Decomposition of Complex Hyperbolic Isometries by Involutions

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Certificate of Examination

This is to certify that the dissertation titled **"Decomposition of Complex Hyperbolic Isometries by Involutions"** submitted by **Cigole Thomas** (Reg. No. MS10029) for the partial fulfillment of BS-MS dual degree programme of the Institute, has been examined by the thesis committee duly appointed by the Institute. The committee finds the work done by the candidate satisfactory and recommends that the report be accepted.

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Dated: April 24, 2015

Declaration

The work presented in this dissertation has been carried out by me under the guidance of Dr. Krishnendu Gongopadhyay at the Indian Institute of Science Education and Research Mohali.
This work has not been submitted in part or in full for a degree, a diploma, or a fellowship to any other university or institute. Whenever contributions of others are involved, every effort is made to indicate this clearly, with due acknowledgement of collaborative research and discussions. This thesis is a bona fide record of original work done by me and all sources listed within have been detailed in the bibliography.

Cigole Thomas
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In my capacity as the supervisor of the candidate's project work, I certify that the above statements by the candidate are true to the best of my knowledge.

Dr. Krishnendu Gongopadhyay
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Contents

Abstract

In a recent work, Basmajian and Maskit have investigated the problem of finding involution and commutator lengths of the isometry group of real space forms. In this thesis we aim to investigate the problem for isometry group of the complex hyperbolic space. A k-reflection of the n-dimensional complex hyperbolic space $\mathrm{H}^n_{\mathbb{C}}$ is an element in $\mathrm{U}(n,1)$ with negative type eigenvalue λ, $|\lambda| = 1$, of multiplicity $k+1$ and positive type eigenvalue 1 of multiplicity $n-k$. We prove that every element in $\mathrm{SU}(n)$ is a product of atmost five involutions using which it can be shown that a holomorphic isometry of $\mathrm{H}^n_{\mathbb{C}}$ is a product of at most four involutions and a complex k-reflection, $k \leq 2$. We also give a short proof of the well-known result that every holomorphic isometry of $\mathrm{H}^n_{\mathbb{C}}$ is a product of two anti-holomorphic involutions.

Chapter 1

An Introduction to Complex Hyperbolic Geometry

1.1 Complex Hyperbolic Space

1.1.1 Hermitian Form

A *Hermitian form* on a complex vector space V is a map $\langle\ ,\ \rangle : V \times V \to \mathbb{C}$ such that

$$\begin{aligned} \langle v, w \rangle &= \overline{\langle w, v \rangle} \\ \langle \lambda_1 v_1 + \lambda_2 v_2, w \rangle &= \lambda_1 \langle v_1, w \rangle + \lambda_2 \langle v_2, w \rangle \end{aligned}$$

where $v_1, v_2, v, w \in V$ and $\lambda_1, \lambda_2 \in \mathbb{C}$

A vector space with a Hermitian form $(V, \langle\ ,\ \rangle)$ is called a *Hermitian space*.
A Hermitian form is called *regular* if $\langle v, w \rangle = 0$ for all $w \in V$ implies $v = 0$.
A Hermitian form which is not regular is called *degenerate.*
A subspace F of $V$is called *regular(or degenerate)* if the Hermitian form restricted to F is regular(or degenerate).

Example 1.1.1. The standard Hermition form on $\mathbb{C}^n$ is given by

$$\langle x, y \rangle = \sum_{i=1}^{n} x_i y_i \quad \text{where } x = (x_1, ..., x_n) \text{ and } y = (y_1, ..., y_n)$$

1.1.2 Classification of vectors and vector subspaces

Since $\langle v, v\rangle = \overline{\langle v, v\rangle}$, it follows that $\langle v, v\rangle$ is real. This provides us a way to classify the vectors. A non-zero vector, $v \in V$ is called *positive vector (or space-like)* if $\langle v, v\rangle > 0$. Similarly v is called *negative vector (or time-like) or null (or light-like) vector* if $\langle v, v\rangle < 0$ or $\langle v, v\rangle = 0$ respectively.
Since $\langle \lambda v, \lambda v\rangle = |\lambda|^2$, any vector λv (where $\lambda \neq 0$) is positive, negative or null iff v is.
It is possible to classify the subspaces in a similar fashion.

1. A subspace F of V is called *positive definite* if every non-zero vector of F is positive.

2. A subspace F is called an *indefinite space* if it contains both positive and negative vectors.

3. A subspace F is called *degenerate* if F is neither positive definite or indefinite.

A subspace $F \subseteq V$ is called *hyperbolic (or indefinite)* if the hermition form, $\langle , \rangle$ restricted to F is non-degenerate and indefinite; it is *elliptic (or space-like)* if $\langle , \rangle|_F$ is positive definite; and it is *parabolic (or light-like)* if $\langle , \rangle|_F$is degenerate.
The *radical* of a subspace F, $Rad(F)$ is defined as

$$Rad(F) = \{v \in F \mid \langle v, w\rangle = 0 \text{ for all } w \in V\}$$

It follows from the definition that a space is regular if and only if $Rad(F) = \mathbf{0}$.

1.1.3 Orthogonal Vectors

Two vectors $v, w \in V$ are called *orthogonal* to each other (denoted by $v \perp w$) if $\langle v, w\rangle = 0$.
A subset B is said to be *orthogonal* to a vector v if $\langle v, v'\rangle = 0$ for all $v' \in B$.
An orthogonal set is called *orthonormal* if $\langle v, v\rangle = 0, 1$ or -1.

The set of all vectors orthogonal to a subset S of V is denoted by $S^\perp$ i.e.

$$S^\perp = \{v \in V \mid \langle v, s\rangle = 0 \text{ for all } s \in S\}$$

1.1.4 Signature of a Hermitian form

Lemma 1.1.2. *If V is a non-trivial vector space equipped with a Hermitian form $\langle\ ,\ \rangle$, then every orthogonal basis of V contains same number of null vectors. Also the number of positive vectors (or negative vectors) in the basis remains invariant as the orthogonal basis varies.*

Proof We will show that the number of null vectors and positive vectors in the basis remains the same for every orthogonal basis. Then it follows that the number of negative vectors are independent of the choice of basis as well.

Step 1: Firstly,we will show that number of null vectors in any orthogonal basis of V is same as the dimension of radical of V. Let V be a non-trivial k-vector space and $\{v_1, ..., v_m\}$ be an orthogonal basis of V. Arrange the basis such that $\langle v_i, v_i\rangle = 0$ for $i \leq s$.
Since any null vector in an orthogonal basis is in Rad(V), $\{v_1, ..., v_s\} \in Rad(V)$. Now let $v \in Rad(V)$. Then

$$v = \lambda_1 v_1 + ... + \lambda_m v_m$$

On taking Hermitian product with each v_i, we obtain

$$0 = \langle v, v_j\rangle = \lambda_j \langle v_j, v_j\rangle \text{ for } j = 1, ..., m$$

Since $\langle v_j, v_j\rangle \neq 0$, we have $\lambda_j = 0$ for $j > s$.
This implies that $Rad(V) \subseteq \text{Span}\{v_1, ..., v_s\}$. Therefore $\{v_1, ...v_s\}$ forms a basis for $Rad(V)$.

