Towards A General Stochastic Integral

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Certificate of Examination

This is to certify that the dissertation titled **"Towards A General Stochastic Integral"** submitted by **Nishant Agrawal** (Reg. No. MS10079) for the partial fulfillment of BS-MS dual degree program of the Institute, has been examined by the thesis committee duly appointed by the Institute. The committee finds the work done by the candidate satisfactory and recommends that the report be accepted.

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Dated: April 22, 2015

Declaration

The work presented in this dissertation has been carried out by me under the guidance of Prof. Abhay G Bhatt at Indian Statistical Institute and Dr. Lingaraj Sahu at the Indian Institute of Science Education and Research Mohali.

This work has not been submitted in part or in full for a degree, a diploma, or a fellowship to any other university or institute. Whenever contributions of others are involved, every effort is made to indicate this clearly, with due acknowledgement of collaborative research and discussions. This thesis is a bonafide record of original work done by me and all sources listed within have been detailed in the bibliography.

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In my capacity as the supervisor of the candidates project work, I certify that the above statements by the candidate are true to the best of my knowledge.

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Nishant Agrawal

Notation

$\{\mathcal{F}_t\}$	Filtration of sub sigma algebras
$\mathcal{M}_2^c$	Space of square integrable continuous martingale
$\mathcal{M}_2$	Space of square integrable martingale
$\langle X, Y \rangle$	cross variation of X and Y. $\langle X, X \rangle = X$
$\mathcal{M}^{c,loc}$	Space of continuous local martingale
$[X]_T^2$	$E \int_0^T X_t^2 d\langle M \rangle_t$
$\mathcal{L}(M)$	measurable with $[X]_T < \infty$
$\mathcal{L}^*(M)$	Progressively measurable with $[X]_T < \infty$
$\mathcal{L}(L^2[0,T], \Omega)$	Random variables adapted and $\int_0^T E(\|X\|^2)dt < \infty$

Abstract

The aim of the project is to understand the construction of Brownian Motion and that of stochastic integral. The construction of stochastic integral with respect to martingales has been carried out rigorously. Further, the stochastic integration developed by Ito was for a nice measurable class of functions; was in 2008 expanded to a larger class by Kuo. In this project I have also studied about the extension of stochastic integration developed by Kuo recently. The idea behind the new stochastic integral has been conveyed through many examples. I have also talked about the existence and uniqueness of solutions to the stochastic differential equations which are also used to study the trajectory of a particle undergoing random motion.

Contents

Chapter 1

Martingales and Brownian Motion

In this chapter we study about Martingales, Brownian motion and properties of Brownian motion. In the next chapter we will see how we integrate measurable functions with respect to some nice martingales. We will observe that Brownian motion is also a Martingale and in many areas integration with respect to Brownian motion is also carried out. The majority of the notions in this chapter can also be referred from [1], [2].

1.1 Discrete Parameter Martingales

We consider a Probability triple $(\Omega, \mathcal{F}, \mathbb{P})$ where $\mathcal{F}$ is the sigma algebra on Ω.

Definition 1.1. ***Filtration*** *can be defined as a sequence of increasing sub σ-algebras of $\mathcal{F}$. We write it as $\mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \mathcal{F}_2 \subseteq ... \subseteq \mathcal{F}$.*

In general we will work on a filtered space $(\Omega, \mathcal{F}, \{\mathcal{F}\}_n, \mathbb{P})$ where n varies over the set of non-negative integers.

Definition 1.2. *A Stochastic Process (also written as Process) $X = \{X_n, n \geq 0\}$ is called an* ***adapted process*** *if for every n, X_n is $\mathcal{F}_n$ measurable.*

Definition 1.3. *A Process X is called a* ***martingale*** *if it satisfies the following:*

- *X is adapted relative to $(\{\mathcal{F}_n\}, \mathbb{P})$*
- $\mathbb{E}(|X_n|) < \infty$
- $\mathbb{E}(X_n|\mathcal{F}_{n-1}) = X_{n-1}. \ \forall \ n$

In above condition if we replace the equality by $\geq$ then we get a sub- martingale. If we replace it by $\leq$ then we get a super-martingale. A martingale is both a sub-martingale and a super-martingale and therefore any result which is valid for a sub-martingale or a super-martingale also holds true for a martingale.

Definition 1.4. *A map $T : \Omega \rightarrow \{1, 2....; \infty\}$ is called a stopping time if $\{T \leq n\} \in \mathcal{F}_n \; \forall \; n$.*

Definition 1.5. *If $X = \{X_n, n \geq 0\}$ is a process and T is a stopping time, then the process $X^T = \{X_{T\wedge n}, n \geq 0\}$ is the stopped process.*

Lemma 1.1. *We show that if $X = \{X_n, \mathcal{F}_n, n \geq 0\}$ is a martingale then the stopped process $X^T = \{X_{T\wedge n}, \mathcal{F}_n, n \geq 0\}$ is also a martingale.*

Proof $X_{T\wedge n}$ is one of the random variables whose index is less than or equal to n hence it is a random variables.
$X_{T\wedge n} = X_0 + \Sigma_{i=0}^{n}(X_i - X_{i-1})\chi_{i\leq T} \implies X_{T\wedge n} - X_{T\wedge n-1} = (X_n - X_{n-1})\chi_{i\leq T}$.
We take conditional expectation of above to get the desired result.

Definition 1.6. *A process $C = \{X_n, \mathcal{F}_n, n \geq 0\}$ is a* ***previsible process*** *if for every n, X_n is $\mathcal{F}_{n-1}$ measurable.*

Theorem 1 (Doob's Optional Sampling Theorem)**.** *Let X be a martingale and T a stopping time. Then X_T is integrable and $E(X_T) = E(X_0)$ in each of the following cases.*

1. *X is uniformly bounded and T is finite*
2. *$E(T) < \infty$ with bounded increments*
3. *T is bounded.*

Proof

1. $X_{T\wedge n}$ is a random variable and since X is a martingale X_T is a stopped process and a martingale. We further have that $E(X_0 - X_{T\wedge n}) = 0$. With $n \rightarrow \infty$ and using Bounded convergence theorem we have the desired result.
2. Using Dominated Convergence Theorem we get the desired result.

3. As T is bounded hence assume that $T(\omega) \leq A$. Therefore let $n = A$ and hence we have the desired result.

Theorem 2 (Doob's Decomposition). *Let $X = \{X_n, \mathcal{F}_n; n \in \mathbb{Z}^+\}$ with $X_n \in \mathcal{L}^1, \forall\, n$. Then X has a decomposition called Doob's decomposition.*

$$X_n = M_n + A_n + X_0 \;;\quad \forall n. \tag{1.1}$$

This decomposition is unique modulo indistinguishability. We further say that if X is a sub-martingale then A is an increasing process.

Proof We define $A_n = \sum_{k=1}^{n}\big(E[X_k \,|\, \mathcal{F}_{k-1}] - X_{k-1}\big)$ and $M_n = X_0 + \sum_{k=1}^{n}\big(X_k - \mathbb{E}[X_k \,|\, \mathcal{F}_{k-1}]\big)$. On proceeding with the algebraic computation we see that $E(M_n - M_{n-1}) = 0$.

For uniqueness we assume that $X_n = M_n + A_n$ and $X_n = M_n' + A_n'$ henceforth the process $Y_n = M_n - M_{n-1} = A_{n-1} - A_n$ which gives us 0 while taking conditional expectation.

To see that A_n is increasing when X_n is a sub martingale, we note that

$$\begin{aligned} E(X_n - X_{n-1}|\mathcal{F}_{n-1}) = E(M_n - M_{n-1}|\mathcal{F}_{n-1}) + E(A_{n-1} - A_n|\mathcal{F}_{n-1}) \\ = 0 + (A_{n-1} - A_n). \end{aligned}$$

Henceforth we have

$$A_n = \sum_{k=1}^{n} E(X_n - X_{n-1}|\mathcal{F}_{k-1}).$$

We further quote a result.

Theorem 3 (Martingale /Doob's Convergence Theorem). *If X is super-martingale and $\sup_n E|X_n| < \infty$ then we have almost surely $X_n \to X_\infty$ where $X_\infty = \lim_{n\to\infty} X_n$ exists and is finite.*

Proof We show that the set

$$S := \{\omega : X_n(\omega) \text{ does not converge to a limit in } [-\infty, \infty]\}$$

has measure 0. Further the notion of Up-crossing lemma is used to prove the theorem.

1.2 Continuous Parameter Martingales

In this section we study about the continuous parameter martingales where the parameter $t \in \mathbb{R}^+ \cup \{0\}$ against the natural numbers as observed in the previous section.

Definition 1.7. *A Process X is called a **martingale** if it satisfies the following:*

- *X is adapted relative to $(\{\mathcal{F}_t\}, \mathbb{P})$*
- $\mathbb{E}(|X_t|) < \infty$
- $\mathbb{E}(X_t|\mathcal{F}_s) = X_s \qquad ; \ s \leq t$

Here t varies over the set non negative real numbers.

Definition 1.8. *A filtration $\{\mathcal{F}_t\}$ is said to satisfy usual conditions if $\mathcal{F}_0$ contains all the null sets and the filtration is right continuous.*

We say that the filtration is right continuous if $\mathcal{F}_t = \mathcal{F}_{t+}$ where $\mathcal{F}_{t+} = \bigcap_{\epsilon>0} \mathcal{F}_{t+\epsilon}$

Definition 1.9. *The stochastic process X is called **progressively measurable** with respect to the filtration $\mathcal{F}_t$ if, for each $t \geq 0$ and $A \in \mathcal{B}(\mathbb{R}^d)$, the set $\{(s,\omega); 0 \leq s \leq t, \omega \in \Omega, X_s(\omega) \in A\}$ belongs to the product σ-field $\mathcal{B}([0,t]) \otimes \mathcal{F}_t$. In other words the mapping $(s,\omega) \to X_s(\omega) : ([0,t] \times \Omega, \mathcal{B}([0,t]) \otimes \mathcal{F}_t) \to (\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ is measurable for each $t \geq 0$.*

Definition 1.10. *A random T is a stopping time of the filtration $\mathcal{F}_t$, if the event $\{T \leq t\} \in \mathcal{F}_t \ \forall \ t$. It is called optional time if $\{T < t\} \in \mathcal{F}_t \ \forall \ t$.*

Theorem 4 (Optional Sampling Theorem)**.** *Let $\{X_t, \mathcal{F}_t; 0 \leq t < \infty\}$ be a right continuous submartingale with a last element X_∞, and let S and T be two optional times of the filtration $\{\mathcal{F}_t\}$. We have*

$$E(X_T|\mathcal{F}_{S+}) \geq X_S \quad a.s\ \mathbb{P}.$$

If S is a stopping time, then $\mathcal{F}_S$ can replace $\mathcal{F}_{S+}$ in the above inequality. In particular we have

$$EX_T \geq EX_0.$$

Definition 1.11. *An adapted process A is called **increasing** if for $\mathbb{P}$ a.e $\omega \in \Omega$ we have*

- $A_0(\omega) = 0$
- $A_t(\omega)$ *is non-decreasing and right continuous in t.*

Definition 1.12. *An increasing process A is called **natural** if for every bounded and right continuous martingale $\{M_t, \mathcal{F}_t; 0 \leq t < \infty\}$ we have*

$$E \int_{(0,t]} M_s dA_s = E \int_{(0,t]} M_{s-} dA_s \quad \forall \;\; t \in (o, \infty).$$

Definition 1.13. *The right continuous process $X = \{X_t, \mathcal{F}_t; 0 \leq t < \infty\}$ is said to be of class S if the family $\{X_T\}_{T \in \varsigma_a}$ is uniformly integrable with $0 < a < \infty$. Here ς_a is a class of all stopping times for the filtration $\mathcal{F}_t$ with $\mathbb{P}(T \leq a) = 1$ for some fixed $a < \infty$.*

Theorem 5 (Doob-Meyer Decomposition)**.** *Let $\{\mathcal{F}_t\}$ satisfy the usual conditions. Let $\{X_t, \mathcal{F}_t; 0 \leq t < \infty\}$ be a right continuous sub-martingale in class S, then it admits the decomposition*

$$X_t = M_t + A_t$$

where $M = \{M_t, \mathcal{F}_t; 0 \leq t < \infty\}$ is a right continuous martingale and $A = \{A_t, \mathcal{F}_t; 0 \leq t < \infty\}$ an increasing process. If the process A is natural then the decomposition is unique.

Definition 1.14. *Let $X = \{X_t, \mathcal{F}_t, 0 \leq t < \infty\}$ be a right continuous (continuous) martingale. We say that X is a square integrable martingale if*

$$EX_t^2 < \infty \qquad \forall \;\; t \geq 0.$$

If also $X_0 = 0$ then $X \in \mathcal{M}_2$ ($X \in \mathcal{M}_2^c$).

Definition 1.15. *If $X \in \mathcal{M}_2$, we define quadratic variation of X to be the process $\langle X \rangle_t = A_t$, where A is a natural increasing process.*

We further define the cross variation of two process in $\mathcal{M}_2$.

Definition 1.16. *For any two martingales $X, Y \in \mathcal{M}_2$, we define their cross variation process $\langle X, Y\rangle$ by*

$$\langle X, Y\rangle =: \frac{1}{4}[\langle X+Y\rangle_t - \langle X-Y\rangle_t]; \qquad 0 \le t < \infty. \tag{1.2}$$

Remark 1.1. *$\langle .,.\rangle$ is a bilinear form. Also*

$$E[(X_t - X_s)(Y_t - Y_s)] = E[(X_tY_t - X_sY_s)|\mathcal{F}_s] = E[\langle X_tY_t\rangle - \langle X_sY_s\rangle|\mathcal{F}_s]$$

as $XY - \langle X, Y\rangle$ is a martingale.

We state a few results which will be used in the proof of Ito's formula in the next chapter.

Definition 1.17. *For a stochastic process we define the pth variation on the interval $[0,t]$ with the partition $\Xi = \{0 = t_0, t_1, ..., t_n = t\}$ by*

$$V_t^{(p)}(\Xi) = \sum_{i=1}^{n} |X_{t_i} - X_{t_{i-1}}|^p.$$

If p=2 then we call it as quadratic variation.

