ON ISOLATED SINGULARTIES

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A dissertation submitted for the partial fulfilment of BS-MS dual degree in Mathematics

Indian Institute of Science Education and Research, Mohali

2015

Certificate of Examination

This is to certify that the dissertation titled **On Isolated Singularities** submitted by **Tanya Kaushal Srivastava** (Reg. No. MS10066) for the partial fulfillment of BS-MS dual degree programme of the Institute, has been examined by the thesis committee duly appointed by the Institute. The committee finds the work done by the candidate satisfactory and recommends that the report be accepted.

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Dated: April 24, 2015

Declaration

The work presented in this dissertation has been carried out by me under the guidance of Prof. Kapil Hari Paranjape at the Indian Institute of Science Education and Research Mohali.
This work has not been submitted in part or in full for a degree, a diploma, or a fellowship to any other university or institute. Whenever contributions of others are involved, every effort is made to indicate this clearly, with due acknowledgement of collaborative research and discussions. This thesis is a bona fide record of original work done by me and all sources listed within have been detailed in the bibliography.

Tanya Kaushal Srivastava
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In my capacity as the supervisor of the candidate's project work, I certify that the above statements by the candidate are true to the best of my knowledge.

Prof. Kapil Hari Paranjape
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Acknowledgement

I would like to thank my supervisor Prof. Kapil Hari Paranjape for his guidance and support. I would like to thank Dr. Alok Maharana and Dr. Yashonidhi Pandey for their helpful comments and advices.

I would also like to thank Prof. Hélène Esnault, Dr. Kay Ruling, Dr. Giulia Battiston and Elena Lavanda for helping me with some basic parts of the proofs.

I would like to express my gratitude to all my teachers at IISER Mohali Mathematics Department for being so supportive and helpful. I would like to thank Prof. I. B. S. Passi and Dr. Varadharaj Srinivasan for teaching me how to do and write mathematics, Dr. Amit Kulsherstha for introducing me to differential geometry and motivating me to choose mathematics major, Dr. Chanchal Kumar for helping me appreciate Combinatorics, Dr. Mahender Singh for introducing me to algebraic topology,and Dr. Dinesh Khurana, Dr. Chandrakant Aribam and Dr. Lingraj Sahu for their helping me in developing a broader perspective of mathematics.

I would also like to thank my teachers at IISER Mohali for always being so helpful, especially Prof. Satyamurthy, Prof. Ramesh Kapoor, Prof. Arvind and Dr. N. G. Prasad for motivating me a lot. I would like to thanks DST for awarding me KVPY fellowship and IISER Mohali for the facilities provided.

I would like to express my gratitude to my parents and Dr. K. P. Yogendran for always encouraging me.

And lastly a special thanks to my friend Soniya Rani for always being there for me.

Contents

Chapter 1

Introduction

The aim of this thesis is to understand the topology of an algebraic set (hypersurface) near a point p belonging to that algebraic set.

We will use the following construction [Mil68] to study the topology. We intersect the hyper-surface V or the algebraic set with a small sphere S_ϵ centered at the given point p. Then the topology of the V within the disc bounded by S_ϵ is closely related to the topology of the set $K = S_\epsilon \cap V$, via the following theorem:

Theorem 1.1 *For small ϵ the intersection of V with D_ϵ is homeomorphic to the cone over $K = V \cap S_\epsilon$. In fact the pair $(D_\epsilon, V \cap D_\epsilon)$ is homeomorphic to the pair consisting of the cone over S_ϵ and the cone over K.*

This provides us the necessary justification for using the construction described above. Therfore, it is sufficient for us to study the topological properties of K, which we do with the help of Milnor fibration.

This thesis is organized into three chapters after introduction. The second chapter deals with the basic definitions and a few preliminary theorems which are in turn used in proving the Milnor fibration theorem in the fourth chapter.In this chapter we also state the basic results from Morse theory which are used in studying the topological properties and we give a proof of the equivalence of the various notion of dimension.

The third chapter deals with the proof of the theorem stated above on the topological properties of a space near a point. This chapter then discusses the results when the point of interest is a regular point:

Theorem 1.2 *If p is a simple point of V, the the intersection $K = V \cap S_\epsilon$ is an unknotted sphere in S_ϵ, for all sufficiently small ϵ.*

We end this chapter by giving an example of the case when the point is singular.

In the first section of fourth chapter we prove the Milnor fibration theorem as stated below:

Theorem 1.3 (Milnor, 1968) *If p is any point of the complex hyper-surface $V = f^{-1}(0)$ and if S_ϵ is a sufficiently small sphere centered at p, then the mapping*

$$\phi(z) = \frac{f(z)}{|f(z)|}$$

from $S_\epsilon - K$ to the unit circle (S^1) is a smooth fiber bundle.

The basic idea of the proof lies in a construction of a non-vanishing vector field locally and then patching it up using partitions of unity to get a global map. The proof that the fibers are smooth manifolds is based on the result that pullback of any regular value gives us a smooth manifold.

In the second section of this chapter we study the topological properties of the fibers of the Milnor fibration as well as the topological properties of the space of our interest near an isolated singular point. The main theorems are:

Theorem 1.4 *The space $K = V \cap S_\epsilon$ is (n-2) connected.*

Theorem 1.5 *Each fiber F_θ has the homotopy type of a bouquet $S^n \vee \ldots \vee S^n$ of sphere, the number of spheres in this bouquet being strictly positive. Each fiber can be realized as the interior of a smooth compact manifold with boundary, $\bar{F}_\theta = F_\theta \cup K$.*

We end with a brief discussion on whether K is a topological sphere or not.

Chapter 2

Algebraic and Analytic Underpinnings

2.1 Basic Definitions

Let Φ be any infinite field, and let Φ^m be the coordinate space consisting of all m-tuples $x = (x_1, \dots, x_m)$ of elements of Φ. (In our case Φ will be either $\mathbb{R}$ or $\mathbb{C}$.)

Definition 2.1 (Algebraic set) *A subset $V \subset \Phi^m$ is called an **algebraic set** if V is the locus of common zeros of some collection of polynomial functions on Φ^m.*

We denote by $\Phi[x_1, \dots, x_m]$ the ring of all polynomial functions from Φ^m to Φ and let $I(V) \subset \Phi[x_1, \dots, x_m]$ be the ideal consisting of those polynomials which vanish throughout V. From Hilbert's basis theorem we can conclude that the ideal $I(V)$ is finitely generated.

We observe that the union $V \cup V'$ of any two algebraic sets V and V' in Φ^m is again an algebraic set since the locus of zeros of the product of two polynomials is precisely the union of locus of zeros of each polynomial.

Definition 2.2 (Irreducible algebraic set) *A non-vacuous algebraic set V is called a **variety or an irreducible algebraic set** if it cannot be expressed as the union of two proper algebraic subsets.*

As an alternate description V is irreducible iff $I(V)$ is a prime ideal for Φ an algebraically closed field. If V is irreducible, then the field of quotients f/g with f and $g \neq 0$ in the integral domain

$$\Phi[x_1, \ldots, x_m]/I(V)$$

is called the **field of rational functions** on V. Its transcendence degree over Φ is called the algebraic dimension of V over Φ. If W is a proper sub variety of V, then the dimension of W is less than dimension of V over Φ.

Let $V \subset \Phi^m$ be any non-vacuous algebraic set. Choose finitely many polynomials $f_1, \ldots, f_k$ which span the ideal $I(V)$ and for each $x \in V$, consider the $k \times m$ matrix $(\partial f_i/\partial x_j)$ evaluated at x. Let ρ be the largest rank which the matrix attains at any point of V.

Definition 2.3 *A point $x \in V$ is called **non-singular** or **regular** if the matrix $(\partial f_i/\partial x_j)$ attains its maximal rank ρ at x and **singular** if*

$$rank(\partial f_i(x)/\partial x_j) < \rho.$$

We note that this definition does not depend on the choice of $\{f_1, \ldots, f_k\}$ since if we add an extra polynomial $f_{k+1} = g_1 f_1 + \ldots + g_k f_k$ the resulting new row will be a linear combination of the old rows.

Lemma 2.4 *The set $\Sigma(V)$ of all singular points of V forms a proper algebraic subset (possibly vacuous) of V.*

Proof A point $x \in V$ will belong to $\Sigma(V)$ iff every $\rho \times \rho$ minor determinant of $(\partial f_i/\partial x_j)$ vanishes at x. Thus $\Sigma(V)$ is an algebraic set determined by algebraic equations.

Remark: If V is a variety then, the dimension of V is $m - \rho$.

Definition 2.5 (Gradient) *We define the **gradient** of an analytic function $f(z_1, \ldots, z_m)$ of m complex variables to be the $m-$vector*

$$grad\ f = (\overline{\partial f/\partial z_1}, \ldots, \overline{\partial f/\partial z_m})$$

whose $j-th$ component is the complex conjugate of $\partial f/\partial z_1$.

Using the above definition, we get the chain rule of the derivative of f along a path $z = \gamma(t)$ to be

$$df(\gamma(t))/dt =< d\gamma/dt, grad\ f >,$$

using the hermitian inner product

$$< a, b >= \Sigma a_j \bar{b_j}.$$

In other words, the directional derivative of f along a vector v at the point z is equal to the inner product $< v, grad\ f(z) >$.

Definition 2.6 (Critical point) *A* ***critical point*** *of a smooth mapping $f : M \to N$ between smooth manifolds is a point of the first manifold ($p \in M$) at which the induced linear mapping between tangent spaces fails to be surjective , i.e., the map*

$$T_p f : T_p M \to T_{f(p)} N$$

is not surjective.

Definition 2.7 (Critical value) *A* ***critical value*** *$f(x) \in N$ is the image under f of a critical point.*

Definition 2.8 (Non-degenerate critical point) *A critical point p is called* ***non-degenerate*** *iff the matrix*

$$\left(\frac{\partial^2 f}{\partial x^i \partial x^j}(p) \right)$$

is non-singular

We now give the definition of a smooth fiber bundle. In our notation a smooth fiber bundle would mean a locally trivial fibration, in the following sense:

Definition 2.9 (Locally trivial fibration) *Let $\phi : E \to M$ be a smooth map, E and M are smooth manifolds . We say that ϕ is a* ***locally trivial fibration*** *if for each $z \in M$ there is an open neighborhood U and a diffeomorphism*

$$h : \phi^{-1}(U) \to U \times \phi^{-1}(z)$$

such that $\begin{tikzcd} \phi^{-1}(U) \arrow[r, "h"] \arrow[dr, "\phi|_{\phi^{-1}(U)}"'] & U \times \phi^{-1}(z) \arrow[d, "pr_U"] \\ & U \end{tikzcd}$ *commutes.*

2.2 Morse Theory

In this section, we have stated the important results from Morse theory [Mil66] which are used in the following sections to compute the topological properties of spaces of our interest.

Let f be a smooth real valued function on a manifold M with critical point at p. If $v, w \in T_pM$ then v and w have extensions $\tilde{v}$ and $\tilde{w}$ to vector fields. We define a symmetric bilinear functional f_{**} on T_pM as

$$f_{**}(v, w) = \tilde{v}_p(\tilde{w}(f)),$$

where $\tilde{v}_p = v$. This functional is called the **Hessian** of f at p.

This symmetric because

$$\tilde{v}_p(\tilde{w}(f)) - \tilde{w}_p(\tilde{v}(f)) = [\tilde{v}, \tilde{w}]_p(f) = 0$$

where $[\tilde{v}, \tilde{w}]$ is the Poisson bracket of $\tilde{v}$ and $\tilde{w}$, and where $[\tilde{v}, \tilde{w}]_p(f) = 0$ since f has a critical point at p.

Hence it is well-defined since $\tilde{v}_p(\tilde{w}(f)) = v(\tilde{w}(f))$ is independent of the extension $\tilde{v}$ of v, while $\tilde{w}_p(\tilde{v}(f))$ is independent of $\tilde{w}$.

Let $(x^1, \ldots, x^n)$ be a local coordinate system and $v = \sum a_i \frac{\partial}{\partial x^i}|_p$, $w = \sum b_j \frac{\partial}{\partial x^j}|_p$, we choose $\tilde{w} = \sum b_j \frac{\partial}{\partial x^j}|_p$, where b_j is a constant function. Then

$$f_{**}(v, w) = v(\tilde{w}(f))(p) = v(\sum b_j \frac{\partial f}{\partial x^j}) = \sum_{i,j} a_i b_j \frac{\partial^2 f}{\partial x^i \partial x^j}(p);$$

so the matrix $(\frac{\partial^2 f}{\partial x^i \partial x^j})$ represents the bilinear function f_{**} with respect to the basis $\frac{\partial}{\partial x^1}|_p, \ldots, \frac{\partial}{\partial x^n}|_p$.

Definition 2.10 (Index) *The* ***index*** *of a bilinear functional H, on a vector space V is defined to be the maximal dimension of a subspace of V on which H is negative definite.*

Definition 2.11 (Nullity) *Nullity is defined to be the dimension of the subspace consisting of all $v \in V$ such that $H(v, w) = 0$ for all $w \in V$.*

We note that the point p is a non-degenerate critical point of f iff f_{**} on T_pM has nullity equal to 0. We refer to the index of f_{**} on T_pM simply as the index of f at p. The following result will show that the behaviour of f at p can be fully described by the index.

Lemma 2.12 (Lemma of Morse) *Let p be a non-degenerate critical point of f. Then there is a local coordinate system $(y^1, \ldots, y^n)$ in a neighborhood U of p with $y^i(p) = 0\ \forall i$ and such that the identity*

$$f = f(p) - (y^1)^2 - \ldots - (y^\lambda)^2 + (y^{\lambda+1})^2 + \ldots + (y^n)^2$$

holds throughout U, where λ is the index of f at p.

Corollary 2.13 *Non- degenerate critical points are isolated.*

Theorem 2.14 (Morse) *[Mor34] Let C be a critical set of f which lies in a coordinate system (x) in which f is analytic. Corresponding to any arbitrarily small neighborhood N of C, there exists a function ψ of class C^2 on a Reimannian manifold M which with its first and second partial derivatives approximates f and its first and second partial derivatives arbitrarily closely over M and which is such that $f \equiv \psi$ on $(M \setminus N)$ while ψ has at most non-degenerate critical points on N.*

If f is a real valued function on a manifold M, we let

$$M^a = f^{-1}(-\infty, a] = \{p \in M | f(p) \leq a\}$$

Theorem 2.15 *Let f be a smooth real valued function on M. Let $a < b$ and suppose that the set $f^{-1}[a, b]$, consisting of all $p \in M$ with $a \leq f(p) \leq b$, is compact, and contains no critical points of f. Then M^a is diffeomorphic to M^b. Furthermore, M^a is a deformation retract of M^b, so the inclusion map $M^a \to M^b$ is an homotopy equivalence.*

Theorem 2.16 *Let $f : M \to \mathbb{R}$ be a smooth function, and let p be a non-degenerate critical point with index λ. Setting $f(p) = c$, suppose that $f^{-1}[c-\epsilon, c+\epsilon]$ is compact, and contain no critical points of f other than p, for some $\epsilon > 0$. Then, for all sufficiently small ϵ, the set $M^{c+\epsilon}$ has the homotopy type of $M^{c-\epsilon}$ with a λ-cell (e^λ) attached.*

From the above theorem it also follows that $M^{c-\epsilon} \cup e^\lambda$ is a deformation retract of M^c.

Theorem 2.17 *If f is a differentiable function on a manifold M with no degenerate critical points, and if each M^a is compact, then M has the homotopy type of a CW-complex, with one cell of dimension λ for each critical point of index λ.*

Let K be a compact subset of the Euclidean space $\mathbb{R}^n$; let U be a neighborhood of K and let

$$f : U \to \mathbb{R}$$

be a smooth function such that all critical points of f in K have index $\geq \lambda_0$.

Theorem 2.18 *If $g : U \to \mathbb{R}$ is any smooth function which is close to f, in the sense that*

$$|\frac{\partial g}{\partial x_i} - \frac{\partial f}{\partial x_i}| < \epsilon, \ |\frac{\partial^2 g}{\partial x_i \partial x_j} - \frac{\partial^2 f}{\partial x_i \partial x_j}| < \epsilon, \ (i, j = 1, \ldots, n)$$

uniformly throughout K, for some sufficiently small constant ϵ, then all critical points of g in K have index $\geq \lambda_0$.

