Topological K-theory

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Certificate of Examination

This is to certify that the dissertation titled **"Topological K-theory"** submitted by **Divya Sharma** (Reg. No. MS10099) for the partial fulfillment of BS-MS dual degree programme of the Institute, has been examined by the thesis committee duly appointed by the Institute. The committee finds the work done by the candidate satisfactory and recommends that the report be accepted.

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Dated: April 23, 2015

Declaration

The work presented in this dissertation has been carried out by me under the guidance of Dr. Mahender Singh at the Indian Institute of Science Education and Research Mohali.

This work has not been submitted in part or in full for a degree, a diploma, or a fellowship to any other university or institute. Whenever contributions of others are involved, every effort is made to indicate this clearly, with due acknowledgement of collaborative research and discussions. This thesis is a bonafide record of work done by me and all sources listed within have been detailed in the bibliography.

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In my capacity as the supervisor of the candidate's project work, I certify that the above statements by the candidate are true to the best of my knowledge.

Dr. Mahender Singh
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I am grateful to my friends and family for giving me all the non-technical support and of course sheer luck because of which I got the opportunity to learn mathematics.

Finally, I would like to acknowledge that the material presented in this thesis is based on other people's work. If I have made then it is the selection, presentation and some basic calculations of the materials from different sources which are listed in the bibliography.

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Notation

- ξ = Real or complex vector bundle over X
- k or $\mathbb{K}$ = Field of real or complex numbers
- $\mathrm{Vect}_k^n(X)$ = Set of isomorphism classes of k-vector bundles of rank n over the topological space X
- $\mathrm{GL}_n^+(k) = n \times n$ matrices of positive determinant
- $\mathrm{Vect}_+^n(X)$ = Set of isomorphism classes of oriented real vector bundles of rank n over X
- $U(n)$ = Group of unitary matrices
- $\pi_p(X)$ = p-th homotopy group of X
- $G_{k,n}$ = Complex or real Grassmann manifold
- $V_{k,n}$ = Stiefel manifold
- $K(X)$ = Grothendieck group of X
- $\tilde{K}(X)$ = Reduced K-group of X
- $P(r,\varphi)$ = Poisson kernel
- CX = Cone of topological space X
- SX = Suspension of topological space X
- ΣX = Reduced suspension of X
- $X \star Y$ = Join of topological spaces X and Y
- $X \wedge Y$ = Smash product of two topological spaces X and Y

Contents

Summary of the report by chapters

- In chapter 1 , we first explain "continuous operations" on vector bundles. For example, direct sum, tensor product, duality and inner product. Clutching theorems are an important technicality to provide the description of tangent bundle of a differentiable manifold and vector bundles over spheres. The Hopf bundle is visualized elegantly using basic quaternion algebra and some diagrams. Finally beautiful construction of classifying spaces is explained in this chapter.

- In chapter 2 , using some "important properties" of locally trivial bundles, we describe bundles in terms of homotopy properties of topological spaces.

- In chapter 3 , starting with simple notion of symmetrization of an abelian monoid, we define the group $K(X)$ of X using the isomorphism classes of vector bundles over X. To extend the study of the properties of the vector bundles, we need further geometric ideas and constructions which lead to deeper properties of vector bundles. One of them is the Bott periodicity theorem, an important result for calculation of K-theory.

- In chapter 4 , for each vector bundle, we define "Chern classes" using cohomology ring of classifying spaces (with suitable coefficient ring) in an axiomatic way. By means of these classes, we construct a fundamental homomorphism, the "Chern character" from $K(X)$ to $H^{even}(X;\mathbb{Q})$.

- In chapter 5 , we explain Gysin sequence for describing the K-groups of spaces by reducing them to a description in terms of the usual cohomology groups of spaces. Then we prove that the only spheres which admit an H-space structure are $\mathbb{S}^1$, $\mathbb{S}^3$ and $\mathbb{S}^7$ and that these are the only parallelizable spheres.

Chapter 1

Introduction to fibre bundles

The sources of this chapter are primarily [1], [2], [3], [4], [5], [10], [18], [23].

1.1 Definitions and examples

Throughout the book we use the word map or morphism to mean a continuous function.

