Curvature

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Certificate of Examination

This is to certify that the dissertation titled **"Curvature"** submitted by **Nitesh Bhardwaj** (Reg. No. MS10087) for the partial fulfillment of BS-MS dual degree programme of the Institute, has been examined by the thesis committee duly appointed by the Institute. The committee finds the work done by the candidate satisfactory and recommends that the report be accepted.

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Dated: April 30, 2015

Declaration

The work presented in this dissertation has been carried out by me under the guidance of Dr. Krishnendu Gongopadhyay at the Indian Institute of Science Education and Research Mohali.

This work has not been submitted in part or in full for a degree, a diploma, or a fellowship to any other university or institute. Whenever contributions of others are involved, every effort is made to indicate this clearly, with due acknowledgement of collaborative research and discussions. This thesis is a bonafide record of work done by me and all sources listed within have been detailed in the bibliography.

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In my capacity as the supervisor of the candidate's project work, I certify that the above statements by the candidate are true to the best of my knowledge.

Dr. Krishnendu Gongopadhyay
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Finally, I would like to acknowledge that the material presented in this thesis is based on other people's work. What I have done is the selection, presentation and some basic calculations of the materials from different sources which are listed in the bibliography.

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Abstract

The Gauss's Theorema Egregium and the Gauss Bonnet theorem are few of the foundational results in Differential geometry that present non trivial hypothesis about curvature. The former asserts that curvature is an invariant of the metric.
In higher dimensions there is a well known theorem by F. Schur :
THEOREM. Let M be a Riemannian manifold with $\dim(M) \geq 3$. If the sectional curvature K of M is constant at each point of M, then K is actually constant on M.

In this thesis we have attempted to give an exposition of a celebrated theorem of R.S Kulkarni that relates the geometry of the curvature with the underlying metric. We consider all manifolds and functions to be smooth. Also all manifolds are assumed to be connected. The theorem is stated as follows:
FUNDAMENTAL THEOREM (R.S Kulkarni, 1970). If dimension ≥ 4, then isocurved manifolds with analytic metric are globally isometric except in the case of diffeomorphic, non-globally isometric manifolds of the same constant curvature.
In other words, under the above hypothesis, a curvature preserving diffeomorphism itself is an isometry.

We will prove the theorem in two chapters. In the first chapter we derive the necessary results required to prove the theorem. One of the result needed is the Weyl's theorem that is stated in the end of the first chapter. In the second chapter we give the complete proof. We have assumed a knowledge of John M. Lee's book on Riemannian geometry for reading the thesis.

Notation

- (M, g)= Riemann Manifold equipped with metric g.
- T_pM= Tangent space at a point $p \in M$.
- $\mathscr{T}(M)$= Tangent bundle of M.
- $\mathscr{T}^*(M)$= Cotangent bundle of M
- $O(n)$= Orthogonal group.
- K= Sectional curvature.
- R= Curvature tensor.

Contents

Chapter 1

Preliminaries

The reference sources for this chapter are primarily [1], [3].

1.1 Curvature tensor

Definition 1.1. *A Riemannian metric on a smooth manifold M is a two tensor field that is symmetric and positive definite, and thus determines an inner product on each tangent space T_pM. We denote $g(X,Y)$ by $\langle X,Y\rangle$ for $X,Y \in T_pM$. We also denote any inner product on a vector space by g.*

Definition 1.2. *Let V be a real vector space with inner product g. A curvature tensor with respect to g is a bilinear map*

$$R : V \times V \longrightarrow End(V)$$

which satisfies the following properties :

1. $R(X,Y) + R(Y,X) = 0$
2. $R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0$
3. $\langle R(X,Y)Z, W\rangle = \langle R(Z,W)X, Y\rangle$.

Remark 1.1. *In the Riemannian case, we shall take $R(X,Y) = \nabla_{[X,Y]} - \nabla_Y\nabla_X - \nabla_X\nabla_Y$, where ∇ is the Levi Civita connection.*

Definition 1.3. *We define sectional curvature as follows: If $\sigma = \{X,Y\}$ is a 2-plane $\subseteq V$ then*

$$K(X,Y) = \frac{\langle R(X,Y)X, Y\rangle}{\langle X,X\rangle\langle Y,Y\rangle - \langle X,Y\rangle^2}.$$

For (M, g), a Riemann manifold let $G_2(M)$ denote the Grassman bundle of 2-plane sections of M and $\pi : G_2(M) \to M$ the canonical projection.