Here f is allowed to have degenerate critical points.

2.3 Whitney's Theorems

We state the following theorems due to Whitney. The proofs can be found in [Whi57].

Theorem 2.19 (Whitney) *If Φ is the field of real (or complex) numbers, then the set $V - \Sigma(V)$ of non-singular points of V forms a smooth, non-vacuous manifold. In fact this manifold is real (or complex) analytic, and has dimension $m - \rho$ over Φ.*

The proof of this theorem will be a corollary of the following result due to Whitney and we will need the following definition for its proof.

Definition 2.20 (Locally algebraic submanifold of $\mathbb{R}^m$) *A **locally algebraic submanifold of** $\mathbb{R}^m$(called as algebraic partial manifold be Whitney in [Whi57]) M is an subset of $\mathbb{R}^m$ associated with a number ρ, with the property that for any $p \in M$, there exists a set of polynomials $f_1, \ldots, f_\rho$ of rank ρ at a point p and a neighborhood U of p, such that $U \cap M$ is the set of zeros in U of these f_i. The number $m - \rho$ is the dimension of the locally lagebraic submanifold of $\mathbb{R}^m$.*

Note that M need not be closed or connected and that any open subset of M is a locally algebraic submanifold of $\mathbb{R}^m$.

The following theorem is stated for the real case but will be proved first in the complex case:

Theorem 2.21 *[Whi57] Let $V \subset \mathbb{R}^n$ be a real algebraic variety and let M_1 be the set of points $p \in V$ where the rank of $(\partial f_i/\partial x_j)$ for f_i the generators of $I(V)$ is its maximum. Then M_1 is a locally algebraic submanifold of $\mathbb{R}^n$ of dimension $n -$ rank $(\partial f_i/\partial x_j)$ and $V_1 = V - M_1$ is void or is a proper algebraic subvariety of V.*

Before proceeding to the proof of the theorem we prove an elementary lemma:

Lemma 2.22 *Let $f_1, \ldots, f_\rho$ have independent differentials at p. Then there is a coordinate system $(x'_1, \ldots, x'_n)$ in a neighborhood U of p such that*

$$x'_i(q) = f_i(q) - f_i(p) \in U \ (i = 1, \ldots, \rho) \tag{2.1}$$

Proof Choose co-vectors $\xi_{\rho+1}, \ldots, \xi_n$ which with $df_i(p)$ form an independent set. Let

$$x'_i(q) = \xi.(q-p) \ for \ i > \rho;$$

then

$$dx'_i(q) = df_i(q) \ (i \leq \rho) \tag{2.2}$$

$$dx'_i(q) = \xi_i \ (i > \rho) \tag{2.3}$$

Hence the Jacobian of the transformation

$$(x_1, \dots, x_n) \to (x'_1, \dots, x'_n)$$

is not zero at p and $(x'_1, \dots, x'_n)$ is a coordinate system.

Proof [Proof of Theorem 2.21] Take any $p \in M_1$. We may choose polynomials $f_1, \dots, f_\rho$ in $I(V)$ with independent differentials at p; we keep these f fixed.

Given any polynomial g and any $\mu = (\mu_1, \dots, \mu_{\rho+1})$, define the polynomial

$$\Psi_\mu g = \frac{\partial(f_1, \dots, f_\rho, g)}{\partial(x_{\mu_1}, \dots, x_{\mu_\rho}, x_{\mu_{\rho+1}})} \tag{2.4}$$

where $\frac{\partial(f_1, \dots, f_\rho, g)}{\partial(x_{\mu_1}, \dots, x_{\mu_\rho}, x_{\mu_{\rho+1}})}$ is the determinant of the matrix of partial derivatives of the function $f_1, \dots, f_\rho, g$ with respect to $x_{\mu_1}, \dots, x_{\mu_\rho}, x_{\mu_{\rho+1}}$.

We choose a coordinate system $(x'_1, \dots, x'_n)$ in a neighborhood of U of p by the above lemma. Now

$$\partial f_i / \partial x'_j = \Sigma_n (\partial f_i / \partial x_n)(\partial x_n / \partial x'_j)$$

and this gives

$$\frac{\partial(f_1, \dots, f_\rho, g)}{\partial(x'_1, \dots, x'_\rho, x'_k)} = \Sigma_{\mu_1 < \dots < \mu_{\rho+1}} \frac{\partial(f_1, \dots, f_\rho, g)}{\partial(x_{\mu_1}, \dots, x_{\mu_\rho}, x_{\mu_{\rho+1}})} \frac{\partial(x_{\mu_1}, \dots, x_{\mu_\rho}, x_{\mu_{\rho+1}})}{\partial(x'_1, \dots, x'_\rho, x'_k)}$$

Because of 2.2 and 2.3, the left hand side is simply $\partial g / \partial x'_k$, if $k > p$. Let T^μ_k denote the last term on the right. Then this relation and 2.4 give

$$\frac{\partial g}{\partial x'_k} = \Sigma_{\mu_1 < \dots < \mu_{\rho+1}} T^\mu_k \Psi_\mu g \ in \ U \ (k > \rho) \tag{2.5}$$

Let M^* be the $(n - \rho)$- dimensional manifold in U defined by the vanishing of the f_i; it is the part of the $(x'_{\rho+1}, \dots, x'_n)$- coordinate plane in U.

Given a function g and a point $p' \in M^*$, $dg(p').v = 0$ for all vector v tangent to M^* at p' iff $dg(p')$ is dependent on the $df_i(p')$, that is, iff all $\Psi_\mu g = 0$ at p or again,

$\partial g / \partial x'_k = 0$ at p' for $k > \rho$.

We may iterate the operations forming $\Psi_\lambda \Psi_\mu g$, $\Psi_\sigma \Psi_\lambda \Psi_\mu g$, etc.

Let $H(g)$ be the set of all such polynomials, and let $J_U(g)$ be the ideal of analytic functions in U generated by $H(g)$, consisting of all functions

$$\Sigma \varphi_i(p) h_i(p) \ with \ h_i \in H(g)$$

and φ_i analytic in U.

We show that each partial derivative of g of any order with respect to the variables $(x'_{\rho+1}, \dots, x'_n)$ is an element of $J_U(g)$.

By 2.5, this holds for the first partial derivatives. Using induction it is sufficient to show that if φ is analytic in U and $h = \Psi_\sigma \dots \Psi_\mu g$, then

$$\partial(\varphi h) / \partial x'_k \in J_U(g) \ , k > \rho$$

Differentiating φh

$$\frac{\partial(\varphi h)}{\partial x'_k} = \frac{\partial(\varphi)}{\partial x'_k} . h + \varphi \frac{\partial(h)}{\partial x'_k}$$

and applying 2.5 with h in place of g shows this to be true.

Now suppose that M^* is connected. To prove the theorem we need to show that $M_1 \cap U = M^*$.

Since V_1 is closed, we may suppose $V_1 \cap U = 0$, hence $M_1 \cap U = V \cap U \subset M^*$ (as $f_i \in I(V)$) and there remains to prove $M^* \subset V$.

Take any polynomial $g \in I(V)$; we must prove that $g = 0$ in M^*. Since $rank(\partial f_i / \partial x_j) = \rho$, the differentials $df_1, \dots, df_\rho, dg$ are dependent through out V; hence all $\Psi_\mu g = 0$ throughout V, and $\Psi_\mu g \in I(V)$.

Repeating the above argument shows that $H(g) \subset I(V)$. In particular, all polynomials in $H(g)$ vanish at p. By the above result, all partial derivatives of g vanish at p.

Now g is analytic in the open connected part of $(x'_{\rho+1}, \ldots, x'_n)$-space which is M^* and $g = 0$ there. This gives us the proof.

Remark: It follows from the theorem 2.21 that we can express V as a disjoint union , $V = M_1 \cup \ldots \cup M_l$, where each M_j is smooth manifold and we call the maximum of dimensions of M_j as the dimension of V.

Theorem 2.23 (Whitney) *For any pair $W \subset V$ of algebraic sets in a real or complex coordinate space, the difference $V - W$ has at most a finite number of topological components.*

Here $V - W$ is itself an algebraic set. It is sufficient to consider the real case, since any complex algebraic set in $\mathbb{C}^m$ can be thought of as a real algebraic set in $\mathbb{R}^{2m}$.

Before giving the proof, we prove a few lemmas which are required in the proof of theorem.

Lemma 2.24 *Let V be an algebraic set in the $m-$dimensional coordinate space over any infinite field, and let $f_1, \ldots, f_m$ be polynomials which vanish on V and at point p. If the matrix $(\partial f_i / \partial x_j)$ is non-singular at a point p of V, then, removing p from V, the complement $V - p$ will still be a closed algebraic subset of V.*

Proof We may assume $p = 0$. Since the f_j vanishes at the origin, we can choose polynomials g_{jk} so that

$$f_j(x) = g_{j1}(x)x_1 + \ldots + g_{jm}(x)x_m.$$

Let W denote the algebraic set consisting of all points $x \in V$ which satisfy the polynomial equation

$$det(g_{jk}(x)) = 0.$$

Then the origin is not a point of W, since the matrix

$$(\partial f_i(0)/\partial x_j) = (g_{jk}(0))$$

is non-singular. But at any point $x \neq 0$ of V the linear dependence realtion

$$\begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} g_{11}(x) \\ \vdots \\ g_{m1}(x) \end{pmatrix} x_1 + \ldots + \begin{pmatrix} g_{1m}(x) \\ \vdots \\ g_{mm}(x) \end{pmatrix} x_m$$

shows that $det(g_{jk}(x)) = 0$. So $V - \{0\} = W$, which proves that $V - \{0\}$ is an algebraic set.

Now we specialize to the field $\mathbb{R}$ of real numbers.

Corollary 2.25 *If an algebraic set $V \subset \mathbb{R}^m$ has topological dimension zero, then V is a finite set.*

Proof Let $f_1, \ldots, f_k$ span the ideal $I(V)$. It is enough to show that every zero dimensional algebraic set V contains at least one point p at which the matrix $(\partial f_i/\partial x_j)$ has rank m. For then the point p can be removed by the above lemma, yielding a proper algebraic subset $V_1 = V - \{p\}$. Iterating our construction, we obtain a chain

$$V_1 \supset V_2 \supset V_3 \supset \ldots$$

of nested algebraic subsets. Since every such chain must terminate by Hilbert's basis theorem, this will prove that V is finite.

But if the matrix $(\partial f_i/\partial x_j)$ has rank at most $\rho \leq m - 1$ at all points of V , then from theorem 2.19 would imply that V contained a smooth manifold $V - \Sigma(V)$ of dimension $m - \rho \geq 1$. Since this would contradict the hypothesis that V has topological dimension zero, this completes the proof of the corollary.

Lemma 2.26 *Any non-singular algebraic set $V \subset \mathbb{R}^m$ has the homotopy type of a finite CW complex.*

Proof Given any point $a \in \mathbb{R}^m$ let

$$r_a : V \to \mathbb{R}$$

denote the squared distance function

$$r_a(x) = ||x - a||^2$$

Then for almost every choice of a, the function r_a on V has only non-degenerate critical points [Mil66].

Let $\Gamma \subset V$ denote the set of all critical points of r_a. Then Γ is an algebraic set. But non-degenerate critical points are clearly isolated, so it follows from the previous corollary that Γ is a finite set.

Every component $V^{(i)}$ of V must intersect the critical set Γ. For the distance from a must be minimized at some point x of the closed set $V^{(i)}$, and clearly this closet point x will belong to Γ. Therefore, V can have only finitely many components.

Alternatively, the main theorem of Morse theory [Mor34], [Mil66] states that the manifold V has the homotopy type of a cell complex with one cell for each critical point of the non-degenerate, proper, non-negative function r_a. Then the finiteness of Γ implies that V has the homotopy type of a finite CW complex.

Corollary 2.27 *For any real algebraic set V, if W is an algebraic subset containing the singular set $\Sigma(V)$, then $V - W$ has the homotopy type of a finite CW complex.*

Proof Suppose that W is defined by polynomial equations $f_1(x) = \ldots = f_k(x) = 0$. setting

$$s(x) = f_1(x)^2 + \ldots + f_k(x)^2,$$

note that W can also be defined by the single polynomial equation s(x) = 0.

Now let G be the graph of the rational function $1/s$ from V to $\mathbb{R}$. That is , let G be the set of all

$$(x, y) \in V \times \mathbb{R} \subset \mathbb{R}^{m+1}$$

for which $s(x)y = 1$.

Then we see that G is an algebraic set, and is homeomorphic to $V - W$. Moreover G has no singular points. This gives us the proof of the lemma.

Proof [Proof of Theorem 2.23] As we have seen, the set V can be expressed as a finite union $M_1 \cup \ldots M_p$, where the manifold M_1 is the set of non-singular points of $V_1 = V$, the manifold M_2 is the set of non-singular points of $V_2 = \Sigma(V_1)$, and so on. Therefore

$$V - W = (M_1 - W) \cup \ldots \cup (M_p - W)$$

where each

$$M_i - W = V_i - (\Sigma(V_i) \cup W)$$

is a manifold which has only finitely many (path-) components by the above lemma.

It follows that the union $V - W$ has only finitely many path-components. This completes the proof of the theorem.

As a consequence of the above theorems we have the following corollary whose proof is just an application of Hilbert's basis theorem.

Corollary 2.28 *Any real or complex algebraic set V can be expressed as a finite disjoint union $V = M_1 \cup M_2 \cup \ldots \cup M_p$, where each M_j is a smooth manifold with only finitely many components. Similarly any difference $V - W$ of varieties can be expressed as such a finite union.*

We end this section with the following theorem:

Theorem 2.29 *The manifold of simple points of a complex variety V is everywhere dense.*

Proof It is sufficient to prove the following statement:

If $V = V(P)$, P is a prime ideal in $\mathbb{C}[x_1, \ldots, x_n]$ and $f \in \mathbb{C}[x_1, \ldots, x_n]$, $f \notin P$, then $V - Z(f)$ is dense in the strong topology on V.

The statement is trivially true for the case when $n = 0$. For the case $n = 1$, we have two subcases: First, if $P = (0)$, then for any $f \in \mathbb{C}[x] - 0$, $Z(f)$ is a finite set and $V(0) = \mathbb{C}$ and $\mathbb{C} - \{finite\ set\}$ is dense in $\mathbb{C}$, so we are done. Second subcase is if $P \neq (0)$, since $\mathbb{C}[x]$ is a principal ideal domain, we have every prime ideal is maximal, so $P = (x - a)$. Thus $V(P) = \{a\}$ and $f \notin P$ iff $f(a) \neq 0$. This gives us

$$Z(f) \cap V(P) = \emptyset.$$

Therefore

$$\overline{V(P) - Z(f)} = V(P) - Z(f) = V(P)$$

and we are done. Now for case $n > 1$, we again have two subcase: First, when $P = (0)$. By induction, we get that $Z(f)$ in $\mathbb{A}^n$ *or* $\mathbb{P}^n$ is not all of $\mathbb{A}^n$ *or* $\mathbb{P}^n$ for k an infinite field. This gives $\mathbb{A}^n - Z(f)$ *or* $\mathbb{P}^n - Z(f)$ is dense in $\mathbb{A}^n$ *or* $\mathbb{P}^n$ respectively as take any point $p \in Z(f)$ and take all the lines passing through that point, then only finitely many lie in $Z(f)$ and $Z(f) \cap L$ is finite. Thus, $L - (Z(f) \cap L)$ is dense in L. Therefore, $\mathbb{A}^n - Z(f)$ *or* $\mathbb{P}^n - Z(f)$ is dense in $\mathbb{A}^n$ *or* $\mathbb{P}^n$.

Second subcase, when $P = (g)$, where g is irreducible. (We can reduce this to the case of curves using induction).Consider

$$k[x] \hookrightarrow k[x, y_1, \ldots, y_n]/P \xrightarrow{\eta} k[x, y_1, \ldots, y_m]/Q.$$

where P is prime and y_i are integral.

Claim 1: There exists η such that $k[x, y_1, \ldots, y_m]/Q$ is integrally closed.