Step 2: Now we shall show that the number of positive vectors remains same even as the basis varies*(Sylvester's Theorem)*.
Let $B_1 = \{v_1, ..., v_m\}$ and $\{w_1, ..., w_m\}$ be two orthogonal basis of V arranged such that first r vectors of B_1 are positive vectors and first s vectors of B_2 are positive vectors. Then, we have

$$\langle v_i, v_i\rangle > 0 \text{ for } i \leq r \quad \text{and} \quad \langle v_j, v_j\rangle \leq 0 \text{ for } i \geq r + 1$$

$$\langle w_i, w_i\rangle > 0 \text{ for } i \leq s \quad \text{and} \quad \langle w_j, w_j\rangle \leq 0 \text{ for } i \geq s + 1$$

We will show that the set $\{v_1, ..., v_r, w_{s+1}, ...w_m\}$ is linearly independent.Consider the relation

$$k_1v_1 + k_2v_2 + ... + l_{s+1}w_{s+1} + ... + l_mw_m = 0$$

Then

$$k_1v_1 + k_2v_2 + ... + k_rv_r = -(l_{s+1}w_{s+1} + ... + l_mw_m) \tag{1.1.1}$$

On taking Hermitian product of the above equation with itself, we get

$$|k_1|^2\langle v_1, v_1\rangle + ... + |k_r|^2\langle v_r, v_r\rangle = |l_{s+1}|^2\langle w_{s+1}, w_{s+1}\rangle + ... + |l_m|^2\langle w_m, w_m\rangle$$

In the above equation the left hand side is greater than or equal to zero and the right hand side is less than or equal to zero. So the equality holds only if both sides equal zero. Therefore,

$$|k_1|^2\langle v_1, v_1\rangle + ... + |k_r|^2\langle v_r, v_s\rangle = 0 \text{ which implies that } k_1 = k_2 = ... = k_r = 0$$

From the linear independency of $\{w_{s+1}, ..., w_m\}$,we get $l_{s+1} = ... = l_m = 0$. Since dim $V = m$, $r + (m - s) \leq k$ and hence $r \leq s$. By considering the set $\{w_1, ...w_s, v_{r+1}, ...v_m\}$ and using a similar argument, we can deduce that $r \geq s$ by which it follows that $r = s$.
Since any orthogonal basis consists of null,positive or negative vectors only, it follows that the number of negative vectors also remains independent of the choice of basis. □

The number of null vectors in an orthogonal basis of a vector space V or equivalently the dimension of $Rad(V)$ is called the *index of nullity.*
Similarly, the number of positive vectors (or negative vectors) in an orthogonal basis of V is called *index of positivity (or index of negativity).*

Definition 1.1.3. The Hermitian form on a vector space V is said to have a *signature* (p, q, r) where p is the index of positivity, q is the index of negativity and r is the index of nullity.

If the index of nullity is zero, the signature can be simply denoted by (p, q) instead of (p, q, r).

1.1.5 Complex Hyperbolic Space

Let $\mathbb{V} = \mathbb{C}^{n,1}$ be a complex vector space equipped with a Hermitian form of signature $(n, 1)$ and $\mathbb{P} : \mathbb{C}^{n,1} \to \mathbb{C}P^n$ be the canonical projection to a complex projective space.Then $\mathbb{P}(\mathbb{V})$ is the projective obtained from $\mathbb{V}$ i.e., $\mathbb{P}(\mathbb{V}) = \mathbb{V} - \{0\} \backslash \sim$, where $u \sim v$ if there exists λ such that $u = \lambda v$ and $\mathbb{P}(\mathbb{V})$ is equipped with quotient topology. Now consider the following subspaces of $\mathbb{C}^{n,1}$.

$$\mathbb{V}_- = \{v \in \mathbb{C}^{n,1} \mid \langle v, v\rangle < 0\}$$

$$\mathbb{V}_0 = \{v \in \mathbb{C}^{n,1} \mid \langle v, v\rangle = 0\}$$

$$\mathbb{V}_+ = \{v \in \mathbb{C}^{n,1} \mid \langle v, v\rangle > 0\}$$

Definition 1.1.4. The n-dimensional complex hyperbolic space $\mathbf{H}^{\mathbf{n}}_{\mathbb{C}}$ is defined as $\mathbb{P}(\mathbb{V}_-)$ and the ideal boundary $\partial\mathbf{H}^{\mathbf{n}}_{\mathbb{C}}$ as $\mathbb{P}(\mathbb{V}_0)$.

In other words $\mathbf{H}^{\mathbf{n}}_{\mathbb{C}}$ is the collection of negative lines and the boundary is the collection of null lines.
Here we are considering $\mathbb{V} = \mathbb{C}^{n+1}$ with the Hermitian form $\langle , \rangle$ of signature $(n, 1)$, given by

$$\langle v, w\rangle = \bar{w}^t J v = -v_0\bar{w}_0 + v_1\bar{w}_1 + ... + v_n\bar{w}_n$$

where $v = (v_0, v_1, ..., v_n)$ and $w = (w_0, w_1, ..., w_n)$ are column vectors in $\mathbb{V}$ and $J = (-1, 1, ..., 1)$ is the diagonal matrix representing the given Hermitian form.

The ball model of $\mathbf{H}^{n}_{\mathbb{C}}$ is obtained by considering the representatives of the homogenous coordinate $W = [(1, w_1, ..., w_n)]$ in $\mathbb{P}(\mathbb{V})$. The vector $(1, w_1, ..., w_n)$ is the *standard lift* of $W \in \mathbf{H}^{n}_{\mathbb{C}}$ to $\mathbb{V}_-$. Further if

$$\langle W, W\rangle = -1 + |w_1|^2 + |w_2|^2 + ... + |w_n|^2 < 0$$

then $|w_1|^2 + |w_2|^2 + ... + |w_n|^2 < 1$ and hence $\mathbb{P}(\mathbb{V}_-)$ can be identified with the unit ball

$$\mathbb{B}^n = \{(w_1, ..., w_n) \in \mathbb{C}^n : |w_1|^2 + |w_2|^2 + ... + |w_n|^2 < 1\}$$

which identifies boundary $\partial\mathbf{H}^{\mathbf{n}}_{\mathbb{C}}$ with the *unit sphere*

$$\mathbb{S}^{2n-1} = \{(w_1, ..., w_n) \in \mathbb{C}^n : |w_1|^2 + |w_2|^2 + ... + |w_n|^2 = 1\}$$

1.1.6 Bergman Metric

Let $\hat{v}, \hat{w} \in \mathbf{H}^{\mathbf{n}}_{\mathbb{C}}$ and $v, w \in \mathbb{C}^{n,1}$ such that $\mathbb{P}(v) = \hat{v}$ and $\mathbb{P}(w) = \hat{w}$.
Then the metric on $\mathbf{H}^{\mathbf{n}}_{\mathbb{C}}$ called Bergman metric is given by the distance function $\rho(\hat{v}, \hat{w})$ defined by:

$$\cosh^2(\frac{\rho(\hat{v}, \hat{w})}{2}) = \frac{\langle v, w\rangle\langle w, v\rangle}{\langle v, v\rangle\langle w, w\rangle}$$

Note that complex mulitiplication on v, w doesnot change the value of distance.

$$\frac{\langle kv, w\rangle\langle w, kv\rangle}{\langle kv, kv\rangle\langle w, w\rangle} = \frac{k\langle v, w\rangle\bar{k}\langle w, v\rangle}{k\bar{k}\langle v, v\rangle\langle w, w\rangle} = \frac{\langle v, w\rangle\langle w, v\rangle}{\langle v, v\rangle\langle w, w\rangle}$$

Therefore the distance function is well defined.

1.2 Isometry group of Hyperbolic Space

1.2.1 Holomorphic Isometries

The isometries of the complex hyperbolic space are the linear transformations on the space $\mathbf{H}^{\mathbf{n}}_{\mathbb{C}}$ which preserves the Bergman metric. Since the Bergman metric is defined entirely on terms of the Hermitian form it is clear that the unitary group $\mathrm{U(n,1)}$ with respect to the Hermitian form, given by

$$\mathrm{U(n,1) = \{A \in GL(n+1, \mathbb{C}) \mid \langle v, w\rangle = \langle Av, Aw\rangle \quad for\ all \quad v, w \in \mathbb{C}^{n,1}\}}$$

acts isometrically on the projective model of complex hyperbolic space. As v and w varies over a basis of $\mathbb{V}$, the unitary group assumes the following characterisation,

$$\mathrm{U(n,1) = \{A \in GL(n+1, \mathbb{C}) : \bar{A}^t JA = J\}}$$

The group $GL(n, \mathbb{C})$ of linear transformations on $\mathbb{C}^{n,1}$ induces the group $PGL(n+1, \mathbb{C})$ of projective transformations on $\mathbb{C}P^n$. The projective unitary group $\mathrm{PU(n, \mathbb{C})}$ is defined as the projection of the unitary group under the projectivization from $GL(n+1, \mathbb{C})$ to $PGL(n+1, \mathbb{C}\}$ i.e.

$$\mathrm{PU(n,1) = U(n,1)/Z(U(n+1))}$$

where the center $Z(\mathrm{U(n,1)})$ can be identified with the circle group $\mathbb{S}^1 = \{\alpha I \mid |\alpha| = 1\}$. PU(n, 1) acts on $\mathbb{C}P^n$ preserving $\mathbf{H}^n_{\mathbb{C}}$ and $\partial\mathbf{H}^n_{\mathbb{C}}$ and hence restriction of its element to $\mathbf{H}^n_{\mathbb{C}}$ gives an isometry. An element of PU(n, 1) is called a *holomorphic isometry*.