Remark 1.2. *The sectional curvature K is a real-valued function on $G_2(M)$.*

Definition 1.4 (Isotropic point)**.** *Let $dim(M) \geqslant 3$. We say a point $p \in M$ is isotropic if $K|_{\pi^{-1}(p)}$ is constant; otherwise, we call p non-isotropic.*

Theorem 1.1. *Let $V, \bar{V}$ be two real vector spaces of dimension $n \geqslant 3$ endowed with inner products $g, \bar{g}$ respectively. let*

$$R : V \times V \longrightarrow End(V)$$

$$\bar{R} : \bar{V} \times \bar{V} \longrightarrow End(\bar{V})$$

be two curvature tensors (with respect to $g, \bar{g}$ respectively), and let $K, \bar{K}$ be corresponding sectional curvatures. Suppose that $K \not\equiv$ constant, and

$$f : V \to \bar{V}$$

a sectional curvature preserving linear isomorphism. Then f is a homothety, i.e.,

$$\bar{g}(f(x), f(y)) = c.g(x, y),$$

where c is any constant.

Lemma 1.1. *Let V be a vector space with inner product g, and curvature tensor R. The following are equivalent:*

1. *The sectional curvature is constant.*
2. *$R(X, Y)Z = c.(\langle X, Z\rangle Y - \langle Y, Z\rangle X)$.*

Definition 1.5 (Acceptable frame)**.** *We define a frame to be an orthonormal basis of a vector space V. A frame would be called acceptable if for every triple of distinct indices i, j, k the values $K(e_i, e_j), K(e_i, e_k), K(e_j, e_k)$ are all distinct. Henceforth, we denote $K(e_i, e_j)$ as K_{ij}.*

Lemma 1.2. *Under the hypothesis of Theorem 1.1, V admits an acceptable frame.*

Proof Let $\dim(V) = 3$. Let $\{e_i, e_j, e_k\}$ be any frame. If K_{ij}, K_{ik}, K_{jk} are not all distinct, we have the following two cases:

1. $K_{ij} = K_{ik} = K_{jk} = K_o$. We suppose R_{ikjk} to be non-zero, otherwise in view of Lemma 1.1 curvature would be constant if all the mixed components of the curvature tensor were zero.

   Let T be an orthogonal transformation which rotates the $\{e_i, e_j\}$-plane through an angle θ, and leaves e_k fixed. Then

$$Te_i = \cos\theta e_i + \sin\theta e_j$$

$$Te_j = -\sin\theta e_i + \cos\theta e_j.$$

   Let $Te_r = f_r, r = i, j, k$. Consider,

$$K(f_i, f_j) = \frac{\langle R(f_i, f_j)f_i, f_j\rangle}{\langle f_i, f_i\rangle\langle f_j, f_j\rangle - \langle f_i, f_j\rangle^2}$$

   substituting for f_i , f_j we evaluate the numerator as

$$\begin{aligned}\langle R(f_i, f_j)f_i, f_j\rangle =& \langle R(\cos\theta e_i + \sin\theta e_j, -\sin\theta e_i + \cos\theta e_j)(\cos\theta e_i + \sin\theta e_j),\\ & -\sin\theta e_i + \cos\theta e_j\rangle\end{aligned}$$

   Using bilinearity of the curvature tensor and metric, we evaluate R.H.S

$$\begin{aligned}&\langle R(\cos\theta e_i, -\sin\theta e_i)\cos\theta e_i, \cos\theta e_j\rangle + \langle R(\cos\theta e_i, \cos\theta e_j)\cos\theta e_i, \cos\theta e_j\rangle\\
&+ \langle R(\sin\theta e_j, \cos\theta e_i)\cos\theta e_i, \cos\theta e_j\rangle + \langle R(\sin\theta e_j, -\sin\theta e_i)\cos\theta e_i, \cos\theta e_j\rangle\\
&+ \langle R(\cos\theta e_i, -\sin\theta e_i)\sin\theta e_j, \cos\theta e_j\rangle + \langle R(\cos\theta e_i, \cos\theta e_j)\sin\theta e_j, \cos\theta e_j\rangle\\
&+ \langle R(\sin\theta e_j, \cos\theta e_i)\sin\theta e_j, \cos\theta e_j\rangle + \langle R(\sin\theta e_j, -\sin\theta e_i)\sin\theta e_j, \cos\theta e_j\rangle\\
&+ \langle R(\cos\theta e_i, -\sin\theta e_i)\cos\theta e_i, -\sin\theta e_i\rangle + \langle R(\cos\theta e_i, \cos\theta e_j)\cos\theta e_i, -\sin\theta e_i\rangle\\
&+ \langle R(\sin\theta e_j, \cos\theta e_i)\cos\theta e_i, -\sin\theta e_i\rangle + \langle R(\sin\theta e_j, -\sin\theta e_i)\cos\theta e_i, -\sin\theta e_i\rangle\\
&+ \langle R(\cos\theta e_i, -\sin\theta e_i)\sin\theta e_j, -\sin\theta e_i\rangle + \langle R(\cos\theta e_i, \cos\theta e_j)\sin\theta e_j, -\sin\theta e_i\rangle\\
&+ \langle R(\sin\theta e_j, \cos\theta e_i)\sin\theta e_j, -\sin\theta e_i\rangle + \langle R(\sin\theta e_j, -\sin\theta e_i)\sin\theta e_j, -\sin\theta e_i\rangle.\end{aligned}$$