Proof of claim 1: We want to show that the integral closure($\tilde{R}$) of $R = k[x, y_1, \ldots, y_n]/P$ in the quotient field is a finitely generated R-module.

Without loss of generality and using Noether normalisation lemma we get

$$\mathbb{C}[x_0, x_1] \hookrightarrow R$$

is an integral extension.This gives us

$$\mathbb{C}(x_0, x_1) \hookrightarrow Q(R)$$

is a finite extension and we know that

$$\mathbb{C}[x_0, x_1] \hookrightarrow \mathbb{C}(x_0, x_1)$$

is integrally closed. Consider $e_1, \ldots, e_d$ be a basis of $Q(R)$ over $\mathbb{C}(x_0, x_1)$ such that e_i are integral over $\mathbb{C}[x_0, x_1]$. This gives $e_i \in \tilde{R}$.

$$M = \{a \in Q(R) | Tr_{Q(R)/\mathbb{C}(x_0,x_1)}(ae_i) \in \mathbb{C}[x_0, x_1]\}$$

is a finitely generated $\mathbb{C}[x_0, x_1]$-module and $\tilde{R} \subset M$. This proves our claim.

Claim 2:$V(Q)$ is smooth at every point. The proof of this can be referred in ([Sha96] Pg 126).

Then again using the fact that complements of finite sets are dense, we get our theorem.

2.4 Application of Whitney's theorem

Let $M_1 = V - \Sigma(V)$ be the manifold of simple (regular, non-singular) points of an algebraic set $V \subset \Phi^m$, where Φ denotes the real or complex numbers, and let g be a polynomial function on Φ^m. The dimension of M_1 is the same as that of the algebraic variety V.

Lemma 2.30 *The set of critical points of the restricted function $g|_{M_1}$ from M_1 to Φ is equal to the intersection of M_1 with the algebraic set W consisting of all points $x \in V$ at which the matrix*

$$\begin{bmatrix} \frac{\partial g}{\partial x_1} & \cdots & \frac{\partial g}{\partial x_m} \\ \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_m} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_k}{\partial x_1} & \cdots & \frac{\partial f_k}{\partial x_m} \end{bmatrix}$$

has rank $\leq \rho$; where $f_1, \ldots, f_k$ denote polynomial spanning $I(V)$.

Proof Using Implicit function theorem or Rank theorem, near any point of M_1 we can choose a (real or complex) smooth system of local coordinates $u_1, \ldots, u_m$ for Φ^m so that M_1 corresponds to the locus $u_1 = 0, \ldots, u_\rho = 0$. Then $u_{\rho+1}, \ldots, u_m$ can be taken as the local coordinates on M_1. We note that $\frac{\partial f_i}{\partial u_j}$, evaluated at a point of M_1, is 0 for $j \geq \rho + 1$ (as f_i vanishes on M_1). Since the matrix $(\partial f_i/\partial u_j)$ is column equivalent to the matrix $(\partial f_i/\partial x_l)$ and therefore has rank ρ, it follows that the first ρ columns of $(\partial f_i/\partial u_j)$ must be linearly independent.

Now the enlarged matrix

$$\begin{bmatrix} \frac{\partial g}{\partial u_1} & \cdots & \frac{\partial g}{\partial u_m} \\ \frac{\partial f_1}{\partial u_1} & \cdots & \frac{\partial f_1}{\partial u_m} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_k}{\partial u_1} & \cdots & \frac{\partial f_k}{\partial u_m} \end{bmatrix}$$

will have the same rank ρ iff

$$\frac{\partial g}{\partial u_{\rho+1}} = \ldots = \frac{\partial g}{\partial u_m} = 0;$$

or one can say iff the given point is a critical point of $g|_{M_1}$. Since this new matrix is column equivalent to the matrix ,

$$\begin{bmatrix} \frac{\partial g}{\partial x_1} & \cdots & \frac{\partial g}{\partial x_m} \\ \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_m} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_k}{\partial x_1} & \cdots & \frac{\partial f_k}{\partial x_m} \end{bmatrix}$$

our proof is complete.

Corollary 2.31 *A polynomial function g on $M_1 = V - \Sigma(V)$ can have at most a finite number of critical values.*

Proof The set of critical points of $g|_{M_1}$ can be expressed as a difference $W - \Sigma(V)$ of algebraic sets, and hence can be expressed as a finite union of smooth manifold,

$$W - \Sigma(V) = M_1' \cup \ldots \cup M_p'$$

where each M_i' has only finitely many components. Each point $x \in M_i'$ is a critical point of the smooth function $g|_{M_1}$, so a fortiori it is a critical point of the restricted function $g|_{M_1'}$. Since all points of M_i' are critical, it follows that g is constant on each connected component. But the union $g(M_2') \cup \ldots \cup g(M_p')$ is precisely the set of all critical values of $g|_{M_1}$.

Let p be either a simple point of V or an isolated point of the singular set $\Sigma(V)$, where V is any real or complex algebraic set.

Corollary 2.32 *Every sufficiently small sphere S_ϵ centered at p intersects V in a smooth manifold (possibly vacuous).*

Proof The complex case reduces to real case, since every complex algebraic set is also a real algebraic set. In the real case this follows by applying the above corollary to the polynomial function $r(x) = ||x - p||^2$. If ϵ^2 is smaller than any positive critical value of $r|_{(V-\Sigma(V))}$ then ϵ^2 will be regular value, hence its inverse image $r^{-1}(\epsilon^2) \cap (V - \Sigma(V)) = S_\epsilon \cap (V - \Sigma(V))$ will be a smooth manifold K. But if ϵ is small enough then S_ϵ will not meet $\Sigma(V)$ (as p is an isolated point), hence K will equal $S_\epsilon \cap V$.

2.5 Lefschetz Theorem

This section has been referred from [Mil68].

We begin by introducing a positive integer μ which measures the amount of degeneracy at the critical point p. This integer μ will be the multiplicity of p as solution to the collection of polynomial equations

$$\frac{\partial f}{\partial z_1} = \ldots = \frac{\partial f}{\partial z_n} = 0.$$

In a general setup, let $g_1(z), \ldots, g_m(z)$ be arbitrary analytic functions of m complex variables, and let p be an isolated solution to the collection of equations

$$g_1(z) = \ldots = g_m(z) = 0.$$

We will say this as: p is an isolated zero of the mapping $g : \mathbb{C}^m \to \mathbb{C}^m$.

Definition 2.33 *The multiplicity μ of the isolated zero p is the degree of the mapping*

$$z \mapsto \frac{g(z)}{||g(z)||}$$

from a small sphere S_ϵ centered at p to the unit sphere of $\mathbb{C}^m$.

The following properties help us in providing a justification for our definition.

Lemma 2.34 *If the Jacobian $\left(\frac{\partial g_j}{\partial z_k}\right)$ is non-singular at p then $\mu = 1$.*

Proof We consider the Taylor series expansion with remainder:

$$g(z) = L(z - p) + r(z),$$

where the linear transformation L is non-singular by hypothesis, and where $\frac{||r(z)||}{||z-p||}$ tends to 0 as $z \to p$. We choose an ϵ small enough so that

$$||r(z)|| < ||L(z - p)||$$

whenever $||z - p|| = \epsilon$. Then the one parameter family of mappings

$$h_t(z) = \frac{(L(z - p) + t.r(z))}{||L(z - p) + t.r(z)||}, \ 0 \leq t \leq 1,$$

from $S_\epsilon(z^0)$ to the unit sphere demonstrates that the degree μ of h_1 is equal to the degree of the mapping $L/||L||$ on $S_\epsilon(p)$. (From Rouche's principle : The degree of $(L + r)/(||L + r||)$ on S_ϵ is equal to the degree of $L/||L||$ whenever $||r|| < ||L||$ throughout S_ϵ.)

Now we deform L continuously to the identity within the group $GL(m, \mathbb{C})$ consisting of all non-singular linear transformations. This is possible since the Lie-group $GL(m, \mathbb{C})$ is connected. It follows easily that the degree the degree of the mapping $L/||L||$ on $S_\epsilon(p)$ is 1. This finishes the proof.

Now let D be a compact region with smooth boundary in $\mathbb{C}^m$ and assume that g has only finitely many zeros in D and no zero on the boundary.

Lemma 2.35 *The number of zeros of g within D, counted with its appropriate multiplicities, is equal to the degree of the mapping*

$$\partial D \to S^{2m-1}$$

given by

$$z \mapsto g(z)/||g(z)||$$

Proof We remove a small open disc about each zero of g from the region D. Then the function $g/||g||$ is defined and continuous throughout the remaining region D_0. The boundary ∂D is homologous to the sum of the small boundary spheres within D_0, it follows that the degree of $g/||g||$ on ∂D is equal to the sum , $\sum \mu$ of the degrees on the small spheres. This gives us our lemma.

Let p be an isolated zero of g with multiplicity μ.

Lemma 2.36 *If D_ϵ is a disc about p containing no other zeros of g then, for almost all points $q \in \mathbb{C}^m$ sufficiently close to the origin, the equation $g(z) = q$ has precisely μ solutions $z \in D_\epsilon$.*

Proof From Sard's Theorem, almost every point q of $\mathbb{C}^m$ is a regular value of the differentiable mapping

$$g : \mathbb{C}^m \to \mathbb{C}^m.$$

That is, for all q not in some set of Lebesgue measure zero, the matrix $(\partial g_j/\partial z_k)$ is non-singular at every point z in the inverse image $g^{-1}(a)$.

Given any such regular value q, the solutions z of the system of analytic equations $g(z) - q = 0$ are all isolated, with multiplicity 1, using 2.34. We choose any regular value q of g which is close to the origin so that $||q|| < ||g(z)||$ for all $z \in \partial D_\epsilon$. Then from 2.35 the number of solutions of the equation $g(z) - q = 0$ inside D_ϵ is equal to the degree of the map $g(z) - q/||g(z) - q||$ on ∂D_ϵ. Thus, by Rouche's principle the degree of the mapping $g(z) - q/||g(z) - q||$ is equal to the degree μ of $g(z)/||g(z)||$. This gives us our result.

Corollary 2.37 *The inequality $\mu \geq 0$ is always satisfied.*

For the special case $g_j(z) = \partial f/\partial z_j$, we see that 2.34 gives us that in case of a non-degenerate critical point of f, where the Hessian matrix is non-singular, the integer μ is 1. 2.36 gives us that if we change f minutely by subtracting almost any small linear polynomial $a_1 z_1 + \ldots + a_m z_m$ from it, then the isolated critical point p will split up into a cluster of μ nearby critical points, all non-degenerate.

Now we come to the main theorem for this section:

Theorem 2.38 (Lefschetz) *The multiplicity μ of an isolated solution of m polynomial equation in m variable is always a positive integer.*

Proof Given a disc D_ϵ about p containing no other zeros of g, we choose a number η which is small enough so that

$$|\eta| < ||g(z)||/\epsilon$$

for all $z \in \partial D_\epsilon$, and which is distinct from all eigenvalues of the matrix $(\partial g_j(p)/\partial z_k)$. Then the perturbed function

$$g'(z) = g(z) - \eta(z - p)$$

has a zero of multiplicity 1 at p, since the matrix

$$\left(\frac{\partial g_j'}{\partial z_k}\right) = \left(\frac{\partial g_j}{\partial z_k} - \eta\delta_{jk}\right)$$

is non-singular at p. Therefore, assuming that g' has only finitely many zeros with D_ϵ, the algebraic number $\sum \mu'$ of the zeros of g' within D_ϵ is certainly ≥ 1 (as $g'(p) = 0$ and all summands being ≥ 0 by 2.37). This sum is equal to the degree of $g'/||g'||$ on ∂D_ϵ, which is equal to the degree μ of $g/||g||$ on ∂D_ϵ by Rouche's principle. Hence $\mu \geq 1$.

We now need to eliminate the possibility of g' having infinitely many zeros inside D_ϵ. In that case we could subtract a small constant vector q from g', where q is a regular value of g'. Then the zeros of $g' - q$ are isolated and hence there are only finitely many zeros of $g' - q$ within D_ϵ. To guarantee that $g' - q$ has at least one zero, we use the inverse function theorem to choose a neighborhood U of p in D_ϵ so that g' maps diffeomorphically onto an open neighborhood of the origin. Choosing q within $g'(U)$ the equation equation $g'(z) - q = 0$ certainly has a solution z within $U \subset D_\epsilon$. This completes the proof that $\mu \geq 1$.

2.6 The Notion of Dimensions

In this section we will show the equivalence of various definition of dimension of a variety. We will begin by the local notions of dimension for a general Noetherian local ring and show that these versions coincide with the notion of dimension of algebraic variety as the transcendence degree of the function field [A M01]. We will be using the concept of Hilbert's function in our discussion.

Let $A = \oplus_{n=0}^{\infty} A_n$ be a Noetherian graded ring. Then A_0 is a Noetherian ring and A is finitely generated as an A_0 algebra by $x_1, \ldots, x_s$ which are homogeneous of degrees $k_1, \ldots, k_s$ (all > 0).
Consider $M = \oplus_{n=0}^{\infty} M_n$ to be a finitely generated graded A-module. Then M is generated by homogeneous element m_j $(1 \leq j \leq t)$ with $r_j = \deg m_j$. Thus elements of M_n have the form $\sum_j f_j(x) m_j$, where $f_j(x) \in A$ is homogeneous of degree $n - r_j$ and is zero if $n < r_j$. Thus M_n is finitely generated as an A_0-module by all $g_j(x) m_j$, where $g_j(x)$ is a monomial in x_i of total degree $n - r_j$.

Definition 2.39 (Additive function) *Let C be a class of A-modules and let λ be a function on C with values in $\mathbb{Z}$, then the function λ is additive if, for each short exact sequence in which all terms belong to C,*

$$0 \to M' \to M \to M'' \to 0$$

we have the relation

$$\lambda(M') - \lambda(M) + \lambda(M'') = 0.$$

Let λ be an additive function on the class of all finitely generated A_0-modules. The Poincaré series of M is the power series

$$P(M, t) = \sum_{n=0}^{\infty} \lambda(M_n) t^n \ \in \mathbb{Z}[[t]]$$

Theorem 2.40 (Hilbert, Serre) *P(M,t) is a rational function in t of the form* $\frac{f(t)}{\prod_{i=1}^{s}(1-t^{k_i})}$, *where* $f(t) \in \mathbb{Z}[t]$.

Proof We proceed by induction on the number of generators of A over A_0. For $s = 0$, we have $A_n = 0$ for all $n > 0$, so that $A = A_0$ and M is a finitely generated A_0 module, hence $M_n = 0$ for all large n. There by giving us that $P(M,t)$ is a polynomial.
Now take $s > 0$ and assume the result for $s-1$. Then multiplication by x_s is an A-module homomorphism of M_n into M_{n+k_s} which gives us the exact sequence

$$0 \to K_n \to M_n \xrightarrow{x_s} M_{n+k_s} \to L_{n+k_s} \to 0. \tag{2.6}$$

Let $K = \oplus_n K_n$, $L = \oplus_n L_n$; these are finitely generated A-modules and both are annihilated by x_s, hence they are $A_0[x_1, \ldots, x_{s-1}]$-modules. On applying λ to equation 2.6 we have

$$\lambda(K_n) - \lambda(M_n) + \lambda(M_{n+k_s}) - \lambda(L_{n+k_s}) = 0$$

multiplying by t^{n+k_s} and summing over n we get

$$(1 - t^{k_s})P(M,t) = P(L,t) - t^{k_s}P(K,t) + g(t) \tag{2.7}$$

where $g(t) = \sum_{i=0}^{k_s} \lambda(M_i)t^i - \sum_{i=0}^{k_s} \lambda(L_i)t^i$ is a polynomial. We get our result by applying the inductive hypothesis.

We denote the order of the pole of $P(M,t)$ at $t = 1$ by $d(M)$. The case when all $k_i = 1$ is given by:

Corollary 2.41 *If each* $k_i = 1$, *then for all sufficiently large* n, $\lambda(M_n)$ *is a polynomial in* n *with rational coefficients of degree* $d - 1$.

Proposition 2.42 *If* $x \in A_k$ *is not a zero divisor in* M *them* $d(M/xM) = d(M) - 1$.