1.2.2 Anti-holomorphic Isometries

A map $g : \mathbb{C}^{n,1} \to \mathbb{C}^{n,1}$ is called an *anti linear map* if

$$g(\lambda v + \mu w) = \bar{\lambda} g(v) + \bar{\mu} g(w)$$

Any anti-linear map is of the form

$$v \mapsto A\bar{v} \quad \text{where } A \in M_{(n+1)\times(n+1)}$$

An anti-linear map is called *anti-unitary* if $A \in \mathrm{U(n,1)}$.

An anti-unitary map g induces an isometry $\hat{g}$ on $\mathbf{H}^n_{\mathbb{C}}$ as

$$\cosh^2\left(\frac{\rho(\hat{g}(\hat{v}), \hat{g}(\hat{w}))}{2}\right) = \frac{\langle A\bar{v}, A\bar{w}\rangle\langle A\bar{w}, A\bar{v}\rangle}{\langle A\bar{v}, A\bar{v}\rangle\langle A\bar{w}, A\bar{w}\rangle} = \frac{\langle w, v\rangle\langle v, w\rangle}{\langle v, v\rangle\langle w, w\rangle}$$

$$= \cosh^2\left(\frac{\rho(\hat{v}, \hat{w})}{2}\right)$$

An isometry induced by an anti-unitary map is called an *anti-holomorphic isometry*.

The group of holomorphic isometries is known to be an index two subgroup of the group of full isometries. $\mathrm{PU}(n,1)$ together with an anti-holomorphic isometry can generate the whole group of isometries.

1.2.3 Conjugacy Classification of Unitary Elements

Let $\overline{\mathbf{H}^n_{\mathbb{C}}}$ denote the closure of $\mathbf{H}^n_{\mathbb{C}}$ in the projective space $\mathbb{P}(V)$. If $g \in \mathrm{U(n,1)}$ then g acts on $\mathbb{P}(V)$ leaving $\overline{\mathbf{H}^n_{\mathbb{C}}}$ invariant. Since $\overline{\mathbf{H}^n_{\mathbb{C}}}$ is a closed ball, g must have fixed points in $\overline{\mathbf{H}^n_{\mathbb{C}}}$. The unitary elements can be classified into three different classes based on their fixed points.

An element f in U(n, 1) is called:

1. *elliptic* if it has a fixed point in $\mathbf{H}^n_{\mathbb{C}}$;

2. *parabolic* if it has exactly one point in $\overline{\mathbf{H}^n_{\mathbb{C}}}$ which lies in $\partial\mathbf{H}^n_{\mathbb{C}}$;

3. *hyperbolic or loxodromic* if it has exactly two fixed points in $\overline{\mathbf{H}^n_{\mathbb{C}}}$ that belongs to $\partial\mathbf{H}^n_{\mathbb{C}}$.

Definition 1.2.1. Let $f \in \mathrm{U(n,1)}$ and λ be an eigenvalue of f. Then λ is said to be of *positive type(or negative type)* if every eigenvector of λ is in $\mathbb{V}_+$(*or* $\mathbb{V}_-$) . The eigenvalue λ is called *null (or indefinite)* if λ-eigenspace,V_λ is light like (or indefinite).

The following theorem from [CG74] classifies the conjugacy classes in $\mathrm{U(n,1)}$.

Theorem 1.2.2. *(a) An elliptic element is semisimple, with eigenvalues of norm one. Two elliptic elements are conjugate if and only if they have the same negative eigenvalue and the same set of* n *positive eigenvalues (with the same multiplicities).*

(b) A loxodromic element is semisimple, with exactly $\mathrm{n}-1$ *eigenvalues of norm one. Two loxodromic elements are conjugate if and only if their eigenvalues are same.*

(c) A parabolic element is not semisimple, and all of its eigenvalues have norm one. It has a unique decomposition $g = pe = ep$*, where* p *is unipotent parabolic and* e *is elliptic. Two parabolic elements are conjugate if and only if their elliptic and unipotent parabolic elements are conjugate.*

The theorem follows from the results proved in this section.

Definition 1.2.3. Let $f \in \mathrm{U(n,1)}$ and F be a f-invariant subspace of $\mathbb{C}^{n,1}$. Then an *eigenbasis* of F for the map f is a basis of F which contains the eigenvectors of f.

Lemma 1.2.4. *If* F *is a positive definite subspce of* $\mathbb{C}^{n,1}$ *which is invariant under* $f \in \mathrm{U(n,1)}$*, then there exists an orthonormal eigenbasis of* F *for* f*.*

Proof We can show this by using induction on the dimension of subspaces.The result is vacuously true when $n = 0$. Now let v be an eigenvector of $f|_F$ and $F' = v^{\perp} \cap F$. Since v is not null, $F = \mathrm{Span}_{\mathbb{C}}\{v\} \oplus F'$. Let $v' \in v^{\perp}$. Then

$$0 = \langle v', v\rangle = \langle f(v'), f(v)\rangle = \langle f(v'), \alpha v\rangle = \bar{\alpha}\langle f(v'), v\rangle$$

This implies $f(v') \in v^{\perp}$ for all $v' \in v^{\perp}$. Therefore, F' is invariant under f and is positive definite. Hence by the induction hypothesis, there exists an orthogonal eigenbasis for F'. Adjoining v to this basis and normalizing, we obtain the desired basis. □

Remark 1.2.5. If v is a non-null eigenvector of f, then $\langle v, v\rangle = \langle f(v), f(v)\rangle = \langle \alpha v, \alpha v\rangle = \alpha\bar{\alpha}\langle v, v\rangle$ implies $|\alpha| = 1$.

Here $\oplus$ always denote the orthogonal sum of subspaces. The direct sum is denoted by $+$.

Lemma 1.2.6. *Let $f \in \mathrm{U(n, 1)}$ be an elliptic element. Then there exists an orthogonal eigenbasis $B = \{v_1, ..., v_{n+1}\}$ for f such that v_1 is a negative vector and v_i is positive where $i = 2, ..., n + 1$ and all eigenvalues of v_i has unit modulus.*

Proof Since $\hat{f}$ has a fixed point in $\mathbf{H}^n_{\mathbb{C}}$, the lift of f has a negative eigenvector in $\mathbb{C}^{n,1}$. Let v_1 be a negative eigen vector of f and $F = \mathrm{Span}_{\mathbb{C}}(v_1)$. As F is non-degenerate, it is possible to write $\mathbb{C}^{(n,1)} = F \oplus F^{\perp}$. Since $F^{\perp}$ is positive definite, by Lemma 1.2.4 there exists an orthogonal eigenbasis $\{v_2, ..., v_{n+1}\}$ of $F^{\perp}$ for f. The set $B = \{v_1, v_2, ...v_{n+1}\}$ is an orthogonal eigenbasis for $\mathbb{C}^{(n,1)}$ where v_1 has a negative type eigenvalue and $v_2, ..., v_{n+1}$ are positive vectors. By Remark 1.2.5, it is clear that each eigenvalue of v_i has unit modulus for $i = 1, ..., n + 1$. Therefore, any elliptic element f is conjugate to a diagonal matrix with entries $(e^{i\theta_1}, ..., e^{i\theta_{n+1}})$ where $e^{i\theta_1}$ is a negative type eigenvalue and the rest of the entries are of positive type. □

Remark 1.2.7. It follows from the lemma that if f is an elliptic element in the unitary group, then distinct eigenspaces of f are orthogonal.

