. From 10.10 we have X is isomorphic to $\mathbb{P}^1$ and hence it has pure transcendence degree 1. $\square$

Corollary 10.12. *Any map from $\mathbb{P}^1$ to a curve C of genus ≥ 1 is constant.*

Example 10.13 (Elliptic Curves) We observe that for curves of genus one, also called elliptic curves, $l(K_X) = 1$ and $\deg K_X = 0$. Hence for elliptic curves $K_X \sim 0$.

Let E be a smooth projective curve of genus $g = 1$. Let P be any closed point on E. Let $D = 3P$ be a divisor. By Riemann-Roch theorem we have $l(D) = 3$. Since $\deg D = 2g + 1$, the map defined by the divisor $\psi_D : E \longrightarrow \mathbb{P}^2$ gives an embedding of degree 3.

Example 10.14 (Group Structure on Elliptic Curves) Let E be an elliptic curve and P_0 be a point on E. Let $\operatorname{Pic}^0 E$ denote the subgroup of $\operatorname{Pic} X$ given by divisors of degree 0. We will show that the map $P \mapsto \mathcal{L}(P - P_0)$ is a one to one correspondence. To show this it is enough to prove that for a given divisor D of degree 0 there exists a unique point $P \in E$ such that $D \sim P - P_0$.
By Riemann-Roch theorem we have

$$l(D + P_0) - l(K_X - D - P_0) = \deg(D + P_0) + 1 - g.$$

Since $\deg K_X = 0$ and $\deg(D+P_0) = 1$, $l(K_X - D - P_0) = 0$ and hence $l(D+P_0) = 1$. and consequently $\dim(D + P_0) = 0$. Therefore there is a unique effective divisor that is linearly equivalent to $D + P_0$. Since $\deg(D + P_0) = 1$, $D + P_0 \sim P$ which is unique.

Theorem 10.15 (Adjunction Formula)**.** *If C is a nonsingular curve of genus g on a nonsingular projective surface S, and if K_S is the canonical divisor on S, then $2g - 2 = C.(C + K)$.*

Proof From 8.13 we have $\omega_C \cong \omega_S \otimes \mathcal{L}(C) \otimes \mathcal{O}_C$. The degree of $div(\omega_C)$ is $2g - 2$ by Riemann-Roch theorem. Also,

$$\deg_C(\omega_S \otimes \mathcal{L}(C) \otimes \mathcal{O}_C) = (C + K_S).C$$

$\square$

Corollary 10.16. $g(\tilde{C}) = g(C) - \frac{1}{2}r(r-1)$

Proof From 10.15 we have $2g - 2 = C.(C+K)$, where K_X is the canonical divisor. Therefore,
$$\begin{aligned} 2g(\tilde{C}) - 2 &= \tilde{C}.(\tilde{C} + K_{\tilde{X}}) \\ &= (\pi^*(C) - rE)(\pi^*(C) - rE + \pi^*(K_X) + E) \\ &= 2g(C) - 2 - r(r-1). \end{aligned}$$
$\square$

The following theorem gives the genus of smooth plane curves.

Theorem 10.17 (Plucker's Formula)**.** *The genus of a nonsingular plane curve of degree d is given by*
$$g = \frac{(d-1)(d-2)}{2}$$

Proof Let $F(x_0 : x_1 : x_2) = 0$ be the defining equation of the curve X such that $x_0 = 0$ intersect X at d distinct points. On an open set where $x_0 \neq 0$ let $x = x_1/x_0$, $y = x_2/x_0$ and $f(x,y) = \frac{1}{x_0^d}F(x_0 : x_1 : x_2)$. Since the curve is nonsingular, we have $f_x(a,b) \neq 0$ or $f_y(a,b) \neq 0$ for point $(1 : a : b)$ on the curve where f_x and f_y are partial derivatives of f with respect to x and y respectively. Also, $f_x(a,b)dx + f_y(a,b)dy = 0$
$$\frac{dx}{f_y(a,b)} = -\frac{dy}{f_x(a,b)}$$
.

Since, $f_x(a,b) \neq 0$ or $f_y(a,b) \neq 0$, above differential form gives canonical divisor K_X regular on $\{x_0 \neq 0\} \cap X$. Near the points of intersections of X with x_0 we can consider the affine neighbourhood of the point by letting $u = x_0/x_1$, $v = x_2/x_1$ and $g(u,v) = \frac{1}{x_1^d}F(x_0 : x_1 : x_2)$ if the point doesn't lie on $x_1 = 0$.