Proof In the sequence 2.6 we replace x_s by an element $x \in A_k$ which is not a zero divisor in M. Then $K = 0$ and equation 2.7 gives us that $d(L) = d(M) - 1$. This gives us our proposition.

Now we prove another proposition which will be an important ingredient for dimension theorem.

Proposition 2.43 *Let A be a Noetherian local ring, $\mathfrak{m}$ its maximal ideal, $\mathfrak{q}$ an $\mathfrak{m}$-primary ideal, M a finitely generated A-module, (M_n) a stable $\mathfrak{q}$-filtration of M. Then*

1. *M/M_n is of finite length, for each $n \geq 0$;*
2. *for all sufficiently large n this length is a polynomial $g(n)$ of degree $\leq s$ in n, where s is the least number of generators of $\mathfrak{q}$;*
3. *the degree and leading coefficient of $g(n)$ depend only on M and $\mathfrak{q}$, not on the filtration chosen.*

Proof

1. Let $G(A) = \oplus_n \mathfrak{q}^n/\mathfrak{q}^{n+1}$, $G(M) = \oplus_n M_n/M_{n+1}$. Then $G_0(A) = A/\mathfrak{q}$ is an Artin local ring, $G(A)$ is Noetherian and $G(M)$ is a finitely generated graded $G(A)$-module. Each $G_n(M) = M_n/M_{n+1}$ is a Noetherian A-module annihilated by q, hence a Noetherian $A/\mathfrak{q}$-module and therefore of finite length (since $A/\mathfrak{q}$ is Artinian). Hence M/M_n is of finite length and
$$l_n = l(M/M_n) = \sum_{r=1}^{n} l(M_{r-1}/M_r) \tag{2.8}$$
as l is an additive function.

2. If $x_1, \ldots, x_s$ generate $\mathfrak{q}$, the images $\bar{x_i}$ of x_i in $\mathfrak{q}/\mathfrak{q}^2$ generate $G(A)$ as an $A/\mathfrak{q}$-algebra and each $\bar{x_i}$ has degree 1. Hence by 2.41 we have $l(M_n/M_{n+1}) = f(n)$, where $f(n)$ is a polynomial in n of degree $\leq s-1 \; \forall \; large \; n$. Since from equation 2.8 we have $l_{n+1} - l_n = f(n)$, it follows that l_n is a polynomial $g(n)$ of degree $\leq s \; \forall \; large \; n$.

3. Let $(M_n{}')$ be another stable $\mathfrak{q}$-filtration of M, and let $g'(n) = l(M/M'_n)$. Then the two filtrations have bounded difference , i.e., there exists an integer n_0 such that $M_{n+n_0} \subseteq M'_n$, $M'_{n+n_0} \subseteq M_n \forall \; n \geq 0$; thus we have $g(n+n_0) \geq g'(n)$, $g'(n+n_0) \geq g(n)$. Since g and g' are polynomials for all large n, we have $\lim_{n\to\infty} g(n)/g'(n) = 1$ and therefore g, g' have the same degree and leading coefficient.

We denote the polynomial $g(n)$ corresponding to the filtration $(\mathfrak{q}^n M)$ by $\chi_{\mathfrak{q}}^M(n)$:

$$\chi_{\mathfrak{q}}^M(n) = l(M/\mathfrak{q}^n M) \ (\forall \ large \ n).$$

If $M = A$, we write $\chi_{\mathfrak{q}}(n)$ for $\chi_{\mathfrak{q}}^A(n)$. In this case from 2.43 we get

Corollary 2.44 *For all large n, the length $l(A/\mathfrak{q}^n)$ is a polynomial $\chi_{\mathfrak{q}}(n)$ of degree $\leq s$, where s is the least number of generators of $\mathfrak{q}$.*

The next proposition shows that the degree of $\chi_{\mathfrak{q}}(n)$ is independent of the choice of the $\mathfrak{m}$-primary ideal $\mathfrak{q}$.

Proposition 2.45 *If $A, \mathfrak{m}, \mathfrak{q}$ are as above then*

$$\deg \chi_{\mathfrak{q}}(n) = \deg \chi_{\mathfrak{m}}(n).$$

Proof Note that $\mathfrak{m} \supseteq \mathfrak{q} \supseteq \mathfrak{m}^r$ for some r, hence $\mathfrak{m}^n \supseteq \mathfrak{q}^n \supseteq \mathfrak{m}^{rn}$ and thus

$$\chi_{\mathfrak{m}}(n) \leq \chi_{\mathfrak{q}}(n) \leq \chi_{\mathfrak{m}}(rn)$$

for all large n. Now let $n \to \infty$, and using the fact that the $\chi's$ are polynomials in n, we have our result.

We denote the common degree of $\chi_{\mathfrak{q}}(n)$ by $d(A)$, by 2.41 this means that we are putting $d(A) = d(G_m(A))$. Let $\delta(A)$ be the least number of generators of an $\mathfrak{m}$-primary ideal of A. Our aim is to prove that $\delta(A) = d(A) = \dim A$, which we do by showing $\delta(A) \geq d(A) \geq \dim A \geq \delta(A)$.

Proposition 2.46 $\delta(A) \geq d(A)$

This we get from 2.44 and 2.45.

Proposition 2.47 *Let M be a finitely generated A-module, $x \in A$ a non-zero divisor in M and $M' = M/xM$. Then*

$$\deg \chi_{\mathfrak{q}}^{M'}(n) \leq \chi_{\mathfrak{q}}^M(n) - 1.$$

Proof Let $N = xM$, then $N \cong M$ as an A-module as x is not a zero divisor. Let $N_n = N \cap \mathfrak{q}^n M$. Then we have the exact sequences

$$0 \to N/N_n \to M/q^n M \to M'/q^n M' \to 0.$$

Hence if $g(n) = l(N/N_n)$, we have

$$g(n) - \chi_{\mathfrak{q}}^{M}(n) + \chi_{\mathfrak{q}}^{M'}(n) = 0$$

for all large n. Now by Artin-Rees, (N_n) is a stable $\mathfrak{q}$-filtration of N. Since $N \cong M$ 2.43 implies that $g(n)$ and $\chi_{\mathfrak{q}}^{M}(n)$ have the same leading terms, hence our result.

Corollary 2.48 *If A is a Noetherian local ring, x is a non zero divisor in A, then $d(A/(x)) \leq d(A) - 1$.*

Proposition 2.49 *$d(A) \geq \dim A$.*

Proof We proceed by induction on $d = d(A)$. If $d = 0$ then $l(A/\mathfrak{m}^n)$ is constant for all large n, hence $\mathfrak{m}^n = \mathfrak{m}^{n+1}$ for some n, hence $\mathfrak{m}^n = 0$ by Nakayama lemma. Thus A is an Artin ring and $\dim A = 0$.
Suppose $d > 0$ and let $\mathfrak{p}_0 \subset \ldots \subset \mathfrak{p}_r$ be any chain of prime ideal in A. Let $x \in \mathfrak{p}_1 - \mathfrak{p}_0$, $A' = A/\mathfrak{p}_0$ and x' be the image of x in A'. Then $x' \neq 0$ and A' is an integral domain, hence by 2.48 we have

$$d(A'/(x')) \leq d(A') - 1.$$

Also, if $\mathfrak{m}'$ is the maximal ideal of A', $A'/\mathfrak{m}'^n$ is a homomorphic image of $A/\mathfrak{m}^n$, hence $l(A/\mathfrak{m}^n) \geq l(A'/\mathfrak{m}'^n)$ and therefore $d(A) \geq d(A')$. Consequently

$$d(A'/(x')) \leq d(A) - 1 = d - 1.$$

Hence by inductive hypothesis, the length of any chain of prime ideals in $A'/(x')$ is $\leq d - 1$. But the images of $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$ in $A'/(x')$ form a chain of length $r - 1$, hence $r - 1 \leq d - 1$ and thus $r \leq d$. Hence $\dim A \leq d$.

Proposition 2.50 *Let A be a Noetherian local ring of dimension d. Then there exists an $\mathfrak{m}$-primary ideal in A generated by d elements $x_1, \ldots, x_d$, and therefore* $\dim A \geq \delta(A)$.

Proof We construct $x_1, \dots, x_d$ inductively in a way such that every prime ideal containing $(x_1, \dots, x_i)$ has height $\geq i$, for each i. Suppose $i > 0$ and $x_1, \dots, x_{i-1}$ constructed. Let $\mathfrak{p}_j$ $(1 \leq j \leq s)$ be the minimal prime ideals of $(x_1, \dots, x_{i-1})$ (if any) which have height exactly $i-1$. Since $i-1 < d = \dim A =$ height $\mathfrak{m}$, we have $\mathfrak{m} \neq \mathfrak{p}_j$ $(i \leq j \leq s)$, hence $\mathfrak{m} \neq \bigcup_{j=1}^{s} \mathfrak{p}_j$. We choose $x_i \in \mathfrak{m} \setminus \bigcup \mathfrak{p}_j$ and let $\mathfrak{q}$ be any prime containing $(x_1, \dots, x_i)$, then $\mathfrak{q}$ contains some minimal prime ideal $\mathfrak{p}$ of $(x_1, \dots, x_{i-1})$. If $\mathfrak{p} = \mathfrak{p}_j$ for some j, we have $x_i \in \mathfrak{q} \setminus \mathfrak{p}$, thus $\mathfrak{q} \supset \mathfrak{p}$ and therefore height $\mathfrak{p} \geq i$, hence height $\mathfrak{q} \geq i$. Thus every prime ideal containing $(x_1, \dots, x_i)$ has height $\geq i$.
We consider then $(x_1, \dots, x_d)$. If $\mathfrak{p}$ is a prime ideal of this ideal, $\mathfrak{p}$ has height $\geq d$, hence $\mathfrak{p} = \mathfrak{m}$ for $\mathfrak{p} \subset \mathfrak{m} \implies$ height $\mathfrak{p} <$ height $\mathfrak{m} = d$. Hence the ideal $(x_1, \dots, x_d)$ is $\mathfrak{m}$-primary.

Theorem 2.51 (Dimension Theorem) *For any Noetherian local ring A the following three integers are equal:*

1. *the maximum length of chains of prime ideals in A;*
2. *the degree of the polynomial $\chi_{\mathfrak{m}}(n) = l(A/\mathfrak{m}^n)$;*
3. *the least number of generators of an $\mathfrak{m}$-primary ideal of A.*

The proof follows from 2.46, 2.49, 2.50.

If $x_1, \dots, x_d$ generate an $\mathfrak{m}$-primary ideal, and $d = \dim A$, we call $x_1, \dots, x_d$ a system of parameters. They satisfy an independence property as described below:

Proposition 2.52 *Let $x_1, \dots, x_d$ be a system of parameters for A and let $\mathfrak{q} = (x_1, \dots, x_d)$ be the $\mathfrak{m}$-primary ideal generate by them. Let $f(t_1, \dots, t_d)$ be a homogeneous polynomial of degree s with coefficients in A, and assume that*

$$f(x_1, \dots, x_d) \in \mathfrak{q}^{s+1}.$$

Then all the coefficients of f lie in $\mathfrak{m}$.

Proof Consider the epimorphism of graded rings

$$\alpha : \frac{A}{\mathfrak{q}}[t_1, \ldots, t_d] \to G_{\mathfrak{q}}(A)$$

given by $t_i \mapsto \bar{x}_i$, where t_i are indeterminates and $\bar{x}_i = x_i \mod \mathfrak{q}$. The assumption on f gives that $\bar{f}(t_1, \ldots, t_d) = f \mod \mathfrak{q}$ is in the kernel of α. We assume if possible that some coefficient of f is a unit, then $\bar{f}$ is not a zero divisor. Then we have

$$\begin{aligned} d(G_{\mathfrak{q}}(A)) &\leq d((A/\mathfrak{q})[t_1, \ldots, t_d]/(\bar{f})) \textit{ as } \bar{f} \in \ker(\alpha) \\ &= d((A/\mathfrak{q})[t_1, \ldots, t_d]) - 1 \textit{ by } 2.42 \\ &= d - 1 \end{aligned}$$

But $d(G_{\mathfrak{q}}(A)) = d$ by 2.51. This gives us a contradiction.

For the case of A containing a field k mapping isomorphically onto the residue field $A/\mathfrak{m}$, we have:

Corollary 2.53 *If $k \subset A$ is a field mapping isomorphically onto $A/\mathfrak{m}$ and if $x_1, \ldots, x_d$ is a system of parameters, then $x_1, \ldots, x_d$ are algebraically independent over k.*

We end this section by proving the equivalence of dimension of local rings and the dimension of varieties defined in terms of the function field. Let V be an irreducible affine variety over k, an algebraically closed field. Then the coordinate ring $A(V)$ is :

$$A(V) = \frac{k[x_1, \ldots, x_n]}{\mathfrak{p}}$$

where $\mathfrak{p}$ is a prime ideal. We denote by $k(V)$ the field of rational functions on V which is the field of fractions of the integral domain $A(V)$. $k(V)$ is a finitely generated extension over k. Its transcendence degree over k is defined to be the dimension of V.

By Hilbert's Nullstellensatz, the points of V correspond bijectively with the maximal ideals of $A(V)$. If P is a point with maximal ideal $\mathfrak{m}$ we call $\dim A(V)_{\mathfrak{m}}$ the local dimension of V at P.

Lemma 2.54 *Let $B \subset A$ be integral domains with B integrally closed and A integral over B. Let $\mathfrak{m}$ be a maximal ideal of A, and let $\mathfrak{n} = \mathfrak{m} \cap B$, then $\mathfrak{n}$ is maximal and* $\dim A_{\mathfrak{m}} = \dim B_{\mathfrak{n}}$.

Proof It follows easily that $\mathfrak{n}$ is maximal. If

$$\mathfrak{m} \supset \mathfrak{q}_1 \supset \ldots \supset \mathfrak{q}_d$$

is a strict chain of primes in A, its intersection with B is a strict chain of primes by going down lemma:

$$\mathfrak{n} \supset \mathfrak{p}_1 \supset \ldots \supset \mathfrak{p}_d$$

This proves $\dim B_{\mathfrak{n}} \geq \dim A_{\mathfrak{m}}$. Conversely, we can go in the other direction using going up lemma and thus $\dim B_{\mathfrak{n}} \leq \dim A_{\mathfrak{m}}$.

Theorem 2.55 *For any irreducible variety V over k the local dimension of V at any point is equal to* $\dim V$.

Proof By 2.53, we have $\dim V \geq \dim A_{\mathfrak{m}}$ for all $\mathfrak{m}$.
By the Noether Normalization Lemma, we can find a polynomial ring $B = k[x_1, \ldots, x_d]$ contained in $A(V)$ such that $d = \dim V$ and $A(V)$ is integral over B. Since B is integrally closed, we apply 2.54 and this reduces to the case for the ring B, i.e., for affine space. But any point of affine space can be taken as the origin of coordinates and as we know that $k[x_1, \ldots, x_d]$ localized at the maximal ideal $(x_1, \ldots, x_d)$ is a local ring of dimension d.

Corollary 2.56 *For every maximal ideal $\mathfrak{m}$ of $A(V)$ we have*

$$\dim A(V) = \dim A(V)_{\mathfrak{m}}$$

Proof By definition we have $\dim A(V) = sup_{\mathfrak{m}} \dim A(V)_{\mathfrak{m}}$ but by 2.55 all $A(V)_{\mathfrak{m}}$ have the same dimension. This gives us the result.

Thus we have shown that various notion of dimension are equal in appropriate settings.

Chapter 3

Knots and Singularities

In this chapter, we will discuss some topological properties of a space near a singular point and we will compute the singularities which give us torus knots.

Let $f(z_1, \ldots, z_{n+1})$ be a non-constant polynomial in $n+1$ complex variables, and let V be the algebraic set consisting of all $(n+1)$-tuples

$$z = (z_1, \ldots, z_{n+1})$$

of complex numbers with $f(z) = 0$. We call such a set as a **complex hyper surface**. We want to study the topology of V in the neighborhood of some point p.

Let D_ϵ denote the closed disk consisting of all x with $||x - p|| \leq \epsilon$ and let p be either a simple point or an isolated singular point of V. Take $K = V \cap S_\epsilon$. This definition is useful only when K is contained in a sphere around p.

