Instabilities in sedimentation at Low Reynolds numbers

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Cerficate of Examination

This is to certify that the dissertation titled ”**Instabilities in sedimentation at Low Reynolds number**” submitted by Mr. RAHUL CHAJWA (Reg No.MS10038) for the partial fulfillment of BS-MS dual degree programme of the Institute, has been examined by the thesis committee duly appointed by the Institute. The committee finds the work done by the candidate satisfactory and recommends the report to be accepted.

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Declaration

I hereby declare that the work presented in this dissertation entitled ”Instabilities in sedimentation at Low Reynolds numbers” is the result of the investigations carried out by me under the guidance of Dr. Abhishek Chaudhuri at the Indian Institute of Science Education, Mohali, jointly under the supervision of Prof. Sriram Ramaswamy and Prof. Narayanan Menon at TIFR Centre for Interdisciplinary Sciences, Hyderabad.

This work has not been submitted in part or in full for a degree, or diploma to other University or Institute. Keeping with the general practice, every effort is made to indicate the contributions of others, with due acknowledgment of collaborative research and discussions. This thesis is a bonafide record of original work done by me and all sources listed within have been detailed in the bibliography.

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Dedicated to my parents,
*w*hose love is the roots of my dreams

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Synopsis

In this thesis we study the sedimentation of particles in a Stokesian fluid, that is, in the limit where viscosity dominates and inertia is ignored. This is a classical n-body problem with long-ranged hydrodynamic interactions which is very difficult to solve. If an analytical form of the interaction between two particles is known, one can do pairwise addition of forces and torques on a particles due to the nearest neighbours and arrive at the discrete form of the equations of motion. But usually it is not at all easy to get the analytical form of interaction by solving the Stokes equation for a particle of general shape.

Our interest is to study the collective behavior of anisotropic sedimenting particles. Taking a different approach to this problem we build up a field theory for the displacement and orientation fields of a lattice of sedimenting particles and construct the mobility for the lattice from general symmetry arguments in the continuum limit. We do this for an array of spherical particles (as done by Lahiri and Ramaswamy, PRL 79 1150 (1997)), apolar axisymmetric particles (disks, rods, ellipsoids or any surface of revolution with up-down symmetry) and polar axisymmetric particles (cones, hemispheres or any surface of revolution with up-down asymmetry). We go back and forth from discrete to continuum version of the equations to get maximum knowledge about the interactions between the particles. In this investigation we also do experiments with disks shaped particles and observe various intriguing dynamics of a pair of disks.

In chapter 1 we give a brief introduction to the hydrodynamic approach for sedimentation and discuss Crowley instability [1].

In chapter 2 we present the continuum dynamical model for the lattice of sedimenting spherical particles and see its consistency with the hydrodynamic results. This is done by defining a displacement field of the lattice ($\vec{u}$) and writing its equations of motion from general symmetry arguments. Lahiri and Ramaswamy write a dispersion relation which incorporates Crowleys instability as a special case. We then study a more complicated problem by adding an orientation degree of freedom to the sedimenting particle. We observe the dynamics of single disks and pair of disks (see chapter 3) and find periodic behavior for a pair of disk for a large set of initial configurations. A detailed study is needed for this.

Once an additional degree of freedom is added to the particles, an obvious question which arises is how the collective behavior of the lattice of particles changes.

We find that the orientation degree of freedom competes with clumping and in certain initial configurations can even lead to lattice dilation and orientation waves.

In chapter 4 of the thesis we construct a continuum dynamical model for an array of apolar axisymmetric particles like disk, rods etc. by defining the orientation field $\vec{K}$, in addition to the displacement field $\vec{u}$. We construct the equation of motion from symmetry arguments and then find the linear dispersion relation. The equation for the orientation variable tells us that there is no rotation of particles if the gradient of the displacement and orientation field is zero. This is ultimately a consequence of the time-reversal symmetry of the system. For array of disks falling one above the other we find the possibility of orientation waves of the type proposed by Wakiya [2]. At the end of this chapter a consistency of the continuum equations with the hydrodynamic solution can be appreciated. One can relax the $K \longrightarrow -K$ symmetry in the system and construct the mobility for an array of polar axisymmetric particles like cones, hemispheres etc. in the continuum limit. We do this in chapter 5 and show the possibility of rotation of cones in a lattice even for the case when the gradient of both the displacement and orientation field is zero. This rotation make the orientation vector asymptotically align with the direction of gravity. A plausible form of this rotation is found just by analyzing the symmetry of the system.

All the accounting required for the construction of mobility tensor for various parts of this thesis is given in the Appendices.

Contents

1 Introduction **1**

1.1 Introduction 1

1.2 Mobility: hydrodynamic approach 2

1.3 Crowley instability 3

2 Lahiri Ramaswamy model: the uses of symmetry **6**

2.1 Continuum Dynamical model 6

2.1.1 Lattice of spherical particles: 6

3 Sedimentation of disks **9**

3.1 Sedimenting disks 9

3.2 Experimental Setup 11

3.3 Pair oscillations 12

3.4 A bigger configuration space 16

4 Lattice of apolar axisymmetric particles **21**

4.1 Lattice of apolar axisymmetric particles (disks, rods etc.): 21

4.2 **Linear dispersion relation:** 25

4.3 Experiments with array of disks 29

4.3.1 Horizontal array of disks: 29

4.3.2 Vertical array of disks and Lattice dilation 30

4.4 Wakiya Screw wave 32

4.4.1 Equations of motion 34

4.4.2 Continuum equations 36

5 Lattice of polar axisymmetric particles **38**

5.1 Polar Particle 38

5.2 Lattice of Polar axisymmetric particles (cones, hemisphere etc.) . . . 38

5.3 Linear dispersion relation for lattice of polar axisymmetric particles, (cones,hemispheres etc.) 41

Appendix A Constructing mobility tensor for spherical particles 44

Appendix B Constructing mobility tensor for spatial part for an array of apolar axisymmetric particles 46

Appendix C Constructing mobility tensor for orientation part for an array of apolar axisymmetric particles 48

Appendix D Constructing mobility tensor for spatial part for an array of polar axisymmetric particle 50

Appendix E Constructing mobility for orientation part for an array of polar axisymmetric particles 53

Chapter 1

Introduction

1.1 Introduction

When a collection of particles are falling through an ideally unbounded viscous medium a non-equilibrium statistical steady state is reached. These particles are slowed down by the viscosity of the medium, which sounds simple, but what this means in practice is that each particle's momentum is transferred to the fluid and thus affects the other particles, which makes this system difficult to study [3]. Even for spherical particles the problem can be very difficult [4]. It is governed by the Stokes equation, which is linear, but each particle is a moving inner boundary, and hence the problem is in effect not linear.

We take an entirely different approach and see the lattice of particles in a continuum limit where we can talk about displacement and orientation field. In long wavelength limit these fields behaves nicely and we can even talk about its derivatives. We avoid the difficulty of solving the stokes equation and try to write a general expression for the mobility tensor based on symmetry of the system.

Though finding a dispersion relation in this case is in general not an easy task, but we can aim at solving it assuming nearest neighbour interaction. For a sedimenting lattice, finding the dispersion relation has not only theoretical but experimental difficulties as well. For small wave numbers, the dispersion relation is known for spherical particles [5]. This work aims at extending this calculations when paricles have an additional dynamical degree of freedom which is orientation. But before we do all this lets understand the hydrodynamic approach to this problem. We proceed in this problem by knowing the interactions between pair of particles. Once this interaction is known we can solve for an array of such particles assuming nearest neighbour interaction.

1.2 Mobility: hydrodynamic approach

When a particle is falling under gravity in a Newtonian fluid it creates a velocity field around it. A precise description of this velocity field is given by Navier Stokes equation. In dimensionless form it reads as:

$$Sl.Re\frac{\partial \vec{u}}{\partial t} + Reu.\nabla u = -\nabla p + \nabla^2\vec{u} + \frac{Re}{Fr}\hat{g}$$

Where $Re = \frac{Ua\rho_f}{\mu}$ is the Reynolds number , $Sl = \frac{a}{U\tau}$ is the Strouhal number, $Fr = \frac{U^2}{ga}$ is the Froude number and $\hat{g}$ denotes the direction of gravity. Here, 'a' denotes a length scale characteristic of the sedimenting structure (in our case its the particle size), U is the velocity scale, $\tau = \frac{a}{U}$ is the characteristic time scale, ρ_f and μ are the density and viscosity respectively of the fluid. We are interested in a case when Reynolds no. is very small ($\ll 1$ i.e the system is inertia less)in which case we obtain unsteady Stokes equation:

$$\nabla^2\vec{u} - \nabla p = Sl.Re\frac{\partial \vec{u}}{\partial t}$$

On long timescales of observation we can drop $\frac{\partial \vec{u}}{\partial t}$ and we get the steady Stokes equation:

$$\nabla^2\vec{u} - \nabla p = 0$$

Given the pressure field we can solve for the velocity field. The level of difficulty involved in solving is determined by the kind of boundary conditions and symmetry of the system. From the velocity and pressure fields we can calculate the stress tensor: $\sigma_{ij} = -p\delta_{ij} + \mu\left(\frac{\partial u_i}{\partial u_j} + \frac{\partial u_j}{\partial u_i}\right)$
Integrating the stress tensor over the particle's surface gives us the net force on the particle. [6]

In the Stokes limit velocity depends linearly on force and we can write $Force = Propulsion \times Velocity$ or $Velocity = Mobility \times Force$, where mobility is the inverse of propulsion (mobility and propulsion are tensors). The well-known Stokes formula, $F = 6\pi\mu aU$ is one such relationship. Such formula for stokes drag on axisymmetric bodies have been studied well. [7]

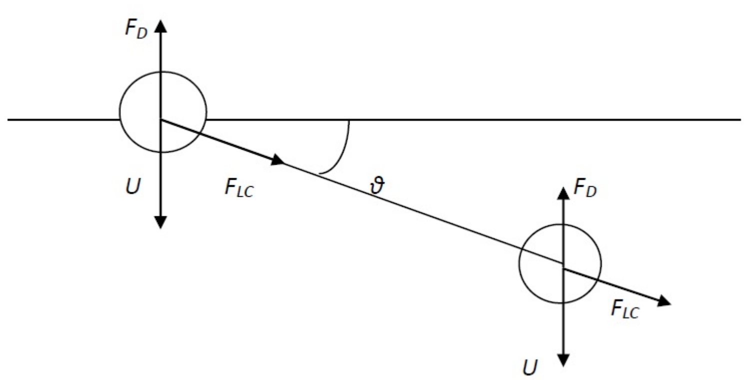
So, this was one of the conventional recipe to find the mobility for a given particle. A general form for the mobility of a particle of arbitrary shape is known [8]. Much less is known for a pair of particles falling in a fluid.

1.3 Crowley instability

J.M. Crowley did theoretical and experimental investigation of the sedimentation of one and two dimensional array of spherical particles at low Reynolds number [1]. He studied this system by perturbing the lattice and observing how the perturbation grows as the lattice sediments. It was found that the lattice of particles is always unstable and any initial perturbation leads to clumping. We have reproduced Crowley's result for a one dimensional lattice of spherical particles and have observed clumping instability. To put it simply there are two things happening (assuming only nearest neighbour interaction):

1) **line of centre force**: when two spherical particles of diameter 'a' and separation 'd' are sedimenting at an angle, it leads to gliding with some horizontal velocity component. The formula for this force directed along the line joining the particle's centre (hence called line of centre force) is:
$F_{LC} = 6\pi\mu a U \left\{\frac{3}{4}\frac{a}{d}\right\} \sin\theta$; here θ is the angle between the line joining the particles and the horizontal axis.

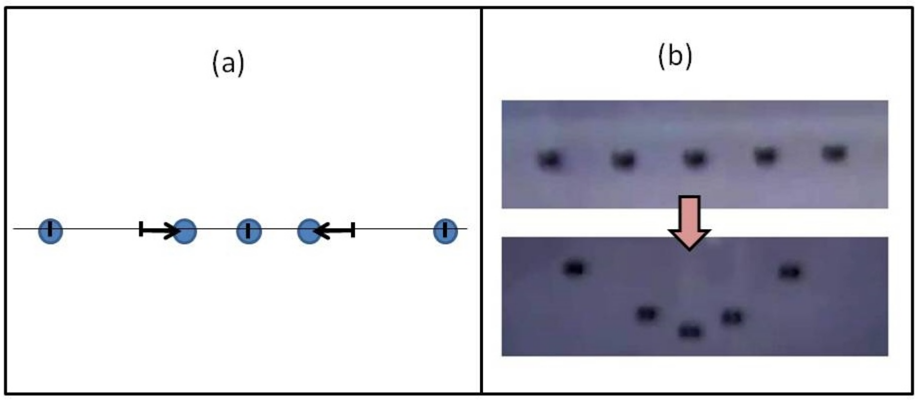
Figure 1.1: Line of Centre force



2) **mutual drag reduction:** Particles fall faster in the presence of neighbouring particles. The formula for stokes drag for a particle in the presence of another looks like:
$F_D = 6\pi\mu a U \left\{1 - \frac{3}{4}\frac{a}{d}\right\}$; where 'a' is the particle size and 'd' is the inter particle distance. This approximation is valid when $\frac{a}{d} \ll 1$

From the above pair interactions we can solve for the dynamics of the lattice by pairwise addition of forces assuming nearest neighbour interaction. We have experimentally observed clumping instability of one dimensional array by dropping an array of steel ball bearings in silicone oil ($Re < 10^{-4}$). The figure below shows the clumping with largest wave number perturbation.

Figure 1.2: (b) shows experimentally observed clumping instability with initial horizontal perturbation of the kind given in (a)



Defining η_n and ξ_n to be the vertical and horizontal perturbation respectively of the n^{th} particle in an array. These are perturbations around the equilibrium positions of the particles in a sedimenting lattice. For small perturbations it is possible to write the perturbation growth equations in the following form:

$$\dot{\xi_n} = \alpha(\eta_{n+1} - \eta_{n-1}) \tag{1.1}$$

$$\dot{\eta_n} = -\alpha(\xi_{n+1} - \xi_{n-1}) \tag{1.2}$$

where $\alpha = \frac{3}{4}\frac{aU}{d^2}$

By solving the above equations in Fourier space one can show that an array of spherical partciles is unstable. Simulating the coupled equations 1.1 and 1.2 using direct numerical integration shows clumping instability as depicted below.

Similar simulation was done for two dimensional array of spherical particles

Figure 1.3: Clumping instability in 1D array with initial horizontal periodic perturbation

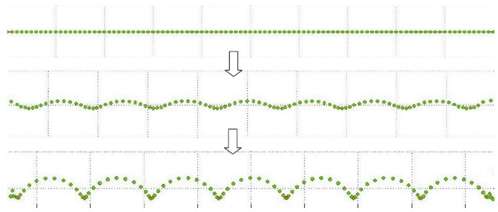
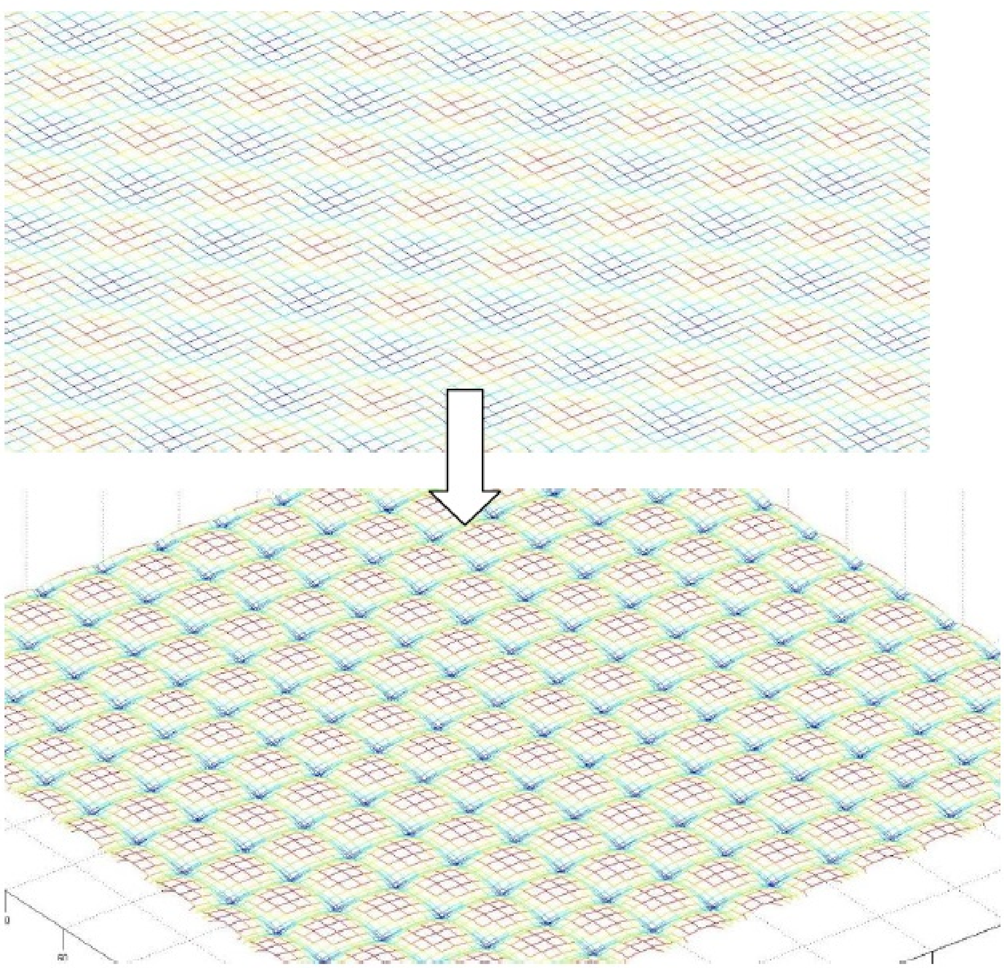


