Characteristic Classes of Vector Bundles

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Certificate of Examination

This is to certify that the dissertation titled **Characteristic Classes of Vector Bundles** submitted by **Neeraj Deshmukh** (Reg. No. MS10085) for the partial fulfillment of BS-MS dual degree program of the Institute, has been examined by the thesis committee duly appointed by the Institute. The committee finds the work done by the candidate satisfactory and recommends that the report be accepted.

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Dated: April 24, 2015

Declaration

The work presented in this dissertation has been carried out by me under the guidance of Dr. Kapil Paranjape at the Indian Institute of Science Education and Research Mohali.

This work has not been submitted in part or in full for a degree, a diploma, or a fellowship to any other university or institute. Whenever contributions of others are involved, every effort is made to indicate this clearly, with due acknowledgement of collaborative research and discussions. This thesis is a bonafide record of work done by me and all sources listed within have been detailed in the bibliography.

Neeraj Deshmukh
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In my capacity as the supervisor of the candidates project work, I certify that the above statements by the candidate are true to the best of my knowledge.

Dr. Kapil Paranjape
(Supervisor)

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Abstract

Given a vector bundle, a natural question to ask is whether it is trivial. This is equivalent to the statement that the bundle admits as many nowhere vanishing, linearly independent vector fields as its rank. Hence, the obstruction to triviality is vanishing of some section. We try to understand this question by studying some well-known topological invariants of real and complex vector bundles. We will construct Stiefel-Whitney classes of real vector bundles and Chern classes of complex vector bundles. These invariants are actually cohomology classes in the cohomology ring the base space B and trivial bundles have trivial invariants. In addition, they also help in distinguishing between different bundles over the same base: in that, bundles with different invariants are different.

We first study Chern-Weil theory which uses differential geometry to construct de Rham cohomology classes for a differential manifold.

We will then study Stiefel-Whitney and Chern classes using algebraic topology for CW complexes.

Contents

Chapter 1

Chern-Weil Theory

In this section we will define Principal Bundles and construct some canonical de Rham cohomology classes for the base. We follow the treatment in [4, 6, 7].

1.1 Principal Bundles

Definition 1.1. *Let $\pi : E \to B$ be a map between smooth manifolds. We say that the quadruple $\xi = (E, \pi, B, F)$ is a* locally trivial *smooth fibre bundle with typical fibre F if for every point $b \in B$,*

1. *There is an open neighbourhood U of b such that $\pi^{-1}(U)$ is diffeomorphic to $U \times F$.*

2. *The fibre over b, viz., $\pi^{-1}(b) := E_b$ is isomorphic to F.*

such that the following diagram commutes.

$$\begin{array}{ccc} \pi^{-1}(U) & \xrightarrow{\quad\phi\quad} & U \times F \\ & {\scriptstyle\pi}\searrow \quad \swarrow{\scriptstyle pr_1} & \\ & U & \end{array}$$

Definition 1.2. *The quadruple (P, π, M, G) with $\pi : P \to M$ is said to be a Principal fibre bundle with Lie group G, if there is a smooth free right action $\mu : P \times G \to P$ such that,*

1. *The action preserves the fibres: $\pi(ug) = \pi(u)$ for all $u \in P$ and $g \in G$.*

2. *For each $p \in M$, there is a bundle chart (U, ϕ) containing p such that if $\phi = (\pi, \Phi) : \pi^{-1}(U) \to U \times G$ is a diffeomorphism and,*

$$\Phi(ug) = \Phi(u)g$$

for all $u \in \pi^{-1}(U)$ and $g \in G$.

We note that the base manifold is in fact the quotient of P by the G action, i.e, $M \simeq P/G$.

Proposition 1.3. *Let (P, π, M, G) be a Principal Bundle. The right G-action restricted to the fibres is transitive.*

Proof. It follows from the definition that any orbit is contained in some fibre. Let $u_1, u_2 \in P_m$ the fibre over the point $m \in M$. Consider a coordinate chart (U, ϕ) containing P_m. Now, let $g := \Phi(u_2)^{-1}\Phi(u_1)$. So that, $\Phi(u_1) = \Phi(u_2)g = \Phi(u_2 g)$. Then,

$$\phi(u_1) = (\pi(u_1), \Phi(u_1)) = (\pi(u_1 g), \Phi(u_2 g)) = \phi(u_2 g)$$

Since, ϕ is a diffeomorphism, $u_1 = u_2 g$ □

For the map $\pi : P \to M$, at any point $u \in P$ we have the sequence,

$$0 \to ker(\pi_{*u}) \to T_u P \to T_{\pi(u)} M \to 0$$

A vector $v \in T_u P$ is said to be vertical if $v \in ker(\pi_{*u})$.

For a fixed $g \in G$, we will the action of this element on P by R_g.

Let $\mathfrak{g}$ be the lie algebra of G. Let $conj_g : G \to G$ denote conjugation by $g \in G$. This is an isomorphism of G and therefore, also induces an isomorphism of $\mathfrak{g}$, which is denoted by $Ad(g)$.

For any $X \in \mathfrak{g}$, we have an associated one-parameter subgroup $exp(tX)$ of G given by the exponential map.

Let $\mu : P \times G \to P$ be the right action of G on P. This gives rise to the action, $P \times exp(tX) \to P$, which is a flow on P. The vector field associated with this flow is called the fundamental vector field generated X and is denoted by $X^\dagger$. Alternately, we may define,

Definition 1.4 (Fundamental Vector Field)**.** *Let $X \in \mathfrak{g}$ be a vector in the lie algebra of G. Define, $(X^\dagger)_p = \mu_{*(p,e)}(0_p, X)$, for $p \in P$ and $e \in G$. Then the vector field, $X^\dagger$ is called the fundamental vector field generated by X.*

Let $P \times \mathfrak{g}$ be the trivial vector bundle with fibre the lie algebra $\mathfrak{g}$ of G. The above discussion also gives rise to a map

$$\psi : P \times \mathfrak{g} \to ker(\pi)$$
$$(p, X) \mapsto (X^\dagger)_p$$

Notice that if $(X^\dagger)_p$ is zero at some point p, then the one parameter subgroup $exp(tX)$ fixes p, i.e, $\mu(p, exp(tX)) = p$ for all t. As the action of G is free, this means that $exp(tX) = e$ for all t. Thus, $X = 0$. This tells us that any non-zero vector in $\mathfrak{g}$ gives rise to a nowhere vanishing vector field on $ker(\pi)$. So, the map ψ gives a trivialisation of $ker(\pi)$.

We can now define the notion of a connection on the principal bundle P.

Definition 1.5. *A connection on P is a $\mathfrak{g}$-valued one form $\omega : TP \to \mathfrak{g}$ satisfying*

1. *If $A^\dagger$ is fundamental vector field on P generated by $A \in \mathfrak{g}$, then $\omega(A^\dagger) = A$*

2. *$R_g^*\omega = Ad(g^{-1})\omega$*

A vector $v \in T_uP$ is said to be horizontal, if $\omega(v) = 0$

Note that a connection on a principle bundle always exists. This is because locally P looks like $U_i \times G$ and we can easily a connection ω_i on every such U_i. The global one form is constructed from these ω_i's by using partitions of unity. In this case the curvature of ω is trivial.

Definition 1.6. *Let ω be a connection on P. We define its curvature to be the $\mathfrak{g}$-valued two form given by,*

$$\Omega(u, v) = d\omega(u, v) + [\omega(u), \omega(v)]$$

We will denote the collection of all k-forms on a manifold M by $\mathcal{A}^k(M)$. A principal bundle $\pi : P \to M$ gives rise to the inclusion $\pi^* : \mathcal{A}^k(M) \subset \mathcal{A}^k(P)$.