Definition 1.1. *A fibre bundle with total space E, base X, fibre F and structure group G is given by the following data:*

1. *A morphism $\pi : E \longrightarrow X$.*
2. *A left G-action on F, given as follows:*

$$G \times F \longrightarrow F$$

$$(g, f) \longmapsto gf.$$

3. *An open covering $\mathcal{U} = \{U_i\}$ of X and morphisms*

$$\phi_i : U_i \times F \longrightarrow \pi^{-1}(U_i)$$

such that

$$\pi(\phi_i(u, f)) = u.$$

4. *Morphisms*

$$t_{ij} : U_{ij} \longrightarrow G,$$

where $U_{ij} := U_i \cap U_j$, *giving transfer functions* T_{ij},

$$T_{ij} : U_{ij} \times F \longrightarrow U_{ij} \times F$$

$$(u, f) \longmapsto (u, t_{ij}(u)f)$$

so that

$$\phi_i = \phi_j \circ T_{ij}.$$

There are many special types of fibre bundles which one may be interested in, where one fixes the type of fibre or the type of structure group. One wellknown class is that of *vector bundles*, where one takes as fibres vector spaces, and as structure group the corresponding general linear group with its natural action. In this case, one usually restricts to those morphisms which act linearly on fibres.

Now we begin with the discussion of *vector bundles*.

Definition 1.2. *A* ***real vector bundle*** *of rank* n *is a continuously varying family of* n*-dimensional real vector spaces which is locally trivial. More formally, a real vector bundle is a triple* $\xi = (E, \pi, X)$ *where* $\pi : E \longrightarrow X$ *is a map such that for every* $x \in X$ *there is a neighbourhood* $x \in U \subseteq X$ *and a homeomorphism* $\varphi : \pi^{-1}(U) \longrightarrow U \times \mathbb{R}^n$ *such that*

1. *the diagram*

$$\begin{array}{ccc} \pi^{-1}(U) & \xrightarrow{\varphi} & U \times \mathbb{R}^n \\ & \searrow^{\pi|_{\pi^{-1}(U)}} & \downarrow pr_1 \\ & & U \end{array}$$

   commutes.

2. $\varphi|_{\pi^{-1}(x)} : \pi^{-1}(x) \longrightarrow \{x\} \times \mathbb{R}^n$ *is a vector space isomorphism for each* $x \in U$.

Here, $\pi^{-1}(x)$ *is called fibre at* x*, spaces* E *and* X *are called total space and base space, respectively.*

Remark 1.1. *Complex vector bundles can be defined in a similar manner.*

Examples

1. Let $E = X \times \mathbb{R}^n$ and $\pi : E \longrightarrow X$ be the projection onto first factor. Then $(E = X \times \mathbb{R}^n, \pi, X)$ is a real vector bundle of rank n. This bundle is called the *trivial bundle.*

2. Let E be the infinite **Möbius band** defined as:

$$E = I \times \mathbb{R}/\sim, \quad \text{where} \quad I = [0,1] \quad \text{and} \quad (0,t) \sim (1,-t) \quad \forall\, t \in \mathbb{R}$$

   The projection $I \times \mathbb{R} \longrightarrow I$ induces a map $\pi : E \longrightarrow \mathbb{S}^1$. So $(E, \pi, \mathbb{S}^1)$ is a real vector bundle of rank 1 or a **line** bundle and is called the Möbius bundle.

3. The tangent bundle $T\mathbb{S}^n = (E, \pi, \mathbb{S}^n)$ of the unit sphere $\mathbb{S}^n$ in $\mathbb{R}^{n+1}$ is a vector bundle. We define the total space E of $T\mathbb{S}^n$ as follows:

$$E = \{(x,v) \in \mathbb{S}^n \times \mathbb{R}^{n+1} \mid x \perp v\}.$$

   The map $\pi : E \longrightarrow \mathbb{S}^n$ is given by

$$\pi(x,v) = x.$$

4. In the case of normal bundle $N\mathbb{S}^n = (E, \pi, \mathbb{S}^n)$ to $\mathbb{S}^n$ in $\mathbb{R}^{n+1}$, E is defined as follows:

$$E = \{(x,v) \in \mathbb{S}^n \times \mathbb{R}^{n+1} \mid v = tx \text{ for some } t \in \mathbb{R}\}.$$