By cancelling terms and furthur simplification we have,

$$\begin{aligned}(\cos^2\theta+\sin^2\theta)\langle R(e_i,e_j)e_i,e_j\rangle &= \langle R(e_i,e_j)e_i,e_j\rangle \\ &= K(e_i,e_j) \\ &= K_{ij}.\end{aligned}$$

It is easy to check that the denominator is equal to 1. Hence,

$$K(f_i,f_j)=K_{ij}=K_o.$$

Similarly we get,

$$\begin{aligned}K(f_i,f_k) &= \cos^2\theta\langle R(e_i,e_k)e_i,e_k\rangle+\sin 2\theta\langle R(e_i,e_k)e_j,e_k\rangle+\sin^2\theta\langle R(e_j,e_k)e_j,e_k\rangle \\ &= \cos^2\theta\ R_{ikik}+\sin 2\theta\ R_{ikjk}+\sin^2\theta R_{jkjk} \\ &= K_o+\sin 2\theta R_{ikjk}.\end{aligned}$$

And, $K(f_j,f_k)=K_o-\sin 2\theta R_{ikjk}$.

So we can have for some value of θ, $\sin 2\theta\neq 0$ for which the new frame would become acceptable.

2. Let $K_{ij}=K_{ik}\neq K_{jk}$ and let $\left|K_{ij}-K_{jk}\right|=\epsilon$. In this case we have

   (a) $K(f_i,f_j)=K_{ij}$

   (b) $K(f_i,f_k)-K_{ik}=\sin^2\theta(K_{jk}-K_{ik})+\sin 2\theta R_{ikjk}$

   (c) $K(f_j,f_k)-K_{jk}=\sin^2\theta(K_{ik}-K_{jk})-\sin 2\theta R_{ikjk}$

   Again to make the frame acceptable, we choose θ to be small such that the R.H.S in (b) and (c) is non-zero and $\left|K(f_i,f_k)-K_{ik}\right|$, $\left|K(f_j,f_k)-K_{jk}\right|$ is $<\frac{\epsilon}{2}$.

The above proof was for a vector space of dimension 3. For a general proof consider a frame $f=\{e_1,e_2,....,e_n\}$ for an n dimensional vector space. To each frame associate a number $N(f)$ given by the number of distinct triples $\{i,j,k\}$ such that the K_{ij},K_{ik},K_{jk} are distinct.

By continuity of curvature, it can be shown that there exists a small variation via $T\in O(n)$ in the neighbourhood of the identity element such that the acceptable triplet remains acceptable i.e $N(f) \leqslant N(Tf)$. We may choose f such that $N(f)$ is maximum. If f is not acceptable then there exists a triplet such that the acceptability condition is not satisfied.

Let that triplet be $\{e_i, e_j, e_k\}$. If the variation of $\{e_i, e_j, e_k\}$ by all transformations in the neighbourhood of the identity in $O(n)$ resulted in the same constant curvature then the curvature itself would be a constant everywhere which is not the case. So we may say that there exists sufficiently small variations of $\{e_i, e_j, e_k\}$ such that the triplet becomes acceptable, which would contradict the choice of f as we started with $N(f)$ being maximum and hence the proof.

■

Now we are ready to give the proof of Theorem 1.1 .