Definition 1.7. *Let $\eta \in \mathcal{A}^k(P)$.*

1. *η is basic if it lies in the image of π^*.*

2. *η is semi-basic or horizontal if $\eta(v_1, \ldots, v_k) = 0$ whenever v_1 is tangent to a fibre.*

As η is a skew-symmetric form, the above condition is sufficient since

$$\eta(v_1, \ldots, v_i, \ldots, v_j, \ldots, v_k) = -\eta(v_1, \ldots, v_j, \ldots, v_i, \ldots, v_k)$$

The following two lemmas will be crucial in our discussion of Chern-Weil Theory. They will, in effect, allow us construct invariants for the bundle in the cohomology of the base space.

Lemma 1.8. *1. Basic forms are horizontal.*

2. η is horizontal $\Leftrightarrow$ each $p \in P$ has a neighbourhood U on which there are basic forms η_i such that $\eta = \sum a_i \eta_i$ for some functions $a_i : U \to \mathbb{R}$.

Proof. **(i)** If η is basic, then $\eta = \pi^* \xi$ for some form ξ on M. If v_1 is vertical, then $\pi_*(v) = 0$, so $\eta(v_1, \ldots, v_k) = \xi(\pi_*(v_1), \ldots, \pi_*(v_k)) = 0$

(ii) $\Leftarrow$: As η_i's are basic, they are also horizontal and so is their linear combination $\eta = \sum a_i \eta_i$.

$\Rightarrow$: Fix $p \in P$. Choose a trivialisation around the point $\pi(p)$. Let V be coordinate neighbourhood of the point $\pi(p)$ so that $\pi^{-1}(V) \simeq V \times G$. Write the point p as $p = (p_M, p_G)$. Then choose a coordinate neighbourhood W around p_G. This gives a local coordinate system (x, y) on the neighbourhood $\psi^{-1}(V \times W)$ of p. η can be written in this coordinate neighbourhood as, $\eta = \sum a_{IJ} dx_I \wedge dy_J$. Let b_I and e_J denote the dual bases to dx_I and dy_J. Since η is horizontal, $a_{KL} = \eta(b_K, e_L) = 0$ if $L \neq \varnothing$. Thus $\eta = \sum a_I dx_I$. □

The above result can be used to prove the following lemma

Lemma 1.9. *Let $\pi : P \to M$ be a principal G-bundle with right action $R_g : P \to P$, $g \in G$. Let η be a k-form on P. Then*

$$\eta \text{ is basic} \Leftrightarrow \eta \text{ is horizontal and right } G\text{-invariant}$$

Proof. $\Rightarrow$: By the previous lemma, we only need to show that η is right G-invariant. As η is basic $\eta = \pi^*(\xi)$ for some form ξ on M. Then $R_g^* \eta = R_g^* \pi^* \xi = (\pi R_g)^* \xi = \pi^* \xi = \eta$.

$\Leftarrow$: Since η is horizontal, we can write it locally as, $\eta = \sum a_i \eta_i$ for some basic forms η_i and functions $a_i : U \to \mathbb{R}$. Now, the G-invariance implies that

$$\sum a_i \eta_i = R_g^* (\sum a_i \eta_i) = \sum (R_g^* a_i) \eta_i.$$

This means that $R_g^*(a_i) = a_i$. Thus, $a_i(pg) = a_i(p)$, i.e, the coefficients a_i are constant along the fibres and so are basic functions. Thus, η is basic. □

1.2 Invariant Polynomials and Characteristic Classes

Let $\pi : P \to M$ be a principal G-bundle with a connection. Let Ω be the curvature. Consider the lie algebra $\mathfrak{g}$ of G. Let $S^k(\mathfrak{g})$ be the collection of symmetric k-multilinear functions on $\mathfrak{g}$, or equivalently,

$$S^k(\mathfrak{g}) = \{f | f : \underbrace{\mathfrak{g} \otimes \ldots \otimes \mathfrak{g}}_{k-times} \to \mathbb{R}, f \text{ is symmetric and linear}\}.$$

We will use these two notions interchangeably. Let $I^k(G)$ denote the collection of all symmetric k-multilinear functions on $\mathfrak{g}$ which are invariant under the adjoint action of G, i.e, for any $a \in G$

$$f(Ad(a)v_1, \ldots, Ad(a)v_k) = f(v_1, \ldots, v_k).$$

For any $f \in I^k(G)$ define,

$$f(\Omega^k)(v_1, \ldots, v_{2k}) := f \circ \underbrace{\Omega \wedge \ldots \wedge \Omega}_{k-times}(v_1, \ldots, v_{2k}),$$

where $v_i's \in T_uP$ and $\Omega^k \in \mathcal{A}^{2k}(E, \mathfrak{g} \otimes \ldots \otimes \mathfrak{g})$. Thus $f(\Omega^k) \in \mathcal{A}^{2k}(P)$.

As $R_h^* \Omega = Ad(h^{-1})\Omega$, and f is invariant, $f(\Omega^k)$ is right invariant. Also, since Ω is horizontal, so is $f(\Omega^k)$. Hence, $f(\Omega^k) \in \pi^* \mathcal{A}(M)$, and gives rise to a 2k-form $\bar{f}(\Omega^k)$ on M.

Proposition 1.10. *$\bar{f}(\Omega^k) \in \mathcal{A}^{2k}(M)$ is a closed form, i.e, $d(\bar{f}(\Omega^k)) = 0$*

Proof. As $\pi^* : \mathcal{A}^k(M) \to \mathcal{A}^k(P)$ is injective, we prove that the form $f(\Omega^k)$ is closed. This result follow from the Bianchi identity, $d\Omega = [\Omega, \omega]$. We have

$$df(\Omega^k) = f(d(\Omega^k)) = k\ f([\Omega, \omega] \wedge \Omega^{k-1}).$$

using the symmetry of f. As the above form is horizontal, it suffices to show that it vanishes on $(2k + 1)$-tuples of horizontal vectors. But, as $[\Omega, \omega]$ vanishes on horizontal vectors, this is true. □

We denote the cohomology class associated with $\bar{f}(\Omega^k)$ by $w(f, P) \in H^{2k}_{DR}(M, \mathbb{R})$.

We will now show that $w(f, P)$ are independent of the choice of a connection. To do this, we will use the following version of the Poincaré lemma.

Lemma 1.11. *Let $i_0, i_1 : M \to M \times \mathbb{R}$ be the inclusions $i_0(p) = (p, 0)$ and $i_1(p) = (p, 1)$. Then, there is an operator $h : \mathcal{A}^k(M \times \mathbb{R}) \to \mathcal{A}^{k-1}(M)$ such that, for $\eta \in \mathcal{A}(M \times \mathbb{R})$,*

$$(dh(\eta) - hd(\eta)) = i_1^*(\eta) - i_0^*(\eta)$$

Theorem 1.12. *The cohomology classes $w(f, P) \in H^{2k}_{DR}(M, \mathbb{R})$ are independent of the choice of a connection on P.*

Proof. Let ω_0 and ω_1 be two connections on P and consider the principal bundle $(P \times \mathbb{R}, \pi \times id, M \times \mathbb{R})$. This has a connection defined by,

$$\tilde{\omega}_{(x,s)} = (1 - s)\omega_{0x} + s\omega_{1x}$$

where $(x, s) \in P \times \mathbb{R}$. Note that $\iota_\nu^*(\tilde{\omega}) = \omega_\nu$ for $\nu = 0, 1$.
Hence, $\iota_\nu^*(\tilde{\Omega}) = \Omega_\nu$.