Proposition 1.1. *If $X \in \mathcal{M}_2$ with $|X_s| \le C < \infty \quad \forall \quad s \in [0,t] \quad$ a.s $\mathbb{P}$. If Ξ be a partition with $t_0 \le t_1 \le .... \le t_n$ then we have*

$$E[V_t^{(2)}(\Xi)]^2 \le 6K^4.$$

Proof Consider $E[\sum_{i=m+1}^{n} |X_{t_i} - X_{t_{i-1}}|^2|\mathcal{F}_{t_m}]$ then using martingale property we get

$$E[\sum_{i=m+1}^{n} (X_{t_i} - X_{t_{i-1}})^2|\mathcal{F}_{t_m}] = E[\sum_{i=m+1}^{n} (X_{t_i}^2 - X_{t_{i-1}}^2)|\mathcal{F}_{t_m}] \le E[X_{t_n}^2|\mathcal{F}_{t_m}] \le K^2$$

therefore,

$$E[\sum_{j=1}^{n-1} \sum_{i=m+1}^{n} (X_{t_i} - X_{t_{i-1}})^2 (X_{t_j} - X_{t_{j-1}})^2] \le K^2 E[\sum_{i=m+1}^{n} (X_{t_i} - X_{t_{i-1}})^2] \le K^4.$$

Further we have

$$E[\sum_{i=1}^{n}(X_{t_i}-X_{t_{i-1}})^4] \le K^2 E[\sum_{i=1}^{n}(X_{t_i}-X_{t_{i-1}})^2] \le 4K^4.$$

Hence due to above inequalities we have

$$E[V_t^{(2)}(\Xi)]^2 \le 6K^4.$$

Lemma 1.2. *If $X \in \mathcal{M}_2^c$ is bounded uniformly by K. Then for partitions Ξ of [0,t], we have*

$$\lim_{\|\Xi \to \infty\|} EV_t^{(4)} = 0.$$

Proof We consider $m_t(X;\delta) := \sup\{|X_p - X_q|; p,q \le t, |p-q| < \delta\}$. We now write

$$V_t^{(4)} \le V_t^{(2)}.m_t(X,\delta).$$

We further proceed using Holder's inequality to obtain the desired result.

Theorem 6. *Let $X \in \mathcal{M}_2^c$. For partitions Ξ we have*

$$\mathbb{P}[|V_t^{(2)}(\Xi) - \langle X\rangle_t| > \lambda] < \epsilon.$$

Proof We divide the proof in two parts

Case 1: X is bounded

$$\begin{aligned} E[V_t^{(2)}(\Xi) - \langle X\rangle_t]^2 &= \sum_{i=1}^{n} E[\{(X_{t_i}-X_{t_{i-1}})^2 - (\langle X\rangle_{t_i} - \langle X\rangle_{t_{i-1}})\}]^2 \\ &\le 2\sum_{i=1}^{n} E[(X_{t_i}-X_{t_{i-1}})^4 + (\langle X\rangle_{t_i} - \langle X\rangle_{t_{i-1}})^2] \\ &\le 2EV_t^{(4)}(\Xi) + 2E[\langle X\rangle_t m_t(\langle X\rangle, \Xi)] \end{aligned}$$

Using the above lemma we get the desired result.

Case 2: X is unbounded In this case we use the technique of localization by defining a sequence stopping time

$$T_n = \inf\{t \geq 0; |X| > n \ \ or \ \ \langle X \rangle_t \geq n\}$$

with this $X_t^{(n)} = X_{t \wedge T_n}$ and $X_{t \wedge T_n}^2 - \langle X \rangle_{t \wedge T_n}$ is a martingale. With the help of this and in limit $T_n \to \infty$ we get the desired result.

Definition 1.18. *Let $X = \{X_t, \mathcal{F}_t, 0 \leq t < \infty\}$ be a (continuous) process. If there exist a non decreasing sequence $\{T_n\}_{n=1}^{\infty}$ such that $T_0 = 0$ and $\lim_{n \to \infty} T_n = \infty$ of stopping time of $\{\mathcal{F}_t\}$ such that $\{X_t^{(n)} := X_{t \wedge T_n}, \mathcal{F}_t, 0 \leq t < \infty\}$ is a martingale for each $n \geq 1$, then we say that X is a (continuous)* ***local martingale****. If in addition $X_0 = 0$ a.s we write $X(\mathcal{M}^{c,loc}) \in \mathcal{M}^{loc}$.*

Remark 1.2. *If X is a sub-martingale then by optional sampling theorem we get $\{X_{T \wedge t}, \mathcal{F}_t, 0 \leq t < \infty\}$ is also a sub-martingale implying the fact that every martingale is a local martingale.*

Lemma 1.3. *Let $X, Y \in \mathcal{M}^{c,loc}$. Then there is a unique adapted, continuous process of bounded variation $\langle X, Y \rangle$ satisfying $\langle X, Y \rangle_0 = 0$ a.s $\mathbb{P}$, such that $XY - \langle X, Y \rangle \in \mathcal{M}^{c,loc}$.*

For $X, Y \in \mathcal{M}^{c,loc}$ we say that the process $\langle X, Y \rangle$ is the **cross variation of X,Y** and $\langle X \rangle$ the **quadratic variation** of X.

1.3 Brownian Motion

Definition 1.19. *An adapted process $W = \{W_t, \mathcal{F}_t, t \geq 0\}$ is said to be a standard* ***Brownian Motion*** *if it satisfies following*

- *$W_0 = 0$ and $\mathcal{W}_\sqcup$ is continuous in t.*
- *Increments are independent, i.e $W_t - W_s$ is independent of $\mathcal{F}_s$.*
- *The increments $W_t - W_s$ are normally distributed with mean 0 and variance $t - s$.*

As Brownian Motion is continuous hence the canonical underlying space which we consider is the $C[0, \infty)$ space. We try to build a measure such that the coordinate mapping process of the form $W_t(\omega) = \omega(t)$ becomes a Brownian motion.

1.3.1 Construction of Brownian Motion

We define a metric on the space of real valued continuous functions on $[0, \infty)$ by

$$\psi(x_1, x_2) = \sum_{n=1}^{\infty} \frac{1}{2^n} \max_{0 \le t \le n} (|x_1(t) - x_2(t)| \wedge 1)$$

due to which the space under this metric is complete and separable.

Definition 1.20.
- *Let $\{\mathbb{P}_n\}_{n=1}^{\infty}$ be a sequence of probability measures on $(S, \mathfrak{B}(S))$, and let $\mathbb{P}$ be another measure on this space. We say $\{\mathbb{P}_n\}_{n=1}^{\infty}$* ***converges weakly*** *to $\mathbb{P}$ iff*

$$\lim_{n \to \infty} \int_S f(s) d\mathbb{P}_n(s) = \int_S f(s) d\mathbb{P}$$

for every bounded, continuous real valued function f on S.

- *Let $\{X_n\}_{n=1}^{\infty}$ be defined on $\{\Omega_n, \mathcal{F}_n, \mathbb{P}_n\}_{n=1}^{\infty}$ with values in (S, ρ). We say $\{X_n\}_{n=1}^{\infty}$ converges to X in distribution, if the sequence of measures $\{\mathbb{P}_n X_n^{-1}\}_{n=1}^{\infty}$ converges weakly to $\{\mathbb{P}X^{-1}\}$. Equivalently X_n converges in distribution to X iff*

$$\lim_{n \to \infty} E_n f(X_n) = Ef(X)$$

for every bounded, continuous real valued function f on S.

Definition 1.21. *Let (S, ρ) be a metric space and let Π be a family of probability measures on $(S, \mathfrak{B}(S))$. We say that Π is* ***relatively compact*** *if every sequence of elements of Π contains a weakly convergent subsequence. We say that Π is* ***tight*** *if for every $\epsilon > 0$, there exist a compact set $K \subseteq S$ such that $\mathbb{P}(K) \ge 1 - \epsilon$, for every $\mathbb{P} \in \Pi$.*

We now need a few results to arrive at Wiener space. Since we will be be talking about the $C[0, \infty)$ which is complete and separable space under the given metric, we take a note of next result due to Prohorov.

Theorem 7. *Let Ξ be a family of probability measures on a separable and complete metric space then Ξ it is tight iff it is relatively compact.*

We define the modulus of continuity essential for the next result.

$$m^T(\omega, \delta) =: \max_{|s-t|<\delta;\ s,t\le T} |\omega(s) - \omega(t)|$$

Theorem 8. *A sequence $\{\mathbb{P}_n\}_{n=1}^{\infty}$ of Probability measures on $C[0,\infty)$, $\Big(\mathfrak{B}(C[0,\infty))\Big)$ is tight iff*

$$\lim_{\lambda\uparrow\infty} \sup_{n\ge 1} \mathbb{P}_n[\omega; |\omega(0)| > \lambda] = 0 \qquad (*)$$

$$\lim_{\delta\downarrow\infty} \sup_{n\ge 1} \mathbb{P}_n[\omega; m^T(\omega,\delta) > \epsilon] = 0 \qquad \forall T > 0, \epsilon > 0 \ \ (**)$$

To prove this result we in turn use a result stated below.

Lemma 1.4. *The set $S \subseteq C[0,\infty)$ has a compact closure iff following are satisfied*

$$\sup_{\omega\in S} |\omega(0)| < \infty$$

$$\lim_{\delta\to 0} \sup_{\omega\in S} m^T(\omega,\delta) = 0 \quad \forall\ T > 0.$$

Theorem 9. *Let $\{X^{(n)}\}_{n=1}^{\infty}$ be a tight sequence of continous processes with the property that whenever $0 \le t_1 < ..... < t_d < \infty$, then the sequence of random vectors $\{X_{t_1}^{(n)}, ..., X_{t_d}^{(n)}\}_{n=1}^{\infty}$ converges in distribution as $n \to \infty$. Let $\mathbb{P}_n$ be the measure induced on $\{C[0,\infty), \mathfrak{B}(C[0,\infty)\}$ by $X^{(n)}$. Then $\{\mathbb{P}_n\}_{n=1}^{\infty}$converges weakly to $\mathbb{P}$.*

Define

$$X_t^{(n)} = \frac{1}{\sigma\sqrt{n}} Y_{nt} \quad \text{where} \quad Y_t = S_{[t]} + (t - [t]), \qquad t \ge 0. \tag{1.3}$$

With $\{X^{(n)}\}$ as defined above (1.3) and $0 \le t_1 < ... < t_d < \infty, d \ge 1$ we have

$$\{X_{t_1}^{(n)}, ..., X_{t_d}^{(n)}\} \xrightarrow{\mathcal{D}} (W_{t_1}, ..., W_{t_d}); \qquad n \to \infty$$

where $\{W_t, \mathcal{F}_t^B; t \ge 0\}$ is a standard one dimensional Brownian Motion.

Theorem 10 (Donsker Invariance Principle)**.** *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a Probability Space on which sequence $\{\xi_j\}$ of i.i.d random variable with 0 mean and finite variance $\sigma^2 > 0$ is given. Define $X^{(n)} = \{X_t^{(n)}; t \ge 0\}$ as in (1.3) above mentioned and let $\mathbb{P}_n$ be the measure induced by $X^{(n)}$ on $\{C[0,\infty), \mathfrak{B}(C[0,\infty)\}$. Then $\{\mathbb{P}_n\}_{n=1}^{\infty}$ converges weakly to $\mathbb{P}_*$ under which the coordinate mapping process $W_t(\omega) = \omega(t)$ on $C[0,\infty)$ is a standard, one dimensional Brownian Motion.*

Definition 1.22. *The probability measure* $\mathbb{P}_*$ *on* $\{C[0,\infty), \mathfrak{B}(C[0,\infty)\}$, *under which the coordinate mapping process* $W_t(\omega) = \omega(t)$ *is a standard, one dimensional Brownian motion is called* ***Wiener measure***.

1.3.2 Properties

$W = \{W_t, \mathcal{F}_t; 0 \leq t < \infty\}$ is a standard Brownian Motion.

1. **Markov Property:** The property of the Brownian motion that it has stationary and independent increments makes it a Markov Process.

2. **Martingale Property: Proof**

$$\begin{aligned} E[W_t|\mathcal{F}_s] &= E[W_t - W_s + W_s|\mathcal{F}_s] \\ &= E[W_t - W_s|\mathcal{F}_s] + E[W_s|\mathcal{F}_s] \\ &= E[W_t - W_s] + W_s = W_s. \end{aligned}$$

3. **Scaling** : $X = \{X_t, \mathcal{F}_{c^2t}; 0 \leq t < \infty\}$ for $c > 0$ defined by

$$X_t = \frac{1}{c} W_{c^2t} \qquad 0 \leq t < \infty.$$

Proof Continuity and stationary increments are preserved. We note that

$$\begin{aligned} Var[X_t - X_s] = Var[c^{(-1)}(W(c^2t) - W(c^2s))] &= c^{(-2)}(c^2t - c^2s) \\ &= t - s. \end{aligned}$$

The expectation is 0. Here $X_t - X_s = c^{(-1)}(W(c^2t) - W(c^2s))$ is distributed as $cN(0, c^2(t-s)) \sim N(0,(t-s))$.

4. **Time inversion** $Y = \{Y_t, \mathcal{F}_t^Y; 0 \leq t < \infty\}$ defined by

$$Y_t = \begin{cases} tW_{1/t} & 0 < t < \infty \\ 0 & t = 0 \end{cases}$$

Proof Here we see that the new process is continuous and 0 at origin. We have $E[Y_t] = tE[W_{1/t}] = 0$ and Y_t is a Gaussian process also we have the covariance function $E[Y_sY_t] = st(\frac{1}{s} \wedge \frac{1}{t}) = s \wedge t$.

5. **Symmetry:**

   Proof If W is a Brownian motion so is -W as continuity and stationary increments are preserved. mean and variance are not affected by the negative sign. The distribution (can be seen with the help of probability law) does not change.