Figure 1.4: Clumping instability in 2D array with initial horizontal periodic perturbation



Chapter 2

Lahiri Ramaswamy model: the uses of symmetry

2.1 Continuum Dynamical model

Getting a bit more abstract with the problem of sedimentation we pose the problem of instability of the lattice in an entirely different way. We say that we have a displacement field which is defined by perturbations from the stationary lattice and we would like to ask, how this field changes as the lattice sediments? Idea is to build the equations of motion for the perturbation based on the symmetry of the system. Let $\vec{u}(\vec{r})$ be the displacement field where r is the position vector. In stokes regime we can write:

$$\dot{u} = MOBILITY \times FORCE \tag{2.1}$$

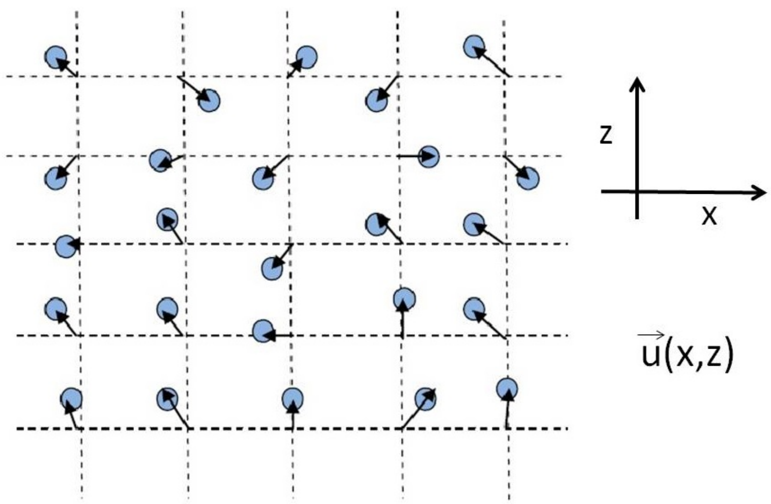
$$\dot{u} = \eta(\nabla u)(K\nabla\nabla u + F + f) \tag{2.2}$$

Here η is the mobility tensor, K is the elastic tensor which is zero in our case, F is the external force and f is the noise term. The mobility cannot depend on the displacement field itself given the translation symmetry of the system. So, we are assuming that the mobility depends on the gradient of the displacement field.

2.1.1 Lattice of spherical particles:

When the particles are spherical we can build the mobility tensors from $\nabla_i u_j$'s and F_i's. Another knowledge that goes in is the time reversal symmetry of the system. Reversing all forces and velocities leaves the Stokes equation invariant. We ensure this by constructing the mobility tensor η_{ij} which is even in F, so when we contract it with F_j we get velocity which is odd in F. A detailed construction is given in Appendix A.

Figure 2.1: Displacement field of a sedimenting lattice



Equations of motion:

As we discussed before, in stokes regime we can write
$\dot{u}_i = \eta_{ij} F^j$
From the mobility tensor which we constructed from the symmetry of the system we can write the above equation in the following two dimensional form which was originally done by Lahiri and Ramaswamy [9].

$$\frac{\partial u_x}{\partial t} = \lambda_1 \frac{\partial u_x}{\partial z} + \lambda_2 \frac{\partial u_z}{\partial x} + O(\nabla\nabla u) + O(\nabla u \nabla u) + f_x \tag{2.3}$$
$$\frac{\partial u_z}{\partial t} = \lambda_3 \frac{\partial u_x}{\partial x} + \lambda_4 \frac{\partial u_z}{\partial z} + O(\nabla\nabla u) + O(\nabla u \nabla u) + f_z \tag{2.4}$$

Where λ's are the coefficients that depend on the external force.
If we drop the z derivatives in these equations we simply get the continuous version of the discrete equations given by Crowley for a one dimensional array of sedimenting spherical particles (1.1) and (1.2).

In a three dimensional lattice we can consider a subspace perpendicular to the direction of external driving force which in our case is gravity along $\hat{z}$. This subspace perpendicular to $\hat{z}$ is assumed to be isotropic which is taken into account by replacing of F_i with δ_{iz} . Considering this symmetry the above result can be easily extended to three dimensions and it looks like.

$$\frac{\partial u_\perp}{\partial t} = \lambda_1 \frac{\partial u_\perp}{\partial z} + \lambda_2 \nabla_\perp u_z + O(\nabla\nabla u) + O(\nabla u \nabla u) + f_x \tag{2.5}$$

$$\frac{\partial u_z}{\partial t} = \lambda_3 \nabla_\perp . u_\perp + \lambda_4 \frac{\partial u_z}{\partial z} + O(\nabla\nabla u) + O(\nabla u \nabla u) + f_z \tag{2.6}$$

Here $u_\perp = (u_x, u_y)$ and $\nabla_\perp$ is the gradient along the perpendicular subspace.

Dispersion relation:

Ignoring the higher orders in $\bigtriangledown u$ the dispersion relation implied by equation (2.3) and (2.4) can be found to be:

$\omega = \frac{-1}{2}\left[(\lambda_1 + \lambda_4)\, k_z \pm \sqrt{(\lambda_1 - \lambda_4)^2 k_z^2 + 4\lambda_2\lambda_3 k_\perp^2}\right]$

This was done by Lahiri, Ramaswamy and Barma [3] [5]. For different values of λ we have the following two possibilities:

1. *Linearly stable case - kinematic waves* : It happens $\lambda_2\lambda_3 > 0$, for $\lambda_2\lambda_3 < 0$ it can still be wavelike if $k_\perp = 0$. These modes are the generalization to the kinematic waves in the moving flux lattice in superconductors [10].

2. *Linearly unstable case - clumping* : The growing mode $\omega \propto -ik$ for $k_z \ll k_\perp$. In case $\lambda_2\lambda_3 < 0$ for wave vectors pointing outwards of the cone pointing along z axis the system is linearly unstable with growth rate linear in wave number. It is the generalization to the Cowley instability of sedimenting lattice of spherical particles.

Chapter 3

Sedimentation of disks

3.1 Sedimenting disks

With the knowledge of Crowley's clumping instability for spherical particles we proceed further, curious to know what will happen if we add another dynamical degree of freedom to the sedimenting particles? How does the collective behaviour of the sedimenting lattice depends on the *shape* of the particles? As far as shape is concerned the additional degrees of freedom could be: orientation, polarity and chirality. However, to be simple we chose to add only orientation degree of freedom to the particles and continued our investigation. But as we proceed further we will see that even such a simple addition makes this system behave in an intriguing fashion. Even though we study the sedimentation of disks, an equally sensible choice of particle could be any axisymmetric particle like rods, ellipsoids etc.

Initial goal was to know the interaction between pair of disks and then assuming nearest neighbour interaction solving for an array by pairwise addition of forces and torques, similar to what Crowley did for spherical particles.

Before trying to approach this problem analytically we studied the problem of sedimenting disks experimentally and found the following:
1) horizontally falling disk falls slower as compared to when released vertically.
2) a single disk falling at an angle glide similar to the two spherical particles at an angle.
3) a disk sedimenting in the presence of another disk rotates.

Figure 3.1: A single coin initially released at an angle with the horizontal axis. The graph below depicts the x-velocity of the coin which is constant as it falls vs the initial angle of release.

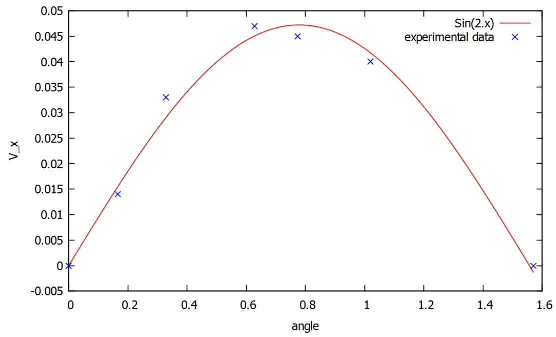
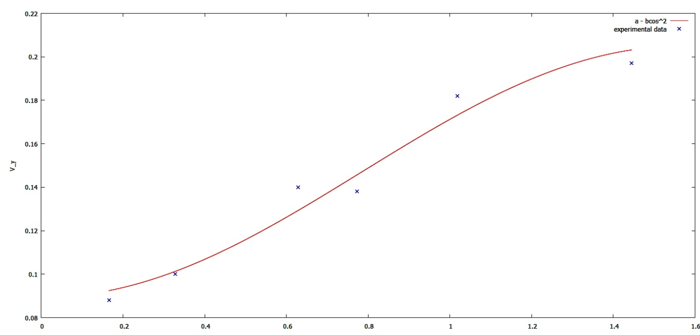


Figure 3.2: Vertical velocity Vs angle of release for a single disk.



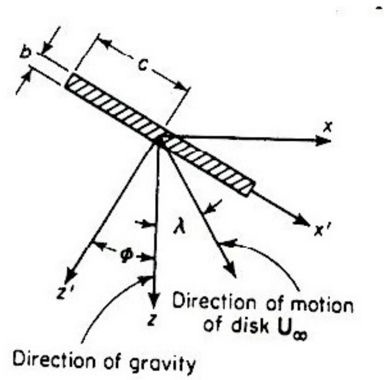
An analytic expression for the horizontal and vertical velocity of isolated settling disk is known in the following form: [2]

$$U_x = \frac{\pi c b g \Delta\rho}{64\mu} sin 2\Phi \tag{3.1}$$

$$U_z = \frac{\pi c b g \Delta\rho}{64\mu} (5 - cos 2\Phi) \tag{3.2}$$

The notations in the above expressions should be clear from Figure [3.3]. We compare the experimental results (Figure [3.1] and [3.2]) with the analytic ones (equations (3.1) and (3.2)) and find a good agreement.

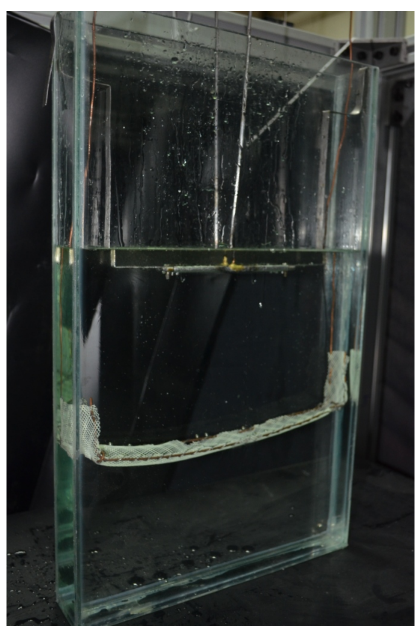
Figure 3.3: Angle convention



3.2 Experimental Setup

The experiments are aimed to video record the particles as they sediment. Videos are then analysed using ImageJ software. We built a container (height = 50cm , length = 30 cm , width = 5 cm).

Figure 3.4: Container for experiments



The particles are thin disks punched out of 1 mm thick aluminium sheet. The diameter of the disk is 1.2 cm. We are using silicone oil of viscosity 60,000 centi-stokes (Crowley used Venice Turpentine). Silicone is a good option for such experiments because it is transparent, non toxic, available in all ranges of viscosity and most importantly its viscosity is almost independent of temperature (at room temperatures). The expected Reynolds no. for given system (disk falling horizontally) is 10^{-4}. The most crucial part is the release mechanism. We wanted to release all

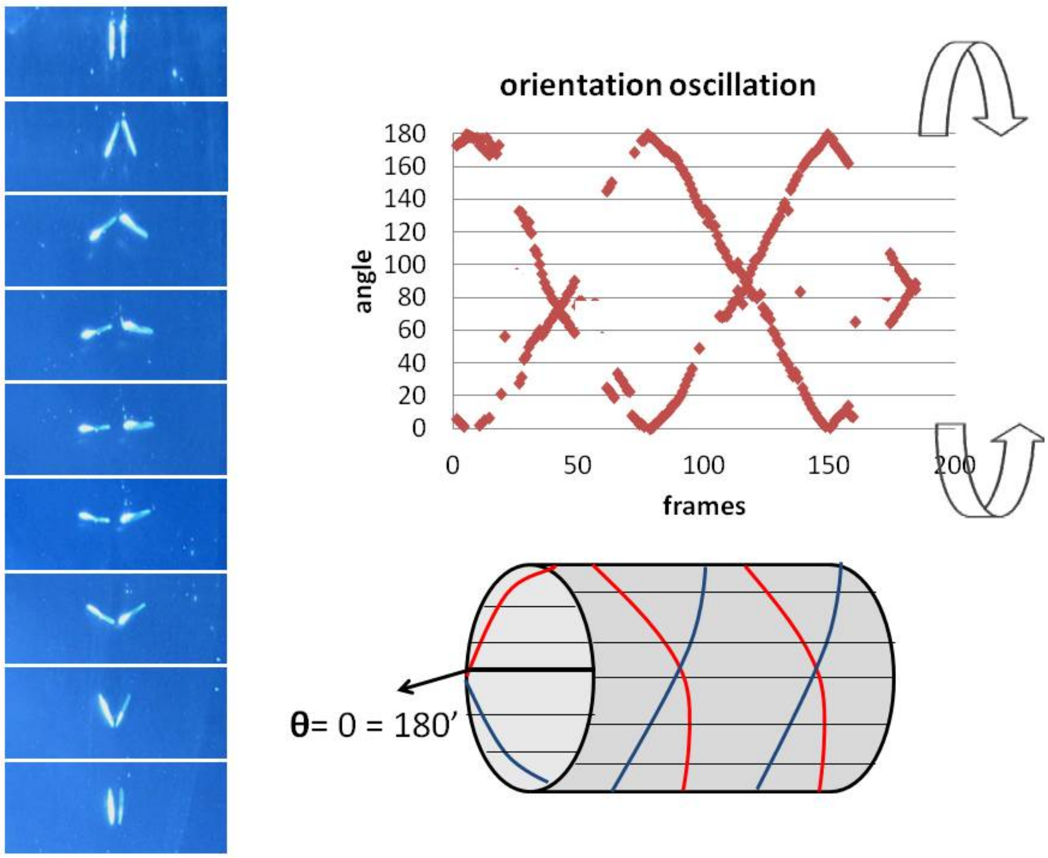
Figure 3.5: disks

the particles at the same time. In various such experiments people have used different kinds of release mechanism. In [12] they simply push the particles down the surface. Following a similar approach we release the particles manually using forceps. Since the fluid is very viscous it is a working technique for release. We did the same thing when we were studying array of spherical particles. Construction of a mechanism that would release an array of disks it in an arbitrary orientation is in progress.

3.3 Pair oscillations

In the presence of the neighbour the disks rotates. To our surprise we saw periodic oscillations of the pair. Sadly this was not new: it was reported in 2006 by [12]. We have substantially extended their study by exploring the dynamics of a pair of disks in a much bigger region of the configuration space. A simplest possible initial condition is to release both the disks at the same height with vertical or horizontal initial orientation. The corresponding periodic trajectory has reflection symmetry in the real space. But there is a bigger set of initial conditions in the configuration space which lead to periodic trajectories. The orientation vector of the discs are assumed to be confined to the x-z plane during the motion.
The following points need to be noted about the pair oscillation :
1) The disks rotates faster when in vertical orientation than horizontal orientation.
2) The disks fall faster when in vertical orientation than horizontal.
The disks undergo periodic oscillations with a characteristic wavelength and time period which depends on the initial separation of the disks which we call 'd'. We

Figure 3.6: Periodic orbits: orientation is plotted such that if θ exceeds π it begins with a new cycle starting from $\theta = 0$. It may seem that the orientation has a turnover and it oscillates in a range of θ but it is actually monotonically increasing for one disc and decreasing for the other. The graph makes more sense if we roll the plot into a cylinder with horizontal axis as is shown schematically.



dropped the disks in vertical orientation with some initial separation ’d’ much greater than the thickness ’b’ of the disks.

We observe that the Time period and wavelength of the oscillation depends strongly on the initial separation which is depicted in the plots given below. In the plots the separation is given in terms of disk diameter which is 1.3 cm.