From 1.11, we obtain

$$\begin{aligned} dh(f(\tilde{\Omega}^k)) &= \iota_1^*(f(\tilde{\Omega}^k)) - \iota_0^*(f(\tilde{\Omega}^k)) \\ &= f(\Omega_1) - f(\Omega_0) \end{aligned}$$

□

The classes $w(f, P)$ have the following important property:

Theorem 1.13. *Let $(\xi, \bar{\xi}) : (E, N, \pi') \to (P, M, \pi)$ be a morphism of principal G-bundles, then $w(f, E) = \tilde{\xi}^* w(f, P)$.*

Proof. Let ω be a connection on P. This induces a connection $\xi^*\omega$ on E, such that the curvature is given by $\xi^*\Omega$. Then,

$$\begin{aligned} f((\xi^*(\Omega))^k) &= \xi^* f(\Omega^k) \\ &= \xi^* \pi^* f(\Omega^k) \\ &= \pi'^* \tilde{\xi}^* f(\Omega^k) \end{aligned}$$

which proves the result.

□

As seen above if the bundle is trivial, then it has a canonical connection such that the curvature is identically zero. Then all classes $w(f, P)$ are trivial.

Chapter 2

Classifying Spaces

From now onwards, we will work in the topological category, at times restricting our attention only to CW complexes
We will prove the following classification theorem. This material follow [2]

Theorem 2.1 (Classification Theorem)**.** *Let B be a paracompact Hausdorff space. There is a one-one correspondence between homotopy classes of maps $f : B \to G_k$ and isomorphism classes of rank k-vector bundles over B, given by,*

$$[B, G_k] \to Vect_k(B)$$
$$f \mapsto f^*(\gamma^k)$$

2.1 Grassmannians

We will start with the definition of a vector bundle.

Definition 2.2. *Let V be a finite dimensional vector space over $\mathbb{R}$ or $\mathbb{C}$. A vector bundle with typical fibre V is a quadruple (E, π, B, V) with $\pi : E \to B$ such that:*

1. *For any $p \in B$ there is an open neighbourhood U of p such that $\phi : \pi^{-1}(U) \to U \times V$ is a homeomorphism.*
2. *For any $x \in U$, $\phi|_{E_x} : E_x \to V$ is a vector space isomorphism.*

The Stiefel manifold $V_k(\mathbb{R}^n)$ is the open subset of $\mathbb{R}^{n\times k}$ defined as,

$$V_k(\mathbb{R}^n) = \{A \in \mathbb{R}^{n\times k} | rk(A) = k\}$$

We write such an A as $A = (v_1, v_2, \ldots, v_k)$. Then, there is natural action of $GL(k, \mathbb{R})$ on $V_k(\mathbb{R}^n)$ given by the usual matrix multiplication on the left.

Then one defines the Grassmannian, $G_k(\mathbb{R}^m)$, as the quotient of $V_k(\mathbb{R}^m)$ by the action of $GL(k, \mathbb{R})$ and we have a map,

$$p : V_k(\mathbb{R}^m) \to G_k(\mathbb{R}^m)$$
$$(v_1, \ldots, v_k) \mapsto span\{v_1, \ldots, v_k\}.$$

Consider,

$$\gamma^k(\mathbb{R}^m) = \{(p, v) | p \in G_k(\mathbb{R}^m),\ v \in p\}.$$

Let $X \in G_k(\mathbb{R}^m)$. Let $\Pi : \mathbb{R}^n \to X$ be the projection onto X.
Choose, $U = \{V \in G_k(\mathbb{R}^m) | V \cap X^\perp = 0\}$. This is an open set of $G_k(\mathbb{R}^m)$.There is a local trivialisation on U given by,

$$\Phi : \pi^{-1}(U) \to U \times \mathbb{R}^n$$
$$(V, v) \mapsto (V, v \cdot v_1, v \cdot v_2, \ldots, v \cdot v_k)$$

Where, $\{v_1, v_2, \ldots, v_k\}$ is an orthogonal basis of X.

Now, for $\mathbb{R}^m \subset \mathbb{R}^{m+1} \subset \ldots \mathbb{R}^n \subset \ldots$, we have the inclusions $G_k(\mathbb{R}^m) \subset G_k(\mathbb{R}^{m+1}) \subset \ldots G_k(\mathbb{R}^n) \subset \ldots$.
We define the infinite Grassmannian as,

$$G_k = \bigcup_{m \geq k} G_k(\mathbb{R}^m)$$

A set $U \subseteq G_k$ is open if and only if its intersection with $G_k(\mathbb{R}^m)$ is open for all m. This is called the weak (or direct limit) topology.

Just as in the finite case, we may construct a tautological bundle γ^k over the infinite Grassmannian. We have,

$$\gamma^k = \bigcup_{m \geq k} \gamma^k(\mathbb{R}^m)$$

Lemma 2.3. *$\pi : \gamma^k \to G_k$ is a vector bundle.*

To prove this, we need to use the following lemma

Lemma 2.4. *Let $A_1 \subset A_2 \subset \ldots$ and $B_1 \subset B_2 \subset \ldots$ be sequences of locally compact spaces with direct limits A and B. Then the product topology on $A \times B$ coincides with the direct limit topology associated with the sequence $A_1 \times B_1 \subset A_2 \times B_2 \subset \ldots$.*

Proof. If W is open in the product topology, then for any $(a, b) \in W$, basic open sets with $a \in U$ and $b \in V$ such that $(a, b) \in U \times V \subset W$. But U is open in A if and only if $U \cap A_i$ is open for all A_i (similarly for V), and note that $U \times V \cap A_i \times B_i = U \cap A_i \times V \cap B_i$ for all i. Hence, $U \times V$ is open in the direct limit topology associated with the sequence $A_1 \times B_1 \subset A_2 \times B_2 \subset \ldots$, and so is W.

Conversely, Let W be open in the direct limit topology, and let (a, b) be any point in W. Suppose that $(a, b) \in A_i \times B_i$. Choose a compact neighbourhood K_i of a in A_i and a compact neighbourhood L_i of b in B_i, so that $K_i \times L_i \subset W$. It is now possible to choose a compact neighbourhood K_{i+1} of K_i in A_{i+1} and L_{i+1} of L_i in B_{i+1}, so that $K_{i+1} \times L_{i+1} \subset W$. Continuing by induction, we get sequences $K_i \subset K_{i+1} \subset \ldots$ with union U and $L_i \subset L_{i+1} \subset \ldots$ with union V. Then U and V are open sets, and $(a, b) \in U \times V \subset W$. Thus, W is open in the product topology. □

Proof of 2.3. As in the finite case, take $X \in G_n$ and let $\Pi : \mathbb{R}^\infty \to X$ be the projection onto X. Choose $U = \{V \in G_n \,|\, V \cap X^\perp = 0\}$. Then,

$$U_k = U \cap G_n(\mathbb{R}^{n+k}) = \{V \in G_n(\mathbb{R}^{n+k}) \,|\, V \cap X^\perp = 0\}$$

so that U is open in the direct limit topology. Taking the map,

$$\begin{aligned} \Phi : \pi^{-1}(U) &\to U \times X \\ (V, v) &\mapsto (V, v \cdot v_1, v \cdot v_2, \ldots, v \cdot v_k) \end{aligned}$$

(where $\{v_1, v_2, \ldots, v_k\}$ is an orthogonal basis of X), we see that Φ is continuous in the direct limit topology and, therefore, by the previous lemma, it is continuous in the product topology.

