K_2 of a rational function field

Sumit Chandra Mishra

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Certificate of Examination

This is to certify that the dissertation titled **$\mathbf{K}_2$ of a rational function field** submitted by **Mr. Sumit Chandra Mishra (Reg. No. MS10065)** for the partial fulfilment of BS-MS dual degree programme of the Institute, has been examined by the thesis committee duly appointed by the Institute. The committee finds the work done by the candidate satisfactory and recommends that the report be accepted.

Dr. Yashonidhi Pandey	Dr. Varadharaj R. Srinivasan	Dr. Amit Kulshrestha (Supervisor)

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Declaration

The work presented in this dissertation has been carried out by me under the guidance of Dr. Amit Kulshrestha at Indian Institute of Science Education and Research Mohali.

This work has not been submitted in part or in full for a degree, a diploma, or a fellowship to any other university or institute. Whenever contributions of others are involved, every effort is made to indicate this clearly, with due acknowledgement of collaborative research and discussions. This thesis is a bonafide record of work done by me and all sources listed within have been detailed in the bibliography.

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(Candidate)

Dated: April 24, 2015

In my capacity as the supervisor of the candidate's project work, I certify that the above statements by the candidate are true to the best of my knowledge.

Dr. Amit Kulshrestha
(Supervisor)

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Sumit Chandra Mishra
MS10065
IISER Mohali

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Chapter 1

Introduction

Let k be a field of characteristic different from 2 and k_s be a separable closure of k. Let L be a finite Galois extension of k contained in k_s. Let G denote $Gal(L/k)$, the Galois group of L over k. Then we have an isomorphism between the second cohomology group $H^2(G, L^\times)$ and the subgroup $Br(L/k)$ of Brauer group $Br(k)$ consisting of Brauer equivalence classes of central simple algebras over k, that are split by L.

$$H^2(G, L^\times) \cong Br(L/k)$$

given by $[f] \mapsto [(k, G, f)]$ denoting the crossed product over L associated to co-cycle f.

The Galois group of k_s over k, denoted by $Gal(k_s/k)$, is a profinite group as it is an inverse limit of finite groups L such that L is a finite Galois extension of k contained in k_s, with ordering on fields L is by inclusion. Let us denote the profinite cohomology group by $H_c^2(G(k_s/k), k_s^\times)$. We have the following isomorphism:

$$H_c^2(G(k_s/k), k_s^\times) := \varinjlim H^2(G(L/k), L^\times) \cong \varinjlim Br(L/k) = Br(k),$$

where the direct limit is taken over all finite Galois extensions L of k contained in k_s. It is also well-known that $Br(k)$ is torsion for any field k.

Brauer groups and Milnor $K-$groups are also related. For example, ${}_2Br(k) \cong K_2(k)/2K_2(k)$. This was proved by Merkurjev in 1981. These relations may be used to get information about central simple algebras over fields using Galois cohomology.

The n-th *Milnor* K*-group* attached to a field k is defined as the quotient of $(k^\times)^{\otimes n}$, the n-th power tensor(over $\mathbb{Z}$) of the multiplicative group of k, by the subgroup generated by those elements $a_1 \otimes \cdots \otimes a_n$ for which $a_i + a_j = 1$ for some $i, 1 \leq i < j \leq n$, where $a_1, \ldots, a_n \in k^\times$. We use $\{a_1, \ldots, a_n\}$ to denote the equivalence class of $(a_1 \otimes \cdots \otimes a_n)$ in $K_n(k)$. We refer to elements of type $\{a_1, \ldots, a_n\}$ as symbols in $K_n(k)$.

Let E be a field and F be the rational function field $E(t)$ in the variable t over E. Let m be a natural number co-prime to characteristic of E. Let $\mathcal{P}$ be the collection of monic irreducible polynomials in $E[t]$. Any $p \in \mathcal{P}$ determines a $\mathbb{Z}$-valuation υ_p on $E(t)$ that is trivial on E with $\upsilon_p(p) = 1$. Other than these, there is a unique $\mathbb{Z}$-valuation υ_∞ on $E(t)$ such that $\upsilon_\infty(f) = -deg(f)$ for any $f \in E[t]\backslash\{0\}$. Let $\mathcal{P}'$ denote $\mathcal{P} \cup \{\infty\}$. For each $p \in \mathcal{P}'$, we write ∂_p to denote the ramification map corresponding to the valuation υ_p and E_p to denote the corresponding residue field. For $p \in \mathcal{P}$, $E_p \cong E[t]/ < p >$ for $p \in \mathcal{P}$, and $E_\infty \cong E$. Then, for every valuation υ_p, we have a homomorphism $\partial_{\upsilon_p} : K_2^m F \to K_1^m E_p$ such that $\partial_{\upsilon_p}(\{f, g\}) = \{(-1)^{\upsilon_p(f)\upsilon_p(g)} \overline{f^{-\upsilon_p(g)} g^{\upsilon_p(f)}}\}$ for $f, g \in F^\times$. Lets denote $\oplus \partial_p$ by ∂, called *ramification map.*

Also, for every $p \in \mathcal{P}'$, we have the norm map of the finite extension E_p over E. These maps induce a group homomorphism from $K_1^{(m)} E_p \to K_1^{(m)} E$ for every $p \in \mathcal{P}'$. Summing over all these induced maps, we get a homomorphism $N : \bigoplus_{p \in \mathcal{P}'} K_1^{(m)} E_p \to K_1^{(m)} E$. Let $\Re'_m(E)$ denote $\bigoplus_{p \in \mathcal{P}'} K_1^{(m)} E_p$.

We then have the following exact sequence [3]

$$0 \to K_2^{(m)} E \to K_2^{(m)} F \xrightarrow{\partial_p} \Re'_m(E) \xrightarrow{N} K_1^{(m)} E \to 0.$$

Let $\Re_m(E)$ denote the kernel of map N, which is same as the image of ∂. For $\rho = (\rho_p)_{p \in \mathcal{P}'} \in \Re'_m(E)$, we shall denote *supp*$(\rho) = \{p \in P' | \rho_p \neq 0\}$ and $\deg(\rho) = \Sigma_{p \in supp(\rho)} [k_p : k]$, and call this the *support* and the *degree* of ρ.

Remark We want to study the relation between the degree of ρ in the image of ∂ and the properties of elements $\xi \in K_2^{(m)}(E(t))$ with $\partial(\xi) = \rho$. More precisely, we want to ask whether given any ρ in the image of ∂ with a fixed degree, does there exist a natural number r such that ξ is a sum of at most r symbols and $\partial(\xi) = \rho$.

In [3], it is shown that for a given element ρ in the image of ∂, how to find a $\eta \in K_2^{(m)}F$ with $\partial(\eta) = \rho$ such that η is a sum of r symbols where r is bounded by half the degree of the support of ρ. All these terms will be defined later. In subsequent chapters, we will define all the terms and basic concepts and later we will prove the results for bound on the minimal length of a sum of symbols in the second Milnor K-group of a rational function field.

Chapter 2

Quaternion Algebras

Unless stated otherwise, k denotes a field of characteristic not 2. We refer to [1] for results of this chapter.

Definition 2.1. (**Quaternion algebra**) : *For any two elements $a, b \in k^{\times}$, we define quaternion algebra (a, b) as the four dimensional k-algebra with basis $1, i, j, ij$, multiplication being determined by*

$$i^2 = a, j^2 = b, ij = -ji.$$

The set $\{1, i, j, ij\}$ is called a quaternion basis of (a, b).

Example The matrix algebra $M_2(k)$ is a quaternion algebra over k.
Indeed, the assignment

$$i \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, j \mapsto \begin{pmatrix} 0 & b \\ 1 & 0 \end{pmatrix}$$

defines an isomorphism (of k-algebras) $(1, b) \cong M_2(k)$.