Figure 3.7: Time period Vs initial separation from experiments

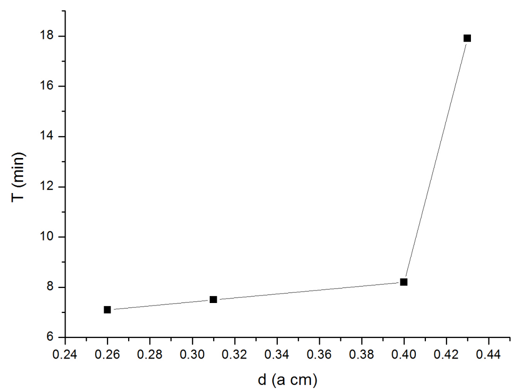
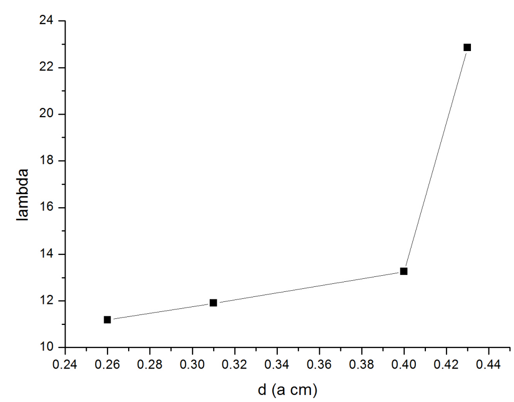


Figure 3.8: Wavelength Vs initial separation from experiments



As an attempt to build a simplistic mathematical model along with some vague guidance from Wakiya's calculation [2] for two disks, we assume that the angular velocity of disks goes as $\frac{1}{d^n}$. From equations (3.1) and (3.2) we know how the velocity of isolated disk depends on the orientation. Our familiarity with Crowley instability tell us that the velocity should also depend on the separation between the particles because of the reduced drag due to the presence of neighbour.

We model this system in terms of the following equations:

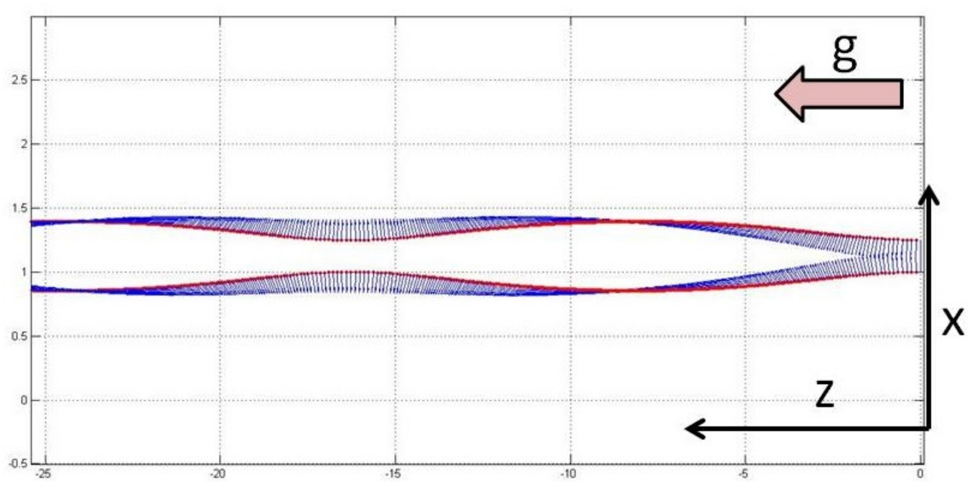
$$\frac{dx_1}{dt} = -\frac{dx_2}{dt} = \frac{\pi cbg\Delta\rho}{64\mu} sin2\Phi_1 \tag{3.3}$$

$$\frac{dz_1}{dt} = \frac{dz_2}{dt} = \frac{\pi cbg\Delta\rho}{64\mu}(5 - cos2\Phi_1)\{1 - \alpha\frac{a}{d}\} \tag{3.4}$$

$$\frac{d\Phi_1}{dt} = -\frac{d\Phi_2}{dt} = \frac{\beta}{d^n} \tag{3.5}$$

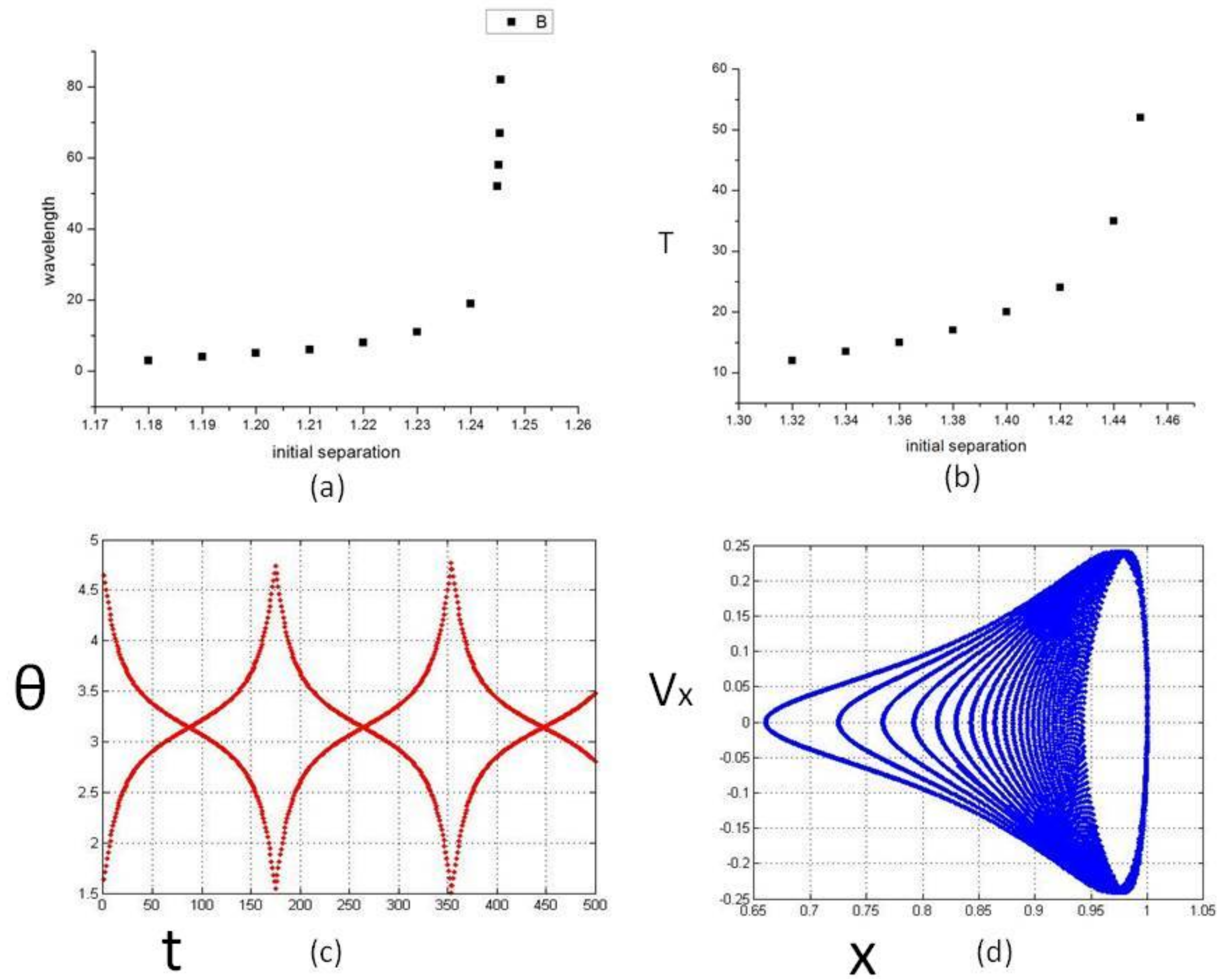
for different values of α and β we did the numerical simulations and found periodic oscillations. These simulations were closest to experiments for n = 4. Such a dependence of angular velocity on separations will also be encountered when we talk about Wakia screw wave.

Figure 3.9: Simulating the equations (3.3) , (3.4) and (3.5) using direct numerical integration. Here blue arrows are the normal vector to the disk from which one can see that both disks were released vertically and then undergo oscillations



Though it is a very simplistic model, it not only captures the qualitative features of the periodic oscillations we observe but also the dependence of wavelength and time period on the initial separation.

Figure 3.10: Model result: (a) initial separation Vs Wavelength graph, (b) initial separation Vs Time period, (c)Orientation as a function of time (d) Phase space trajectories (X - Vx) in the COM frame with initial separations varying by 0.01 units

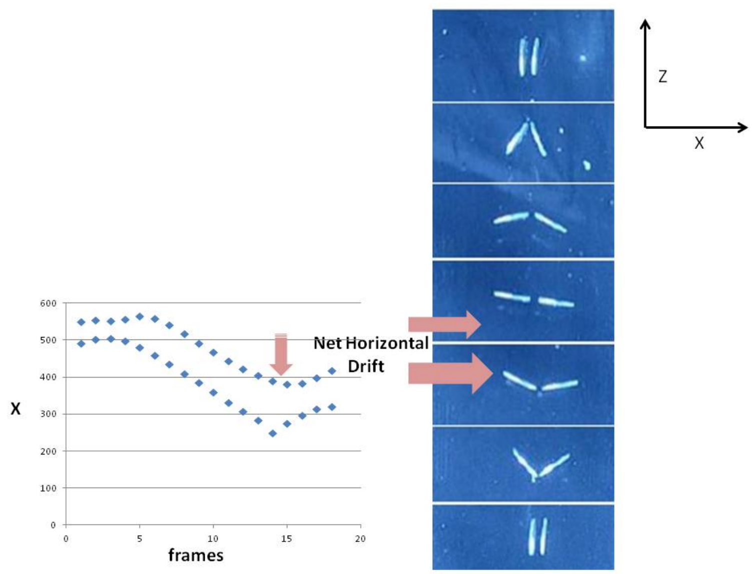


3.4 A bigger configuration space

Experimenting with disks we found that the periodic trajectory of the pair of disks is exhibited by a wide range of initial configurations. We attempt to classify the set of these initial configurations which lead to periodic motion. If the initial condition is such that the normal vector to the disks is in the x-z plane then throughout the dynamics the normal vectors remains in that plane. So, we are not covering the whole of the configuration space but only the subset in which the normal to the disks lies in x-z plane. It is still amazing that such a wide range of initial configurations lead to a closed trajectories in the configuration space.

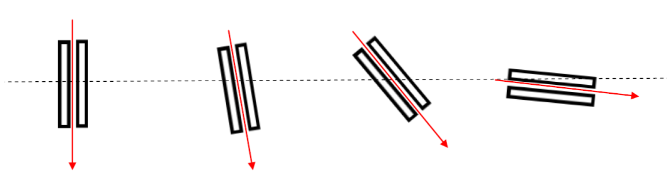
To start with we perturbed the vertical initial configuration by having it at an angle with the direction of gravity. For a small initial tilt of the pair the motion is still periodic. In addition to this there is a net horizontal drift of the pair.

Figure 3.11: A small initial tilt



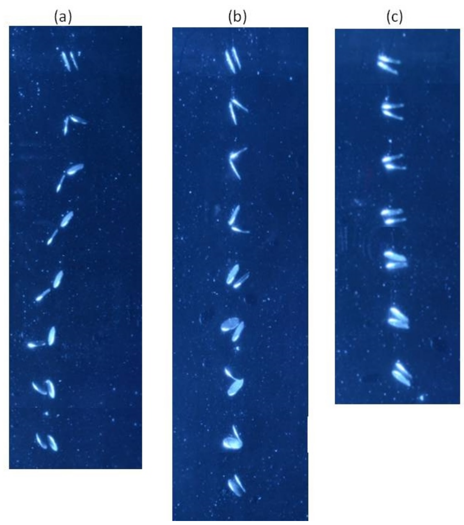
This tilt angle can be varied from zero to $\pi/2$. Beyond a certain angle of initial tilt the orientation of the disks oscillates between a range of angles and doesn't undergo complete cycles (we call this flipping transition). We find periodic motion

Figure 3.12: Varying the initial tilt angle



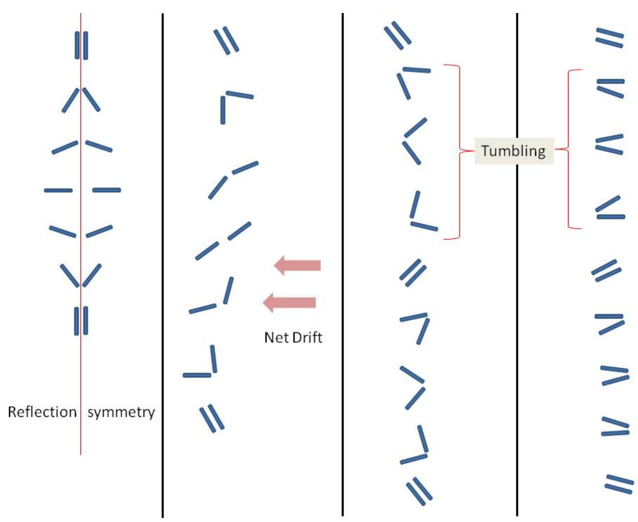
for all the angles but beyond a certain angle the qualitative feature of the orbit changes dramatically. Also after the flipping transition there is no net horizontal drift.

Figure 3.13: flipping transition: (a) tilted initial configuration which retains the qualitative feature of vertical oscillation with net horizontal drift. (b) beyond a critical angle of tilt the disks oscillates in a range of angles and there is no net horizontal drift. (c) this feature sustains for angle very close to horizontal



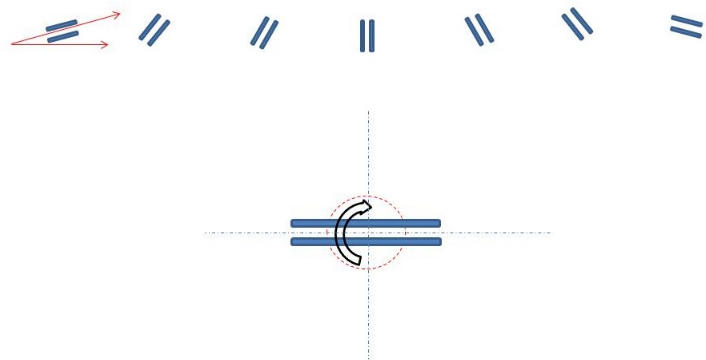
A schematic representation of the above experimental result is given below:

Figure 3.14: increasing the initial tilt angle from left to right



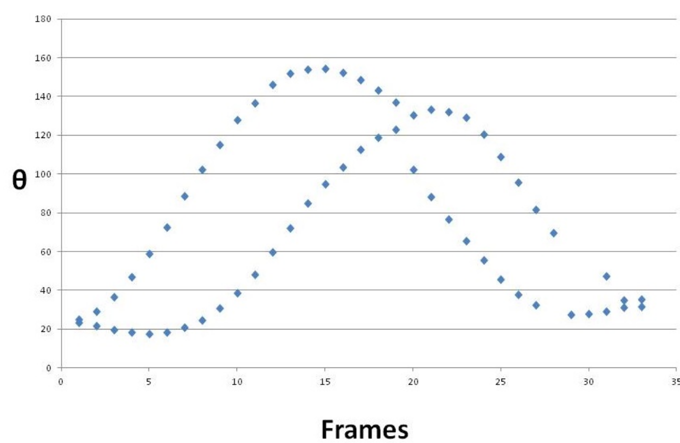
All the following initial configurations lead to closed trajectories in the configuration space which defines the various set of configurations, P_{θ_1},P_{θ_2},P_{θ_3}, . . .