□

2.2 Classification Theorem

We will now prove the following theorem,

Theorem 2.5. *Let B be a paracompact space and $(E, p, B \times I)$ be a vector bundle. Then, the restrictions of E over $B \times \{0\}$ and $B \times \{1\}$ are isomorphic.*

Proof. **Compact Case**: Let us first prove this in the compact case. The general case is similar.
By compactness of $B \times I$, we can find finitely many open sets, $U_1, U_2, \ldots, U_m$ such

that $p^{-1}(U_i \times I)$ is trivial.
Choose a partition of unity $\{\phi_i\}$ subordinate to U_i such that $supp(\phi_i) \subset U_i$. Define $\psi_i : B \to I$ by $\psi_i(x) := \sum_{\nu=1}^{i} \phi_\nu$.
Let $X_i := \{(x, \psi_i)|x \in B\}$ be the graph of ψ_i. Then we have a homeomorphism,

$$\pi_i : X_i \to X_{i-1}$$
$$(x, \psi_i(x)) \mapsto (x, \psi_{i-1}(x)).$$

We will construct a homeomorphism $h_i : E_i \to E_{i-1}$.
Since $supp(\phi_i) \subset U_i$, on $W = B \setminus U_i$, $\psi_i = \psi_{i-1}$.
Thus, $p^{-1}(X_i|_W) = p^{-1}(X_{i-1}|_W)$ and the homeomorphism over W is just the identity map. Now, by construction, the bundle is trivial over $X_i|_{U_i}$. Hence, we have,

$$p^{-1}(X_i|_{U_i}) \simeq_{\theta_i} U_i \times I \times \mathbb{R}^m \xrightarrow{f_i} U_{i-1} \times I \times \mathbb{R}^m \simeq_{\theta_{i-1}} p^{-1}(X_{i-1}|_{U_{i-1}})$$

where θ_i's are the trivialisations. Patching these maps together we obtain a bundle isomorphism, $h_i : E_i \to E_{i-1}$.
Notice that $E_m = p^{-1}(B \times \{1\})$ and $E_0 = p^{-1}(B \times \{0\})$.
Then, the composition $h := h_1 \circ h_2 \circ \ldots \circ h_m$ is the required isomorphism.

General Case: As B is paracompact, we can find a countable open cover $\{V_i\}_{i\in I}$ and a partition of unity $\{\phi_i\}_{i\in I}$ subordinate to $\{V_i\}_{i\in I}$, such that each $V_i = \sqcup(V_i \cap U_\alpha)$. Thus, E is trivial over V_i. Also, choose a well-ordering for the indexing set I. Then, by similar arguments as before, we can construct the infinite composition $h := h_1 \circ h_2 \circ \ldots$, using the well-ordering on I. By paracompactness, for any point $x \in B$ has a neighbourhood U such that only finitely may V_i's intersect it. Let $\{V_{i_1}, V_{i_2}, \ldots, V_{i_n}\}$ be the collection of V_i's which intersect U, so that $U = \sqcup_k(V_{i_k} \cap U)$. Then, on U, the above map takes the form $h = h_{i_1} \circ h_{i_2} \circ \ldots h_{i_n}$, which is well-defined, continuous and therefore the required isomorphism.

□

Definition 2.6. *Let $\pi : E \to B$ be a rank k vector bundle and $f : A \to B$ be a continuous map. Then the pullback of E along f is defined as,*

$$f^*(E) := \{(a, e) \in A \times E|f(a) = \pi(e)\}$$

is a vector bundle over A of rank k such that the following diagram commutes,

$$\begin{array}{ccc} f^*(E) & \xrightarrow{prj_2} & E \\ \Big\downarrow{\scriptstyle prj_1} & & \Big\downarrow{\scriptstyle \pi} \\ A & \xrightarrow{f} & B \end{array}$$

Let $f_0, f_1 : A \to B$ two homotopic maps. This gives us a homotopy $F : A \times I \to B$ such that $F|_0 = f_0$ and $F_1 = f_1$. Now, if E is a vector bundle over B, we get a pullback $F^*(E)$ over $A \times I$. Using the previous theorem, we have proved:

Theorem 2.7 (Homotopy Invariance of Pullbacks)**.** *Let $f_0, f_1 : A \to B$ be homotopic maps and $E \to B$ be a vector bundle. Then, $f_0^*(E) \cong f_1^*(E)$.*

We can now prove the classification theorem.
Let $E \to B$ be any vector bundle of rank k with B paracompact Hausdorff.

Theorem 2.8 (Classification Theorem)**.** *Let B be a paracompact Hausdorff space. There is a one-one correspondence between homotopy classes of maps $f : B \to G_k$ and isomorphism classes of rank k-vector bundles over B, given by,*

$$[B, G_k] \to Vect_k(B)$$
$$f \mapsto f^*(\gamma^k)$$

Proof. Note that an isomorphism $E \simeq f^*(\gamma_k)$ is equivalent to a map $g : E \to \mathbb{R}^\infty$ which is a linear injection on each fibre of E. Similarly, given such a map $g : E \to \mathbb{R}^\infty$, we define $f : B \to G_k$ by $f(x) = g(p^{-1}(E_x))$.

Surjectivity: As before, we take a countable cover $\{U_i\}$ of B with a partition of unity $\{\phi_i\}$ subordinate to $\{U_i\}$, such that E is trivial over each U_i. Let, g_i be the composition of the local trivialization on U_i with the projection onto $\mathbb{R}^k$, i.e,

$$g_i : p^{-1}(U_i) \to U_i \times \mathbb{R}^k \overset{prj}{\to} \mathbb{R}^k$$

Then define, $g : E \to \mathbb{R}^\infty$ by,

$$v \mapsto (\phi_1(p(v))g_1(v), \phi_2(p(v))g_2(v), \ldots)$$

This is well-defined by local finiteness of the cover and is a linear injection on each fibre.

Injectivity: Let $E \simeq f_0^*(\gamma_k) \simeq f_1^*(\gamma_k)$, for two maps $f_0, f_1 : B \to G_k$.
Then we have maps $g_0, g_1 : E \to \mathbb{R}^\infty$ which are linear injections on the fibres.
Let, $g_0(v) = (x_1, x_2, \ldots)$. Then, g_0 is homotopic to the map $g_0'(v) = (x_1, 0, x_2, 0, \ldots)$.
Similarly, g_1 is homotopic to the map $g_1'(v) = (0, y_1, 0, y_2, 0, \ldots)$. And, $g_t' = (1-t)g_0' +$

tg_1 is a homotopy from g'_0 to g'_1.

Hence, g_0 is homotopic to g_1. Let g_t be such a homotopy.

Then, $f_t(x) = g_t(p^{-1}(E_x))$ is a homotopy between f_0 and f_1.

$\square$

Chapter 3

Cell Structure for Grassmannians

This section mirrors the treatment in [1].

Definition 3.1. *A CW-complex consists of a Hausdorff space K, together with a partition of K into a collection $\{e_\alpha\}$ of disjoint subsets, such that,*

1. *Each e_α is homeomorphic to an open ball of dimension $n(\alpha) \geq 0$. Furthermore, for each e_α there exists a continuous map,*

$$f : D^{n(\alpha)} \to K$$

which carries the interior of the closed ball $D^{n(\alpha)}$ homeomorphically onto e_α

2. *Each point $x \in (\bar{e_\alpha} \setminus e_\alpha)$ lies in an e_β of lower dimension.*

3. *Each point of K is contained in a finite subcomplex. A subcomplex of K is a closed subset which is a union of finitely many e_α's.*

4. *K is topologised as the direct limit of its finite subcomplexes.*

We shall now exhibit a cell structure for the Grassmannian, $G_n(\mathbb{R}^m)$. As the infinite Grassmannian G_n is a direct limit of $\{G_n(\mathbb{R}^m\}_{m\geq n}$, the cell structure for $G_n(\mathbb{R}^m)$ induces a cell structure for G_n.

Fix a collections of subspaces for $\mathbb{R}^m$,

$$\mathbb{R}^0 \subset \mathbb{R}^1 \subset \ldots \mathbb{R}^{m-1} \subset \mathbb{R}^m$$

Then any n-plane $X \subset \mathbb{R}^m$ gives rise to a sequence of integers,

$$0 \leq dim(X \cap \mathbb{R}^1) \leq dim(X \cap \mathbb{R}^2) \leq \ldots \leq dim(X \cap \mathbb{R}^m) = n$$

Any two consecutive integers in this sequence differ by at most 1.