Definition 2.2. (**Conjugate and Norm**) *: Let (a, b) be a quaternion algebra with basis $\{1, i, j, ij\}$, then for an element $q \in (a, b)$ with $q = x + yi + zj + wij$, its conjugate is defined as $\bar{q} = x - yi - zj - wij$.*
And, its norm is defined as

$$N(q) = q\bar{q}.$$

Lemma 2.3. *An element q of the quaternion algebra (a, b) is invertible if and only if it has non-zero norm. Hence, (a, b) is a division algebra if and only if the norm map $N : (a, b) \to k$ does not vanish outside 0.*

Definition 2.4. *A quaternion algebra over k is called split if it is isomorphic to $M_2(k)$ as a k-algebra.*

Proposition 2.5. *For a quaternion algebra (a,b) , the following statements are equivalent.*

1. *The algebra (a,b) is split.*
2. *The algebra (a,b) is not a division algebra.*
3. *The norm map $N:(a,b)\to k$ has a non-trivial zero.*
4. *The element b is a norm from the field extension $k(\sqrt{a})|k$.*

Lemma 2.6. *Let D be a four dimensional central division algebra over k. Suppose D contains commutative k- sublagebra isomorphic to a non-trivial quadratic field extension $k(\sqrt{a})$ of k, then D is isomorphic to a quarternion algebra (a,b) for suitable $b\in k^{\times}$.*

Proposition 2.7. *A four dimensional central division algebra D over k is isomorphic to a quaternion algebra.*

Proposition 2.8. *Consider a quaternion algebra A over k, and fix an element $a\in k^{\times}$ which is not in $k^{\times 2}$. The following statements are equivalent.*

1. *A is isomorphic to the quaternion algebra (a,b) for some $b\in k^{\times}$.*
2. *The $k(\sqrt{a})$- algebra $A\otimes_k k(\sqrt{a})$ is split.*
3. *A contains a commutative $k-$sublagebra isomorphic to $k(\sqrt{a})$.*

2.1 The associated Conic

To every quaternion algebra (a,b), we can associate a conic, called the associated conic $C(a,b)$, which is the projective plane curve defined by the homogeneous equation

$$ax^2+by^2=z^2 \tag{2.1}$$

where x,y,z are the homogeneous coordinates in the projective plane $\mathbb{P}^2$.

Remark The conic $C(a,b)$ associated to (a,b) doe not depend on the choice of a basis. The conic $C(a,b)$ is isomorphic to the conic $ax^2+by^2=abz^2$ via the substitution $x\mapsto by, y\mapsto ax, z\mapsto abz$ and division of the equation by ab. Now, $ax^2+by^2-abz^2$ is exactly the square of the pure quaternion $xi+yj+zij$ and hence is intrinsically defined. (A pure quaternion is an element q of (a,b) such that $q^2\in k$ but q does not belong to k. Hence, the notion of pure quaternion is intrinsic and does not depend on the basis of the quaternion. It is easy to see that $q=x=yi+zj+wij$ is a pure quaternion if and only if $x=0$).

We recall that the conic $C(a,b)$ defined above is said to have a $k-$rational point if there exist $x_0, y_0, z_0 \in k$, not all zero, that satisfy equation (2.1) above.
We have following proposition.

Proposition 2.9. *The quaternion algebra (a,b) is split if and only if the conic $C(a,b)$ has a k-rational point.*

Proof If $x_0, y_0, z_0 \in k$ is a $k-$rational point on $C(a,b)$ with $y_0 \neq 0$, then $b = (z_0/y_0)^2 - a(x_0/y_0)^2$ and a part (4) of 2.5 is satisfied. If y_0 happens to be 0, then x_0 must be nonzero and we get similarly that a is a norm from the extension $k(\sqrt{b})|k$. Conversely, if $b = r^2 - as^2$ for some $r, s \in k$, then $(s, 1, r)$ is a $k-$ rational point on $C(a,b)$. $\square$

Remark Let $a \in k \backslash \{0,1\}$, then the projective conic $ax^2 + (1-a)y^2 = z^2$ has $(1,1,1)$ as a k-rational point. So, the quaternion algebra $(a, 1-a)$ is isomorphic to the matrix algebra $M_2(k)$. This suggest that there may be some connection between $K_2(k)$ and quaternions. Indeed, there is a connection. The map from $k^\times \times k^\times$ to the 2-torsion subgroup of Brauer group (see next chapter for definition of Brauer group of a field) of k, ${}_2Br(k)$ which is given by $(a,b) \mapsto (a,b)$ induces a homomorphism $\alpha_k : K_2(k) \to {}_2Br(k)$. This further induces a homomorphism $\tilde{\alpha_k} : K_2(k)/2K_2(k) \to {}_2Br(k)$.

2.2 Tensor Product of quaternion algebras

Lemma 2.10. *The tensor product of two matrix algebras $M_n(k)$ and $M_m(k)$ over kis isomorphic to the matrix algebra $M_{nm}(k)$.*

Lemma 2.11. *Given elements $a, b, b' \in k^\times$, we have an isomorphism*

$$(a,b) \otimes_k (a,b') \cong (a,bb') \otimes_k M_2(k).$$

Proof We denote by $(1, i, j, ij)$ and $(1, i', j', i'j')$ quaternion bases of (a,b) and (a,b'), respectively, and consider the k-subspaces

$$A_1 = k(1 \otimes_k 1) \oplus k(i \otimes_k 1) \oplus k(j \otimes_k j') \oplus k(ij \otimes_k j')$$
$$A_2 = k(1 \otimes_k 1) \oplus k(1 \otimes_k j') \oplus k(i \otimes_k i'j') \oplus k((-b'i) \otimes_k i')$$

of $(a,b) \otimes_k (a,b')$. One checks that A_1 and A_2 are both closed under multiplication and hence are sub-algebras of $(a,b) \otimes_k (a,b')$. By squaring the basis elements $i \otimes_k 1$,$j \otimes_k j'$ and $1 \otimes_k j'$,$i \otimes_k i'j'$, we see that A_1 and A_2 are isomorphic to the quaternion algebras (a, bb') and $(b', -a^2b')$, respectively. But this latter algebra is isomorphic to $(b', -b')$,

which is split because the conic $C(b', -b')$ has the $k-$rational point $(1, 1, 0)$.
Now, we consider the map $\rho : A_1 \otimes_k A_2 \to (a, b) \otimes_k (a, b')$ induced by the k-bilinear map $(x, y) \to xy$. It can be checked that all basis elements of $(a, b) \otimes_k (a, b')$ lie in the image of ρ, so it is surjective and hence induces the required isomorphism for dimension reasons. $\square$

Remark Let $[(x, y)]$ be the Brauer equivalence class (please see next chapter for definition) of the quaternion algebra (x, y), for $x, y \in k^{\times}$. Then, the above proposition implies that $[(a, b)][(a, b')] = [(a, bb')]$, where the operation is group multiplication in Brauer group of k. Similarly, $[(b, a)][(b', a)] = [(bb', a)]$. And, we already showed in the previous remark that $(a, 1 - a)$ is a matrix algebra, which means $[(a, 1 - a)]$ is the identity of the Brauer group of k(see next chapter for definition of Brauer group of a field). We will see later(chapter 4) that the definition of K_2 of a field essentially captures all these relations.

Corollary 2.12. *For a quaternion algebra (a, b) over k, the tensor product algebra $(a, b) \otimes_k (a, b)$ is isomorphic to the matrix algebra $M_4(k)$.*

Remark In terms of Brauer group of k, it means that Brauer equivalence class of quaternion algebra (a, b) is of order 2.

Chapter 3

Central simple algebras and Brauer groups

In this chapter k will denote a field of characteristic not equal to 2. We refer to [8] and [1] for results of this chapter.