$u = 1/x$ and $v = y/x$
$$f(x,y) = \frac{1}{x_0^d}F(x_0 : x_1 : x_2) = \frac{x_1^d}{x_0^d}\frac{1}{x_1^d}F(x_0 : x_1 : x_2)$$
$$f(x,y) = \frac{1}{u^d}g(u,v)$$
$$f_y(x,y) = \frac{1}{u^d}\frac{\partial g(u,v)}{\partial v}\frac{\partial v}{\partial y} = \frac{1}{u^d}g_v(u,v)\frac{1}{x} = \frac{1}{u^{d-1}}g_v(u,v)$$
$$\omega = \frac{dx}{f_y(x,y)} = \frac{d(\frac{1}{u})}{\frac{1}{u^{d-1}}g_v(u,v)} = -\frac{u^{d-3}du}{g_v(u,v)} = \frac{u^{d-3}dv}{g_u(u,v)}$$

Clearly K_X has zero of order $(d-3)$ at every point of intersection of X with $x_0 = 0$ which are of the form $(0 : 1 : c)$. Similarly, we can prove this to be true for points of the form $(0 : e : 1)$. So we have

$$K_X = (d-3)\Sigma_i P_i$$

where P_i are the points of intersection of X and $x_0 = 0$.

$$\deg K_X = d(d-3)$$

Since $\deg K_X = 2g - 2$ we get

$$2g - 2 = d(d-3)$$

and therefore

$$g = \frac{(d-1)(d-2)}{2}$$

. □

Example 10.18 The curve $y^2z - x^3 + xz^2 = 0$ is a nonsingular planar curve of degree $d = 3$. From the above proposition we get genus of the genus of the curve is $g = (3-1)(3-2)/2 = 1$.

Example 10.19 (Conics) Every conic in $\mathbb{P}^2$ is isomorphic to $\mathbb{P}^1$ if it is smooth, hence has genus zero, or else is either a union of two lines intersecting transversally or it is a double line. This is easily seen by writing the homogeneous degree two equation for a conic in the projective plane.

Example 10.20 (Fermat curves) The Fermat curves defined by the affine equation $x^n + y^n = 1$, $n \geq 2$, are non-singular curves: the derivatives are $nx^{n-1} = 0$ and $ny^{n-1} = 0$ which have no common solution with the curve. Homogenising, we get $x^n + y^n = z^n$ in the projective plane where it is again smooth at the points at infinity by the same argument: in the yz plane the equation is $1 + y^n = z^n$ etc. For $n = 1, 2$ the Fermat curve is isomorphic to $\mathbb{P}^1$ and for $n \geq 3$ the genus of Fermat curve is $\frac{(n-1)(n-2)}{2}$.

Example 10.21 (Not all curves are planar) It is clear from the degree-genus formula that not all positive integers appear as the genus of a non-singular plane curve since not all positive integers can be written in the form $\frac{(d-1)(d-2)}{2}$. In fact countably many integers are not of this form. As we show later that all curves are embeddable in space, this implies that countably many different genus curves are space curves.

Chapter 11

Riemann-Hurwitz

Definition 11.1. *The degree of a finite morphism $f : X \longrightarrow Y$ is given by the degree of extension of their function fields i.e.* $[K(X) : K(Y)]$.

Definition 11.2 (Ramification index)**.** *Let P be any point of X and $Q = f(P)$. If $t \in \mathcal{O}_X$ is a local parameter at Q, then we define ramification index $e_P = v_P(f^\sharp(t))$ where v_P is the valuaton of $\mathcal{O}_P$ and $f^\sharp$ is the natural map from $\mathcal{O}_Q \mapsto \mathcal{O}_P$.*

Definition 11.3. *If $e_P > 1$ we say that f is ramified at P and Q is the branch point. If $e_P = 1$ then f is said to be unramified at P.*

Proposition 11.4. *Let $f : X \longrightarrow Y$ be a finite morphism of curves. Then, following is an exact sequence of sheaves on X*

$$0 \longrightarrow f^*\Omega_Y \longrightarrow \Omega_X \longrightarrow \Omega_{X/Y} \longrightarrow 0.$$