6. **Finite Quadratic variation** Let $\{\Pi_n\}_{n=1}^{\infty}$ be a sequence of partitions of the interval [0,t] with $\lim_{n\to\infty} \|\Pi_n\| = 0$.

$$\sum_{i=1}^{n}[(W_{t_i} - W_{t_{i-1}})^2 - (t_i - t_{i-1})] = \sum_{i=1}^{n} X_i$$

with

$$X_i = (W_{t_i} - W_{t_{i-1}})^2 - (t_i - t_{i-1})$$

We have $E(X_iX_j) = 0$ for $i \neq j$ since the increments are independent; also $E[(W_{t_i} - W_{t_{i-1}})^2] = t_i - t_{i-1}$. We also see using computations that $E[(W_t - W_s)^4] = 3(t-s)^2$

$$E(X_i^2) = E\{(W_{t_i} - W_{t_{i-1}})^4 - 2t(W_{t_i} - W_{t_{i-1}})^2 - t^2\}$$
$$= 3\sum_{k=1}^{n}(t_j - t_{j-1})^2 + 2\sum_{1\leq j<k\leq n}(t_j - t_{j-1})(t_k - t_{k-1}) - t^2$$
$$= 2\sum_{k=1}^{n}(t_k - t_{k-1})^2$$
$$\leq 2t \mid \Delta_n[0,t] \mid \to_{n\to+\infty} 0.$$

7. For almost every $\omega \in \Omega$, the sample path $W(\omega)$ is **monotone in no interval**

   Proof Our idea is to show that the measure of the set

$$S = \{\omega \in \Omega; W(\omega) \text{ is non decreasing on } [0,1]\}$$

is 0 and the set belongs to $\mathcal{F}$. We can write $S = \bigcap_{n=1}^{\infty} S_n$ where

$$S_n = \bigcap_{i=0}^{n-1} \{\omega \in \Omega; W_{(i+1)/n}(\omega) - W_{i/n}(\omega)\} \in \mathcal{F}.$$

has probability $\mathbb{P}(S_n) = \prod_{i=0}^{n-1} \mathbb{P}[W_{(i+1)/n}(\omega) - W_{i/n}(\omega) \geq 0] = 2^{(-n)}$. Therefore, $\mathbb{P}(S) \leq \lim_{n\to\infty} \mathbb{P}(S_n) = 0$

8. For almost every $\omega \in \Omega$, the Brownian sample path $W(\omega)$ **is nowhere differentiable.**

Proof To prove this we show the set

$$\{\omega \in \Omega;\ \text{for each } t \in [0,\infty),\ \text{either } D^+W_t(\omega) = \infty \text{ or } D_+W_t(\omega) = -\infty\}$$

has a subset of measure 1.

Chapter 2

Stochastic Integration

In this chapter we will discuss about the integrals with respect to martingales followed by integration with respect to local martingale. After this we will also see the celebrated Ito's formula which uses the previous discussed notions. We will end this chapter by looking at the Girsanov's theorem. Majority of the text in this chapter can be referred from [2].

2.1 Construction of Stochastic Integration

In calculus courses we have seen Riemann integration as area under the curve. We will now see Stochastic Integration where we integrate a progressively measurable adapted process with respect to a square integrable continuous martingale. We will deal with the class of square integrable continuous martingales and integrate appropriate X with respect to the square integrable continuous martingale.

[2] For any $X \in \mathcal{M}_2$ and $0 \leq t < \infty$ we define

$$\parallel X \parallel = \sum_{n=1}^{\infty} \frac{\parallel X \parallel_n \wedge 1}{2^n}$$

where (A)

$$\parallel X \parallel_t =: \sqrt{E(X_t^2)}.$$

With the just defined metric $\mathcal{M}_2$ is complete and $\mathcal{M}_2^c$ is a closed subspace of $\mathcal{M}_2$. We define

$$[X]_T^2 =: E\int_0^T X_t^2 d\langle M\rangle_t.$$

Definition 2.1. *Let $\mathcal{L}(M)$ denote the set of equivalence class of all measurable $\{\mathcal{F}_t\}$ adapted process X, for which $[X]_T < \infty$ for all $T > 0$. We define a metric on $\mathcal{L}$ by [X-Y], where*

$$[X] = \sum_{n=1}^{\infty} \frac{[X]_n \wedge 1}{2^n}.$$

Consider an equivalence class $\mathcal{L}^*(M)$ of progressively measurable processes having property that $[X]_T < \infty$ for all $T > 0$ and a metric in a similar way. When $\langle M\rangle_t(\omega)$ is absolutely continuous in t then the stochastic integral can be constructed for all X in $\mathcal{L}(M)$, if not then for all X in $\mathcal{L}^*(M)$.

We define a measure on $([0,\infty)\times\Omega, \mathcal{B}([0,\infty)\otimes\mathcal{F})$ by

$$\mu_M(A) = E\int_0^\infty 1_A(t,\omega) d\langle M\rangle_t(\omega).$$

Our space on which we are about to work is a subspace of the Hilbert space $\mathcal{H} = L^2([0,\infty)\times\Omega, \mathcal{B}([0,\infty)\otimes\mathcal{F},\mu)$; the space of square integrable functions. This fact can be seen that if $\{X^{(n)}\}_{n=1}^\infty$ is a convergent sequence in $\mathcal{L}^*(M)$ and it converges to X in $\mathcal{H}$ then as it is measurable and adapted it has a progressively measurable modification Y. X and Y are equivalent with respect to the measure μ_M.
We will start this section by first defining the stochastic integral for simple process..

Definition 2.2. *A process X is called a simple process if there exist a sequence of random variables $\{\xi_n\}_{n=0}^\infty$ with $\sup_{n\geq 0}|\xi_n(\omega)| \leq L < \infty$ as well as strictly increasing sequence of real numbers $\{t_n\}_{n=0}^\infty$ with $t_0 = 0$ and $\lim_{n\to\infty} t_n = \infty$, ξ_n is $\mathcal{F}_{t_n}$ measurable and*

$$X_t(\omega) = \xi_0(\omega)1_0(t) + \sum_{t=0}^{\infty} \xi_t(\omega)1_{(t_i,t_{i+1}]}(t) \qquad 0 \leq t < \infty.$$

The class of all simple processes is denoted by $\mathcal{L}_0$ and for X in $\mathcal{L}_0$ we define the martingale transform as

$$I_t(X) =: \sum_{i=1}^{n-1} \xi_i(M_{t_{i+1}} - M_{t_i}) + \xi_n(M_t - M_{t_n}) \tag{2.1}$$

$$= \sum_{i=1}^{\infty} \xi_i(M_{t\wedge t_{i+1}} - M_{t\wedge t_i}) \qquad 0 \leq t < \infty. \tag{2.2}$$

After observing the integral for simple process we would like to extend this to the class $\mathcal{L}^*(M)$. To get there we require a couple of results.

Lemma 2.1. *For every X in $\mathcal{L}(M)$ there exist a sequence $\{X^{(n)}\}_{n=0}^{\infty}$ in $\mathcal{L}^0$such that*

$$\sup_{T>0} \lim_{n\to\infty} E \int_0^T |X_t^{(n)} - X_t| dt = 0$$

We divide the proof of this lemma into three cases namely when X is continuous, X is progressively measurable and X is measurable and adapted.

Proposition 2.1. *If $\langle M \rangle_t(\omega)$ is absolutely continuous in t with respect to Lebesgue measure then $\mathcal{L}^0$ is dense in $\mathcal{L}(M)$ with respect to the metric in definition 2.1.*

The proof can be divided into the case of bounded process and unbounded process and then using the above lemma and localization technique respectively.

We can further say that $\mathcal{L}^0$ is dense in $\mathcal{L}^*(M)$ with respect to the metric in definition 2.1.

Lemma 2.2. *Let $\{A_t, \mathcal{F}_t, t \geq 0\}$ be an increasing process, $M = \{M_t, \mathcal{F}_t, t \geq 0\}$ a martingale and $X = \{X_t, \mathcal{F}_t, t \geq 0\}$ a progressively increasing process, agreeing to*

$$E \int_0^T X_t^2 dA_t < \infty.$$

Then there exist a sequence of simple process $\{X^{(n)}\}_{n=1}^{\infty}$ such that

$$\sup_{T>0} \lim_{m\to\infty} E \int_0^T |X_t^{(n)} - X_t| dA_t = 0.$$

We now have the required tools to begin with the construction of stochastic integral. The basic idea is to satisfy the below mentioned equations for the simple process and then for X $\in \mathcal{L}^*(M)$ in limiting sense.

$$I_0(X) = 0 \qquad a.s\ \mathbb{P} \tag{2.3}$$

$$I(\alpha X + \beta Y) = \alpha I(X) + \beta I(Y) \qquad \alpha, \beta \in \mathbb{R} \tag{2.4}$$

$$E[I_t(X)|\mathcal{F}_s] = I_s(X) \tag{2.5}$$

$$E(I_t(X))^2 = E\int_0^t X_u^2 d\langle M\rangle_u \tag{2.6}$$

$$\|I(X)\| = [X] \tag{2.7}$$

$$E[(I_t(X) - I_s(X))^2|\mathcal{F}_s] = E[\int_s^t X_u^2 d\langle M\rangle_u|\mathcal{F}_s] \tag{2.8}$$

We have defined through equations 2.1 the integral for simple process and using that definition we prove the above mentioned properties.

$$\text{As } \ I_t(X) = \sum_{i=1}^{i=n-1} \xi_i(M_{t_{i+1}} - M_{t_i}) + \xi_n(M_t - M_{t_n}).$$

Hence for $t = 0$ we have $I_0(X) = 0$. Therefore it satisfies equation 2.3. We now show how the integral satisfy 2.4.

$$I_t(\alpha X + \beta Y) = \sum_{i=1}^{i=n-1} (\alpha\xi_i + \beta\rho_i)(M_{t_{i+1}} - M_{t_i}) + (\alpha\xi_n + \beta\rho_n)(M_t - M_{t_n})$$

$$= \sum_{i=1}^{i=n-1} (\alpha\xi_i)(M_{t_{i+1}} - M_{t_i}) + (\alpha\xi_n)(M_t - M_{t_n}) + \sum_{i=1}^{i=n-1} (\beta\rho_i)(M_{t_{i+1}} - M_{t_i}) + (\beta\rho_n)(M_t - M_{t_n})$$

and as $\alpha, \beta \in \mathbb{R}$ equation 2.4 holds true for simple stochastic integrals.
Taking conditional expectation in equation (2.1) with respect to $\mathcal{F}_s$ we have

$$\sum_{i=1}^{\infty} E[\xi_i(M_{t\wedge t_{i+1}} - M_{t\wedge t_i})|\mathcal{F}_s] = \sum_{i=1}^{\infty} \xi_i(M_{s\wedge t_{i+1}} - M_{s\wedge t_i}) \qquad 0 \le s < t < \infty.$$

Hence we have shown that simple process satisfy equation 2.5. We can observe this by considering for cases $i < s < t$; $i < s \le i + 1 < t$ and $s \le i < t$. Hence we also see that $I(X) = \{I_t(X), \mathcal{F}_t, t \ge 0\}$ is a continuous martingale.
We proceed to prove the next property for simple integrals. Using the fact that for $p < q < r < s$ we have $E[(M_s - M_r)(M_q - M_p)] = 0$. For $s < t$ and $t_{k-1} < s < t_k$ and

$t_n < t < t_{n+1}$ we have

$$E[(I_t(X) - I_s(X))^2|\mathcal{F}_s] = E[\{\xi_{k-1}(M_{t_k} - M_s) + \sum_{i=k}^{n-1} \xi_i(M_{t_{i+1}} - M_{t_i}) + \xi_n(M_t - M_{t_n})\}^2|\mathcal{F}_s]$$
$$= E[\xi_{k-1}^2(M_{t_k} - M_s)^2 + \Sigma_{i=k}^{n-1}\xi_i^2(M_{t_{i+1}} - M_{t_i})^2 + \xi_n^2(M_t - M_{t_n})^2|\mathcal{F}_s].$$

Using the Doob - Meyer decomposition and the fact that $E[M_0] = 0$ we write the preceding equation as

$$= E[\xi_{k-1}^2(\langle M\rangle_{t_k} - \langle M\rangle_s)^2 + \sum_{i=k}^{n-1} \xi_i^2(\langle M\rangle_{t_{i+1}} - \langle M\rangle_{t_i})^2 + \xi_n^2(\langle M\rangle_t - \langle M\rangle_{t_n})^2|\mathcal{F}_s]$$
$$= E[\int_s^t X_u^2 d\langle M\rangle_u|\mathcal{F}_s]$$

which proves 2.8 and also implying that

$$\langle I(X)\rangle_t = \int_0^t X_u^2 d\langle M\rangle_u.$$

We set $s = 0$ to get $E[(I_t(X))^2] = E[\int_0^t X_u^2 d\langle M\rangle_u]$. With this the we also get equation 2.6. From (A) we get equation 2.7 as well.

For $X \in \mathcal{L}^*(M)$ by lemma 2.2 there exist a sequence $\{X^{(n)}\}_{n=1}^{\infty}$ in $\mathcal{L}^0$ such that $[X^{(n)} - X] \longrightarrow 0$ as $n \to \infty$ By 2.7 we say that $\|I(X^{(n)}) - I(X^{(k)})\| = \|I(X^{(n)} - X^{(k)})\| = [X^{(n)} - X^{(k)}]$. Therefore as $n, k \to \infty$ $[X^{(n)} - X^{(k)}] \to 0$. This helps us in concluding that $\{I(X^{(n)})\}$ forms a cauchy sequence in $\mathcal{M}_2^c$. From the discussion preceding definition 2.1 we know that $\mathcal{M}_2^c$ is closed in $\mathcal{M}_2$ and therefore the limit $I(X) = \{I_t(X); 0 \le t < \infty\}$ (say) exists and belongs to $\mathcal{M}_2^c$.

We now satisfy the properties 2.3 to 2.8 for $I(X)$ so that we can call it as a stochastic integral for any $X \in \mathcal{L}^*(M)$. Equations 2.3 and 2.5 are valid as $I(X) \in \mathcal{M}_2^c$. As equation 2.4 holds true for $X, Y \in \mathcal{L}^0$; in the limiting sense it also holds for $X, Y \in \mathcal{L}^*$. Consider $\{I(X_s^{(n)})\}$, $\{I(X_t^{(n)})\}$ converging to $I_s(X)$ and $I_t(X)$ respectively in square norm. Then for $A \in \mathcal{F}_s$

$$\begin{aligned} E[1_A(I_t(X) - I_s(X))^2] &= \lim_{n\to\infty} E[1_A(I_t(X^{(n)}) - I_s(X^{(n)}))^2] \\ &= \lim_{n\to\infty} E[1_A \int_s^t ((X_u^{(n)}))^2 d\langle M\rangle_u] \\ &= E[1_A \int_s^t ((X_u^2 d\langle M\rangle_u]. \end{aligned}$$

Showing that $I(X)$ also satisfies equation 2.8. By taking s=0 it also satisfies equation 2.6 and consequently equation 2.7.

Since X and M are progressively measurable $\int_s^t ((X_u^{(n)}))^2 d\langle M\rangle_u$ is also $\mathcal{F}_t$ measurable for fixed $s < t$ and therefore it also implies

$$\langle I(X)\rangle = \int_0^t X_u^2 d\langle M\rangle_u$$

Hence we define

Definition 2.3. *For $X \in \mathcal{L}^*$, the **stochastic integral** of X with respect to the martingale M in $\mathcal{M}_2^c$ is the unique square integrable martingale $I(X) = \{I_t(X), \mathcal{F}_t; 0 \le t < \infty\}$ (limit of cauchy sequence as discussed above) which satisfies $\|I(X^{(n)}) - I(X)\| = 0$ for every sequence $\{X^{(n)}\}_{n=1}^{\infty} \subseteq \mathcal{L}^0$ with $\lim_{n\to\infty}[X^{(n)} - X] = 0$. We write*

$$I_t(X) = \int_0^t X_s dM_s; \qquad 0 \le t < \infty \tag{2.9}$$

2.2 Integration with respect to Local Martingale

Let $M \in \mathcal{M}^{c,loc}$. We then define a class of processes.