Definition 3.1. *An algebra A over k is called central simple if the centre of A is k, $[A : k] < \infty$ and the only two sided ideals of A are 0 and A.*

Definition 3.2. *An extension K/k of fields is called a splitting field for A if $K \otimes_k A \cong M_n(K)$, where A is a central simple algebra over k.*

Lemma 3.3. *Let A be a central simple algebra over k and B a k-algebra whose only two sided ideals are 0 and B. Then the only two-sided ideals of $A \otimes_k B$ are 0 and $A \otimes_k B$.*

Proof By Wedderburn's theorem, $A \cong M_r(D)$, where D is a division ring, and centre(D) =centre$M_r(D)$ =centre$(A) = k$. Since, every two-sided ideal of $M_r(D \otimes_k B)$ comes from a two-sided ideal of $D \otimes_k B$, we replace A by D and assume that A is finite dimensional central division algebra over k. Let $C \neq 0$ be an two-sided ideal of $A \otimes_k B$. Let $\{e_i\}_{i \in I}$ be a $k-$basis for B. Every element $a \in C, a \neq 0$, can be uniquely written as $\Sigma_{i \in J} a_i \otimes_k e_i, J \subset I, a_i \in A$. We call $l(a) = |J|$. We choose $a \in C$ with $l(a)$ minimal. Replacing a by $(a_{j_0}^{-1} \otimes_k 1)a$, for some $j_0 \in J$, we may assume $a_{j_0} = 1$. For any $d \in A, a' = (d \otimes_k 1)a - a(d \otimes_k 1) = \sigma(da_i - a_i d) \otimes_k e_i \in C$ and $l(a') < l(a), a_{j_0}$ being 1, unless $a' = 0$. Since $l(a)$ is minimal, $a' = 0 \implies da_i = a_i d$ for all $i \in J \implies a_i \in k$ for all $i \in J \implies a \in C \cap 1 \otimes_k B$. Since, B is simple, $C \cap (1 \otimes_k B) = 1 \otimes B \implies 1 \otimes_k 1 \in C \implies C = A \otimes_k B$.

□

Lemma 3.4. *Let A and B be k-algebras, then centre$(A \otimes_k B) =$ centre$A \otimes_k$ centre$B$.*

Proof Clearly, centre$A\otimes_k$ centre$B \subset$ centre$(A\otimes_k B)$. Let $x \in centre(A\otimes_k B)$. Write $x = \Sigma_i e_i \otimes b_i, \{e_i\}_{i\in I}$ a basis of A over k, the condition $(1 \otimes_k b)x = x(1 \otimes_k b)$ for all $b \in B$ implies, by the linear independence of e_i, that $bb_i = b_i b$ for all $b \in B$. Thus, the centre$(A \otimes_k B) \subset A\otimes_k$ centre(B). Similarly, centre$(A \otimes_k B) \subset$ centre$(A) \otimes_k B$. Thus, centre$(A \otimes_k B) \subset (A\otimes_k$ centre $(B)) \cap (centre(A) \otimes_k B) \subset$ centre$(A)\otimes_k$ centre(B). $\square$
Therefore, we have following proposition.

Proposition 3.5. *If A and B are central simple algebras over k, then $A \otimes_k B$ is a central simple algebra over k.*

Proposition 3.6. *Let A be an algebra over k. Then, the following are equivalent.*

1. *A is a central simple algebra over k.*
2. *There exists a field extension L/k such that $L \otimes_k A \cong M_n(L)$ for some natural number n. When this happens, we say A is form over k for the matrix algebra.*

Proof Let A be a form over k for the matrix algebra and let L be a field extension such that $L \otimes_k A \cong M_n(L)$. Then, $[A : k] = [M_n(L) : L] = n^2$. By 3.4,

$$\text{centre}(L \otimes_k A) = L\otimes_k \text{centre } A = \text{centre } M_n(L) = L.$$

Thus, $[centreA : k] = [L \otimes_k centreA : L] = 1$ and centre $A = k$. If C is a two-sided ideal of A, then $L \otimes C \neq 0$ is a two-sided ideal of $L \otimes_k A \cong M_n(L)$. Since $M_n(L)$ is simple, we must have $L \otimes C = L \otimes_k A$. Hence, $C = A$. Suppose now that A is a central simple algebra over k. Let $\bar{k}$ denote the algebraic closure of k. By lemmas above, $\bar{k} \otimes k$ is central simple over $\bar{k}$. Since the only finite dimensional division algebra over an algebraically closed field is itself, it follows by Wedderburn's theorem that $\bar{k} \otimes k \cong M_n(\bar{k})$. $\square$

Proposition 3.7. *Every central simple algebra A over k admits of a splitting field L which is a finite extension of k.*

Proof Let $\bar{k}$ denote the algebraic closure of k and $\phi : \bar{k} \otimes_k A \cong M_n(\bar{k})$ be an isomorphism of $\bar{k} - algebras$. If $e_i, 1 \leq i \leq n^2$ is a k-basis of A and $\phi(1 \otimes e_i) = \Sigma\lambda_{ijk}e_{jk}, e_{jk}, 1 \leq j, k \leq n$ denoting the standard basis of $M_n(\bar{k})$, we set $L = k(\lambda_{ijk}), 1 \leq i \leq n^2, 1 \leq j, k \leq n$. Then, ϕ induces a L-algebra homomorphism $\tilde{\phi} : L\otimes_k A \to M_n(L)$. Since $L \otimes A$ is simple , $\tilde{\phi}$ is injective. Since $n^2 = [A : k] = [M_n(L); L], \tilde{\phi}$ is an isomorphism. $\square$

We consider the set S of isomorphism classes of all central simple algebras over k. The set S is a commutative monoid with tensor product over k as operation, and the class of k as the identity element. If A is a central simple algebra over k, then $A \cong M_n(D_A)$, where D_A is a central division algebra over k, whose isomorphism class is uniquely determined by A. We define $A \sim B$(*Brauer equivalent*) if and only if $D_A \cong D_B$. We denote by $[A]$ the class of A in $S/_\sim$. We note that if $A \sim B$ and $[A : k] = [B : k]$, then $A \cong B$. Also, two central simple algebras A and B are Brauer equivalent if and only if $M_r(A) = M_s(B)$ for some integers r and s. The equivalence relation on S is compatible with the monoid structure of S i.e. $A \sim A', B \sim B' \implies A \otimes_k B \sim A' \otimes B'$. Thus, the set $S/_\sim$ is again a commutative monoid with class of all matrix algebras over k as identity element. The following proposition shows that , in fact, $S/_\sim$ is a group.

Proposition 3.8. *For a central simple algebra A over k, if A^{op} denotes the opposite algebra, then A^{op} is central simple and $[A][A^{op}] = [k]$ in $S/_\sim$.*

Proof If A is central simple, clearly A^{op} is again central simple. The maps $A \to End_k A, a \mapsto L_a$ and $A^{op} \to R_a, L_a, R_a$ denoting the left and right multiplications, induce a homomorphism $\phi : A \otimes_k A^{op} \to End_k A$, since $L_a \circ R_b = R_b \circ L_a, a, b \in A$. Since $A \otimes_k A^{op}$ is simple, ϕ is injective. Further, $[A \otimes_k A^{op}] = [A : k]^2 = [End_k A : k]$ so that ϕ is surjective and hence an isomorphism. For a choice of basis of A over k, $End_k A$ is isomorphic to a matrix algebra over k. $\square$

The group $S/_\sim$ is called the *Brauer group* of k, denoted by $Br(k)$. The assignment $A \mapsto D_A$ yields a bijection between $Br(k)$ and the set of isomorphism classes of central division algebras over k. Thus, the Brauer group classifies finite dimensional central division algebras over k. If k is algebraically closed, $Br(k)$ is trivial, since the only finite dimensional division algebra over k is itself. Also, if k is finite, in view of a celebrated theorem by Wedderburn, $Br(k)$ is trivial.
For any field k, the Brauer group $Br(k)$ is torsion i.e every element of the group has a finite order. Also, it is known that $2-$torsion Brauer group of a field k is generated by quaternion algebras. The assignment $K \mapsto Br(k)$ is functorial. In fact, if $k \hookrightarrow K$ is an injection of fields, we have an induced functorial homomorphism $Br(k) \to Br(K)$ defined by $[A] \mapsto [K \otimes_k A]$.