By a Schubert symbol $\sigma = (\sigma_1, \sigma_2 \ldots, \sigma_n)$ we mean a sequence of n integers satisfying

$$1 \leq \sigma_1 \leq \sigma_2 \leq \ldots \leq \sigma_n \leq m.$$

For each Schubert symbol σ, let $e(\sigma) \subset G_n(\mathbb{R}^m)$ denote the set of all n-plane X such that,

$$e(\sigma) = \{X \,|\, dim(X \cap \mathbb{R}^{\sigma_i}) = i,\, dim(X \cap \mathbb{R}^{\sigma_{i-1}}) = i - 1 \,\forall\, i\}$$

Each $X \subset G_n(\mathbb{R}^m)$ belongs to precisely one of the sets $e(\sigma)$. We will now show that each $e(\sigma)$ is an open cell of dimension $d(\sigma) = (\sigma_1 - 1) + (\sigma_2 - 2) + \ldots + (\sigma_n - n)$.

Let $\mathbb{H}^k \subset \mathbb{R}^k$ denote the upper half-space consisting of all $x = (\xi_1, \ldots, \xi_k, 0, \ldots, 0)$ with $\xi_k > 0$. Then, it is easy to see that an n-plane $X \in e(\sigma)$ if an only if it possesses a basis $x_1, \ldots, x_n$ so that $x_1 \in \mathbb{H}^{\sigma_1}, \ldots, x_n \in \mathbb{H}^{\sigma_n}$.

Lemma 3.2. *Each n-plane $X \in e(\sigma)$ possesses a unique orthonormal basis $(x_1, \ldots, x_n) \in \mathbb{H}^{\sigma_1} \times \ldots \times \mathbb{H}^{\sigma_n}$*

Proof. Let x_1 lie in the one dimensional space $X \cap \mathbb{R}^{\sigma_1}$. Then, the conditions that x_1 is a unit vector and that its σ_1-th coordinate is positive determines it. Now x_2 is a unit vector in the two dimensional space $X \cap \mathbb{R}^{\sigma_2}$, orthogonal to x_1. Again, it is determined by the condition that the σ_2-th coordinate is positive. Continuing by induction, we get the required basis.

□

Definition 3.3. *Let $e'(\sigma)$ denote the set of all orthonormal n-frames $(x_1, \ldots, x_n)$ such that each $x_i \in \mathbb{H}^{\sigma_i}$. Let $\overline{e}'(\sigma)$ denote the set of all orthonormal frames $(x_1, \ldots, x_n)$ such that $x_i \in \overline{\mathbb{H}}^{\sigma_i}$.*

Lemma 3.4. *The set $\overline{e}'(\sigma)$ is topologically a closed cell of dimension $d(\sigma) = (\sigma_1 - 1) + (\sigma_2 - 2) + \ldots + (\sigma_n - n)$, with the interior $e'(\sigma)$. Furthermore, there is a homeomorphism $e'(\sigma) \simeq e(\sigma)$.*

Proof. Note that the map $q : \overline{e}'(\sigma) \to G_n(\mathbb{R}^m)$ just sends an n-frame to the subspace spanned by it.

The proof will be by induction on n. For $n = 1$, the set $\overline{e}'(\sigma_1)$ consists of all vectors

$$x_1 = (x_{11}, x_{12}, \ldots, x_{1\sigma_1}, 0, \ldots, 0).$$

with $\sum x_{1i}^2 = 1$, $x_{1\sigma_1} \geq 1$. Evidently, $\overline{e}'(\sigma_1)$ is a closed hemisphere of dimension $\sigma_1 - 1$, and therefore is homeomorphic to the disc $D^{\sigma_1 - 1}$.

Given unit vectors $u, v \in \mathbb{R}^m$ consider the map,

$$T(u,v)x = x - \frac{(u+v)\cdot x}{1+u\cdot v}(u+v) + 2(u\cdot x)v.$$

Evidently, $T(u,u) = Id$ and $T(v,u) = T(u,v)^{-1}$. This map carries u to v and leaves everything orthogonal to u and v fixed. It follow that $T(u,v)$ is a continuous function in three variables and that if $u, v \in \mathbb{R}^k$ then $T(u,v)x \equiv x \,(mod\, \mathbb{R}^k)$.

Let $b_i \in \mathbb{H}^{\sigma_i}$ denote the vector with the σ_i-th coordinate equal to 1, and all other coordinates equal to zero. Thus $(b_1, \dots, b_n) \in e'(\sigma)$. For any n-frame $(x_1, \dots, x_n) \in \overline{e}'(\sigma)$ consider the rotation,

$$T = T(b_n, x_n) \circ T(b_{n-1}, x_{n-1} \circ \dots \circ T(b_1, x_1))$$

of $\mathbb{R}^m$. This rotation carries the n-frame $(b_1, \dots, b_n)$ to the n-frame $(x_1, \dots, x_n)$.

Given an integer $\sigma_{n+1} > \sigma_n$, let D denote the set of all unit vectors $u \in \overline{\mathbb{H}}^{\sigma_{n+1}}$ with $b_i \cdot u = 0 \,\forall\, i$. Then D is closed hemisphere of dimension $\sigma_{n+1} - n - 1$, and hence is topologically a closed cell. We will now a homeomorphism,

$$f : \overline{e}'(\sigma_1, \dots, \sigma_n) \times D \to \overline{e}'(\sigma_1, \dots, \sigma_{n+1}).$$

To do this, define f by the formula,

$$f((x_1, \dots, x_n), u) = (x_1, \dots, x_n, Tu),$$

Note that $x_i \cdot Tu = Tb_i \cdot Tu = b_i \cdot u = 0,\ \forall\ i$, and that $Tu \cdot Tu = u \cdot u = 1$, where $Tu \in \overline{H}^{\sigma_{n+1}}$ since $Tu \equiv u \,(mod\, \mathbb{R}^{\sigma_n})$. The map f is continuous because T is continuous. One may, similarly, define the inverse of f by sending the last coordinate $x_{n+1} \mapsto T^{-1}x_{n+1}$.

Thus, we have $\overline{e}'(\sigma_1, \dots, \sigma_{n+1})$ is homeomorphic to the product $\overline{e}'(\sigma_1, \dots, \sigma_n) \times D$. By induction, it follows that each $\overline{e}'(\sigma)$ is a closed cell of dimension $d(\sigma)$.

To see that q is a homeomorphism, observer that q carries $e'(\sigma)$ in one-one fashion onto $e(\sigma)$. If $(x_1, \dots, x_n)$ belongs to the boundary $\overline{e}'(\sigma) \setminus e'(\sigma)$, then the n-plane X spanned by this n-frame does not belong to $e(\sigma)$, for one of the vectors x_i must lie in the boundary $\mathbb{R}^{\sigma_i - 1}$ of the half-space $\overline{\mathbb{H}}^{\sigma_i}$. This means that, $dim(X \cap \mathbb{R}^{\sigma_i - 1}) \geq i$, and so X does not lie in $e(\sigma)$.

Now let $A \subset e'(\sigma)$ be a relatively compact subset. Then $\overline{A} \cap e'(\sigma) = A$, where the closure $\overline{A} \subset \overline{e}'(\sigma)$ is compact. Hence, $q(\overline{A})$ is closed. Then, $q(\overline{A}) \cap e(\sigma) = q(A)$, and it follows that $q(A) \subset e(\sigma)$ is a relatively closed set. Thus q is a homeomorphism.

□

Theorem 3.5. *The* $\binom{m}{n}$ *sets* $e(\sigma)$ *form the cells of a CW complex with the underlying space* $G_n(\mathbb{R}^m)$*. Further, taking the direct limit one obtains a CW complex with the underlying space* G_n*.*

Proof. We need to show that a point in the boundary of an $e(\sigma)$ lies in an $e(\tau)$ of lower dimension. As $\overline{e}'(\sigma)$ is compact, the image $q(\overline{e}'(\sigma)) = \overline{e}(\sigma)$. Hence, every n-plane X in the boundary $\overline{e}(\sigma) \setminus e(\sigma)$ has a basis $(x_1, \ldots, x_n)$ belonging to $\overline{e}'(\sigma) \setminus e'(\sigma)$. Also, the Schubert symbol associated with X must satisfy $\tau_i \leq \sigma_i \; \forall \; i$.