Proof Using Lemma 1.1 , let $\{e_1, \ldots, e_n\}$ be an acceptable frame. Set

$$fe_i = \bar{e}_i$$

and

$$\bar{g}(\bar{e}_i, \bar{e}_j) = a_{ij}.$$

We denote the components of $\bar{R}$ with respect to the basis $\{\bar{e}_i\}$. Under the hypothesis we have

$$K(e_i, e_j) = \bar{K}(\bar{e}_i, \bar{e}_j)$$

i.e,

$$\langle R(e_i, e_j)e_i, e_j\rangle = \frac{\langle \bar{R}(\bar{e}_i, \bar{e}_j)\bar{e}_i, \bar{e}_j\rangle}{a_{ii}a_{jj} - (a_{ij})^2}$$

Hence, we have

$$(a_{ii}a_{jj} - (a_{ij})^2)R_{ijij} = \bar{R}_{ijij}. \tag{1.1}$$

Now let $\{i, j, k\}$ be a distinct triplet and x, y be two real numbers such that atleast one is non-zero. Now we compute the following:

$$K(xe_i + ye_j, e_k) = \frac{\langle R(xe_i + ye_j, e_k)(xe_i + ye_j), e_k\rangle}{\langle(xe_i + ye_j), (xe_i + ye_j)\rangle\langle e_k, e_k\rangle - (\langle xe_i + ye_j, e_k\rangle)^2}$$
$$= \frac{\langle R(xe_i + ye_j, e_k)(xe_i + ye_j), e_k\rangle}{x^2 + y^2} \tag{1.2}$$

Numerator of Equation (1.2) is given as follows:

$$\begin{aligned}\langle R(xe_i + ye_j, e_k)xe_i, e_k\rangle + \langle R(xe_i + ye_j, e_k)ye_j, e_k\rangle &= \langle R(xe_i, e_k)xe_i, e_k\rangle + \langle R(ye_j, e_k)xe_i, e_k\rangle + \\ &\langle R(xe_i, e_k)ye_j, e_k\rangle + \langle R(ye_j, e_k)ye_j, e_k\rangle \\ &= x^2\langle R(e_i, e_k)e_i, e_k\rangle + 2xy\langle R(e_i, e_k)e_j, e_k\rangle + \\ &y^2\langle R(e_j, e_k)e_i, e_k\rangle \\ &= x^2 R_{ikik} + 2xyR_{ikjk} + y^2 R_{jkjk}\end{aligned}$$

Therefore, we have

$$K(xe_i + ye_j, e_k) = \frac{x^2 R_{ikik} + 2xyR_{ikjk} + y^2 R_{jkjk}}{x^2 + y^2} = \frac{N}{D}. \tag{1.3}$$

Similar calculations yield,

$$\bar{K}(x\bar{e_i} + y\bar{e_j}, \bar{e_k}) = \frac{x^2 \bar{R}_{ikik} + 2xy\bar{R}_{ikjk} + y^2 \bar{R}_{jkjk}}{(x^2 a_{ii} + 2xya_{ij} + y^2 a_{jj})a_{kk} - (xa_{ik} + ya_{jk})^2} = \frac{\bar{N}}{\bar{D}}. \tag{1.4}$$

From the hypothesis, we have

$$K(xe_i + ye_j, e_k) = \bar{K}(x\bar{e_i} + y\bar{e_j}, \bar{e_k}).$$

So computing,

$$\begin{aligned}\bar{D}N &= ((x^2 a_{ii} + 2xya_{ij} + y^2 a_{jj})a_{kk} - (xa_{ik} + ya_{jk})^2)(x^2 R_{ikik} + 2xyR_{ikjk} + y^2 R_{jkjk}) \\ &= x^4(a_{ii}a_{kk} - a_{ik}^2)R_{ikik} + x^3 y(2a_{ii}a_{kk}R_{ikjk} + 2a_{ij}a_{kk}R_{ikik} - 2a_{ik}^2 R_{ikjk} - 2a_{ik}a_{jk}R_{ikik}) + \\ &x^2y^2(a_{ii}a_{kk}R_{jkjk} + 4a_{ij}a_{kk}R_{ikjk} + a_{jj}a_{kk}R_{ikik} - a_{ik}^2 R_{jkik} - a_{jk}^2 R_{ikik} - 4a_{ik}a_{jk}R_{ikjk}) + \\ &xy^3(2a_{ij}a_{kk}R_{jkjk} + 2a_{jj}a_{kk}R_{ikjk} - 2a_{jk}^2 R_{ikjk} - 2a_{ik}a_{jk}R_{jkjk}) + \\ &y^4(a_{jj}a_{kk}R_{jkjk} - a_{jk}^2 R_{jkjk})\end{aligned}$$

and

$$
\begin{aligned}
D\bar{N} &= (x^2+y^2)(x^2\bar{R}_{ikik}+2xy\bar{R}_{ikjk}+y^2\bar{R}_{jkjk}) \\
&= x^4\bar{R}_{ikik}+2x^3y\bar{R}_{ikjk}+x^2y^2(\bar{R}_{jkjk}+\bar{R}_{ikik})+2xy^3\bar{R}_{ikjk}+y^4\bar{R}_{jkjk}
\end{aligned}
$$