Proposition 3.9. *Let $k(X)$ denote the rational function field in the variable X. The inclusion $k \hookrightarrow k(X)$ induces an injection $Br(k) \mapsto Br(k(X))$.*

We have these important results:

Proposition 3.10. *If A is a central simple algebra over k, there exists a finite Galois extension L over k which splits A.*

Proposition 3.11. *Let L be a finite Galois extension of k with Galois group G. Then, we have an isomorphism $H^2(G, L^\times) \cong Br(L/k)$, where $Br(L/k)$ is a subgroup of $Br(k)$ consisting of Brauer classes that are split by L.*

Proposition 3.12. *The two torsion Brauer group of k is generated by quaternion algebras over k.(for $char(k) = 2$ by Albert and for $char(k) \neq 2$ follows from Merkurjev's theorem)*

3.1 Crossed Products

Let L be a finite Galois extension of k with Galois group $G(L/k) = G$. Then, L and $L^\times = L \backslash \{0\}$ are $\mathbb{Z}[G]$ modules , via the action of G on L.

A (normalised) 2- cocycle of G with values in $L^\times$ is a map $f : G \times G \to L^\times$ with the property $f(1,1) = 1$, and for $\sigma_1, \sigma_2, \sigma_3 \in G$,

$$\sigma_1 f(\sigma_2, \sigma_3) f(\sigma_1\sigma_2, \sigma_3)^{-1} f(\sigma_1, \sigma_2\sigma_3) f(\sigma_1, \sigma_2)^{-1} = 1.$$

It is easy to see that $f(1, \sigma) = f(\sigma, 1) = 1$ for all $\sigma \in G$. The 2- cocycles form an abelian group under the operation

$$(f + g)(\sigma_1, \sigma_2) = f(\sigma_1, \sigma_2) g(\sigma_1, \sigma_2)$$

This group is denoted by $\mathbb{Z}^2(G, L^\times)$.

A (normalised)2- coboundary is a map $\delta h : G \times G \to L^\times$ of the form $(\sigma_1, \sigma_2) \mapsto \sigma_1(h(\sigma_2))h(\sigma_1\sigma_2)^{-1}$, where $h : G \to L^\times$ is a map with $h(1) = 1$. δh is a 2- cocycle and 2-coboundaries form a subgroup denoted by $\mathbb{B}^2(G, L^\times)$ of $\mathbb{Z}^2(G, L^\times)$. Let $H^2(G, L^\times) = \mathbb{Z}^2(G, L^\times)/\mathbb{B}^2(G, L^\times)$. We call $H^2(G, L^\times)$ the second cohomology group of G with coefficients in $L^\times$.

Let $f \in \mathbb{Z}^2(G, L^\times)$. For every $\sigma \in G$, let e_σ denote a symbol. We let (k, G, f) be the free L- vector space on the set $\{e_\sigma\}, \sigma \in G$, as a basis. We , then, define a multiplication on (k, G, f) by setting $(\lambda e_{\sigma_1})(\mu e_{\sigma_2}) := \lambda\sigma_1(\mu) f(\sigma_1, \sigma_2) e_{\sigma_1\sigma_2}$ for all $\sigma_1, \sigma_2 \in G$ and $\lambda, \mu \in L$, and, then extending it to (k, G, f) by distributivity. The algebra (k, G, f) is called a *crossed-product* over L.

Remark The multiplication defined above makes (k, G, f) into a central simple algebra over k. Also, if for $f, g \in \mathbb{Z}^2(G, L^\times)$, then, (k, G, f) and (k, G, g) are isomorphic if and only if $f - g \in \mathbb{B}^2(G, L^\times)$. And, for $f \in \mathbb{Z}^2(G, L^\times)$, (k, G, f) is isomorphic to matrix algebra over k if and only if $f \in \mathbb{B}^2(G, L^\times)$. Proofs of these facts can be found in [8].

Theorem 3.13. *Let L be a Galois extension of k with Galois group G. Then, we have an isomorphism $c : H^2(G, L^\times) \cong Br(L/k)$ given by $[f] \mapsto [(k, G, f)]$.*

Chapter 4

Milnor K- theory

Throughout this chapter, the unadorned tensor product denotes the tensor product over $\mathbb{Z}$. Milnor K-groups were defined in [5] and we recall them here. We refer to [1] for results of this chapter.

The n-th *Milnor K-group* attached to a field k is defined as the quotient of $(k^\times)^{\otimes n}$, the n-th power tensor(over $\mathbb{Z}$) of the multiplicative group of k, by the subgroup generated by those elements $a_1 \otimes \cdots \otimes a_n$ for which $a_i + a_j = 1$ for some $i, 1 \leq i < j \leq n$, where $a_1, \ldots, a_n \in k^\times$. Thus, $K_0(k) = \mathbb{Z}$ and $K_1(k) = k^\times$. Elements of $K_n(k)$ are called *symbols*. We write $\{a_1, \ldots, a_n\}$ for the image of $a_1 \otimes \cdots \otimes a_n$ in $K_n(k)$. The relation $a_i + a_j = 1$ will be often referred to as the *Steinberg relation*. *Milnor K-groups* are functorial with respect to field extensions: given an inclusion $\phi : k \hookrightarrow K$, there is a natural map $i_{K|k} : K_n(k) \to K_n(K)$ induced by ϕ. Given $\alpha \in K_n(k)$, we shall abbreviate $i_{K|k}(\alpha)$ by α_K. There is a natural product structure

$$K_n(k) \times K_m(k) \to K_{n+m}(k), (\alpha, \beta) \mapsto \{\alpha, \beta\} \tag{4.1}$$

coming from the tensor product pairing $(k^\times)^{\otimes n} \times (k^\times)^{\otimes n}$ which obviously preserves the Steinberg relation. This product operation equips the direct sum

$$K_*(k) = \bigoplus_{n \geq 0} K_n(k)$$

with the structure of a graded ring indexed by the non-negative integers. The ring $K_*(k)$ is commutative in graded sense.

Proposition 4.1. *The product operation as defined in equation* 4.1 *is graded-commutative, i.e. it satisfies*

$$\{\alpha, \beta\} = (-1)^{mn}\{\beta, \alpha\}$$

for $\alpha \in K_n(k), \beta \in K_m(k)$.

For the proof, we establish an easy lemma :

Lemma 4.2. *The group* $K_2(k)$ *satisfies the relations*

$$\{x, -x\} = 0 \text{ and } \{x, x\} = \{x, -1\}.$$

Proof For the first relation, we compute in $K_2(k)$

$$\{x, -x\} + \{x, -(1-x)x^{-1}\} = \{x, 1-x\} = 0,$$

and, so

$$\{x, -x\} = -\{x, -(1-x)x^{-1}\} = \{x, 1-x^{-1}\} = \{x^{-1}, 1-x^{-1}\} = 0.$$

The second one follows by bilinearity as $\{x, x\} = \{x, -1\} + \{x, -x\}$. □

Proof of Proposition 4.1 : By the previous lemma, in $K_2(k)$, we have the equalities

$$0 = \{xy, -xy\} = \{x, -x\} + \{x, y\} + \{y, x\} + \{y, -y\} = \{x, y\} + \{y, x\},$$

which establishes the proposition in the case $n = m = 1$. The proposition follows from this by a straightforward induction. □

We recall that a $\mathbb{Z}$ valuation υ on k, where k is a field, is a surjective map $\upsilon : k^\times \to \mathbb{Z}$ such that:

- $\upsilon(xy) = \upsilon(x) + \upsilon(y)$
- $\upsilon(x+y) \geq \min(\upsilon(x), \upsilon(y))$ for all $x, y \in k^\times$

The set $A := A_\upsilon = \{x \in k^\times : \upsilon(x) \geq 0\}$ is a ring, called *valuation ring*. This ring has $I = \{x \in k^\times : \upsilon(x) > 0\}$ as its unique maximal ideal. The field $\kappa := \kappa_\upsilon := A/I$ is called its residue field. For $x \in A$, let $\bar{x}$ denote the image of x in κ_υ under the natural surjection map.