   The map $\pi : E \longrightarrow \mathbb{S}^n$ is again given by

$$\pi(x,v) = x.$$

5. The canonical line bundle $\gamma^1_{n,\mathbb{R}} = (E, \pi, \mathbb{RP}^n)$ has as its total space E the subspace of $\mathbb{RP}^n \times \mathbb{R}^{n+1}$, defined as follows:

$$E = \{(l,v) \in \mathbb{RP}^n \times \mathbb{R}^{n+1} \mid v \in l\}.$$

   The map $\pi : E \longrightarrow \mathbb{RP}^n$ is given by

$$\pi(l,v) = l.$$

Similarly, we define canonical line bundle $\gamma^1_{n,\mathbb{C}}$ over $\mathbb{CP}^n$.

6. The canonical line bundle $\gamma^1_{n,\mathbb{R}} = (E, \pi, \mathbb{RP}^n)$ has an orthogonal complement, denoted by $(\gamma^1_{n,\mathbb{R}})^\perp = (E^\perp, \pi, \mathbb{RP}^n)$, where $E^\perp$ is defined as follows:

$$E^\perp = \{(l, v) \in \mathbb{RP}^n \times \mathbb{R}^{n+1} \mid v \perp l\}.$$

The map $\pi : E^\perp \longrightarrow \mathbb{RP}^n$ is given by

$$\pi(l, v) = l.$$

Definition 1.3. *A* ***section*** *of a vector bundle $\xi = (E, \pi, X)$ is a map $s : X \longrightarrow E$ assigning to each $x \in X$ a vector $s(x)$ in the fibre E_x.*

Every fibre E_x of E is a vector space and thus has a distinguished element, the zero-vector in E_x, which we denote by 0_x. It follows that every vector bundle admits a section, given as follows:

$$s_0(x) = 0_x \in E_x.$$

Definition 1.4. *1. Suppose $\xi = (E, \pi, X)$ and $\xi' = (E', \pi', X')$ are real or complex vector bundles. A map $\tilde{f} : E \longrightarrow E'$ is a vector bundle homomorphism if $\tilde{f}$ desends to a map $f : X \longrightarrow X'$, i.e. the diagram*

$$\begin{array}{ccc} E & \xrightarrow{\tilde{f}} & E' \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \pi'} \\ X & \xrightarrow[f]{} & X' \end{array}$$

commutes, and the restriction $\tilde{f}|_{E_x} : E_x \longrightarrow E'_x$ is linear for all $x \in X$.

2. If $\xi = (E, \pi, X)$ and $\xi' = (E', \pi', X)$ are vector bundles, a vector bundle homomorphism $\tilde{f}$ is an isomorphism of vector bundles if the diagram

$$\begin{array}{ccc} E & \xrightarrow{\tilde{f}} & E' \\ & {\scriptstyle \pi}\searrow & \downarrow{\scriptstyle \pi'} \\ & & X \end{array}$$

commutes, and its restriction to each fibre is an isomorphism of vector spaces.

We use the notation $E_1 \approx E_2$ to indicate that E_1 and E_2 are isomorphic.

Examples

1. The tangent bundle $T\mathbb{S}^1$ and the trivial bundle over $\mathbb{S}^1$ are isomorphic and isomorphism is given as:

$$\phi : T\mathbb{S}^1 \longrightarrow \mathbb{S}^1 \times \mathbb{R}$$

$$(e^{\iota\theta}, \iota t e^{\iota\theta}) \longmapsto (e^{\iota\theta}, t)$$

for $e^{\iota\theta} \in \mathbb{S}^1$ and $t \in \mathbb{R}$.

2. The *Möbius* bundle M is not isomorphic to the trivial line bundle over $\mathbb{S}^1$. We show that M is not even homeomorphic to the trivial line bundle.

$$\mathbb{S}^1 \times \mathbb{R} - s_0(\mathbb{S}^1) = \mathbb{S}^1 \times \mathbb{R} - \mathbb{S}^1 \times \{0\} = \mathbb{S}^1 \times \mathbb{R}^- \sqcup \mathbb{S}^1 \times \mathbb{R}^+,$$

The space above is not connected but $M - s_0(\mathbb{S}^1)$ is connected.