Since, X lies in the boundary, one of the vectors x_i must actually belong to $\mathbb{R}^{\sigma_i - 1}$, hence the corresponding τ_i must be strictly less than σ_i. Therefore, $d(\tau) \leq d(\sigma)$. Then, the map q gives the required boundary map on various cells. This completes the proof that $G_n(\mathbb{R}^m)$ is a finite CW complex.

In the case of G_n, the topology is by definition the direct limit topology. Also, any n-plane X in $\mathbb{R}^\infty$ is contained in some finite dimensional $\mathbb{R}^m$ for m sufficiently large, so that $X \in G_n(\mathbb{R}^m)$. This gives the closure finiteness condition. □

The above CW structure may be easily adapted to the complex case for $G_n(\mathbb{C}^m)$. In that case all dimensions will increase by a multiple of 2.

Chapter 4

Characteristic Classes

This section is based on the approach described in [2, 3]

4.1 Leray-Hirsch Theorem

In this section, we will give a topological construction of Stiefel-Whitney classes of real vector bundle and Chern classes of complex vector bundles. Our construction will depend on the Leray-Hirsch Theorem (4.2). We will prove this result for CW complexes, though it is valid in general. We begin with the following lemma which will be used in the proof of that result.

Note: All spaces are assumed to be CW complexes.

Lemma 4.1. *Given a fibre bundle $p : E \to B$ and a subspace $A \subset B$ such that (B, A) is k-connected, then $(E, p^{-1}(A))$ is also k-connected.*

Proof. For a map $g : (D^i, \partial D^i) \to (E, p^{-1}(A))$ with $i \leq k$, there is by hypothesis a homotopy $f_t : (D^i, \partial D^i) \to (B, A)$ of $f_0 = pg$ to a map f_1 with image in A. The homotopy lifting property then gives a homotopy $g_t : (D^i, \partial D^i) \to (E, p^{-1}(A))$ of g to a map with image in $p^{-1}(A)$. □

Theorem 4.2 (Leray-Hirsch Theorem)**.** *Let $E \to B$ be a fibre bundle with fibre F such that for some commutative coefficient ring R:*

1. *$H^n(F; R)$ is a finitely generated free R-module for each n.*

2. *There exist classes $c_j \in H^{k_j}(E; R)$ whose restrictions $i^*(c_j)$ form a basis for $H^n(F; R)$ in each fibre F, where $i : F \to E$ is the inclusion map for some chosen base point.*

Then the map $\Phi : H^*(B;R) \otimes H^*(F;R) \to H^*(E;R)$, given by

$$\Sigma_{ij} b_i \otimes i^*(c_j) \mapsto \Sigma_{ij} p^*(b_i) \smile c_j$$

is an isomorphism of $H^*(B;R)$-modules.

Proof. We prove the result in two step.
Step 1: Let B be a finite dimensional CW complex. We proceed by induction on dimension of B. If B is zero dimensional, then the claim holds trivially.
So, assume that the dimension of B is n and let $B' \subset B$ obtained by deleting a point x_α from each n-cell e^n_α, that is, $B' = B^{n-1} \bigsqcup (\sqcup_\alpha (e^n_\alpha \setminus \{x_\alpha\}))$. Let $E' = p^{-1}(B')$. Then we have a commutative diagram,

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & H^*(B,B') \otimes_R H^*(F) & \longrightarrow & H^*(B) \otimes_R H^*(F) & \longrightarrow & H^*(B') \otimes_R H^*(F) & \longrightarrow & \cdots \\ & & \downarrow \Phi & & \downarrow \Phi & & \downarrow \Phi & & \\ \cdots & \longrightarrow & H^*(E,E') & \longrightarrow & H^*(E) & \longrightarrow & H^*(E,E') & \longrightarrow & \cdots \end{array}$$

The map Φ is defined exactly as in the absolute case, as

$$H^*(B,B') \otimes_R H^*(F) \to H^*(B,B') \otimes_R H^*(F) \xrightarrow{\smile} H^*(E)$$

where the last map is the relative cup product. As $H^*(F)$ is a free module, the first row is exact. The second row is also exact. By construction, the two squares are commutative, for the other square involving coboundary maps, take
$b \otimes i^*(c_j) \in H^*(B') \otimes_R H^*(F)$. Mapping this horizontally, we get $\delta b \otimes c_j$ which maps vertically onto $p^* \delta b \smile c_j$. On the other hand, first mapping $b \otimes i^*(c_j)$ vertically we get $p^* b \smile c_j$ which maps horizontally onto $\delta p^* b \smile c_j = p^* \delta(b) \smile c_j$ since $\delta c_j = 0$.
Note that B' deformation retracts onto B^{n-1} and hence, the right-hand vertical arrow is an isomorphism. Thus, we need to show show that the left-hand arrow is an isomorphism. Then the theorem will follow by the five-lemma.
By the fibre bundle property there are open disc neighbourhood's $U_\alpha \subset e^n_\alpha$ of the points x_α such that the bundle is a product over each U_α. Let $U = U_\alpha$ and $U' = U \cap B'$. By excision we have $H^*(B,B') \simeq H^*(U,U')$, and $H^*(E,E') \simeq H^*(p^{-1}(U), p^{-1}(U'))$. This reduces the problem to showing that $\Phi : H^*(U,U') \otimes_R H^*(F) \to H^*(U \times F, U' \times F)$ is an isomorphism. As U, U' deformation retract onto complexes of dimensions 0 and $n-1$, respectively, applying five-lemma to long exact cohomology sequence for the pair (U,U'), we get $H^*(U,U') \otimes_R H^*(F) \simeq H^*(U \times F, U' \times F)$ and therefore the theorem in the finite

case.
Step 2: Next we look at the case where B is an infinite dimensional CW complex. In this case we have the commutative diagram,

$$\begin{array}{ccc} H^*(B)\otimes_R H^*(F) & \longrightarrow & H^*(B^n)\otimes_R H^*(F) \\ \downarrow \Phi & & \downarrow \Phi \\ H^*(E) & \longrightarrow & H^*(p^{-1}(B^n)) \end{array}$$

Since (B, B^n) is n-connected, the lemma implies that the same is true for $(E, p^{-1}(B^n))$. Thus the horizontal maps are isomorphism below dimension n. Then as the right-hand Φ is an isomorphism, we have that the left-hand is also an isomorphism below dimension n. As n was arbitrary this gives the theorem.

□

4.2 Stiefel-Whitney and Chern Classes

We will use the Theorem 4.2 to construct Stiefel-Whitney classes (Chern classes, respectively) of real (complex, respectively) vector bundles.