Proof From 8.11 we have

$$f^*\Omega_Y \longrightarrow \Omega_X \longrightarrow \Omega_{X/Y} \longrightarrow 0.$$

To prove the proposition we only need to prove $f^*\Omega_Y \longrightarrow \Omega_X$ is injective. Since f is separable $\Omega_{X/Y} = 0$ on generic point and hence,$f^*\Omega_Y \longrightarrow \Omega_X \longrightarrow 0$ is exact on generic point. Therefore $f^*\Omega_Y \longrightarrow \Omega_X$ is surjective. Since $f^*\Omega_Y$ and Ω_X are both invertible sheaves on X the map is injective. $\square$

Definition 11.5 (Ramification Divisor)**.** *Let $f : X \longrightarrow Y$ be a finite separable morphism of curves. Ramification divisor of f is defined to be*

$$R = \Sigma_{P \in X} length(\Omega_{X/Y})_P P.$$

Theorem 11.6 (Hurwitz)**.** *Let $f : X \longrightarrow Y$ be a finite separable morphism of curves and n be the degree of f. Then,*

$$2g(X) - 2 = n(2g(Y) - 2) + degR$$

where R is the ramification divisor.

Proof Let $P \in X$ and $f(P) = Q$. Let t be local parameter at Q and dt a generator of $\Omega_{Y,Q}$. $(\Omega_{X/Y})_P = 0$ iff f^*dt is generator for $\Omega_{X,P}$ or in other words t is local parameter $\mathcal{O}_P$, i.e., f is unramified at P. Therefore, $\mathcal{O}_R \cong \Omega_{X/Y}$. On tensoring the exact sequence in previous proposition we get

$$0 \longrightarrow f^*\Omega_Y \otimes \Omega_X^{-1} \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_R \longrightarrow 0$$

Ideal sheaf of R is isomorphic to $\mathcal{L}(-R)$. Hence

$$f^*\Omega_Y \otimes \Omega_X^{-1} \cong \mathcal{L}(-R).$$

The corresponding divisors give

$$f^*K_Y - K_X = -R$$

$$2g(X) - 2 = n.(2g(Y) - 2) + degR.$$

□

Corollary 11.7. $\mathbb{P}^1$ *has no everywhere unramified cover of degree $n > 1$.*

Proof Let $f : X \longrightarrow \mathbb{P}^1$ be unramified map of degree n. Then

$$2g(X) - 2 = (2g(\mathbb{P}^1) - 2)n + \deg R$$

Since $\deg R = 0$ and $g(\mathbb{P}^1) = 0$, we get $2g(X) - 2 = -2n$ which is possible iff $g(X) = 0$ and $n = 1$. Since $g(X) = 0$, $X \cong \mathbb{P}^1$. Therefore the only unramified map to $\mathbb{P}^1$ is the identity map. $\square$

Corollary 11.8. *If $f : X \longrightarrow Y$ be a finite morphism of curves then $g(X) \geq g(Y)$.*

Proof If $g(Y) = 0$ there is nothing to prove. We can rewrite Riemann-Hurwitz formula as

$$g(X) = g(Y) + (n-1)(g(Y) - 1) + \frac{1}{2}\deg R$$

. Since $n - 1 \geq 0$, $g(Y) - 1 \geq 0$ and $\deg R \geq 0$, we have $g(X) \geq g(Y)$. $\square$

Example 11.9 (Hyperelliptic curves) The plane curve C given by the affine equation $y^2 = f(x)$ where f is a reduced polynomial, is called a hyperelliptic curve. Let deg(f) = n > 4, then one checks that the point at infinity is smooth if n is odd and singular if n is even. In the even case let $n = 2g + 2$ and in the odd case $n = 2g + 1$. The projection of the curve on the x-axis gives a $2 : 1$ rational and ramified cover of $\mathbb{P}^1$ which extends to a $2 : 1$ morphism to the smooth model of the curve. In the odd n case there are $n + 1$ points of ramification since the point at infinity is smooth and ramified, while in the even case there is a simple node at infinity which resolves to give two points in the normalisation of the curve hence both these points are unramified. We calculate by Riemann-Hurwitz in the odd case: $2g_C - 2 = -4 + n + 1$ which implies $n = 2g_C + 1$ or in other words $g_C = g = (n-1)/2$. The case of even n is similar and gives $g_C = g$.