Corollary 1.2.8. *Two elliptic elements in* $\mathrm{U(n, 1)}$ *are conjugate if and only if they have the same negative type eigenvalue and same set of positive eigenvalues.*

Proof Suppose f and g in $(\mathrm{U(n,1)}$ belong to the same conjugacy class. Then both are conjugate to the diagonal matrix given in the proof of Lemma 1.2.6 which means that both f and g have the same set of eigenvalues with $e^{i\theta_1}$ being the negative type of eigenvalue for both.
Conversely, suppose both f and g has the same set of negative and positive type eigenvalues. Then with respect to the eigenbasis given in Lemma 1.2.6, f and g has the same diagonal matrix by which it follows that these two are conjugate to each other. □

Lemma 1.2.9. *Let $f \in \mathrm{U(n,1)}$ be hyperbolic (or loxodromic). Then there exists an eigen-basis $B = \{v_1, ..., v_{n+1}\}$ of $\mathbb{C}^{n,1}$ for f such that v_1 and v_2 are null vectors that has non-unit eigenvalues and $\{v_3, ..., v_{n+1}\}$ are postive vectors with unit eigenvalues.*

Proof Since $\hat{f}$ has two fixed points in $\partial\mathbf{H}^n_{\mathbb{C}}$, their lifts v_1 and v_2 in $\mathbb{C}^{n,1}$ will give two distinct null eigenvectors of f. Let $F = Span_{\mathbb{C}}\{v_1, v_2\}$. Since the dimension of a light like space cannot exceed one, $\langle v_1, v_2\rangle \neq 0$. If α and β are the eigenvalues of v_1 and v_2 respectively, then $\langle v_1, v_2\rangle = \langle f(v_1), f(v_2)\rangle = \alpha\bar{\beta}\langle v_1, v_2\rangle \neq 0$ implies that $\alpha\bar{\beta} = 1$. It then follows that $|\alpha|$ and $|\beta|$ cannot be one. Otherwise multiplying by $\bar{\alpha}$ or β on both sides will give $\alpha = \beta$ which is a contradiction. Therefore, $\alpha = re^{i\theta}$ and $\beta = \frac{1}{r}e^{i\theta}$ where $r \neq 1$. Since F is indefinite, $F^{\perp}$ is positive definite and hence by Lemma 1.2.4 there exists an orthogonal eigenbasis $\{v_3, ..., v_{n+1}\}$ of $F^{\perp}$ for f where each basis element has unit eigenvalue. As $\mathbb{C}^{n,1} = F \oplus F^{\perp}$, by adjoining v_1 and v_2 to the above basis, we obtain the desired basis. By Remark 1.2.5 it follows that the eigenvalues of v_i has modulus one for $i = 3, ..., n+1$. □

Corollary 1.2.10. *Two loxodromic elements of $\mathrm{U(n,1)}$ are conjugate if and only if they have the same set of eigenvalues.*

Proof From Lemma 1.2.9 we have that two loxodromic elements f and g are conjugate if and only if both can be diagonalised to the same matrix with diagonal entries $(re^{i\theta}, \frac{1}{r}e^{i\theta}, e^{i\theta_3}, ..., e^{i\theta_n+1})$ $(r \neq 1)$which is possible if and only if f and g has the same eigenvalues. □

The elliptic and hyperbolic elements are semisimple i.e., their minimal polynomial is a product of linear factors whereas parabolic elements are not semisimple.

Let $T \in \mathrm{U(n,1)}$ be parabolic. Then T have the unique Jordan decomposition $T = AN$ where A is elliptic, N is nilpotent and A commutes with N. In adddition, all the eigenvalues of T has modulus 1. Suppose T is unipotent i.e., all eigenvalues are unipotent. Then T has the minimal polynomial $(x-1)^k$ where $k = 2$ or 3. When $k = 2$, T is called *vertical* and when $k = 3$, T is called *non-vertical translation*.
If T is not nilpotent, then it has a null eigenvalue λ and a factor of the form $(x-\lambda)^k$ in its minimal polynomial. If $k = 2$, then T is called *ellipto-translation* and when $k = 3$, T is called *ellipto-parabolic*. This implies that $\mathbb{C}^{n,1}$ has a T-invariant orthogonal decomposition,

$$\mathbb{C}^{n,1} = \mathbb{U} \oplus \mathbb{W} \tag{1.2.1}$$

where $T|_{\mathbb{W}}$ is semisimple, U is indefinite with dimension $k = 2$ or 3 and $T|_{\mathrm{U}}$ has minimal polynomial $(x-\lambda)^k$. Without loss of generality, it is possible to assume, $T|_{\mathbb{W}}$ as an element of $U(n-k+1)$ by identifying $U(\langle , \rangle|_{\mathbb{W}})$ with $U(n-k+1)$.

Lemma 1.2.11. *If f is a prabolic element in* $\mathrm{U(n,1)}$*, then there exists a basis of $C^{n,1}$ which contains all distinct eigenvectors(upto scalar multiplication) of f.*

Proof Let $f \in \mathrm{U(n,1)}$ be parabolic with a factor $(x-\lambda)^2$ in its minimal polynomial. Then $C^{n,1}$ has the f-invariant decomposition as in (1.2.1). Let $\{v_2, ..., v_n\}$ be the eigenbasis for the positive definite basis for $\mathbb{W}$ given by Lemma 1.2.4 and v be the null eigenvector with eigenvalue 1. Let u be a negative vector in $\mathbb{U}$ such that$\langle v, u\rangle \in \mathbb{R}\backslash\{0\}$, then $\mathbb{U} = \mathrm{span}_{\mathbb{C}}\{v, u\}$. By replacing u with a scalar multiple if necessary, it is possible to assume that $f(u) = kv + u$. And $\langle u, u\rangle = \langle f(u), f(u)\rangle = \langle kv + u, kv + u\rangle$. On expanding, we obtain $k + \bar{k} = 0$. Therefore the set $\{v, u, v_2, ...v_n\}$ gives the desired basis.

Suppose f is parabolic with $(x-\lambda)^3$ in the minimal polynomial giving the decomposition as in (1.2.1) with $\dim \mathbb{U} = 3$. Let $\{v, v_3, ..., v_n\}$ be a linear independent orthogonal set of eigenvectors of f with $f(v) = v$ and v_i is the unit eigenvaule of $f|_{\mathbb{W}}$ for $i = 3, ..., n$. Since $\dim_{\mathbb{C}}(v^{\perp} \cap \mathbb{U}) = 2$, choose $w_0 \in v^{\perp} \cap \mathbb{W}$, linearly independent to v. Also $\langle v, w_0\rangle = 0$. Then $f(w_0) = \lambda v + w_0$. Since v is the only eigenvector in $\mathbb{U}$, $\lambda \neq 0$. Then $\mathbb{U} = \mathrm{span}_{\mathbb{C}}\{w_0\} \perp (w_0^{\perp} \cap \mathbb{W})$. The subspace $w_0^{\perp} \cap \mathbb{W}$ is a 2-dimensional indefinite space and hence we can choose a negative vector u_0 from $w_0^{\perp} \cap \mathbb{W}$. Since the $trf|_{\mathbb{U}} = 3$, we can write $f(u_0) = mv + nw_0 + u_0$.

Therefore, $0 = \langle w_0, u_0 \rangle = \langle f(w_0), f(u_0) \rangle = \lambda \langle v, u_0 \rangle + \bar{n} \langle w_0, w_0 \rangle$.
$\lambda \langle v, u_0 \rangle = -\bar{n} \langle w_0, w_0 \rangle \neq 0 \qquad \Rightarrow n\lambda \langle v, u_0 \rangle = -|n^2| \langle w_0, w_0 \rangle \in \mathbb{R} \setminus 0$.
By substituting, $w = \frac{w_0}{\lambda}, \quad u = \frac{u_0}{n\lambda}, \quad k = \frac{m}{n\lambda}$ we obtain,

$$f(v) = v, \quad f(w) = v + w \quad \text{and} \quad f(u) = kv + w + u$$

Also, $0 = \langle f(w), f(u) \rangle = \langle v, u \rangle + \langle w, w \rangle$. Substituting for the value of $f(u)$ in the equation $\langle u, u \rangle = \langle f(u), f(u) \rangle$ and using the identity $\langle w, w \rangle = -\langle v, u \rangle$, we get $k + \bar{k} = 1$. Since $F = Span_{\mathbb{C}}\{v, w, u\}$ and $F \perp H = Span_{\mathbb{C}}\{v_2, ..., v_{n-1}\}$, the vectors v, w, u are orthogonal to v_i where $i = 2, ..., n-1$. And it is clear that $\langle v, w \rangle = \langle w, u \rangle = 0$. For our purpose, we will be considering the basis $\{v, iw, u, v_2, ..., v_{n-1}\}$. □

Lemma 1.2.12. *Let f be a parabolic element in* $\mathrm{U(n, 1)}$.

1. *There exists a unique unipotent parabolic element p and a unique elliptic element e such that $f = pe = ep$.*

2. *f is not semisimple.*

3. *All eigenvalues of f has norm one.*

4. *Two parabolic elements are conjugate if and only if their elliptic and unipotent parabolic elements are conjugate.*

Proof

1. From the proof of Lemma 1.2.11 it is clear that any parabolic element in which the minimal polynomial contains the factor $(x - \lambda)^m$ is of the form $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$ where $A = \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}$ or $\begin{pmatrix} 1 & i & k \\ 0 & 1 & -i \\ 0 & 0 & 1 \end{pmatrix}$ when $m = 2$ or 3 respectively and $B \in \mathrm{U(n + 1 - m, \mathbb{C})}$.