Definition 3.1 (Cone over K) *Cone over K is defined to be the union of all line segments*

$$tk + (1-t)p, \ 0 \leq t \leq 1$$

joining points $k \in K$ to the base point p. It is denoted as Cone(K).

The set Cone(S_ϵ), defined similarly is equal to D_ϵ.

Theorem 3.2 *For small ϵ the intersection of V with D_ϵ is homeomorphic to the cone over $K = V \cap S_\epsilon$. In fact the pair $(D_\epsilon, V \cap D_\epsilon)$ is homeomorphic to the pair consisting of the cone over S_ϵ and the cone over K.*

Proof It is sufficient to consider the real case. Let ϵ be small enough so that the disc D_ϵ contains no singular points of V and no critical points of $r|_{V-\Sigma(V)}$, other than p itself.

We will construct a smooth vector field $v(x)$ on the punctured disk $D_\epsilon - p$ with two properties:

First, the vector $v(x)$ will point away from p for all x; that is the euclidean inner product

$$< v(x), x - p >$$

will be strictly positive.

Second, the vector $v(x)$ will be tangent to the manifold $M_1 = V - \Sigma(V)$ whenever $x \in M_1$.

We begin by constructing the vector field locally. Given any point q of $D_\epsilon - p$ we will construct a vector field $v^q(x)$ throughout a neighborhood U^q of q so that the above two properties are satisfied.

If q does not belong to V then we can simply set

$$v^q(x) = x - p$$

for all x in some neighborhood $U^q \subset \mathbb{R}^m - V$.

If q belongs to V, and hence belongs to M_1, we choose a system of local coordinates $u_1, \ldots, u_m$ throughout a neighborhood of q so that M_1 corresponds to the locus $u_1 = \ldots = u_\rho = 0$. Since q is not a critical point of the function $r|_{M_1}$, where $r(x) = ||x - p||^2$, it follows that at least one of the partial derivatives

$$\partial r/\partial u_{\rho+1}, \ldots, \partial r/\partial u_m$$

must be non-zero at q. Suppose $\partial r/\partial u_h$ is non-zero at q, then let U^q be a small connected neighborhood throughout which $\partial r/\partial u_h \neq 0$ and , let $v^q(x)$ be the vector

$$\pm(\partial x_1/\partial u_h, \ldots, \partial x_m/\partial u_h)$$

tangent to the u_h-coordinate curve through x; choosing the plus sign or the minus sign according as $\partial r/\partial u_h$ is positive or negative. This vector $v^q(x)$ is certainly tangent to M_1, whenever $x \in M_1$, since the entire u_h- coordinate curve is contained in M_1. Furthermore

$$\begin{aligned} 2 < v^q(x), x - p > &= \sum 2(x_i - p_i)v_i^q(x) && (3.1) \\ &= \sum \partial r/\partial x_i(\pm\partial x_i/\partial u_h) && (3.2) \end{aligned}$$

is equal to $\pm\partial r/\partial u_h > 0$ for all $x \in U^q$.

Now choose a smooth partition of unity λ^q on $D_\epsilon - p$, with $support(\lambda^q) \subset U^q$. Then the vector field

$$v(x) = \Sigma\lambda^q(x)v^q(x)$$

on $D_\epsilon - p$ clearly has the required properties.

We normalize by setting

$$w(x) = v(x)/ < 2(x-p), v(x) >$$

and consider the differential equation

$$dx/dt = w(x).$$

In other words, we look for smooth curves $x = \gamma(t)$, defined say for $\alpha < t < \beta$, which satisfy

$$d\gamma(t)/dt = w(\gamma(t)).$$

Given any solution $\gamma(t)$, note that the derivative of the composition $r(\gamma(t))$ is given by

$$\begin{aligned} dr/dt &= \Sigma(\partial r/\partial x_i)w_i(x) \\ &= < 2(x-p), w(x) > \\ &= 1 \end{aligned}$$

where $x = \gamma(t)$. So the function $r(\gamma(t))$ must be equal to $t + constant$. Thus, subtracting a constant from the parameter t if necessary, we may assume that

$$r(\gamma(t)) = ||\gamma(t) - p||^2 = t.$$

Claim: This solution $\gamma(t)$ can certainly be extended throughout the interval $0 < t \leq \epsilon^2$.

Proof of Claim: We assume that the vector field $w(x)$ has been constructed over an open set slightly larger than $D_\epsilon - p$, so that the boundary points of D_ϵ do not behave weirdly.

By Zorn's lemma and compactness the given solution can be extended over some maximal open interval $\alpha' < t < \beta'$. Suppose for example that $\beta' \leq \epsilon^2$. Then we will extend the solution $\gamma(t)$ over a slightly larger interval, thus contradicting the definition of β'. Since the points $\gamma(t)$ with $\alpha' < t < \beta'$ all belong to the compact set D_ϵ, there exists at least one limit point x' of $\{\gamma(t)\}$ as $t \to \beta'$; clearly $r(x') = \beta' \neq 0$ so that $x' \in D_\epsilon - p$.

We will use the local existence, uniqueness and the smoothness theorem for the differential equation $dx/dt = w(x)$ near x'. This theorem asserts that for each x'' in some neighborhood U of x' and each t'' in some arbitrarily small open interval I containing β' there exists a unique solution

$$x = \gamma'(t), \ t \in I$$

satisfying the initial condition $\gamma'(t'') = x''$; and further more , that $\gamma'(t)$ is a smooth function of x'' , t'' and t. To apply this theorem, we choose $t'' \in (\alpha', \beta') \cap I$. So the solutions γ and γ' can be put together to yield a solution which is defined for all t in the larger interval $(\alpha', \beta') \cup I$. This contradiction proves that $\beta' > \epsilon^2$ and a similar argument shows that $\alpha' = 0$. This finishes the proof of the claim.

We note that the solution $\gamma(t)$, $0 < t \leq \epsilon^2$ is uniquely determined by the initial value

$$\gamma(\epsilon^2) \in S_\epsilon$$

For each $a \in S_\epsilon$ let

$$\gamma(t) = P(a, t), \ 0 < t \leq \epsilon^2,$$

be the unique solution which satisfies the initial condition

$$\gamma(\epsilon^2) = P(a, \epsilon^2) = a.$$

Clearly this function P maps the product $S_\epsilon \times (0, \epsilon^2]$ diffeomorphically onto the punctured disk $D_\epsilon - p$. Furthermore, since the vector field $w(x)$ is tangent to M_1 for all $x \in M_1$, it follows that every solution curve which touches M_1 must be contained in M_1. Hence P maps the product $K \times (0, \epsilon^2]$ diffeomorphically onto $V \cup (D_\epsilon - p)$.

Finally, we note that $P(a, t)$ tends uniformly to p as $t \to 0$. Therefore the correspondence

$$ta + (1-t)p \mapsto P(a, t\epsilon^2),$$

defined for $0 < t \leq 1$, extends uniquely to a homeomorphism from $Cone(S_\epsilon)$ to D_ϵ. Moreover, this homeomorphism carries $Cone(K)$ onto $V \cap D_\epsilon$. This proves our theorem.

Now we study the specific case when p is a simple point of V.

Theorem 3.3 *If p is a regular point of V, the the intersection $K = V \cap S_\epsilon$ is an unknotted sphere in S_ϵ, for all sufficiently small ϵ.*

Proof We note that the smooth function $r(x) = ||x - p||^2$ restricted to $M_1 = V - \Sigma(V)$ has a non-degenerate critical point at p. Then from the lemma of Morse, there exists a system of local coordinates $u_1, \ldots, u_k$ for M_1 near p so that

$$r(x) = u_1^2 + \ldots + u_k^2.$$

It follows immediately that $K = V \cap S_\epsilon$ is diffeomorphic to the sphere consisting of all $u_1, \ldots, u_k$ with $u_1 + \ldots + u_k^2 = \epsilon^2$.

We can extend the Morse's argument to be used for the pair of manifolds $M_1 \subset \mathbb{R}^m$. That is: there exists local coordinates $u_1, \ldots, u_m$ for $\mathbb{R}^m$ near p so that

$$r(x) = u_1^2 + \ldots + u_k^2.$$

and so that V corresponds to the locus $u_{k+1} = \ldots = u_m = 0$.

Thus the pair (S_ϵ, K) is diffeomorphic to the pair consisting of a sphere and a great sub sphere in the u-coordinate space. This proves the theorem.

Thus if we can identify the manifold K, and the way in which K is embedded in S_ϵ, then we will have completely determined the topology of V, and the embedding of V in its coordinate space, throughout a neighborhood of p. As in the case of K being a topological sphere, then V must be a topological manifold near p.

Now we consider the special case of a regular point p of a complex hypersurface

$$V = f^{-1}(0) \subset \mathbb{C}^{n+1}.$$

We want to study the set

$$F_0 = \phi^{-1}(1) = f^{-1}(\mathbb{R}_+) \cap S_\epsilon.$$

Theorem 3.4 *If the center p of S_ϵ is a regular point of f, then this fiber F_0 is diffeomorphic to $\mathbb{R}^{2n}$.*

Proof Applying the Morse argument to the pair of manifolds $V \subset f^{-1}(\mathbb{R})$ near p we obtain local coordinates $u_1, \ldots, u_{2n+1}$ for $f^{-1}(\mathbb{R})$ so

$$||z - p||^2 = u_1^2 + \ldots + u_{2n+1}^2;$$

and so that V corresponds to the locus $u_1 = 0$. Then

$$\phi^{-1} = f^{-1}(\mathbb{R}_+) \cap S_\epsilon$$

will correspond to the open hemisphere

$$\pm u_1 > 0, u_1^2 + u_2^2 + \ldots + u_{2n+1}^2 = \epsilon^2;$$

which clearly is diffeomorphic to $\mathbb{R}^{2n}$. This proves our theorem.

Now we consider a particular example of a polynomial function in two variables for which the hypersurface has a singular point and we compute the space K near that singular point, which in our case will turn out to be a knot unlike the smooth case.

Consider the following polynomial

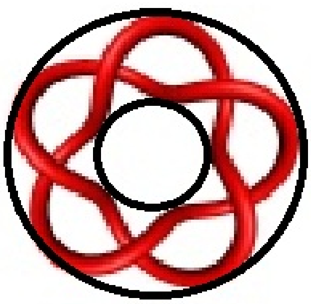
$$f(z_1, z_2) = z_1^p + z_2^q$$

in two variables, with a critical point ($\partial f/\partial z_1 = \partial f/\partial z_2 = 0$) at the origin and assume that the integers p, q are relatively prime and ≥ 2. This point will be a singular point in $V = (z_1, z_2)|f(z_1, z_2) = z_1^p + z_2^q = 0$.

Theorem 3.5 (Brauner) *The intersection of $V = f^{-1}(0)$ with a sphere S_ϵ centered at the origin is a knotted circle of the type known as "torus knots of the type (p, q)" in the 3-sphere S_ϵ.*

Proof We note that the intersection $K = V \cap S_\epsilon$ lies in the torus consisting of all (z_1, z_2) with $|z_1| = \xi, |z_2| = \eta$, where ξ and η are positive constants. In fact, K consists of all pairs $(\xi e^{qi\theta}, \eta e^{pi\theta + \pi i/q})$ as the parameter θ ranges from 0 to 2π: Thus K sweeps around the torus q times in one coordinate and p times in the other.

We end this section by giving an example of the torus knot of the type (3,5) as illustrated in the figure.



The higher dimensional analogues of these torus knots are know as Brieskorn spheres [Mil68].

Chapter 4

Milnor Fibre and their properties

This chapter has been referred from [Mil68].

4.1 Milnor Fibration Theorem

This section details the main Fibration theorem due to Milnor.

Let $V \subset \mathbb{R}^m$ be a real algebraic set, and let $U \subset \mathbb{R}^m$ be an open set defined by finitely many polynomial inequalities:

$$U = \{x \in \mathbb{R}^m | g_1(x) > 0, \ldots, g_l(x) > 0\}.$$

Before proceeding ahead we state a result about the parametrization of algebraic curves which is used in the proof of the curve selection lemma.

Lemma 4.1 (Parametrization of algebraic curves) *Let p be a non-isolated point of a real (or complex) 1-dimensional variety V. Then a suitably chosen neighborhood of p in V is the union of finitely many "branches" which intersect only at p. Each branch is homeomorphic to an open interval of real numbers (or to an open disc of complex numbers) under a homeomorphism $x = \gamma(t)$ which is given by a power series*

$$\gamma(t) = p + a_1 t + a_2 t^2 + a_3 t^3 + \ldots,$$

convergent for $t < \epsilon$.

Proof [Sketch of proof] The full proof for case of complex curve in $\mathbb{C}^2$ can be found in [Per08]. The main idea of the proof depends on showing that: If $f(X,Y)$ is the polynomial which defines the curve in the affine space $\mathbb{A}^2(k)$ and if p is an element of k, then the irreducible factors of f as a polynomial in Y with coefficients in $k((X-p))$ correspond bijectively to the different branches of the curve C which have center at the points (x_0, y_i) of the curve.

The case of a complex curve in $\mathbb{C}^m$, with $m > 2$, can be considered in a similar way.

For the case of real 1-dimensional variety $V \subset \mathbb{R}$ we see that:

Let $V_{\mathbb{C}}$ be the smallest complex algebraic set in $\mathbb{C}^m$ which contains V. Then we see that $V_{\mathbb{C}}$ is irreducible, of complex dimension 1, and that the set of $\mathbb{R}^m \cap V_{\mathbb{C}}$ of real points in $V_{\mathbb{C}}$ is equal to V. Now for each branch of $V_{\mathbb{C}}$ we can form the complex parametrization

$$x = \gamma(t) = p + (0, \ldots, 0, t^{\mu}, \Sigma_i a_{k+1,i} t^i, \ldots, \Sigma_i a_{mi} t^i).$$

We want to know for which values of the complex parameter t will the vector $\gamma(t)$ be real. Clearly the $k-th$ component t^{μ} is real iff t can be expressed as the product of $\mu - th$ root of unity ξ and a real number s. But, for each choice of ξ, substituting $t = \xi s$ in the power series γ, we obtain a new complex power series $p + \Sigma(a_i \xi^i) s^i$ in the real variable s. If the coefficients $a_i \xi^i$ are all real, then $\gamma(\xi s) \in \mathbb{R}^m$. But if some coefficient vector $a_i \xi^i$ is not real, then $\gamma(\xi s) \notin \mathbb{R}^m$ for all small non-zero values of s. Therefore each branch of $V_{\mathbb{C}}$ intersects $\mathbb{R}^m$ in at most a finite number of branches of the real variety V. This sketches the proof of our lemma.

Lemma 4.2 (The Curve Selection Lemma) *[Milnor, 1968] If $U \cap V$ contains points arbitrarily close to the origin (i.e. if $0 \in \overline{(U \cap V)}$) then there exists a real analytic curve*

$$\gamma : [0, \epsilon) \to \mathbb{R}^m$$

with $\gamma(0) = 0$ and with $\gamma(t) \in U \cap V$ for $t > 0$.

Proof We may assume that dimension of $V \leq 1$ as if the dimension of $V \geq 2$, then we can construct a proper algebraic subset $V_1 \subset V$ so that $0 \in \overline{(U \cap V_1)}$. We iterate this procedure inductively until we find an algebraic subset V_q of dimension ≤ 1 with $0 \in \overline{(U \cap V_q)}$. We also assume that V is irreducible and that the open set U does not contain any points of the singular set $\Sigma(V)$, within some neighborhood D_η of 0. For if V was a union of two proper algebraic subsets, then one of these subsets serves as V_1 and if U would contain some point of $\Sigma(V)$, we could choose V_1 to be $\Sigma(V)$.

Let $f_1, \ldots, f_k$ span the ideal $I(V)$. The singular set $\Sigma(V)$ is the set of all $x \in V$ for which

$$rank\{df_1(x), \ldots, df_k(x)\} < \rho;$$

where the dimension of the variety V is $m - \rho$. Consider the following auxiliary functions:

$$r(x) = ||x||^2, \; g(x) = g_1(x)g_2(x)\ldots g_l(x).$$

Let V' be the set of all $x \in V$ with

$$rank\{df_1(x), \ldots, df_k(x), dr(x), dg(x)\} \leq \rho + 1.$$

Claim: The intersection $U \cap V'$ also contains points arbitrarily close to 0.

