The Leray-Serre Spectral Sequence and its Applications

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Certificate of Examination

This is to certify that the dissertation titled **"The Leray-Serre Spectral Sequence and its Applications"** submitted by **Ms. Monika** (Reg. No. MS10069) for the partial fulfillment of BS-MS dual degree programme of the Institute, has been examined by the thesis committee duly appointed by the Institute. The committee finds the work done by the candidate satisfactory and recommends that the report be accepted.

Prof. I.B.S. Passi	Dr. Varadharaj R. Srinivasan	Dr. Mahender Singh (Supervisor)

Dated: 24^{th} *April*, 2015

Declaration

The work presented in this dissertation has been carried out by me under the guidance of Dr. Mahender Singh at the Indian Institute of Science Education and Research Mohali.

This work has not been submitted in part or in full for a degree, a diploma, or a fellowship to any other university or institute. Whenever contributions of others are involved, every effort is made to indicate this clearly, with due acknowledgement of collaborative research and discussions. This thesis is a bonafide record of work done by me and all sources listed within have been detailed in the bibliography.

Monika
(Candidate)

Dated: 24^{th} April, 2015

In my capacity as the supervisor of the candidate's project work, I certify that the above statements by the candidate are true to the best of my knowledge.

Dr. Mahender Singh
(Supervisor)

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Finally, I would like to acknowledge that the material presented in this thesis is based on other people's work. If I have made any contribution, then it is the selection and presentation of the material from different sources which are listed in the bibliography.

Notations

- $B_\infty^{p,q} = im\ d \cap F^p A^{p+q}$
- $\mathcal{C} =$ exact couple
- $\mathcal{C}' =$ derived couple of $\mathcal{C}$
- $\mathbb{C}P(n) = n$-dimensional complex projective space
- $C^*(X;R) =$ cochain complex of X with coefficients in R
- $E_0^*(A) =$ associated graded module of A
- $\Lambda =$ exterior algebra
- $\Omega X =$ space of based loops in X
- $PX =$ space of based paths in X
- $\mathbb{R}P(n) = n$-dimensional real projective space
- $SU(n) =$ group of $(n \times n)$ special unitary matrices
- $Total(M^{*,*}) =$ total complex of $M^{*,*}$
- $U(n) =$ group of $(n \times n)$ the unitary matrices
- $V_k(\mathbb{C}^n) =$ Stiefel manifold of k-frames in $\mathbb{C}^n$
- $WX =$ free path space of X
- $X \simeq Y = X\ \ and\ \ Y$ are homotopy equivalent
- $[X, Y] =$ set of homotopy class of maps from $X\ \ to\ \ Y$
- $Z_\infty^{p,q} = ker\ d \cap F^p A^{p+q}$

Contents

Summary

The aim of this thesis is to understand the **Leray-Serre spectral sequence** of a fibration and use it to compute cohomology of some interesting manifolds. To do that, it is important to understand spectral sequences first. So the first part of my thesis is devoted to an introduction to spectral sequences and then some background in topology to understand the Leray-Serre spectral sequence of a fibration.

A cohomology **spectral sequence** is a collection of differential bigraded R-modules $\{E_r^{*,*}, d_r\}$; $r = 1, 2, \ldots$, where the differentials are all of bidegree $(r, 1-r)$ such that for all r, the $E_{r+1}^{*,*}$-term is given as the cohomology of the $E_r^{*,*}$-term. Pictorially one can imagine this as a three dimensional lattice with each lattice point an R-module, the differentials as arrows between them and each page is obtained by taking the cohomology of the previous page. One can observe that, the knowledge of $E_r^{*,*}$ and d_r determines $E_{r+1}^{*,*}$, but not d_{r+1}. So if a differential is not known then one needs some other method to proceed.
The first property that a spectral sequence admits is that it can be represented as an infinite tower of submodules of the E_2-term and conversely. Thus one can define the limit term of this sequence which we call the E_∞-term. Now the ultimate goal is to compute this E_∞-term. It is interesting to note that if a spectral sequence 'collapses' at, say N, then the computation of E_∞-term becomes easy as the sequence becomes constant after $(N-1)$th page.
Once we know what a spectral sequence is, the natural question that one can ask is how can we construct one? In this direction, there are two general algebraic settings in which spectral sequences arise naturally. First is a filtered differential graded module and second is an exact couple.
In the first case, each filtered differential graded module A determines a spectral sequence with differential of bidegree $(r, 1-r)$ and if the filtration is bounded then the spectral sequence converges to $H(A, d)$ (the homology of A with respect to d). This result first appeared in the work of Koszul [3] and Cartan [1]. There are also weaker conditions which ensure the convergence and uniqueness of the target. Thus

if the filtration is exhaustive and weakly convergent, the same result will still hold true.

Second case is that of an exact couple. This idea was introduced by Massey [5]. An exact couple also determines a spectral sequence of cohomological type. It is interesting to observe that one can also associate a tower of submodules of E and an E_∞-term to an exact couple, just as we can do for any spectral sequence.

The next question one could ask is that if the two approaches are related in any way? and if yes then how do the two spectral sequences compare? The answer to the above question is yes and it is not very difficult to see that a filtered differential graded module gives rise to an exact couple. And in fact, the two spectral sequences, one associated to the filtered differential graded module and the other associated to the exact couple derived from the filtered differential graded module, turn out to be same.

There is another algebraic object namely double complex which gives rise to two spectral sequences, which in turn help in the calculation of the homology of the total complex associated to a double complex. Double complexes offer an example of the filtered differential graded module construction of a spectral sequence.

Finally, with enough background on spectral sequences, one can talk about fibrations and the spectral sequence associated to them. A map satisfying the homotopy lifting property with respect to all spaces is called a Hurewicz fibration (or just a fibration), while a map with the homotopy lifting property with respect to all n-cells is called a Serre fibration. It was Leray [4] who solved the problem of relating the cohomology rings of spaces making up a fiber space by developing a powerful computational gadget called a spectral sequence. This spectral sequence has many applications such as computation of cohomology of various Lie groups, homogeneous spaces and loop spaces.

Chapter 1

Introduction

1.1 Definition of a spectral sequence

In this chapter, we will give some basic definitions and properties of a spectral sequence. Let us start by defining the key element of a spectral sequence.

Definition 1.1 *A* ***differential bigraded module*** *over a ring R, is a sequence of bigraded R-modules $\{E^{p,q}\}$, where p and q are integers, together with a R-linear map $d: E^{*,*} \to E^{*,*}$, the* ***differential****, of bidegree $(s, 1-s)$ or $(-s, s-1)$, for some integer s, and satisfying $d \circ d = 0$.*

With the differential, it makes sense to take the **cohomology** of a differential bigraded module as follows,

$$H^{p,q}(E^{*,*}, d) = \frac{ker(d: E^{p,q} \to E^{p+s,q-s+1})}{im(d: E^{p-s,q+s-1} \to E^{p,q})}.$$

Now we can define what a spectral sequence is.

Definition 1.2 *A* ***Spectral sequence*** *is a collection of differential bigraded R-modules $\{E_r^{*,*}, d_r\}$, where $r = 1, 2, \ldots$; the differentials are either all of bidegree $(-r, r-1)$ (for a spectral sequence of* ***homological type****) or all of bidegree $(r, 1-r)$ (for a spectral sequence of* ***cohomological type****) and $E_{r+1}^{p,q} \cong H^{p,q}(E_r^{*,*}, d_r)$ for all p, q, r.*

The r^{th} stage of this sequence is called the E_r-**term** (or r^{th}-page) and it may be pictured as a lattice with each lattice point an R-module and the differentials as arrows. Figure 1.1 below depicts 3^{rd} page of the spectral sequence.

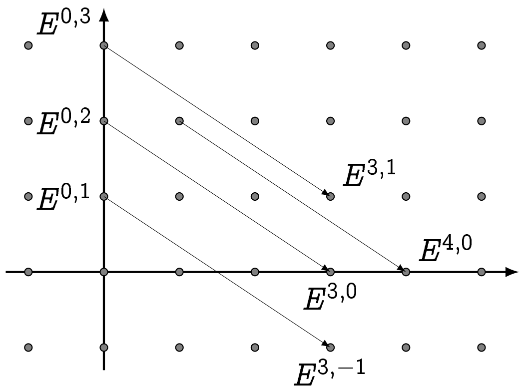


Figure 1.1: E_3-term with differential of bidegree $(3, -2)$.

Remark 1.1 *Knowledge of $E_r^{*,*}$ and d_r can determine $E_{r+1}^{*,*}$ but not d_{r+1}. So if some differential is not known then one needs some other method to proceed.*

The first property of a spectral sequence is that it can be represented as an infinite tower of submodules of the E_2-term. To see this, let $\{E_r^{*,*}, d_r\}_{r \geq 2}$ be a spectral sequence. From now on, we will suppress the bigrading.

Let $Z_2 := ker\ d_2$ and $B_2 := im\ d_2$, then $d_2 \circ d_2 = 0$ implies that $B_2 \subset Z_2 \subset E_2$. By definition, we have $E_3 \cong Z_2/B_2$. Let $\bar{Z}_3 := ker(d_3 : E_3 \to E_3)$. Then $\bar{Z}_3$ is a submodule of $E_3 \cong Z_2/B_2$, so we can write $\bar{Z}_3 = Z_3/B_2$ for a submodule $B_2 \subset Z_3 \subset Z_2$. Similarly, if $\bar{B}_3 := im\ d_3$, then $\bar{B}_3 = B_3/B_2$ for a submodule $B_2 \subset B_3 \subset Z_2$. Again, $E_4 \cong \bar{Z}_3/\bar{B}_3 \cong Z_3/B_3$, so we get the following tower of inclusions

$$B_2 \subset B_3 \subset Z_3 \subset Z_2 \subset E_2.$$

Iterating this process, we present the spectral sequence as an infinite tower of submodules of E_2 as follows

$$B_2 \subset B_3 \subset \cdots \subset B_n \subset \cdots\cdots \subset Z_n \subset \cdots \subset Z_3 \subset Z_2 \subset E_2,$$

with the property that $E_{n+1} \cong Z_n/B_n$ and the differential d_{n+1}, can be taken as a mapping $Z_n/B_n \to Z_n/B_n$ which has the kernel Z_{n+1}/B_n and image B_{n+1}/B_n. The differential d_{n+1} induces the following short exact sequence

$$0 \to Z_{n+1}/B_n \to Z_n/B_n \xrightarrow{d_{n+1}} B_{n+1}/B_n \to 0,$$

which gives rise to isomorphisms $Z_n/Z_{n+1} \cong B_{n+1}/B_n$ for all n.

Conversely, given a tower of submodules of E_2,

$$B_2 \subset B_3 \subset \cdots \subset B_n \subset \cdots\cdots \subset Z_n \subset \cdots \subset Z_3 \subset Z_2 \subset E_2$$

together with a set of isomorphisms $Z_r/Z_{r+1} \cong B_{r+1}/B_r$, we get a spectral sequence $\{E_r^{*,*}, d_r\}$.

In general, the submodule B_r of E_2 is the set of elements that are **boundaries by the r^{th} stage**. Similarly, the submodule Z_r of E_2 is the set of elements that are in the kernel of all the previous $(r-2)$ differentials but not in their image. These elements are said to have **survived to the r^{th} stage**.

Let $Z_\infty = \bigcap_n Z_n$ be the submodule of E_2 of elements that **survive forever**, that is the elements that are cycles at every stage. Similarly, let $B_\infty = \bigcup_n B_n$ be the submodule with elements that **eventually bound**. Clearly, from the tower of inclusions, we have $B_\infty \subset Z_\infty$ and hence $E_\infty := Z_\infty/B_\infty$ is the bigraded module which remains after the computation of successive homologies. In general, the aim of a computation is to determine the E_∞-term of the spectral sequence. Sometimes under the best possible conditions the computation of E_∞-term becomes easy as the sequence ends at some finite stage, which happens, for example, under the following condition.

Definition 1.3 *A spectral sequence $\{E_r^{*,*}, d_r\}$ is said to **collapse at the N^{th} term** if the differentials $d_r = 0$ for all $r \geq N$.*

The immediate result of collapse at the N^{th} term is that $E_N^{*,*} \cong E_{N+1}^{*,*} \cong \cdots \cong E_\infty^{*,*}$. From the following short exact sequence,

$$0 \to Z_r/B_{r-1} \to Z_{r-1}/B_{r-1} \xrightarrow{d_r} B_r/B_{r-1} \to 0,$$

the condition $d_r = 0$ implies that $Z_r = Z_{r-1}$ and $B_r = B_{r-1}$. Thus, the tower of submodules becomes

$$\begin{aligned} &B_2 \subset B_3 \subset \cdots \subset B_{N-1} \\ &\qquad = B_N = \cdots = B_\infty \subset Z_\infty = \cdots = Z_N \\ &\qquad\qquad = Z_{N-1} \subset \cdots \subset Z_3 \subset Z_2 \subset E_2 \end{aligned}$$

and hence $E_\infty = \cdots = E_{N+1} = E_N$.

Below we discuss some examples.

Example 1.1 Suppose n_1 and n_2 are natural numbers and $E_2^{p,q} = \{0\}$ for $p > n_1$ or $q > n_2$. Then the spectral sequence collapses at the N^{th} term where $N = min(n_1 + 1, n_2 + 2)$.

Case (1) When $n_1 \leq n_2 + 1$, that is $N = n_1 + 1$.

For $r \geq N$, $d_r : E_r^{p,q} \to E_r^{p+r,q-r+1}$ and $p + r \geq p + N = p + n_1 + 1 \geq n_1$. Therefore $d_r = 0$.

Case (2) When $n_2 + 1 < n_1$, that is $N = n_2 + 2$.

Since $q - r + 1 \leq q - n_2 - 1 < 0$, so $d_r = 0$. Hence the spectral sequence collapses at N^{th} term.

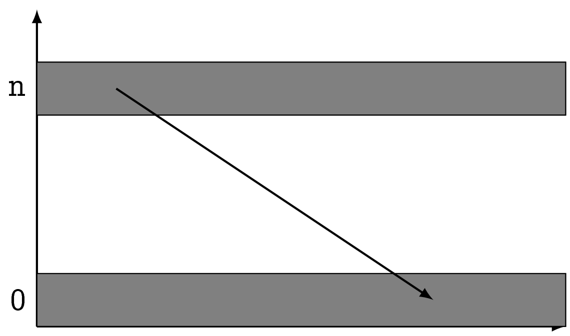
Example 1.2 Suppose $E_2^{p,q} = \{0\}$ whenever p is even or q is odd. Then the spectral sequence collapses at E_2.

To see this, note that for any $r \geq 2$, $E_r^{p,q}$ will be non-zero only if $E_2^{p,q} \neq \{0\}$, which can happen only if $p + q \equiv 1 (mod\, 2)$. So, $E_r^{p,q} \neq \{0\}$ implies that $p + q \equiv 1 (mod\, 2)$. But, the differential $d_r : E_r^{p,q} \to E_r^{p+r,q-r+1}$ changes the total degree, $p+q$, by 1. That is $p + r + q - r + 1 = p + q + 1 \equiv 0 (mod\, 2)$. Therefore, $E_r^{p+r,q-r+1} = \{0\}$. Thus, $d_r = o$ for $r \geq 2$.

Example 1.3 Suppose $E_2^{p,q} = \{0\}$ unless $q = 0$ or $q = n$, for some $n \geq 2$. Then there is a long exact sequence

$$\cdots \to H^{p+n} \to E_2^{p,n} \xrightarrow{d_{n+1}} E_2^{p+n+1,0} \to$$
$$H^{p+n+1} \to E_2^{p+1,n} \xrightarrow{d_{n+1}} E_2^{p+n+2,0} \to \cdots.$$

Since the E_2-term is non-zero only when $q = 0$ or $q = n$, thus we can depict this pictorially as follows.



Then observe that the only non-zero differential is d_{n+1}. Therefore $E_2 \cong \cdots \cong E_{n+1}$ and $E_{n+2} = H(E_{n+1}, d_{n+1}) \cong E_\infty$. Now, since d_{n+1} is zero homomorphism on the bottom stripe, so $E_\infty^{*,0} \cong E_2^{*,0}/im\, d_{n+1}$. Similarly, since everything above the strip $q = n$ is trivial, thus $E_\infty^{*,n} \cong ker\, d_{n+1}$. Thus we obtain the following short exact sequence

$$0 \to E_\infty^{p,n} \to E_2^{p,n} \xrightarrow{d_{n+1}} E_2^{p+n+1,0} \to E_\infty^{p+n+1,0} \to 0$$

for each p. Since $E_\infty^{*,q}$ is non-trivial only when $q = 0$ or $q = n$, thus the filtration on H^* takes the following form

$$\begin{aligned} H^{p+n} = F^0 H^{p+n} = \cdots = &F^n H^{p+n} \supset F^{n+1} H^{p+n} \\ &= F^{n+2} H^{p+n} = \cdots = F^{p+n} H^{p+n} \supset \{0\}. \end{aligned}$$

Further, $E_\infty^{p,n} = H^{p+n}/F^{p+1}H^{p+n} \cong H^{p+n}/E_\infty^{p+n,0}$ gives us the following short exact sequence

$$0 \to E_\infty^{p+n,0} \to H^{p+n} \to E_\infty^{p,n} \to 0,$$

for each p. Finally to obtain the long exact sequence in the example, we splice together these short exact sequences as in the following diagram

$$\begin{tikzcd}
{} \arrow[dr, dashed] & {} \arrow[d] & & & & & \\
& H^{p+n-1} \arrow[d] \arrow[dr, dashed] & & & 0 \arrow[d] & & \\
0 \arrow[r] & E_\infty^{p-1,n} \arrow[r] \arrow[d] & E_2^{p-1,n} \arrow[r] & E_2^{p+n,0} \arrow[r] \arrow[dr, dashed] & E_\infty^{p+n,0} \arrow[r] \arrow[d] & 0 & \\
& 0 & & & H^{p+n} \arrow[d] \arrow[dr, dashed] & & \\
& & & 0 \arrow[r] & E_\infty^{p,n} \arrow[r] \arrow[d] & E_2^{p,n} & \\
& & & & 0 & &
\end{tikzcd}$$

This long exact sequence is called the **Gysin sequence**. We will come back to this sequence again in chapter 4.