Definition 2.4. *Let $\mathcal{P}$ be the equivalence class of measurable, adapted process $X = \{X_t, \mathcal{F}_t; 0 \le t < \infty\}$ satisfying*

$$\mathbb{P}([\int_0^T X_t^2 d\langle M\rangle_t < \infty]) = 1 \qquad \forall\ T \in [0, \infty)$$

Let $\mathcal{P}^*$ be the class of progressively measurable process agreeing to above condition.

[2] Since $M \in \mathcal{M}^{c,loc}$ hence there exists a sequence of stopping times $\{T_n\}_{n=1}^{\infty}$ such that $\lim_{n\to\infty} T_n = \infty$ and $M_{t\wedge T_n}$ is a martingale for every n. For $X \in \mathcal{P}$ we construct a sequence

$$R_n = n \wedge \inf\{0 \le t < \infty; \int_0^t X_t^2 d\langle M\rangle_t \ge n\}.$$

Note that $\lim_{n\to\infty} R_n = \infty$. Set

$$S_n = R_n \wedge T_n$$

$$M_t^{(n)}(\omega) = M_{t\wedge S_n}(\omega); \quad X_t^{(n)}(\omega) = X_t(\omega)1_{\{T_n(\omega)\ge n\}} \qquad 0 \le t < \infty$$

By constructing such a sequence we can say that $M^{(n)} \in \mathcal{M}_2^c$ and $X^{(n)} \in \mathcal{L}^*(M^{(n)})$ hence we define for $X \in \mathcal{P}^*$

$$I_t(X) = I_t^{(M^{(n)})}(X^{(n)}). \tag{2.10}$$

This process is a local martingale. By the discussion above we define

Definition 2.5. *For $M \in \mathcal{M}^{c,loc}$ and $X \in \mathcal{P}^*$ the stochastic integral of X with respect to M is $I_t(X) = \{I_t(X), \mathcal{F}_t; 0 \le t < \infty\} \in \mathcal{M}^{c,loc}$ defined by 2.10.*

We state couple of results which will be required for the next section

Proposition 2.2. *Let $M \in \mathcal{M}^{c,loc}$ and $\{X^{(n)}\}_{n=1}^{\infty} \in \mathcal{P}^*(M)$ and $X \in \mathcal{P}^*(M)$ and let for some stopping time T of $\{\mathcal{F}_t\}$ we have $\lim_{n\to\infty} \int_0^T |X_s^{(n)} - X_s| d\langle M\rangle_s = 0$ in probability. Then*

$$\lim_{n\to\infty} \sup_{0\le t\le T} |\int_0^t X_s^{(n)} dM_s - X_s dM_s| \xrightarrow{\mathbb{P}} 0. \tag{2.11}$$

Proposition 2.3. *Let $M \in \mathcal{M}^{c,loc}$ and $X \in \mathcal{P}^*(M)$. Then there exist a sequence of simple process $\{X^{(n)}\}_{n=1}^{\infty}$ such that, for every $T > 0$,*

$$\lim_{n\to\infty} \int_0^T |X_s^{(n)} - X_s| d\langle M\rangle_s = 0$$

also

$$\lim_{n\to\infty} \sup_{0\le t<T} |I_s(X^{(n)}) - I(X)| = 0$$

holds a.s $\mathbb{P}$

2.3 Ito's Formula

This Formula provides us with simple rules through which we can compute stochastic integral.

Definition 2.6. *A continuous semi martingale* $X = \{X_t, \mathcal{F}_t, 0 \leq t < \infty\}$ *is an adapted process which has a decomposition as*

$$X_t = X_0 + M_t + B_t; \qquad 0 \leq t < \infty \tag{2.12}$$

where $M = \{M_t, \mathcal{F}_t, 0 \leq t < \infty\} \in \mathcal{M}^{c,loc}$ *and* $B_t = \{B_t, \mathcal{F}_t, 0 \leq t < \infty\}$ *is a process of bounded variation i.e for every t,* $B_t = A_t^{'} - A_t^{''}$ *where* $A_t^{'},\ A_t^{''}$ *are increasing process.*

Theorem 11 (Ito's Rule)**.** *Let f be a* $C^2(\mathbb{R})$ *function. Let X be a continuous semi-martingale with above decomposition. Then*

$$f(X_t) = f(X_0) + \int_0^t f^{'}(X_s)dM_s + \int_0^t f^{'}(X_s)dB_s + \frac{1}{2}\int_0^t f^{''}(X_s)d\langle M\rangle_s \tag{2.13}$$

Proof We divide the proof into three steps

Step 1: We define the stopping time of 2.13 for the purpose of localisation

$$T_n = \begin{cases} 0 & \text{if } |X_0| \geq n \\ \inf\{t \geq 0; |M_t| \geq n \text{ or } B_t \geq n \text{ or } \langle M\rangle_t \geq n\} & \text{if } |X_0| < n \\ \infty & \text{if } |X_0| < n \text{ and } \inf\{t \geq 0; |M_t| \geq n \text{ or } B_t \geq n \text{ or } \langle M\rangle_t \geq n\} = \emptyset \end{cases} \tag{2.14}$$

Thus we have a sequence of stopping time $\{T_n\}_{n=1}^{\infty}$ with $\lim_{n\to\infty} T_n = \infty$ for 2.13. If we prove the theorem for $X_{t\wedge T_n}(\omega), M_{t\wedge T_n}(\omega), \langle M\rangle_t(\omega)$ then we will be done. We hence make a legitimate assumption that all our process are uniformly bounded by some constant $K \in [0, \infty)$. Therefore $|X_t(\omega)| \leq 3K$. We also make an assumption that f has a compact support on $[-3K, 3K]$ making $f\prime$ and $f\prime\prime$ bounded.

Step 2 : Using the Taylor expansion. We fix a $t > 0$ and consider the partition $\{t_0, t_1, \cdots, t_m\}$ with $\Delta = \{t_0 = 0 < t_1 < \cdots < t_m = t\}$. For the partition we get

$$f(X_t) - f(X_0) = \sum_{k=1}^{m} \{f(X_{t_k}) - f(X_{t_{k-1}})\}$$
$$= \sum_{k=1}^{m} f'(X_{t_k})\{X_{t_k} - X_{t_{k-1}}\} + \frac{1}{2}\sum_{k=1}^{m} f''(\eta_k)\{X_{t_k} - X_{t_{k-1}}\}^2$$

Here $\eta_k = X_{t_{k-1}}(\omega) + \theta_k(\omega)\{X_{t_k} - X_{t_{k-1}}\}$. where $0 \leq \theta_k(\omega) \leq 1$where$\omega \in \Omega$.

We write the above equation as

$$f(X_t) - f(X_0) = J_1(\Delta) + J_2(\Delta) + J_3(\Delta) \tag{2.15}$$

where

$$J_1(\Delta) = \sum_{k=1}^{m} f'(X_{t_{k-1}})\{B_{t_k} - B_{t_{k-1}}\}.$$
$$J_2(\Delta) = \sum_{k=1}^{m} f'(X_{t_{k-1}})\{M_{t_k} - M_{t_{k-1}}\}.$$
$$J_3(\Delta) = \sum_{k=1}^{m} f''(\eta_k)\{X_{t_k} - X_{t_{k-1}}\}^2.$$

Step 3 : Convergence of J's

We start with $J_1(\Delta)$, which can be also realized as a Lebesgue-Stieltjes integral hence this converges to $\int_0^t f'(X_s)dB_s$ almost surely as the size of partition tends to 0 (or size of mesh goes to 0).

We next take $J_2(\Delta)$ and observe that $f'(X_{t_{k-1}})$ is an adapted, bounded and continuous. We consider a simple process

$$Y_s^{(\Delta)} = f'(X_0(\omega)1_{\{0\}})(s) + \sum_{k=1}^{m} f'(X_{t_{k-1}}(\omega))1_{(t_{k-1},t_k]}(s).$$

Hence due to bounded convergence theorem we have

$$EI_t^2(Y^\Delta - Y) = E\int_0^t |Y_s^\Delta - Y_s|^2 d\langle M\rangle_s \longrightarrow 0$$

as the size of partition tends to 0. Therefore as $I_t^2(Y^\Delta - Y) \longrightarrow 0$ in quadratic mean we get $J_2(\Delta) \longrightarrow \int_0^t Y_s dM_s$

Finally we consider $J_3(\Delta)$ and write it as

$$J_3(\Delta) = J_a(\Delta) + J_b(\Delta) + J_c(\Delta)$$

where

$$J_a(\Delta) = \sum_{k=1}^{m} f''(\eta_k)\{B_{t_k} - B_{t_{k-1}}\}^2$$

$$J_b(\Delta) = \sum_{k=1}^{m} f''(\eta_k)\{B_{t_k} - B_{t_{k-1}}\}\{M_{t_k} - M_{t_{k-1}}\}$$

$$J_c(\Delta) = \sum_{k=1}^{m} f''(\eta_k)\{M_{t_k} - M_{t_{k-1}}\}^2$$

As the total variation of B is bounded by K we have for

$$J_a(\Delta) + J_b(\Delta) \leq 2K\|f''\|_\infty(\max_{1\leq k\leq m} |B_{t_k} - B_{t_{k-1}}| + \max_{1\leq k\leq m} |M_{t_k} - M_{t_{k-1}}|)$$

Due to continuity of B and the bounded convergence theorem this converges to 0.

We now consider

$$J_c^* = \Sigma_{k=1}^m f''(X_{t_{k-1}})\{M_{t_k} - M_{t_{k-1}}\}^2$$

We observe using Theorem 6 in previous chapter that

$$|J_c - J_c^*| \leq V_t^2(\Delta) \cdot \max_{1\leq k\leq m} |f''(\eta_k) - f''(X_{t_{k-1}})|$$

Hence by the proposition 1.1 in previous chapter we have

$$\begin{aligned} E|J_c(\omega) - J_c^*(\omega)| &\leq E(V_t^2(\Delta) \cdot \max_{1\leq k\leq m} |f''(\eta_k) - f''(X_{t_{k-1}})|) \\ &\leq \sqrt{6K^4}\sqrt{E(\max_{1\leq k\leq m} |f''(\eta_k) - f''(X_{t_{k-1}})|)} \end{aligned}$$

This converges to zero as the size of mesh decreases to 0. This is due to the fact that X is continuous. From here we focus on the term

$$J_d(\omega) = \sum_{k=1}^{m} f''(X_{t_{k-1}})\{(\langle M\rangle_{t_k} - \langle M\rangle_{t_{k-1}})\}.$$

$$\begin{aligned}
E|J_c^*(\omega) - J_d(\omega)|^2 &= E|\sum_{k=1}^{m} f''(X_{t_{k-1}})\{(M_{t_k} - M_{t_{k-1}})^2 - (\langle M\rangle_{t_k} - \langle M\rangle_{t_{k-1}})\}|^2 \\
&= E|\sum_{k=1}^{m}[f''(X_{t_{k-1}})]^2\{(M_{t_k} - M_{t_{k-1}})^2 - (\langle M\rangle_{t_k} - \langle M\rangle_{t_{k-1}})\}^2| \\
&\leq 2\|f''\|_\infty^2 \cdot E[\sum_{k=1}^{m}(M_{t_k} - M_{t_{k-1}})^4 + \sum_{k=1}^{m}(\langle M\rangle_{t_k} - \langle M\rangle_{t_{k-1}})^2] \\
&\leq 2\|f''\|_\infty^2 \cdot E[(V_t^{(4)}(\Delta)) + \langle M\rangle_t \max_{1\leq k\leq m}(\langle M\rangle_{t_k} - \langle M\rangle_{t_{k-1}})]
\end{aligned}$$

by lemma 1.2 $E[(V_t^{(4)}(\Delta)] \longrightarrow 0$ as size of mesh goes to 0. Hence we have shown the convergence in L^2 which implies convergence in L^1 which lets us conclude that

$$\lim_{\|\Delta\|\to 0} J_3(\Delta) = \lim_{\|\Delta\|\to 0} J_c(\Delta) = \lim_{\|\Delta\|\to 0} J_c^*(\Delta) = \lim_{\|\Delta\|\to 0} J_d(\Delta) = \int_0^t f''(X_s)d\langle M\rangle_s$$

Further if we have a sequence of partitions $\{\Delta_{(n)}\}_{n=1}^\infty$ of $[0,t]$ such that $\|\Delta_{(n)}\| \longrightarrow 0$ then for some subsequence $\{\Delta_{(n_k)}\}_{k=1}^\infty$ we have

$$\begin{aligned}
\lim_{k\to\infty} J_1(\Delta_{(n_k)}) &= \int_0^t f'(X_s)dB_s \quad a.s \\
\lim_{k\to\infty} J_2(\Delta_{(n_k)}) &= \int_0^t f'(X_s)dM_s \quad a.s \\
\lim_{k\to\infty} J_3(\Delta_{(n_k)}) &= \int_0^t f''(X_s)d\langle M\rangle_s \quad a.s
\end{aligned}$$

Hence in the limiting sense we see that (2.16) holds. □

This result can be generalized for a d-dimensional local martingale also.