Let k be a field equipped with a $\mathbb{Z}$ valuation $\upsilon : k^\times \to \mathbb{Z}$. We denote by A the associated discrete valuation ring and by κ its residue field. Once a local parameter π (i.e. an element with $\upsilon(\pi) = 1$) is fixed, each element $x \in k^\times$ can be uniquely written as a product $u\pi^i$ for some unit u of A and integer i. From this, it follows by bilinearity and graded-commutativity of symbols that the groups $K_n(k)$ are generated by symbols of the form $\{\pi, u_2, \ldots, u_n\}$ and $\{u_1, u_2, \ldots, u_n\}$, where the u_i are the units in A.

Proposition 4.3. *For each $n \geq 1$, there exist a unique homomorphism*

$$\partial : K_n(k) \to K_{n-1}(\kappa)$$

satisfying

$$\partial(\pi, u_2, \ldots, u_n) = \{\bar{u_2}, \ldots, \bar{u_n}\} \tag{4.2}$$

for all local parameters π and all $(n-1)$- tuples $(u_2, \ldots, u_n)$ of units of A, where $\bar{u}_i$ denotes the image of u_i in κ.
Moreover, once a local parameter π is fixed, there is a unique homomorphism

$$s_\pi : K_n(k) \to K_n(\kappa)$$

with the property

$$s_\pi(\{\pi^{i_1}u_1, \ldots, \pi^{i_n}u_n\}) = \{\bar{u_1}, \ldots, \bar{u_n}\} \tag{4.3}$$

for all n-tuples of integers $(i_1, \ldots, i_n)$ and units $(u_1, \ldots, u_n)$ of A.

The map ∂ is called the *tame symbol* or the *residue map* for Milnor $K-$ theory; the maps s_π are called *specialisation maps.* We note that s_π depends on the choice of π, whereas ∂ does not, as seen from its definition.

Proof Uniqueness for s_π is obvious, and that of ∂ follows from the above remark on generators of $K_n(k)$, in view of the fact that a symbol of the form $\{u_1, \ldots, u_n\}$ can be written as a difference $\{\pi u_1, u_2, \ldots, u_n\} - \{\pi, u_2, \ldots, u_n\}$ with local parameters π and πu_1, and hence must be annihilated by ∂.

We prove the existence simultaneously for ∂ and the s_π via a construction due to Serre. Consider the free graded-commutative $K_*(\kappa)$- algebra $K_*(\kappa)[x]$ on one generator of degree 1. By definition, its elements can be identified with polynomials with coefficients in $K_*(\kappa)$, but the multiplication is determined by $\alpha x = -x\alpha$ for $\alpha \in K_1(\kappa)$. Now, take the quotient $K_*(\kappa)[\eta]$ of $K_*(\kappa)[x]$ by the ideal $(x^2 - -1x)$, where $\{-1\}$ is regarded as a symbol in $K_1(\kappa)$. The image of η in the quotient satisfies $\eta^2 = -1\eta$. The ring $K_*(\kappa)[\eta]$ has a natural grading in which η has a degree 1: one has

$$K_*(\kappa)[\eta] = \bigoplus_{n \geq 0} L_n,$$

where $L_n = K_n(\kappa) \oplus \eta K_{n-1}(\kappa)$ for $n > 0$ and $L_0 = K_0(\kappa) = \mathbb{Z}$.
Now, we fix a local parameter π and consider the group homomorphism

$$d_\pi : k^\times \to L_1 = \kappa^\times \bigoplus \eta\mathbb{Z}$$

given by $\pi^i u \mapsto (\bar{u}, \eta i)$. Taking tensor powers and using the product structure in $K_*(\kappa)[\eta]$, we get maps

$$d_\pi^{\otimes n} : (k^\times)^{\otimes n} \to L_n = K_n(\kappa) \oplus \eta K_{n-1}(\kappa).$$

Denoting by $\pi_1 : L_n \to K_n(\kappa)$ and $\pi_2 : L_n \to K_{n-1}(\kappa)$ the natural projections, put

$$\partial := \pi_2 \circ d_\pi^{\otimes n} \text{ and } s_\pi := \pi \circ d_\pi^{\otimes n}.$$

One sees that these maps satisfy the required properties. Therefore the construction will be complete if we show that $d_\pi^{\otimes n}$ factors through $K_n(k)$, for then so do ∂ and s_π. Concerning our claim about $d_\pi^{\otimes n}$, it is enough to establish the Steinberg relations $d_\pi(x)d_\pi(1-x) = 0$ in L_2. To do so, note first that the multiplication map $L_1 \times L_1 \to L_2$ is given by

$$(x, \eta i)(y, \eta j) = (x, y, \eta\{(-1)^{ij} x^j y^i\}), \tag{4.4}$$

where apart from the definition of the L_i we have used the fact that the multiplication map $K_0(\kappa) \times K_1(\kappa) \to K_1(\kappa)$ is given by $(i, x) \mapsto x^i$.

Now, take $x = \pi^i u$. If $i > 0$, the element $1 - x$ is a unit, hence $d_\pi(1 - x) = 0$ and the Steinberg relation holds trivially. If $i < 0$, then $1 - x = (-u + \pi^{-1}\pi^i)$ and $d_\pi(1 - x) = (-\bar{u}, \eta i)$. It follows from (4.4) that

$$d_\pi(x)d_\pi(1 - x) = (\bar{u}, \eta i)(-\bar{u}, \eta i) = (\{\bar{u}, -\bar{u}\}, \eta\{(-1)^{i^2} \bar{u}^{-i}(-\bar{u})^i\}),$$

which is 0 in L_2. It remains to treat the case $i = 0$. If $\upsilon(1 - x) \neq 0$, then replacing x by $1 - x$, we arrive at one of the above cases. If $\upsilon(1 - x) = 0$, i.e. x and $1 - x$ both are units, then $d_\pi(x)d_\pi(1 - x) = (\{\bar{u}, 1 - \bar{u}\}, 0 \cdot \eta) = 0$, and the proof is complete. $\square$

Example The tame symbol $\partial : K_1(k) \to K_0(\kappa)$ is none but the valuation map $\upsilon : k^\times \to \mathbb{Z}$.

Chapter 5

The second Milnor K-group of a rational function field

The results of this chapter are based on [3]. We have already introduced Milnor K-groups in the previous chapters.

Let E denote a field, $E(t)$ denote the rational function field in one variable t over E and $\mathcal{P}$ denote the set of monic irreducible polynomials p in $E[t]$. Let E_p be the quotient $E[t]/(p)$. Any $p \in \mathcal{P}$ determines a $\mathbb{Z}-$ valuation υ_p on $E(t)$ that is trivial on E and such that $\upsilon_p(p) = 1$. We describe the valuation υ_p explicitly. Let $\phi(t) \in E(t)\backslash\{0\}$, then we can write $\phi(t) = p(t)^{n_1} f(t)/p(t)^{n_2} g(t)$ for n_1, n_2 non-negative integers, where $p(t)$ neither divides $f(t)$ nor $g(t)$. Then $\upsilon_p(\phi) = n_1 - n_2$. Apart from these valuations, there is a unique $\mathbb{Z}-$valuation, denoted by υ_∞, on $E(t)$ such that $\upsilon_\infty(f(t)/g(t)) = -\deg(f(t)) + \deg(g(t))$ for any $f(t), g(t) \in E[t]\backslash\{0\}$.(We may call it the degree valuation)

Let $\mathcal{P}'$ denote $\mathcal{P} \cup \{\infty\}$. For each $p \in \mathcal{P}'$, we write ∂_p to denote the ramification map corresponding to the valuation υ_p and E_p to denote the corresponding residue field. The map $\partial_p : K_2^{(m)} E(t) \to K_1^{(m)} E_p$ is given by $\partial_p : K_2^m E(t) \to K_1^m E_p$ such that $\partial_{\upsilon_p}(\{f, g\}) = \{(-1)^{\upsilon_p(f)\upsilon_p(g)} \overline{f^{-\upsilon_p(g)} g^{\upsilon_p(f)}}\}$ for $f, g \in F^\times$. For $p \in \mathcal{P}$, $E_p \cong E[t]/ < p >$ for $p \in \mathcal{P}$, and $E_\infty \cong E$.