1.2 Convergence of spectral sequence

In order to define convergence of a spectral sequence let us first define what a filtration is.

Definition 1.4 *A **filtration** F^* on an R-module A is a family of submodules $\{F^pA\}$ for p in $\mathbb{Z}$ so that*

$$\cdots \subset F^{p+1}A \subset F^pA \subset F^{p-1}A \subset \cdots \subset A \ \ \textit{(decreasing filtration)}$$
$$\textit{or} \ \ \cdots \subset F^{p-1}A \subset F^pA \subset F^{p+1}A \subset \cdots \subset A \ \ \textit{(increasing filtration).}$$

Example 1.4 $\mathbb{Z}$ is a filtered $\mathbb{Z}$-module, together with the decreasing filtration

$$F^p\mathbb{Z} = \begin{cases} \mathbb{Z}, & \text{if } p \leq 0, \\ 2^p\mathbb{Z}, & \text{if } p > 0. \end{cases}$$

$$\cdots \subset 16\mathbb{Z} \subset 8\mathbb{Z} \subset 4\mathbb{Z} \subset 2\mathbb{Z} \subset \mathbb{Z} \subset \mathbb{Z} \subset \cdots \subset \mathbb{Z}.$$

Example 1.5 If H^* is a graded vector space with $H^n = \{0\}$ for $n < 0$, then there is an obvious filtration, induced by the grading, and given by

$$F^pH^* = \bigoplus_{n \geq p} H^n.$$

Example 1.6 A filtration is induced on the cohomology of a CW-complex X, by filtering the space itself by successive skeleta

$$X \supset \cdots \supset X^{(n)} \supset X^{(n-1)} \supset X^{(n-2)} \supset \cdots \supset X^{(0)} \supset \{*\}$$

and defining $F^pH^*(X) = ker\ i^*_{(p-1)}$ where $i^*_{(p-1)} : H^*(X) \to H^*(X^{(p-1)})$ is induced by the following $(p-1)^{st}$ inclusion map

$$X^{(n-1)} \xhookrightarrow{i_{(p-1)}} X.$$

Definition 1.5 *A filtration of A, say F^*, can be collapsed into another graded module called the **associated graded module**, $E_0^*(A)$ given by*

$$E_0^p(A) = \begin{cases} F^pA/F^{p+1}A, & \textit{when } F \textit{ is decreasing}, \\ F^pA/F^{p-1}A, & \textit{when } F \textit{ is increasing}. \end{cases}$$

Example 1.7 In the example 1.4 above, we have

$$E_0^p(\mathbb{Z}) = \begin{cases} \{0\}, & \text{if } \; p < 0, \\ \mathbb{Z}/2\mathbb{Z}, & \text{if } \; p \geq 0. \end{cases}$$

Example 1.8 In the example 1.5 above, the associated graded module is given by $E_0^p(H^*, F) = \dfrac{\bigoplus_{n \geq p} H^n}{\bigoplus_{n \geq p+1} H^n} = H^p.$

Thus, one can easily collapse a filtered module to its associated graded module as shown above.

But, what about the converse? The reconstruction of a filtered module from its associated graded module may be difficult. For instance, in the case of a locally finite graded vector space H^* (that is, H^n is finite dimensional for each n), H^* can be recovered up to isomorphism from the **associated graded vector space**, $E_0^p(H^*) = F^pH^*/F^{p+1}H^*$, by taking direct sums, that is,

$$H^* \cong \bigoplus_{p=0}^{\infty} E_0^p(H^*).$$

However, for a graded module A over an arbitrary commutative ring R, there may be extension problems that prevent one from reconstructing A from the associated graded module $E_0^*(A)$.

Let A be a filtered R-module with a bounded (decreasing) filtration:

$$\{0\} \subset F^nA \subset F^{n-1}A \subset \cdots \subset F^1A \subset F^0A \subset F^{-1}A = A.$$

Then the associated graded module $E_0^*(A)$, is nontrivial only in degrees $-1 \leq k \leq n$, and we obtain a sequence of short exact sequences

$$\begin{array}{ccccccccc}
 & & 0 & \longrightarrow & F^nA & \longrightarrow & E_0^n(A) & \longrightarrow & 0 \\
0 & \rightarrow & F^nA & \rightarrow & F^{n-1}A & \rightarrow & E_0^{n-1}(A) & \rightarrow & 0 \\
 & & \vdots & & \vdots & & \vdots & & \\
0 & \rightarrow & F^kA & \rightarrow & F^{k-1}A & \rightarrow & E_0^{k-1}(A) & \rightarrow & 0 \\
 & & \vdots & & \vdots & & \vdots & & \\
0 & \longrightarrow & F^1A & \longrightarrow & F^0A & \longrightarrow & E_0^0(A) & \longrightarrow & 0 \\
0 & \longrightarrow & F^0A & \longrightarrow & A & \longrightarrow & E_0^{-1}(A) & \longrightarrow & 0
\end{array}$$

Then $E_0^n(A)$ determines F^nA. But at each step $n-1 \geq p \geq 0$, F^pA is only determined up to choice of extension of $F^{p+1}A$ by E_0^pA. Which means, we know A only up to a sequence of choices.

Before moving forward, we observe a simple property of the associated graded module. In case of a graded R-module H^*, the associated graded module $E_0^p(H^*)$ with respect to filtration F^* of H^* is **bigraded**. Using the degree in H^*, we can define $F^pH^n = F^pH^* \cap H^n$ and thus we obtain a bigrading on E_0 by defining

$$E_0^{p,q}(H^*,F^*) = \begin{cases} F^pH^{p+q}/F^{p+1}H^{p+q}, & \text{if } F^* \text{ is decreasing,} \\ F^pH^{p+q}/F^{p-1}H^{p+q}, & \text{if } F^* \text{ is increasing.} \end{cases}$$

Now, how does this associated graded module play any role in the determination of H^*? To understand this, we give the following definition.

Definition 1.6 *A spectral sequence $\{E_r^{*,*}, d_r\}$ is said to* ***converge*** *to H^*, a graded R-module, if there is a filtration F^* on H^* such that*

$$E_\infty^{p,q} \cong E_0^{p,q}(H^*,F^*),$$

where $E_\infty^{*,*}$ is the limit term of the spectral sequence.

Finally, we can describe the first general setting in which a spectral sequence arises. We will talk about the spectral sequence associated to this setting in chapter 2.

Definition 1.7 *An R-module A is a* ***filtered differential graded module*** *if:*

1. *A is a direct sum of submodules, $A = \bigoplus_{n=0}^\infty A^n$.*

2. *There is an R-linear mapping, $d: A \to A$, of degree 1 ($d: A^n \to A^{n+1}$) or degree -1 ($d: A^n \to A^{n-1}$) satisfying $d \circ d = 0$.*

3. *A has a filtration F^* and the differential d respects the filtration, that is, $d: F^pA \to F^pA$ for all p.*

Recall the definition of convergence. Now we would like to define some weaker conditions on the filtration of a filtered differential module which guarantee the convergence and uniqueness of the target of its spectral sequence, as will be discussed in chapter 2. Let us consider a filtered differential module A over a ring R. Let (A, d, f) denote a decreasing filtration on (A, d),

$$\cdots \subset F^{p+1}A \subset F^{p}A \subset F^{p-1}A \subset \cdots \subset A.$$

Note that an inclusion $F^{s}A \subset F^{t}A$ need not induce an inclusion in homology, $H(\subset) : H(F^{s}A) \to H(F^{t}A)$. Define (suppressing one of the bidegrees)

$$Z_{r}^{p} := F^{p} \cap d^{-1}(F^{p+r}A) \quad and \quad B_{r}^{p} := F^{p} \cap d(F^{p-r}A)$$

and obtain the tower of submodules

$$B_{0}^{p} \subset B_{1}^{p} \subset \cdots \subset B_{r}^{p} \subset \cdots\cdots \subset Z_{r}^{p} \subset \cdots \subset Z_{1}^{p} \subset Z_{0}^{p}.$$

The E_{∞}-term of the associated spectral sequence is given by $E_{\infty}^{p} = \bigcap_{r} Z_{r}^{p} / \bigcup_{r} B_{r}^{p}$. Define $Z_{\infty}^{p} := F^{p}A \cap ker\ d$ and $B_{\infty}^{p} := F^{p}A \cap im\ d$ to obtain the induced filtration on $H(A, d)$. Since these modules Z_{∞}^{p} and B_{∞}^{p} need not come from the tower, we extend the tower as follows

$$B_{0}^{p} \subset B_{1}^{p} \subset \cdots \subset \bigcup_{r} B_{r}^{p} \subset B_{\infty}^{p} \subset Z_{\infty}^{p} \subset \bigcap_{r} Z_{r}^{p} \subset \cdots \subset Z_{1}^{p} \subset Z_{0}^{p}.$$

Note that, the equality $B_{\infty}^{p} = \bigcup_{r} B_{r}^{p}$, that is $F^{p}A \cap im\ d = \bigcup_{r}(F^{p} \cap d(F^{p-r}A))$, can fail if $\bigcup_{s} F^{s}A \neq A$. To avoid this pathology, we require the filtration to be **exhaustive** as defined below.

Definition 1.8 *A filtration F of a differential graded module (A, d), is said to be* ***exhaustive*** *if $A = \bigcup_{s} F^{s}A$. It is called* ***weakly convergent*** *if, for all p, $Z_{\infty}^{p} = \bigcap_{r} Z_{r}^{p}$ that is $F^{p}A \cap ker\ d = \bigcap_{r}(F^{p}A \cap d^{-1}(F^{p+r}A))$.*

The significance of these conditions will be clear in chapter 2 when we talk about the convergence of the spectral sequence associated to a filtered differential graded module. The following conditions on the filtration imply that it is weakly convergent:

1. The filtration is **bounded above**, that is, for each n, there is a value $s(n)$ with $F^{s(n)}A = 0$.

2. $\bigcap_{p} F^{p}A = \{0\}$.

But the definition of weak convergence looks to be dependent on the explicit knowledge of (A, d, f). So let us give some equivalent definition for it involving homologies of subquotients of A or its filtration.

Proposition 1.9 *[6, Proposition 3.3] The following conditions are equivalent on a filtration F of a differential graded module (A, d):*

1. *F is weakly convergent.*

2. *$\bigcap_{r \geq 1} im(H(F^pA/F^{p+r}A) \to H(F^{p+1}A)) = \{0\}$.*

3. *For all p, the mappings induced by the filtration, $R^{p+1} \to R^p$, are monomorphisms, where $R^p = \bigcap_r im(H(F^{p+r}A) \to H(F^pA))$.*

1.3 Morphisms of spectral sequences

In this section, we will deal with the question if two spectral sequences are isomorphic then how do the targets of the spectral sequences compare?

Definition 1.10 *Given two spectral sequences $\{(E_r^{*,*}, d_r)\}$ and $\{(\bar{E}_r^{*,*}, \bar{d}_r)\}$, we define a **morphism of spectral sequences** to be a sequence of homomorphisms of bigraded modules $f_r : (E_r^{*,*}, d_r) \longrightarrow (\bar{E}_r^{*,*}, \bar{d}_r)$ for all r, of bidegree $(0,0)$, such that f_r commutes with the differentials, that is $f_r \circ d_r = \bar{d}_r \circ f_r$, and each f_{r+1} is induced by f_r on homology, that is f_{r+1} is the composite*

$$f_{r+1} : E_{r+1}^{*,*} \cong H(E_r^{*,*}, d_r) \stackrel{H(f_r)}{\longrightarrow} H(\bar{E}_r^{*,*}, \bar{d}_r) \cong \bar{E}_{r+1}^{*,*}.$$

The class of spectral sequences, with morphisms as defined above, constitutes a category called **SpecSeq**.

Suppose we have a morphism $\{f_r\} : \{(E_r, d_r)\} \longrightarrow \{(\bar{E}_r, \bar{d}_r)\}$ of spectral sequences. Recall that each spectral sequence can be represented as a tower of submodules of its E_2-term. By restricting $f_2 : E_2 \to \bar{E}_2$, we get the diagram:

$$\begin{tikzcd}
B_2 \arrow[r, hook] \arrow[d] & B_3 \arrow[r, hook] \arrow[d] & \cdots \arrow[r, hook] & \bigcap_r B_r \arrow[r, hook] \arrow[d] & \bigcap_r Z_r \arrow[r, hook] \arrow[d] & \cdots \arrow[r, hook] & Z_3 \arrow[r, hook] \arrow[d] & Z_2 \arrow[r, hook] \arrow[d] & E_2 \arrow[d, "f_2"] \\
\bar{B}_2 \arrow[r, hook] & \bar{B}_3 \arrow[r, hook] & \cdots \arrow[r, hook] & \bigcap_r \bar{B}_r \arrow[r, hook] & \bigcap_r \bar{Z}_r \arrow[r, hook] & \cdots \arrow[r, hook] & \bar{Z}_3 \arrow[r, hook] & \bar{Z}_2 \arrow[r, hook] & \bar{E}_2
\end{tikzcd}$$

The condition $f_r \circ d_r = \bar{d}_r \circ f_r$ allows us to identify f_{r+1} with the following map induced by f_2

$$f_{r+1} : E_{r+1} \cong Z_r/B_r \to \bar{Z}_r/\bar{B}_r \cong \bar{E}_{r+1}.$$

Furthermore, such a morphism induces a map $f_\infty : E_\infty \to \bar{E}_\infty$.

The second condition, that f_{r+1} is induced by f_r on homology, can be expressed as:

$$\begin{tikzcd}
0 \arrow[r] & b_r \arrow[r] \arrow[d, "f_r"] & c_r \arrow[r] \arrow[d, "f_r"] & E_{r+1} \arrow[r] \arrow[d, "f_{r+1}"] & 0 \\
0 \arrow[r] & \bar{b}_r \arrow[r] & \bar{c}_r \arrow[r] & \bar{E}_{r+1} \arrow[r] & 0
\end{tikzcd}$$

where $c_r = ker\ d_r : E_r \to E_r$ and $b_r = im\ d_r$. As a consequence of all this, we have the following result.

Theorem 1.11 *If $\{f_r\} : \{(E_r, d_r)\} \longrightarrow \{(\bar{E}_r, \bar{d}_r)\}$ is a morphism of spectral sequences and, for some n, $f_n : E_n \to \bar{E}_n$ is an isomorphism of bigraded modules, then for all r, $n \leq r \leq \infty$, $f_r : E_r \to \bar{E}_r$ is an isomorphism.*

Proof For $r = n$, as seen above we have the following diagram

$$\begin{tikzcd}
0 \arrow[r] & b_n \arrow[r] \arrow[d, "f_n"] & c_n \arrow[r] \arrow[d, "f_n"] & E_{n+1} \arrow[r] \arrow[d, "f_{n+1}"] & 0 \\
0 \arrow[r] & \bar{b}_n \arrow[r] & \bar{c}_n \arrow[r] & \bar{E}_{n+1} \arrow[r] & 0
\end{tikzcd}$$

where $c_n = ker\ d_n : E_n \to E_n$ and $b_n = im\ d_n$. Since f_n is an isomorphism thus applying the Five lemma [2, p.129], we obtain f_{n+1} is also an isomorphism. Proceeding in the same manner we can show that f_r is an isomorphism for all $n \leq r \leq \infty$.

■

Thus an isomorphism at some stage of the spectral sequences gives an isomorphism of E_∞-term. Morphisms of spectral sequences also arise in the case of filtered differential graded modules.

Definition 1.12 *A map $\phi : (A, d, F) \to (\bar{A}, \bar{d}, \bar{F})$ with $\phi : A \to \bar{A}$ a morphism of graded modules, such that $\phi \circ d = \bar{d} \circ \phi$ and $\phi(F^pA) \subset \bar{F}^p\bar{A}$ is called a **morphism of filtered differential graded modules**.*

Now given a morphism of filtered differential graded modules, we would like to compare the targets $H(A,d)$ and $H(\bar{A},\bar{d})$ of the two spectral sequences respectively.

Theorem 1.13 (Moore [6, Theorem 3.5]) *A morphism of filtered differential graded modules*

$$\phi : (A,d,F) \to (\bar{A},\bar{d},\bar{F}),$$

determines a morphism of the associated spectral sequences. If for some n, $\phi_n : E_n \to \bar{E}_n$ is an isomorphism of bigraded modules, then $\phi_r : E_r \to \bar{E}_r$ is an isomorphism for all r, $n \leq r \leq \infty$. If the filtrations are bounded, then ϕ induces an isomorphism $H(\phi) : H(A,d) \to H(\bar{A},\bar{d})$.