Proof of Claim: By hypothesis there exist arbitrarily small spheres S_ϵ centered at 0 which contains points of $U \cap V$. We choose any such sphere S_ϵ and consider the compact set consisting of all $x \in V \cap S_\epsilon$ with

$$g_1(x) \geq 0, \ldots, g_l(x) \geq 0.$$

From the extreme value theorem, the continuous function $g(x)$ must be maximized at some point x' of this compact set; and clearly $x' \in U$. Now we show that $x' \in V'$. We first observe that S_ϵ intersect $U \cap V$ in a smooth manifold of dimension $m - \rho - 1$; and that

$$rank\{df_1(x), \ldots, df_k(x), dr(x)\} = \rho + 1$$

at every point x of $U \cap V \cap S_\epsilon$. Observe that the critical points of $g|_{U \cap V \cap S_\epsilon}$ are just those points of $U \cap V \cap S_\epsilon$ which lie in V' as V' is the set of all $x \in V$ with

$$rank\{df_1(x), \ldots, df_k(x), dr(x), dg(x)\} \leq \rho + 1$$

and in $U \cap V \cap S_\epsilon$ we have $rank\{df_1(x), \ldots, df_k(x), dr(x)\} = \rho + 1$.

But $g|_{U \cap V \cap S_\epsilon}$ attains its maximum at x'. So x' is certainly a critical point and therefore belong to V'. This gives us our claim.

Thus, if V' is a proper subset of V, then it satisfies our requirements. The case that remains is when $V = V'$. In that case we carry out the above proceduce with the following function in place of g :

$$(x_1, \ldots, x_m) \to x_i g(x_1, \ldots, x_m).$$

Let V_i' be the set of all $x \in V$ with

$$rank\{df_1(x), \ldots, df_k(x), dr(x), d(x_i g)(x)\} \le \rho + 1$$

Then in a similar way one can show that $0 \in \overline{(U \cap V_i')}$.

Thus we have found a suitable algebraic subset $V_1 \subset V$ except in the case $V = V' = V_1' = \cdots = V_m'$.

Claim: $V = V' = V_1' = \cdots = V_m'$ occurs only when the dimension $m - \rho$ of V is equal to 1.

Proof of claim: We choose a point $x' \in U \cap V$ so that

$$rank\{df_1(x'), \ldots, df_k(x'), dr(x')\} = \rho + 1\ .$$

If $V = V'$, then $x' \in V'$ and hence the differential $dg(x') \in \{df_1(x'), \ldots, df_k(x'), dr(x')\}$ a $\rho + 1$ dimensional vector space. Similarly, if $V = V_i'$, then $d(x_i g)(x')$ must belong to this vector space. Using the identity

$$d(x_i g) = (dx_i)g + x_i(dg)$$

and the fact that $g(x') \neq 0$ (since $x' \in U$) it follows that $dx_i(x')$ also belongs to this $\rho + 1$ dimensional vector space. But the differentials $dx_1, \ldots, dx_m$ form a basis of for the entire m-dimensional vector space of differentials at x'. So the subspace spanned by $df_1(x'), \ldots, df_k(x'), dr(x')$ must be the whole of the space i.e. , $\rho + 1 = m$. This gives $m - \rho = 1$, which is the dimension of V. This proves our claim.

Now suppose that V contains points x arbitrarily close to 0 with $x \in U$, that is with

$$g_1(x) > 0, \ldots, g_l(x) > 0;$$

and that V has dimension 1. Then using the parametrization of algebraic curves (varieties of dimension 1), one of the finitely many branches of V through 0 must contain points of U arbitrarily close to 0. Let

$$x = \gamma(t), \quad |t| < \epsilon$$

be a real analytic parametrization of this branch. For each g_i note that the real analytic function $g_i(\gamma(t))$ must be either > 0 for all t in some interval $0 < t < \epsilon'$, or ≤ 0 for all t with $0 < t < \epsilon'$. So the half-branch $\gamma(0, \epsilon')$ is either contained in U or disjoint from U, for ϵ' sufficiently small. Similarly for the half-branch $\gamma(-\epsilon', 0)$. But we have assumed that $\gamma(-\epsilon, \epsilon)$ contains points of U arbitrarily close to 0, so at least one of these two half-branches must be contained in U. We are done.

Here is an application of curve selection lemma.

Corollary 4.3 *If $f \geq 0$ and $g \geq 0$ are non-negative polynomial functions on $\mathbb{R}^m$ which vanish at p, then for x in some neighborhood of D_ϵ of p, the two differentials $df(x)$ and $dg(x)$ cannot point in exactly opposite directions unless at least one of them vanishes.*

Proof Let U be the open set consisting of all x for which the inner product

$$\Sigma_i(\partial f(x)/\partial x_i)(\partial g(x)/\partial x_i)$$

is negative, and let V be the algebraic set consisting of all x for which

$$rank\{df(x), dg(x)\} \leq 1$$

Thus $U \cap V$ is the set of all x for which $df(x)$ and $dg(x)$ point exactly in opposite directions. If $U \cap V$ contained points arbitrarily close to p , then there would exist an entire real analytic curve

$$x = \gamma(t), \quad 0 \leq t < \epsilon,$$

which consisted entirely of such points, except for $p = \gamma(0)$.

For every $x \in U$ note that $f(x) > 0$ and $g(x) > 0$. For if the non-nefative function f were to vanish at x then the differential $df(x)$ would have to vanish also, hence x could not belong to U. Therefore

$$f(\gamma(t)) > 0 \ for \ t > 0;$$

and since $f \circ \gamma$ is real analytic this implies that $df(\gamma(t))/dt > 0$ also for small positive values of t. Similarly $dg(\gamma(t))/dt$ is positive for small positive t. But

$$df/dt = \Sigma(\partial f/\partial x_i) d\gamma_i/dt, \ dg/dt = \Sigma(\partial g/\partial x_i) d\gamma_i/dt,$$

where the row vector $(\partial f/\partial x_1, \ldots, \partial f/\partial x_m)$ is a negative multiple of $(\partial g/\partial x_1, \ldots, \partial g/\partial x_m)$, for all $t > 0$. Hence df/dt and dg/dt must have opposite sign. This is a contradiction which shows that the condition must be false: p cannot be te limit point of $U \cap V$.

Theorem 4.4 (Milnor, 1968) *If p is any point of the complex hyper-surface $V = f^{-1}(0)$ and if S_ϵ is a sufficiently small sphere centered at p, then the mapping*

$$\phi(z) = \frac{f(z)}{|f(z)|}$$

from $S_\epsilon - K$ to the unit circle (S^1) is a locally trivial fibration.

Proof The idea of the proof is to create a non-vanishing vector field on the space $S_\epsilon - K$ and using that we construct a local diffeomorphism using the inverse function theorem. For creating such a vector field we will use the curve selection lemma.

Our aim is now to show that the fiber is a smooth manifold and for that we will show that the mapping ϕ has no critical points for ϵ small enough. For that part we will proceed through the following series of claims:
Let K denote the intersection of the set of zeros of f with the sphere S_ϵ, consisting of all $z \in \mathbb{C}^m$ with $||z|| = \epsilon$.

Claim 1: The critical points of the mapping $\Phi : S_\epsilon - K \to S^1$ defined as above are precisely those points $z \in S_\epsilon - K$ for which the vector $igrad \log f(z)$ is a real multiple of the vector z.

Proof of Claim 1: Locally one can take the logarithm function as a single-valued function; and its gradient $grad \log f(z) = (grad\ f(z))/\bar{f}(z)$, is well defined everywhere.
Now setting $f(z)/|f(z)| = e^{i\theta(z)}$, we see that $\theta(z) = Re(-i\log f(z))$ as we can multiply the equation

$$i\theta = \log(f/|f|) = \log f - \log|f|$$

by $-i$ to get

$$\theta = -i\log(f/|f|) = -i\log f + i\log|f|$$

and we take the real part on both the sides and observe that $i\log|f|$ is always a purely imaginary number. Differentiating $\theta(z) = Re(-i\log f(z))$ along the curve $z = \gamma(t)$ we obtain

$$\begin{aligned} d\theta(\gamma(t))/dt &= Re(d(-\log f)/dt) \\ &= Re < d\gamma/dt, grad(-i\log f) > \\ &= Re < d\gamma/dt, igrad \log f > . \end{aligned}$$

In other words, the directional derivative of the function $\theta(z)$ in the direction $v = d\gamma/dt$ is equal to

$$Re < d\gamma/dt, igrad \log f >$$

We note that the hermitian vector space $\mathbb{C}^m$ can be thought of as a euclidean vector space (of dimension 2m) over $\mathbb{R}$, defining the euclidean inner-product of two vectors $\mathbf{a}$ and $\mathbf{b}$ be the real part

$$Re < \mathbf{a}, \mathbf{b} >= Re < \mathbf{b}, \mathbf{a} >$$

We observe that a vector $\mathbf{v}$ is tangent to the sphere S_ϵ at z iff the real inner product $Re < \mathbf{v}, \mathbf{z} >$ is zero.

Now if the vector $igrad(\log f(z))$ happens to be a real multiple of z, i.e., this vector is normal to S_ϵ, then for every vector $\mathbf{v}$ tangent to S_ϵ at z the directional derivative

$$Re < \mathbf{v}, igrad \log f(z) >$$

of θ in the direction $\mathbf{v}$ will be 0. Hence, z is a critical point of the mapping ϕ.

For the converse, if the vectors $igrad \log f(z)$ and z are linearly independent over $\mathbb{R}$, then there exists a vector $\mathbf{v}$ in our euclidean vector space so that

$$Re < \mathbf{v}, z >= 0$$

$$Re < \mathbf{v}, igrad \log f(z) >= 1.$$

Thus $\mathbf{v}$ is tangent to S_ϵ and the directional derivative of θ along $\mathbf{v}$ is $1 \neq 0$; hence z is not a ccritical point of ϕ. This proves our claim.

Let V denote the hyper surface $f^{-1}(0) \subset \mathbb{C}^m$.

Assume that f is a polynomial which vanishes at the origin.

Claim 2: Let $\gamma : [0, \epsilon) \to \mathbb{C}^m$ be a real analytic path with $\gamma(0) = 0$ such that, for each $t > 0$, the number $f(\gamma(t))$ is non-zero and the vector $grad \log f(\gamma(t))$ is a complex multiple $\lambda(t)\gamma(t)$. Then the argument of the complex number $\lambda(t)$ tends to zero as $t \to 0$.

Proof of claim 2: We want to show that $\lambda(t)$ is non-zero for small positive values of t and $\lim_{t\to 0} \lambda(t)/|\lambda(t)| = 1$.

Consider the Taylor expansions

$$\gamma(t) = at^\alpha + a_1 t^{\alpha+1} + \dots,$$
$$f(\gamma(t)) = bt^\beta + b_1 t^{\beta+1} + \dots,$$
$$grad\ f(p(t)) = ct^\delta + c_1 t^{\delta+1} + \dots,$$

where the leading coefficients a, b, c are non-zero. The identity $df/dt = < d\gamma/dt, grad\ f >$ shows that $grad\ f(\gamma(t))$ cannot be identically zero. The leading exponents α, β, δ are integers with $\alpha \geq 1, \beta \geq 1, \delta \geq 0$. These series are all convergent say for $|t| < \epsilon'$.

For each $t > 0$ we have

$$grad \log f(\gamma(t)) = \lambda(t)\gamma(t),$$

hence

$$grad\ f(\gamma(t)) = \lambda(t)\gamma(t)\bar{f}(\gamma(t)),$$

or in other words

$$(ct^{\delta} + c_1 t^{\delta+1} + \ldots) = \lambda(t)(a\bar{b}t^{\alpha+\beta} + \ldots).$$

Comparing corresponding components of these two vector valued functions, we see that $\lambda(t)$ is the quotient of real analytic functions, and therefore has a Laurent series expansion of the form

$$\lambda(t) = \lambda_0 t^{\delta-\alpha-\beta}(1 + k_1 t + k_2 t^2 + \ldots).$$

Furthermore the leading coefficients must satisfy the equation

$$c = \lambda_0 a\bar{b}.$$

Substituting this equation in the power series expansion of the identity

$$df/dt = < d\gamma/dt, grad\ f >$$

we get

$$\begin{aligned} (\beta b t^{\beta-1} + \ldots) &= < \alpha a t^{\alpha-1} + \ldots, \lambda_0 a\bar{b}t^{\delta} + \ldots > \\ &= \alpha ||a||^2 \bar{\lambda_0} b t^{\alpha-1+\delta} + \ldots. \end{aligned}$$

Comparing the leading coefficients it follows that

$$\beta = \alpha ||a||^2 \bar{\lambda_0}$$

which proves that $\lambda_0$is a positive real number. Therefore

$$lim_{t\to 0} argument \lambda(t) = 0,$$

which proves our claim 2.

Claim 3: Given any polynomial f which vanishes at the origin, there exists a number $\epsilon_0 > 0$ so that , for all $z \in \mathbb{C}^m - V$ with $||z|| \leq \epsilon_0$, the two vectors z and $grad \log f(z)$ are either linearly independent over the complex numbers or else

$$grad \log f(z) = \lambda z$$

where λ is a non-zero complex number whose argument has absolute value less than $\pi/4$. (Here argument of $\lambda \neq 0$ will mean the unique number $\theta \in (-\pi, \pi]$ such that $\lambda/|\lambda| = e^{i\theta}$.

In other words, λ lies in the open quadrant of the complex plane which is centered about the positive real axis. It follows that

$$Re(\lambda) > 0;$$

so that λ cannot be purely imaginary. We see an immediate consequence of this in the following specialization of claim 3.

Claim 4: For every $z \in \mathbb{C}^m - V$ which is sufficiently close to the origin, the two vectors z and $i grad \log f(z)$ are linearly independent over $\mathbb{R}$.

Proof of claim 3: Suppose there were points $z \in \mathbb{C}^m - V$ arbitrarily close to the origin with

$$grad \log f(z) = \lambda z \neq 0$$

and with $|arg \lambda| > \pi/4$. In other words, assume that λ lies in the open half-plane

$$Re((1+i)\lambda) < 0$$

or the open half-plane

$$Re((1-i)\lambda) < 0$$

We would like to change these conditions into algebraic statements so that we can use the curve selection lemma.

Let W be the set of all $z \in \mathbb{C}^m$ for which the vectors $grad\ f(z)$ and z are linearly dependent. Thus $z \in W$ iff the following holds.

$$z_j\overline{(\partial f/\partial z_k)} = z_k\overline{(\partial f/\partial z_j)}$$

Set $z_j = x_j + iy_j$, and then taking the real and imaginary parts, we obtain a collection of real polynomial equations in the real variables x_j and y_j. This shows that $W \subset \mathbb{C}^m$ is a real algebraic set.

Note that a point $z \in \mathbb{C}^m - V$ belong to W iff

$$(grad\ f(z))/\bar{f}(z) = \lambda z$$

for some complex number λ. Multiplying by $\bar{f}(z)$ and taking the inner product with $\bar{f}(z)z$, this yields

$$< (grad\ f(z)), \bar{f}(z)z >= \lambda||\bar{f}(z)z||^2.$$

In other words, the number λ, multiplied by a positive real number, is equal to

$$\lambda'(z) =< grad\ f(z), \bar{f}(z)z >$$

Hence

$$arg\lambda = arg\lambda'.$$

Clearly, λ' is a complex valued polynomial function of the real variables x_j and y_j.

Now let U_+ (resp. U_-) denote the open set consisting of all z satisfying the real polynomial inequality

$$Re((1+i)\lambda'(z)) < 0 \tag{4.1}$$

respectively

$$Re((1-i)\lambda'(z)) < 0$$

We have assumed that there exist points z arbitrarily close to the origin with $z \in W \cap (U_+ \cup U_-)$. Hence by the curve selection lemma, there must exist a real analytic path

$$\gamma : [0, \epsilon) \to \mathbb{C}^m$$

with $\gamma(0) = 0$ and with either

$$\gamma(t) \in W \cap U_+$$

for all $t > 0$ or

$$\gamma(t) \in W \cap U_-$$

for all $t > 0$. In either case, for each $t > 0$ we get

$$grad \log f(\gamma(t)) = \lambda(t)\gamma(t)$$

with

$$|arg\lambda(t)| > \pi/4;$$

which contradicts Claim 2. To complete the proof we still have to show that $W - (V \cap W)$ does not contain points arbitrarily close to the origin with either

$$\lambda'(z) = 0 \ or \ |arg\lambda'(z)| = \pi/4.$$

But we can proceed similarly as above, substituting the polynomial equality

$$Re((1+i))\lambda'(z))Re((1-i)\lambda'(z)) = 0,$$

together with the polynomial inequality

$$||f(z)||^2 > 0,$$

for the inequality (4.1). Again we would obtain a path $p(t)$ which would contradict claim 2. This completes the proof of claim 3 and 4.