3. The normal bundle $N\mathbb{S}^n$ is isomorphic to the product bundle $\mathbb{S}^n \times \mathbb{R}$ by the map

$$(x, tx) \longmapsto (x, t).$$

Let $E = X \times V$ and $E' = X \times V'$ be trivial vector bundles with base X. We want to explicitly describe the morphisms from E to E'. Since the diagram

$$\begin{array}{ccc} X \times V & \xrightarrow{\tilde{f}} & X \times V' \\ & \searrow^{\pi} & \downarrow^{\pi'} \\ & & X \end{array}$$

is commutative, for each point x of X, $\tilde{f}$ induces a linear map $\tilde{f}_x : V \longrightarrow V'$. Let $\check{\tilde{f}} : X \longrightarrow \mathscr{L}(V, V')$ be defined by $\check{\tilde{f}}(x) = \tilde{f}_x$.

Theorem 1.1. *The map $\check{\tilde{f}} : X \longrightarrow \mathscr{L}(V, V')$ is continuous relative to the natural topology of $\mathscr{L}(V, V')$. Conversely, let $h : X \longrightarrow \mathscr{L}(V, V')$ be a continuous map, and let $\hat{h} : E \longrightarrow E'$ be the map which induces $h(x)$ on each fibre. Then $\hat{h}$ is a morphism of vector bundles.*

Proof To prove this theorem, we choose a basis $\{e_1, \ldots, e_n\}$ of V and a basis $\{e'_1, \ldots, e'_n\}$ of V'. With respect to this basis, $\tilde{f}_x$ may be regarded as matrix $(\alpha_{ij}(x))$, where $\alpha_{ij}(x)$ is the i^{th} coordinate of vector $\tilde{f}_x(e_j)$. Hence the function $x \longmapsto \alpha_{ij}(x)$ is obtained from the composition of the following continuous maps

$$X \xrightarrow{\beta_j} X \times V \xrightarrow{(\mathrm{Id}_X, \tilde{f}_x)} X \times V' \xrightarrow{\gamma} V' \xrightarrow{p_i} k,$$

where $\beta_j(x) = (x, e_j)$, $\gamma(x, v') = v'$, and p_i is the i^{th} projection of V' on k. Since the functions $x \longmapsto \alpha_{ij}(x)$ are continuous, the map $\check{\tilde{f}}$ which they induce is also continuous according to the definition of the topology of $\mathscr{L}(V, V')$.

Conversely, let $h : X \longrightarrow \mathscr{L}(V, V')$ be a continuous map. Then $\hat{h}$ is obtained from the composition of the continuous maps

$$X \times V \xrightarrow{\delta} X \times \mathscr{L}(V, V') \times V \xrightarrow{\epsilon} X \times V',$$

where $\delta(x, v) = (x, h(x), v)$ and $\epsilon(x, \alpha, v) = (x, \alpha(v))$. Hence $\hat{h}$ is continuous and defines a morphism of vector bundles.

$\blacksquare$

To say whether a real or complex vector bundle is isomorphic to the trivial bundle or not, the following lemma will be used.

Lemma 1.1. *If $\xi = (E, \pi, X)$ is a real (or a complex) vector bundle of rank n, it is isomorphic to the trivial real (or a complex) bundle of rank n over X iff ξ admits n sections $s_1, \ldots, s_n$ such that the vectors $s_1(x), \ldots, s_n(x)$ are linearly independent over $\mathbb{R}$ (or $\mathbb{C}$, respectively) in E_x for all $x \in X$.*

Examples

1. Let M be a smooth manifold, and let TM be its tangent bundle. So the fibre over each $x \in M$ is the tangent space at x. Let $x \in M$ and let U be a local coordinate patch about x. Let $x_1, \ldots, x_n$ be local coordinate in U, and let $\partial_1, \ldots, \partial_n$ be the associated vector fields. Then $\partial_1, \ldots, \partial_n$ are independent sections of TM, and hence give a local trivialization of TM.