Figure 3.15: Simple way of classifying disjoint sets of configuration



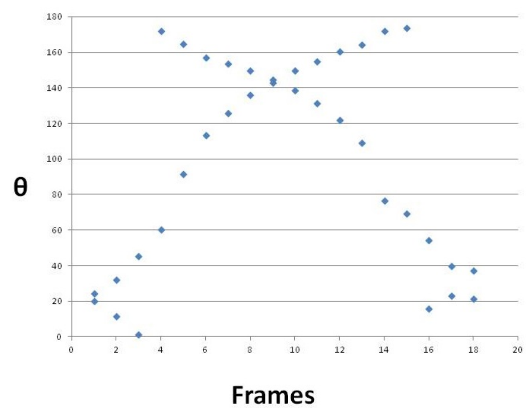
The set P_{θ_n} consists of points in the closed trajectory with the initial pair tilted at angle θ_n. Amazing part of the story is that all these sets are disjoint.
As mentioned earlier after the flipping transition the disks oscillates in a range of values $0 < \theta < \pi$. This is depicted in the experimental plot given below (Fig. [3.16]):

Figure 3.16: θ window



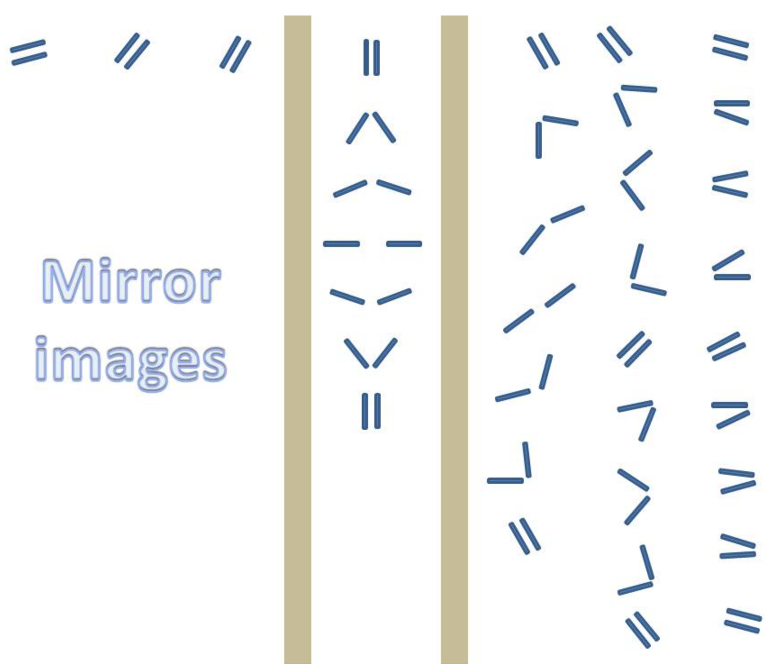
Contrasting this with the case of small tilt oscillation in which each disk undergoes complete orientation cycle (Fig. [3.17]):

Figure 3.17: Complete orientation cycles in case of small tilt



Since the system is inertia less, any point in the trajectory can be thought of as the initial condition. In that sense any initial configuration which is close to any of the configurations depicted in figure [3.18] leads to periodic motion. Getting periodic trajectories for such a big region in the configuration space as initial condition is indeed remarkable.

Figure 3.18: Periodic initial configurations



Chapter 4

Lattice of apolar axisymmetric particles

4.1 Lattice of apolar axisymmetric particles (disks, rods etc.):

The idea is to extent the results of a sedimenting lattice of spherical particles to a case when the particles are axisymmetric (disks , rods etc.). We now have an additional dynamical variable viz. the orientation of the particles. The mobility tensor (Equation 2.1) is now expected to be a function of this dynamical variable. Lets define an orientation field by unit vectors, $\vec{K}(\vec{r})$ pointing along the direction of the symmetry axis of the particle positioned at $\vec{r}$. In case of disk it is just a unit normal at the centre of the disk. The system is invariant when the direction of this unit vector is reversed since it is still pointing along the symmetry axis and is the same system. Given this when we construct the mobility tensor from ∇u, F_i and K_i , we have to make sure that the mobility is even in K when writing the equation of motion of the horizontal and vertical perturbation in position of the particles. We can do that by imposing that the K terms always appear in pairs when constructing the mobility tensor. Following the same recipe as we did for spherical particles we construct the mobility tensor for the spatial perturbation. The detailed construction is given in Appendix B.

The equations of motion: From the above mobility tensor we can build the equations of motion for the horizontal and vertical perturbation. A physical picture could be thought of as building the equations of motion for an array of disks sedimenting at low Reynolds no. (though the equations are more general).

$$\frac{\partial u_i}{\partial t} = \eta^u(\nabla\vec{u},\vec{K})_{ij}F^j$$
$$= \left[\eta^u{}_{a_{ij}} + \eta^u{}_{b_{ij}} + \eta^u{}_{c_{ij}} + \eta^u{}_{d_{ij}}\right] F^j$$

Using **I , II , III and IV** from Appendix B it is straightforward to arrive at the following equation of motion by substituting δ_{iz} in place of F_i and discarding all the z derivatives since we are essentially looking at only one dimensional array sedimenting in x-z plane.

$$\dot{u_x} = \left\{ A_1 + \left(A_2 + C_1 K_x^2 + C_2 K_z^2\right) \frac{\partial u_z}{\partial x} + \left(B_1 + D_1 \frac{\partial u_x}{\partial x}\right) K_x K_z \right\} \times \sum_{n=0}^{\infty} \alpha_n (K_z)^{2n} \quad \textbf{(4.1)}$$

$$\dot{u_z} = \left\{ \acute{A_1} + \left(\acute{A_2} + \acute{C_1} K_x^2 + \acute{C_2} K_z^2\right) \frac{\partial u_x}{\partial x} + \acute{B_1} \frac{\partial u_z}{\partial x} K_x K_z + + \acute{D_1} {K_z}^2 \right\} \times \sum_{n=0}^{\infty} \alpha_n (K_z)^{2n} \quad \textbf{(4.2)}$$

All the above constants depend on the external force. By assuming that the mobility is not having higher orders in $F(\leq 4)$ we can take only the first term in the right summation (n=0), which is just a constant. The above equations then becomes:

$$\dot{u_x} = A_1 + \left(A_2 + C_1 K_x^2 + C_2 K_z^2\right) \frac{\partial u_z}{\partial x} + \left(B_1 + D_1 \frac{\partial u_x}{\partial x}\right) K_x K_z \quad \textbf{(4.3)}$$

$$\dot{u_z} = \acute{A_1} + \left(\acute{A_2} + \acute{C_1} K_x^2 + \acute{C_2} K_z^2\right) \frac{\partial u_x}{\partial x} + \acute{B_1} \frac{\partial u_z}{\partial x} K_x K_z + \acute{D_1} K_z^2 \quad \textbf{(4.4)}$$

One can see that the above equations has the following symmetry:
when $x \longrightarrow -x$, $u_x \longrightarrow -u_x$ and $K_x \longrightarrow -K_x$ which is similar to saying that the perpendicular subspace is isotropic. The knowledge of Crowley instability and the experimental plot (figure [3.1] and [3.2]) will makes the above equations quite intuitive.

From the experimental plot and analytic results we know that for an isolated settling disk $u_x \propto \alpha \sin 2\theta$ which implies that the horizontally and vertically released disks have zero x velocity. It is non zero when $0 \leq \theta \leq \frac{\pi}{2}$. Since K is a unit vector, we can write $\sin 2\theta = 2K_x K_z$ where $K_x = \sin\theta$ and $K_z = \cos\theta$ (figure [4.1]). So we shouldn't be surprised to see a $K_x K_z$ term in the above equations.

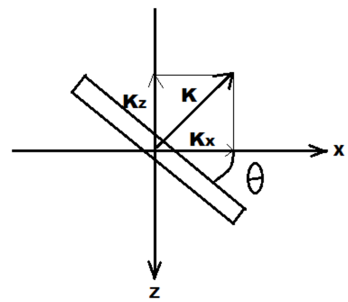
We would like to make one more modification in the above equations by replacing K_x^2 by $1 - K_z^2$.

$$\dot{u_x} = A_1 + \left(A_2 + C_1 K_z^2\right) \tfrac{\partial u_z}{\partial x} + \left(B_1 + D_1 \tfrac{\partial u_x}{\partial x}\right) K_x K_z$$

$$\dot{u_z} = \acute{A_1} + \left(\acute{A_2} + \acute{C_1} K_z^2\right) \tfrac{\partial u_x}{\partial x} + \acute{B_1} \tfrac{\partial u_z}{\partial x} K_x K_z + \acute{D_1} K_z^2$$

The K_z^2 term takes care of the fact that the drag on the disk moving along the plane of the disk is less as compared to when it is moving in the direction perpen-

Figure 4.1: Side view of a sedimenting disk



dicular to the plane of the disk. The $\cos^2\theta$ dependence of drag is consistent with the analytic solution.(3.2)

Throughout this analysis we have not kept track of the constants while carrying them forward from one equation to the other, just assuming that these are some unknown constants which depends on external force. These constants may vary from system to system. In case of sedimenting disks these constants can be determined by solving the hydrodynamics of two sedimenting disks. We can't even say anything about their signs from the present analysis, though we can get some idea about it from the experiments. I suggest the reader to not be too worried about the choice of constants and redundancy in such analysis.

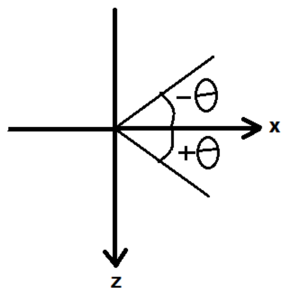
Small angle approximation: By replacing $K_x = \sin\theta \simeq \theta$ and $K_z = \cos\theta \simeq 1 - \frac{\theta^2}{2}$, as $\theta \to 0$. We can write the above equations as follows:

$$\dot{u_x} = A_1 + (A_2 + C_1\theta^2)\frac{\partial u_z}{\partial x} + \left(B_1 + D_1\frac{\partial u_x}{\partial x}\right)2\theta$$

$$\dot{u_z} = \acute{A}_1 + \left(\acute{A}_2 + \acute{C}_1\theta^2\right)\frac{\partial u_x}{\partial x} + \acute{B}_1\frac{\partial u_z}{\partial x}2\theta + \acute{D}_1\theta^2$$

If the physical picture for the dynamics of single disk is clear we can appreciate the above equations because we would expect the second term on the right to be even in theta and the third term to be odd in theta (figure[4.2]). Which is what we get. The drag in both the below configuration is the same in both the x and z direction. Where as the glide force reverses direction.

Figure 4.2: Angle convention



Constructing mobility tensor for the Orientation variable

So far we have done all our analysis with the axisymmetric particles by defining a new dynamical variable $\vec{K}$ and said that the mobility for the spatial perturbation depends on it. But this is not the complete story. A pair of axisymmetric particles rotates as it falls, so we have to construct the equation of motion for the orientation variable $\vec{K}$ from symmetry principle as we have done till now. Defining the mobility η^K_{ij} in the following manner:

$$\dot{K}_i = \eta^K_{ij} F^j \tag{4.5}$$

Again, we would demand the dynamics to be invariant under $\vec{K} \longrightarrow -\vec{K}$. To ensure that η^K_{ij} has to be odd in $\vec{K}$. We argue that the mobility cannot depend independently on $\vec{K}$. The argument is the following: *Lets say we have a lattice of disks , with all their axis of symmetry pointing in the same direction (need not be vertical or horizontal only). We prepare the system such that $\nabla u = 0$ and $\nabla K = 0$. We ask the following question: Is $\dot{K}$ nonzero? i.e. whether the disk rotate or not.*
Answer: From translation symmetry if one disk rotates all rotates in the same direction, lets say clockwise $\circlearrowright$. If we now look at the time reversed picture the rotation would be anticlockwise $\circlearrowleft$, which is not allowed if we are modelling a system in stokesian regime. The equations of motion have time reversal symmetry.

Also to ensure $\vec{K} \longrightarrow -\vec{K}$ symmetry the mobility cannot depend independently on ∇u. Even though these terms can't appear independently these can of course couple with a term which is odd in K. The simplest possible such term is ∇K . It would be valid to ask why not KKK terms as it is also odd in K. We leave that as an exercise for the reader to try to construct a second order tensor which is odd in K and even in F using just K, F and ∇u (*Hint: It may not be possible*). A detailed construction is demonstrated in Appendix C.

Equations of motion for orientation variable: From **I , II and III** in Appendix C, we can write the equation of motion. These expression can be simplified by substituting δ_{iz} in place of F_i and dropping all the terms involving z derivative as we are essentially looking at a 1 dimensional array of particles sedimenting in x-z plane. We arrive at the following equation:

$$\frac{\partial K_x}{\partial t} = \left\{ (B_1 K_x K z) \frac{\partial K_x}{\partial x} + \left(A_1 + C_1 K_x^2 + C_2 K_z^2 \right) \frac{\partial K_z}{\partial x} \right\} \times \sum_{n=0}^{\infty} \alpha_n (K_z)^{2n} \quad (4.6)$$

$$\frac{\partial K_z}{\partial t} = \left\{ \left(\acute{B}_1 K_x K z \right) \frac{\partial K_z}{\partial x} + \left(\acute{A}_1 + \acute{C}_1 K_x^2 + \acute{C}_2 K_z^2 \right) \frac{\partial K_x}{\partial x} \right\} \times \sum_{n=0}^{\infty} \alpha_n (K_z)^{2n} \quad (4.7)$$

As before we can ignore terms involving higher orders of F which just gives the first term in the summation which is a constant. Above equation becomes:

$$\tfrac{\partial K_x}{\partial t} = (B_1 K_x K z) \tfrac{\partial K_x}{\partial x} + (A_1 + C_1 K_x^2 + C_2 K_z^2) \tfrac{\partial K_z}{\partial x} \quad (4.8)$$

$$\tfrac{\partial K_z}{\partial t} = \left(\acute{B}_1 K_x K z \right) \tfrac{\partial K_z}{\partial x} + \left(\acute{A}_1 + \acute{C}_1 K_x^2 + \acute{C}_2 K_z^2 \right) \tfrac{\partial K_x}{\partial x} \quad (4.9)$$

4.2 Linear dispersion relation:

Small angle approximation: By replacing $K_x = \sin\theta \simeq \theta$ and $K_z = \cos\theta \simeq 1 - \frac{\theta^2}{2}$, as $\theta \to 0$. We can write the equations (4.3), (4.4) and (4.6) as follows:

$$\frac{\partial u_x}{\partial t} = A_1 + \left(A_2 + C_1 \theta^2 \right) \frac{\partial u_z}{\partial x} + \left(B_1 + D_1 \frac{\partial u_x}{\partial x} \right) 2\theta \quad (4.10)$$

$$\frac{\partial u_z}{\partial t} = \acute{A}_1 + \left(\acute{A}_2 + \acute{C}_1 \theta^2 \right) \frac{\partial u_x}{\partial x} + \acute{B}_1 \frac{\partial u_z}{\partial x} 2\theta + \acute{D}_1 \theta^2 \quad (4.11)$$

$$\frac{\partial \theta}{\partial t} = (B_1" - A_1")\, \theta \frac{\partial \theta}{\partial x} \equiv \lambda \theta \frac{\partial \theta}{\partial x} \quad (4.12)$$

Equation (4.12) has only non-linear term which suggests that rotation is a slow variable in this analysis. To linearise it we assume there is a stationary θ , around which we can perturb the orientation. We call this stationary orientation θ^o.

$$\theta = \theta^o + \delta\theta \quad (4.13)$$

We know that a state where ∇K is zero is stationary, i.e $\frac{\partial \theta^o}{\partial x} = 0$ implies that $\frac{\partial \theta^o}{\partial t} = 0$

(Time reversibility) . This θ^o we take is close to zero but not zero, such that θ small approximation holds. So, we have successfully defined a stationary state θ^o such that $\frac{\partial \theta^o}{\partial x} = 0$ and we have a perturbation $\delta\theta$ around this state.
Substituting equation (4.13) in equations (4.10),(4.11) and (4.12) and retaining only linear terms we get:

$$\frac{\partial u_x}{\partial t} = A_2 \frac{\partial u_z}{\partial x} + 2B_1\theta^o + 2B_1\delta\theta + 2D_1\theta^o \frac{\partial u_x}{\partial x} \tag{4.14}$$

$$\frac{\partial u_z}{\partial t} = \acute{A}_1 + \acute{A}_2 \frac{\partial u_x}{\partial x} + 2\acute{B}_1\theta^o \frac{\partial u_z}{\partial x} \tag{4.15}$$

$$\frac{\partial \delta\theta}{\partial t} = \lambda_{\parallel}\theta^o \frac{\partial \delta\theta}{\partial x} \tag{4.16}$$

Analysis in Fourier space: We assume that the solutions to above equations are of form,
$u_x = a_1 e^{i(kx-\omega t)}$; $u_z = a_2 e^{i(kx-\omega t)}$; $\delta\theta = a_3 e^{i(kx-\omega t)}$.
Constant terms in equation (4.14),(4.15) and (4.16) can be made zero by choosing an appropriate reference frame so we ignore them. Substituting our trial solution we get three equations in three variables a_1, a_2 and a_3.

$$-i\omega a_1 = -2iD_1\theta^o k a_1 - A_2 ik a_2 + 2B_1 a_3 \tag{4.17}$$

$$-i\omega a_2 = -i\acute{A}_2 k a_1 - 2i\acute{B}_1 k\theta^o a_2 \tag{4.18}$$

$$-i\omega a_3 = -i\lambda_{\parallel}\theta^o k a_3 \tag{4.19}$$

We have essentially written it in the form $AX = 0$, where

$$X = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$$

And,

$$A = \begin{pmatrix} (i\omega - 2iD_1\theta^o k) & -A_2 ik & 2B_1 \\ -i\acute{A}_2 k & (i\omega - 2i\acute{B}_1 k\theta^o) & 0 \\ 0 & 0 & (i\omega - i\lambda_{\parallel}\theta^o k) \end{pmatrix}$$

For solutions to exist $|A| = 0$, which gives us the relation between ω and k. Carrying out the above calculation gives us the following cubic equation:

$$\omega^3 - (2D_1 + 2\acute{B}_1 + \lambda_{\parallel})\theta^o k\omega^2 - A_2\acute{A}_2 k^2\omega + \lambda_{\parallel}\theta^o A_2\acute{A}_2 k^3 = 0$$

Solving the above equation gives us the dispersion relation $\omega(k)$. An interesting case is when we take $\theta^o = 0$,i.e horizontal array of discs, we get:

$\omega^3 - A_2\acute{A}_2k^2\omega = 0$

$\omega(\omega^2 - A_2\acute{A}_2k^2) = 0$; which gives a linear dispersion relation

$$\omega = \pm\sqrt{A_2\acute{A}_2}k \tag{4.20}$$

Correspondence with Crowley's calculation gives $A_2 = -\acute{A}_2$. This makes ω imaginary, which is the case of linear instability similar to Crowley.