Remark 2.1. *Let $f(t,x) : [0,\infty) \times \mathbb{R} \longrightarrow \mathbb{R}$ be of class $C^{1,2}(\mathbb{R})$ function. Let X be a continuous semi-martingale with similar decomposition. Then*

$$f(t,X_t) = f(0,X_0)+\int_0^t \frac{\delta}{\delta t} f(s,X_s)ds+\int_0^t \frac{\delta}{\delta x} f(t,X_s)dM_s+\int_0^t \frac{\delta}{\delta x} f(t,X_s)dB_s+\frac{1}{2}\int_0^t \frac{\delta^2}{\delta t^2} f(t,X_s)d\langle M\rangle_s. \tag{2.16}$$

The proof of this much like proof of Ito's formula (above). In step 2 we apply Taylor's expansion to

$$f(t_k,X_{t_k}) - f(t_{k-1}X_{t_{k-1}}) = [f(t_k,X_{t_k}) - f(t_{k-1},X_{t_k})] + [f(t_{k-1},X_{t_k}) - f(t_{k-1},X_{t_{k-1}})].$$

Example 1 In the Ito's Formula if $f(x) = x^2$ and $B_t = 0\ \forall t$ and M = W = Brownian Motion then we have

$$W_t^2 = 2\int_0^t W_s dW_s + t.$$

Example 2 Let $M = W$ a standard brownian motion and $X \in \mathcal{P}$. We define

$$\zeta_t^s(X) := \int_s^t X_u dW_u - \frac{1}{2}\int_s^t X_u^2 du$$

The process $\{\exp(\zeta_t(X)), \mathcal{F}_t; 0 \le t < \infty\}$ is a super martingale.
We see here that $\exp(\int_0^t X_u dW_u)$ is a martingale since the process $\{\int_0^t X_u dW_u, \mathcal{F}_t; t \geq 0\}$ is a martingale and $\exp(x)$ is a continuous function. Also

$$\frac{1}{2}\int_0^t X_u^2 du > \frac{1}{2}\int_0^s X_u^2 du$$

for $0 \le s < t < \infty$.
We further see that by Ito's rule that process $\{\exp(\zeta_t(X)), \mathcal{F}_t; 0 \le t < \infty\}$ satisfies the stochastic integral equation

$$Z_t = 1 + \int_0^t Z_s X_s dW_s; \qquad 0 \le t < \infty$$

where $Z_t = \exp(\zeta_t(X))$. $\{\zeta_t(X), \mathcal{F}_t; 0 \le t < \infty\}$ is a semimartingale with $M_t = \int_s^t X_u dW_u$ as a local martingale part and $B_t = -\frac{1}{2}\int_s^t X_u^2 du$ as bounded variation part. To apply the Ito's formula we take $f(x) = \exp(x)$ hence we have

$$
\begin{aligned}
Z_t = f(\zeta_t) = f(\zeta_0) + \int_0^t f'(\zeta_s)dM_s + \int_0^t f'(\zeta_s)dB_s + \frac{1}{2}\int_0^t f''(\zeta_s)d\langle M\rangle_s \\
= 1 + \int_0^t Z_sX_sdW_s + \int_0^t Z_s(-\frac{1}{2}X_s^2)ds + \frac{1}{2}\int_0^t Z_sX_s^2ds \\
= 1 + \int_0^t Z_sX_sdW_s
\end{aligned}
$$

Note: We have a **martingale characterisation theorem** which states that if X is a continuous local martingale with $\{X_0 = 0\}$, then, the following are equivalent.

- X is standard Brownian motion on the underlying filtered probability space.
- X has quadratic variation $\{[X]_t = t\}$.

2.4 Girsanov's theorem

In this section we show that if we have a Brownian motion on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ then we construct a new probability measure under which a translated process becomes a brownian motion.
We consider a d-dimensional brownian motion $W = \{W_t, \mathcal{F}, 0 \le t < \infty\}$ on $(\Omega, \mathcal{F}, \mathbb{P})$.
We take a adapted d-dimensional stochastic process $X = \{X_t = \{X_t^1, X_t^2, ...X_t^d\}, \mathcal{F}_t, 0 \le t < \infty$ process satisfying

$$\mathbb{P}[\int_0^T (X_t^{(i)})^2 < \infty] = 1. \tag{2.17}$$

Consider

$$Z_t(X) := \exp(\sum_{i=1}^d \int_s^t X_u dW_u - \frac{1}{2}\int_s^t X_u^2 du) \tag{2.18}$$

it is a local martingale by Example 2 of previous section.
As seen in example 2.3 we have this in the form

$$Z_t = 1 + \int_0^t \sum_{i=1}^d Z_sX_sdW_s; \qquad 0 \le t < \infty$$

Using the Radon -Nikodym theorem we have an absolutely continuous measure given by

$$\tilde{\mathbb{P}} = \int_A Z_T(X) \qquad A \in \mathcal{F}_T$$

We use a couple of results before proving the main result of this section. In these we assume that $Z(X)$ is a martingale.

Lemma 2.3. *For a $\mathcal{F}_t$ measurable random variable Y satisfying $E_T|Y| < \infty$, where $0 < s \leq t \leq T$ is a non-negative but fixed constant, then we have the Bayes Rule*

$$\tilde{E}_T[Y|\mathcal{F}_s] = \frac{1}{Z_s(X)} E[YZ_t(X)|\mathcal{F}_s] \quad \text{almost surely } \mathbb{P},\ \tilde{\mathbb{P}}. \tag{2.19}$$

Proof

$$\begin{aligned} &\tilde{E}_T\{1_A \frac{1}{Z_s(X)} E[YZ_t(X)|\mathcal{F}_s]\} \\ &= E\{1_A E[YZ_t(X)|\mathcal{F}_s]\} = E[1_A Y Z_t(X)] = \tilde{E}_T[1_A Y]. \end{aligned}$$

Our main result will is a corollary of the next proposition.

Proposition 2.4. *With a fixed non-negative T if $M \in \mathcal{M}_T^{c,loc}$ then*

$$\tilde{M} := M_t - \sum_{i=1}^{d} \int_0^t X_s^{(i)} d\langle M, W^{(i)} \rangle_s \quad \mathcal{F}_s; \qquad 0 \leq t \leq T \tag{2.20}$$

is in $\tilde{\mathcal{M}}_T^{c,loc}$. If $N \in \mathcal{M}_T^{c,loc}$ and

$$\tilde{N} := M_t - \sum_{i=1}^{d} \int_0^t X_s^{(i)} d\langle N, W^{(i)} \rangle_s \quad \mathcal{F}_s; \qquad 0 \leq t \leq T \tag{2.21}$$

then

$$\langle \tilde{M}, \tilde{N} \rangle_t = \langle M, N \rangle_t; \quad 0 \leq t \leq T \ a.s \ \ \mathbb{P} \ \ and \ \ \tilde{\mathbb{P}}_T.$$

Here we say that $\mathcal{M}_T^{c,loc}$ is the class of continuous local martingale on $(\Omega, \mathcal{F}, \mathbb{P})$ with $\mathbb{P}[M_0 = 0] = 1$ and similarly the class $\tilde{\mathcal{M}}_T^{c,loc}$ on $(\Omega, \mathcal{F}, \tilde{\mathbb{P}})$. In this proof we use a few things.

Proposition 2.5 (Kunita Watanabe Inequality)**.** *If* $M, N \in \mathcal{M}_2^c$ *and* $X \in \mathcal{L}^*(M)$ *and* $Y \in \mathcal{L}^*(N)$ *then*

$$\int_0^t |X_s Y_s| d\langle M, N\rangle \leq (\int_0^t X_s^2 d\langle M\rangle_s)^{1/2}(\int_0^t Y_s^2 d\langle N\rangle_s)^{1/2} \tag{2.22}$$

Lemma 2.4 (Integration by Parts)**.** *For X, Y semi martingales we have*

$$\int_0^t X_s dY_s = X_t Y_t - X_0 Y_0 - \int_0^t Y_s dX_s - \langle X, Y\rangle \tag{2.23}$$

Proof

$$(X_t - Y_t)^2 - (X_0 - Y_0)^2 = 2\int_0^t (X_s - Y_s)d(X_s - Y_s) - \langle X - Y\rangle_t$$
$$(X_t + Y_t)^2 - (X_0 + Y_0)^2 = 2\int_0^t (X_s + Y_s)d(X_s + Y_s) - \langle X + Y\rangle_t$$

We get on subtraction

$$4X_tY_t - 4X_0Y_0 = 4\int_0^t X_s dY_s + 4\int_0^t Y_s dX_s + (\langle X+Y\rangle_t - \langle X-Y\rangle_t).$$

By rearranging the terms we get the desired result.

Proof [2.4] As W is of bounded quadratic variation and we require this result for W hence we assume that M, N are of bounded variation. We also assume that $Z_t(X)$ and $\sum_{i=1}^d \int_0^t (X_s^{(i)})^2 ds$ are bounded in ω and t. Using Kunita Watanabe Inequality we can say that $\tilde{M}$ is also bounded. Using the 2.23 Integration by parts formula we say that

$$Z_t(X)\tilde{M}_t = \int_0^t Z_u(X)dM_u + \sum_{i=1}^d \int_0^t \tilde{M}_u X_u^{(i)} Z_u(X) dW_u^i.$$

This is a martingale under $\mathbb{P}$. By the lemma 2.3 we have

$$\tilde{E}_T[\tilde{M}_t|\mathcal{F}_s] = \frac{1}{Z_s(X)} E[Z_t(X)\tilde{M}_t|\mathcal{F}_s] = \tilde{M}_s, \quad \text{a.s P and .}$$

This shows local (because of T) martingale property and because of continuity it belongs $\tilde{\mathcal{M}}^{c,loc}$.

$$\tilde{M}_t\tilde{N}_t - \langle M, N\rangle = \int_0^t \tilde{M}_u dN_u + \int_0^t \tilde{N}_u dM_u$$

$$+\sum_{i=1}^{d}[\int_0^t \tilde{M}_u X_s^{(i)} d\langle N, W^{(i)}\rangle_u + \int_0^t \tilde{N}_u X_s^{(i)} d\langle M, W^{(i)}\rangle_u]$$

Hence also

$$Z_t(X)[\tilde{M}_t\tilde{N}_t - \langle M, N\rangle] = \int_0^t Z_u(X)\tilde{M}_u dN_u + \int_0^t Z_u(X)\tilde{N}_u dM_u$$

$$+\sum_{i=1}^{d}[\int_0^t Z_u(X)\tilde{M}_u X_s^{(i)} d\langle N, W^{(i)}\rangle_u + \int_0^t Z_u(X)\tilde{N}_u X_s^{(i)} d\langle M, W^{(i)}\rangle_u]$$

The above process turns out to be a martingale and hence lemma 2.3 can be applied to yield

$$\tilde{E}_T[\tilde{M}_t\tilde{N}_t - \langle M, N\rangle_t|\mathcal{F}_s] = \tilde{M}_s\tilde{N}_s - \langle M, N\rangle_s$$

Theorem 12 (Girsanov's Theorem)**.** *Let $Z(X)$ as in 2.18 be a martingale and W a Brownian motion as discussed in the beginning of the section. We now let $\tilde{W} = \{\tilde{W}_t = (\tilde{W}_t^{(1)}, . \ , \ , \ \tilde{W}_t^{(d)})\}$ where*

$$\tilde{W}_t^{(i)} = W_t^{(i)} - \int_0^t X_s^{(i)} ds; \quad 1 \le i \le d \ \ t \in [0, \infty). \tag{2.24}$$

This new process for each fixed $0 \le T < \infty$ is a d-dimensional brownian motion with respect to the changed probability measure (due to Radon Nikodym theorem).

Proof This follows by putting $M = W$ in 2.4 and by martingale characterization theorem.

2.5 Some Interesting Results

The condition under which the process $Z_t(X)$ as in 2.18 is a martingale. The condition is known as **Novikov Condition**. We had seen in the examples discussed earlier that $Z_t(X)$ as described here is a super martingale.

Theorem 13. *If W is a d dimensional Brownian Motion and X is a d dimensional measurable adapted process satisfying* 2.17*. If*

$$E[\exp(1/2\int_0^T \|X_s\|^2 ds)] < \infty \quad 0 \le T < \infty$$

then Z(X) from 2.18 is a martingale.

We also have representations of martingale in terms of Brownian motion.

Theorem 14. *If M is a d dimensional local martingale on* $(\Omega, \mathcal{F}, \mathbb{P})$ *and the cross variation* $\langle M^{(i)}, M^{(j)}\rangle_t(\omega)$ *is an absolutely continuous function in t for almost every* ω*. Then there exist an extension* $(\Omega, \mathcal{F}', \mathbb{P}')$ *of* $(\Omega, \mathcal{F}, \mathbb{P})$ *on which is defined a d dimensional Brownian Motion* $W = \{(W_t^i)_{i=1}^d, \mathcal{F}'; 0 \le t < \infty\}$ *and a matrix* $X = \{(X_t^{i,j})_{i,j=1}^d, \mathcal{F}'; 0 \le t < \infty\}$ *of adapted processes with* $\mathbb{P}'[\int_0^t (X_s^{i,j})^2 ds < \infty] = 1$ *for* $1 \le i, j \le d$ *with* $t \in [0, \infty)$ *such that a.s* $\mathbb{P}'$ *we have the representations*

$$M_t^i = \sum_{j=1}^d \int_0^t X_s^{i,j} dW_s^j \quad 1 \le i \le d$$

$$\langle M^{(i)}, M^{(j)}\rangle_t = \sum_{j=1}^d \int_0^t X_s^{i,j} X_s^{k,j} dW_s^j \quad 1 \le i, j \le d.$$

Chapter 3

Stochastic Differential Equation

3.1 Introduction

Stochastic Differential Equations (SDEs) are used to model diverse phenomena such as fluctuating stock prices or physical systems subject to thermal fluctuations. The stochastic differential equation is of the form

$$dX_t = b(t, X_t)dt + \sigma(t, X_t)dW_t. \tag{3.1}$$

This may be interpreted as the stochastic process X_t changes in time δ_t its value by an amount that is normally distributed with expectation $b(t, X_t)\delta_t$ and variance $\sigma(t, X_t)^2\delta_t$. The solutions to 3.1 are also known as diffusion process and are written as

$$X_t = X_0 + \int_0^t b(s, X_s)ds + \int_0^t \sigma(s, X_s)dW_s; \qquad 0 \leq t < \infty.$$

The Stochastic differential equations has many applications in mathematical economics like in option pricing theory where SDE's are used to model fluctuating stock prices.

Consider

$$dX_t = X_t^2 dW_t + X_t^3 dt, \quad X_0 = 1$$

which can also be written as

$$X_t = 1 + \int_0^t X_s^2 dW_s + \int_0^t X_s^3 ds.$$

We now apply Ito's formula

$$d(1/X_t) = \frac{1}{2}(2/X_t^3)(dX_t)^2 - (1/X_t^2)dX_t$$
$$= \frac{1}{2}(2/X_t^3)(dX_t)^2 - (1/X_t^2)(X_t^2 dW_t + X_t^3 dt)$$
$$= X_t dt - (1/X_t^2)(X_t^2 dW_t + X_t^3 dt)$$
$$= -dW_t$$

Now with a constant C=1 and $X_0 = 1$ we can say that $X_t = \dfrac{1}{1 - W_t}$.
In the subsequent sections we discuss the uniqueness and existence of solutions. Majority of the text in this chapter can be referred from [2].