Finally, we set $\mathcal{P}' := \mathcal{P} \cup \{\infty\}$.

The Faddeev Exact Sequence It is well known that for m co-prime to the characteristic of the field E, the sequence

$$0 \to K_n(m)E \to K_n^{(m)} E(t) \xrightarrow{\oplus \partial_p} \bigoplus_{p \in \mathcal{P}} K_{n-1}^{(m)} E_p \to 0, \tag{5.1}$$

is split exact.

In what follows, we take $n = 2$ and set $\Re'_m(E) = \bigoplus_{p \in \mathcal{P}'} K_1^{(m)} E_p$.
For $p \in \mathcal{P}'$, the norm map of the finite field extension yields a group homomorphism from $K_1^{(m)} E_p$ to $K_1^{(m)} E$. Summing over all these maps for all $p \in \mathcal{P}'$ yields a homomorphism $N : \Re'_m(E) \to K_1^{(m)} E$.. We set $\partial = \bigoplus_{p \in \mathcal{P}'} \partial_p$, where ∂_p is the residue map corresponding to the valuation v_p. By [1, (7.2.4), (7.2.5)], we obtain an exact sequence

$$0 \to K_2^{(m)} E \to K_2^{(m)} F \xrightarrow{\partial} \Re'_m(E) \xrightarrow{N} K_1^{(m)} E \to 0. \tag{5.2}$$

Let $\Re_m$ denote the kernel of N, which is same as the image of $\partial : K_2^{(m)} E(t) \to \Re'_m(E)$. We will work with this sequence.

Notation For $\rho = (\rho_p)_{p \in \mathcal{P}'} \in \Re'_m(E)$, the *support* of ρ, denoted by $\mathrm{supp}(\rho)$, is defined as $\mathrm{supp}(\rho) = \{p \in \mathcal{P}' | \rho_p \neq 0\}$. And, *degree* of ρ, denoted by $\deg(\rho)$, is defined as $\deg(\rho) = \Sigma_{p \in \mathrm{supp}(\rho)} [E_p : E]$. We note that the degree of an element of $\Re'_m(E)$ is invariant under $E-$ automorphisms of $E(t)$.

We want to study the relation between the degree of $\rho \in \Re_m(E)$ and the properties of elements $\xi \in K_2^{(m)} E(t)$ with $\partial(\xi) = \rho$. More precisely, we want to ask whether given any $\rho \in \Re_m(E)$ with a fixed degree, does there exist a natural number r such that ξ is a sum of at most r symbols and $\partial(\xi) = \rho$. We will show that this is indeed the case with r being the integral part of $\deg(\rho)/2$. So, we will prove following result:

Theorem 5.1. *For $\rho \in \Re_m(E)$ and $r = \lfloor \deg(\rho)/2 \rfloor$, there exists symbols $\sigma_1, \ldots, \sigma_n$ in $K_2^{(m)} E(t)$ such that $\rho = \partial(\sigma_1 + \cdots + \sigma_r)$.*

5.1 Example showing the bound obtained is sharp

The example here is adapted from [4, Proposition 2].
For any $a \in A$, we have a unique homomorphism $s_a : K_n^{(m)} E(t) \to K_n^{(m)} E$ such that $s_a(\{f_1, \ldots, f_n\}) = \{f_1(a), \ldots, f_n(a)\}$ for any $f_1, \ldots, f_n \in E[t]$ prime to $t - a$ and such that $s_a(\{t - a, ., \ldots, .\}) = 0$ [1, (7.1.4)].

Lemma 5.2. *The homomorphism* $s := s_0 - s_1 : K_n^{(m)}E(t) \to K_N^{(m)}E$ *satisfies following properties*:

(*a*) $s(K_n^{(m)}E) = 0$.

(*b*) $s(\{(1-a)t + a, b_2, \ldots, b_n\}) = \{a, b_2, \ldots, b_n\}$ *for any* $a, b_2, \ldots, b_n \in E^\times$.

(*c*) *Any symbol in* $k_n^{(m)}E(t)$ *is mapped under* s *to a sum of two symbols in* $K_n^{(m)}E$.

Proof (*a*) is clear. (*b*) is true as $s_1(\{(1-a)t + a, b_2, \ldots, b_n\}) = 0$. (*c*) is true as s_0 and s_1 map symbols to symbols, by definition. □

Proposition 5.3. *Let* d *be a natural number,* $a_1 \ldots, a_d$*, and* $\sigma_1, \ldots, \sigma_d$ *symbols in* $K_{n-1}^{(m)}E$*. Assume that* $\Sigma_{i=1}^d \{a_i\}\dot{\sigma}_i \in K_n^{(m)}E$ *is not equal to a sum of less than d symbols and let*

$$\eta = \Sigma_{i=1}^d \{(1-a_i)t + a_i\}\dot{\sigma}_i \in K_n^{(m)}E(t).$$

Then, $\deg(\partial(\eta)) = d+1$*, and if* $r \in \mathbb{N}$ *is such that* $\partial(\eta) = \partial(\tau_1 + \cdots + \tau_r)$ *for symbols* $\tau_1, \ldots, \tau_r$ *in* $K_n^{(m)}E(t)$*, then* $r \geq \lfloor (d+1)/2 \rfloor$.

Proof For $i = 1, \ldots, d$, we would have $\{a_i\}\dot{\sigma}_i \neq 0$ and hence, $a_i \neq 1$. So, we consider $p = t + [a_i/(1-a_i)]$.
$\partial_p(\eta) = \sigma_i \neq 0$ in $K_{n-1}^{(m)}E$.

Furthermore, since

$$\Sigma_{i=1}^d \{a_i\}\dot{\sigma}_i \neq \Sigma\{a_i a_d^{-1}\}\dot{\sigma}_i,$$

we have $\partial_\infty(\eta) = -\Sigma_{i=1}^d \sigma \neq 0$ in $K_{n-1}^{(m)}E$. Therefore, we obtain

$$\mathrm{supp}(\partial(\eta)) = \{t + [a_i/1 - a_i] | 1 \leq i \leq d\} \cup \{\infty\}$$

and thus $\deg(\partial(\eta)) = d+1$.

Now, let $r \in \mathbb{N}$ and $\partial(\eta) = \partial(\tau_1 + \cdots + \tau_r)$ for symbols $\tau_1, \ldots, \tau_r$ in $K_n^{(m)}E(t)$. Then, $\tau_1 + \cdots + \tau_r - \eta$ is defined over E. Let s be the map as described in the lemma above. Then, we obtain $s(\tau_1 + \cdots + \tau_r - \eta) = 0$ and thus

$$\Sigma_{i=1}^d \{a_i\}\dot{\sigma}_i = s(\eta) = s(\tau_1) + \cdots + s(\tau_r) \in K_n^{(m)}E,$$

which is a sum of $2r$ symbols. Hence, $2r \geq d$, by the hypothesis on d. □

Example We take a prime p dividing m. Also, let k be a field containing a primitive p-th root of unity ω and $a_1, \ldots, a_d \in k^\times$ such that the Kummer extension $k(\sqrt[p]{a_1}, \ldots, \sqrt[p]{a_d})$ of k has degree p^d. Let $b_1, \ldots, p_d$ be the indeterminates over k and set $E = k(b_1, \ldots, b_d)$. Using [T1987,(2.10)] and [2, (2.10)] it follows that $\Sigma_{i=1}^d \{a_i, b_i\}$ is not equal to a sum of less than d symbols in $K_2^{(p)}E$. Since $p|m$, it follows immediately that $\Sigma_{i=1}^d \{a_i, b_i\} \in K_2^{(m)}E$ is not a sum of less than d symbols in $K_2^{(m)}E$. We consider

$$\eta = \Sigma_{i=1}^d \{(1 - a_i)t + a_i, b_i\}$$

in $K_2^{(m)}E(t)$. By previous proposition, for $\rho = \partial(\eta)$, we have that $\deg(\rho) = d + 1$ and $\rho \neq \partial(\eta')$ for any $\eta' \in K_2^{(m)}E(t)$ that is a sum of less than $r = \lfloor \deg(\rho)/2 \rfloor$ symbols.