   Then A is unipotent and hence $f = pe$ where $p = \begin{pmatrix} A & 0 \\ 0 & I_{n+1-m} \end{pmatrix}$ and $e = \begin{pmatrix} I_m & 0 \\ 0 & B \end{pmatrix}$ such that I_k is the $k \times k$ matrix representing the identity transformation gives us the desired decomposition. This is the multiplicative Jordan decomposition of f into its semisimple and unipotent composition and hence is unique.

2. Since the unipotent part in the Jordan decomposition is non-zero, it follows that f is not semi-simple.

3. From the above decomposition, it is clear that the eigenvalues of f are the eigenvalues of e and one each of which has norm one.

4. Suppose $f = pe$ and $g = p'e'$ are two conjugate parabolic transformations such that $f = xfx^{-1}$. Then $pe = xp'e'x^{-1} = xp'x^{-1}xe'x^{-1}$. The conjugates xpx^{-1} and xex^{-1} of p and e are unipotent parabolic and elliptic respectively. Since the only unipotent part and elliptic part in LHS is p and e respectively, we have $p = xp'x^{-1}$ and $e = xe'x^{-1}$. Conversely, suppose $f = pe$ and $g = p'e'$ are parabolic such that the elliptic and unipotent components are conjugate. If the elliptic components are considered in their diagonal form then they commutes with every matrix. Consequently we have

$$g = p'e' = xp'x^{-1}ye'y^{-1} = xp'e'x^{-1} = xfx^{-1}$$

□

Proof of Theorem 1.2.2:

The theorem follows from Lemma 1.2.6, Lemma 1.2.8, Lemma 1.2.9, Lemma 1.2.10, Lemma 1.2.11 and Lemma 1.2.12

1.2.4 Reversible elements and Involutions

Definition 1.2.13. An element g in a group G is called an *involution* if g^2 equals identity of G.

Definition 1.2.14. An element f in a group G is called *reversible* or *real* if there exists $h \in G$ such that $f^{-1} = hfh^{-1}$. $f \in G$ is called *strongly reversible* if it can be written as a product of two involutions in G.

Suppose $f \in \mathrm{U(n,1)}$ is reversible such that f^{-1} is conjugate to f by an involution h then, $f^{-1} = hfh$ or equivalently $(hf)^2 = hfhf = e$ which implies that f is strongly reversible. If $f \in U(n,1) = hg$ where h and g are involutions, then $f^{-1} = hg = hfh = hfh^{-1}$. It follows that if $f \in U(n,1)$ is strongly reversible, then f is reversible.
We can characterize the involutions of $\mathrm{U(n,1)}$ as product of Hermitian matrices from which it follows that Hermitian matrices in $U(n,1)$ are reversible.

Lemma 1.2.15. *An element $A \in \mathrm{U}(n,1)$ is an involution iff $A = HJ$ where $H \in \mathrm{U}(n,1)$ is Hermitian and $J = diag(-1, 1, \cdots, 1)$ is the matrix corresponding to the Hermitian form on $\mathbb{C}^{n,1}$.*

Proof Let $A \in \mathrm{U}(n,1)$ be an involution. Then $A = A^{-1}$ and it follows from $AJ\bar{A}^t = J$ that $J\bar{A}^t = AJ$. As $\overline{(J\bar{A}^t)}^t = AJ$, it follows that $J\bar{A}^t$ is hermitian. Hence, $A = HJ$ where $H = J\bar{A}^t$.
Conversely, let $A = HJ$ where $H \in \mathrm{U}(n,1)$ is Hermitian. Then $A^2 = HJHJ = HJ\bar{H}^tJ = HH^{-1} = I$.

□

In particular it follows that:

Corollary 1.2.16. *If A is Hermitian in $\mathrm{U}(n,1)$, then it is strongly reversible. In particular, every Hermitian element in $\mathrm{U}(n,1)$ is reversible.*

Proof As $HJ = A$ is an invoution, we have $H = AJ$ as a product of two involutions in $\mathrm{U}(n,1)$. Hence it is strongly reversible.

□

Chapter 2

Decomposition of Complex Hyperbolic Isometries by Involutions

2.1 Complex Reflections

Definition 2.1.1. An element f in $\mathrm{U}(n,1)$ is called a *complex k-reflection* if it has a negative eigenvalue λ of multipilicity $k+1$ and $n-k$ eigenvalues 1.

A complex k-reflection pointwise fixes a k-dimensional totally geodeic subspace S of $\mathbf{H}^n_{\mathbb{C}}$ and acts as a rotation in the co-dimension k orthogonal complement of S.

Example 2.1.2. Consider the ball model of $\mathbf{H}^n_{\mathbb{C}}$. Then a 0- reflection is of the form $Z \mapsto \lambda Z$ where $|\lambda| = 1$.

A 0-relection is called *complex rotation*; 1-reflection is called *complex-line reflection* and 2-reflection is called *complex plane-reflection*.

2.2 Product of Involutions in SU(n)

Any element of $\mathrm{SU}(n)$ can be written as a product of atmost five involutions. The actual theroem goes as follows.

Theorem 2.2.1. *Let $n > 1$. If $n \neq 2 \mod 4$, an unitary transformation in* $\mathrm{SU}(n)$ *is a product of at most four involutions. If $n = 2 \mod 4$, then every element in* $\mathrm{SU}(n)$ *is a product of at most five involutions.*

That is, the involution length of $\mathrm{SU}(n)$ *is four, resp. five, if $n \neq 2 \mod 4$, resp. $n = 2 \mod 4$.*

The proof of the theorem will follow from the following lemmas.

Lemma 2.2.2. *[GP13] Let $n \neq 2 \mod 4$. An element $T \in \mathrm{SU}(n)$ is reversible if and only if it is a product of two involutions.*

Lemma 2.2.3. *If $n = 2 \mod 4$, then a reversible element T in $\mathrm{SU}(n)$ that has no eigenvalue ± 1, can be written as a product $T = J_1 J_2$, where J_1 and J_2 are involutions in $\mathrm{U}(n)$, each of determinant -1. If A has eigenvalue ± 1, it can be written as a product of two involutions in $\mathrm{SU}(n)$.*

Proof Let $n = 4m+2$. If $T \in \mathrm{SU}(n)$ be reversible. Then if λ is a root, so is λ^{-1} with the same multiplicity. Thus we can decompose $\mathbb{C}^n$ into two-dimensional subspaces

$$\mathbb{C}^n = \mathbb{W}_1 \oplus \mathbb{W}_2 \oplus \cdots \oplus \mathbb{W}_{2m+1}, \tag{2.2.1}$$

where each $\mathbb{W}_i$ has an orthonormal basis w_{i1}, w_{i2} such that $T(w_{i1}) = \lambda w_{i1}$ and $T(w_{i2}) = \lambda^{-1} w_{i2}$. Define J_1 and J_2 such that their restrictions on $\mathbb{W}_i$ is given by

$$J_{i1}(w_{i1}) = \lambda w_{i2},\ J_{i1}(w_{i2}) = \lambda^{-1} w_{i1}; \quad J_{i2}(w_{i1}) = w_{i2},\ J_{i2}(w_{i2}) = w_{i1}.$$

Then for each $i = 1, 2, \ldots, 2m+1$, J_{i1} and J_{i2} are involutions each with determinant -1. Let $J_1 = J_{11} \oplus \cdots J_{(2m+1)1}$ and $J_2 = J_{12} \oplus \cdots J_{(2m+1)2}$. Then $T = J_2 J_1$ and $\det J_1 = -1 = -\det J_2$, $J_1^2 = I = J_2^2$.

If T has an eigenvalue ± 1, then $\mathbb{C}^n$ has a T-invariant orthogonal decomposition

$$\mathbb{C}^n = \mathbb{U}_1 \oplus \mathbb{U}_{-1} \oplus \mathbb{W},$$

where $\dim \mathbb{U}_{-1}$ is even, say 2l, $T|_{\mathbb{U}_{-1}} = -1_{2l}$; $\dim \mathbb{U}_1 = k$, $T|_{\mathbb{U}_1} = 1_k$ and, $T|_{\mathbb{W}}$ has no eigenvalue ± 1. By the above method, $T|_{\mathbb{W}} = j_1 j_2$ for involutions j_1, j_2 on $\mathbb{W}$. Define $J_1 = -1 \oplus 1_{k-1} \oplus -1_{2l} \oplus j_1$, $J_2 = -1 \oplus 1_{2l+k-1} \oplus j_2$. Then J_1 and J_2 are involutions such that each has determinant one and $T = J_2 J_1$. This completes the proof.