Chapter 12

Embedding of Curves

Let X be a curve in $\mathbb{P}^n$. This chapter aims at showing that any curve can be embedded in $\mathbb{P}^3$.

Definition 12.1 (Secant Line)**.** *A line in $\mathbb{P}^n$ joining two distinct points of X is said to be a secant line.*

Definition 12.2 (Secant Variety)**.** *The union of all secant lines of X is secant variety of X. We will denote it by* $\operatorname{Sec} X$.

Definition 12.3 (Tangent Variety)**.** *The tangent variety of X is defined to be the union of all tangent lines of X.*

Let $\varphi : X \longrightarrow \mathbb{P}^n$ be a morphism. We state without proof the local criteria for φ to be a closed immersion.

Theorem 12.4. *Let $\varphi : X \longrightarrow \mathbb{P}^n$ be a morphism corresponding to invertible sheaf $\mathcal{L}$ and $s_0, s_1, \cdots, s_n \in \Gamma(X, \mathcal{L})$. Let $V \in \Gamma(X, \mathcal{L})$ be the subspace. Then φ is a closed immersion iff*

(i) elements of V separates points, i.e., for any two distinct points P, Q on X we have $s \in V$ such that $s \in \mathtt{m}_P\mathcal{L}_P$ but $s \notin \mathtt{m}_Q\mathcal{L}_Q$

(ii) elements of V separates tangents, i.e., for every $P \in X$ set $\{s \in V | s_P \in \mathtt{m}_P\mathcal{L}_P\}$ spans the vector space $\mathtt{m}_P\mathcal{L}_P/\mathtt{m}_P^2\mathcal{L}_P$

Proof *cf.* Hartshorne [Ha] [II, Proposition 7.3] □

We have seen there is a 1-1 correspondence between invertible sheaves and corresponding linear system of divisors. Hence the following remark.

Remark 12.5 Let $\varphi : X \longrightarrow \mathbb{P}^n$ be a morphism corresponding to the linear system $L(D)$. Then φ is a closed immersion if and only if

(i) $L(D)$ separates points, i.e., for distinct closed points $P, Q \in X$, there exists $D \in L(D)$ such that $P \in Supp(D)$ and $Q \notin Supp(D)$

(ii) $L(D)$ separates tangents, i.e., for $P \in X$ and $t \in T_P(X) = \mathtt{m}_P/\mathtt{m}_P^2$ we have $D \in L(D)$ such that $P \in Supp(D)$ but $t \notin T_P(D)$.

Proposition 12.6. *Let X be a curve in $\mathbb{P}^n$ and let O be a point not on X. Let $\varphi : X \longrightarrow \mathbb{P}^{n-1}$ be the morphism determined by projection from O. Then φ is a closed immersion if and only if*

(i) O is not on any secant line of X, and

(ii) O is not on any tangent line of X.

Proof The morphism φ corresponds to the linear system given by the hyperplanes H of $\mathbb{P}^n$ passing through O. From the above remark φ is a closed immersion if and only if this linear system separates points and separates tangent vectors on X. Let P and Q be two distinct points on X, then φ separates them if and only if there is a hyperplane H containing O and P but not Q. This is possible if and only if O is not on the line PQ. Also, φ separates tangent vectors at P if and only if there is a hyperplane H containing O and P, and meeting X at P with multiplicity 1. This is possible if and only if O is not on the tangent line at P. □

Theorem 12.7. *Any curve can be embedded in $\mathbb{P}^3$.*

Proof The variety $\operatorname{Sec} X$ is the image of $X \times X \setminus \Delta \times \mathbb{P}^1 \mapsto \mathbb{P}^n$. The image is a locally closed subset of dimension $\leq 2\dim X + 1 = 3$. Similarly, $\operatorname{Tan} X$ is locally an image of $X \times \mathbb{P}^1 \mapsto \mathbb{P}^n$ of dim ≤ 2. For $n \geq 4$ we have $\operatorname{Sec} X \cup \operatorname{Tan} X \neq \mathbb{P}^n$. So there exists a point O which does not lie either on a secant line or a tangent line of X and hence the image of the curve under the projection from O determines the embedding of X in the hyperplane $\mathbb{P}^{n-1}$ of $\mathbb{P}^n$. By induction on n we can embed X in $\mathbb{P}^3$. □

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