Proof By Theorem 1.11, it is enough to prove the last part of the theorem. By definition, a bounded filtration implies that there exist functions $s = s(n)$ and $t = t(n)$ such that

$$\{0\} = F^s H^n \subset F^{s-1} H^n \subset \cdots \subset F^{t+1} H^n \subset F^t H^n = H^n$$

where $H^n := H^n(A,d)$ and $E_\infty^{p,q} \cong F^p H^{p+q} / F^{p+1} H^{p+q}$. Similarly for $\bar{H}^n := H^n(\bar{A},\bar{d})$. Now since ϕ_∞ is an isomorphism, by the boundedness of the filtration, we have for the same $s = s(n)$,

$$F^{s-1} H^n = F^{s-1} H^n / F^s H^n \cong E_\infty^{s-1,n-s+1} \cong \bar{E}_\infty^{s-1,n-s+1} \cong \bar{F}^{s-1} H^n.$$

We now apply induction downward to H^n and $\bar{H}^n$. Consider the following commutative diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & F^p H^n & \longrightarrow & F^{p-1} H^n & \longrightarrow & E_\infty^{p-1,n-p+1} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle H(\phi)} & & \downarrow{\scriptstyle H(\phi)} & & \downarrow{\scriptstyle \phi_\infty} & & \\
0 & \longrightarrow & \bar{F}^p H^n & \longrightarrow & \bar{F}^{p-1} H^n & \longrightarrow & \bar{E}_\infty^{p-1,n-p+1} & \longrightarrow & 0
\end{array}$$

Let $p = s - 1 - i$. Applying induction on i we assume that $H(\phi)$ induces an isomorphism $F^p H^n \cong \bar{F}^p \bar{H}^n$. Since ϕ_∞ is an isomorphism, thus the Five lemma [2, p.129] implies that $H(\phi) : F^{p-1} H^n \to \bar{F}^{p-1} \bar{H}^n$ is also an isomorphism. Thus we have $H^n = F^t H^n \cong \bar{F}^t \bar{H}^n = \bar{H}^n$. This completes the proof of the theorem.

■

Chapter 2

Construction of spectral sequences

Now that we know what a spectral sequence is, the next natural question that arises is, how can we construct one? In this chapter, we present two general algebraic settings in which spectral sequences arise naturally: one is a filtered differential module and another is an exact couple.

2.1 Filtered differential graded modules

Recall the definition of a filtered differential graded module as mentioned in chapter 1. Since the differential respects the filtration, $H(A, d) = ker\ d/\ im\ d$ also inherits a filtration as given below

$$F^p H(A, d) = image(H(F^p A, d) \stackrel{H(inclusion)}{\longrightarrow} H(A, d)).$$

Thus we can now describe what a spectral sequence associated to a filtered differential graded module would look like. Suppose that A is a filtered differential graded module with differential of degree $+1$ and a descending filtration.

Theorem 2.1 *[6, Theorem 2.6] Each filtered differential graded module* (A, d, F^*) *determines a spectral sequence,* $\{E_r^{*,*}, d_r\}$*,* $r = 1, 2, \cdots$ *with* d_r *of bidegree* $(r, 1-r)$ *and*

$$E_1^{p,q} \cong H^{p+q}(F^p A/F^{p+1} A).$$

Suppose further that the filtration is **bounded***, that is, for each dimension* n*,* *there are values $s = s(n)$ and $t = t(n)$, so that*

$$\{0\} \subset F^s A^n \subset F^{s-1} A^n \subset \cdots \subset F^{t+1} A^n \subset F^t A^n = A^n,$$

then the spectral sequence converges to $H(A,d)$, that is,

$$E_\infty^{p,q} \cong F^p H^{p+q}(A,d)/F^{p+1} H^{p+q}(A,d).$$

Proof Consider the following decreasing filtration for A

$$\cdots \subset F^p A^{p+q} \subset F^{p-1} A^{p+q} \subset F^{p-2} A^{p+q} \subset \cdots.$$

Then by definition of a filtered differential graded module, we know that d respects the filtration, that is $d(F^p A^{p+q}) \subset F^p A^{p+q+1}$. Before proceeding further, we set the following notation.

$$\begin{aligned}
Z_r^{p,q} &:= \text{ elements in } F^p A^{p+q} \text{ that have boundaries in } F^{p+r} A^{p+q+1} \\
&= F^p A^{p+q} \cap d^{-1}(F^{p+r} A^{p+q+1}) \\
B_r^{p,q} &:= \text{ elements in } F^p A^{p+q} \text{ that form the image of } d \text{ from } F^{p-r} A^{p+q-1} \\
&= F^p A^{p+q} \cap d(F^{p-r} A^{p+q-1}) \\
Z_\infty^{p,q} &:= ker d \cap F^p A^{p+q} \\
B_\infty^{p,q} &:= im d \cap F^p A^{p+q}.
\end{aligned}$$

We claim that there is a tower of submodules

$$B_0^{p,q} \subset B_1^{p,q} \subset \cdots \subset B_\infty^{p,q} \subset Z_\infty^{p,q} \subset \cdots \subset Z_1^{p,q} \subset Z_0^{p,q}.$$

The decreasing filtration and the stability of the differential give us the desired tower of submodules. First observe that, $Z_0^{p,q} = F^p A^{p+q} \cap d^{-1}(F^p A^{p+q+1}) = F^p A^{p+q}$, since $F^p A^{p+q} \subset d^{-1}(F^p A^{p+q+1})$ using the stability of the differential. Obviously $Z_1^{p,q} \subset F^p A^{p+q}$ and hence $Z_1^{p,q} \subset Z_0^{p,q}$. In general, since we have a decreasing filtration, we get $F^{p+r} A^{p+q+1} \subset F^{p+r-1} A^{p+q+1}$, which further implies that $d^{-1}(F^{p+r} A^{p+q+1}) \subset d^{-1}(F^{p+r-1} A^{p+q+1})$.

Thus we have $F^p A^{p+q} \cap d^{-1}(F^{p+r} A^{p+q+1}) \subset F^p A^{p+q} \cap d^{-1}(F^{p+r-1} A^{p+q+1})$. Therefore $Z_r^{p,q} \subset Z_{r-1}^{p,q}$ for any $r \geq 0$. Finally, to see that $Z_\infty^{p,q} \subset Z_r^{p,q}$, note that $d(Z_\infty^{p,q}) = 0 \in F^{p+r} A^{p+q+1}$ and $Z_\infty^{p,q} \subset F^p A^{p+q}$ by definition. Therefore $Z_\infty^{p,q} \subset F^p A^{p+q} \cap d^{-1}(F^{p+r} A^{p+q+1})$.

Similarly, since $F^{p-r+1}A^{p+q-1} \subset F^{p-r}A^{p+q-1}$, we obtain $B^{p,q}_{r-1} \subset B^{p,q}_r$.

Finally, since $d \circ d = 0$ implies that $im\ d \subset ker\ d$, we obtain $B^{p,q}_\infty \subset Z^{p,q}_\infty$. Hence we obtain the tower of submodules as claimed above.

We also observe that

$$\begin{aligned} d(Z^{p-r,q+r-1}_r) &= d(F^{p-r}A^{p+q-1} \cap d^{-1}(F^pA^{p+q})) \\ &= F^pA^{p+q} \cap d(F^{p-r}A^{p+q-1}) \\ &= B^{p,q}_r. \end{aligned}$$

Now boundedness of filtration implies that, for $r > s(p+q+1) - p$ and $r \geq p - t(p+q-1)$, that is, for

$$\begin{array}{ccc} r+p > s(p+q+1) & and & p-r \leq t(p+q-1) \\ F^{p+r}A^{p+q+1} = 0 & & F^{p-r}A^{p+q-1} = A^{p+q-1} \\ \Rightarrow d^{-1}(F^{p+r}A^{p+q+1}) = ker\ d & & \Rightarrow d(F^{p-r}A^{p+q-1}) = im\ d \end{array}$$

Therefore, by definition $Z^{p,q}_r = Z^{p,q}_\infty$ and $B^{p,q}_r = B^{p,q}_\infty$. This ensures the convergence. Now define,

$$E^{p,q}_r := Z^{p,q}_r / (Z^{p+1,q-1}_{r-1} + B^{p,q}_{r-1})$$

for all $0 \leq r \leq \infty$, and define $\eta^{p,q}_r : Z^{p,q}_r \to E^{p,q}_r$ to be the canonical projection with $ker\ \eta^{p,q}_r = (Z^{p+1,q-1}_{r-1} + B^{p,q}_{r-1})$. Observe that

$$\begin{aligned} d(Z^{p,q}_r) &= d(F^pA^{p+q} \cap d^{-1}(F^{p+r}A^{p+q+1})) \\ &= F^{p+r}A^{p+q+1} \cap d(F^pA^{p+q}) \\ &= B^{p+r,q-r+1}_r \\ &\subset Z^{p+r,q-r+1}_r \end{aligned}$$

and

$$\begin{aligned} d(Z^{p+1,q-1}_{r-1} + B^{p,q}_{r-1}) &= d(Z^{p+1,q-1}_{r-1}) + d(B^{p,q}_{r-1}) \\ &\subset B^{p+r,q-r+1}_{r-1} + 0 \qquad \text{(since } d \circ d = 0) \\ &\subset Z^{p+r+1,q-r}_{r-1} + B^{p+r,q-r+1}_{r-1}. \end{aligned}$$

Thus the differential, as a map $d : Z^{p,q}_r \to Z^{p+r,q-r+1}_r$, induces a homomorphism d_r

such that the following diagram commutes

$$\begin{tikzcd} Z_r^{p,q} \arrow[r, "d"] \arrow[d, "\eta"] & Z_r^{p+r,q-r+1} \arrow[d, "\eta"] \\ E_r^{p,q} \arrow[r, "d_r"] & E_r^{p+r,q-r+1} \end{tikzcd}$$

Since $d \circ d = 0$, we have $d_r \circ d_r = 0$. Next we claim the following:

I. $H^*(E_r^{*,*}, d_r) \cong E_{r+1}^{*,*}$,

II. $E_1^{p,q} \cong H^{p+q}(F^pA/F^{p+1}A)$,

III. $E_\infty^{p,q} \cong F^pH^{p+q}(A,d)/F^{p+1}H^{p+q}(A,d)$.

I. Consider the following diagram

$$\begin{tikzcd} Z_r^{p+1,q-1} + B_r^{p,q} \arrow[r, "\subset"] & Z_{r+1}^{p,q} \arrow[r, "\subset"] \arrow[d, "\eta|"] & Z_r^{p,q} \arrow[r, "d"] \arrow[d, "\eta_r^{p,q}"] & Z_r^{p+r,q-r+1} \arrow[d, "\eta_r^{p+r,q-r+1}"] \\ & ker\ d_r \arrow[r] \arrow[d] & E_r^{p,q} \arrow[r, "d_r"'] & E_r^{p+r,q-r+1} \\ & H^{p,q}(E_r^{*,*}, d_r) \arrow[d] & & \\ & 0 & & \end{tikzcd}$$

where the first inclusion is true, since $F^{p+1}A^{p+q} \subset F^pA^{p+q}$ implies that $F^{p+1}A^{p+q} \cap d^{-1}(F^{p+r+1}A^{p+q+1}) \subset F^pA^{p+q} \cap d^{-1}(F^{p+r+1}A^{p+q+1})$ which further implies that $Z_r^{p+1,q-1} \subset Z_{r+1}^{p,q}$. Thus $Z_r^{p+1,q-1} + B_r^{p,q} \subset Z_{r+1}^{p,q}$.

Firstly we observe that $\eta_r^{p,q}(Z_{r+1}^{p,q}) = ker\ d_r$. For that, we consider $\eta^{-1}(ker\ d_r)$.

$$\begin{aligned} d_r(\eta z) = 0 &\Leftrightarrow dz \in Z_{r-1}^{p+r+1,q-r} + B_{r-1}^{p+r,q-r+1} && (\text{since } d_r \circ \eta = \eta \circ d) \\ &\Leftrightarrow z \in Z_{r+1}^{p,q} + Z_{r-1}^{p+1,q-1} && (\text{by definitions of } Z_*^{*,*} \ and\ B_*^{*,*}). \end{aligned}$$

Thus, $\eta^{-1}(ker\ d_r) = Z_{r+1}^{p,q} + Z_{r-1}^{p+1,q-1}$. Therefore $ker\ d_r = \eta(Z_{r+1}^{p,q} + Z_{r-1}^{p+1,q-1}) = \eta(Z_{r+1}^{p,q})$, since $Z_{r-1}^{p+1,q-1} \subset ker\ \eta_r^{p,q}$.

Secondly, $Z_r^{p+1,q-1} + B_r^{p,q} = Z_{r-1}^{p,q} \cap ((\eta_r^{p,q})^{-1}(im\ d_r))$. We know that $im\ d_r = \eta_r^{p,q}(d(Z_r^{p-r,q+r-1})) = \eta_r^{p,q}(B_r^{p,q})$ and hence

$$\begin{aligned} (\eta_r^{p,q})^{-1}(im\ d_r) &= B_r^{p,q} + ker\ \eta_r^{p,q} \\ &= B_r^{p,q} + B_{r-1}^{p,q} + Z_{r-1}^{p+1,q-1} && (\text{since } B_{r-1}^{p,q} \subset B_r^{p,q}) \\ &= B_r^{p,q} + Z_{r-1}^{p+1,q-1}. \end{aligned}$$

Also,

$$
\begin{aligned}
Z_{r-1}^{p+1,q-1} \cap Z_{r+1}^{p,q} &= F^{p+1}A^{p+q} \cap d^{-1}(F^{p+r}A^{p+q+1}) \cap F^pA^{p+q} \cap d^{-1}(F^{p+r+1}A^{p+q+1}) \\
&= F^{p+1}A^{p+q} \cap d^{-1}(F^{p+r}A^{p+q+1}) \cap d^{-1}(F^{p+r+1}A^{p+q+1}) \\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{(since } F^{p+1}A^{p+q} \subset F^pA^{p+q}) \\
&= F^{p+1}A^{p+q} \cap d^{-1}(F^{p+r}A^{p+q+1}) \\
&\qquad\qquad\qquad\qquad \text{(since } F^{p+r+1}A^{p+q+1} \subset F^{p+r}A^{p+q+1}) \\
&= Z_r^{p+1,q-1}.
\end{aligned}
$$

Therefore, $Z_{r+1}^{p,q} \cap (\eta_r^{p,q})^{-1}(im\ d_r) = Z_r^{p+1,q-1} + B_r^{p,q}$.

Finally, let $\gamma : Z_{r+1}^{p,q} \to H^{p,q}(E_r^{*,*}, d_r)$ be the composition of the maps

$$
Z_{r+1}^{p,q} \xrightarrow{\eta_r^{p,q}} ker\ d_r \longrightarrow H^{p,q}(E_r^{*,*}, d_r).
$$

Then, $ker\ \gamma = Z_{r+1}^{p,q} \cap (\eta_r^{p,q})^{-1}(im\ d_r) = Z_r^{p+1,q-1} + B_r^{p,q}$. Since γ is an epimorphism, we obtain

$$
H^{p,q}(E_r^{*,*}, d_r) \cong Z_{r+1}^{p,q}/(Z_r^{p+1,q-1} + B_r^{p,q}) =: E_{r+1}^{p,q}.
$$

II. Observe that $E_0^{p,q} = Z_0^{p,q}/(Z_{-1}^{p+1,q-1} + B_{-1}^{p,q})$, where

$$
Z_{-1}^{p+1,q-1} = F^{p+1}A^{p+q} \quad \text{and} \quad B_{-1}^{p,q} = d(F^{p+1}A^{p+q-1}).
$$

Since d respects the filtration, $F^pA^{p+q} \subset d^{-1}(F^pA^{p+q+1})$ and $d(F^{p+1}A^{p+q-1}) \subset F^{p+1}A^{p+q}$. Thus

$$
\begin{aligned}
E_0^{p,q} &= F^pA^{p+q} \cap d^{-1}(F^pA^{p+q+1})/(F^{p+1}A^{p+q} + d(F^{p+1}A^{p+q-1})) \\
&= F^pA^{p+q}/F^{p+1}A^{p+q}.
\end{aligned}
$$

The differential $d_0 : E_0^{p,q} \to E_0^{p,q+1}$ is induced by the differential $d : F^pA^{p+q} \to F^pA^{p+q+1}$, therefore we have

$$
E_1^{p,q} \cong H^{p+q}(F^pA/F^{p+1}A).
$$

III. Let $\eta_\infty^{p,q} : Z_\infty^{p,q} \to E_\infty^{p,q}$ and $\pi : ker\ d \to H(A,d)$ denote the canonical projections. Then

$$
F^pH^{p+q}(A,d) = H^{p+q}(im(F^pA \to A), d) = \pi(F^pA^{p+q} \cap ker\ d) = \pi(Z_\infty^{p,q}).
$$

Since, $\pi(ker\ \eta_\infty^{p,q}) = \pi(Z_\infty^{p+1,q-1} + B_\infty^{p,q}) = F^{p+1}H^{p+q}(A,d)$, we have that π induces a map $d_\infty : E_\infty^{p,q} \to F^pH^{p+q}(A,d)/F^{p+1}H^{p+q}(A,d)$. Further,

$$\begin{aligned}
ker\ d_\infty &= \eta_\infty^{p,q}(\pi^{-1}(F^{p+1}H^{p+q}(A,d)) \cap Z_\infty^{p,q}) \\
&= \eta_\infty^{p,q}(Z_\infty^{p+1,q-1} \cap d(A) \cap Z_\infty^{p,q}) \\
&\subset \eta_\infty^{p,q}(Z_\infty^{p+1,q-1} + B_\infty^{p,q}) = \{0\}.
\end{aligned}$$

Thus d_∞ is an isomorphism, hence

$$E_\infty^{p,q} \cong F^pH^{p+q}(A,d)/F^{p+1}H^{p+q}(A,d).$$

This completes the proof.

■

Thus, as a consequence of the above theorem, if A can be filtered and some term of the associated spectral sequence is something calculable, then we can compute $H(A,d)$ up to the calculation of the successive homologies and reconstruction from the associated graded module.

Now coming to the weaker conditions, we talked about in chapter 1, that guarantee convergence and uniqueness of the target. With slight modifications in the proof of Theorem 2.1 we have the following result.