Lemma 2.2.4. *Every element in* $\mathrm{SU}(n)$*, can be written as a product of two reversible elements.*

Proof Suppose A is an element of $\mathrm{SU}(n)$. Let $\lambda_1, \dots, \lambda_n$ are the eigenvalues of A. Note that $|\lambda_i| = 1$ for all i. Then $\mathbb{C}^n$ has an orthogonal decomposition into eigenspaces:

$$\mathbb{C}^n = \mathbb{V}_1 \oplus \cdots \oplus \mathbb{V}_n,$$

where each $\mathbb{V}_i$ has dimension 1 and $T|_{\mathbb{V}_i}(v) = \lambda_i v$ for $v \in \mathbb{V}_i$. Choose an orthonormal basis $v_1, v_2, \dots, v_n$ of $\mathbb{C}^n$, where $v_i \in \mathbb{V}_i$ for each i. Consider the unitary transformations $R_1 : \mathbb{V} \to \mathbb{V}$ and $R_2 : \mathbb{V} \to \mathbb{V}$ defined as follows: for each $k = 0, 1, 2 \dots,$

$$R_1(v_{2k}) = \prod_{j=0}^{2(k-1)} \overline{\lambda}_{2k-j-1}\ v_{2k}, \qquad R_1(v_{2k+1}) = \prod_{j=0}^{2k} \lambda_{2k-j+1}\ v_{2k+1}, \tag{2.2.2}$$

$$R_2(v_{2k}) = \prod_{j=0}^{2k-1} \lambda_{2k-j}\ v_{2k}, \qquad R_2(v_{2k+1}) = \prod_{j=0}^{2k-1} \overline{\lambda}_{2k-j}\ v_{k}, \tag{2.2.3}$$

with the convention $\lambda_0 = 1 = \lambda_{-1},\ v_0 = 0$. Note that $k \leq [\frac{n}{2}] + 1$ and $\max k = \frac{n}{2}$ or $\frac{n-1}{2}$ depending on n is even or odd. For each i, $R_1 R_2(v_i) = \lambda_i v_i = T(v_i)$, and hence $T = R_1 R_2$. Note that both R_1 and R_2 has the property that if λ is an eigenvalue, then so is $\bar{\lambda} = \lambda^{-1}$. This shows that R_1 and R_2 are reversible, cf. [GP13]. Further, if $T \in \mathrm{SU}(n)$, then $\lambda_1 \dots \lambda_n = 1$ and hence, both R_1 and R_2 have determinants 1. Hence the result follows.

In matrix form, up to conjugacy, if $T = \mathrm{diag}(\lambda_1, \lambda_2, \dots, \lambda_n)$, then

$$R_1 = \mathrm{diag}(\lambda_1, \overline{\lambda}_1,\ \lambda_1\lambda_2\lambda_3, \overline{\lambda}_1\overline{\lambda}_2\overline{\lambda}_3,\ \dots,\ \lambda_1\lambda_2 \dots \lambda_{2k+1}, \overline{\lambda}_1\overline{\lambda}_2 \dots \overline{\lambda}_{2k+1}, \dots) \tag{2.2.4}$$

$$R_2 = \mathrm{diag}(1,\ \lambda_1\lambda_2, \overline{\lambda}_1\overline{\lambda}_2,\ \dots,\ \lambda_1\lambda_2 \dots \lambda_{2k}, \overline{\lambda}_1\overline{\lambda}_2 \dots \overline{\lambda}_{2k}, \dots). \tag{2.2.5}$$

Note that R_2 has always an eigenvalue 1. Hence it can be written as a product of two involutions, see [GP13, Proposition 3.3].

Lemma 2.2.5. *Let $n = 2 \mod 4$, $n > 2$. Let $T \in \mathrm{SU}(n)$ be a reversible element that can not be written as a product of two involutions in $\mathrm{SU}(n)$. Then T can be written as a product of three involutions in $\mathrm{SU}(n)$.*

Proof Let $n = 4m + 2$. We have the decomposition of $\mathbb{C}^n$ as in (2.2.1). Further we see that $T|_{\mathbb{W}_i} = J_{i1}J_{i2}$, where J_{i1} and J_{i2} are involutions each with determinant -1. Now define involutions I_1, I_2, I_3 as follows.

$$I_1|_{\mathbb{W}_1} = J_{11}, \ \ I_2|_{\mathbb{W}_2} = 1, \ \ I_2|_{\mathbb{W}_i} = J_{i1}, \ i = 3, \ldots, 2m+1.$$

$$I_2|_{\mathbb{W}_1} = 1, \quad J_2|_{\mathbb{W}_2} = J_{21}, \quad J_2|_{\mathbb{W}_i} = J_{i2}, \ i = 3, \ldots, 2m+1.$$

$$I_3|_{\mathbb{W}_1} = J_{12}, \ \ J_2|_{\mathbb{W}_2} = J_{22}, \quad J_2|_{\mathbb{W}_i} = 1, \ i = 3, \ldots, 2m+1.$$

Then each I_1, I_2, I_3 has determinant 1 and they are involutions.

Combining the above lemmas we have Theorem 2.2.1 .

Lemma 2.2.6. *Let T be a reversible element in* $\mathrm{SU}(n)$*. Then T is a commutator.*

Proof We can choose S in $\mathrm{SU}(n)$ such that $S^2 = T$ and S is also reversible. If $n \neq 2 \mod 4$, then $S = i_1 i_2$ for involutions i_1 and i_2. Consequently, $T = S^2 = [i_1, i_2]$. If $n = 2 \mod 4$, then $S = i_1 i_2 i_3$. Consequently, $T = [i_1 i_3, i_3 i_2]$.

Using the above lemma it follows from Theorem 2.2.1 that:

Corollary 2.2.7. $\mathrm{SU}(n)$ *has commutator length two.*

2.3 Decomposition of Complex Hyperbolic Isometries

By using the results of the previous section, we can prove the following theorem.

Theorem 2.3.1. *Let T be a holomorphic isometry of $\mathrm{H}^n_{\mathbb{C}}$, that is, $T \in \mathrm{PU}(n,1)$. Then T is a product of at most four involutions and a complex k-reflection, where $k \leq 2$; $k = 0$ if T is elliptic; $k = 1$ if T is ellipto-translation or hyperbolic; $k = 2$ if T is ellipto-parabolic and $n > 2$.*

Since an isometry that is a product of two involutions is also reversible, we have the following.

Corollary 2.3.2. *Let T be a holomorphic isometry of $\mathrm{H}^n_{\mathbb{C}}$, that is, $T \in \mathrm{PU}(n,1)$. Then T is a product of at most two reversible elements and a complex k-reflection, where $k \leq 2$; $k = 0$ if T is elliptic; $k = 1$ if T is ellipto-translation or hyperbolic; $k = 2$ if T is ellipto-parabolic and $n > 2$.*

The theorem will follow from several lemmas that we prove below. The following theorem from [GP13] will be used in the proof.

Theorem 2.3.3. *[GP13, Theorem 4.2]*

(i) Let T be an element of $\mathrm{U}(n,1)$. Then T is strongly reversible if and only if it is reversible.

(ii) Let T be an element of $\mathrm{SU}(n,1)$ whose characteristic polynomial is self-dual. Then the following conditions are equivalent

(a) T is reversible but not strongly reversible.

(b) T is hyperbolic, $n = 4m+1$ for $m \in \mathbb{Z}$ and ± 1 is not an eigenvalue of T.

Suppose that T is in $\mathrm{PU}(n,1)$. Let $\widetilde{T}$ be a lift of T to $\mathrm{U}(n,1)$ and note that $e^{i\theta}\widetilde{T}$ corresponds to the same element of $\mathrm{PU}(n,1)$ for all $\theta \in [0,2\pi)$. For simplicity, from hereon we will not differentiate between $\tilde{T}$ and T. Both will be represented by T.