By comparing coefficients of x^3y, xy^3, x^2y^2, we get the following:

1. $\bar{R}_{ikjk} = R_{ikjk}(a_{ii}a_{kk}-a_{ik}^2)+R_{ikik}(a_{ik}a_{kk}-a_{ik}a_{jk})$.
2. $\bar{R}_{ikjk} = R_{ikjk}(a_{jj}a_{kk}-a_{jk}^2)+R_{jkjk}(a_{ij}a_{kk}-a_{ik}a_{jk})$.
3. $\bar{R}_{ikik}+\bar{R}_{jkjk} = R_{ikik}(a_{jj}a_{kk}-a_{jk}^2)+R_{jkjk}(a_{ii}a_{kk}-a_{ik}^2)+$
   $4R_{ikjk}(a_{ij}a_{kk}-a_{ik}a_{jk})$.

(1)-(2) gives

$$(a_{ij}a_{kk}-a_{ik}a_{jk})(R_{ikik}-R_{jkjk})+((a_{ii}a_{kk}-a_{ik}^2)-(a_{jj}a_{kk}-a_{jk}^2))R_{ikjk}=0. \tag{1.5}$$

Using (1.1), we get

$$4(a_{ij}a_{kk}-a_{ik}a_{jk})R_{ikjk}-((a_{ii}a_{kk}-a_{ik}^2)-(a_{jj}a_{kk}-a_{jk}^2))(R_{ikik}-R_{jkjk})=0. \tag{1.6}$$

Since the frame is acceptable, $R_{ikik}-R_{jkjk}\neq 0$, so from the Equations (1.5) and (1.6) we get

$$(a_{ii}a_{kk}-a_{ik}^2)-(a_{jj}a_{kk}-a_{jk}^2)=a_{ij}a_{kk}-a_{ik}a_{jk}=0. \tag{1.7}$$

Let

$$
\begin{aligned}
\angle(\bar{e}_i,\bar{e}_j) &= \theta, \\
\angle(\bar{e}_j,\bar{e}_k) &= \phi, \\
\angle(\bar{e}_k,\bar{e}_i) &= \psi.
\end{aligned}
$$

Then

$$\cos\theta = \frac{\langle \bar{e}_i,\bar{e}_j\rangle}{\langle \bar{e}_i,\bar{e}_i\rangle\langle \bar{e}_j,\bar{e}_j\rangle} = \frac{a_{ij}}{a_{ii}a_{jj}}$$

From Equation (1.7), it follows

$$\frac{a_{ij}}{a_{ii}a_{jj}} - \frac{a_{jk}}{a_{jj}a_{kk}}\frac{a_{ki}}{a_{ii}a_{kk}} = \cos\theta-\cos\phi\cos\psi=0.$$

By symmetry, we also have,

$$\cos\phi - \cos\theta\cos\psi = 0$$

$$\cos\psi - \cos\theta\cos\phi = 0.$$

This is possible only if $\{\bar{e}_i, \bar{e}_j, \bar{e}_k\}$ are orthogonal. From Equation (1.7), we also have

$$a_{ii}a_{kk} - a_{ik}^2 = a_{jj}a_{kk} - a_{jk}^2 = a_{ii}a_{jj} - a_{ij}^2.$$

This shows that $\{\bar{e}_i, \bar{e}_j, \bar{e}_k\}$ is an orthogonal frame whose vectors are of the same length i.e f is a homothety.

■

As an immediate consequence of Theorem 1.1 we have,

Theorem 1.2. *Let*

$$f : M \longrightarrow \bar{M}$$

be a curvature preserving diffeomorphism of two Riemann manifolds of dimension ≥ 3. *Then f is conformal on the closure of set of non-isotropic points.*

1.2 Conformal change of a metric

In this section we state the propositions that compare different tensors and connections obtained by a conformal change of a metric. We also state the celebrated Weyl's theorem and conformal invariant curvature tensor.