Theorem 2.2 *Let (A,d,f) be a filtered differential graded module such that the filtration is exhaustive and weakly convergent. Then the associated spectral sequence with $E_1^{p,q} \cong H^{p+q}(F^pA/F^{p+}A)$ converges to $H(A,d)$, that is,*

$$E_\infty^{p,q} \cong F^pH^{p+q}(A,d)/F^{p+1}H^{p+q}(A,d).$$

So we know now, that a weakly convergent and exhaustive filtration guarantees that the spectral sequence from a filtered differential graded module (A,d,f), converges to $H(A,d)$, in the sense that the E_∞-term is related directly to a filtration of $H(A,d)$.

2.2 Exact couples

So far we discussed the algebraic setting of a filtered differential graded module. But not always our objects of study are explicitly filtered or come from a filtered differential module. In this direction, we present exact couples, another general setting giving rise to a spectral sequence naturally. The idea of an exact couple was introduced by Massey [5]. The target of the spectral sequence coming from an exact couple may be difficult to identify, unlike the case of a filtered differential graded module.

Let D and E denote R-modules and let $i : D \to D$, $j : D \to E$ and $k : E \to D$ be module homomorphisms.

$$\begin{tikzcd} D \arrow[rr, "i"] & & D \arrow[dl, "j"] \\ & E \arrow[ul, "k"] & \end{tikzcd}$$

Definition 2.3 *We say that $C = \{D, E, i, j, k\}$ is an* ***Exact Couple*** *if the above diagram is exact at each group, that is, $im\ i = ker\ j$, $im\ j = ker\ k$ and $im\ k = ker\ i$.*

Example 2.1 Let $0 \to \mathbb{Z} \xrightarrow{\times p} \mathbb{Z} \to \mathbb{Z}/p\mathbb{Z} \to 0$ be the short exact sequence associated to the 'times p' map. Suppose (C^*, d) is a differential graded abelian group that is free in each degree. When we tensor C^* with the coefficients, the 'times p' map results in the following short exact sequence

$$0 \to C^* \xrightarrow{\times p} C^* \to C^* \otimes \mathbb{Z}/p\mathbb{Z} \to 0$$

which on taking homology gives the following exact couple.

$$\begin{tikzcd} H(C^*) \arrow[rr, "H(\times p)"] & & H(C^*) \arrow[dl] \\ & H(C^* \otimes \mathbb{Z}/p\mathbb{Z}) \arrow[ul] & \end{tikzcd}$$

The spectral sequence associated to this exact couple is known as the **Bockstein spectral sequence**.

Note that, an immediate consequence of the exactness of a couple is that, E becomes a differential R-module with differential $d : E \to E$ given by $d = j \circ k$. To see this, we compute $d \circ d = (j \circ k) \circ (j \circ k) = j \circ (k \circ j) \circ k = 0$.

A fundamental operation on exact couples is the formation of the **derived couple**. Let $E' = H(E, d) = ker\ d/im\ d = ker(j \circ k)/im(j \circ k)$ and $D' = i(D) = ker\ j$. Also define,

$i' = i|_{iD} : D' \to D'$, $j' : D' \to E'$ given by $j'(i(x)) = j(x) + dE \in E'$, *where* $x \in D$ and $k' : E' \to D'$ given by $k'(e + dE) = k(e)$. We note the following.

Firstly, j' is well-defined. If $i(x) = i(x')$, then $(x - x') \in ker\ i = im\ k$ and there is a $y \in E$ with $k(y) = x - x'$. Thus $(j \circ k)(y) = d(y) = j(x) - j(x')$ and $j(x) = j(x') + d(y)$, that is, $j(x) + dE = j(x') + dE$ as cosets in E'.

Secondly, k' is also well defined. If $e + dE = e' + dE$, then $e' = e + d(x)$ for some $x \in E$ which implies that $k(e') = k(e) + k(d(x)) = k(e) + (k \circ j \circ k)(x) = k(e)$. Also, since $d(e) = 0$, we have $k(e) \in ker\ j = im\ i = D'$.

We call $\mathcal{C}' = \{D', E', i', j', k'\}$ the **Derived couple** of $\mathcal{C}$.

Proposition 2.4 *$\mathcal{C}' = \{D', E', i', j', k'\}$ is an exact couple.*

Proof We first show the exactness at the left D'. We have

$$\begin{aligned} ker\ i' &= im\ i \cap ker\ i = ker\ j \cap im\ k \\ &= k(k^{-1}(ker\ j)) = k(ker\ d) = k'(ker\ d/im\ d) \\ &= im\ k'. \end{aligned}$$

Note that $D' = iD = D/ker\ i$. Thus we can write

$$\begin{aligned} ker\ j' &= j^{-1}(im\ d)/ker\ i = j^{-1}(j(im\ k))/ker\ i \\ &= (im\ k + ker\ j)/ker\ i = (ker\ i + ker \jmath)/ker\ i \\ &= i(ker\ j) = i(im\ i) = im\ i'. \end{aligned}$$

Finally, since $j \circ i = 0$, we have $ker\ k' = ker\ k/im\ d = im\ j/im\ d = jD/im\ d = im\ j'$.

■

We can iterate this process to obtain the n^{th} **derived couple** of $\mathcal{C}$,

$$\mathcal{C}^{(n)} = \{D^{(n)}, E^{(n)}, i^{(n)}, j^{(n)}, k^{(n)}\} = (\mathcal{C}^{(n-1)})'.$$

Note that $E^{(n+1)} = H(E^{(n)}, d^{(n)})$. Now, we can describe the spectral sequence associated to an exact couple.

Theorem 2.5 *[6, Theorem 2.8] Suppose $D^{*,*} = \{D^{p,q}\}$ and $E^{*,*} = \{E^{p,q}\}$ are bigraded modules over R equipped with homomorphisms: i of bidegree $(-1,1)$,*

j of bidegree $(0,0)$ and k of bidegree $(1,0)$.

$$\begin{tikzcd} D^{*,*} \arrow[rr, "i"] & & D^{*,*} \arrow[dl, "j"] \\ & E^{*,*} \arrow[ul, "k"] & \end{tikzcd}$$

These data determine a spectral sequence $\{E_r, d_r\}$ for $r = 1,2,\ldots,$ of cohomological type, with $E_r = (E^{*,*})^{(r-1)}$, the $(r-1)$-st derived module of $E^{*,*}$ and $d_r = j^{(r-1)} \circ k^{(r-1)}$.

Proof It is enough to check that the differentials d_r, have the correct bidegree $(r, 1-r)$. We will prove this by induction. Let $r = 1$, so $E_1 = E^{*,*}$, $d_1 = j \circ k$ and hence d_1 has bidegree $(1,0)+(0,0) = (1,0)$. Thus the statement is true for $r = 1$. Now assuming it to be true for $r-1$, that is, given that bidegree of $d_{r-1} = (r-1, 2-r)$, we will show that the result is true for r, that is we claim that bidegree of $d_r = (r, 1-r)$. Since bidgree of $k^{(r-2)} = (1,0)$ and $d_{r-1} = j^{(r-2)} \circ k^{(r-2)}$, thus bidgree of $j^{(r-2)} = (r-2, 2-r)$. Now, since the map

$$j^{(r-1)} : (D^{(*,*)})^{(r-1)} \longrightarrow (E^{p,q})^{(r-1)} \text{ is defined as}$$

$$j^{(r-1)}(i^{(r-2)}(x)) = j^{(r-2)}(x) + d^{(r-2)}E^{(r-2)} \in (E^{p,q})^{(r-1)},$$

where $j^{(r-2)}(x) \in (E^{p,q})^{(r-2)}$.
The image in $(E^{p,q})^{(r-1)}$ must come from $i^{(r-2)}(D^{p-r+2,q+r-2})^{(r-2)}$. To see this, let us assume that the image comes from $i^{(r-2)}(D^{\alpha,\beta})^{(r-2)}$. Then since $j^{(r-2)}(D^{\alpha,\beta})^{(r-2)}$ is in $(E^{p,q})^{(r-2)}$ and by induction hypothesis bidegree of $j^{(r-2)} = (r-2, 2-r)$, therefore we have $\alpha + r - 2 = p$ and $\beta - r + 2 = q$. That is, the image comes from $i^{(r-2)}(D^{p-r+2,q+r-2})^{(r-2)}$.
But note that $i^{(r-2)}(D^{p-r+2,q+r-2})^{(r-2)} = (D^{p-r+1,q+r-1})^{(r-1)}$, therefore we have

$$j^{(r-1)} : (D^{p-r+1,q+r-1})^{(r-1)} \longrightarrow (E^{p,q})^{(r-1)},$$

that is, the bidegree of $j^{(r-1)} = (p-(p-r+1), q-(q+r-1)) = (r-1, 1-r)$. Also, since $k^{(r-1)}(e + d^{(r-2)}E^{(r-2)}) = k^{(r-2)}(e)$ and $k^{(r-2)}$ has bidegree $(1,0)$, so does $k^{(r-1)}$. Thus, bidegree of $d_r = (r-1, 1-r) + (1,0) = (r, 1-r)$, which is what we wanted to prove. Therefore the statement is true for all $r = 1,2,\ldots$.

■

A bigraded exact couple can be represented as in the following diagram:

$$
\begin{tikzcd}
 & {} \arrow[d, "i"] & & {} \arrow[d, "i"] & \\
{} \arrow[r, "k"] & D^{p+2,q-1} \arrow[r, "j"] \arrow[d, "i"] & E^{p+2,q-1} \arrow[r, "k"] & D^{p+3,q-1} \arrow[r, "j"] \arrow[d, "i"] & {} \\
{} \arrow[r, "k"] & D^{p+1,q} \arrow[r, "j"] \arrow[d, "i"] & E^{p+1,q} \arrow[r, "k"] & D^{p+2,q} \arrow[r, "j"] \arrow[d, "i"] & {} \\
{} \arrow[r, "k"] & D^{p,q+1} \arrow[r, "j"] \arrow[d, "i"] & E^{p,q+1} \arrow[r, "k"] & D^{p+1,q+1} \arrow[r, "j"] \arrow[d, "i"] & {} \\
 & {} & & {} &
\end{tikzcd}
$$

Another useful presentation of exact couples is the **unrolled exact couple**:

$$
\begin{tikzcd}
{} \arrow[r, "i"] & D^{p+1,*} \arrow[rr, "i"] & & D^{p,*} \arrow[rr, "i"] \arrow[dl, "j"] & & D^{p-1,*} \arrow[r, "i"] \arrow[dl, "j"] & {} \\
 & & E^{p,*} \arrow[ul, "k"] & & E^{p-1,*} \arrow[ul, "k"] & &
\end{tikzcd}
$$

It is interesting to observe that we can also associate a tower of submodules of E and an E_∞-term to an exact couple, just as we can do for any spectral sequence. Another expression for the E_r-terms of spectral sequence associated to an exact couple is given in the following proposition and the corollary following it.

Proposition 2.6 *[6, Proposition 2.9] Let* $Z_r^{p,*} = k^{-1}(im\ i^{r-1} : D^{p+r,*} \to D^{p+1,*})$ *and* $B_r^{p,*} = j(ker\ i^{r-1} : D^{p,*} \to D^{p-r+1,*})$ *be submodules of* $E^{p,*}$. *Then these submodules determine the spectral sequence associated to the exact couple:*

$$E_r^{p,*} = (E^{p,*})^{(r-1)} \cong Z_r^{p,*}/B_r^{p,*}.$$

Furthermore,

$$
\begin{aligned}
E_\infty^{p,*} \cong \bigcap_r Z_r^{p,*} / \bigcup_r B_r^{p,*} \cong & \\
& \bigcap_r k^{-1}(im\ i^{r-1} : D^{p+r,*} \to D^{p+1,*}) / \bigcup_r j(ker\ i^{r-1} : D^{p,*} \to D^{p-r+1,*})
\end{aligned}
$$

Proof We prove this by induction. For $r = 2$, $E_2^{*,*} = (E^{*,*})' = ker\ d/im\ d = ker(j \circ k)/im(j \circ k)$. Now $im(j \circ k) = j(im\ k) = j(ker\ i)$. Also $ker(j \circ k) = k^{-1}(ker\ j) = k^{-1}(im\ i)$. So

$$E_2^{p,*} = k^{-1}(im\ i : D^{p+2,*} \to D^{p+1,*})/j(ker\ i : D^{p,*} \to D^{p-1,*}).$$

Then by induction,

$$E_r^{p,*} = (E_{r-1}^{p,*})' \cong Z_r^{p,*}/B_r^{p,*}$$

since i^r, j^r and k^r are induced by i, j and k with the appropriate images in $D^{(r)} = im\ i^{r-1}$. Note that by composing with the map i the following inclusions follow: $Z_r^{p,*} \subset Z_{r-1}^{p,*}$ and $B_r^{p,*} \subset B_{r+1}^{p,*}$. Also, the map $j \circ k$ induces the differential $d\colon Z_{r-1}/B_{r-1} \to B_r/B_{r-1} \subset Z_{r-1}/B_{r-1}$. Thus we obtain the tower of submodules. Hence the E_∞-term of the associated spectral sequence follows immediately.

■

Corollary 2.7 *[6, Corollary 2.10] For, $r \geq 1$, there is an exact sequence:*

$$0 \to D^{p,*}/(ker\ i^r(D^{p,*} \to D^{p-r,*}) + iD^{p+1,*}) \xrightarrow{\bar{j}} E_{r+1}^{p,*}$$
$$\xrightarrow{\bar{k}} im\ i^r(D^{p+r+1,*} \to D^{p+1,*}) \bigcap ker\ i(D^{p+1,*} \to D^{p,*}) \longrightarrow 0$$

2.3 The equivalence of the two approaches

Having investigated two algebraic settings in which a spectral sequence arises naturally, the next question that one can ask is that, are these two related in any way? And, if yes, then how do their spectral sequences compare? In this direction we observe that, a filtered differential graded R-module (A, d, f) gives rise to an exact couple. This can be seen as follows.
For each filtration degree p, there is a short exact sequence of graded modules

$$0 \longrightarrow F^{p+1}A \longrightarrow F^pA \longrightarrow F^pA/F^{p+1}A \longrightarrow 0.$$

Since the differential respects the filtration, we get a short exact sequence of differential graded modules. On applying the homology functor, we obtain the long exact sequence, for each p:

$$\cdots H^{p+q}(F^{p+1}A) \xrightarrow{i} H^{p+q}(F^pA) \xrightarrow{j} H^{p+q}(F^pA/F^{p+1}A)$$
$$\xrightarrow{k} H^{p+q+1}(F^{p+1}A) \xrightarrow{i} H^{p+q+1}(F^pA) \xrightarrow{j} \cdots$$

where k is the connecting homomorphism. Define the bigraded modules:

$$E^{p,q} = H^{p+q}(F^pA/F^{p+1}A) \quad and \quad D^{p,q} = H^{p+q}(F^pA).$$

This gives us an exact couple from the long exact sequences:

$$\begin{tikzcd}
 & D^{p+1,q-1} \arrow[rr, "i"] & & D^{p,q} \arrow[dl, "j"] \\
D^{p+1,q} & & E^{p,q} \arrow[ll, "k"] &
\end{tikzcd}$$

Note that the bigradings agree with Theorem 2.5 to yield a spectral sequence, so we have the following result.

Proposition 2.8 *For a filtered differential graded R-module (A, d, f), the spectral sequence associated to the (decreasing) filtration and the spectral sequence associated to the exact couple are the same.*

Proof It is enough to show that, in the spectral sequence for the exact couple, the E_r-term, as a subquotient of $F^pA/F^{p+1}A$, coincides with the subquotient given in the proof of Theorem 2.1. That is, we need to show that

$$E_r^{p,q} := k^{-1}(im\ i^{r-1})/j(ker\ i^{r-1}) \cong Z_r^{p,q}/(Z_{r-}^{p+1,q-1} + B_{r-1}^{p,q}).$$

Suppose $z \in E_1^{p,q} = H^{p+q}(F^pA/F^{p+1}A)$, that is z is represented by $[x + F^{p+1}A]$ where $x \in F^pA$ and $d(x) \in F^{p+1}A$. Then the boundary homomorphism k in the long exact sequence associated to the exact couple can be defined as:

$$k([x + F^{p+1}A]) = [d(x)] \in H^{p+q+1}F^{p+1}A.$$

Thus,

$$\begin{aligned}
[x + F^{p+1}A] \in k^{-1}(im\ i^{r-1}) &\Leftrightarrow k([x + F^{p+1}A]) = [d(x)] \in im\ i^{r-1} \\
&\Leftrightarrow d(x) \in F^{p+r}A^{p+q+1}.
\end{aligned}$$

But since $x \in F^pA^{p+q}$, then $x \in F^pA^{p+q} \cap d^{-1}(F^{p+r}A^{p+q+1}) = Z_r^{p,q}$. Therefore we have, $k^{-1}(im\ i^{r-1}) = Z_r^{p,q}/F^{p+1}A^{p+q}$. Now, in order to determine $j(ker\ i^{r-1})$ consider $ker\ i^{r-1} \subset H^{p+q}(F^pA)$. Then, $[u] \in ker\ i^{r-1} \Leftrightarrow u \in F^pA^{p+q}$ and u is a boundary in $F^{p-r+1}A^{p+q}$ which implies that $u \in F^pA^{p+q} \cap d(F^{p-r+1}A^{p+q}) = B_{r-1}^{p,q}$. Since j assigns to a class in $H^{p+q}(F^pA)$ its relative class modulo $F^{p+1}A$, we obtain

$$j(ker\ i^{r-1}) = B_{r-1}^{p,q}/F^{p+1}A^{p+q}.$$

Thus we have

$$
\begin{aligned}
E_r^{p,q} &= k^{-1}(im\ i^{r-1})/j(ker\ i^{r-1}) \\
&= Z_r^{p,q}/F^{p+1}A^{p+q}/B_{r-1}^{p,q}/F^{p+1}A^{p+q} \\
&= Z_r^{p,q}/F^{p+1}A^{p+q}/(B_{r-1}^{p,q} + Z_{r-1}^{p+1,q-1})/F^{p+1}A^{p+q} \quad (\text{since } Z_{r-1}^{p+1,q-1} \subset F^{p+1}A^{p+q}) \\
&\cong Z_r^{p,q}/(Z_{r-1}^{p+1,q-1} + B_{r-1}^{p,q}).
\end{aligned}
$$

■

2.4 Double complexes

As an example of the construction of a spectral sequence from a filtered differential graded module, we present another algebraic setting called double complexes.