Lemma 2.3.4. *Let T be an elliptic element of $\mathrm{SU}(n,1)$ with negative type eigenvalue 1. Then T is a product of at most four involutions.*

Proof Since T has negative type eigenvalue 1, $\mathbb{C}^{n,1}$ has a T-invariant decomposition $\mathbb{C}^{n,1} = L \oplus \mathbb{W}$, where $T|_L = 1$, $\dim L = 1$ and $\dim \mathbb{W} = n$, $T|_{\mathbb{W}} \in \mathrm{SU}(n)$. By Theorem 2.2.1, if $n \neq 2 \mod 4$, then $T|_{\mathbb{W}}$ can be written as a product of four involutions. Assume $T|_{\mathbb{W}}$ has no eigenvalue ± 1. If $n = 2 \mod 4$, it follows from Lemma 2.2.3 and Lemma 2.2.4 that $T|_{\mathbb{W}} = j_1 j_2 j_3 j_4$, where j_i are involutions in $\mathrm{U}(n)$ each of determinant -1. For each $i = 1,2,3,4$, define $J_i = -1 \oplus j_i$. Then J_i is an element of $\mathrm{SU}(n,1)$ and $T = J_1 J_2 J_3 J_4$. When $T|_{\mathbb{W}}$ has eigenvalue ± 1, then it can be seen using Lemma 2.2.3 that it is a product of four involutions. This proves the lemma.

Lemma 2.3.5. *Let T be an elliptic element in $\mathrm{PU}(n,1)$. Then T is a product of a k-reflection, $k \geq 0$, and four involutions.*

Proof Choose a lift of T in $\mathrm{U}(n,1)$ such that $\mathbb{C}^{n,1}$ has a T-invariant orthogonal decomposition $\mathbb{C}^{n,1} = \mathbb{U} \oplus \mathbb{W}$, where $\dim \mathbb{U} = k+1 \geq 1$, $T|_{\mathbb{W}} \in \mathrm{SU}(n-k)$ and $T|_{\mathbb{U}}(v) = \lambda v$. Thus we have $T = RK$, where R is a k-reflection and $K \in \mathrm{SU}(n,1)$ with negative type eigenvalue 1 of multiplicity $k+1$. By the above lemma it follows that $T = RJ_1J_2J_3J_4$. This completes the proof.

Corollary 2.3.6. *Let T be an elliptic element in $\mathrm{PU}(n,1)$. Then T is a product of a complex rotation and four involutions.*

Proof Since T is semisimple, we can choose a lift T such that $\mathbb{C}^{n,1}$ has the decomposition $T = RK$, where $K \in \mathrm{SU}(n,1)$ be an elliptic with negative type eigenvalue 1 and R is an elliptic with one negative type eigenvalue λ, $|\lambda| = 1$, and one positive type eigenvalue 1 of multiplicity n. Note that R represents a complex rotation. The proof now follows as above.

Lemma 2.3.7. *Let T be a hyperbolic element in $\mathrm{SU}(n,1)$, $n > 2$, with real null eigenvalues. Then T can be written as a product of four involutions.*

Proof Since T has null eigenvalues real numbers r, r^{-1}, hence $\mathbb{C}^{n,1}$ has a T-invariant decomposition

$$\mathbb{C}^{n,1} = H \oplus \mathbb{W},$$

where $H = \mathbb{V}_r + \mathbb{V}_{r^{-1}}$, $\dim \mathbb{V}_r = 1 = \dim \mathbb{V}_{r^{-1}}$ and $T|_{\mathbb{W}} \in \mathrm{SU}(n-1)$. By Lemma 2.2.4, $T|_{\mathbb{W}} = r_1 r_2$, where r_1 and r_2 are reversible elements in $\mathrm{SU}(n-1)$ and are of the form given by (2.2.2) and (2.2.3). Let $R_1 = 1|_H \oplus r_1$ and $R_2 = T|_H \oplus r_2$. Then R_1 and R_2 are reversible elements in $\mathrm{SU}(n,1)$. Note that R_1 is elliptic and R_2 is hyperbolic with an eigenvalue 1. By Theorem 2.3.3, it follows that both R_1 and R_2 can be expressed as a product of two involutions in $\mathrm{SU}(n,1)$. Hence T can be written as a product of four involutions in $\mathrm{SU}(n,1)$.

Corollary 2.3.8. *A hyperbolic element in $\mathrm{PU}(n,1)$ is a product of a complex line-reflection and four involutions.*

Proof A hyperbolic element T in $\mathrm{U}(n,1)$ can be written as $T = DK$, where $K \in \mathrm{SU}(n,1)$ is a hyperbolic element with real null eigenvalues and D, up to conjugacy, is a diagonal matrix of the form $\lambda 1_2 \oplus 1_{n-1}$. D is clearly a complex line-reflection. The result now follow from the above lemma.

Lemma 2.3.9. *A vertical-translation in* $\mathrm{PU}(n,1)$*,* $n \geq 2$ *is a product of four involutions. A non-vertical translation is a product of two involutions.*

Proof The statement concerning vertical translation follows from the theorem of Djoković and Malzan [DM82]. It follows from [GP13, Theorem 4.1] that a non-vertical translation is reversible. Now using Theorem 2.3.3, the result follows.

Lemma 2.3.10. *Let* T *be an ellipto-translation in* $\mathrm{PU}(n,1)$*. Then it is a product of a complex line-reflection and four involutions.*

Proof Choose a lift in $\mathrm{U}(n,1)$ such that $T = DP$, where P is a ellipto-translation in $\mathrm{SU}(n,1)$ with null eigenvalue 1 and, D is elliptic with characteristic polynomial $(x-\lambda)^2(x-1)^{n-1}$, $|\lambda| = 1$. Now, $\mathbb{C}^{n,1}$ has a P-invariant decomposition $\mathbb{C}^{n,1} = \mathbb{U} \oplus \mathbb{W}$, where $\dim \mathbb{U} = 2$, $P|_{\mathbb{U}}$ has minimal polynomial $(x-1)^2$ and $P|_{\mathbb{W}} \in \mathrm{SU}(n-1)$. By Djoković and Malzan's theorem, $P|_{\mathbb{U}}$ is a product $i_1 i_2 i_3 i_4$ of involutions and, by Lemma 2.2.3 and Lemma 2.2.4, $P|_{\mathbb{W}}$ is a product of four involutions $r_1 r_2 r_3 r_4$. Thus P is product of four involutions $R_k = i_k \oplus r_k$ in $\mathrm{U}(n,1)$. Clearly, D is a complex line-reflection. Hence the lemma is proved.

Corollary 2.3.11. *Let* T *be an ellipto-translation in* $\mathrm{SU}(n,1)$ *with null eigenvalue* 1*. Then* T *is a product of four involutions in* $\mathrm{U}(n,1)$*.*

Lemma 2.3.12. *Let* T *be an ellipto-parabolic in* $\mathrm{PU}(n,1)$*. Then it is a product of a complex plane-reflection and four involutions.*

Proof Choose a lift, again denoted by T, in $\mathrm{U}(n,1)$ such that $T = KP$, where K is elliptic with characteristic polynomial $(x-\lambda)^3(x-1)^{n-2}$ and $P \in \mathrm{SU}(n,1)$ is a ellipto-parabolic with null eigenvalue 1. Then $\mathbb{C}^{n,1}$ has a P-invariant decomposition $\mathbb{C}^{n,1} = \mathbb{U} \oplus \mathbb{W}$, where $\dim \mathbb{U} = 3$, $P|_{\mathbb{U}}$ has minimal polynomial $(x-1)^3$ and, $\dim \mathbb{W} = n-2$, $P|_{\mathbb{W}} \in \mathrm{SU}(n-2)$. Now by Lemma 2.3.9, $P|_{\mathbb{U}} = i_1 i_2$, where i_1, i_2 are involutions and by Lemma 2.2.4, $P|_{\mathbb{W}}$ is a product of two reversible elements $P|_{\mathbb{W}} = r_1 r_2$. Let $R_1 = i_1 \oplus r_1$ and $R_2 = i_2 \oplus r_2$. Then $P = R_1 R_2$. Note that, R_1 and R_2 are reversible elements in $\mathrm{U}(n,1)$ and hence by Theorem 2.3.3, each of them is a product of two involutions. The elliptic element K is clearly a complex reflection that fixes a totally geodesic two dimensional subspace of $\mathrm{H}^n_{\mathbb{C}}$. This completes the proof.