Dispersion relation for nearly horizontal array

When θ_o is small but not zero i.e. all disks are tilted at very small angle with the horizontal and the perturbation is around this stationary state, the problem now is to solve the third degree quadratic equation, the roots of which can be written in a closed but complicated form first found by Cardano in 1545. But we go about knowing what happens in long wavelength limit by dropping down terms in the equation with $O(k^3)$. We arrive at the following dispersion relation:

$$\omega = \frac{1}{2}\left[(2D_1 + 2\acute{B}_1 + \lambda_{\parallel})\theta_o \pm \sqrt{(2D_1 + 2\acute{B}_1 + \lambda_{\parallel})^2\theta_o^2 + 4A_2\acute{A}_2}\right]k \tag{4.21}$$

For small θ_o , the second term in the $\sqrt{\ldots}$ dominates and we can write:

$$\omega = \frac{1}{2}\left[(2D_1 + 2\acute{B}_1 + \lambda_{\parallel})\theta_o \pm \sqrt{4A_2\acute{A}_2}\right]k \tag{4.22}$$

As discussed before the second term on right is imaginary which means ω has both real and imaginary parts. The real part is smaller than the imaginary as it is a product of two small numbers θ_o and k. The real part leads to travelling waves and imaginary part leads to instability. It is worth comparing this dispersion relation to the dispersion relation of the LR model [9].

Dispersion relation for nearly vertical array

Now lets ask what happens with vertically settling array of disks. Again for perfectly vertical stationary state we can't have a linear term which gives rotation. So, we assume a stationary state as disks making a very small angle θ_o with the vertical axis. We find exactly the same dispersion relation except that the λ is different in this case. This can be made transparent from the relations $\sin(\frac{\pi}{2} - \theta) = \cos(\theta)$ and $\cos(\frac{\pi}{2} - \theta) = \sin(\theta)$. To distinguish between different λs we use the notation $\lambda_{\parallel}$ and $\lambda_{\perp}$.

$$\omega = \frac{1}{2}\left[(2D_1 + 2\acute{B}_1 + \lambda_{\perp})\theta_o \pm \sqrt{(2D_1 + 2\acute{B}_1 + \lambda_{\perp})^2\theta_o^2 + 4A_2\acute{A}_2}\right] k \tag{4.23}$$

The above analysis tells us that addition of orientation degree of freedom to the particles can lead to a combination of waves and instability. The relative magnitude of $(2D_1 + 2\acute{B}_1 + \lambda)$ and $i\sqrt{4A_2\acute{A}_2}$ decides whether the dominant feature of the settling lattice is waves or instability.

Note: a critique

-The above calculation assumes only nearest neighbour interaction which is of course not true for hydrodynamic interactions. But it should always be there in the back of our minds. Also, this can more or less be achieved by confinement between planar walls which screens the hydrodynamics.

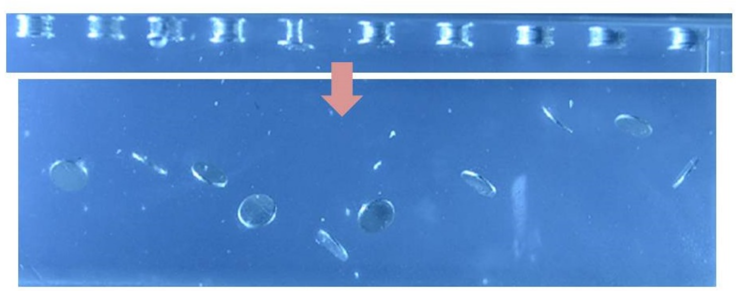
4.3 Experiments with array of disks

Though we didn't have a proper release mechanism for releasing an array of disks we crudely tried to study the sedimentation of disk array by releasing them using forceps over the fluid surface and then pushing down and inducing perturbation using needle. By doing so we now have some qualitative idea of how the array of settling disks behaves when released horizontally and vertically.

4.3.1 Horizontal array of disks:

It is to be noted that since there is a rotational symmetry along the disk axis it is almost impossible to release all the disks horizontally with their normal vectors lying in the same plane. Even for small perturbation of the normal vectors out of the x-z plane leads to growing instability of the disks which disrupts the array out of the plane. This can be explained simply by the gliding force of a single disk. It is shown in the figure below:

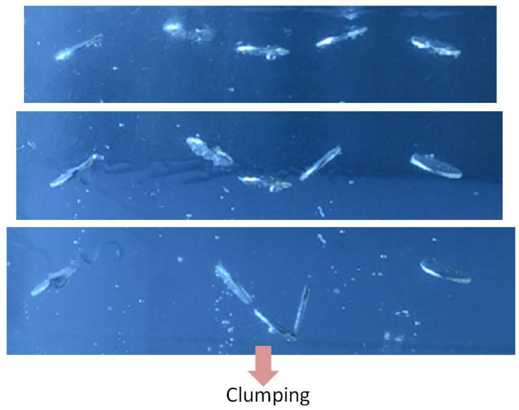
Figure 4.3: Instability of horizontal array



Longest wave number perturbation was induced in an array of 5 disks and clumping was observed similar to Crowley's clumping:

This observation is consistent with the dispersion relation that we found for horizontally sedimenting disks.

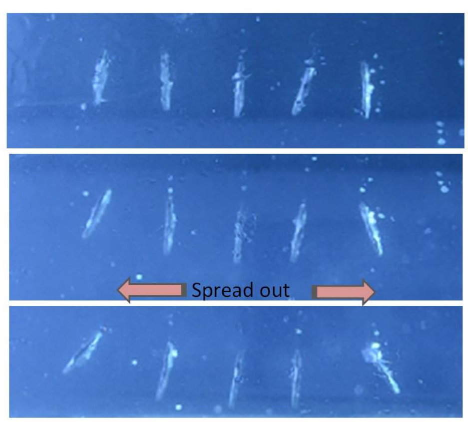
Figure 4.4: Clumping instability of horizontal array



4.3.2 Vertical array of disks and Lattice dilation

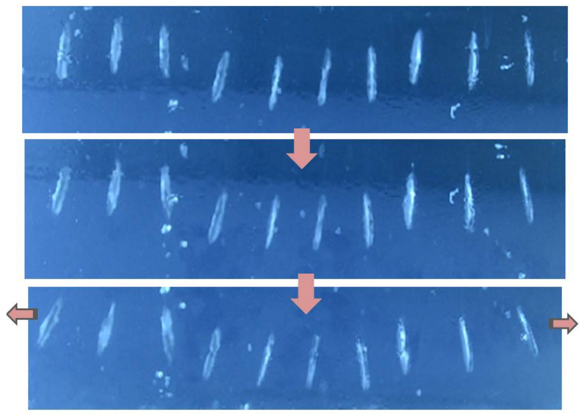
Vertically settling array of disks with open boundary seems to repel each other such that the whole lattice dilates. This is opposite of what one observes in Crowley's clumping instability. This is a classic example which shows that orientation degree of freedom can compete with clumping. This may be our most qualitatively important result. An observation for largest wave number perturbation is shown below:

Figure 4.5: Lattice dilation for largest wave number perturbation



It was found that the vertical settling array was more stable than horizontal one. There is spreading because of open boundary. We didn't have a good release mechanism to see the travelling waves as expected from the dispersion relation but we hope to do that very soon. The stability of vertical array can be seen in the observation below:

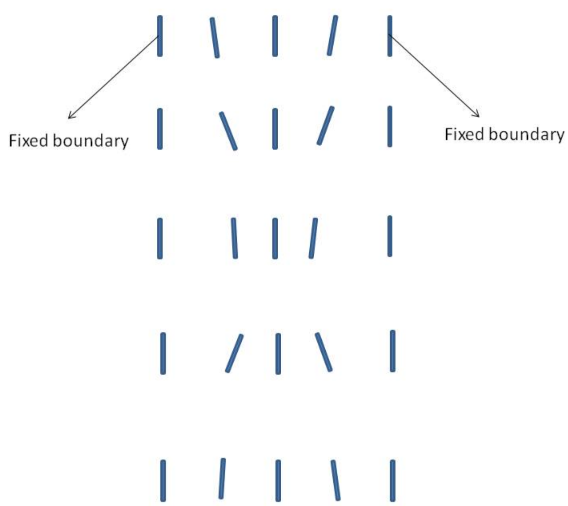
Figure 4.6: Stability of vertical disks



A physical argument for the presence of lattice waves

Looking at figure [4.5] and thinking what would happen if the boundary is fixed. This is true if we have an infinite array of particles with largest wave number perturbation and we just observe one wavelength. Having the knowledge of pair dynamics for vertical sedimentation we can deduce that the array should behave in the following manner:

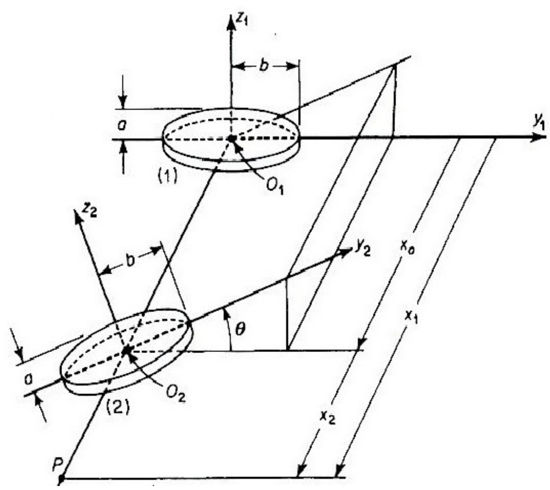
Figure 4.7: Schematic representation of waves for largest wave number with ends fixed



4.4 Wakiya Screw wave

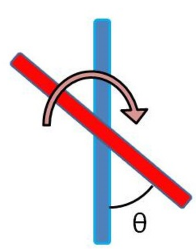
Wakiya gives an expression for the drag and torque on a disk in the presence of the neighbouring disk [2]. We proceed by doing a pairwise addition of forces and torque in order to solve for an array of disks, similar to what Crowley does for array of spherical paricles [1].

Figure 4.8: Interaction between disk pair



The disks are tilted at an angle θ with respect to the neighbour and both are moving along the negative x direction with a velocity U. If two disks are settling one above the other with an angle θ between them then they both will rotate.

Figure 4.9: Skew rotation of two disks as viewed from above



The following expressions gives the drag and torque on each disks:

$$D_{x_1} = 16\pi\mu A \tag{4.24}$$

$$T_{x_1} = \frac{16}{3}\pi\mu B \tag{4.25}$$

$$A = -\frac{2aU}{3\pi}\left[1 - \frac{8}{3\pi}\frac{a}{d} + \left(\frac{8}{3\pi}\right)^2\frac{a^2}{d^2} - \left(\frac{8}{3\pi}\right)^3\frac{a^3}{d^3} + \left(\frac{8}{3\pi}\right)^2\left\{\left(\frac{8}{3\pi}\right)^2 + \frac{7}{8}\right\}\frac{a^4}{d^4}\right] \tag{4.26}$$

$$B = -\frac{4}{3\pi^2}a^2U\frac{a^4}{d^4}\left[1 - \frac{8}{3\pi}\frac{a}{d} + \left\{\left(\frac{8}{3\pi}\right)^2 + \frac{1}{2}\right\}\frac{a^2}{d^2}\right] \times \frac{x_o}{|x_o|}\sin\theta\cos\theta\left(6\cos^2\theta - 1\right) \tag{4.27}$$

$\sin\theta\cos\theta(6\cos^2\theta - 1) > 0$ for $\theta = 0$ to 65.8’ and negative for $\theta = 65.8'$ to 90’

Similar to what Crowley did for array of spherical particles, we now we consider an array of disks all moving along the negative x direction with velocity U. The normal vectors of all the disks lies in the y-z plane. We induce a spatial perturbation along x direction. Lets say ξ_n is the perturbation of the n^{th} disk.

Drag force on n^{th} disk because of the $n+1^{th}$ and $n-1^{th}$ disk

Assuming nearest neighbour interaction there will be a drag reduction on the n^{th} disk because of the $n+1^{th}$ and $n-1^{th}$ disk. Using binomial expansion and ignoring terms of $O(\xi^2)$ and higher orders, the Drag force on the n^{th} disk can be written as:

$$D_n = -\frac{32\pi\mu aU}{3\pi}[1 - \frac{8a}{3\pi}\left(\frac{1}{d + (\xi_{n+1} - \xi_n)} + \frac{1}{d + (\xi_n - \xi_{n-1})}\right) + \left(\frac{8a}{3\pi}\right)^2 a^2\left(\frac{1}{d^2 + 2d(\xi_{n+1} - \xi_n)} + \frac{1}{d^2 + 2d(\xi_n - \xi_{n-1})}\right) + O\left(\frac{1}{d^6}\right)] \tag{4.28}$$

For small spatial perturbation ξ the perturbation force on the n^{th} disk can be written as:

$$F_D^n = -\frac{32\mu aU}{3}\left[\frac{8a}{3\pi d^2} + \left(\frac{8a}{3\pi d^2}\right)^2\right](\xi_{n+1} - \xi_{n-1}) \tag{4.29}$$

torque on the n^{th} disk because of the $n+1^{th}$ and $n-1^{th}$ disk

Ignoring terms of $O(\frac{a^6}{d^6})$ and higher orders, the torque on n^{th} disk because of $n+1^{th}$ disk can be written as:

$$T^n_{n+1} = -\frac{64\mu U}{9\pi}\frac{a^6}{[d^4 + 4d^3(\xi_{n+1} - \xi_n)]}\sin(\theta_{n+1} - \theta_n)\cos(\theta_{n+1} - \theta_n)\left(6\cos^2(\theta_{n+1} - \theta_n) - 1\right)$$
$$T^n_{n+1} = -\frac{64\mu U}{9\pi}\frac{a^6}{d^4}\frac{1}{\left(1 + 4\frac{(\xi_{n+1} - \xi_n)}{d}\right)}\sin(\theta_{n+1} - \theta_n)\cos(\theta_{n+1} - \theta_n)\left(6\cos^2(\theta_{n+1} - \theta_n) - 1\right)$$

When angle between adjacent disks is small i.e. when $\theta_{n+1} - \theta_n$ is small we can write:

$$T^n_{n+1} = -\frac{64\mu U}{9\pi}\frac{a^6}{d^4}\left(1 - 4\frac{(\xi_{n+1} - \xi_n)}{d}\right)5(\theta_{n+1} - \theta_n)$$

Ignoring non-linear terms in θ and ξ we get

$$T^n_{n+1} = -\frac{320\mu U}{9\pi}\frac{a^6}{d^4}(\theta_{n+1} - \theta_n) \tag{4.30}$$

By doing similar analysis to find the torque on n^{th} disk because of $n-1^{th}$ disk, we get:

$$T^n_{n-1} = -\frac{320\mu U}{9\pi}\frac{a^6}{d^4}(\theta_n - \theta_{n-1}) \tag{4.31}$$

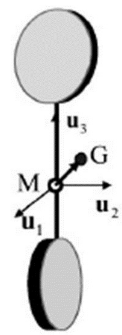
The net torque on n^{th} disk because of the $n+1^{th}$ and $n-1^{th}$ disk is found to be:

$$T^n = -\frac{320\mu U}{9\pi}\frac{a^6}{d^4}(\theta_{n+1} - \theta_{n-1}) \tag{4.32}$$

4.4.1 Equations of motion

To find the angular velocity given the torque we need to know the mobility of the disks about the required axis. Makino and Doi writes an expression for the mobility of the following particle [13]:

Figure 4.10: Doi's disks connected by thin massless rod



$$V = a.F + b.T \tag{4.33}$$

$$\Omega = b.F + c.T \tag{4.34}$$

$$a = \frac{3}{64a\mu}\left[\frac{4a^2+5h^2}{5a^2+6h^2}(\hat{u_1}\hat{u_1}+\hat{u_2}\hat{u_2}+\hat{u_3}\hat{u_3})\right]$$

$$b = \frac{3}{64a\mu}\left[-\frac{h}{5a^2+6h^2}(\hat{u_1}\hat{u_2}+\hat{u_2}\hat{u_1})\right]$$

$$c = \frac{3}{64a\mu}\left[\frac{5}{5a^2+6h^2}(\hat{u_1}\hat{u_1}+\hat{u_2}\hat{u_2})+\frac{1}{a^2}\hat{u_3}\hat{u_3}\right]$$

In the above equations h = $\frac{d}{2}$. Since we are doing pairwise addition of forces and torque we can make use the above equations.
In our case both F and T points along $\hat{u_3}$, which gives $b.F = b.T = 0$. Therefore the equations of motion for array of disks can be written down as:

$$V = a.F$$

$$\Omega = c.T$$

$$\frac{d\xi_n}{dt} = V_n = -\frac{U}{2}\left[\frac{8a}{3\pi d^2}+\left(\frac{8a}{3\pi d^2}\right)^2\right](\xi_{n+1}-\xi_{n-1})\hat{u_3} \tag{4.35}$$

$$\frac{d\theta_n}{dt} = \Omega_n = -\frac{5}{3\pi}\frac{a^3}{d^4}U(\theta_{n+1}-\theta_{n-1})\hat{u_3} \tag{4.36}$$

Dispersion relation for Spatial perturbation

Let, $\xi_n \propto e^{i(\mathbb{k}nd-\omega t)}$, substituting it into equation for ξ_n gives us the dispersion relation:

$$\omega = U\left[\frac{8a}{3\pi d^2}+\left(\frac{8a}{3\pi d^2}\right)^2\right]\sin(\mathbb{k}d) \tag{4.37}$$

which corresponds to spatial waves.