Definition 2.9 *A* ***double complex,*** $\{M^{*,*}, d', d''\}$*, is a bigraded R-module with two R-linear maps* $d' : M^{*,*} \to M^{*,*}$ *of bidegree* $(1,0)$ *and* $d'' : M^{*,*} \to M^{*,*}$ *of bidegree* $(0,1)$ *satisfying*

$$d' \circ d' = 0,\ d'' \circ d'' = 0 \text{ and } d' \circ d'' + d'' \circ d' = 0.$$

The ***total complex,*** $Total(M)$*, is the differential graded R-module defined by* $Total(M)^n = \bigoplus_{p+q=n} M^{p,q}$ *with total differential* $d = d' + d''$ *satisfying* $d \circ d = 0$.

Example 2.2 If we let $K^{m,n} = A^m \otimes_R B^n$, $d' = d_A \otimes 1$ and $d'' = (-1)^m 1 \otimes d_B$, then we have a double complex such that $(Total(K), d) = (A \otimes_R B, d_\otimes)$.

Now our aim is to compute homology of the total complex with respect to the differential d i.e. $H(Total(M), d)$. For this, we construct two spectral sequences by taking homologies in two different directions, namely, horizontal and vertical.

Definition 2.10 (Horizontal and Vertical homology) *The Horizontal homology is defined as:*

$$H_I^{n,m}(M) := \frac{ker\ d' : M^{n,m} \to M^{n+1,m}}{im\ d' : M^{n-1,m} \to M^{n,m}} = H(M^{n,m}, d')$$

and the Vertical homology as:

$$H_{II}^{n,m}(M) := \frac{ker\ d'' : M^{n,m} \to M^{n,m+1}}{im\ d'' : M^{n,m-1} \to M^{n,m}} = H(M^{n,m}, d'').$$

Then $H_I^{*,*}(M)$ and $H_{II}^{*,*}(M)$ are differential bigraded modules with respective differentials

$$\bar{d}'': H_I^{n,m}(M) \to H_I^{n,m+1}(M), \quad \bar{x} \mapsto \bar{d}''(x) \text{ and}$$
$$\bar{d}': H_{II}^{n,m}(M) \to H_{II}^{n+1,m}(M), \quad \bar{y} \mapsto \bar{d}'(y)$$

and respective homology groups

$$H_I^{*,*} H_{II}(M) = H(H_{II}^{*,*}(M), \bar{d}') \quad \text{and} \quad H_{II}^{*,*} H_I(M) = H(H_I^{*,*}(M), \bar{d}'').$$

Now we can describe the spectral sequence for a double complex.

Theorem 2.11 *[6, Theorem 2.15] Given a double complex $\{M^{*,*}, d', d''\}$, there are two spectral sequences, $\{{}_IE_r^{*,*}, {}_Id_r\}$ and $\{{}_{II}E_r^{*,*}, {}_{II}d_r\}$ with*

$${}_IE_2^{*,*} \cong H_I^{*,*} H_{II}(M) \text{ and } {}_{II}E_2^{*,*} \cong H_{II}^{*,*} H_I(M)$$

If $M^{p,q} = 0$ when $p < 0$ or $q < 0$, then both spectral sequences converge to $H^*(Total(M), d)$.

Proof We prove the case of $\{{}_IE_r^{*,*}, {}_Id_r\}$; the other case will follow by symmetry. To prove this we will appeal to Theorem 2.1. To do that, let us first define filtrations of $(Total(M), d)$ in the following two ways:

$$F_I^p(Total(M))^t = \bigoplus_{r \geq p} M^{r,t-r} \qquad \textit{and} \qquad F_{II}^p(Total(M))^t = \bigoplus_{r \geq p} M^{t-r,r}.$$

where F_I^* is called the *column-wise filtration* and F_{II}^* the *row-wise filtration.* Note that both are decreasing filtrations and d, the total differential, respects each filtration since $d = d' + d''$ and both d' and d'' respect the filtrations. Now because $M^{p,q} = \{0\}$ when $p < 0$ or $q < 0$, this filtration is bounded. Thus by Theorem 2.1, we obtain two spectral sequences converging to $H(Total(M), d)$. In the case of F_I we have:

$${}_IE_1^{p,q} = H^{p+q}(F_I^p Total(M)/F_I^{p+1} Total(M), d).$$

It only remains to identify the E_2-term as described. We claim that ${}_IE_1^{p,q} \cong H_{II}^{p,q}(M)$. Note that

$$\begin{aligned}(F_I^p Total(M)/F_{II}^p Total(M))^{p+q} &= (\bigoplus_{r \geq p} M^{r,p+q-r} / \bigoplus_{r \geq p+1} M^{r,p+q-r}) \\ &\cong M^{p,q}\end{aligned}$$

where the differential is d'' since $d = d' + d''$ and $d'(F_I^p Total(M)) \subset F_I^{p+1} Total(M)$. Therefore ${}_I E_1^{p,q} \cong H_{II}^{p,q}(M)$. Now following Proposition 2.8, consider the following diagram (where we write F^p for $F_I^p Total(M)$)

$$
\begin{tikzcd}
\cdots \arrow[rr, "i"] & & H^{p+q}(F^{p+1}) \arrow[r, "i"] & H^{p+q}(F^p) \arrow[r, "i"] \arrow[dd, "j"] & {} \\
& \cdots H^{p+q+1}(F^{p+2}) \arrow[r, "i"] & H^{p+q+1}(F^{p+1}) \arrow[dll, "j"'] \arrow[r, "i"] & {} & \\
H^{p+q+1}(F^{p+1}/F^{p+2}) \arrow[d, "="] & & & H^{p+q}(F^p/F^{p+1}) \arrow[ul, "k"'] \arrow[d, "="] & \\
H_{II}^{p+1,q}(M) & & & H_{II}^{p,q}(M) \arrow[lll, "d_1"] &
\end{tikzcd}
$$

Now, an element in $H^{p+q}(F^p/F^{p+1})$ is represented by $[x + F^{p+1}]$, where $x \in F^p$ such that $d(x) \in F^{p+1}$ or it can also be written as a class, $[z] \in H_{II}^{p,q}(M)$ with $z \in M^{p,q}$. Now,

$$k([x + F^{p+1}]) = [dx] \in H^{p+q+1}(F^{p+1}).$$

Taking z as the representative, this determines $[d'z] \in H^{p+q+1}(F^{p+1})$ since $d''z = 0$. The morphism j assigns a class in $H^{p+q+1}(F^{p+1})$ to its representative mod F^{p+2}. Thus we can consider $d'z$ as an element of $M^{p+1,q}$. Hence $d_1 := j \circ k = \bar{d}'$, the induced mapping of d' on $H_{II}^{p,q}(M)$. Therefore ${}_I E_2^{p,q} = H_I^{p,q} H_{II}(M)$.
Similarly, to obtain the second spectral sequence from $F_{II}^* Total(M)$ we can reindex the double complex as its transpose: ${}^t M^{p,q} = M^{q,p}$, ${}^t d' = d''$ and ${}^t d'' = d'$. Then we have $Total({}^t M) = Total(M)$ and $F_{II}^* Total(M) = F_I^* Total({}^t M)$. By a similar procedure as above we obtain the required result. This completes the proof of the theorem.

■

2.5 Spectral sequences of algebras

Let $(E^{*,*}, d_E)$ and $(\bar{E}^{*,*}, d_{\bar{E}})$ be differential bigraded modules over R.

Definition 2.12 *The tensor product of differential bigraded modules over*

R is defined as:

$$(E \otimes_R \bar{E})^{p,q} = \bigoplus_{\substack{r+t=p \\ s+u=q}} E^{r,s} \otimes_R \bar{E}^{t,u}$$

with $d_{\otimes}(e \otimes \bar{e}) = d_E(e) \otimes \bar{e} + (-1)^{r+s} e \otimes d_{\bar{E}}(\bar{e})$, *where* $e \in E^{r,s}$ *and* $\bar{e} \in \bar{E}^{t,u}$. *Further, a* ***differential bigraded algebra*** *over R is defined to be a differential bigraded module* $(E^{*,*}, d)$ *together with a morphism* $\psi : (E \otimes E)^{*,*} \to E^{*,*}$ *such that it is associative.*

Now we can define a "tensor product of spectral sequences" by forming the tensor product of differential bigraded modules at each term in the sequences.

Definition 2.13 *A* ***spectral sequence of algebras*** *over R is a spectral sequence,* $\{E_r^{*,*}, d_r\}$ *together with algebra structures* $\psi_r : E_r \otimes_R E_r \to E_r$ *for each r, such that* ψ_{r+1} *can be written as the composite*

$$\psi_{r+1} : E_{r+1} \otimes_R E_{r+1} \overset{\cong}{\to} H(E_r) \otimes_R H(E_r) \overset{p}{\to} H(E_r \otimes_R E_r) \overset{H(\psi_r)}{\to} H(E_r) \overset{\cong}{\to} E_{r+1},$$

where the homomorphism p is given by $p([u] \otimes [v]) = [u \otimes v]$.

For example the Leray-Serre spectral sequence that we will encounter in chapter 4 is also a spectral sequence of algebras.

Remark 2.1 *[6, p.25] If there is a spectral sequence converging to* H^* *as an algebra and the* E_∞*-term is a free, graded-commutative, bigraded algebra, then* H^* *is a free, graded commutative algebra isomorphic to* $Total E_\infty^{*,*}$.

Chapter 3

Fibrations

In this chapter, we give an overview of the topological background required to understand the Leray-Serre spectral sequence of a fibration, which is our ultimate aim.

3.1 Definition of a fibration

Definition 3.1 *A map $p: E \to B$ has the* ***homotopy lifting property*** *(HLP), with respect to a space Y if, given a homotopy $G: Y \times I \to B$ and a map $g: Y \times \{0\} \to E$ such that $p \circ g(y,0) = G(y,0)$, then there is a homotopy $\tilde{G}: Y \times I \to E$ such that $\tilde{G}(y,0) = g(y,0)$ and $p \circ \tilde{G} = G$.*

$$\begin{tikzcd} Y \times \{0\} \arrow[r, "g"] \arrow[d] & E \arrow[d, "p"] \\ Y \times I \arrow[r, "G"] \arrow[ur, dashed, "\tilde{G}"] & B \end{tikzcd}$$

Definition 3.2 *A mapping with the HLP with respect to all spaces is called a* ***Hurewicz fibration****. A mapping with the HLP with respect to all n-cells is called* ***Serre fibration****.*

From now on, by a **fibration**, we mean a Hurewicz fibration.

Definition 3.3 *If $p: E \to B$ is a fibration, then we call the space B as the* ***base space*** *and E as the* ***total space*** *of the fibration. If b is a point in B, then we call $F_b = p^{-1}(b)$ as the* ***fiber*** *of p over b.*

Proposition 3.4 *Suppose $p : E \to B$ is a fibration and that B is path-connected. Then, for $b_0, b_1 \in B$, F_{b_0} is homotopy equivalent to F_{b_1}.*

Proof Given a fibration $E \xrightarrow{p} B$, let the **free path space** be denoted by $WB := map([0,1], B) = \{\lambda : [0,1] \to B | \lambda \text{ is continuous}\}$, with the compact-open topology. The evaluation map $ev_0 : WB \to B$, defined by $ev_0(\lambda) = \lambda(0)$, is continuous. Let $U_p = \{(\lambda, e) \in WB \times E | \lambda(0) = p(e)\}$ denote the pullback of $p : E \to B$ over ev_0 as in the following diagram

$$\begin{tikzcd} U_p \arrow[r, "pr_2"] \arrow[d, "pr_1"'] & E \arrow[d, "p"] \\ WB \arrow[r, "ev_0"] & B \end{tikzcd}$$

The homotopy $H : U_p \times I \to B$ given by $H((\lambda, e), t) = \lambda(t)$ poses the homotopy lifting problem:

$$\begin{tikzcd} U_p \arrow[r, "pr_2"] \arrow[d, "(-{,}0)"'] & E \arrow[d, "p"] \\ U_p \times I \arrow[r, "H"] \arrow[ur, dashed, "?"] & B \end{tikzcd}$$

When $p : E \to B$ is a fibration , we get a solution $\tilde{H} : U_p \times I : E$:

$$\begin{tikzcd} U_p \arrow[r, "pr_2"] \arrow[d, "(-{,}0)"'] & E \arrow[d, "p"] \\ U_p \times I \arrow[r, "H"] \arrow[ur, dashed, "\exists \tilde{H}"] & B \end{tikzcd}$$

Let

$$\begin{aligned} \Lambda : \ & U_p \to WE \\ & (\lambda, e) \mapsto \Lambda(\lambda, e) \end{aligned}$$

denote the **adjoint** of $\tilde{H}$ given by:

$$\begin{aligned} &\Lambda(\lambda, e) : [0,1] \to E \\ &\Lambda(\lambda, e)(t) = \tilde{H}((\lambda, e), t) \end{aligned}$$

where $WE := map([0,1], E)$. Λ is called a **lifting function** for p. Note that the map Λ satisfies the following properties

$$p \circ \Lambda(\lambda, e) = \lambda, \qquad \text{and} \qquad \Lambda(\lambda, e)(0) = e.$$

Which are easy to see since

$$\begin{aligned} p \circ \Lambda(\lambda, e)(t) &= p(\tilde{H}((\lambda, e), t)) \\ &= H((\lambda, e), t) \qquad (\text{since } \ p \circ \tilde{H} = H) \\ &= \lambda(t) \end{aligned}$$

and

$$\begin{aligned} \Lambda(\lambda, e)(0) &= \tilde{H}((\lambda, e), 0) \\ &= pr_2(\lambda, e) \qquad (since \ \ \tilde{H} \circ (-, 0) = pr_2 \ \text{ by HLP}) \\ &= e. \end{aligned}$$

Now suppose $\alpha : I \to B$ is a path with $\alpha(0) = b_0$ and $\alpha(1) = b_1$. Consider the following composite.

$$\begin{aligned} \Phi_\alpha : & F_{b_0} \to U_p \xrightarrow{\Lambda} WE \xrightarrow{ev_1} E \\ & x \mapsto (\alpha, x) \mapsto \Lambda(\alpha, x) \mapsto \Lambda(\alpha, x)(1) \end{aligned}$$

Then since, $p \circ \Lambda(\alpha, x) = \alpha$, so we have

$$\begin{aligned} p \circ \Lambda(\alpha, x)(1) &= \alpha(1) = b_1 \\ \Rightarrow \Lambda(\alpha, x)(1) &\in p^{-1}(b_1) =: F_{b_1}. \end{aligned}$$

Therefore, the following map

$$\begin{aligned} \Phi_\alpha : & F_{b_0} \to F_{b_1} \\ & x \mapsto \Lambda(\alpha, x)(1) \end{aligned}$$

is continuous. The adjoint of the composite $F_{b_0} \to WE$ gives a homotopy $h : F_{b_0} \times I \to E$ between the inclusion of F_{b_0} and $\Phi_\alpha : F_{b_0} \to F_{b_1} \hookrightarrow E$. Similarly, if we reverse the path we can obtain the homotopy inverse of the mapping. Thus F_{b_0} and F_{b_1} are homotopy equivalent. This completes the proof of the proposition.

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Thus, as a consequence of above Proposition we have a unique fiber upto homotopy of a fibration over a path-connected base space.

The lifting function provides some further structure. For $b \in B$, let $\Omega B = \Omega(B,b)$ denote the loops in B based at b and let $F = F_b$. Then $\Omega B \times F \subset U_p$ and for the map $\mu = ev_1 \circ \Lambda : \Omega B \times F \to E$ we have $im\ \mu \subset F$. Since $p \circ ev_1 \circ \Lambda(\beta, e) = p \circ \Lambda(\beta, e)(1) = \beta(1) = b$, we have $ev_1 \circ \Lambda(\beta, e) \in p^{-1}(b) = F$. Thus, $im\ \mu \subset F$ which determines an action F,

$$\mu = ev_1 \circ \Lambda : \Omega B \times F \to F$$

Let $\alpha^{-1}(t) := \alpha(1-t)$.

Proposition 3.5 *To the action $\mu = ev_1 \circ \Lambda : \Omega B \times F \to F$ and a loop $\alpha \in \Omega B$ associate the mapping $h_\alpha = \mu(\alpha^{-1}, -) : F \to F$. Then*

(1) If $\alpha \simeq \beta$, then $h_\alpha \simeq h_\beta$.

(2) If α is homomorphic to a constant map, $h_\alpha \simeq id_F$.

(3) If $\alpha * \beta$ denotes the loop multiplication of α and β, then $h_{\alpha*\beta} \simeq h_\alpha \circ h_\beta$.