Let (M, g) be a Riemann manifold. Let f be a positive real-valued function on M. Consider the new Riemann metric

$$\bar{g} = fg.$$

Proposition 1.1. *For any two vector fields X, Y, we have*

$$\bar{\nabla}_X Y = \nabla_X Y + S(X, Y)$$

where

$$S(X,Y) = \frac{1}{2f}\{(Xf)Y + (Yf)X - \langle X,Y\rangle grad\, f\}.$$

Here, grad $f=$ gradient of f with respect to the metric g, such that for a vector field X we have,

$$\langle X, grad\, f\rangle = Xf.$$

Now we define *hessian* of a real-valued function ϕ on M.

Definition 1.6. *For any two vector fields X, Y,*

$$hess_\phi(X,Y) = XY\phi - (\nabla_X Y)\phi.$$

or

$$hess_\phi(X,Y) = \langle Y, \nabla_X(grad\, \phi)\rangle. \tag{1.8}$$

Remark 1.3. *Hessian is a symmetric, bilinear $(0,2)$-tensor field.*

Using hessian of a real-valued function ϕ on M, we define another symmetric, bilinear $(0,2)$-tensor field as follows:

$$Q(X,Y) = \mathrm{hess}_\phi(X,Y) - X\phi Y\phi$$

where Q defines a bundle map

$$Q_0 : \mathscr{T}(M) \longrightarrow \mathscr{T}^*(M)$$

such that for any two vector fields X, Y, we have

$$(Q_0(X))(Y) = Q(X,Y).$$

If $G = \mathrm{grad}\ \phi$, then from Equation (1.8), we have

$$Q_0(X) = \nabla_X G - X\phi G.$$

Now we state the propositions that describe the deformation of various tensors under the considered conformal change.

Proposition 1.2. *For any three vector fields X, Y, Z,*

$$\bar{R}(X,Y)Z = R(X,Y)Z + T(X,Y)Z$$

where

$$T(X,Y)Z = \{Q(Y,Z)+\langle Y,Z\rangle\|G\|^2\}X-\{Q(X,Z)+\langle X,Z\rangle\|G\|^2\}Y+\langle Y,Z\rangle Q_0(X)-\langle X,Z\rangle Q_0(Y).$$

Definition 1.7. *Ricci tensor is a symmetric, bilinear* $(0,2)$*-tensor field defined as: for* $X, Z \in T_p(M)$

$$Ric(X,Z) = Trace\{Y \longrightarrow R(X,Y)Z\}.$$

Remark 1.4. *Ric defines an endomorphism of the tangent bundle:*

$$Ric_0 = \mathscr{T}(M) \xrightarrow{c} \mathscr{T}^*(M) \xrightarrow{i} \mathscr{T}(M)$$

where c *is the canonical map defined by Ric and* i *is the usual identification of* $\mathscr{T}^*(M)$ *with* $\mathscr{T}(M)$.

Definition 1.8. *Scalar curvature:*

$$Sc : M \longrightarrow \mathbb{R}$$

at a point $p \in M$ *is defined as*

$$Sc(p) = Trace\ Ric_0(p).$$

Definition 1.9 (Weyl conformal curvature tensor)**.** *Weyl conformal curvature tensor is the conformal invariant defined as follows:*

$$\begin{aligned} C(X,Y)Z = {} & R(X,Y)Z + \\ & \frac{1}{n-2}\{Ric(Y,Z)X - Ric(X,Z)Y + \langle Y,Z\rangle Ric_0(X) - \langle X,Z\rangle Ric_0(Y)\} - \\ & \frac{Sc}{(n-1)(n-2)}\{\langle Y,Z\rangle X - \langle X,Z\rangle Y\}. \end{aligned}$$

Theorem 1.3 (Weyl)**.** *Let* (M,g) *be a Riemann manifold of dimension* ≥ 4*, then* M *is conformally flat iff* $C \equiv 0$.

Chapter 2

The Fundamental theorem

The reference sources for this chapter are primarily [1], [2], [3].