3.2 Strong Solutions

Equation (3.1) can also be written as

$$dX_t^{(i)} = b_i(t, X_t)dt + \sum_{j=1}^{r} \sigma_{ij}(t, X_t)dW_t^{(j)}; \qquad 1 \leq i \leq d. \tag{3.2}$$

We treat here $b_i(t,x), \sigma_{ij}(t,x),\ 1 \leq i \leq d$ as borel measurable functions. $b(t,x) = \{b_i(t,x)\}_{1\leq i\leq d}$ is the drift vector and $\sigma(t,x) = \{\sigma_{ij}(t,x)\}_{1\leq i\leq d\ 1\leq j\leq r}$ is the dispersion matrix. Here X is the solution of the equation and W is the r-dimensional Brownian Motion.
To define the notion of strong solution we require a suitable filtration . We consider a probability triple $(\Omega, \mathcal{F}, \mathbb{P})$ and an r-dimensional Brownian Motion
$W = \{W_t, \mathcal{F}_t^W, 0 \leq t < \infty\}$ defined on it. We also consider a random vector ξ with values in $\mathbb{R}^d$ independent of $\mathcal{F}_\infty^W$. This vector helps us in defining the notion of strong solution.
We start with construction of the filtration. We consider

$$\mathcal{G}_t := \sigma(\xi, W_s, 0 \leq s \leq t)$$

We augment it with null sets

$$\mathcal{N} := \{N \subseteq \Omega; \exists G \in \mathcal{G}_\infty \text{ with } N \subseteq G \text{ and } \mathbb{P}(G) = 0\}$$

to get

$$\mathcal{F}_t = \sigma(\mathcal{G}_t \cup \mathcal{N}) \tag{3.3}$$

Note: The Brownian motion W will remain a Brownian motion with respect to the new filtration.

Definition 3.1. *A strong solution of the stochastic differential equation 3.1, on* $(\Omega, \mathcal{F}, \mathbb{P})$ *and with respect to the fixed Brownian motion W and initial condition* ξ*, is a process* $X = \{X_t; 0 \leq t < \infty\}$ *with continuous sample paths and with the following properties :*

1. *X is adapted to the filtration* $\{\mathcal{F}_t\}$ *of 3.3*
2. $\mathbb{P}[X_0 = \xi] = 1$
3. $\mathbb{P}[\int_0^t \{|b_i(s, X_s)| + \sigma_{ij}^2(s, X_s)\} < \infty]$ *holds for every* $1 \leq i \leq d, \;\; 1 \leq j \leq r$ *and* $0 \leq t < \infty$
4. *The integral version of 3.1*

$$X_t^i = X_0^i + \int_0^t b_i(s, X_s)ds + \Sigma_{j=1}^r \int_0^t \sigma_{ij}(s, X_s)dW_s^{(j)}$$

holds almost surely.

Lets us define the notion of uniqueness as follows

Definition 3.2. *If X and* $\tilde{X}$ *are two strong solutions of 3.1 with respect to the r-dimensional Brownian Motion W, initial condition* ξ*, drift vector* $b_i(s, x)$ *dispersion matrix and* $\sigma_{ij}(s, x)$ *then* $\mathbb{P}[X = \tilde{X}; 0 \leq t < \infty] = 1$*. In such a case strong uniqueness holds for the SDE.*

We now need conditions for the uniqueness and existence of solutions. We first see the uniqueness of the solution. We use Gronwall inequality in the proofs of these.

Proposition 3.1 (Gronwall Inequality)**.** *Let continuous* $g(t)$ *satisfy*

$$0 \leq g(t) \leq f(t) + a\int_0^t g(s)ds; \qquad 0 \leq t \leq T \tag{3.4}$$

with $a \geq 0$ and $f : [0,T] \to \mathbb{R}$ integrable. Then we have

$$g(t) \leq f(t) + a\int_0^t f(t)\exp(a(t-s))ds$$

Proof We have

$$\frac{d}{dt}(\exp(-at)\int_0^t g(s)ds) = (g(t) - a\int_0^t g(s)ds)\exp(-at) \leq f(t)\exp(-at).$$

This will give us the required result

Theorem 15. *Let coefficients $b_i(s, X_s)$ and $\sigma_{ij}(s, X_s)$ be locally Lipschitz continuous that is for every $n \geq 1$ there exist a constant C_n such that for every $t \geq 0$, $\|x\| \leq n$ and $\|y\| \leq n$:*

$$\|b(t,x) - b(t,y)\| + \|\sigma(t,x) - \sigma(t,y)\| \leq C_n\|x - y\| \tag{3.5}$$

Then strong uniqueness holds for 3.1.

Proof Let X and $\tilde{X}$ be the solution of 3.2 as in definition 3.1. Define the stopping time be $\tau_n = \inf\{t \geq 0; \|X\| \geq n\} \quad n \geq 1$ and
$\tilde{\tau_n} = \inf\{t \geq 0; \|\tilde{X}\| \geq n; n \geq 1\}$. We further define $S_n = \tau_n \wedge \tilde{\tau}_n$.
We have $\lim_{n\to\infty} S_n = \infty$.

$$X_{t\wedge S_n} - \tilde{X}_{t\wedge S_n} = \int_0^{t\wedge S_n}\{b(u, X_u) - b(u,(\tilde{X_u}))\} + \int_0^{t\wedge S_n}\{\sigma(u, X_u) - \sigma(u,(\tilde{X_u})).$$

Using the vector inequality, Holder inequality and 3.5 we have for $0 \leq t \leq T$

$$\begin{aligned}
E\|X_{t\wedge S_n} - \tilde{X}_{t\wedge S_n}\|^2 &\leq E[\int_0^{t\wedge S_n}\|\{b(u, X_u) - b(u,(\tilde{X_u}))du\}\|]^2 \\
+4E\sum_{i=1}^{d}[\sum_{i=1}^{r}\int_0^{t\wedge S_n}&\|\{\sigma_{ij}(u, X_u) - \sigma_{ij}(u,(\tilde{X_u}))dW_u^{(j)}\}\|]^2 \\
&= E\int_0^{t\wedge S_n}\|\{b(u, X_u) - b(u,(\tilde{X_u}))\}\|^2 du \\
+4E\int_0^{t\wedge S_n}&\|\{\sigma(u, X_u) - \sigma(u,(\tilde{X_u}))\}\|^2 du \\
&\leq 4(T+1)C_n^2\int_0^t E\|X_{t\wedge S_n} - \tilde{X}_{t\wedge S_n}\|^2 du.
\end{aligned}$$

Apply Gronwall inequality to get that X and $\tilde{X}$ are modifications of one another.

For existence of a solution we impose stronger conditions.

Theorem 16. *Let coefficients* $b_i(s, X_s)$ *and* $\sigma_{ij}(s, X_s)$ *satisfy global Lipschitz and linear growth conditions*

$$\|b(t,x) - b(t,y)\| + \|\sigma(t,x) - \sigma(t,y)\| \leq C_n \|x - y\| \tag{3.6}$$

$$\|b(t,x)\|^2 + \|\sigma(t,x)\|^2 \leq C^2(1 + \|x\|^2) \tag{3.7}$$

every $0 \leq t < \infty$, $x \in \mathbb{R}^d$, $y \in \mathbb{R}^d$ *and* $C \in [0, \infty)$. *Now on some probability triple* $(\Omega, \mathcal{F}, \mathbb{P})$ *let* $\mathcal{F}_t$ *be as in 3.3, let* ξ *be an* $\mathbb{R}^d$ *valued random variable which is independent of the r-dimensional Brownian motion* $W = \{W_t, \mathcal{F}_t^W, 0 \leq t < \infty\}$, *and with finite second moment*

$$E\|\xi\|^2 < \infty. \tag{3.8}$$

Then there exist a continuous adapted process $X = \{X_t, \mathcal{F}_t, 0 \leq t < \infty\}$ *which is a strong solution of the 3.1 relative to initial condition* ξ *and Brownian motion W. This process is also a square integrable process.*

We use the idea of iterations and make it converge to a solution of 3.3. The iterations look like

$$X_t^{(k+1)} := \xi + \int_0^t b(s, X_s^{(k)})ds + \int_0^t \sigma(s, X_s^{(k)})dW_s; \quad 0 \leq t < \infty. \tag{3.9}$$

Proposition 3.2. *Let* M *be a d-dimensional vector with* $M^{(i)} \in \mathcal{M}_{\in}^{\rfloor}$. *Let*

$$\|M\|_t^* = \max_{s \leq t} \|M_s\| \ ; \qquad\qquad A_t = \sum_{i=1}^{d} \langle M^{(i)} \rangle_t \tag{3.10}$$

Then for every stopping time T *we have the inequality*

$$\lambda_m E(A_T^m) \leq E(\|M\|_T^*)^{2m} \leq \Lambda_m E(A_T^m). \tag{3.11}$$

Proposition 3.3. *If* $M_t^{(i)} = \sum_{i=1}^r \int_0^t X_s^{i,j} dW_s^{(j)}$ *where* W *is a* r *dimensional brownian motion and*

$$X = \{X_t = X_t^{i,j};\ 1 \leq i \leq d;\ 1 \leq j \leq r;\ t \geq 0\}.$$

Here X_t is $\mathcal{F}_t$ measurable and

$$\|X_t\|^2 = \sum_{i=1}^{d}\sum_{j=1}^{r}\left(X_t^{i,j}\right)^2.$$

Then with

$$A_T = \int_0^T \|X_t\|^2$$

the above proposition holds

Proof

$$B_t = \int_0^t \{b_i(s, X_s^k) - b_i(s, X_s^{k-1})\}ds \qquad M_t = \int_0^t \{\sigma_i(s, X_s^k) - \sigma_i(s, X_s^{k-1})\}dW_s$$

Then from the just stated results (above) and Lipschitz condition we have

$$E[\max_{s\le t}\|M_s\|^2] \le \Lambda E\int_0^t \|\sigma(s, X_s^k) - \sigma(s, X_s^{k-1})\|ds \le \Lambda C^2 E\int_0^t \|X_s^k - X_s^{k-1}\|^2 ds.$$

We also have

$$E\|B_t\|^2 \le C^2 tE\int_0^t \|X_s^k - X_s^{k-1}\|^2 ds.$$

Hence

$$E[\max_{s\le t}\|X_s^{k+1} - X_s^k\|^2] \le 4C^2(\Lambda + T)E\int_0^t \|X_s^k - X_s^{k-1}\|^2 ds.$$

Hence using successive iterations we get

$$E[\max_{s\le t}\|X_s^{k+1} - X_s^k\|^2] \le \max_{s\le T} E\|X_t^1 - \xi\|^2 \frac{(4C^2(\Lambda + T)t)^k}{k!}.$$

By Chebyshev's inequality we now get

$$\mathbb{P}\Big[\max_{s\le T}\|X_s^{k+1} - X_s^k\|^2 \ge \frac{1}{2^k}\Big] \le 4\max_{s\le t}\|X_t^1 - \xi\|^2 \cdot \frac{(4\cdot 4C^2(\Lambda + T)t)^k}{k!}.$$

From Borel Cantelli lemma get that $\exists\ \Omega^*$ such that $\mathbb{P}(\Omega^*) = 1$ and an integer valued random $N(\omega)$ variable such that $\forall\ \omega \in \Omega^*$

$$\max_{s\le T}\|X_s^{k+1}-X_s^k\|\le 1/2^{k+1}\quad\forall\ k\ge N(\omega)$$
$$\max_{s\le T}\|X_s^{k+m}-X_s^k\|\le 1/2^{k+1}\quad\forall\ k\ge N(\omega)\ \forall\ m\ge 1$$

Hence by this we can say that the sequence of sample paths are convergent in sup norm from where the existance of continuous $\{X_t; t\le T\}$ is established. Since T is arbitrary and the process is continuous the sample paths uniformly converge on the compact sets. Finally we show that

$$X_t=\lim X_t^k\quad t\ge 0$$

satisfies the 4th condition of the definition. Since the process is square integrable and satisfy linear growth condition we have property 3 being satisfied of the definition. The X which we get also clearly satisfies condition 1 and 2 of the definition.

Proposition 3.4. *Let $d=r=1$. Let us suppose that the coefficients of the one dimensional equation satisfy*

$$|b(t,x)-b(t,y)|\le C|x-y| \tag{3.12}$$
$$|\sigma(t,x)-\sigma(t,y)|\le h(|x-y|) \tag{3.13}$$

$\forall\ t\in[0,\infty)$ and $x,y\in\mathbb{R}$, where C is a positive constant and 'h' is a strictly increasing function with h(0) =0 and

$$\int_{(0,\epsilon)} h^{-2}(u)du=\infty;\qquad\forall\epsilon>0.$$

Then we have strong uniqueness for 3.1.

Proof There exists a decreasing sequence $\{a_n\}_{n=0}^{\infty}$ in [0,1) such that $\lim_{n\to\infty}a_n=0$ and $\int_{a_n}^{a_{n-1}}h^{-2}(u)du=n\ \forall\ n\ge 1$. For each n there exist a continuous function ρ_n on $\mathbb{R}$ with support in (a_{n-1},a_n) such that $0\le\rho_n(x)\le(\dfrac{2}{nh^2(x)});\quad\forall x\in(0,\infty)$ and

$\int_{a_n}^{a_{n-1}} \rho_n(x) = 1$. Then we have

$$\psi_n(x) := \int_0^{|x|} \int_0^y \rho_n(u) du dy; \quad x \in \mathbb{R}. \tag{3.14}$$

This is a $C^2(\mathbb{R})$ function such that $|\psi_n'(x)| \leq 1$ and the sequence $\{\psi_n\}_{n=1}^{\infty}$ is non decreasing. Let us take two solutions of 3.1 $X^{(1)}, X^{(2)}$. By definition we take

$$E \int_0^t |\sigma(s, X_s)|^2 ds < \infty; \quad 0 \leq t < \infty.$$

$$\Xi_t = X_t^{(1)} - X_t^{(2)} = \int_0^t \{b(s, X_s^{(1)}) - b(s, X_s^{(2)})\} ds + \int_0^t \{\sigma(s, X_s^{(1)}) - \sigma(s, X_s^{(2)})\} ds$$

We further employ Ito's Formula to get

$$\begin{aligned} \psi_n(\Xi_t) = \int_0^t \psi_n'(\Xi_s)[b(s, X_s^{(1)}) - b(s, X_s^{(2)})] ds + \frac{1}{2} \int_0^t \psi_n''(\Xi_s)[\sigma(s, X_s^{(1)}) - \sigma(s, X_s^{(2)})]^2 ds \\ + \int_0^t \psi_n'(\Xi_s)[\sigma(s, X_s^{(1)}) - \sigma(s, X_s^{(2)})]^2 dW_s. \end{aligned}$$

We now have $E[\int_0^t \psi_n'(\Xi_s)[\sigma(s, X_s^{(1)}) - \sigma(s, X_s^{(2)})]^2 dW_s] = 0$ as W is a brownian motion and $E[\int_0^t |\sigma(s, X_s)|^2 ds] < \infty; 0 \leq t < \infty$. Further we have for the second term

$$E[\int_0^t \psi_n''(\Xi_s)[\sigma(s, X_s^{(1)}) - \sigma(s, X_s^{(2)})]^2 ds] \leq E[\int_0^t \psi_n''(\Xi_s)[h|\Xi_s|]^2 ds] \leq 2t/n.$$

Hence forth

$$\begin{aligned} E\psi_n(\Xi_t) = E \int_0^t \psi_n'(\Xi_s)[b(s, X_s^{(1)}) - b(s, X_s^{(2)})] ds + t/n \\ \leq C \int_0^t E|\Xi_s| ds + t/n \end{aligned}$$

as $n \to \infty$ gives us $E|\Xi_t| \leq C \int_0^t E|\Xi_s| ds$. Now by Gronwall inequality our result follows.