Proof (1) Since $\alpha \simeq \beta$ so there exists a homotopy:

$$K : I \times I \to B$$
$$K(s,0) = \alpha^{-1}(s)$$
$$K(s,1) = \beta^{-1}(s)$$
$$K(0,t) = b = K(1,t).$$

Now, consider the adjoint of K, $\hat{K} : I \to WB$ given as $t \mapsto \hat{K}(t)$ where,

$$\hat{K}(t) : [0,1] \to B$$
$$\hat{K}(t)(s) = K(s,t).$$

Then, $\hat{K}(0)(s) = K(s,0) = \alpha^{-1}(s)$ and $\hat{K}(1)(s) = K(s,1) = \beta^{-1}(s)$ which implies that $\hat{K}(0) = \alpha^{-1}$ and $\hat{K}(1) = \beta^{-1}$. Also, $\hat{K}(t)(0) = K(0,t) = b = K(1,t) = \hat{K}(t)(1)$, therefore $\hat{K}(t)$ is a loop in B based at b, $\hat{K}(t) \in \Omega B$. Thus,

$$F \times I \to F$$
$$(x,t) \mapsto \mu(\hat{K}(t), x)$$
$$(x,0) \mapsto \mu(\alpha^{-1}, x) = h_\alpha(x)$$
$$(x,1) \mapsto \mu(\beta^{-1}, x) = h_\beta(x)$$

is a homotopy between h_α and h_β. Therefore $h_\alpha \simeq h_\beta$.

(2) Note that if $c =$ a constant map, then $\Omega(c,x)(1) = x$ since the lifting problem is solved by constant maps. Now since $\alpha \simeq c$, thus by part (1) we have $h_\alpha \simeq h_c$ and $h_c(x) = \mu(c^{-1},x) = \Lambda(c^{-1},x)(1) = x$. Therefore $h_c = id_F$. Hence $h_\alpha \simeq id_F$.

(3) Observe that, $h_{\alpha*\beta} = \mu((\alpha*\beta)^{-1},-) = \Lambda((\alpha*\beta)^{-1},-)(1)$ and $h_\alpha \circ h_\beta = \mu(\alpha^{-1},\mu(\beta^{-1},-)) = \Lambda(\alpha^{-1},\Lambda(\beta^{-1},-)(1))(1)$. But since,

$$\begin{aligned}\Lambda((\alpha*\beta)^{-1},x)(1) &= \Lambda(\beta^{-1}*\alpha^{-1},x)(1) \quad (\text{since } (\alpha*\beta)^{-1} = \beta^{-1}*\alpha^{-1}) \\ &= \Lambda(\alpha^{-1},\Lambda(\beta^{-1},x)(1))(1),\end{aligned}$$

we obtain $h_{\alpha*\beta} \simeq h_\alpha \circ h_\beta$.

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Corollary 3.6 *Let G denote an abelian group. If $p : E \to B$ is a fibration, $b \in B$, a path-connected space, then there is an action of the fundamental group $\pi_1(B,b)$ on $H_*(F;G)$ and on $H^*(F;G)$ induced by $[\alpha] \mapsto h_{\alpha^*}$ and h_α^*, respectively.*

Example 3.1 Let X be a path-connected space and x_0 a basepoint in X. Define $PX = \{\lambda : [0,1] \to X \mid \lambda \text{ is continuous and } \lambda(0) = x_0\}$ to be the **space of based paths** in X. Then the continuous map $p : PX \to X$, given by $p(\lambda) = \lambda(1)$, is a fibration.

We prove this as follows. Let $g : Y \to PX$ and $G : Y \times I \to X$ be maps such that $G(y,0) = p(g(y))$, then define $\tilde{G} : Y \times I \to PX$ by

$$\tilde{G}(y,t)(s) = \begin{cases} g(y)(s(t+1)), & \text{for } 0 \le s \le 1/(t+1), \\ G(y,s(t+1)-1), & \text{for } 1/(t+1) \le s \le 1. \end{cases}$$

Thus p has the homotopy lifting property with respect to all spaces, hence is a fibration. Note that, the fiber over x_0 is the space of based loops $\Omega X = \{\lambda : [0,1] \to X \mid \lambda(0) = x_0 = \lambda(1)\}$.

Definition 3.7 *The above fibration, $\Omega X \hookrightarrow PX \to X$, is called the* ***path-loop fibration*** *over X.*

Example 3.2 For a pair of spaces B and F, the projection map $p : B \times F \to B$, is called the **trivial fibration** with base B and fiber F.

Example 3.3 From the definition of a covering space, $f : E \to X$, the covering map is a fibration. For example the real projective spaces yield an interesting example of this. The covering map $\mathbb{S}^n \to \mathbb{R}P(n)$ is a fibration with fiber $\mathbb{S}^0$.

The complex analog of this also gives us a fibration:

Example 3.4 $\mathbb{S}^1 \to \mathbb{S}^{2n+1} \to \mathbb{C}P(n)$ is a fibration, where $\mathbb{S}^{2n+1}$ is the unit sphere in $\mathbb{C}^{n+1}$ and $\mathbb{C}P(n)$ is viewed as the quotient space of $\mathbb{S}^{2n+1}$ under the equivalence relation $(z_0, \cdots, z_n) \sim \lambda(z_0, \cdots, z_n)$ for $\lambda \in \mathbb{S}^1$, the unit circle in $\mathbb{C}$. The projection $p : \mathbb{S}^{2n+1} \to \mathbb{C}P(n)$ sends $(z_0, \cdots, z_n)$ to its equivalence class $[z_0, \cdots, z_n]$, so the fibers are copies of $\mathbb{S}^1$. This construction also works for $n = \infty$.

Next is the special case of above fibration for $n = 1$.

Example 3.5 Since $\mathbb{C}P(1)$ is homeomorphic to $\mathbb{S}^2$, the above fibration becomes $\mathbb{S}^1 \to \mathbb{S}^3 \to \mathbb{S}^2$ with fiber, total space and base all spheres. This is known as the **Hopf fibration**. The projection $\mathbb{S}^3 \to \mathbb{S}^2$ is defined as $(z_0, z_1) \mapsto z_0/z_1 \in \mathbb{C} \cup \{\infty\} = \mathbb{S}^2$.

We will also see some of the above examples of fibrations in Chapter 4 where we will look at some applications of the key result: The Leray-Serre spectral sequence.

Definition 3.8 *A pair of maps $(\tilde{f}, f) : (E', B') \to (E, B)$ is called a* ***morphism of fibrations*** *if the following diagram commutes:*

$$\begin{tikzcd}
E' \arrow[r, "\bar{f}"] \arrow[d, "p'"'] & E \arrow[d, "p"] \\
B' \arrow[r, "f"] & \bar{B}
\end{tikzcd}$$

with p and p' fibrations.

Definition 3.9 *A fibration is called* ***locally trivial*** *if there is a covering of B by open sets $\{V_\alpha\}_{\alpha \in J}$ and a set of homomorphisms $\{\phi_\alpha : V_\alpha \times F \to p^{-1}(V_\alpha)\}_{\alpha \in J}$, each of which induces a morphism of fibrations*

$$\begin{tikzcd}
V_\alpha \times F \arrow[r, "\phi_\alpha"] \arrow[d, "pr_1"] & p^{-1}(V_\alpha) \arrow[r, "\subset"] \arrow[d, "p"] & E \arrow[d, "p"] \\
V_\alpha \arrow[r, "="] & V_\alpha \arrow[r, "\subset"] & B
\end{tikzcd}$$

Suppose $p: E \to B$ is fibration and $f: X \to B$ a continuous map. Then we can define the **pullback** of p over f which will give rise to the following fibration.

Definition 3.10 *Define the* **pullback** *of p over f by letting E_f denote the set $\{(x,e) \in X \times E \mid f(x) = p(e)\}$. The projection maps on E_f give the following diagram*

$$\begin{tikzcd} E_f \arrow[r] \arrow[d, "p_f"'] & E \arrow[d, "p"] \\ X \arrow[r, "f"] & B \end{tikzcd}$$

The universal property of a pullback has as input data maps $u: Z \to X$ and $v: Z \to E$ such that $f \circ u = p \circ v$ and associates to them a unique map $w: Z \to E_f$ with all the triangles and squares in the following diagram commutative

$$\begin{tikzcd} Z \arrow[drr, bend left, "v"] \arrow[dr, "w"] \arrow[ddr, bend right, "u"'] & & \\ & E_f \arrow[r] \arrow[d, "p_f"'] & E \arrow[d, "p"] \\ & X \arrow[r, "f"] & B \end{tikzcd}$$

The universal property of pullbacks and the fact that p is a fibration implies that the map p_f satisfies the homotopy lifting property, thus $p_f: E_f \to X$ is a fibration and the fiber space is same as the fiber of p.

3.2 Long exact sequence of homotopy groups

In this section we consider the exactness properties of fibrations.

Theorem 3.11 *Let $E \xrightarrow{p} B$ be a fibration and b_0 a basepoint in B. If Z is a space, then for $F = p^{-1}(b_0)$*

$$[Z,F] \xrightarrow{i_*} [Z,E] \xrightarrow{p_*} [Z,B]$$

is an exact sequence of pointed sets. The same conclusion holds if p is a Serre fibration and Z has the homotopy type of a finite CW-complex.

Proof We are given that the maps i_* and p_* are defined as

$$[Z,F] \xrightarrow{i_*} [Z,E] \xrightarrow{p_*} [Z,B]$$

$$g \mapsto i \circ g \mapsto p \circ i \circ g$$

where $[X,Y]$ denotes the homotopic class of maps from X to Y and

$$\begin{aligned} ker\, p_* &:= \{h \in [Z,E] \mid p_*(h) \simeq c_{b_0} \in [Z,B]\} \\ &= \{h : Z \to E \mid p \circ h \simeq c_{b_0} : Z \to B\} \end{aligned}$$

where $\simeq$ denotes homotopic equivalence of maps. Now in order to show that this is an exact sequence we need to show that $ker\, p_* = im\, i_*$. So let us start by considering a map $g \in ker\, p_*$. So we have the map $g : Z \to E$ such that $p \circ g \simeq c_{b_0} : Z \to B$. Then we need to show that it is also in the image of the map i_*. But first note that since $p \circ g$ is homotopic equivalent to c_{b_0}, so there exists a homotopy, say $G : Z \times I \to B$ such that $G(y,0) = p \circ g(y,0)$ and $G(y,1) = c_{b_0}(y,1) = b_0$. Thus we have the following diagram

$$\begin{tikzcd} Z \times \{0\} \arrow[r, "g"] \arrow[d, "inc"'] & E \arrow[d, "p"] \\ Z \times I \arrow[r, "G"] & \bar{B} \end{tikzcd}$$

Then since p is a fibration, therefore by HLP there exists a map $\tilde{G}$:

$$\begin{tikzcd} Z \times \{0\} \arrow[r, "g"] \arrow[d, "inc"'] & E \arrow[d, "p"] \\ Z \times I \arrow[r, "G"] \arrow[ur, dashed, "\exists\, \tilde{G}"] & \bar{B} \end{tikzcd}$$

such that $p \circ \tilde{G} = G$ and $\tilde{G}(y,0) = g(y,0)$. Then consider the map $f : \tilde{G}\mid_{Z\times\{1\}}$: $Z \times \{1\} \to B$. Since $G(y,1) = c_{b_0}(y,1) = b_0$ and $p \circ \tilde{G} = G$, thus $p \circ \tilde{G}(y,1) = b_0$ which implies that $\tilde{G}(y,1) \in p^{-1}(b_0) =: F$ and hence $f(y,1) \in F$. Therefore f determines a map into F, $f : Z \times \{1\} \to F$. Further, it is clear that $\tilde{G} : Z \times I \to E$ is a homotopy from g to $f_* = i \circ f : Z \to F \hookrightarrow E$ since, $\tilde{G}(y,0) = g(y,0)$ and $\tilde{G}(y,1) = f(y,1) = f_*(y,1)$. So, $f_* \simeq g : Z \to E$ and since $f_* \in im\ i_*$ therefore $[g] \in im\ i_*$. Hence $ker\, p_* \subseteq im\, i_*$.

Now for the other direction $im\, i_* \subseteq ker\, p_*$ we let $\alpha_* \in im\ i_*$, then

$$\begin{aligned} p_*(\alpha_*)(x) &= p_*(i \circ \alpha)(x) = p \circ (i \circ \alpha)(x) = p(\alpha(x)) \\ &\subseteq p(p^{-1}(b_0)) = b_0. \end{aligned}$$

Therefore, $p_* \circ \alpha_* \simeq c_{b_0} \Rightarrow \alpha_* \in ker\, p_*$. Hence $ker\, p_* = im\, i_*$. This finishes the proof of the theorem.

■

The following theorem shows how we can extend a fibration to a sequence of fibrations . The key point of the construction is that, PX the space of based paths is contractible which is clear since we have the following contraction

$$H : PB \times [0,1] \to PB$$
$$H(\lambda, t) = \lambda(1-t).$$

Theorem 3.12 *[6, Theorem 4.30] Given a fibration $p : E \to B$ with B path connected and fiber F, there is a sequence of fibrations up to homotopy*

$$\cdots \to \Omega^n F \xrightarrow{\Omega^n i} \Omega^n E \xrightarrow{\Omega^n p} \Omega^n B \to \Omega^{n-1} F \to \cdots$$
$$\to \Omega B \to F \xrightarrow{i} E \xrightarrow{p} B.$$

Proof First step in proving this theorem is to form the pullback of the path-loop fibration over B with respect to the fibration $p : E \to B$.

$$\begin{tikzcd}
\Omega B \arrow[r, "="] \arrow[d] & \Omega B \arrow[d] \\
F \simeq E_p \arrow[r] \arrow[d, "pr_1"'] & PB \arrow[d, "ev_1"] \\
E \arrow[r, "p"'] & B
\end{tikzcd}$$

where by definition $E_p := \{(\lambda, e) \mid \lambda : (I, 0) \to (B, b_0), \lambda(1) = p(e)\}$ and $\Omega B := \{\lambda : (I, 0) \to (B, b_0) \mid \lambda(0) = b_0 = \lambda(1)\}$ is the loop-space at $b_0 \in B$. The map $: \Omega B \to E_p$ is given by $\omega \mapsto (\omega, e_0)$ where e_0 denotes some choice of basepoint for E in the fiber over b_0. Let $c = c_{b_0}$ be the constant loop at $b_0 \in B$ then define the map $: F \to E_p$ by $x \mapsto (c, x)$. Then since PB is contractible, this map induces the homotopy equivalence of F and E_p. Hence $F \simeq E_p$. Thus we obtain a fibration upto homotopy: $\Omega B \hookrightarrow F \to E$. This gives us the following last part of the sequence of fibrations in the theorem

$$\Omega B \to F \xrightarrow{i} E \xrightarrow{p} B.$$

Proceeding in the same manner as above, we obtain the pullback of the path-loop fibration over E with respect to the map $pr_2 : E_p \to E$

$$\begin{tikzcd}
\Omega E \arrow[r, "="] \arrow[d] & \Omega E \arrow[d] \\
\Omega B \simeq E' \arrow[r] \arrow[d, "pr_1"'] & PE \arrow[d] \\
F \simeq E_p \arrow[r, "pr_2"'] & E
\end{tikzcd}$$

where $E' := \{(\eta, \lambda, e) \mid \eta : (I, 0) \to (E, e_0), \lambda : (I, 0) \to (B, b_0), \eta(1) = e, \lambda(1) = p(e)\}$ and ΩE is the loop-space. Then again, the space E' is homotopy equivalent to ΩB by the map: $(\eta, \lambda, e) \mapsto \lambda * (p \circ \eta)$, where $*$ is loop multiplication. The map of ΩE to E' is given by $\tilde{\omega} \mapsto (\tilde{\omega}, c, e_0)$ and continuing through to ΩB we get

$$\begin{aligned} &\Omega E \to \Omega B \\ &\quad \tilde{\omega} \mapsto c * (p \circ \tilde{\omega}) \simeq p \circ \tilde{\omega} \end{aligned}$$

Thus we obtain the fibration $\Omega E \hookrightarrow \Omega B \to F$ with Ωp as the 'inclusion' of the fiber ΩE in ΩB. This gives us the next part of the sequence of fibrations given in theorem $\Omega E \xrightarrow{\Omega p} \Omega B \to F \xrightarrow{i} E \xrightarrow{p} B$. Finally, we iterate the above process to obtain the sequence of fibrations as defined in theorem. This completes the proof of the theorem.

■

Thus as a consequence of above theorem we obtain the following corollary (to be used in the proof of Theorem 4.5).

Corollary 3.13 *For a Serre fibration, $F \hookrightarrow E \to B$ with B path connected, there is a long exact sequence,*

$$\begin{aligned} \cdots \to \pi_n(F) \xrightarrow{i_*} \pi_n(E) \xrightarrow{p_*} \pi_n(B) \to \pi_{n-1}(F) \to \cdots& \\ \to \pi_1(B) \to \pi_0(F) \xrightarrow{i_*} \pi_0(E) \xrightarrow{p_*} \pi_0(B).& \end{aligned}$$

Proof Apply $[\mathbb{S}^0, -]$ to the sequence in Theorem 3.12 and use $\pi_n(X) = [\mathbb{S}^0, \Omega^n(X)]$ to obtain the sequence of homotopy groups as described. Note that the exactness of the sequence follows by Theorem 3.11.

■

The above long exact sequence shows how a fibration is a sort of exact sequence in the category of topological spaces, up to homotopy.

Remark 3.1 *Note that the last part of the sequence need not be onto in general since B could have path-components B_α such that $p^{-1}(B_\alpha)$ is empty. However, in the case when E and B are path-connected the sequence typically ends with $\pi_1(B) \to \pi_0(F) \to 0$.*

Chapter 4

The Leray-Serre spectral sequence

It was an important problem to relate the cohomology of spaces making up a fibration. In 1946, Leray [4] developed the powerful technique of spectral sequence to solve this problem. The theory was later developed for singular homology by Serre [7] in 1950.