5.2 Proof of the theorem

We start with propositions and lemmas which lead to the above mentioned result.

Proposition 5.4. *If* $\rho \in \Re_m(E)$*, then* $\deg(\rho) \neq 1$.

Proof Let $\rho \in \Re'_m$ with $\deg(\rho) = 1$. The support of ρ consists of one rational point $p \in \mathcal{P}'$. Thus, $N(\rho) = \rho_p \neq 0$ in $K_1^{(m)}E$, hence ρ cannot belong to $\Re_m(E)$.

□

Definition 5.5. *We say* $p \in \mathcal{P}'$ *is rational if* $[E_p : E] = 1$*. We call a subset of* $\mathcal{P}'$ *rational if all its elements are rational.*

Remark We want to find a bound for the number of symbols r so that any $\rho \in \Re_m(E)$ is an image of $\xi \in K_2^{(m)}E(t)$ under the map ∂, where ξ is a sum of at most r symbols. A natural way would be to use the principle of mathematical induction and it is the method we will use for the proof. But, for applying induction, we need to find some way to reduce the degree of a given element in $\Re_m(E)$ upto image of a symbol in $K_2^{(m)}E(t)$. This is what we do in the next lemma.

Lemma 5.6. *Let* ρ *be an element of* $\Re'_m(E)$ *with* $\deg(\rho) \geq 2$*. Then there exists a symbol* σ *in* $K_2^{(m)}E(t)$ *such that* $\deg(\rho - \partial(\sigma)) \leq \deg(\rho) - 1$ *and where this inequality is strict if* $\deg(\rho) \geq 3$ *and* $\rho_\infty \neq 0$*. More precisely, one may choose* $\sigma = \{fh, g\}$ *where* f *is the product of polynomials in supp(*ρ*) and where* $g, h \in E[t]\backslash\{0\}$ *are such that* $\deg(g) < \deg(f)$ *and, either* $\deg(h) < \deg(g)$*, or* $gh \in E^\times$.

Idea of the Proof We will consider f, which is the product of the polynomials in supp(ρ). Then, we choose $g \in E[t]$ prime to f with $\deg(g) < \deg(f)$ such that $\partial_p(\{f, g\}) = \rho_p$ for all monic irreducible polynomials $p \in$supp(ρ). Finally, depending on whether g is constant or square free or neither of them, we choose a suitable h and get a σ which satisfies the properties as desired in the lemma.

Proof Let f be the product of the polynomials in supp(ρ). We claim that it is possible to choose $g \in E[t]$ prime to f with $\deg(g) < \deg(f)$ such that $\partial_p(\{f, g\}) = \rho_p$ for all monic irreducible polynomials $p \in$supp(ρ).

Suppose there exists such a g, then we have

$$\partial_p(\{f, g\}) = \rho_p$$

$$\iff \{(-1)^{v_p(f)v_p(g)}\overline{f^{-v_p(g)}g^{v_p(f)}}\} = \rho_p \iff \{\overline{g}\} = \rho_p$$

for all monic irreducible polynomials $p \in$ supp(ρ).

Using the Chinese Remainder Theorem, we see that the natural projection map from $E[t]$ to $\bigoplus\limits_{p\in supp(\rho)} E_p$ is a surjection. So, we can find a g such that $\{\overline{g}\} = \rho_p$ for all p monic irreducible polynomials in the support of ρ. We can ensure that $\deg(g) < \deg(f)$, for we can divide g by f and consider the remainder , if needed. As $\{g, lg\} = 0$ for any $l \in E[t]$ and ∂_p is a homomorphism, such a g will satisfy $\partial_p(\{f, g\}) = \rho_p$. Also, by construction, g and f are co-prime as $\rho_p \neq 0$ for all $p \in$ supp(ρ).

Now, we consider three cases. In each case, we would choose a $h \in E[t]\backslash\{0\}$ carefully such that if we take $\sigma := \{fh, g\}$, we have $\deg(\rho - \partial(\sigma)) \leq \deg(\rho) - 1$.

Case 1 g is a constant i.e.$g = c \in E^\times$.

In this case, we simply take $h = 1$. So,

$$\sigma = \{fh, g\} = \{f, g\}.$$

Now, from the definition of ∂_p, it is clear that the monic irreducible polynomials for which $\partial_p(\{f, c\})$ can take non-zero values lie in supp(ρ). And, by the choice of g, we have $\partial_p(\sigma) = \rho_p$ for all monic irreducible polynomials $p \in$ supp(ρ), therefore,

$(\rho - \partial(\sigma))_p = 0.$

So, only $p \in \mathcal{P}'$ for which $(\rho - \partial(\sigma))_p$ can be possibly non-zero is for $p = \infty$. And, so, $\deg(\rho - \partial(\sigma)) \leq 1$. And, so $\deg(\rho - \partial(\sigma)) \leq \deg(\rho) - 1$ as $\deg(\rho) \geq 2$.

Case 2 g is not constant and g is not square-free.

In this case, we take h to be the product of all monic irreducible factors of g. Clearly, then, $\deg(h) < \deg(g)$ (as g is not square-free).
Then, the only possible p for which $(\rho - \partial(\sigma))_p$ may be non-zero can be p which belong to supp$(\rho) \cup \{\infty\} \cup$ set of monic irreducible factor of h .
For $p \in$supp(ρ),

$$(\rho - \partial(\{fh, g\}))_p = \rho_p - \partial_p(\{f, g\}) - \partial_p\{h, g\}\} = -\partial_p\{h, g\} = -\{1\} = 0$$

(as both g and h are coprime to f.)
So, now,

$$\deg(\rho - \partial(\sigma)) \leq 1 + \deg(h) \implies \deg(\rho - \partial(\sigma)) \leq \deg(\rho) - 1$$

as

$$\deg(h) < \deg(g) < \deg(f) \implies \deg(f) - \deg(h) \geq 2$$

$$\implies \deg(f) - 1 \geq 1 + \deg(h) \implies \deg(\rho) - 1 \geq 1 + \deg(h).$$

since $\deg(f) = \deg(\rho)$, by construction of f.

Case 3 g is not constant and g is square-free.

Now again, by the Chinese Remainder Theorem, the natural projection map from $E[t]$ to $\bigoplus_p E_p$, with p varying over all monic irreducible factors of g, is a surjection.
Thus, we can choose $h \in E[t]^\times$ such that $\partial_p(\{f, g\}) - \rho_p = \{\overline{h}\}$ in $K_1^{(m)} E_p$ for every monic irreducible factor p of g. We can ensure that $\deg(h) < \deg(g)$, for we can divide h by g and consider the remainder, if needed. Such a h is co-prime to g. Here, if $(\rho - \partial(\sigma))_p \neq 0$, then p must be a monic irreducible factor of h. And, as $\deg(h) < \deg(g) < \deg(f)$, we have $\deg(\rho - \partial(\sigma)) \leq \deg(\rho) - 1$.