Dispersion relation for orientation perturbation

Let $\theta_n \propto e^{i(\Bbbk nd-\omega t)}$, substituting it into equation for Ω_n gives us the dispersion relation:

$$\omega = \frac{10}{3\pi}\frac{a^3}{d^4}U\sin(\Bbbk d) \tag{4.38}$$

This corresponds to orientation waves. In long wavelength limit the dispersion relation becomes:

$$\omega = \frac{10}{3\pi}\frac{a^3}{d^3}U\Bbbk \tag{4.39}$$

These orientation waves travel with group velocity $\frac{10}{3\pi}\frac{a^3}{d^3}U$.

Our array of disks simply behaves like a screw, so we call it Wakiya Screw wave.

4.4.2 Continuum equations

We can construct the equations of motion for Wakiya's case from symmetry principles the way we did it for horizontal array of disks. We now need to do it for a vertical array of disks along the z direction with $\vec{K}$ lying in the x-y plane. The construction is straightforward from the result of horizontal case. In this case x derivative will drop down and the terms K_iF_i are all zero. The spatial perturbation is only along the vertical direction. For this case the equation of motion from symmetry turns out to be (keeping only leading orders in ∇K and ∇u):

Equations of motion

$$\dot{u_z} = \lambda_1\frac{\partial u_z}{\partial z} \tag{4.40}$$

$$\dot{K_x} = \lambda_2\frac{\partial K_x}{\partial z} + \lambda_3 K_x K_i\frac{\partial K_i}{\partial z} \tag{4.41}$$

The term $K_i\frac{\partial K_i}{\partial z} = 0$ since the magnitude of K is a constant. In terms of θ the orientation equation simplifies to:

$$\frac{\partial \theta}{\partial t} = -\lambda_2\frac{\partial \theta}{\partial z} \tag{4.42}$$

As expected this is a continuous version of the equation we derived for Wakia Screw wave. The phenomenological constant λ_2 can be found by comparing this equation

with the discrete case:
$\lambda_2 = \frac{10}{3\pi}\left(\frac{a}{d}\right)^3 U$; this is just the group velocity of the orientation waves in long wavelength limit.

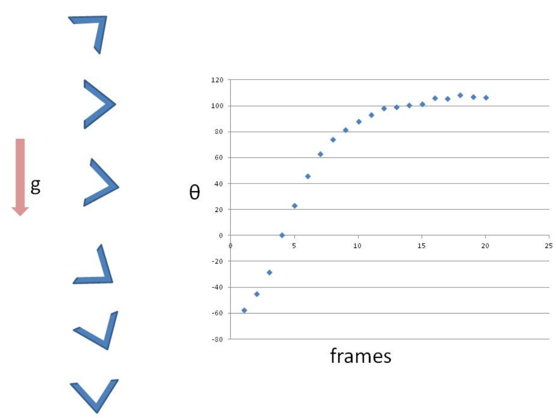
Chapter 5

Lattice of polar axisymmetric particles

5.1 Polar Particle

We know that a single rod doesn't rotate while settling in stokes regime. But does a cone shaped particle rotate? We would try to answer this question as we go along in this section. We did experiment with V shaped particle and found that a single particle rotates till it attains a stable configuration. It was found that the stable configuration is when the V points along the gravity.

Figure 5.1: Rotation of a single V shaped particle



5.2 Lattice of Polar axisymmetric particles (cones, hemisphere etc.)

Given that we have already constructed the mobility for lattice axisymmetric particles from symmetry principles, it takes very little effort to construct the mobility for the lattice of polar axisymmetric particles in similar manner. For polar axisymmmetric particles $\vec{K} \longrightarrow -\vec{K}$ symmetry is broken. After analysing the construction given in Appendix B and C , one finds that the mobility for spatial part can now have a ∇K dependence in linear order. And the mobility of the orientation variable can now have a $K_i K_j$ and ∇u dependence in linear order allowed by symmetry. A detailed contruction of the mobility tensor is given in Appendix D and E.

Equations of Motion:
The equation on motion for the spatial part can be written down as:

$$\begin{aligned}
\dot{u_x} =& \{\acute{A_1} + \left(A_1 + C_1 K_z^2\right)\frac{\partial u_z}{\partial x} + \left(B_1 + D_1\frac{\partial u_x}{\partial x}\right)K_x K_z + \\
& \left(E_1 K_x Kz\right)\frac{\partial K_x}{\partial x} + \left(F_1 + G_1 K_z^2\right)\frac{\partial K_z}{\partial x}\} \times \sum_{n=0}^{\infty}\alpha_n (K_z)^{2n} \qquad (5.1)\\
\dot{u_z} =& \{\acute{A_2} + \left(A_2 + C_2 K_z^2\right)\frac{\partial u_x}{\partial x} + B_2 {K_z}^2 + D_2\frac{\partial u_z}{\partial x}K_x K_z + \\
& \left(E_2 K_x Kz\right)\frac{\partial K_z}{\partial x} + \left(F_2 + G_2 K_z^2\right)\frac{\partial K_x}{\partial x}\} \times \sum_{n=0}^{\infty}\alpha_n (K_z)^{2n} \qquad (5.2)
\end{aligned}$$

One can check that the above equations has the following symmetry:
when $x \longrightarrow -x$, $u_x \longrightarrow -u_x$ and $K_x \longrightarrow -K_x$ which is similar to saying that the perpendicular subspace is isotropic.

The equation for the x component of orientation can be written down as:

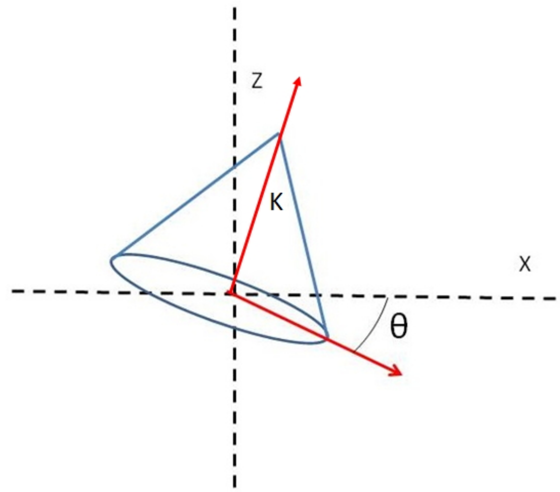
$$\begin{aligned}
\frac{\partial K_x}{\partial t} =& \{\left(A_3 K_x Kz\right)\frac{\partial K_x}{\partial x} + \left(B_3 + C_3 K_z^2\right)\frac{\partial K_z}{\partial x} + \\
& \left(D_3 + E_3 K_z^2\right)\frac{\partial u_z}{\partial x} + \left(F_3 + G_3\frac{\partial u_x}{\partial x}\right)K_x K_z\} \times \sum_{n=0}^{\infty}\alpha_n (K_z)^{2n} \qquad (5.3)
\end{aligned}$$

From the above equation let's see whether we can say something about the dynamics of isolated polar axisymmetric particle. Since the gradient terms marks the presence of interactions, for the case when $\nabla K = 0$ and $\nabla u = 0$, i.e. all the cones are equidistant and point along the same direction. Then in the co-moving frame the leading term which contributes to rotation can be written as:

$$\frac{\partial K_x}{\partial t} = F_3 K_x K_z \qquad (5.4)$$

by our convention $K_x = \sin\theta$ and $K_y = \cos\theta$ which can be depicted in the image below:

Figure 5.2: Angle convention for a cone

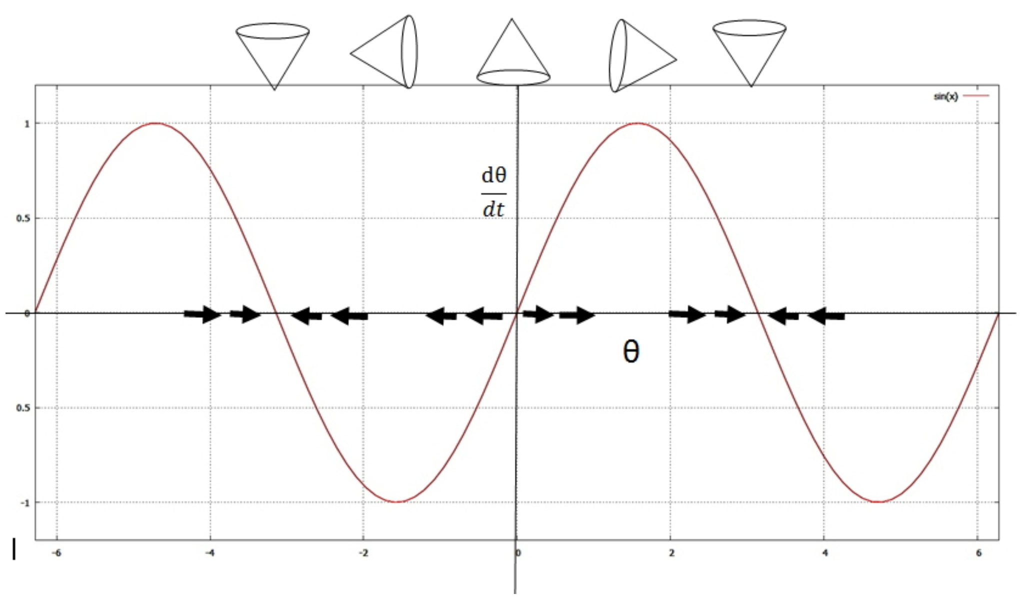


With the above convention, orientation equation can be written down as:

$$\frac{\partial \theta}{\partial t} = \lambda \sin \theta \tag{5.5}$$

From the above equation the phase diagram of a single cone in a co-moving frame looks like:

Figure 5.3: Phase diagram of a cone



The above equation can be integrated and assuming at t=0 we start with $\theta = 2\epsilon$

we get:

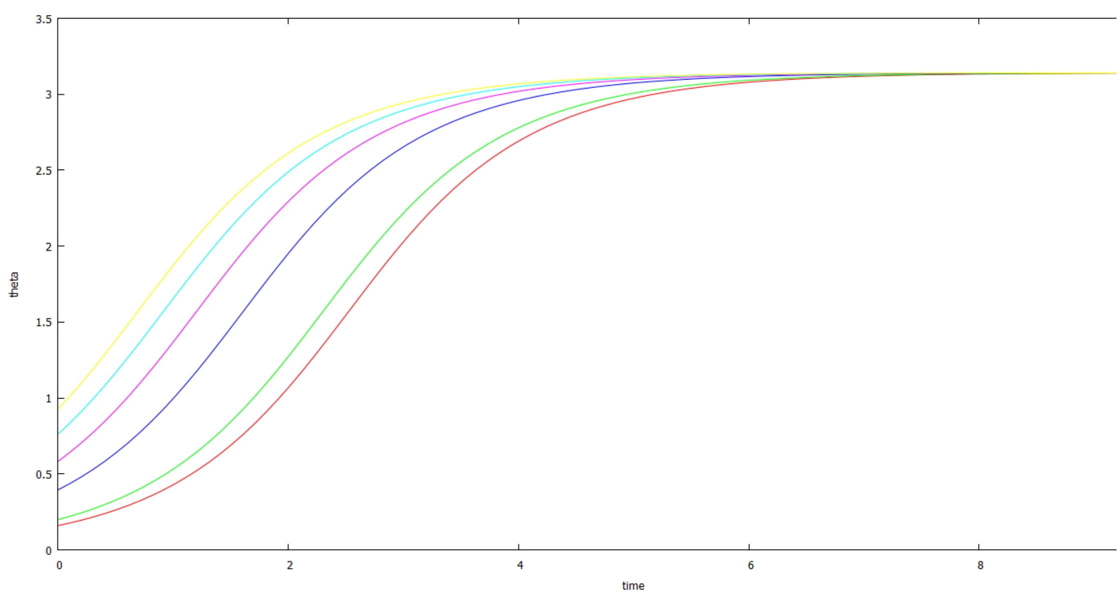
$$\log \tan \frac{\theta}{2} = \lambda t + \log \tan \epsilon \tag{5.6}$$

$$\log \left[\frac{\tan \frac{\theta}{2}}{\tan \epsilon} \right] = \lambda t$$

$$\theta(t) = 2 \tan^{-1} \left[\tan \epsilon . \exp^{\lambda t} \right] \tag{5.7}$$

setting $\lambda = 1$ and plotting the above solution for different values of initial angles: It implies that the cone becomes stable for $\theta \longrightarrow \pi$ i.e. when the tip points down-

Figure 5.4: Rotation of cone for different initial tilts



wards. This is consistent with the experimental observation of the rotation of a V shaped particle which has the same symmetry as that of a cone in two dimensions. The above plot also tells us something more amazing. If we start with an array of cones all pointing in the same direction , all will rotate in the same sense to point along gravity.

5.3 Linear dispersion relation for lattice of polar axisymmetric particles, (cones,hemispheres etc.)

As we did for the lattice of axisymmetric particles the simplest analysis we can do with the above equations is to find the linear dispersion relation. For that we first write the small angle approximation of the above equations.

Small angle approximation

By replacing $K_x = \sin\theta \simeq \theta$ and $K_z = \cos\theta \simeq 1 - \frac{\theta^2}{2}$, as $\theta \to 0$. We can write the above equations as follows:

$$
\begin{aligned}
\frac{\partial u_x}{\partial t} =& \acute{A}_1 + \left(A_1 + C_1\theta^2\right)\frac{\partial u_z}{\partial x} + \left(B_1 + D_1\frac{\partial u_x}{\partial x}\right)\theta + \\
& E_1\theta\frac{\partial \theta}{\partial x} + G_1\theta^3\frac{\partial \theta}{\partial x} \qquad (5.8)\\
\frac{\partial u_z}{\partial t} =& \acute{A}_2 + \left(A_2 + C_2\theta^2\right)\frac{\partial u_x}{\partial x} + B_2\theta^2 + D_2\frac{\partial u_z}{\partial x}\theta + \\
& E_2\theta^2\frac{\partial \theta}{\partial x} + \left(F_2 + G_2\theta^2\right)\frac{\partial \theta}{\partial x} \qquad (5.9)\\
\frac{\partial \theta}{\partial t} =& A_3\theta\frac{\partial \theta}{\partial x} + C_3\theta^3\frac{\partial \theta}{\partial x} + \left(D_3 + E_3\theta^2\right)\frac{\partial u_z}{\partial x} + \left(F_3 + G_3\frac{\partial u_x}{\partial x}\right)\theta
\end{aligned}
$$

(5.10)

Linearizing the above equations by assuming $\theta = 0$ as stationary state we get:

$$
\begin{aligned}
\frac{\partial u_x}{\partial t} =& \acute{A}_1 + A_1\frac{\partial u_z}{\partial x} + B_1\theta \qquad (5.11)\\
\frac{\partial u_z}{\partial t} =& \acute{A}_2 + A_2\frac{\partial u_x}{\partial x} + F_2\frac{\partial \theta}{\partial x} \qquad (5.12)\\
\frac{\partial \theta}{\partial t} =& D_3\frac{\partial u_z}{\partial x} + F_3\theta \qquad (5.13)
\end{aligned}
$$

(5.14)

Analysis in Fourier space: We assume that the solutions to above equations are of form,

$u_x = a_1 e^{i(kx-\omega t)}$; $u_z = a_2 e^{i(kx-\omega t)}$; $\theta = a_3 e^{i(kx-\omega t)}$.