Theorem 4.3. *Let $E \to B$ be a real vector bundle of rank n. There exist a sequence of functions $w_1, w_2, \ldots$, each assigning to E a class $w_i(E) \in H^i(B; \mathbb{Z}_2)$, such that,*

1. *$w_i(f^*(E)) = f^*(w_i(E))$ for the pullback $f^*(E)$.*
2. *Let $w = 1 + w_1 + w_2 + \ldots \in H^*(B; \mathbb{Z}_2)$, then $w(E_1 \oplus E_2) = w(E_1) \smile w(E_2)$.*
3. *$w_i(E) = 0$ if $i > n$.*
4. *For the canonical line bundle $\gamma^1 \to \mathbb{R}P^\infty$, $w_1(\gamma^1)$ is a generator of $H^1(\mathbb{R}P^\infty : \mathbb{Z}_2)$.*

Similarly, for complex vector bundles, we have:

Theorem 4.4. *Let $E \to B$ be a complex vector bundle of rank n. There exist a sequence of functions $c_1, c_2, \ldots$ each assigning to E a class $c_i(E) \in H^{2i}(B; \mathbb{Z})$, such that,*

1. *$c_i(f^*(E)) = f^*(c_i(E))$ for the pullback $f^*(E)$.*
2. *Let $c = 1 + c_1 + c_2 + \ldots \in H^*(B; \mathbb{Z})$, then $c(E_1 \oplus E_2) = c(E_1) \smile c(E_2)$.*
3. *$c_i(E) = 0$ if $i > n$.*

4. *For the canonical line bundle $\gamma^1 \to \mathbb{C}P^\infty$, $c_1(\gamma^1)$ is a generator of $H^1(\mathbb{C}P^\infty : \mathbb{Z})$, specified in advance.*

Notice that for Chern classes, the generator of $H^1(\mathbb{C}P^\infty; \mathbb{Z})$ be specified in advance, while no such condition is required for Stiefel-Whitney classes. This is because, in the case of Stiefel-Whitney classes, the coefficient ring is $\mathbb{Z}_2$, hence there is a unique generator for $H^1(\mathbb{R}P^\infty; \mathbb{Z}_2)$.

We will only prove 4.3. The proof of 4.4 is exactly the same obtained by replacing $\mathbb{R}P^\infty$ with $\mathbb{C}P^\infty$ and taking cohomology with $\mathbb{Z}$ coefficients.

But first, we note an important result which will be required to establish the uniqueness of these characteristic classes.

Lemma 4.5 (The Splitting Principle)**.** *Let $E \xrightarrow{\pi} B$ be a vector bundle. Then, there exists a space $F(E)$ and map $p : F(E) \to B$ such the $p^*(E)$ splits as a direct sum of line bundles and $p^* : H^*(B; \mathbb{Z}_2) \to H^*(F(E); \mathbb{Z}_2)$ is injective.*

Proof. Let $E \to B$ be a vector bundle. Consider the deleted bundle $E' = E \setminus \{0\}$ with the zero section removed. Then, we may construct $P(E)$ as the space of all lines through the origin in all the fibres of E. This has a natural projection map $P(p)$ to B sending each line in $p^{-1}(b)$ to $b \in B$. Note that $P(E) = E' \diagup \{v \sim \lambda v | v \in p^{-1}(b), \lambda \in \mathbb{R}\}$. Thus, $P(E)$ inherits the quotient topology from E'. Locally, if U is a neighbourhood of a point $x \in B$ then $p^{-1}(U) \simeq U \times \mathbb{R}^n$. So, on $E'|_{p^{-1}(U)}$ the above quotient is $U \times \mathbb{R}P^{n-1}$. This gives a fibre bundle structure on $P(E)$ over B. Now, we would like to apply 4.2 to the bundle $P(E)$ for $\mathbb{Z}_2$ coefficients. Recall that there is a map $g : E \to \mathbb{R}^\infty$, that is a linear injection on each fibre. Projectivising this map we get $P(g) : P(E) \to \mathbb{R}P^\infty$. Let α be the generator of $H^1(\mathbb{R}P^\infty; \mathbb{Z}_2)$ and let $x = P(g)^*(\alpha) \in H^1(P(E); \mathbb{Z}_2)$. Also, the injection $\mathbb{R}^n \hookrightarrow E \hookrightarrow \mathbb{R}^\infty$ gives an embedding $\mathbb{R}P^{n-1} \hookrightarrow P(E) \hookrightarrow \mathbb{R}P^\infty$ for which α pull back to a generator of $H^1(\mathbb{R}P^{n-1}; \mathbb{Z}_2)$, so that $\alpha^i \in H^i(\mathbb{R}P^\infty; \mathbb{Z}_2)$ pulls back to a generator of $H^i(\mathbb{R}P^{n-1}; \mathbb{Z}_2)$. Thus, from the maps above, the classes x^i restrict to generators of $H^i(\mathbb{R}P^{n-1})$. Note that any two linear maps $\mathbb{R}^n \to \mathbb{R}^\infty$ are homotopic through linear injections, so the induced embeddings $\mathbb{R}P^{n-1} \hookrightarrow \mathbb{R}P^\infty$ of different fibres are all homotopic. As any two choices of the map g are homotopic through maps that are linear injections on fibres, so the classes x^i are independent of the choice of g.

Consider the pullback $P(\pi)^*(E)$ of the map $P(\pi) : P(E) \to B$. This contains a natural one-dimensional sub-bundle

$$L = \{(\ell, v) \in P(E) \times E | \, v \in \ell\}$$

. Equipping $P(\pi)^*(E)$ with an inner product we get a decomposition $P(\pi)^*(E) = L \oplus L^\perp$. Now, 4.2 applied to $P(E) \to B$, gives $H^*(P(E);\mathbb{Z}_2)$ as a free $H^*(B;\mathbb{Z}_2)$-module with 1 as basis element. So, in particular, the induced map $p^* : H^*(B;\mathbb{Z}_2) \to H^*(P(E);\mathbb{Z}_2)$ is injective. Repeating this construction with $L^\perp \to P(E)$ in place $E \to B$, finitely many times gives the desired result. □

We can now prove 4.3.

Existence and Uniqueness of Stiefel-Whitney Classes. The idea is to simply apply 4.2 to the bundle $P(E) \to B$. Doing so, we can write $x^n \in H^*(P(E);\mathbb{Z}_2)$ uniquely as

$$x^n = \sum_{i=1}^{n} (-1)^{i-1} w_i(E) x^{n-i},$$

for certain classes $w_i(E) \in H^i(B : \mathbb{Z}_2)$. Here,

$$w_i(E)x^{n-i} = P(p)^*(w_i(E)) \smile x^{n-i}.$$

For completeness, we define $w_i(E) = 0$ for $i > n$ and $w_0(E) = 1$.

We now check that these $w_i(E)$ indeed satisfy the given properties.

Consider the diagram,

$$\begin{array}{ccc} E' & \xrightarrow{\tilde{f}} & E \\ \downarrow{\scriptstyle \pi'} & & \downarrow{\scriptstyle \pi} \\ B' & \xrightarrow[f]{} & B \end{array}$$

If $g : E \to \mathbb{R}^\infty$ is a linear injection on fibres then so is $g\tilde{f}$. It follows that $P(\tilde{f})^*$ take the canonical class $x = x(E)$ of $P(E)$ to the canonical class $c(E')$ for $P(E')$. Then,

$$\begin{aligned} &P(\tilde{f})^*\Big(\sum_{i=1}^{n} (-1)^{i-1} P(\pi)^*(w_i(E)) \smile x(E)^{n-i}\Big) \\ &= \sum_{i=1}^{n} (-1)^{i-1} P(\tilde{f})^* P(\pi)^* w_i(E) \smile P(\tilde{f})^* x(E)^{n-i} \\ &= \sum_{i=1}^{n} (-1)^{i-1} P(\pi')^* P(f)^* w_i(E) \smile x(E')^{n-i} \end{aligned}$$

so the defining equation for $x(E)^n$ pulls back to an equation for $x(E')^n$ with coefficients $f^*(w_i(E))$. By our construction, we get $w_i(E') = f^*(w_i(E))$.