Throughout we assume that R is a commutative ring with unit and we will only consider the cohomology spectral sequence.

Theorem 4.1 (The cohomology Leray-Serre spectral sequence) *[6, Theorem 5.2] Suppose $F \hookrightarrow E \xrightarrow{\pi} B$ is a fibration, where B is path-connected and F is connected. Then there is a first quadrant spectral sequence of algebras, $\{E_r^{*,*}, d_r\}$, converging to $H^*(E;R)$ as an algebra, with*

$$E_2^{p,q} \cong H^p(B; \mathcal{H}^q(F;R)),$$

the cohomology of the space B with local coefficients in the cohomology of the fiber of π. This spectral sequence is natural with respect to fiber-preserving maps of fibrations. Furthermore, the cup product $\smile$ on cohomology with local coefficients and the product $\cdot_2$ on $E_2^{*,*}$ are related by $u \cdot_2 v = (-1)^{p'q} u \smile v$ when $u \in E_2^{p,q}$ and $v \in E_2^{p',q'}$.

Thus the above theorem defines the spectral sequence associated to a fibration. Further, it also defines the ring structure on the cohomology ring.

4.1 Construction of the spectral sequence

We will prove the above theorem in parts. First, we determine the E_1-term.

Let us begin by motivating the ideas for the construction of the above spectral sequence and defining the E_1-term and the differential d_1. For that we have the following proposition.

Proposition 4.2 *Given a fibration $F \hookrightarrow E \xrightarrow{\pi} B$ with base space a CW-complex, there is a first quadrant spectral sequence, $\{E_r^{*,*}, d_r\}$, with*

$$E_1^{p,q} \cong H^{p+q}(\pi^{-1}(B^{(s)}), \pi^{-1}(B^{(s-1)}); R)$$

and $d_1 = \Delta$, the boundary homomorphism in the exact sequence in cohomology for the triple $(\pi^{-1}(B^{(s+1)}), \pi^{-1}(B^{(s)}), \pi^{-1}(B^{(s-1)})$. The spectral sequence converges to $H^*(E; R)$.

Proof Consider the fibration $F \hookrightarrow E \xrightarrow{\pi} B$ and assume that B **is homotopic equivalent to a CW-complex**. Then B has an obvious filtration induced by the successive skeleta

$$B \supset \cdots \supset B^{(s)} \supset X^{(s-1)} \supset B^{(s-2)} \supset \cdots \supset B^{(0)} \supset \phi.$$

Now we can lift this filtration to a filtration of E by defining $J^s = \pi^{-1}(B^{(s)})$ to be the subspace of E lying over the s-skeleton of B. Thus we have the following situation

$$\begin{array}{ccccccccccccc} E & \supset & \cdots & \supset & J^s & \supset & J^{s-1} & \supset & \cdots & \supset & J^0 & \supset & \emptyset \\ \downarrow & & & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow \\ B & \supset & \cdots & \supset & B^{(s)} & \supset & B^{(s-1)} & \supset & \cdots & \supset & B^{(0)} & \supset & \emptyset \end{array}$$

Now we will obtain an exact couple from this filtration which will give us a spectral sequence using the construction we did in Chapter 2. So consider the long exact sequence of cohomology groups for the pair (J^s, J^{s-1})

$$\cdots \to H^n(J^s, J^{s-1}; R) \xrightarrow{j^*} H^n(J^s; R) \xrightarrow{i^*} H^n(J^{s-1}; R) \xrightarrow{\delta} H^{n+1}(J^s, J^{s-1}; R) \to \cdots$$

where

$$H^n(J^s, J^{s-1}; R) := \text{the cohomology groups of the cochain complex } \{C^n(J^s, J^{s-1}; R), \delta\}$$

and $C^n(J^s, J^{s-1}; R) := Hom(C_n(J^s, J^{s-1}); R)$. Now if we define,
$D^{p,q} := H^{p+q-1}(J^{p-1})$ and $E^{p,q} := H^{p+q}(J^p, J^{p-1})$ then we obtain the following exact couple

$$\begin{tikzcd}
 & H^{p+q-1}(J^p;R) \arrow[rr, "i^*"] & & H^{p+q-1}(J^{p-1};R) \arrow[dl, "\delta"] \\
H^{p+q}(J^p;R) & & H^{p+q}(J^p, J^{p-1};R) \arrow[ll, "j^*"] &
\end{tikzcd}$$

Thus by Theorem 2.5 we have a first quadrant spectral sequence with, $E_1^{p,q} = H^{p+q}(J^p, J^{p-1}; R)$ and

$$d_1 = \delta \circ j^* : H^{p+q}(J^p, J^{p-1}; R) \xrightarrow{j^*} H^{p+q}(J^p; R) \xrightarrow{\delta} H^{p+q+1}(J^{p+1}, J^p; R)$$

which is the boundary homomorphism in the long exact sequence in cohomology for the triple (J^{p+1}, J^p, J^{p-1}).

Another way of doing this is by directly defining the filtration for $C^*(E; R)$, the cochain complex of E with coefficients in R, and hence appealing to the first method of construction we did in Chapter 2, that is spectral sequence via filtered differential graded module. So define,

$$F^s C^*(E; R) := ker(C^*(E; R) \to C^*(J^{s-1}; R))$$

Then this is a decreasing filtration. Thus by Theorem 2.1, we get a spectral sequence with

$$E_1^{p,q} \cong H^{p+q}(F^p C^*(E; R)/F^{p+1} C^*(E; R)) = H^{p+q}(J^p, J^{p-1}; R)$$

Also note that the above mentioned filtration is bounded, hence by the second part of the Theorem 2.1 it follows that this spectral sequence converges to $H^*(E; R)$. It can also be seen by Proposition 2.8, that the two spectral sequences we obtained above turn out to be the same.

■

The next step in proving Theorem 4.1 is to determine the E_2-term. We consider the simplest case of a trivial fibration where $E = B \times F$,

$$F \hookrightarrow B \times F \xrightarrow{\pi} B.$$

Then, since $J^s = \pi^{-1}(B^{(s)})$ is homeomorphic to $B^{(s)} \times F$ and $B^{(s)} \supset B^{(s-1)}$, so we have $(J^s, J^{s-1}) = (B^{(s)}, B^{(s-1)}) \times F$. Thus by **Künneth theorem** [6, Theorem 2.12] we have that

$$H^{p+q}(J^p, J^{p-1}) = H^{p+q}((B^{(p)}, B^{(p-1)}) \times F) \cong H^p(B^{(p)}, B^{(p-1)}) \otimes H^q(F).$$

Therefore, using Proposition 4.2 we conclude

$$E_1^{p,q} \cong Cell^p(B) \otimes H^q(F; R)$$

where the cellular cochain complex of X: $(Cell^*(X), \delta^{cell})$ is defined as $Cell^n(X) = H^n(X^{(n)}, X^{(n-1)})$ and d_1 is given by $d_1 = \delta^{cell} \otimes 1$. Thus $E_2^{p,q} = H^p(B; H^q(F; R))$. In the case of an arbitrary filtration, the 'twisting' of the fiber and base spaces in the total space leads to a non-trivial complex expression for E_1 and hence for E_2. The following result defines the E_2-term in the case of simplified hypothesis:

Proposition 4.3 *Suppose that the system of local coefficients on B determined by the fiber is simple, F is connected and F and B are of finite type; then for a field k, we have*

$$E_2^{p,q} \cong H^p(B; k) \otimes_k H^q(F; k).$$

Proof By **Universal coefficient theorem** [6, Theorem 2.16] we have

$$\begin{aligned} E_2^{p,q} &\cong H^p(B; H^q(F; R)) \\ &\cong H^p(B; k) \otimes_k H^q(F; k) \bigoplus Tor_1^{\mathbb{Z}}(H^{p-1}(B; k), H^q(F; k)) \\ &\cong H^p(B; k) \otimes_k H^q(F; k) \end{aligned}$$

■

Suppose that the system of local coefficients is simple and that B and F are connected. Then

$$E_2^{p,0} \cong H^p(B; H^0(F; R)) \cong H^p(B; R),$$
$$E_2^{0,q} \cong H^0(B; H^q(F; R)) \cong H^q(F; R).$$

For any bigraded algebra, like $E_2^{*,*}$, both $E_2^{*,0}$ and $E_2^{0,*}$ are subalgebras.

Proposition 4.4 *[6, Proposition 5.6] When restricted to the subalgebras $E_2^{*,0}$ and $E_2^{0,*}$, the product structure in the spectral sequence on $E_2^{*,*}$ coincides with the cup product structure on $H^*(B;R)$ and $H^*(F;R)$, respectively. Furthermore, if for all p,q, $H^p(B;R)$ and $H^q(F;R)$ are free R-modules of finite type, and the system of local coefficients on B is simple, then $E_2^{*,*} \cong H^*(B;R) \otimes_R H^*(F;R)$ as a bigraded algebra.*

The first part of this proposition is a consequence of theorem 4.6 which we will talk about in the next section. The second part is again an application of the Universal coefficient theorem.

4.2 Some formal consequences

Theorem 4.5 *[6, Theorem 5.7] Suppose $F \hookrightarrow \mathbb{R}^n \xrightarrow{\pi} B$ is a locally trivial fibration with B a polyhedron and F connected. Then B and F are* ***acyclic spaces****. By an acyclic space we mean that $\tilde{H}^*(B) \cong \{0\} \cong \tilde{H}^*(F)$.*

Proof By Corollary 3.13, we have the following part of homotopy exact sequence

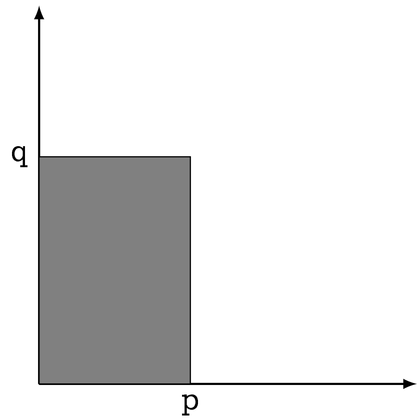
$$\cdots \to \pi_1(\mathbb{R}^n) \to \pi_1(B) \to \pi_0(F) \to \cdots$$

Now since $\mathbb{R}^n$ is contractible so $\pi_1(\mathbb{R}^n) = \{0\}$ and also F being connected and locally path-connected (which follows because B is a polyhedron and F is pullback under π) is therefore path-connected. Hence $\pi_0(F)$ is also trivial. Then it follows from the above sequence that $\pi_1(B) = \{0\}$. Therefore the system of local coefficients induced by F on B is simple.

Now we claim that $H^i(B)$ and $H^i(F)$ are trivial for all $i > n$. First note that F is a subset of $\mathbb{R}^n$, so for any $i > n$, $H^i(F)$ is trivial. Secondly, let B has a system of neighborhoods, $\{U\}$, so that $\pi^{-1}(U)$ is homeomorphic to $U \times F$ for each U in the system. Then since $\pi^{-1}(U)$ is also a subset of $\mathbb{R}^n$, thus U must have dimension $\leq n$. Therefore the claim follows.

Suppose p and q are the greatest non-zero dimensions in which B and F have non-trivial integral cohomology, respectively. Then by **Universal coefficient theorem** [6, Theorem 2.16] we have that $E_2^{r,s} \cong H^r(B) \otimes H^s(F) \bigoplus Tor_1^{\mathbb{Z}}(H^{r-1}(B), H^s(F))$.

And since $H^r(B) = \{0\}$ for $r > p$ and $H^s(F) = \{0\}$ for $s > q$, thus $E_2^{r,s}$ is non-zero only for $r \leq p$ and $s \leq q$ that is only in the box pictured below.



Now consider the differential $d_2 : E_2^{p,q} \to E_2^{p+2,q-1} = \{0\}$, therefore $d_2 = 0$ and similarly all subsequent differentials are also zero maps. Thus our spectral sequence collapses at $r = 2$ and hence $E_\infty^{p,q} \cong E_2^{p,q} \cong H^p(B) \otimes H^q(F)$. And since the spectral sequence converges to $H^*(\mathbb{R}^n)$ and $H^m(\mathbb{R}^n) \cong \bigoplus_{r+s=m} E_\infty^{r,s}$ as a vector space implies that $H^i(\mathbb{R}^n) = \{0\}$ for $i > p+q$. $E_\infty^{p,q}$ is the only vector space contributing to $H^{p+q}(\mathbb{R}^n)$, therefore

$$H^{p+q}(\mathbb{R}^n) \cong H^p(B) \otimes H^q(F).$$

Finally $\mathbb{R}^n$ is acyclic implies that $p+q = 0$ and since $p, q > 0$ therefore B and F are also acyclic. This finishes the proof of the theorem.

■

Example 4.1 (The Gysin Sequence) Suppose $F \hookrightarrow E \xrightarrow{\pi} B$ is a fibration with B path-connected and the system of local coefficients on B induced by the fiber is simple. Suppose further that F is a homology n-sphere, that is, $H_*(F) \cong H_*(\mathbb{S}^n)$, for some $n \geq 1$. Then,

(i) there exists following exact sequence called the **Gysin Sequence**

$$\to H^k(B;R) \xrightarrow{\gamma} H^{n+k+1}(B;R) \xrightarrow{\pi^*} H^{n+k+1}(E;R) \xrightarrow{Q} H^{k+1}(B;R) \to$$

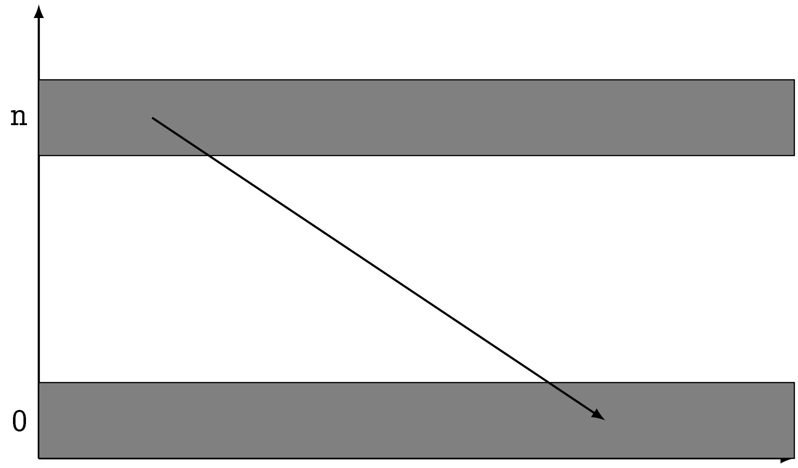
where $\gamma(u) = z \smile u$ for some $z \in H^{n+1}(B;R)$ and,

(ii) if n is even and $2 \neq 0$ in R, then $2z = 0$.

Proof (i) We are given that

$$H^m(F;R) \cong \begin{cases} R, & \text{when } m = 0 \text{ or } n, \\ \{0\}, & \text{otherwise} \end{cases}$$

Thus, with B path-connected, the E_2-term of the spectral sequence which is given by $E_2^{p,q} \cong H^p(B;R) \otimes H^q(F;R)$ (using Proposition 4.4) is non-trivial only for $q = 0$ or n as pictured below



Then since the only non-zero differential is d_{n+1}, therefore $E_2 \cong E_3 \cong \cdots \cong E_{n+1}$ and $H(E_{n+1}, d_{n+1}) \cong E_{n+2} \cong \cdots \cong E_\infty$. Thus,

$$E_{n+1}^{*,*} \cong H^*(B;R) \otimes H^*(F;R).$$

Recall example 1.3, so we also have the following exact sequence

$$\to E_2^{k,n} \xrightarrow{d_{n+1}} E_2^{k+n+1,0} \to H^{k+n+1}(E;R) \to E_2^{k+1,n} \xrightarrow{d_{n+1}} E_2^{k+n+2,0} \to$$

Now suppose h is the generator of $H^n(F;R) = R$ (note that, in general h is different from 1, the unity of R). Then we claim that,

- There exists a $z \in H^{n+1}(B;R)$ such that $d_{n+1}(1 \otimes h) = z \otimes 1$.
- $d_{n+1}(x \otimes 1) = 0$

Now we prove our claims. Consider the map $d_{n+1} : E_{n+1} \to E_{n+1}$. Recall that $E_{n+1}^{p,q}$ is non-trivial only for $q = 0$ or n. Now for $q = 0$ we have

$$d_{n+1} : E_{n+1}^{p,0} \to E_{n+1}^{p+n+1,-n}$$

where $E_{n+1}^{p+n+1,-n} = \{0\}$ since by Theorem 4.1 we have a first quadrant spectral sequence. Hence, $d_{n+1}^{p,0} = 0$. Thus the only non-zero differential is when $q = n$. So let $q = n$ and $p = 0$. Then we have the following map

$$d_{n+1} : E_{n+1}^{0,n} \to E_{n+1}^{n+1,0}.$$

But since by Proposition 4.4, $E_{n+1}^{p,q} \cong H^p(B;R) \otimes_R H^q(F;R)$, therefore

$$\begin{aligned} E_{n+1}^{0,n} &\cong H^0(B;R) \otimes_R H^n(F;R) \\ &\cong R \otimes_R H^n(F;R) \end{aligned} \qquad \text{and} \qquad \begin{aligned} E_{n+1}^{n+1,0} &\cong H^{n+1}(B;R) \otimes_R H^0(F;R) \\ &\cong H^{n+1}(B;R) \otimes_R R. \end{aligned}$$

Thus we have the differential $d_{n+1} : R \otimes_R H^n(F;R) \to H^{n+1}(B;R) \otimes R$. So consider $1 \otimes h \in E_{n+1}^{0,n}$ then $d_{n+1}(1 \otimes h) \in H^{n+1}(B;R) \otimes R$, that is $\exists\ z \in H^{n+1}(B;R)$ such that $d_{n+1}(1 \otimes h) = z \otimes 1$. Hence the claim.
Now using the property, Leibniz rule, of a derivation, that is:

$$d(a \cdot a') = d(a) \cdot a' + (-1)^{deg\, a} a \cdot d(a')$$

and the claim above, we have

$$\begin{aligned} d_{n+1}(x \otimes h) &= d_{n+1}((1 \smile x) \otimes (h \smile 1)) \\ &= d_{n+1}((1 \otimes h) \smile (x \otimes 1)) \\ &= d_{n+1}(1 \otimes h) \smile (x \otimes 1) + (-1)^n (1 \otimes h) \smile d_{n+1}(x \otimes 1) \\ &= (z \otimes 1) \smile (x \otimes 1) \\ &= (z \smile x) \otimes 1 \end{aligned}$$

where $1 \otimes h \in E_{n+1}^{0,n} \Rightarrow deg(1 \otimes h) = n$. Thus we have the following map

$$\begin{aligned} d_{n+1} :\ E_2^{k,n} \cong H^k(B;R) \otimes h &\longrightarrow H^{n+k+1}(B;R) \otimes 1 \cong E_2^{n+k+1,0} \\ d_{n+1}(x \otimes h) &\longmapsto (z \smile x) \otimes 1. \end{aligned}$$

So define the map $\gamma :\ H^k(B;R) \longrightarrow H^{n+k+1}(B;R)$ by $\gamma(x) = d_{n+1}(x \otimes h)$. This gives us the first part of the long exact sequence. Now for the second part note that the fibration $E \xrightarrow{\pi} B$ induces the following map

$$H^*(B) \xrightarrow{\pi^*} H^*(E)$$

which gives us the second part of the long exact sequence. Finally making use of the exact sequence from example 1.3, we obtain:

$$Q : H^{k+n+1}(E;R) \longrightarrow E_2^{k+1,n} \cong H^{k+1}(B;R).$$

Thus we obtain the Gysin sequence as described.