Now, combining Claim 1 and 4 we get that

If $\epsilon \leq \epsilon_0$ then the map

$$\phi : S_\epsilon - K \to S^1$$

has no critical points at all.

It follows that, for each $e^{i\theta} \in S^1$, the inverse image

$$F_\theta = \phi^{-1}(e^{i\theta}) \subset S_\epsilon - K$$

is a smooth $(2m-2)$-dimensional manifold.

Now we only need to prove that ϕ is the projection map of a locally trivial fibration. We do this by the help of following claims:

Claim 5: If $\epsilon \leq \epsilon_0$ then there exists a smooth tangential vector field $v(z)$ on $S_\epsilon - K$, then the complex inner product

$$< v(z), igrad \log f(z) >$$

is non-zero, and has argument less than $\pi/4$ in absolute value.

Proof of Claim 5: It is sufficient to construct such a vector field locally, in a neighborhood of some given point z^α, as we can make the field global by using partition of unity.

CASE 1: If the vectors w and $grad \log f(w)$ are linearly independent over $\mathbb{C}$, then the linear equations

$$< v, w >= 0,$$

$$< v, igrad \log f(w) >= 1$$

have a simultaneous solution v. The first equation guarantees that $Re(< v, w >) = 0$, so that v is the tangent to S_ϵ at w.

CASE 2: If $grad \log f(w)$ is equal to the multiple λw, then set $v = iw$. Clearly

$$Re(< iw, w >) = 0;$$

and by claim 3 the number

$$< iw, igrad \log f(w) >= \bar{\lambda}||w||^2$$

has argument less than $\pi/4$ in absolute value.

In either case we can choose a local tangential vector field $v^w(z)$ which takes the constructed value v at w. The condition

$$|arg < v^w(z), igrad \log f(z) > | < \pi/4$$

will then certainly hold throughout a neighborhood of w. Using partition of unity, we obtain a global vector field $v(z)$ having the same property. This gives us claim 5.

Next we normalize by setting

$$u(z) = v(z)/Re(< v(z), igrad \log f(z) >).$$

Thus we obtain a smooth tangential vector field u on $S_\epsilon - K$ which satisfies two conditions:

$$Re(< u(z), igrad \log f(z) >) = 1$$

and the corresponding imaginary part satisfies

$$|Re(< v(z), grad \log f(z) >)| < 1.$$

Now consider the trajectories of the differential equation $dz/dt = u(z)$.

Claim 6: Given any $p \in S_\epsilon - K$ there exists a unique smooth path

$$\gamma : \mathbb{R} \to S_\epsilon - K$$

which satisfies the differential equation

$$d\gamma/dt = w(\gamma(t))$$

with initial condition $p(0) = p$.

Proof of Claim 6: We know that such a solution $z = \gamma(t)$ exists locally, and can be extended over some maximal open interval of real numbers. Then only problem, which arises since $S_\epsilon - K$ is non-compact, is to insure that $\gamma(t)$ cannot tend towards

K as t tends toward some finite limit t_0. That is we must guarantee that $f(\gamma(t))$ cannot tend to zero, or

$$Re(\log f(\gamma(t))) \to -\infty$$

as $t \to t_0$. But the derivative

$$\begin{aligned} d(Re(\log f(\gamma(t)))/dt) &= Re < d\gamma/dt, grad \log f > \\ &= Re < w(\gamma(t)), grad \log f > \end{aligned}$$

has absolute value less than 1. Hence $|f(\gamma(t))|$ is bounded away from zero as t tends to any finite limit. This proves claim 6.

Setting $\phi(z) = e^{i\theta(z)}$, we note that

$$d\theta(\gamma(t))/dt = Re < d\gamma/dt, igrad \log f >= 1.$$

Hence

$$\theta(\gamma(t)) = t + constant.$$

In other words the path $\gamma(t)$ projects under ϕ to a path which winds around the unit circle in the positive direction with unit velocity.

We note that the point $p(t)$ is a smooth function both of t and of the initial value

$$p = \gamma(0).$$

We express this dependence by setting

$$\gamma(t) = h_t(p).$$

We note that each h_t is a diffeomorphism mapping $S_\epsilon - K$ to itself and it maps each fiber $F_\theta = \phi^{-1}(e^{i\theta})$ onto the fiber $F_{\theta+t}$.

Now, given $e^{i\theta} \in S^1$ let U be a small neighborhood of $e^{i\theta}$. Then the correspondence

$$(e^{i(\theta+t)}, z) \mapsto h_t(z),$$

for $|t| <$ constant, and $z \in F_\theta$, maps the product $U \times F_\theta$ diffeomorphically onto $\phi^{-1}(U)$. This proves our theorem.

4.2 Topological Properties

In this section we describe the topological properties of the Milnor Fiber

$$F_\theta = \phi^{-1}(e^{i\theta})$$

where ϕ was the Milnor fibration map and thereby derive the topological properties of $K = V \cap S_\epsilon$. We will use tools from Morse theory to understand these fibers.

If we set $m - n + 1 \geq 1$, we have already seen that each F_θ will be a smooth manifold of dimension 2n. We associate to this manifold a real valued function $|f|$. We proceed to compute the critical points of this map. For this purpose we work with the smooth function $a_\theta : F_\theta \to \mathbb{R}$ defined by

$$a_\theta(z) = \log|f(z)|$$

An easy computation shows that the critical points of a_θ are same as those of $|f|$ on F_θ.

Lemma 4.5 *The critical points of the smooth real valued function $a_\theta = \log|f(z)|$ on F_θ are those point $z \in F_\theta$ for which the vector $grad \log f(z)$ is a complex multiple of z.*

Proof The directional derivative of the function

$$\log|f(z)| = Re \log f(z)$$

in any direction v is equal to the real inner product

$$Re < v, gradlogf(z) > .$$

Thus z will be a critical point of this function restricted to F_θ if and only if the vector $grad \log|f(z)|$ is normal to F_θ at z, where normal means orthogonal to all tangent vectors using the real inner product. The space of normal vectors to the submanifold $F_\theta \subset \mathbb{C}^m$, of real codimension 2, is spanned by the two linearly independent vectors z and $igrad \log f(z)$. Thus z is a critical point of a_θ iff there is a real linear dependence between the vectors $grad \log f(z)$, z and $igrad \log f(z)$. This gives us our lemma.

We want to compute the Morse index of the function a_θ, for which we will compute the Hessian, at a critical point. We use the following formulation for this step.

Given a tangent vector v at the critical point z we choose a smooth path

$$\gamma : \mathbb{R} \to F_\theta$$

with velocity vector $\frac{d\gamma}{dt} = v$ at $\gamma(0) = z$. Then the second derivative

$$\ddot{a_\theta} = \frac{d^2 a_\theta(\gamma(t))}{dt^2}$$

at $t = 0$ can be expressed as a quadratic function of v as described below. This quadratic function is the Hessian.

Lemma 4.6 *The second derivative of $a_\theta(\gamma(t))$ at $t = 0$ is given by an expression of the form*

$$\ddot{a_\theta} = \sum Re(B_{jk} v_j v_k) - c||v||^2$$

where (B_{jk}) is a matrix of complex numbers and c is real positive number.

Proof The path $\gamma(t)$ lies within the manifold F_θ on which $f/|f| = e^{i\theta}$ is constant. On differentiating

$$a_\theta(\gamma(t)) = \log|f(\gamma(t))| = \log f(\gamma(t))$$

we get

$$\dot{a_\theta}(\gamma(t)) = d\log f(\gamma(t))/dt = \sum(\partial \log f/\partial z_j)(d\gamma_j/dt)$$

On differentiating again we get

$$\ddot{a_\theta}(\gamma(t)) = \sum(\partial \log f/\partial z_j)(d^2\gamma_j/dt^2) + \sum(\partial^2 \log f/\partial z_j \partial z_k)(d\gamma_j/dt)(d\gamma_k/dt).$$

We set $t = 0$, $grad \log f(z) = \lambda z$ (by previous lemma) and introduce the notation

$$D_{jk} = \partial^2 \log f/\partial z_j \partial z_k,$$

we can rewrite the above relation as

$$\ddot{a_\theta}(\gamma(t)) = < \ddot{\gamma}, \lambda z > + \sum D_{jk} v_j v_k.$$

where the left side $\ddot{a}_\theta$ is clearly real. Now we multiply both the sides by λ and take the real part:

$$\ddot{a}_\theta Re(\lambda) = |\lambda|^2 Re < \ddot{\gamma}, z > + \sum Re(\lambda D_{jk} v_j v_k).$$

We differentiate the following equation twice

$$< \gamma(t), \gamma(t) > = constant,$$

this gives us the identity $Re < \ddot{\gamma}, z > = -||v||^2$, which we substitute into the relation above to get

$$\ddot{a}_\theta Re(\lambda) = \sum Re(\lambda D_{jk} v_j v_k) - ||\lambda v||.$$

On dividing by $Re(\lambda)$, which is positive as was shown in the proof of the fibration theorem , we get our lemma.

Now we estimate the index as:

Lemma 4.7 *The Morse index of $a_\theta : F_\theta \to \mathbb{R}$ at a critical point is $\geq n$.*

Proof The Morse Index I of the quadratic function

$$H(v) = Re \sum Re(B_{jk} v_j v_k) - c||v||^2$$

where $v \in T_z F_\theta$, is the maximum dimension of a subspace on which H is negative definite.

If $H(v) \geq 0$ for any non-zero vector v, then $H(iv) < 0$; as the first term in $H(v)$ changes sign while the second term remains negative and iv is also a tangent vector to F_θ.

We split the tangent space at z as a real direct sum $T_0 \oplus T_1$ where the Hessian is negative definite on T_0 and positive semidefinite on T_1. Clearly, $dimT_0 = I$, the Morse index. But H is also negative definite on iT_1. Thus

$$I \geq dim(iT_1) = dim(T_1) = 2n - I.$$

This gives us that $I \geq n$.

Remark We can do the same estimations for a smooth function on the total space of the locally trivial fribration (Milnor Fibration). For that consider the function $a : S_\epsilon \setminus K \to \mathbb{R}$ defined as

$$a(z) = \log |f(z)|$$

We observe that on a fiber F_θ the function $a_\theta(z) = a(z)$ and the critical points of a are same as those of $|f|$ on $S_\epsilon \setminus K$. The corresponding statement for a as the above lemma i.e., the Morse index of $a : S_\epsilon \setminus K \to \mathbb{R}$ at any critical point is $\geq n$, follows as every critical point of a is also a critical point of the appropriate a_θ and the index of a at z is clearly the index greater than or equal to the index of a_θ at z.

Now we show that the critical points all lie within a compact subset of F_θ or if $S_\epsilon \setminus K$.

Lemma 4.8 *There exists a constant $\eta_\theta > 0$ so that the critical points of a_θ all lie within the compact subset $|f(z)| \geq \eta_\theta$ of F_θ. Similarly, there exists $\eta > 0$ so that the critical points z of a all satisfy $|f(z)| \geq \eta$.*

Proof Let us assume that there were critical points z of $a_\theta = \log |f|$ on F_θ with $|f(z)|$ arbitrarily close to zero, then these critical points will have a limit point p on the compact set S_ϵ. Using the curve selection lemma, there would exist a real analytic curve

$$\gamma : (0, \epsilon') \to F_\theta$$

consisting completely of critical points with $\gamma(0) = p$. Clearly, the function a_θ is constant along this path, hence $|f|$ is constant and cannot tend to $|f(p)| = 0$. This is a contradiction to our assumption. Hence our lemma holds.

Lemma 4.9 *There exists a smooth mapping*

$$s_\theta : F_\theta \to \mathbb{R}_+$$

so that all critical points of s_θ are non- degenerate, with Morse index $\geq n$, and so that $s_\theta(z) = |f(z)|$ whenever $|f(z)|$ is sufficiently close to zero. Similarly there exists a smooth mapping

$$s : S_\epsilon \setminus K \to \mathbb{R}_+$$

with all critical points non-degenerate, of index $\geq n$ and with $s(z) = |f(z)|$, whenever $|f(z)|$ is sufficiently close to 0.

Proof From 2.14, we can choose s_θ (or s) so as to be equal to $|f|$ except on a compact neighborhood of the critical set, have only non-degenerate critical points and so that the first and second derivative s of s_θ on any compact coordinate patch uniformly approximates those of $|f|$. Since the critical points of $|f|$ all have index $\geq n$, it follow that if the approximation is sufficiently close, that critical points of s_θ also have index $\geq n$, using 2.18. This gives the lemma.

As the critical points of s_θ are isolated, and all lie within a compact set. Hence there are only finitely many critical points of s_θ.

Now we proceed to prove the main theorems for this section which describe the topological properties of the fibers.

Theorem 4.10 *Each fiber F_θ has the homotopy type of a finite CW-complex of dimension n.*

Proof Consider the function $g : F_\theta \to \mathbb{R}$ defined as

$$g(z) = -\log s_\theta(z)$$

This function has the property that the set $\{z \in F_\theta | g(z) \leq c\}$ is compact, for every constant c.

The index I of s_θ or $\log s_\theta$ at a critical point is $\geq n$. Hence the index of $-\log s_\theta$ is $2n - I \leq n$. Thus from 2.17, the manifold F_θ has the homotopy type of a CW-complex of dimension $\leq n$, made up of one cell for each critical point of g. This gives us the theorem.

A similar argument shows that the total space $S_\epsilon \setminus K$ has the homotopy type of a finite complex of dimension $n + 1$.

We now give an alternative description of the fibers in terms of nearby hyperplanes. Let D_ϵ denote the closed disk bounded by S_ϵ.

Lemma 4.11 *There exists a smooth vector field v on $D_\epsilon \setminus V$ so that the inner product*

$$< v(z), grad \log f(z) >$$

is real and positive, for all $z \in D_\epsilon \setminus V$, and so that the inner product $< v(z), z >$ has positive real part.

Proof It is sufficient to construct such a vector field locally, in the neighborhood of some point w.
Case 1: The vectors w and $grad \log f(w)$ are linearly independent over $\mathbb{C}$, then the linear equations

$$< v, w > = r, \ r \in \mathbb{R}_+$$

$$< v, grad \log f(w) > = 1$$

have a simultaneous solution. The first equation gives us that $Re < v, w >> 0$.
Case 2: If $grad \log f(w) = \lambda w$, then we take $v = w$. Clearly,

$$Re < w, w > = ||w||^2 > 0;$$

and as from the proof of fibration theorem claim 3, the number

$$< w, \lambda w > = \bar{\lambda} ||w||^2$$

has argument less than $\pi/4$ in absolute value.
Therefore in both the cases one can choose a local vector field $v(z)$ which takes the constructed value v at w. The conditions required hold throughout a neighborhood of w. Using partition of unity, we obtain a global vector field $v(z)$ having the same property. This gives the proof of lemma.

Let c be a small complex constant, and let $c/|c| = e^{i\theta}$.

Lemma 4.12 *The intersection of the hyperplane $f^{-1}(c)$ with the open ϵ-disc is diffeomorphic to the portion of the fiber F_θ defined by the inequality $|f(z)| > |c|$.*

Proof Next we consider the solutions of the differential equation

$$\frac{d\gamma}{dt} = v(\gamma(t))$$

on $D_\epsilon \setminus V$. The condition that

$$< \frac{d\gamma}{dt}, grad \log f(\gamma(t)) > \in \mathbb{R}_+$$

gives us that the argument of $f(\gamma(t))$ is constant, and that $|f(\gamma(t))|$ is strictly monotone as a function of t because

$$d \log f(\gamma(t))/dt = < \frac{d\gamma}{dt}, grad \log f(\gamma(t)) > = d|\log f(\gamma(t))|/dt + idarg(f(\gamma(t)))/dt.$$

The condition

$$2Re < \frac{d\gamma(t)}{dt}, \gamma(t) > = d||\gamma(t)||^2/dt > 0$$

ensures that $||\gamma(t)||$ is a strictly monotone function of t.