To prove property (2), note that the inclusions of E_1 and E_2 into $E_1 \oplus E_2$ give inclusions of $P(E_1)$ and $P(E_1)$ into $P(E_1 \oplus E_2)$ with $P(E_1) \cap P(E_2) = \varnothing$. Let $U_1 = P(E_1 \oplus E_2) \setminus P(E_1)$ and $U_2 = P(E_1 \oplus E_2) \setminus P(E_2)$. These are open sets in $P(E_1 \oplus E_2)$ that deformation retract onto $P(E_1)$ and $P(E_2)$, respectively. Let $g : E_1 \oplus E_2 \to \mathbb{R}^\infty$ be a continuous map which is linear injection on each fibre, then the canonical class $x \in H^1(P(E_1 \oplus E_2); \mathbb{Z}_2)$ for $E_1 \oplus E_2$ restricts to canonical classes for E_1 and E_2, following the natural injections. If E_1 and E_2 have dimensions m and n, respectively, consider classes $\omega_1 = \sum_i (-1)^i w_i(E_1) x^{m-i}$ and $\omega_2 = \sum_j (-1)^{j-1} w_j(E_2) x^{n-j}$ in $H^*(P(E_1 \oplus E_2); \mathbb{Z}_2)$. Taking cup product, $\omega_1\omega_2 = \sum_r (-1)^r [\sum_{i+j=r} w_i(E_1) w_j(E_2)] x^{m+n-r}$. By the definition, ω_i pull back to zero in $H^*(P(E_i); \mathbb{Z}_2)$. This means that, on considering the long exact sequence associated the pairs $P(E_1 \oplus E_2), P(E_1)$, ω_1 gives a class in the relative group $H^m(P(E_1 \oplus E_2), P(E_1); \mathbb{Z}_2) \simeq H^m(P(E_1 \oplus E_2), U_2; \mathbb{Z}_2)$, and similarly for ω_2. Then, we have the following commutative diagram with $\mathbb{Z}_2$ coefficients,

$$\begin{array}{ccc} H^m(P(E_1 \oplus E_2), U_2) \times H^n(P(E_1 \oplus E_2), U_2) & \xrightarrow{\smile} & H^{m+n}(P(E_1 \oplus E_2), U_1 \cup U_2) = 0 \\ \downarrow & & \downarrow \\ H^m(P(E_1 \oplus E_2)) \times H^n(P(E_1 \oplus E_2)) & \xrightarrow{\smile} & H^{m+n}(P(E_1 \oplus E_2)) \end{array}$$

Thus, $\omega_1\omega_2 = 0$ and so is the defining relation for the Stiefel-Whitney classes for $E_1 \oplus E_2$, so that $w_r = \sum_{i+j=r} w_i(E_1) w_j(E_2)$, and this proves property (2).

Property (3) is true by definition. For property (4), take the canonical line bundle over $\mathbb{R}P^\infty$, $\gamma^1 = \{(\ell, v) \in \mathbb{R}P^\infty \times \mathbb{R}^\infty | \, v \in \ell\}$ and the map $P(\pi)$ is identity. Then, the map $g(\ell, v) = v$ is a linear injection on fibres, and this gives that $P(g)$ is identity. So, $x(\gamma^1)$ is a generator of $H^1(\mathbb{R}P^\infty; \mathbb{Z}_2)$. From the defining relation, $x(\gamma^1) + w_1(\gamma^1) \cdot 1 = 0$, we have that $w_1(\gamma^1)$ is a generator of $H^1(\mathbb{R}P^\infty; \mathbb{Z}_2)$.

To prove uniqueness, we use the splitting principle. Let $w(E)$ and $v(E)$ be classes satisfying the above properties. By the splitting principle, there is a space $F(E)$ with a map $F(E) \xrightarrow{p} B$, such that $p^*(E) = \ell_1 \oplus \ldots \oplus \ell_n$. Then,

$$\begin{aligned} p^*(w(E)) &= w(\ell_1 \oplus \ldots \oplus \ell_n) \\ &= w(\ell_1) \ldots w(\ell_n) \\ &= v(\ell_1) \ldots v(\ell_n) \\ &= v(\ell_1 \oplus \ldots \oplus \ell_n) \\ &= p^*(v(E)) \end{aligned}$$

as the induced map $p^* : H^*(B; \mathbb{Z}_2) \to H^*(F(E); \mathbb{Z}_2)$ is injective, we get $w(E) = v(E)$. □

We now prove the result we stated earlier,

As immediate consequence of the above theorem is the following,

Corollary 4.6. *Let $\Theta^k \to B$ be a trivial (real or complex) vector bundle. Then $x(\Theta^k) = 1$ (here, $x(\Theta^k)$ is total Stiefel-Whitney or Chern class, depending on the context). Further, if $E \to B$ is another (real or complex) vector bundle, then $x(E) = x(E \oplus \Theta^k)$.*

Proof. The trivial bundle is induced by a constant map of B into the infinite grassmannian. Thus, map induced in cohomology is trivial. Hence, $x(\Theta^k) = 1$. The second part simply follows from the product formula for the direct sum.

□

4.3 Cohomology of G_n

We now use the above result to compute the cohomology of the infinite grassmannian, $H^*(G_n)$. This was original done in [8].

Theorem 4.7. *$H^*(G_n; \mathbb{Z}_2) = \mathbb{Z}_2[w_1, \ldots, w_n]$, where $w_i = w_i(\gamma^n)$ of the universal bundle $\gamma^n \to G_n$.*

Proof. Consider the bundle , $\xi = (\gamma^1)^n \to (\mathbb{R}P^\infty)^n$. Then $w(\xi) = (1 + a_1)(1 + a_2) \ldots (1 + a_n)$. Thus, each $w_k(\xi)$ is a symmetric polynomial in the $a_i's$. Note that, by Ḱunneth formula $H^*((\mathbb{R}P^\infty)^n; \mathbb{Z}_2) = \mathbb{Z}_2[a_1, \ldots, a_n]$. So, we have that $w_1(\xi), \ldots, w_n(\xi)$ are algebraically independent over $\mathbb{Z}_2$.

If $f : (\mathbb{R}P^\infty)^n \to G_n$ such that $f^*(\gamma^n) = \xi$, then $f^*(w_i(\gamma^n)) = w_i(\xi)$, so that each $w_i(\gamma^n)$ is also algebraically independent over $\mathbb{Z}_2$. Thus, $H^*(G_n; \mathbb{Z}_2)$ contains the polynomial algebra $\mathbb{Z}_2[w_1(\gamma^n), \ldots, w_n(\gamma^n)]$. We will show that this subalgebra actually coincides with $H^*(G_n; \mathbb{Z}_2)$.

Now, the number of r-cells in the CW complex G_n is equal to the number of partitions of n into at most n integers and the rank of $H^r(G_n; \mathbb{Z}_2)$ is bounded by this number.

On the other hand, the number of distinct monomials $w_1(\gamma^k)^{r_1} \ldots w_n(\gamma^n)^{r_n}$ is also equal to number of partitions of n into at most n integers. For each sequence $r_1, r_2, \ldots, r_n$ we may associate

$$(r_1, r_2, \ldots, r_n) \leftrightarrow r_n \leq r_n + r_{n-1} \leq \ldots \leq r_n + r_{n-1} + \ldots + r_1$$

Such a partition becomes the sequence $\sigma_1 - 1 \leq \sigma_2 - 2 \leq \ldots \leq \sigma_n - n$, corresponding to the strictly increasing sequence $\sigma_1 < \sigma_2 < \ldots < \sigma_n$. But such sequences precisely give a CW structure on G_n and, since the monomial are known to be linearly independent, the theorem follows.

□

Note that the above discussion can be easily adapted to prove the analogous statement in the complex case, i.e, $H^*(G_n(\mathbb{C}^\infty); \mathbb{Z}) = \mathbb{Z}[c_1(\gamma^n(\mathbb{C}^\infty)), \ldots, c_n(\gamma^n(\mathbb{C}^\infty))]$. In this case, the $c_i's$ are $2i$ dimensional and the CW structure has an extra factor of 2 in the dimension of its cells. Moreover, all cells are even dimensional, so the cellular boundary maps are zero. Hence, each non-zero cohomology group consists of as many copies $\mathbb{Z}$ as the number of cells in that dimension.

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