The constant terms in the equations are zero in the co-moving frame. Substituting our trial solution we get three equations in three variables $a_1, a_2 and a_3$.

$$-i\omega a_1 = A_1 ika_2 + B_1 a_3 \qquad (5.15)$$

$$-i\omega a_2 = A_2 ika_1 + F_2 ika_3 \qquad (5.16)$$

$$-i\omega a_3 = B_3 ika_2 + F_3 a_3 \qquad (5.17)$$

We have essentially written it in the form $AX = 0$, where

$$X = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$$

And,

$$A = \begin{pmatrix} i\omega & A_1 ik & B_1 \\ ikA_2 & i\omega & F_2 ik \\ 0 & B_3 ik & (i\omega + F_3) \end{pmatrix}$$

For solutions to exist $|A| = 0$, which gives us the relation between ω and k. It can be found by solving the following quadratic equation in ω.

$$\omega^3 - iF_3\omega^2 - (F_2B_3 + A_1A_2)k^2\omega + (A_1A_2F_3 - B_1A_2B_3)k^2 = 0$$

One can see the behavior of the array in the long wavelength limit by ignoring k^2 terms.
It gives $\omega = iF_3$ which is the case of linear instability. And since for cones F is positive as known from experiment it is an unstable mode (Equation (5.4) and (5.5)).

The most interesting information that we got from the above analysis is that a horizontal array of cones all pointing in some direction and equidistant from one another rotates to point along gravity.

Appendix A

Constructing mobility tensor for spherical particles

When the particles are spherical we can build the mobility tensors from $\nabla_i u_j$'s and F_i's. Another knowledge that goes in is the time reversibility of the system. Reversing the direction of forces reverses the direction of velocities. We ensure this by constructing the mobility tensor η_{ij} which is even in F, so when we contract it with F_j we get velocity which is odd in F.

$$\eta\left(\nabla\vec{u}\right)_{ij} = a_1\delta_{ij} + a_2F_iF_j + a_3F_iF_j\frac{\partial u_k}{\partial x_l}\delta_{kl} + a_4\frac{\partial u_i}{\partial x_j} + \acute{a_4}\frac{\partial u_j}{\partial x_i} + a_5F_iF_l\frac{\partial u_j}{\partial x_l} + \acute{a_5}F_jFl\frac{\partial u_i}{\partial x_l} + \\ a_6F_iFl\frac{\partial u_l}{\partial x_j} + \acute{a_6}F_jF_l\frac{\partial u_l}{\partial x_i} + a_7F_iF_jF_kF_l\frac{\partial u_k}{\partial x_l} + a_8F_iF_kF_kF_l\frac{\partial u_j}{\partial x_l} + \acute{a_8}F_jF_kF_kF_l\frac{\partial u_i}{\partial x_l} + \\ a_9F_iF_kF_kF_l\frac{\partial u_l}{\partial x_j} + \acute{a_9}F_jF_kF_kF_l\frac{\partial u_l}{\partial x_i} + \text{higher orders in } \nabla u... \tag{A.1}$$

There can be higher orders in F's and $\delta's$ contracting among themselves, but that would just give a constant term. So, above given terms exhaust all the possible terms that can form using only $F's, \delta's$ and $\nabla u's$. We have expanded only in linear orders in ∇u .

Since we are assuming that the external driving force is only along the z direction. We can replace the F_i terms with a constant times δ_{iz} and the constant can be exhausted in the coefficients of the term.

$$\eta\left(\nabla\vec{u}\right)_{ij} = a_1\delta_{ij} + a_2\delta_{iz}\delta_{jz} + a_3\delta_{iz}\delta_{jz}\frac{\partial u_k}{\partial x_l}\delta_{kl} + a_4\frac{\partial u_i}{\partial x_j} + \acute{a_4}\frac{\partial u_j}{\partial x_i} + a_5\delta_{iz}\delta_{lz}\frac{\partial u_j}{\partial x_l} + \acute{a_5}\delta_{jz}\delta lz\frac{\partial u_i}{\partial x_l} + a_6\delta_{iz}\delta lz\frac{\partial u_l}{\partial x_j} + \acute{a_6}\delta_{jz}\delta_{lz}\frac{\partial u_l}{\partial x_i} + a_7\delta_{iz}\delta_{jz}\delta_{kz}\delta_{lz}\frac{\partial u_k}{\partial x_l} + a_8\delta_{iz}\delta_{kz}\delta_{kz}\delta_{lz}\frac{\partial u_j}{\partial x_l} + \acute{a_8}\delta_{jz}\delta_{kz}\delta_{kz}\delta_{lz}\frac{\partial u_i}{\partial x_l} + a_9\delta_{iz}\delta_{kz}\delta_{kz}\delta_{lz}\frac{\partial u_l}{\partial x_j} + \acute{a_9}\delta_{jz}\delta_{kz}\delta_{kz}\delta_{lz}\frac{\partial u_l}{\partial x_i} + \text{higher orders in } \nabla u... \quad \text{(A.2)}$$

Appendix B

Constructing mobility tensor for spatial part for an array of apolar axisymmetric particles

Since there is an additional dynamical variable which is the orientation of the particle, the mobility depends on this variable. An additional symmetry of the system is the inversion of unit vector pointing along the symmetry axis of the particle.

Constructing the mobility tensor for spatial perturbation:

In the following analysis we are assuming the dynamics is happening in 2 dimensions only. We can later on extend the results to 3 dimensions
$\frac{\partial \vec{u}}{\partial t} = \eta^u(\nabla \vec{u}, \vec{K})\vec{F}$

For convenience in handling we have divided the mobility tensor into four parts
$\eta^u(\nabla \vec{u}, \vec{K})_{ij} = \eta^u_{a_{ij}} + \eta^u_{b_{ij}} + \eta^u_{c_{ij}} + \eta^u_{d_{ij}}$
I) $\eta^u_{a_{ij}} \longrightarrow$ terms coupling $\nabla u, \vec{F}$
II) $\eta^u_{b_{ij}} \longrightarrow$ terms coupling $\vec{K}, \vec{F}$
III) $\eta^u_{c_{ij}} \longrightarrow$ terms coupling $\nabla u, \vec{K}$
IV) $\eta^u_{a_{ij}} \longrightarrow$ terms coupling $\nabla u, \vec{K}, \vec{F}$

The superscript 'u' in η is to remind that we are writing the equation of motion for u, it is to distinguish this mobility from the one which we would construct when we write the equations of motion for the $\vec{K}$. Now one by one we will construct the mobility parts given above.

I)

$$\eta^u_{a_{ij}} = a_1\delta_{ij} + a_2F_iF_j + a_3F_iF_j\frac{\partial u_k}{\partial x_l}\delta_{kl} + a_4\frac{\partial u_i}{\partial x_j} + \acute{a_4}\frac{\partial u_j}{\partial x_i} + a_5F_iF_l\frac{\partial u_j}{\partial x_l} + \acute{a_5}F_jF_l\frac{\partial u_i}{\partial x_l} +$$
$$a_6F_iF_l\frac{\partial u_l}{\partial x_j} + \acute{a_6}F_jF_l\frac{\partial u_l}{\partial x_i} + a_7F_iF_jF_kF_l\frac{\partial u_k}{\partial x_l} + a_8F_iF_kF_kF_l\frac{\partial u_j}{\partial x_l} + \acute{a_8}F_jF_kF_kF_l\frac{\partial u_i}{\partial x_l} +$$
$$a_9F_iF_kF_kF_l\frac{\partial u_l}{\partial x_j} + \acute{a_9}F_jF_kF_kF_l\frac{\partial u_l}{\partial x_i} + \text{higher orders in } \nabla u \quad \text{(B.1)}$$

*This term is the same as for spherical particles. Again we will ignore the higher orders in ∇u . The higher order contraction of **F** with itself would just yield a constant term which simplifies the above expression further.*

$$\eta^u_{a_{ij}} = a_1\delta_{ij} + a_2F_iF_j + a_3F_iF_j\frac{\partial u_k}{\partial x_l}\delta_{kl} + a_4\frac{\partial u_i}{\partial x_j} + \acute{a_4}\frac{\partial u_j}{\partial x_i} + a_5F_iF_l\frac{\partial u_j}{\partial x_l} + \acute{a_5}F_jF_l\frac{\partial u_i}{\partial x_l} +$$
$$a_6F_iF_l\frac{\partial u_l}{\partial x_j} + \acute{a_6}F_jF_l\frac{\partial u_l}{\partial x_i} \quad \text{(B.2)}$$

II)

$$\eta^u_{b_{ij}} = \Big\{b_1K_iK_j + b_2F_iF_jK_lK_l + b_3F_iF_lK_lK_j + \acute{b_3}F_jF_lK_lK_i\Big\} \times \sum_{n=0}^{\infty}\alpha_n(K_z)^{2n} \quad \text{(B.3)}$$

*All the K_lK_l terms would just give 1 since $\vec{K}$ is a unit vector. Though K_i can contract with F_i which is given by the summation above on the right. The F_lK_l term gives K_z owing to the force being only in the z direction and since both these terms always appear in pairs we have the summation of K_z^2. As you will see later we will ignore all terms involving higher orders of **F** and hence confine ourselves with n=0,1 terms only.*

III)

$$\eta^u_{c_{ij}} = c_1K_iK_j\frac{\partial u_k}{\partial x_l}\delta_{kl} + c_2\frac{\partial u_i}{\partial x_j} + \acute{c_2}\frac{\partial u_j}{\partial x_i} + c_3K_iK_l\frac{\partial u_j}{\partial x_l} + \acute{c_3}K_jK_l\frac{\partial u_i}{\partial x_l} + c_4K_iK_l\frac{\partial u_l}{\partial x_j} + \acute{c_4}K_jK_l\frac{\partial u_l}{\partial x_i} \quad \text{(B.4)}$$

IV)

$$\eta^u{}_{d_{ij}} = \{d_1F_iF_nK_nK_l\frac{\partial u_l}{\partial x_j} + \acute{d_1}F_jF_nK_nK_l\frac{\partial u_l}{\partial x_i} + d_2F_iF_nK_nK_l\frac{\partial u_j}{\partial x_l} + \acute{d_2}F_jF_nK_nK_l\frac{\partial u_i}{\partial x_l} +$$
$$d_3F_iF_nK_jK_m\frac{\partial u_n}{\partial x_m} + \acute{d_3}F_jF_nK_iK_m\frac{\partial u_n}{\partial x_m} + d_4F_iF_nK_jK_m\frac{\partial u_m}{\partial x_n} + \acute{d_4}F_jF_nK_iK_m\frac{\partial u_m}{\partial x_n} +$$
$$d_5F_lF_nK_lK_n\frac{\partial u_i}{\partial x_j} + \acute{d_5}F_lF_nK_lK_n\frac{\partial u_j}{\partial x_i} + d_6F_iF_jK_lK_m\frac{\partial u_l}{\partial x_m} + d_7F_lF_mK_iK_j\frac{\partial u_l}{\partial x_m} +$$
$$d_8F_iF_nK_jK_n\frac{\partial u_l}{\partial x_l}\} \times \sum_{n=0}^{\infty}\alpha_n(K_z)^{2n} \quad \text{(B.5)}$$

Appendix C

Constructing mobility tensor for orientation part for an array of apolar axisymmetric particles

Idea is to construct the mobility tensor which depends on ∇K. To begin with lets say that it doesn't contain ∇u term (we can add this complexity later on). So, ∇K can couple with the terms δ_{ij},F_i,K_i to give the mobility tensor. In this analysis we are ignoring any dependence on the higher orders in ∇K.

$\frac{\partial \vec{K}}{\partial t} = \eta^K(\nabla \vec{K})\vec{F}$

$\eta^K(\nabla \vec{K})_{ij} = \eta^K_{a_{ij}} + \eta^K_{b_{ij}} + \eta^K_{c_{ij}}$
I) $\eta^K_{a_{ij}} \longrightarrow$ terms coupling $\nabla K, \vec{F}$
II) $\eta^K_{b_{ij}} \longrightarrow$ terms coupling $\nabla K, \vec{K}$
III) $\eta^K_{c_{ij}} \longrightarrow$ terms coupling $\nabla K, \vec{K}$ and $\vec{F}$
The superscript K is to remind that we are writing the equation of motion for K and this mobility is not to be confused withη^u_{ij}.

I)

$$\begin{aligned}\eta^K_{a_{ij}} =& a_1 F_i F_j \frac{\partial K_k}{\partial x_l}\delta_{kl} + a_2\frac{\partial K_i}{\partial x_j} + \acute{a_2}\frac{\partial K_j}{\partial x_i} + a_3 F_i F_l \frac{\partial K_j}{\partial x_l} + \acute{a_3} F_j F_l \frac{\partial K_i}{\partial x_l} + \\ & a_4 F_i F_l \frac{\partial K_l}{\partial x_j} + \acute{a_4} F_j F_l \frac{\partial K_l}{\partial x_i}\end{aligned} \tag{C.1}$$

II)

$$\eta^K_{b_{ij}} = b_1 K_i K_j \frac{\partial K_k}{\partial x_l}\delta_{kl} + b_2 \frac{\partial K_i}{\partial x_j} + b_2' \frac{\partial K_j}{\partial x_i} + b_3 K_i K_l \frac{\partial K_j}{\partial x_l} + b_3' K_j K_l \frac{\partial K_i}{\partial x_l} + \\ b_4 K_i K_l \frac{\partial K_l}{\partial x_j} + b_4' K_j K_l \frac{\partial K_l}{\partial x_i} \quad \text{(C.2)}$$

III)

$$\eta^K{}_{c_{ij}} = \{c_1 F_i F_n K_n K_l \frac{\partial K_l}{\partial x_j} + c_1' F_j F_n K_n K_l \frac{\partial K_l}{\partial x_i} + c_2 F_i F_n K_n K_l \frac{\partial K_j}{\partial x_l} + c_2' F_j F_n K_n K_l \frac{\partial K_i}{\partial x_l} + \\ c_3 F_i F_n K_j K_m \frac{\partial K_n}{\partial x_m} + c_3' F_j F_n K_i K_m \frac{\partial K_n}{\partial x_m} + c_4 F_i F_n K_j K_m \frac{\partial K_m}{\partial x_n} + c_4' F_j F_n K_i K_m \frac{\partial K_m}{\partial x_n} + \\ c_5 F_l F_n K_l K_n \frac{\partial K_i}{\partial x_j} + c_5' F_l F_n K_l K_n \frac{\partial K_j}{\partial x_i} + c_6 F_i F_j K_l K_m \frac{\partial K_l}{\partial x_m} + c_7 F_l F_m K_i K_j \frac{\partial K_l}{\partial x_m} + \\ c_8 F_i F_n K_j K_n \frac{\partial K_l}{\partial x_l}\} \times \sum_{n=0}^{\infty} \alpha_n (K_z)^{2n} \quad \text{(C.3)}$$

Appendix D

Constructing mobility tensor for spatial part for an array of polar axisymmetric particle

Since there is an additional dynamical variable which is the orientation of the particle, the mobility depends on this variable. In addition to this the symmetry allows a linear order dependence on ∇K along with ∇u. Knowing that the symmetry $K \longrightarrow -K$ is not the symmetry of the system anymore we can start writing the general form of mobility allowed by the symmetry to linear order in $\nabla \vec{K}$ and $\nabla \vec{u}$.

Constructing the mobility tensor for spatial perturbation:

In the following analysis we are assuming the dynamics is happening in 2 dimensions only. We can later on extend the results to 3 dimensions
$\frac{\partial \vec{u}}{\partial t} = \eta^u(\nabla \vec{u}, \vec{K}, \nabla \vec{K})\vec{F}$

For convenience in handling we have divided the mobility tensor into following parts
$\eta^u(\nabla \vec{u}, \vec{K}, \nabla \vec{K})_{ij} = {\eta^u}_{a_{ij}} + \eta^u_{b_{ij}} + \eta^u_{c_{ij}} + \eta^u_{d_{ij}} + \eta^u_{e_{ij}} + \eta^u_{f_{ij}} + \eta^u_{g_{ij}}$
I) $\eta^u_{a_{ij}} \longrightarrow$ terms coupling $\nabla u, \vec{F}$
II) $\eta^u_{b_{ij}} \longrightarrow$ terms coupling $\vec{K}, \vec{F}$
III) $\eta^u_{c_{ij}} \longrightarrow$ terms coupling $\nabla u, \vec{K}$
IV) $\eta^u_{d_{ij}} \longrightarrow$ terms coupling $\nabla u, \vec{K}, \vec{F}$
V) $\eta^K_{e_{ij}} \longrightarrow$ terms coupling $\nabla K, \vec{F}$
VI) $\eta^K_{f_{ij}} \longrightarrow$ terms coupling $\nabla K, \vec{K}$
VII) $\eta^K_{g_{ij}} \longrightarrow$ terms coupling $\nabla K, \vec{K}$ and $\vec{F}$

The superscript 'u' in η is to remind that we are writing the equation of motion for u, it is to distinguish this mobility from the one which we would construct when we write the equations of motion for the $\vec{K}$. Now one by one we will construct the mobility parts given above.