Definition 2.1. *Let $(M, g), (\bar{M}, \bar{g})$ be two Riemann manifolds. We say, $M, \bar{M}$ are isocurved if there exists a sectional-curvature preserving diffeomorphism*

$$f : M \longrightarrow \bar{M}$$

i.e., for every $p \in M$ and for every σ, a 2-plane section of tangent space T_pM, we have

$$K(\sigma) = \bar{K}(f_*\sigma).$$

Theorem 2.1 (Fundamental theorem)**.** *If dim ≥ 4, then isocurved manifolds with analytic metric are globaly isometric except in the case of diffeomorphic but non-globally isometric manifolds of the same constant curvature.*

To give a complete proof of the fundamental theorem we consider the following cases:

2.1 Conformally non-flat case

Under the "curvature preserving" hypothesis the following proposition relates the second order tensors with respect to g and $\bar{g}$ respectively.

Proposition 2.1. *Let V be a real vector space equipped with two inner products g, $\bar{g}$ and two curvature tensors $R, \bar{R}$ such that*

1. *$\bar{g} = \lambda g$ for some $\lambda (> 0) \in \mathbb{R}$.*

2. $\bar{K} = K$

Then $\bar{R} = \lambda R$, $\bar{Ric} = \lambda Ric$, $\bar{Ric}_0 = Ric_0$, $\bar{Sc} = Sc$, $\bar{C} = \lambda C$.

Proof The condition

$$\bar{K} = K$$

implies

$$\frac{\overline{\langle \bar{R}(X,Y)X,Y\rangle}}{\overline{\langle X,X\rangle}\ \overline{\langle Y,Y\rangle} - \overline{\langle X,Y\rangle}^2} = \frac{\langle R(X,Y)X,Y\rangle}{\langle X,X\rangle\langle Y,Y\rangle - \langle X,Y\rangle^2}.$$

or

$$\langle \bar{R}(X,Y)X,Y\rangle = \lambda\langle R(X,Y)X,Y\rangle.$$

Replacing Y by $Y+Z$, we get

$$\langle \bar{R}(X,Y)X,Z\rangle = \lambda\langle R(X,Y)X,Z\rangle.$$

Since this holds for all Z, we have

$$\bar{R}(X,Y)X = \lambda R(X,Y)X.$$

By replacing X by $X+Z$, we get

$$\bar{R}(X,Y)Z - \bar{R}(Y,Z)X = \lambda\{R(X,Y)Z - R(Y,Z)X\}.$$

By symmetry, we also have

$$\bar{R}(Y,Z)X - \bar{R}(Z,X)Y = \lambda\{R(Y,Z)X - R(Z,X)Y\},$$

$$\bar{R}(Z,X)Y - \bar{R}(X,Y)Z = \lambda\{R(Z,X)Y - R(X,Y)Z\}.$$

From the usual property of the curvature tensor, we have

$$\bar{R}(X,Y)Z + \bar{R}(Y,Z)X + \bar{R}(Z,X)Y = R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0.$$

Hence

$$\bar{R}(X,Y)Z = \lambda R(X,Y)Z.$$

The rest of the propositions follow from this.

■

Now we state the fundamental theorem incase of conformally non-flat manifold.

Theorem 2.2. *Let (M, g), $(\bar{M}, \bar{g})$ be isocurved manifolds. Suppose dim(M) ≥ 4 and M not conformally flat. Then a curvature preserving diffeomorphism is an isometry.*

Proof Let

$$f : M \longrightarrow \bar{M}$$

be a sectional curvature preserving diffeomorphism.

We identify M with $\bar{M}$ via f so that we have the situation where M is equipped with two metrics $g, f^*\bar{g}$ such that the identity map,

$$\mathrm{Id}_M : (M, g) \longrightarrow (\bar{M}, f^*\bar{g})$$

is curvature preserving. Since M is assumed to be conformally non-flat and dim $(M) \geq 4$, it follows from the Theorem 1.3 that $C \not\equiv 0$ on an open dense subset of M and that if a point is isotropic then $C = 0$. Hence, non-isotropic points are dense.

And by Theorem 1.2 , we know that f is conformal on the closure of the set of non-isotropic points. So we may write $f^*\bar{g} = \phi g$ for some real valued positive function ϕ on M. In such a case by the previous proposition we have $\bar{C} = \phi C$. But C is a conformal invariant we have $\bar{C} = C$ and since $C \not\equiv 0$ on a dense subset, we have $\phi \equiv 1$. Hence, f is an isometry.