Thus beginning from any interior point z of $D_\epsilon \setminus V$ and we follow a path through Z going away from the origin, in a direction of increasing $|f|$, until we reach a point z' on $S_\epsilon \setminus K$, which must satisfy

$$\frac{f(z')}{|f(z')|} = \frac{f(z)}{|f(z)|}.$$

Using the correspondence $z \mapsto z'$ we have proved our lemma.

But if $|c|$ is sufficiently small, then from 4.8 and 2.15 it follows that this portion of F_θ is diffeomorphic to all of F_θ.Thus we have shown that

Theorem 4.13 *If the complex number $c \neq 0$ is sufficiently close to zero, then the complex hypersurface $f^{-1}(c)$ intersects the open ϵ-disc in a smooth manifold which is diffeomorphic to the fiber F_θ.*

Now we proceed on to describe a result about the topological property of K which will be later used to give the homotopy type of the fibers.

Theorem 4.14 *The space $K = V \cap S_\epsilon$ is (n-2) connected.*

Thus for $n \geq 2$ the space K is connected, and for $n \geq 3$ it is simply connected.

Proof Let $N_\eta(K)$ denote the neighborhood of K consisting of all $z \in S_\epsilon$ with $|f(z)| \leq \eta$. It follows from 4.8 that $N_\eta(K)$ is a smooth manifold with boundary, for η sufficiently small. Using the smooth non-degenerate function s on $S_\epsilon \setminus$ *Interior* $N_\eta(K)$, we note that the entire sphere S_ϵ has the homotopy type of a complex built up from $N_\eta(K)$ by adjoining finitely many cells of dimension $\geq n$, one $I-$cell for each critical point of s of index I.

Clearly, the adjunction of a cell of dimension $\geq n$ cannot alter the homotopy groups in dimension $\leq n-2$. Therefore

$$\pi_i(N_\eta(K)) \cong \pi_i(S_\epsilon) = 0$$

for $i \leq n-2$.

Now we use the fact that K is a absolute neighborhood retract [Hu65] and as it is a real algebraic set , it can be triangulated.

Therefore K is a retract of the neighborhood $N_\eta(K)$ when η is sufficiently small. It follows that $\pi_i(K)$ is also trivial for $i \leq n-2$, which completes the proof.

Next we put an additional hypothesis that the polynomial $f(z_1, \ldots, z_{n+1})$ has no critical points in some neighborhood of the origin, except possibly the origin itself. Thus the origin is either an isolated singular point, or a non-singular point, of the hypersurface $V = f^{-1}(0)$. We know that the intersection $K = V \cap S_\epsilon$ is a smooth $(2n-1)$-dimensional manifold, provided ϵ is small enough. We improve this statement as follows:

Lemma 4.15 *For ϵ sufficiently small, the closure of each fiber F_θ in S_ϵ is a smooth $2n$-dimensional manifold with boundary, the interior of this manifold being F_θ and the boundary K.*

Proof From our assumption the mapping $f|_{S_\epsilon}$ to $\mathbb{C}$ has no critical points on K, for ϵ sufficiently small. In other words, the number zero is regular value of $f|_{S_\epsilon}$. This can be derived as follows using the curve selection lemma:

The critical points of $f|_{S_\epsilon}$ are those points z in S_ϵ at which the (non-zero) vector $gradf(z)$ is a complex multiple of z. Given a non-zero smooth path

$$\gamma : [0, \epsilon') \to \mathbb{C}^{n+1}$$

consisting only of such points, with

$$\gamma(0) = 0 \text{ and } f(\gamma(t)) \equiv 0,$$

we would have

$$< \frac{d\gamma}{dt}, gradf > = \frac{df(\gamma(t))}{dt} \equiv 0$$

hence

$$2Re < \frac{d\gamma}{dt}, \gamma(t) > = \frac{d||\gamma||^2}{dt} \equiv 0$$

and therefore $\gamma(t) \equiv 0$, which contradicts the hypothesis.

Now let p be any point of K. We choose a real local coordinate system $u_1, \ldots, u_{2n+1}$ for S_ϵ in a neighborhood U of p so that

$$f(z) = u_1(z) + iu_2(z)$$

for all z in U. A point of U belongs to the fiber $F_0 = \phi^{-1}(1)$ iff $u_1 > 0, u_2 = 0$. Hence the closure $\bar{F_0}$ intersects U in the set $u_1 \geq 0, u_2 = 0$. Clearly this is a smooth $2n$-dimensional manifold, with $F_0 \cap U$ as interior and with $K \cap U$ as boundary. Similarly we can do this for other fibers F_θ. This completes the proof.

Corollary 4.16 *The compact manifold with boundary $\bar{F_\theta}$ is embedded in S_ϵ in such a way as to have the same homotopy type as its complement $S_\epsilon \setminus \bar{F_\theta}$.*

Proof The complement is a locally trivial fiber space over the contractible manifold $S^1 - (e^{i\theta})$. Hence $S_\epsilon \setminus \bar{F_\theta}$ has any other fiber $F_{\theta'}$, as a deformation retract. Thus the complement has the same homotopy type as the fiber $\bar{F_{\theta'}}$. But $F_{\theta'}$ is diffeomorphic to F_θ and so has the same homotopy type as $\bar{F_{\theta'}}$.

Corollary 4.17 *The fiber F_θ has the homotopy type of a point in dimension less than n.*

Proof Alexander duality theorem gives us that the reduced homology group $\tilde{H}_i(S_\epsilon \setminus \bar{F}_\theta) \cong \tilde{H}^{2n-i}(\bar{F}_\theta)$, which is zero for $2n - i > n$ using 4.10.

The above statement can be made more precise in the following way:

Lemma 4.18 *The fiber F_θ is $(n-1)$-connected.*

Proof From the previous corollary we only need to verify F_θ is simply connected, provided that $n \geq 2$.

For $n \geq 3$, we can prove this lemma using 4.9. Using the smooth function s_θ on $\bar{F}_\theta$ note that $\bar{F}_\theta$ can be constructed, starting with a neighborhood $K \times [0, \eta]$ of the boundary by adjoining handles (I-dim cell) of index $\geq n$, there being one cell(handle) corresponding to each critical point of s_θ. Since adjoining such cells cannot change the homotopy groups in dimension $\leq n-2$, it follows that

$$\pi_i(\bar{F}_\theta) \cong \pi_I(K \times [0, \eta]) = 0$$

for $i \leq n-2$, using 4.14.

We can prove the above statment in another way as well using the Morse function $-s_\theta$ on $\bar{F}_\theta$. We can build $\bar{F}_\theta$ starting with a disc D_0^{2n} and successively adjoining handles of index $\leq n$. All of these handles (I-cells) can be attached within the containing space S_ϵ but the complement $S_\epsilon \setminus D_0^{2n}$ is certainly simply connected, and the adjunction of handles of index $\leq dim(S_\epsilon) - 3 = 2n-2$ cannot change the fundamental group of the complementary set. So it follows inductively that the complement $S_\epsilon \setminus \bar{F}_\theta$ is simply connected, provided $n \leq 2n-2$. Together with 4.16, this gives us the proof.

Theorem 4.19 *Each fiber has the homotopy type of a bouquet $S^n \vee \ldots \vee S^n$ of spheres.*

Proof The homology group $H_n(F_\theta)$ must be free abelian, since any torsion element would give rise to a cohomology classes in dimension $n+1$, contradicting 4.10. Hence $\pi_n(F_\theta) \cong H_n(F_\theta)$ is free abelian, using the Hurewicz theorem and assuming $n \geq 2$. Thus we can choose finitely many maps

$$(S^n, basepoint) \rightarrow (F_\theta, basepoint)$$

representing a basis. These combine to give a map

$$S^n \vee \ldots \vee S^n \to F_\theta$$

which induces an isomorphism of homology groups and hence, by Whitehead theorem is a homotopy equivalence. This finishes the proof for the case $n \geq 2$. For the case of $n = 1$, the proof follows from the fact that the first homology group is the abelianization of the fundamental group, thus it is a free group. This completes the proof of our theorem.

The above theorem can be made more precise in the following result which we state without proof.

Theorem 4.20 *For $n \neq 2$ the manifold $\bar{F}_\theta$ is diffeomorphic to a handle body, obtained from the disc D^{2n} by simultaneously attaching a number of handles of index precisely equal to n.*

Now we proceed on to understanding the Betti numbers of the fiber.

We want to compute the degree of a smooth map

$$v : S^k \to S^k$$

of a sphere into itself in terms of fixed points of v. Let M be a compact region with smooth boundary on the sphere $S^k \subset \mathbb{R}^{k+1}$ and for each boundary point $x \in M$ let $n(x)$ denote the inward normal vector, the unique unit vector which is tangent to S^k and normal to ∂M at x and points into M.

Lemma 4.21 *If the following properties are satisfied*

1. *Every fixed point of the mapping $v : S^k \to S^k$ lies in the interior of M*
2. *No point $x \in M$ is mapped into the antipode $-x$ by v ,and*
3. *The euclidean inner product $< v(x), n(x) >$ is positive for every $x \in \partial M$,*

then the euler number $\chi(M)$ is related to the degree d of v by the equality

$$\chi(M) = 1 + (-1)^k d.$$

Proof After perturbing v slightly we may assume that the fixed point of v are isolated. From the Lefschetz fixed point theorem one can assign an $\imath(x)$ to each fixed point so that the the sum of the indices is equal to the Lefschetz number

$$\sum(-1)^j Trace(v_* : H_j(S^k) \to H_j(S^k)) = 1 + (-1)^k d.$$

Consider the one parameter family of mappings

$$v_t : M \to S^k$$

defined by

$$v_t(x) = \frac{((1-t)x + tv(x))}{||((1-t)x + tv(x))||}$$

This is well defined since $v(x) \neq -x$ for $x \in M$. Clearly, v_0 is the identity and v_t maps M into itself, by a mapping homotopic to the identity, for small values of t. So, the Lefschetz number of $v_t : M \to M$ must be equal to the Euler number $\chi(M)$, for say $0 < t \leq \epsilon$.

But the fixed points of v_t are precisely the same as the fixed points of $v : S^k \to S^k$, for $t > 0$. Since the Lefschetz index of the fixed point x of v_t is an integer which varies continuously with t, it follows that the Lefschetz number $\chi(M)$ of v_ϵ must be equal to the Lefschetz number of v. This proves our lemma.

We will now show that

Theorem 4.22 *The middle Betti number of the fiber F_0 is equal to the multiplicity μ. Hence the middle homology group $H_n(F_0)$ is free abelian of rank μ.*

Proof Let M be the region consisting of all points $z \in S_\epsilon$ which satisfy the inequality $Re(f(z)) \geq 0$. Thus, M is the union of the fibers F_θ as θ ranges over the interval $[-\pi/2, \pi/2]$, together with the common boundary K. Clearly, $\partial M = F_{-\pi/2} \cup K \cup F_{\pi/2}$ is a smooth manifold. Note that M has the homotopy type of F_θ. In fact the interior of M is fibered over an open semicircle with F_θ as fiber.

Consider the smooth function

$$v(z) = \epsilon \, \frac{gradf(z)}{||gradf(z)||}$$

from the sphere S_ϵ to itself. We will show that v satisfies the three hypotheses of 4.21.

1. Clearly z is a fixed point of $v = \epsilon\ gradf/||gradf||$ iff $gradf(z)$ is a positive real multiple of z. But if $gradf(z) = cz$, $c > 0$,then $f(z) \neq 0$, and $grad \log f(z) = cz/\bar{f}(z)$, where the coefficient $c/\bar{f}(z)$ must have positive real part. Hence $Re(f(z)) > 0$ and z is an interior point of M.

2. it follows similarly

3. Given any boundary point z of M we can choose a smooth path $p(t)$ crossing into M with velocity vector $dp/dt = n(z)$ at $p(0) = z$. Clearly the derivative of $Ref(p(t)) > 0$ at $t = 0$ from the definition of M. So the identity
$$\frac{d\ Ref}{dt} = Re < \frac{dp}{dt}, gradf >$$
shows that the euclidean inner product $Re < n(z), v(z) >$ is positive.

Hence from 4.21 we get that
$$\chi(F_\theta) = \chi(M) = 1 - deg(v), \tag{4.2}$$
since the dimension $2n + 1$ of S_ϵ is odd.
But the degree of the mapping v is equal to $(-1)^{n+1}$ times the multiplicity μ of the origin as solution to the set of polynomial equations
$$\frac{\partial f}{\partial z_1} = \ldots = \frac{\partial f}{\partial z_{n+1}} = 0.$$
For μ was defined as the degree of the mapping
$$z \mapsto \frac{g(z)}{||g(z)||}$$
on S_ϵ where $g(z)$ is the complex conjugate of $gradf(z)$. And the conjugation map
$$(g_1, \ldots, g_{n+1}) \to (\bar{g}_1, \ldots, \bar{g}_{n+1})$$
clearly takes S_ϵ into itself with degree $(-1)^{n+1}$. Substituting this into 4.2 we obtain $\chi(F_\theta) = 1 + (-1)^n\mu$. But by definition the Euler number $\chi(F_\theta)$ is equal to
$$\sum(-1)^j \text{rank } H_j(F_\theta) = 1 + (-1)^n \text{rank } H_n(F_\theta).$$

Therefore $\mu = \text{rank } H_n(F_\theta)$, which gives us the result.

Since $\mu > 0$, we have

Corollary 4.23 *If the origin is an isolated critical point of f, then the fibers F_θ are not contractible and the manifold $K = V \cap S_\epsilon$ is not an unknotted sphere in S_ϵ.*

Proof If K was a topologically unknotted sphere in S_ϵ, then $S_\epsilon \setminus K$ would have the homotopy type of a circle. The homotopy exact sequence

$$\ldots \to \pi_{n+1}(S^1) \to \pi_n(F_0) \to \pi_n(S_\epsilon \setminus K) \to \ldots$$

of the fibration then leads to a contradiction.

We end this section with a discussion on the question whether or not the compact $(2n-1)$-dimensional manifold $K = f^{-1} \cap S_\epsilon$ is a topological sphere? This is answered partly for the case $n \neq 2$ by the following lemma

Lemma 4.24 *If $n \neq 2$ then K is homeomorphic to the sphere S^{2n-1} iff K has the homology of a sphere.*

Proof If $n \geq 3$ then K is simply connected by 4.14. and has dimension ≥ 5, so we can apply generalized Poincaré hypothesis. Since the statement is trivially true for $n = 1$ this completes the proof.

The above criterion can be sharpened as follows:

Lemma 4.25 *For $n \neq 2$ the manifold K is a topological sphere if and only if the reduced homology group $\tilde{H}_{n-1}$ is trivial.*

Proof For if this group is trivial, then using the Universal coefficient theorem, Poincaré duality and the fact that K is $n-2$ connected, we get that K is a homology sphere.

Bibliography

[A M01] Atiyah, M. F.; Mcdonald, I., *Commutative Algebra*, 2001.

[Dud04] Dundas, B. I., *Differential Topology*, 2004.

[Har06] Hartshorne, R., *Algebraic Geometry*, Springer, 2006.

[Hat02] Hatcher, A., *Algebraic Toplogy*, Cambridge University Press, 2002.

[Hu65] Hu, S. T., *Theory of Retracts*, Wayne State Univ. Press, 1965.

[Mil66] Milnor, J., *Morse Theory*, Princeton University Press, 1966.

[Mil68] Milnor, J., *Singular Points of Complex Hypersurface*, Princeton University Press, 1968.

[Mor34] Morse, M., *The Calculus of Variation in the Large*, American Math. Soc. Colloq. Publ. 18, 1934.

[Per08] Peruginelli, G., *Formal Parametrizations of Algebraic Curves*, 2008.

[Sha96] Shafarevich, I., *Algebraic Geometry*, Springer, 1996.

[Whi57] Whitney, H., *Elementary Structure of Real Algebraic Varieties*, Annals of Mathematics, 1957.

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