I)

$$\eta^u_{a_{ij}} = a_1\delta_{ij} + a_2F_iF_j + a_3F_iF_j\frac{\partial u_k}{\partial x_l}\delta_{kl} + a_4\frac{\partial u_i}{\partial x_j} + \acute{a}_4\frac{\partial u_j}{\partial x_i} + a_5F_iF_l\frac{\partial u_j}{\partial x_l} + \acute{a}_5F_jF_l\frac{\partial u_i}{\partial x_l} + a_6F_iF_l\frac{\partial u_l}{\partial x_j} + \acute{a}_6F_jF_l\frac{\partial u_l}{\partial x_i} + a_7F_iF_jF_kF_l\frac{\partial u_k}{\partial x_l} + a_8F_iF_kF_kF_l\frac{\partial u_j}{\partial x_l} + \acute{a}_8F_jF_kF_kF_l\frac{\partial u_i}{\partial x_l} + a_9F_iF_kF_kF_l\frac{\partial u_l}{\partial x_j} + \acute{a}_9F_jF_kF_kF_l\frac{\partial u_l}{\partial x_i} + \text{higher orders in } \nabla u \tag{D.1}$$

*This term is the same as for spherical particles. Again we will ignore the higher orders in ∇u . The higher order contraction of **F** with itself would just yield a constant term which simplifies the above expression further.*

$$\eta^u_{a_{ij}} = a_1\delta_{ij} + a_2F_iF_j + a_3F_iF_j\frac{\partial u_k}{\partial x_l}\delta_{kl} + a_4\frac{\partial u_i}{\partial x_j} + \acute{a}_4\frac{\partial u_j}{\partial x_i} + a_5F_iF_l\frac{\partial u_j}{\partial x_l} + \acute{a}_5F_jF_l\frac{\partial u_i}{\partial x_l} + a_6F_iF_l\frac{\partial u_l}{\partial x_j} + \acute{a}_6F_jF_l\frac{\partial u_l}{\partial x_i} \tag{D.2}$$

II)

$$\eta^u_{b_{ij}} = \left\{b_1K_iK_j + b_2F_iF_jK_lK_l + b_3F_iF_lK_lK_j + \acute{b}_3F_jF_lK_lK_i\right\} \times \sum_{n=0}^{\infty}\alpha_n(K_z)^{2n} \tag{D.3}$$

*All the K_lK_l terms would just give 1 since $\vec{K}$ is a unit vector. Though K_i can contract with F_i which is given by the summation above on the right. The F_lK_l term gives K_z owing to the force being only in the z direction and since both these terms always appear in pairs we have the summation of K_z^2. As you will see later we will ignore all terms involving higher orders of **F** and hence confine ourselves with n=0,1 terms only.*

III)

$$\eta^u_{c_{ij}} = c_1K_iK_j\frac{\partial u_k}{\partial x_l}\delta_{kl} + c_2\frac{\partial u_i}{\partial x_j} + \acute{c}_2\frac{\partial u_j}{\partial x_i} + c_3K_iK_l\frac{\partial u_j}{\partial x_l} + \acute{c}_3K_jK_l\frac{\partial u_i}{\partial x_l} + c_4K_iK_l\frac{\partial u_l}{\partial x_j} + \acute{c}_4K_jK_l\frac{\partial u_l}{\partial x_i} \tag{D.4}$$

IV)

$$
\begin{aligned}
\eta^{u}_{d_{ij}} =&\{d_1 F_i F_n K_n K_l \frac{\partial u_l}{\partial x_j} + \acute{d}_1 F_j F_n K_n K_l \frac{\partial u_l}{\partial x_i} + d_2 F_i F_n K_n K_l \frac{\partial u_j}{\partial x_l} + \acute{d}_2 F_j F_n K_n K_l \frac{\partial u_i}{\partial x_l} + \\
& d_3 F_i F_n K_j K_m \frac{\partial u_n}{\partial x_m} + \acute{d}_3 F_j F_n K_i K_m \frac{\partial u_n}{\partial x_m} + d_4 F_i F_n K_j K_m \frac{\partial u_m}{\partial x_n} + \acute{d}_4 F_j F_n K_i K_m \frac{\partial u_m}{\partial x_n} + \\
& d_5 F_l F_n K_l K_n \frac{\partial u_i}{\partial x_j} + \acute{d}_5 F_l F_n K_l K_n \frac{\partial u_j}{\partial x_i} + d_6 F_i F_j K_l K_m \frac{\partial u_l}{\partial x_m} + d_7 F_l F_m K_i K_j \frac{\partial u_l}{\partial x_m} + \\
& d_8 F_i F_n K_j K_n \frac{\partial u_l}{\partial x_l}\} \times \sum_{n=0}^{\infty} \alpha_n (K_z)^{2n}
\end{aligned} \tag{D.5}
$$

V)

$$
\begin{aligned}
\eta^{K}_{e_{ij}} =& e_1 F_i F_j \frac{\partial K_k}{\partial x_l} \delta_{kl} + e_2 \frac{\partial K_i}{\partial x_j} + \acute{e}_2 \frac{\partial K_j}{\partial x_i} + e_3 F_i F_l \frac{\partial K_j}{\partial x_l} + \acute{e}_3 F_j F_l \frac{\partial K_i}{\partial x_l} + \\
& e_4 F_i F_l \frac{\partial K_l}{\partial x_j} + \acute{e}_4 F_j F_l \frac{\partial K_l}{\partial x_i}
\end{aligned} \tag{D.6}
$$

VI)

$$
\begin{aligned}
\eta^{K}_{f_{ij}} =& f_1 K_i K_j \frac{\partial K_k}{\partial x_l} \delta_{kl} + f_2 \frac{\partial K_i}{\partial x_j} + \acute{f}_2 \frac{\partial K_j}{\partial x_i} + f_3 K_i K_l \frac{\partial K_j}{\partial x_l} + \acute{f}_3 K_j K_l \frac{\partial K_i}{\partial x_l} + \\
& f_4 K_i K_l \frac{\partial K_l}{\partial x_j} + \acute{f}_4 K_j K_l \frac{\partial K_l}{\partial x_i}
\end{aligned} \tag{D.7}
$$

VII)

$$
\begin{aligned}
\eta^{K}_{g_{ij}} =&\{g_1 F_i F_n K_n K_l \frac{\partial K_l}{\partial x_j} + \acute{g}_1 F_j F_n K_n K_l \frac{\partial K_l}{\partial x_i} + g_2 F_i F_n K_n K_l \frac{\partial K_j}{\partial x_l} + \acute{g}_2 F_j F_n K_n K_l \frac{\partial K_i}{\partial x_l} + \\
& g_3 F_i F_n K_j K_m \frac{\partial K_n}{\partial x_m} + \acute{g}_3 F_j F_n K_i K_m \frac{\partial K_n}{\partial x_m} + g_4 F_i F_n K_j K_m \frac{\partial K_m}{\partial x_n} + \acute{g}_4 F_j F_n K_i K_m \frac{\partial K_m}{\partial x_n} + \\
& g_5 F_l F_n K_l K_n \frac{\partial K_i}{\partial x_j} + \acute{g}_5 F_l F_n K_l K_n \frac{\partial K_j}{\partial x_i} + g_6 F_i F_j K_l K_m \frac{\partial K_l}{\partial x_m} + g_7 F_l F_m K_i K_j \frac{\partial K_l}{\partial x_m} + \\
& g_8 F_i F_n K_j K_n \frac{\partial K_l}{\partial x_l}\} \times \sum_{n=0}^{\infty} \alpha_n (K_z)^{2n}
\end{aligned} \tag{D.8}
$$

Appendix E

Constructing mobility for orientation part for an array of polar axisymmetric particles

Idea is to construct the mobility tensor which depends on ∇K, ∇u. Unlike the case of non-polar axisymmetric particles, now the symmetry allows a linear order dependence on ∇u. So, ∇K and ∇u can couple with the terms δ_{ij},F_i,K_i to give the mobility tensor. In this analysis we are ignoring any dependence on the higher orders in ∇K and ∇u.

$\frac{\partial \vec{K}}{\partial t} = \eta^K(\nabla\vec{K}, \nabla\vec{u}, \vec{K})\vec{F}$

$\eta^K(\nabla\vec{K}, \nabla\vec{u}, \vec{K})_{ij} = \eta^K_{a_{ij}} + \eta^K_{b_{ij}} + \eta^K_{c_{ij}} + \eta^K_{d_{ij}} + \eta^K_{e_{ij}} + \eta^K_{f_{ij}} + \eta^K_{g_{ij}} + \eta^K_{h_{ij}}$

I) $\eta^K_{a_{ij}} \longrightarrow$ terms coupling $\nabla K, \vec{F}$
II) $\eta^K_{b_{ij}} \longrightarrow$ terms coupling $\nabla K, \vec{K}$
III) $\eta^K_{c_{ij}} \longrightarrow$ terms coupling $\nabla K, \vec{K}$ and $\vec{F}$
IV $\eta^K_{d_{ij}} \longrightarrow$ terms coupling $\vec{K}, \vec{F}$
V) $\eta^K_{e_{ij}} \longrightarrow$ terms coupling $\nabla u, \vec{F}$
VI) $\eta^K_{f_{ij}} \longrightarrow$ terms coupling $\vec{K}, \vec{F}$
VII) $\eta^K_{g_{ij}} \longrightarrow$ terms coupling $\nabla u, \vec{K}$
VIII) $\eta^K_{h_{ij}} \longrightarrow$ terms coupling $\nabla u, \vec{K}, \vec{F}$

The superscript K is to remind that we are writing the equation of motion for K and this mobility is not to be confused with η^u_{ij}.

I)

$$\eta^K_{a_{ij}} = a_1 F_i F_j \frac{\partial K_k}{\partial x_l}\delta_{kl} + a_2 \frac{\partial K_i}{\partial x_j} + \acute{a_2} \frac{\partial K_j}{\partial x_i} + a_3 F_i F_l \frac{\partial K_j}{\partial x_l} + \acute{a_3} F_j F_l \frac{\partial K_i}{\partial x_l} + a_4 F_i F_l \frac{\partial K_l}{\partial x_j} + \acute{a_4} F_j F_l \frac{\partial K_l}{\partial x_i} \quad \textbf{(E.1)}$$

II)

$$\eta^K_{b_{ij}} = b_1 K_i K_j \frac{\partial K_k}{\partial x_l}\delta_{kl} + b_2 \frac{\partial K_i}{\partial x_j} + \acute{b_2} \frac{\partial K_j}{\partial x_i} + b_3 K_i K_l \frac{\partial K_j}{\partial x_l} + \acute{b_3} K_j K_l \frac{\partial K_i}{\partial x_l} + b_4 K_i K_l \frac{\partial K_l}{\partial x_j} + \acute{b_4} K_j K_l \frac{\partial K_l}{\partial x_i} \quad \textbf{(E.2)}$$

III)

$$\eta^K_{c_{ij}} = \{c_1 F_i F_n K_n K_l \frac{\partial K_l}{\partial x_j} + \acute{c_1} F_j F_n K_n K_l \frac{\partial K_l}{\partial x_i} + c_2 F_i F_n K_n K_l \frac{\partial K_j}{\partial x_l} + \acute{c_2} F_j F_n K_n K_l \frac{\partial K_i}{\partial x_l} + c_3 F_i F_n K_j K_m \frac{\partial K_n}{\partial x_m} + \acute{c_3} F_j F_n K_i K_m \frac{\partial K_n}{\partial x_m} + c_4 F_i F_n K_j K_m \frac{\partial K_m}{\partial x_n} + \acute{c_4} F_j F_n K_i K_m \frac{\partial K_m}{\partial x_n} + c_5 F_l F_n K_l K_n \frac{\partial K_i}{\partial x_j} + \acute{c_5} F_l F_n K_l K_n \frac{\partial K_j}{\partial x_i} + c_6 F_i F_j K_l K_m \frac{\partial K_l}{\partial x_m} + c_7 F_l F_m K_i K_j \frac{\partial K_l}{\partial x_m} + c_8 F_i F_n K_j K_n \frac{\partial K_l}{\partial x_l}\} \times \sum_{n=0}^{\infty} \alpha_n (K_z)^{2n} \quad \textbf{(E.3)}$$

IV

$$\eta^K_{d_{ij}} = \{d_1 F_i F_j K_k K l \delta_{kl} + d_2 K_i K_j + d_3 F_i F_l K_j K_l + \acute{d_3} F_j F_l K_i K_l + d_4 F_i F_l K_l K_j + \acute{d_4} F_j F_l K_l K_i\} \times \sum_{n=0}^{\infty} \alpha_n (K_z)^{2n} \quad \textbf{(E.4)}$$

V)

$$\eta^K_{e_{ij}} = e_1 \delta_{ij} + e_2 F_i F_j + e_3 F_i F_j \frac{\partial u_k}{\partial x_l}\delta_{kl} + e_4 \frac{\partial u_i}{\partial x_j} + \acute{e_4} \frac{\partial u_j}{\partial x_i} + e_5 F_i F_l \frac{\partial u_j}{\partial x_l} + \acute{e_5} F_j F_l \frac{\partial u_i}{\partial x_l} + e_6 F_i F_l \frac{\partial u_l}{\partial x_j} + \acute{e_6} F_j F_l \frac{\partial u_l}{\partial x_i} + e_7 F_i F_j F_k F_l \frac{\partial u_k}{\partial x_l} + e_8 F_i F_k F_k F_l \frac{\partial u_j}{\partial x_l} + \acute{e_8} F_j F_k F_k F_l \frac{\partial u_i}{\partial x_l} + e_9 F_i F_k F_k F_l \frac{\partial u_l}{\partial x_j} + \acute{e_9} F_j F_k F_k F_l \frac{\partial u_l}{\partial x_i} + \text{higher orders in } \nabla u \quad \textbf{(E.5)}$$

This term is the same as for spherical particles. Again we will ignore the higher orders in ∇u *. The higher order contraction of F with itself would just yield a* *constant term which simplifies the above expression further.*

$$\eta^K_{e_{ij}} = e_1\delta_{ij} + e_2F_iF_j + e_3F_iF_j\frac{\partial u_k}{\partial x_l}\delta_{kl} + e_4\frac{\partial u_i}{\partial x_j} + \acute{e_4}\frac{\partial u_j}{\partial x_i} + e_5F_iF_l\frac{\partial u_j}{\partial x_l} + \acute{e_5}F_jF_l\frac{\partial u_i}{\partial x_l}+ e_6F_iF_l\frac{\partial u_l}{\partial x_j} + \acute{e_6}F_jF_l\frac{\partial u_l}{\partial x_i} \tag{E.6}$$

VI)

$$\eta^K_{f_{ij}} = \left\{f_1K_iK_j + f_2F_iF_jK_lK_l + f_3F_iF_lK_lK_j + \acute{f_3}F_jF_lK_lK_i\right\} \times \sum_{n=0}^{\infty}\alpha_n(K_z)^{2n} \tag{E.7}$$

All the K_lK_l terms would just give 1 since $\vec{K}$ is a unit vector. Though K_i can contract with F_i which is given by the summation above on the right. The F_lK_l term gives K_z owing to the force being only in the z direction and since both these terms always appear in pairs we have the summation of K_z^2. As you will see later we will ignore all terms involving higher orders of ***F*** *and hence confine ourselves with n=0,1 terms only.*

VII)

$$\eta^K_{g_{ij}} = g_1K_iK_j\frac{\partial u_k}{\partial x_l}\delta_{kl}+g_2\frac{\partial u_i}{\partial x_j}+\acute{g_2}\frac{\partial u_j}{\partial x_i}+g_3K_iK_l\frac{\partial u_j}{\partial x_l}+\acute{g_3}K_jK_l\frac{\partial u_i}{\partial x_l}+g_4K_iK_l\frac{\partial u_l}{\partial x_j}+\acute{g_4}K_jK_l\frac{\partial u_l}{\partial x_i} \tag{E.8}$$

VIII)

$$\begin{aligned}\eta^K_{h_{ij}} =\{&h_1F_iF_nK_nK_l\frac{\partial u_l}{\partial x_j} + \acute{h_1}F_jF_nK_nK_l\frac{\partial u_l}{\partial x_i} + h_2F_iF_nK_nK_l\frac{\partial u_j}{\partial x_l} + \acute{h_2}F_jF_nK_nK_l\frac{\partial u_i}{\partial x_l}+ \\ &h_3F_iF_nK_jK_m\frac{\partial u_n}{\partial x_m} + \acute{h_3}F_jF_nK_iK_m\frac{\partial u_n}{\partial x_m} + h_4F_iF_nK_jK_m\frac{\partial u_m}{\partial x_n} + \acute{h_4}F_jF_nK_iK_m\frac{\partial u_m}{\partial x_n}+ \\ &h_5F_lF_nK_lK_n\frac{\partial u_i}{\partial x_j} + \acute{h_5}F_lF_nK_lK_n\frac{\partial u_j}{\partial x_i} + h_6F_iF_jK_lK_m\frac{\partial u_l}{\partial x_m} + h_7F_lF_mK_iK_j\frac{\partial u_l}{\partial x_m}+ \\ &h_8F_iF_nK_jK_n\frac{\partial u_l}{\partial x_l}\} \times \sum_{n=0}^{\infty}\alpha_n(K_z)^{2n}\end{aligned} \tag{E.9}$$

Adding all the above terms I, II, III, IV, V, VI, VII and VIII will give the general expression for mobility allowed by symmetry (to linear orders in ∇K and ∇u)

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