Corollary 2.3.13. *Let T be an ellipto-parabolic in $\mathrm{SU}(n,1)$ with null eigenvalue 1. Then T is a product of four involutions.*

Proof of Theorem 2.3.1

Combining Corollary 2.3.6, Corollary 2.3.8, Lemma 2.3.9, Lemma 2.3.10 and Lemma 2.3.12, we have Theorem 2.3.1.

2.4 Product of anti-holomorphic Involutions

Theorem 2.4.1. *Every holomorphic isometry of $\mathbf{H}^{\mathbf{n}}_{\mathbb{C}}$ can be written as a product of two antiholomorphic involutions.*

Proof Let $f \in \mathrm{U}(n,1)$ be elliptic and A be the matrix of f with respect to the basis B as defined in Lemma 1.2.6. Now,consider the maps $\alpha : v \mapsto A\bar{v}$ and $\beta : v \mapsto \bar{v}$. Then $\alpha^2(v) = \alpha(A\bar{v}) = A\bar{A}v = v$ and $\beta^2(v) = \beta(\bar{v}) = v$. Also $\alpha\beta(v) = \alpha(\bar{v}) = A(v)$ by which the result follows.

If $f \in \mathrm{U}(n,1)$ is a hyperbolic element and A is the matrix of f with respect to the basis B given in Lemma 1.2.9, then by defining α and β as in the elliptic case it is possible to write $A = \alpha\beta$.

Suppose f is a parabolic element in $\mathrm{U}(n,1)$ such that the minimal polynomial of f contains a factor of the form $(x-\lambda)^2$ where $|\lambda| = 1$. Note that when $\lambda = 1$, f is unipotent. Consider the basis B of $\mathbb{C}^{n,1}$ as defined in Lemma 1.2.11. Then $\mathbb{C}^{n,1} = \mathbb{U} \oplus \mathbb{W}$ as in (1.2.1) where $\mathbb{U} = \mathrm{Span}_{\mathbb{C}}\{v, u\}$ and W is a positive definite space which contains unit-eigenvectors of f. Then with respect to B, $f|_{\mathbb{U}}=\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}$ and $f|_{\mathbb{W}}$ is the identity map. For $w \in \mathbb{U}$ define $\mu(w) = f\bar{w}$ and $\nu(w) = \bar{w}$. Since $\mu^2(w) = f|_{\mathbb{U}}\bar{f}|_{\mathbb{U}}(w) = w$, we have involutions μ and ν such that $f|_{\mathbb{U}} = \nu\mu$. Extending μ and ν to whole of $\mathbb{C}^{n,1}$ by composing the map $v \mapsto \bar{v}$ on $\mathbb{W}$, we obtain the required involutions.

If $f \in \mathrm{U}(n,1)$ is parabolic with a factor of $(x-\lambda)^3$ in its minimal polynomial, then considering the basis B for $\mathbb{C}^{n,1}$ as in Lemma 1.2.11, we have $\mathbb{C}^{n,1} = \mathbb{U} \oplus \mathbb{W}$ where $\mathbb{U} = \mathrm{Span}\{v, iw, u\}$ and $\mathbb{W}$ is positive definite. Then $f|_{\mathbb{U}} = \begin{pmatrix} 1 & i & k \\ 0 & 1 & -i \\ 0 & 0 & 1 \end{pmatrix}$

By defining μ and ν on U and then extending it to the whole space as above, f can be written as the product of two involutions.

Lemma 2.4.2. *Let $T \in \mathrm{U}(n,1)$. Then there exists $S \in \mathrm{U}(n,1)$ such that $S^2 = T$.*

Supppose $f \in \mathrm{U}(n,1)$ be elliptic and T be its matrix with respect to basis B in Lemma 1.2.6. Let $S = \begin{pmatrix} e^{i\frac{\theta_1}{2}} & 0 & \cdots & 0 \\ 0 & e^{i\frac{\theta_2}{2}} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e^{i\frac{\theta_{n+1}}{2}} \end{pmatrix}$ Then clearly $B \in \mathrm{U}(n,1)$ and $B^2 = A$.

Let $f \in \mathrm{U}(n,1)$ be hyperbolic with T as its matrix corresponding to the basis as given in Lemma 1.2.9. Then define $S = \begin{pmatrix} \sqrt{r}e^{i\frac{\theta}{2}} & 0 & \cdots & 0 \\ 0 & \frac{1}{\sqrt{r}}e^{i\frac{\theta}{2}} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e^{i\frac{\theta_{n+1}}{2}} \end{pmatrix}$ such that $S^2 = T$.

If T is parabolic with a decomposition $\mathbb{C}^{n,1} = \mathbb{U} \oplus \mathbb{W}$ where dim $\mathbb{U} = 2$ with the basis $\{v, u\}$ as explained in the proof of above theorem, then $T|_{\mathbb{W}}$ is elliptic. Therefore define $S|_{\mathbb{W}}$ by considering S as an elliptic element of $\mathrm{U}(n-1)$ such that $(S|_{\mathbb{W}})^2 = T|_{\mathbb{W}}$. Now let $S|_{\mathbb{U}} = \begin{pmatrix} 1 & \frac{k}{2} \\ 0 & 1 \end{pmatrix}$. Then as $k + \bar{k} = 0$, $S^2 = T$.

When T is parabolic with a factor of $(x-\lambda)^3$ in its minimal polynomial, dim $\mathbb{U} = 3$ where $\mathbb{C}^{n,1} = \mathbb{U} \oplus \mathbb{W}$ and $\mathbb{U} = \mathrm{Span}_{\mathbb{C}}\{v, iw, u\}$ where v, w, and u are defined as in Lemma 1.2.11.

Here, define $T|_{\mathbb{U}} = \begin{pmatrix} 1 & \frac{i}{2} & (\frac{k}{2} - \frac{1}{2}) \\ 0 & 1 & \frac{-i}{2} \\ 0 & 0 & 1 \end{pmatrix}$ and $S|_{\mathbb{W}}$ such that $(S|_{\mathbb{W}})^2 = T|_{\mathbb{W}}$ when considered as an elliptic element in $\mathrm{U}(n-2)$. Since $k + \bar{k} = 1$ here, it follows that $S^2 = T$.

Corollary 2.4.3. *Every holomorphic isometry of $\mathbf{H}^n_{\mathbb{C}}$ is a commutator in the isometry group of $\mathbf{H}^n_{\mathbb{C}}$.*

Proof Let $T \in \mathrm{U}(n, 1)$. By the above lemma, there exists $S \in \mathrm{U}(n, 1)$ such that $S = \alpha\beta$ where α and β are antiholomorphic involutions. Then $T = [\alpha, \beta]$.

Bibliography

[BM12] A. Basmajian and B. Maskit, *Space form isometries as commutators and products of involutions*. Trans. Amer. Math. Soc. 364, No. 9, 5015–5033 (2012).

[BP12] A. Basmajian and K. Puri, *Generating the Möbius group with involution conjugacy classes*. Proc. Amer. Math. Soc. 140, No. 11, 4011–4016 (2012).

[CG74] S. S. Chen and L. Greenberg, *Hyperbolic spaces*. Contributions to analysis, New York: Academic Press, 49–87 (1974).

[Cho07] Hyounggyu Choi, *Product of two involutions in complex and real hyperbolic geometry*, (preprint), summary available in Abstract of the Korean mathematics Society, no. 1 (2007).

[DM82] D. Djoković and J. Malzan, *Products of reflections in* $\mathrm{U}(p,q)$, Mem. Amer. Math. Soc. 259 (1982).

[DM79] D. Djoković and J. Malzan, *Products of reflections in the unitary group*. Proc. Amer. Math. Soc. 73, 157–162 (1979).

[Gon11] K. Gongopadhyay, *Conjugacy classes in Möbius groups*. Geom. Dedicata, Volume 151, Number 1, 245–258 (2011).

[GT15] K.Gongopadhyay and C. Thomas, *Decomposition of Complex Hyperbolic Isometries by Involutions*, preprint available on arxiv.org/1503.05660 (2015)

[GP13] K. Gongopadhyay and J.R. Parker, *Reversible complex hyperbolic isometries*, Linear Algebra and its Applications. Volume 438, Issue 6, 2728 –2739 (2013).

[Par12] J. R. Parker, *Traces in complex hyperbolic geometry*, Geometry, Topology and Dynamics of Character Varieties, Lecture Notes Series 23, Institute for Mathematical Sciences, National University of Singapore, 191-245. World Scientific Publishing Co, 2012.

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