■

2.2 Conformally flat case

Theorem 2.3. *Let (M, g), $(\bar{M}, \bar{g})$ be isocurved manifolds with dim ≥ 4, and M conformally flat. Moreover, assume that the set of non-isotropic points is dense. Then the curvature preserving diffeomorphism is an isometry.*

Remark 2.1. *In this case, since M is conformally flat we take*

$$g = e^{2\sigma} g_0$$

where g_0 is the flat metric and σ is a smooth function on M. Like in the conformally non-flat case we reduce the situation to the one in which M is equipped with two *metrics $g, \bar{g}$ and the Identity map*

$$Id_M : (M, g) \longrightarrow (\bar{M}, \bar{g})$$

is curvature preserving. Moreover, we have assumed the set of non-isotropic points to be dense so by Theorem 1.2 we may write

$$\bar{g} = e^{2\varphi} g$$

where $\varphi : M \longrightarrow \mathbb{R}$.

Before giving the proof for the above Theorem we make the following observations in view of the Propositions 1.1 and 1.2 stated in the Section 1.2 :

Observations

1. For any three orthonormal vector fields X, Y, Z with respect to the metric g_0, we have
$$Q(X, Z)Y\varphi - Q(Y, Z)X\varphi = 0.$$

2. Let $\{X_1, \ldots, X_n\}$ be a set of orthonormal vector fields with repect to g_0, so that,
$$\mathrm{grad}_0(\varphi) = \sum_i (X_i\varphi)X_i.$$
Again for any three orthonormal vector fields X, Y, Z with respect to the metric g_0, we have
$$Q(X, Z)X\varphi + Q(Y, Z)Y\varphi - \{Q(X, X) + Q(Y, Y)\}Z\varphi + 2\sum_i Q(Z, X_i)X_i\varphi = 0.$$

Lemma 2.1. *Consider (M, g) be conformally flat with dim(M) ≥ 4. Suppose for a point $p \in M$, $(grad_0\varphi)p \neq 0$, where $grad_0$ is the gradient with respect to the flat metric g_0. Then p is isotropic.*

Proof Let

$$X_1 = \frac{\mathrm{grad}_0\varphi}{\|\mathrm{grad}_0\varphi\|_0},$$

and let $\{X_2, \ldots, X_n\}$ be orthonormal vector fields with respect to g_0 which are tangent to the level surface of φ in a neighbourhood of p. Then X_i are orthonormal,

$$(X_1\varphi)(p) = \|\mathrm{grad}_0\varphi\|_0 \neq 0, \quad X_i\varphi = 0 \text{ for } i > 1$$

In Observations (1) and (2), we set $X = X_1, Y = X_i, Z = X_j$, where $\{1, i, j\}$ are distinct. Then we have the following:

1. $Q(X_i, X_j) = 0, i \neq j, i > 1, j > 1.$

2. $Q(X_1, X_i) = 0, i > 1.$

3. $Q(X_i, X_i) = Q(X_1, X_1).$

From the above conditions we conclude that p is isotropic [1] .

Now we are ready to give a proof of the Theorem 2.3 .

Proof In view of the situation to which we have reduced the hypothesis of Theorem 2.3, if grad $\varphi \not\equiv 0$, then by the above Lemma, the set of isotropic points has non-empty interior which contradicts our hypothesis of non-isotropic points being dense.

Hence grad $\varphi \equiv 0$ or $\varphi \equiv c$, where c is a constant. Since we started with

$$\bar{g} = e^{2\varphi} g$$

which implies,

$$\bar{K} = \frac{K}{e^{2\varphi}}$$

And since we our not considering the case of $K \equiv 0$, the "curvature-preserving" hypothesis implies

$$\varphi \equiv 0.$$

■

[1] We have used the fact that for orthonormal vector fields $\{X_1, \ldots, X_n\}$, $Q(X_i, X_j) = \lambda\delta_{ij}$, where $\lambda : M \longrightarrow \mathbb{R}$, the condition implies M is of constant curvature.

2.3 Complete proof

So far we have shown that a sectional curvature preserving diffeomorphism f is an isometry on the closure of the set of non-isotropic points. Thus to show that f is indeed a global isometry, a sufficient condition is that the set of non-isotropic points is dense.
We state the fundamental theorem again and the following argument will establish the complete proof.

Theorem 2.4 (Fundamental theorem)**.** *If dim ≥ 4, then isocurved manifolds with analytic metric are globaly isometric except in the case of diffeomorphic but not globally isometric manifolds of the same constant curvature.*

Proof If the set of isotropic points has a non-empty interior, then by Schur's theorem we would have the sectional curvature to be constant on each of its connected components. In particular, it would be constant on an open subset of $G_2(M)$. Hence by the analyticity of the sectional curvature (equivalently, that of the metric) it would be constant everywhere.

■

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