The proof of the existence of the solution of 3.1 follows in a similar fashion as in the proof of Theorem 16.

3.3 Weak Solutions

Definition 3.3. (X, W) *on* $(\Omega, \mathcal{F}, \mathbb{P})$ *is a weak solution of the equation 3.1 if following happens*

1. $\mathcal{F}_t$ *is a filtration of* $\mathcal{F}$ *satisfying usual conditions.*
2. $W = \{W_t, \mathcal{F}_t, 0 \leq t < \infty\}$ *is an r-dimensional brownian motion and* $X = \{X_t, \mathcal{F}_t, 0 \leq t < \infty\}$ *is an adapted* $\mathbb{R}^d$ *valued process.*
3. $\mathbb{P}[\int_0^t \{|b_i(s, X_s)| + \sigma_{ij}^2(s, X_s)\} < \infty]$ *holds for every* $1 \leq i \leq d$ $1 \leq j \leq r$ *and* $1 \leq t < \infty$.
4. *The integral version of 3.1*

$$X_t^i = X_0^i + \int_0^t b_i(s, X_s) ds + \Sigma_{j=1}^r \int_0^t \sigma_{ij}(s, X_s) dW_s^{(j)}$$

holds almost surely.

With this definition one can say that strong solution satisfies all the conditions of being a weak solution.

Chapter 4

New Stochastic Integral

4.1 Introduction and Motivation

In chapter two we constructed the stochastic integral where the integrand was an adapted process. The process is adapted to the filtration to which the Brownian motion is adapted. But if the integrand is not adapted with respect to the filtration (upto that time) then we cannot integrate using that definition. This chapter presents the idea of the Kuo's paper [3], [4],[5] published after 2008 to integrate the anticipating integrals with respect to the Brownian motion. Let $W = \{W_t, \mathcal{F}_t, 0 \leq t < \infty\}$ be a Brownian motion and let $X = \{X_t, 0 \leq t < \infty\}$ be a stochastic process. Then how we define a stochastic integral such that $\int_0^a X_t(\omega)dW_t(\omega)$ is a martingale. Our idea is to make this a stochastic integral.

We draw some ideas from the authors book [6]. Let $X_t = W_t;\ \forall t$ and consider a partition $\Delta_n = \{0 = t_0, t_1, ..., t_n = t\}$ of $[0, t]$. Let L_n, R_n be

$$L_n = \sum_{i=1}^{n} W_{t_{i-1}}(W_{t_i} - W_{t_{i-1}}) \tag{4.1}$$

$$R_n = \sum_{i=1}^{n} W_{t_i}(W_{t_i} - W_{t_{i-1}}) \tag{4.2}$$

Further

$$R_n - L_n = \sum_{i=1}^{n} (W_{t_i} - W_{t_{i-1}})^2 \tag{4.3}$$

$$R_n + L_n = \sum_{i=1}^{n} (W_{t_i}^2 - W_{t_{i-1}}^2) = W_t^2 - W_0^2 \tag{4.4}$$

Henceforth

$$R_n = \frac{1}{2}\Big(\sum_{i=1}^{n} (W_{t_i} - W_{t_{i-1}})^2 + W_t^2 - W_0^2 \Big)$$

$$L_n = \frac{1}{2}\Big(W_t^2 - W_0^2 - \sum_{i=1}^{n} (W_{t_i} - W_{t_{i-1}})^2 \Big).$$

4.3 is regarded as the quadratic variation and hence by the properties of Brownian motion we say that it is non zero and hence $L_n \neq R_n$ as $\|\Delta_n\| \to 0$
We note that as $\|\Delta_n\| \to 0$ we have

$$R_t = \lim_{\|\Delta_n\| \to 0} R_n = \frac{1}{2}((W_t^2 - W_0^2) + t)$$

$$L_t = \lim_{\|\Delta_n\| \to 0} L_n = \frac{1}{2}(W_t^2 - W_0^2 - t)$$

As mentioned previously we want to assign meaning to the integral so that it satisfies martingale property and becomes a stochastic integral. Here $ER_t = t$ which is not constant as required in the case of martingales, on the other hand EL_t comes out to be a constant which also falls in place with expectation of any martingale. We now check that is L_t a martingale

$$\begin{aligned} E(W_t^2|\mathcal{F}_s) &= E((W_t - W_s + W_s)^2|\mathcal{F}_s) \\ &= E((W_t - W_s)^2 + 2W_s(W_t - W_s) + W_s^2|\mathcal{F}_s) \\ &= E((W_t - W_s)^2) + 2W_sE((W_t - W_s)) + W_s^2 \\ &= t - s + W_s^2. \end{aligned}$$

Taking this into account we have $E(L_t|\mathcal{F}_s) = L_s$ and hence we can conclude that we should take left end point of each subinterval as the evaluation point as it gives us the martingale property. With L_t other properties of integral discussed in chapter 2 are also satisfied. Hence we can use L_t to compute the stochastic integral as it goes by the definition.

We know that the integrand should be adapted to the the filtration but what if it is not adapted. Take for example

$$\int_0^t W_1 dW_t; \quad 0 \leq t \leq 1.$$

We note that $E[W_1 W_t] = \min\{1, t\}; \quad 0 \leq t \leq 1$ which is not a constant. This by definition is not a 'defined' integral as W_1 is not adapted to the filtration $\sigma\{W_s; s \leq t; 0 \leq t \leq 1\}$.

The new idea defined in the Kuo's paper is something like: we break the above integrand W_1 into $W_1 = (W_1 - W_s) + W_s$. Now $\int_0^t W_1 dW_s = \int_0^t (W_1 - W_s) dW_s + \int_0^t W_s dW_s$. Here we have the second integral as the stochastic integral which through the Ito's formula gives us $\frac{1}{2}(W_t^2 - t)$. The term $W_1 - W_s$ is not adapted (i.e anticipating). This integral is then taken forward by defining

$$\int_0^t (W_1 - W_s) dW_s = \lim_{\Delta \to \infty} \sum_{i=1}^{n} (W_1 - W_{s_i})(W_{s_i} - W_{s_{i-1}})$$

We pause to note that here we have taken the right end points whereas in case of the stochastic integral of adapted integrand we had taken the left end points to evaluate the integral.

$$\begin{aligned} &\sum_{i=1}^{n} (W_1 - W_{s_i})(W_{s_i} - W_{s_{i-1}}) \\ &= W_1 \sum_{i=1}^{n} (W_{s_i} - W_{s_{i-1}}) - \sum_{i=1}^{n} W_{s_i}(W_{s_i} - W_{s_{i-1}}) \\ &= W_1 W_t - \{\sum_{i=1}^{n} (W_{s_i} - W_{s_{i-1}})^2 + \sum_{i=1}^{n} W_{s_{i-1}}(W_{s_i} - W_{s_{i-1}})\} \\ &\longrightarrow W_1 W_t - \{t + \frac{1}{2}(W_t^2 - t)\}. \end{aligned}$$

Which gives us

$$\int_0^t W_1 W_t = W_1 W_t - \{t + \frac{1}{2}(W_t^2 - t)\} + \frac{1}{2}(W_t^2 - t) = W_1 W_t - t.$$

We further note that $E[W_1W_t - t] = \min\{1, t\} - t = 0$ for $0 \le t \le 1$. This is one of the properties of stochastic integral that $E[\int X_t dW_t] = 0$.

4.2 Stochastic Integral

To take this idea forward we need a few notions. We fix a brownian motion $W = \{W_t, \mathcal{F}_t, 0 \le t < \infty\}$.

Definition 4.1. *A stochastic process ξ_t and a filtration $\{\mathcal{F}_t\}$is said to be instantly independent with respect to each other if ξ_t and $\mathcal{F}_t$ are independent for every instant.*

Both $W_1 - W_t$ and W_1 are not adapted for $0 \le t \le 1$ but $W_1 - W_t$ is instantly independent process with respect to $\mathcal{F}_t$.

Definition 4.2. *Let $X = \{X_s, \mathcal{F}_s, 0 < s < t\}$, be adapted and let $\xi_s, 0 < s < t$ be instantly independent. Define the stochastic integral of $X_s\xi_s$ by*

$$I(X\xi) = \int_0^t X_s\xi_s dW_s = \lim_{\|\Delta\|\to 0} \sum_{i=1}^{n} X_{t_{i-1}}\xi_{t_i}(W_{t_i} - W_{t_{i-1}}). \tag{4.5}$$

If $\xi_t = 1 \;\forall\; t$ then we have the stochastic integration defined in chapter 2. Carrying on further we observe that

$$\begin{aligned} E\{X_{t_{i-1}}\xi_{t_i}(W_{t_i} - W_{t_{i-1}})\} &= E\{E[X_{t_{i-1}}\xi_{t_i}(W_{t_i} - W_{t_{i-1}})|\mathcal{F}_{t_i}]\} \\ & E\{X_{t_{i-1}}(W_{t_i} - W_{t_{i-1}})E[\xi_{t_i}|\mathcal{F}_{t_i}]\} \end{aligned}$$

$E[\xi_{t_i}|\mathcal{F}_{t_i}] = E[\xi_{t_i}]$ as ξ_t is instantly independent. Hence we have

$$\begin{aligned} &= E[\xi_{t_i}]E\{E[X_{t_{i-1}}(W_{t_i} - W_{t_{i-1}})|\mathcal{F}_{t_{i-1}}]\} \\ &= 0. \end{aligned}$$

Hence $E(I(X\xi)) = 0$. This is also one of the properties of the stochastic integral defined in chapter 2.

We now discuss some examples.

Example 1 We now evaluate $\int_0^t W_1^2 dW_s$ We do it case by case

Case1: $0 \leq t \leq 1$

$W_1^2 = (W_1 - W_s)^2 + 2W_s(W_1 - W_s) + W_s^2$. By definition of the corresponding stochastic Integral this is

$$\sum_{i=1}^{n} \{(W_1 - W_{s_i}^2) + 2W_{s_{i-1}}(W_1 - W_{s_i}) + W_{s_{i-1}}^2\}(W_{s_i} - W_{s_{i-1}}) \tag{4.6}$$

$$= W_1^2 W_t - 2W_1 \sum_{i=1}^{n}(W_{s_i} - W_{s_{i-1}})^2 + \sum_{i=1}^{n}(W_{s_i} - W_{s_{i-1}})^3. \tag{4.7}$$

Here we make use of the fact that if $X = \{X_t, \mathcal{F}_t, 0 \leq t < \infty\}$ is a continuous stochastic process with the property that

$$\lim_{\|\Delta\| \to 0} V_t^p(\Delta) = A_t$$

where $A_t \in [0, \infty)$ is a random variable then $\lim_{\|\Delta\| \to 0} V_t^q(\Delta) = 0$ where $0 < p < q$. We prove this fact by using that

$$\lim_{\|\Delta\| \to 0} V_t^q(\Delta) \leq \lim_{\|\Delta\| \to 0} V_t^p(\Delta) \sup(|W_t - W_s|; 0 \leq r \leq s < t)^{q-p}.$$

Hence as partition goes to zero $V_t^q(\Delta) \to 0$

Therefore the second summation in 4.6 goes to t and third one goes to 0. By this we have

$$\int_0^t W_1^2 dW_s = W_1^2 W_t - 2W_1 t, \qquad 0 \leq t \leq 1.$$

Case 2: $t > 1$

$$\begin{aligned} \int_0^t W_1^2 dW_s &= \int_0^1 W_1^2 dW_s + \int_1^t W_1^2 dW_s \\ &= W_1^3 - 2W_1 + W_1^2(W_t - W_1) \\ &= W_1^2 W_t - 2W_1. \end{aligned}$$

Example 2 Let us evaluate $\int_0^t W_1 W_s dW_s$

Case 1 $0 \le t \le 1$**:** We write $W_1 W_s$ as

$$W_1 W_s = W_s(W_1 - W_s) + W_s^2$$

Once again we look at

$$\sum_{i=1}^{n} \{W_{s_{i-1}}(W_1 - W_{s_i}) + W_{s_{i-1}}^2\}(W_{s_i} - W_{s_{i-1}})$$
$$= W_1 \sum_{i=1}^{n} W_{s_{i-1}}(W_{s_i} - W_{s_{i-1}}) - \sum_{i=1}^{n} W_{s_{i-1}}(W_{s_i} - W_{s_{i-1}})^2$$

This converges in probability to

$$W_1 \int_0^t W_s dW_s - \int_0^t W_s ds.$$

Hence

$$\int_0^t W_1 W_s dW_s = \frac{1}{2} W_1(W_t^2 - t) - \int_0^t W_s ds \qquad 0 \le t \le 1$$
$$\int_0^t W_1 W_s dW_s = \frac{1}{2} W_1(W_t^2 - t) - \int_0^1 W_s ds \qquad t > 1.$$

Example 3

Let $\xi(x)$ be a continuous function. we evaluate $\int_0^t W_1 \xi(W_s) dW_s$ where $t \in [0,1]$. We write the integrand as

$$W_1 \xi(W_s) = (W_1 - W_s)\xi(W_s) + W_s \xi(W_s).$$

Then we have

$$\sum_{i=1}^{n} \{(W_1 - W_{s_i})\xi(W_{s_{i-1}}) + W_{s_{i-1}}\xi(W_{s_i})\}(W_{s_i} - (W_{s_{i-1}}))$$
$$= W_1 \sum_{i=1}^{n} \xi(W_{s_{i-1}})(W_{s_i} - W_{s_{i-1}}) - \sum_{i=1}^{n} \xi(W_{s_{i-1}})(W_{s_i} - W_{s_{i-1}})^2.$$

This converges in probability to

$$W_1 \int_0^t \xi(W_s)dW_s - \int_0^t \xi(W_s)ds$$

which gives us for $0 \leq t \leq 1$

$$\int_0^t W_1\xi(W_s)dW_s = W_1 \int_0^t \xi(W_s)dW_s - \int_0^t \xi(W_s)ds.$$

For $t > 1$

$$\int_0^t W_1\xi(W_s)dW_s = W_1 \int_0^t \xi(W_s)dW_s - \int_0^1 \xi(W_s)ds$$

4.3 Ito's Formula

In this section we try and give Ito's formula for the newly defined stochastic integral [4]. Consider a partition $\Delta = \{s_0, ..., s_n\}$ of [0,T].