Also, in all three cases, clearly, if $\deg(\rho) \geq 3$ and $\rho_\infty \neq 0$, then, $\deg(\rho - \partial(\sigma)) \leq \deg(\rho) - 2$. $\square$

Lemma 5.7. *Let d be a non-negative integer and $f \in E[t]$ be non-constant and square-free such that $\deg(p) \geq d$ for every monic irreducible factor p of f. Let $F = E[t]/(f)$ and let ϑ denote the class of t in F. Then for any $a \in F^{\times}$, there exist non-zero polynomials $g, h \in E[t]$ with $\deg(h) \leq d-1$ and $\deg(g) \leq \deg(f) - d$ such that $a = g(\vartheta)/h(\vartheta)$.*

Proof Let

$$V = \bigoplus_{i=0}^{d-1} E\vartheta^i \quad \textit{and} \quad W = \bigoplus_{i=0}^{e-d} E\vartheta^i,$$

where $e = \deg(f)$.

Let $g \in E[t]$ be a polynomial of degree $m \leq d-1$. Then, $g(t)$ is co-prime to $f(t)$. So, there exists $l(t)$, $q(t) \in E[t]$ such that $g(t)l(t) + f(t)q(t) = 1$, which implies $g(\vartheta)l(\vartheta) = 1$. So, $g(\vartheta)$ is invertible in F. Now, we have $V\backslash\{0\} \subset F^{\times}$, where $F^{\times}$ denotes the group of invertible elements of F. As $a \in F^{\times}$, we have $dim_E(Va) = dim_E(V) = d$ and $dim_E(Va) + dim_E(W) = e + 1 > e = [F : E]$(dimension of F as a vector space over E). So $Va \cap W \neq 0$. Consequently, $h(\vartheta)a = g(\vartheta)$ for certain $h, g \in E[t]\backslash\{0\}$ with $\deg(h) \leq d-1$ and $\deg(g) \leq e-d$. Thus, $h(\vartheta) \in V\backslash\{0\} \subset F^{\times}$ and $a = g(\vartheta)/h(\vartheta)$. □

Lemma 5.8. *Let $\rho \in \Re'_m(E)$ and $q \in supp(\rho)$ be such that $\deg(q) = 2n+1$ with $n \geq 1$. Then there exists a symbol σ in $K_2^{(m)}E(t)$ such that $\deg(\rho - \partial(\sigma)) \leq \deg(\rho) - 2$. More precisely, one may choose $\sigma = \{qhf^{-2}g^{-2}, g^{-1}f\}$ with $f, g, h \in E[t]\backslash\{0\}$ such that $\deg(f), \deg(g) \leq n$ and $\deg(h) \leq 2n-1$.*

Proof We apply the previous lemma for $d = n+1$ and thus get f, $g \in E[t]\backslash\{0\}$ with $\deg(f), \deg(g) \leq n$ such that $\partial_q(\{q,^{-1} f\}) = \rho_q$. Then, q is co-prime to fg as $\deg(fg) = 2n \leq deg(q)$ and q is irreducible. We make three cases. Let $\sigma = \{qhf^{-2}g^{-2}, g^{-1}f\}$, where $h \in E[t]$ will vary according to the case we consider.

Case 1 fg is constant.

Let $h = 1$ and consider $\sigma = \{qf^{-2}g^{-2}, g^{-1}f\}$. Then, $\partial_q(\sigma) = \rho_q$ and $\partial_p(\sigma) = 0$ for every monic irreducible polynomial $p \in E[t]$ not contained in $supp(\rho)$ and prime to g and f. So, $\deg(\rho - \partial(\sigma)) \leq \deg(\rho) - (2n-1) + 1 \leq \deg(\rho) - 2n \leq \deg(\rho) - 2$.

Case 2 fg is not constant, not square-free.

Let h to be the product of the different monic irreducible factors of fg. Then, $\deg(h) \leq 2n-1$. We consider two sub-cases.

Case 2A $\deg(h) = 2n-1$. This also implies $\deg(f) = \deg(g) = n$. So, $\deg(qh) = 4n = 2\deg(fg)$. Hence, $\partial_\infty(\sigma) = 0$. So, $\deg(\partial - \partial(\sigma)) \leq \deg(\rho) - 2$.

Case 2B $\deg(h) \leq 2n-2$. Then, $\deg(\rho - \partial(\sigma)) \leq \deg(\rho) - 3 + 1$.

So, we are done with Case2.

Case 3 fg is not square-free, not constant.

We choose $h \in E[t]$ co-prime to fg with $\deg(h) < \deg(fg)$ such that $\partial_p(\{h, g^{-1}f\}) = \partial_p(\{q^{-1}f^2g^2, g^{-1}f\})$ for every monic irreducible factor p of fg. So, $\deg(h) \leq 2n-1 = \deg(q) - 2$. Again, clearly, for $p \in E[t]$ co-prime to h and not contained in $supp(\rho)$, $\partial_p(\sigma) = 0$. Again, we have two sub-cases.

Case 3A $\deg(h) = 2n-1$. Then, $\partial_\infty = 0$. So, we are done.

Case 3B $\deg(h) \leq 2n-2$. We are done whether ∂_∞ is zero or not. □

Proposition 5.9. *Let ρ be an element of $\Re'_m$ with* $\deg(\rho) \geq 2$. *Then there exists a symbol σ in $K_2^{(m)}E(t)$ such that* $\deg(\rho - \partial(\sigma)) \leq \deg(\rho) - 1$. *Moreover, if* $\deg(\rho) \geq 3$ *and $supp(\rho)$ contains an element of odd degree, then there exists a symbol σ in $K_2^{(m)}E(t)$ such that* $\deg(\rho - \partial(\sigma)) \leq \deg(\rho) - 2$.

Proof We have already proved the first part in earlier results. So, we just need to prove the second part of the statement. Again, if $supp(\rho)$ contains a non-rational point of odd degree, the statement follows from the previous lemma. Now, suppose $supp(\rho)$ contains a rational support. We note that the statement is invariant under E- automorphisms of $E(t)$. Hence, we may assume that $\infty \in supp(\rho)$, in which case the statement follows from one of the earlier lemmas.

Let i denote an E-automorphism of $E[t]$ and $\tilde{i}$ denote induced automorphism of $K_2^{(m)}E(t)$. Further, let i' denote induced automorphism of $\Re'_m(E)$. Now, we take i such that $i'(\rho)$ has ∞ in its support. Then, it follows that there exists a symbol σ $inK_2^{(m)}E(t)$ such that $\deg(i'(\rho) - \partial(\sigma)) \leq \deg(i'(\rho)) - 2$. Now, applying inverse automorpisms, we get that $\deg(\rho - \partial(\tilde{i}^{-1}(\sigma))) \leq \deg(\rho) - 2$. So, the result follows. □

Theorem 5.10. *For $\rho \in \Re'_m(E)$ and $r = \lfloor \deg(\rho)/2 \rfloor$, there exist symbols $\sigma_1, \ldots, \sigma_r$ in $K_2^{(m)}E(t)$ such that $\rho = \partial(\sigma_1 + \cdots + \sigma_r)$.*

Proof We induct on r. If $r = 0$, then $\rho = 0$ and the statement is trivial. So, we assume that $r > 0$. We have either $\deg(\rho) = 2r + 1$, in which case ρ contains a point of odd degree, or $\deg(\rho) = 2r$. Hence, by the previous proposition, there exists a symbol σ in $K_2^{(m)}E(t)$ with $deg(\rho - \partial(\sigma)) \leq 2r - 1$. By the induction hypothesis, there exist symbols $\sigma_1, \ldots, \sigma_{r-1}$ in $K_2^{(m)}E(t)$ with $\rho - \partial(\sigma) = \partial(\sigma_1 + \cdots + \sigma_{r-1})$. Then, $\rho = \partial(\sigma_1 + \cdots + \sigma_{r-1} + \sigma)$.

□

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[8] Notes by R. Sridharan (in collaboration with R. Parimala) on 2- *Torsion in Brauer Groups: A Theorem of Merkurjev.*