(ii) If n is even then again using Leibniz rule we obtain:

$$\begin{aligned}0 &= d_{n+1}(1 \otimes (h \smile h)) = d_{n+1}((1 \otimes h) \smile (1 \otimes h)) \\ &= (z \otimes 1) \smile (1 \otimes h) + (-1)^n (1 \otimes h) \smile (z \otimes 1) \\ &= 2((z \otimes 1) \smile (1 \otimes h)) = 2(z \otimes h) \\ &= 2z \otimes h = z \otimes 2h.\end{aligned}$$

Now since $2h \neq 0$, hence $2z = 0$.

This completes the proof.

■

Example 4.2 (An application of Gysin sequence) We use the Gysin sequence to compute the cohomology of n-dimensional complex projective space as an algebra which is given by

$$H^*(\mathbb{C}P(n);R) \cong R[x_2]/(x_2^{n+1}),$$

the **truncated polynomial algebra** of height $(n+1)$ on one generator of degree 2.

Proof Since the n-dimensional complex projective space is the quotient of $\mathbb{S}^{2n+1}$, $\mathbb{C}P(n) \cong \mathbb{S}^{2n+1}/\mathbb{S}^1$, therefore it sits in the following fibration with spherical fiber

$$\mathbb{S}^1 \hookrightarrow \mathbb{S}^{2n+1} \to \mathbb{C}P(n).$$

Thus $\mathbb{C}P(n)$ is simply-connected for all n. Hence the system of local coefficients is always simple. So we obtain a Gysin sequence associated to this fibration. Also observe that $\mathbb{C}P(1) \cong \mathbb{S}^2$. Note that since $\mathbb{C}P(n)$ is a $2n$-dimensional manifold, therefore $H^k(\mathbb{C}P(n);R)$ is trivial for all $k \geq 2n$. Therefore,

$$H^*(\mathbb{C}P(n);R) \cong \bigoplus_{k=0}^{2n} H^k(\mathbb{C}P(n);R).$$

Now for $k \leq 2n$ we have two cases, $k =$ even and $k =$ odd. We claim the following:

- $(x_2)^k$ generates $H^{2k}(\mathbb{C}P(n);R)$, the even dimensional cohomology for all $k \leq n$, where $x_2 = \gamma(1)$.

- The odd dimensional cohomology, $H^{2k+1}(\mathbb{C}P(n);R)$ is trivial for all k.

Now we prove our claims. We begin with the case $k=1$. Consider the following initial part of the Gysin sequence

$$H^0(\mathbb{C}P(n);R) \xrightarrow{\gamma} H^2(\mathbb{C}P(n);R) \longrightarrow H^2(\mathbb{S}^{2n+1};R) \longrightarrow \cdots$$

Note that since $H^0(\mathbb{C}P(n);R) = R$ therefore $x_2 = \gamma(1)$ generates $H^2(\mathbb{C}P(n);R)$. We also have the following short exact sequence from the Gysin sequence

$$\begin{aligned} H^2(\mathbb{S}^{2n+1};R) \cong \{0\} \longrightarrow H^1(\mathbb{C}P(n);R) &\xrightarrow{\gamma} H^3(\mathbb{C}P(n);R) \\ &\longrightarrow \{0\} \cong H^3(\mathbb{S}^{2n+1};R). \end{aligned}$$

Therefore γ is an isomorphism. Now since $\mathbb{C}P(n)$ is simply-connected, $H^1(\mathbb{C}P(n);R)$ is trivial and hence $H^3(\mathbb{C}P(n);R)$ is also trivial. Now by induction suppose that $(x_2)^k$ generates $H^{2k}(\mathbb{C}P(n);R)$. Then consider the following portion of the Gysin sequence

$$\begin{aligned} H^{2k+1}(\mathbb{S}^{2n+1};R) \longrightarrow H^{2k}(\mathbb{C}P(n);R) &\xrightarrow{\gamma} H^{2k+2}(\mathbb{C}P(n);R) \\ &\longrightarrow \{0\} \cong H^{2k+2}(\mathbb{S}^{2n+1};R) \longrightarrow \end{aligned}$$

Then since $H^{2k+1}(\mathbb{S}^{2n+1};R)$ is trivial for $k < n$, therefore γ is an isomorphism for $k < n$. So, $\gamma((x_2)^k) = x_2 \smile (x_2)^k = (x_2)^{k+1}$ generates $H^{2k+2}(\mathbb{C}P(n);R)$. Similarly, in odd dimensions, the pattern of trivial modules continues. Finally for $k = n$, we have the following short exact sequence

$$\begin{aligned} H^{2n+1}(\mathbb{C}P(n);R) \cong \{0\} \longrightarrow H^{2n+1}(\mathbb{S}^{2n+1};R) &\xrightarrow{Q} H^{2n}(\mathbb{C}P(n);R) \\ &\xrightarrow{\gamma} \{0\} \cong H^{2n+2}(\mathbb{C}P(n);R). \end{aligned}$$

Thus γ is the trivial homomorphism and Q is an isomorphism. Since $\gamma((x_2)^n) = (x_2)^{n+1}$, we conclude that $(x_2)^{n+1} = 0$. Therefore,

$$H^*(\mathbb{C}P(n);R) \cong \bigoplus_{k=0}^{n} H^{2k}(\mathbb{C}P(n);R) \cong R[x_2]/(x_2^{n+1}).$$

$\blacksquare$

Theorem 4.6 *Given a fibration* $F \xrightarrow{i} E \xrightarrow{\pi} B$ *with* B *path-connected,* F *connected, and for which the system of local coefficients on* B *is simple; then the composites*

$$H^q(B;R) = E_2^{q,0} \twoheadrightarrow E_3^{q,0} \twoheadrightarrow \cdots$$
$$\twoheadrightarrow E_q^{q,0} \twoheadrightarrow E_{q+1}^{q,0} = E_\infty^{q,0} \subset H^q(E;R)$$

and $H^q(E;R) \twoheadrightarrow E_\infty^{0,q} = E_{q+1}^{0,q} \subset E_q^{0,q} \subset \cdots \subset E_2^{0,q} = H^q(F;R)$ *are the homomorphisms*

$$\pi^* : H^q(B;R) \to H^q(E;R) \quad \textit{and} \quad i^* : H^q(E;R) \to H^q(F;R).$$

Thus, the above Theorem establishes a relation of the cohomology of the total space E with the base space B and the fiber F, respectively.
An immediate consequence of this theorem is Proposition 4.4. Since the induced maps are morphisms of algebras, thus this establishes Proposition 4.4 on the product structure of $E_2^{0,*}$ and $E_2^{*,0}$.

Definition 4.7 *F is said to be **totally nonhomologous to zero** in E with respect to the ring R if the homomorphism $i^* : H^*(E;R) \to H^*(F;R)$ is onto.*

Note that in the case of a trivial fibration, F is totally nonhomologous to zero in E for obvious reasons. Now the following theorem gives a converse (in some sense).

Theorem 4.8 (The Leray-Hirsch theorem [6, Theorem 5.10]) *Given a fibration* $F \xrightarrow{i} E \xrightarrow{\pi} B$ *with* F *connected,* B *of finite type, path-connected, and for which the system of local coefficients on* B *is simple; then, if* F *is totally nonhomologous to zero in* E *with respect to a field* k*, we have*

$$H^*(E;k) \cong H^*(B;k) \otimes_k H^*(F;k)$$

as vector spaces.

Thus, the Leray-Hirsch theorem above, gives us the cohomology of the total space E in terms of the cohomology of the base space B and fiber F and this isomorphism is an isomorphism of vector spaces.

4.3 Some computations

(1) Computation of $H^*(SU(n);R)$

Let $SU(n)$ be the group of $(n \times n)$ complex-valued unitary $(A\bar{A}^t = I)$ matrices of determinant 1. Then the consecutive quotients are spheres: that is $\mathbb{S}^{2n-1}$ is the quotient $SU(n)/SU(n-1)$. So we obtain a fibration:

$$SU(n-1) \hookrightarrow SU(n) \xrightarrow{\rho} \mathbb{S}^{2n-1}$$

by defining ρ for a fixed $\vec{x}_0 \in \mathbb{C}^n$ as $\rho(A) = A\vec{x}_0$ with fiber the subgroup of $SU(n)$ that fixes $\vec{x}_0$, which is $SU(n-1)$. Also $SU(2)$ can be identified with $\mathbb{S}^3$. We claim that $H^*(SU(n);R) \cong \Lambda(x_3, x_5, \ldots, x_{2n-1})$, the exterior algebra on generators x_i, where $deg x_i = i$.

Proof We prove this by induction on n. For $n = 2$, we have

$$H^*(SU(2);R) \cong H^*(\mathbb{S}^3;R) \cong R[x]/(x^2) = \Lambda(x_3).$$

Consider the fibration

$$SU(n-1) \hookrightarrow SU(n) \xrightarrow{\rho} \mathbb{S}^{2n-1},$$

then since $\mathbb{S}^{2n-1}$ is path-connected and $SU(n-1)$ is connected, so we can apply the Leray-Serre spectral sequence to obtain

$$E_2^{p,q} \cong H^p(B;\mathcal{H}^q(F;R)) \cong H^p(\mathbb{S}^{2n-1};H^q(SU(n-1);R)).$$

Let y_{2n-1} be generator of $H^*(\mathbb{S}^{2n-1};R)$ as an exterior algebra. By induction suppose that $H^*(SU(n-1);R) \cong \Lambda(x_3, x_5, \ldots, x_{2n-3})$. Since we obtain a spectral sequence of algebras according to Theorem 4.1, so we can consider only the algebra generators while describing the differentials. Since for $n \geq 2$, $\mathbb{S}^{2n-1}$ is simply-connected, thus the system of local coefficients on the base space is simple. So we can apply the Universal coefficient theorem [6, Theorem 2.16] to obtain

$$\begin{aligned} E_2^{*,*} &\cong H^*(\mathbb{S}^{2n-1};R) \otimes H^*(SU(n-1);R)) \\ &\cong \Lambda(y_{2n-1}) \otimes \Lambda(x_3, x_5, \ldots, x_{2n-3}) \\ &\cong \Lambda(x_3, x_5, \ldots, x_{2n-3}, y_{2n-1}). \end{aligned}$$

Since the algebra generators are found in bidegrees so all the differentials are zero and the spectral sequence collapses at the E_2-term. Thus,

$$E_\infty^{*,*} \cong \Lambda(x_3, x_5, \ldots, x_{2n-3}, y_{2n-1}).$$

Then,

$$H^*(SU(n); R) \cong Total E_\infty^{*,*} \cong \Lambda(x_3, x_5, \ldots, x_{2n-3}, y_{2n-1})$$

where $H^n(SU(n); R) \cong (Total E_\infty^{*,*})^n = \bigoplus_{p+q=n} E_\infty^{p,q}$ and hence the claim.

■

(2) Computation of $H^*(U(n); R)$

$U(n)$ is the group of linear transformations of $\mathbb{C}^n$ which preserve the complex inner product. Note that $U(1) \cong \mathbb{S}^1$. Consider the determinant map:

$$det : U(n) \to U(1) \cong \mathbb{S}^1$$

which gives a group homomorphism and the kernel of this homomorphism is the set of unitary matrices with determinant 1, that is $SU(n)$. So we have the following fibration

$$SU(n) \xrightarrow{inc} U(n) \xrightarrow{det} \mathbb{S}^1$$

Further, $U(n) \simeq U(1) \times SU(n) \simeq \mathbb{S}^1 \times SU(n)$. Thus,

$$\begin{aligned} H^*(U(n); R) &\cong H^*(\mathbb{S}^1 \times SU(n); R) \cong H^*(\mathbb{S}^1; R) \otimes H^*(SU(n); R) \\ &\cong \Lambda(x_1) \otimes \Lambda(x_3, x_5, \cdots, x_{2n-1}) \\ &\cong \Lambda(x_1, x_3, x_5, \cdots, x_{2n-1}). \end{aligned}$$

(3) Computation of $H^*(V_k(\mathbb{C}^n); R)$

The Stiefel manifold $V_k(\mathbb{C}^n)$ consists of the orthonormal k-frames in $\mathbb{C}^n$. One can see that $V_k(\mathbb{C}^n)$ is homeomorphic to $SU(n)/SU(n-k)$, thus we have the following fibration

$$SU(n-k) \xrightarrow{inc} SU(n) \longrightarrow V_k(\mathbb{C}^n)$$

where the map $SU(n) \to V_k(\mathbb{C}^n)$ is given by $A \mapsto (Ae_1, Ae_2, \ldots Ae_k)$ where $e_i \in \mathbb{C}^n$ denote the i^{th} elementary vector, $e_i = (0, \ldots, 0, 1, 0, \ldots 0)$ with i in the i^{th} place.

It is immediate from the definitions that $V_1(\mathbb{C}^n) = \mathbb{S}^{2n-1}$ and $V_n(\mathbb{C}^n) = SU(n)$. Now the inclusion: $SU(n-k-1) \subset SU(n-k) \subset SU(n)$ gives us the fibration:

$$0 \to SU(n-k)/SU(n-k-1) \to SU(n)/SU(n-k-1) \to SU(n)/SU(n-k) \to 0$$

which may be identified as:

$$\mathbb{S}^{2(n-k)-1} \xrightarrow{inc} V_{k+1}(\mathbb{C}^n) \longrightarrow V_k(\mathbb{C}^n).$$

Proposition 4.9 $H^*(V_k(\mathbb{C}^n); R) \cong \Lambda(x_{2(n-k)+1}, x_{2(n-k)+3}, \cdots, x_{2n-1})$.

Proof We prove this by induction on k. For $k = 1$, we have

$$H^*(V_1(\mathbb{C}^n); R) \cong H^*(\mathbb{S}^{2n-1}; R) \cong \Lambda(x_{2n-1})$$

thus the proposition holds. Now suppose it is true for k, then the Leray-Serre spectral sequence for the following fibration

$$\mathbb{S}^{2(n-k)-1} \xrightarrow{inc} V_{k+1}(\mathbb{C}^n) \longrightarrow V_k(\mathbb{C}^n)$$

has E_2-term given by

$$E_2^{*,*} = H^*(V_k(\mathbb{C}^n); H^*(\mathbb{S}^{2(n-k)-1}; R)).$$

But since the system of local coefficients is simple and applying the Universal coefficient theorem [6, Theorem 2.16] gives

$$\begin{aligned} E_2^{*,*} &= H^*(V_k(\mathbb{C}^n); R) \otimes H^*(\mathbb{S}^{2(n-k)-1}; R) \\ &\cong \Lambda(x_{2(n-k)+1}, x_{2(n-k)+3}, \ldots, x_{2n-1}) \otimes \Lambda(y_{2(n-k)-1}) \\ &\cong \Lambda(y_{2(n-k)-1}, x_{2(n-k)+1}, x_{2(n-k)+3}, \ldots, x_{2n-1}). \end{aligned}$$

Again for dimensional reasons, the spectral sequence collapses at E_2-term and so

$$E_\infty^{*,*} \cong \Lambda(y_{2(n-k)-1}, x_{2(n-k)+1}, x_{2(n-k)+3}, \ldots, x_{2n-1}).$$

Finally we have that

$$H^n(V_{k+1}(\mathbb{C}^n); R) \cong \bigoplus_{p+q=n} E_\infty^{p,q}$$

which completes the induction.

■

Remark 4.1 *We can also reproduce the cohomology ring $H^*(U(n); R)$, of the unitary group $U(n)$ by putting $k = n$ in the above computation and using $V_k(\mathbb{C}^n) \simeq U(n)/U(n-k)$.*

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