Lemma 4.1. *Let f(x) be a continuous and g(x) be a C^1-function. Let $h(x,y) = f(x)g(y-x)$. Then for each $t \in [0,T]$*

$$\sum_{i=1}^{n} h(W_{s_{i-1}}, W_T)(W_{s_i} - W_{s_{i-1}})$$
$$\longrightarrow \int_0^t h(W_s, W_T)dW_s + \int_0^t \frac{\delta h}{\delta y}(W_s, W_T)ds$$

in probability as $\|\Delta\| \to 0$.

Proof

$$\sum_{i=1}^{n} h(W_{s_{i-1}}, W_T)(W_{s_i} - W_{s_{i-1}}) = \sum_{i=1}^{n} f(W_{s_{i-1}})g(W_T - W_{s_{i-1}})(W_{s_i} - W_{s_{i-1}})$$
$$\approx \sum_{i=1}^{n} f(W_{s_{i-1}})\{g(W_T - W_{s_i}) + g'(W_T - W_{s_i})(W_{s_i} - W_{s_{i-1}})\}(W_{s_i} - W_{s_{i-1}}).$$

The above approximation is due to Euler's method. As $\|\Delta\| \to 0$ this converges to

$$\to \int_0^t f(W_s)\{g(W_T - W_s)\}dW_s + \int_0^t f(W_s)g'(W_T - W_s)ds$$

which gives the desired result.

Using the above rule we give a proof of a formula which can be considered as the Ito's integral of the new stochastic integral.

Theorem 17. *let* $f(x), g(x) \in C^2$ *and h(x) be as defined above. Then the following equality holds for* $0 \leq t \leq T$

$$h(W_t, W_T) = h(W_0, W_T) + \int_0^t \frac{\delta h}{\delta x}(W_s, W_T) dW_s + \int_0^t \{\frac{1}{2}\frac{\delta^2 h}{\delta x^2}(W_s, W_T) + \frac{\delta^2 h}{\delta x \delta y}(W_s, W_T)\} ds \tag{4.8}$$

Proof With the same partition we proceed

$$\begin{aligned} h(W_t, W_T) &= \sum_{i=1}^{n} \{h(W_{s_i}, W_T) - h(W_{s_{i-1}}, W_T)\} \\ &\approx \sum_{i=1}^{n} \Bigg(\frac{\delta h}{\delta x}(h(W_{s_{i-1}}, W_T))(W_{s_i} - W_{s_{i-1}}) \\ &\quad + \frac{1}{2}\frac{\delta^2 h}{\delta x^2}(W_{s_{i-1}}, W_T)(W_{s_i} - W_{s_{i-1}})^2 \Bigg). \end{aligned}$$

The above approximation is due to Taylors expansion. Using the lemma 4.1 to the function $\frac{\delta h}{\delta x}$ we get

$$\int_0^t \frac{\delta h}{\delta x}(W_s, W_T) dW_s + \int_0^t \frac{\delta^2 h}{\delta x \delta y}(W_s, W_T) ds$$

and the second sum converges in probability to

$$\int_0^t \frac{1}{2}\frac{\delta^2 h}{\delta x^2}(W_s, W_T)$$

with these two limits together we get the desired result.

This theorem can only be applied to the case where h is a function of W_t. Even if we want to integrate

$$\int_0^t W_1 dW_s$$

then we have to decompose this as $W_1 = (W_1 - W_s) + W_s$

We try and define a Ito's formula which is valid for a bigger class.

Definition 4.3. *An* ***Ito's Process*** *is a stochastic process of the form*

$$X_t = X_0 + \int_0^t p(s)dW_s + \int_0^t q(s)ds, \qquad 0 \leq t \leq T. \tag{4.9}$$

Here X_t is $\mathcal{F}_t$ measurable $\forall t$ and $p \in \mathcal{L}(\Omega, L^2[0,T])$ and $q \in \mathcal{L}(\Omega, L^1[0,T])$.

Here for any stochastic process $X = \{X_t, 0 \leq t \leq T\}$ in $\mathcal{L}(\Omega, L^2[0,T])$ it is $\mathcal{F}_t$ adapted such that $\int_0^T |X_t|^2 < \infty$. Similarly for $\mathcal{L}(\Omega, L^1[0,T])$.
Consider another process which we require for the next result which is generalized version of earlier Ito's theorem.

$$Y_t = Y_T + \int_t^T u(s)dW_s + \int_t^T v(s)ds \tag{4.10}$$

where Y_T is independent of $\mathcal{F}_T$ and the functions $u \in L^2[0,T]$ and $v \in L^1[0,T]$.
Our motive is to replace $f(W_t)$ with $f(X_t)$ where X_t is of the form 4.9 and $g(W_t)$ with $g(Y_t)$ where Y_t is of the form 4.10.

Theorem 18. *Let h(x,y) = f(x)g(y), where $f, g \in C^2(\mathbb{R})$. Let X_t be as in 4.9 and Y_t as in 4.10. Then for $0 \leq t \leq T$.*

$$h(X_t, Y_t) = h(X_0, Y_0) + \int_0^t \frac{\delta h}{\delta x}(X_s, Y_s)dX_s + \frac{1}{2}\int_0^t \frac{\delta^2 h}{\delta x^2}(X_s, Y_s)(dX_s)^2 \tag{4.11}$$

$$\int_0^t \frac{\delta h}{\delta y}(X_s, Y_s)dY_s + \frac{1}{2}\int_0^t \frac{\delta^2 h}{\delta y^2}(X_s, Y_s)(dY_s)^2. \tag{4.12}$$

We make a note of a few things required for the proof.

$$(dW_t)^2 = dt \quad (dt)^2 = 0 \quad dW_t dt = 0.$$

Proof We use the partition as introduced in the beginning.

$$h(X_t, Y_t) - h(X_0, Y_0) = \sum_{i=1}^n [h(X_{s_i}, Y_{s_i}) - h(X_{s_{i-1}}, Y_{s_{i-1}})] \tag{4.13}$$

$$= \sum_{i=1}^n [f(X_{s_i})g(Y_{s_i}) - f(X_{s_{i-1}})g(Y_{s_{i-1}})] \tag{4.14}$$

We note that we can write 4.9 and 4.10 in the differential form.

$$dX_t = p(t)dW_t + q(t)dt$$
$$dY_t = -u(t)dW_t - v(t)dt$$

Using taylors expansion of f and g to obtain

$$f(X_{s_i}) \approx f(X_{s_{i-1}}) + f'(X_{s_{i-1}})(X_{s_i} - X_{s_{i-1}}) + \frac{1}{2}f''(X_{s_{i-1}})(X_{s_i} - X_{s_{i-1}})^2 \quad (4.15)$$
$$g(Y_{s_{i-1}}) \approx g(Y_{s_{i-1}}) + g'(Y_{s_{i-1}})(-(Y_{s_i} - Y_{s_{i-1}})) + \frac{1}{2}g''(Y_{s_{i-1}})(-(Y_{s_i} - Y_{s_{i-1}}))^2 \quad (4.16)$$

by 4.15, 4.16 and 4.14 we have

$$\begin{aligned} & h(X_t, Y_t) - h(X_0, Y_0) \\ \approx & \sum_{i=1}^{n}[\left(f(X_{s_{i-1}}) + f'(X_{s_{i-1}})(X_{s_i} - X_{s_{i-1}}) + \frac{1}{2}f''(X_{s_{i-1}})(X_{s_i} - X_{s_{i-1}})^2 \right) g(Y_{t_i}) \\ & -f(X_{s_{i-1}})\left(g(Y_{s_i}) + g'(Y_{s_i})(-(Y_{s_i} - Y_{s_{i-1}})) + \frac{1}{2}g''(Y_{s_i})(-(Y_{s_i} - Y_{s_{i-1}}))^2 \right)] \\ = & \sum_{i=1}^{n}[\left(f'(X_{s_{i-1}})g(Y_{s_i})(X_{s_i} - X_{s_{i-1}}) + \frac{1}{2}f''(X_{s_{i-1}})g(Y_{s_i})(X_{s_i} - X_{s_{i-1}})^2 \right) \\ & -\left(f(X_{s_{i-1}})g'(Y_{s_i})(-(Y_{s_i} - Y_{s_{i-1}}))^2 + \frac{1}{2}f(X_{s_{i-1}})g''(Y_{s_i})(-(Y_{s_i} - Y_{s_{i-1}}))^2 \right)] \\ = & \sum_{i=1}^{n}[\frac{\delta h}{\delta x}(X_{s_{i-1}}, Y_{s_i})(X_{s_i} - X_{s_{i-1}}) + \frac{1}{2}\frac{\delta^2 h}{\delta x^2}(X_{s_{i-1}}, Y_{s_i})(X_{s_i} - X_{s_{i-1}})^2 \\ & \frac{\delta h}{\delta y}(X_{s_{i-1}}, Y_{s_i})(Y_{s_i} - Y_{s_{i-1}}) + \frac{1}{2}\frac{\delta^2 h}{\delta y^2}(X_{s_{i-1}}, Y_{s_i})(Y_{s_i} - Y_{s_{i-1}})^2]. \end{aligned}$$

Now as $\|\Delta\|_n \to 0$ this expression converges in probability to the RHS of 4.12.

The above integral does not apply directly to the integrals of the form $\int_0^1 W_1 dW_s$. We give the next result from [4]. In this theorem we can evaluate the integrals whose integrand is not adapted or not instantly independent. The difference from the last theorem will be that the instantly independent process Y_t as given in 4.10 can be just Y_k (k is constant) because of which it does not have to depend on t parameter. In the proof the notion of infinite radius of convergence of function g is used so as to move freely from Y_k to Y_t.

Theorem 19. *Suppose that* $h(x,y) = f(x)g(y)$*, where* $f \in C^2(\mathbb{R})$*, and* $g \in C^\infty(\mathbb{R})$ *has Maclaurin expansion with infinite radius of convergence. Let* X_t *be as in Equation 4.9 and* Y_t *be as in Equation 4.10. Then for* $a \leq t \leq b$

$$h(X_t, Y_a) = h(X_a, Y_a) + \int_a^t \frac{\delta h}{\delta x}(X_s, Y_a) dX_s \tag{4.17}$$

$$+\frac{1}{2}\int_a^t \frac{\delta^2 h}{\delta x^2}(X_s, Y_a)(dX_s)^2 - \int_a^t \frac{\delta^2 h}{\delta x \delta y}(X_s, Y_a)(dX_s)(dY_s) \tag{4.18}$$

Example: Let $h(x,y) = \dfrac{x^{n+1}y^m}{n+1}$ with m, n as integers. If we take X_t and Y_a as W_t and $W_1 - W_0$ respectively on the interval $[0,1]$

$$\frac{W_1^{n+1}W_1^m}{n+1} = \frac{W_0^{n+1}W_1^m}{n+1} + \int_0^1 W_t^n W_1^m dW_t + \frac{n}{2}\int_0^1 W_t^{n-1}W_1^m dt$$
$$+m\int_0^1 W_t^n W_1^{m-1} dt. \tag{4.19}$$

Hence we have

$$\int_0^1 W_t^n W_1^{m-1} dW_t = \frac{W_1^{n+m+1}}{n+1} - W_1^{m-1}\int_0^1 W_t^{n-1}\big(\frac{n}{2}W_1 + mW_t\big)dt.$$

4.4 Stochastic Differential Equation

We try to solve a SDE in which the initial condition is not adapted. Consider

$$dX_t = X_t dW_t + \frac{1}{W_1}X_t dt \quad X_0 = W_1 \quad 0 \leq t \leq 1.$$

which can be written in the integral form as

$$X_t = W_1 + \int_0^t X_s dW_s + \int_0^t \frac{1}{W_1}X_s ds. \tag{4.20}$$

We try and solve this equation using iteration technique. Let $X_t^{(1)} = W_1$. Then using the integral computed in the first section and 4.20

$$X_t^{(2)} = W_1 + \int_0^t X_s dW_s + \int_0^t \frac{1}{W_1}X_s ds = W_1(1 + W_t).$$

Using one of the examples of section 2 we write

$$X_t^{(3)} = W_1 + \int_0^t X_s^{(2)} dW_s + \int_0^t \frac{1}{W_1} X_s^{(2)} ds = W_1(1 + W_t + \frac{1}{2}W_t^2 - \frac{1}{2}t).$$

Due to the last example of section 2 and $\xi(x) = x^2$ we have

$$X_t^{(4)} = W_1(1 + W_t + \frac{1}{2}W_t^2 - \frac{1}{2}t + \frac{1}{3!}W_t^3 - \frac{1}{2}tW_t).$$

We also note that this is a Hermite polynomial where

$$\exp(W_t - \frac{1}{2}t) = \sum_{i=0}^{\infty} \frac{H_i(W_t, t)}{n!}$$

$$\text{where } \; H_i(W_t, t) = \sum_{k=0}^{n/2} \begin{pmatrix} n \\ 2k \end{pmatrix} (2k-1)!!(-t)^k W_t^{n-2k}.$$

In general we can write

$$X_t^{(m)} = W_1 \sum_{i=0}^{m-1} \frac{1}{n!} H_n(W_t, t).$$

For say

$$\begin{aligned} X_t^{(2)} &= W_1[H_0(W_t, t) + H_1(W_t, t)] \\ &= W_1[1 + W_1] \end{aligned}$$

$$\begin{aligned} X_t^{(3)} &= W_1[H_0(W_t, t) + H_1(W_t, t) + \frac{1}{2}H_2(W_t, t)] \\ &= W_1[1 + W_1 + \frac{1}{2}W_t^2 - t]. \end{aligned}$$

Hence by the above discussion we can say that

$$X_t = W_1 \exp(W_t - \frac{1}{2}t)$$

which is the desired solution.

4.5 Some open problems

1. What is the class of stochastic processes ξ_t for which the new stochastic integral exists.

2. Can we have Conditions of Existence and Uniqueness for general stochastic differential equation.

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