2. Let V be a vector space and fix an integer $k > 0$. Consider the Grassmann

manifold [1]$G_{k,V}$ of k-planes in V. Let

$$\eta = \{(W, x) \mid W \in G_{k,V}, x \in W\}.$$

The projection to the first coordinate

$$\pi : \eta \longrightarrow G_{k,V}$$

is given by

$$(W, x) \longmapsto W$$

makes η into a vector bundle of rank k. Let $U \subseteq G_{k,V}$ be the collection of all k-planes whose orthogonal projection onto W is surjective (equivalently, an isomorphism). For each $J \in U$, let $s_1(J), \ldots, s_k(J)$ be the unique vectors in J that orthogonally project onto $e_1, \ldots, e_k$. These are continuous sections of $\eta|_U$, and are linearly independent.

1.2 Transition data

Suppose $\xi = (E, \pi, X)$ is real vector bundle of rank n. By Definition 1.2 , there exists a collection $\{(U_\alpha, \varphi_\alpha)\}_{\alpha\in\mathcal{A}}$ of trivializations for ξ such that $\bigcup_{\alpha\in\mathcal{A}} U_\alpha = X$. Thus, for all $\alpha, \beta \in \mathcal{A}$

$$\varphi_{\beta\alpha} := \varphi_\alpha \circ \varphi_\beta^{-1} : \big(U_\alpha \cap U_\beta\big) \times \mathbb{R}^n \longrightarrow \big(U_\alpha \cap U_\beta\big) \times \mathbb{R}^n$$

is a homeomorphism, and the restriction of $\varphi_{\beta\alpha}$ to $\{x\} \times \mathbb{R}^n$ defines an isomorphism of $\{x\} \times \mathbb{R}^n$ with itself. Such an isomorphism must be given by

$$(x, v) \longrightarrow (x, g_{\beta\alpha}(x)(v)) \quad \forall\, v \in \mathbb{R}^n,$$

for a unique element $g_{\beta\alpha}(x) \in GL_n(\mathbb{R})$. The map $\varphi_{\beta\alpha}$ is then given by

$$\varphi_{\beta\alpha}(x, v) = (x, g_{\beta\alpha}(x)(v)) \quad \forall\, x \in U_\alpha \cap U_\beta, v \in \mathbb{R}^n,$$

and is completely determined by the map $g_{\beta\alpha} : U_\alpha \cap U_\beta \longrightarrow GL_n(\mathbb{R})$.

[1] We will have detailed explaination of Grassmann manifolds and Grassmann bundles in the Subsection 2.1.1 .

By the previous paragraph, starting with a rank n real vector bundle $\xi = (E, \pi, X)$, we obtain an open cover $\{U_\alpha\}_{\alpha \in \mathcal{A}}$ of X and a collection of transition maps

$$\{g_{\beta\alpha} : U_\alpha \cap U_\beta \longrightarrow GL_n(\mathbb{R})\}_{\alpha,\beta \in \mathcal{A}}$$

Let I_n denote the identity element in $GL_n(\mathbb{R})$. These transition maps satisfy the following equalities

1. $g_{\alpha\alpha} = I_n$;
2. $g_{\beta\alpha} \circ g_{\alpha\beta} = I_n$;
3. $g_{\beta\alpha} \circ g_{\alpha\gamma} \circ g_{\gamma\beta} = I_n$.

The last condition is called the *Čech cocycle* condition.

Conversely, given an open cover $\{U_\alpha\}_{\alpha \in \mathcal{A}}$ of X and a collection of maps

$$\{g_{\beta\alpha} : U_\alpha \cap U_\beta \longrightarrow GL_n(\mathbb{R})\}_{\alpha,\beta \in \mathcal{A}}$$

that satisfy the above three conditions, we can assemble a rank-n vector bundle over X.

1.3 Operations on vector bundles

1.3.1 Direct sum or Whitney sum

Given two vector bundles $\xi = (E, \pi, X)$ and $\xi' = (E', \pi', X)$ over the same base space X, we would like to create a third vector bundle over X whose fibre over each point x of X is the direct sum of the fibres E_x and E'_x. So, we define *direct sum* of E and E' as the space

$$E \oplus E' = \{(e_1, e_2) \in E \times E' \mid \pi(e_1) = \pi'(e_2)\}$$

Examples

1. The direct sum of two trivial bundles is again a trivial bundle.
2. The direct sum of tangent and normal bundle over $\mathbb{S}^n$ is the trivial bundle and the isomorphism is given by the map

$$(x, v, tx) \longmapsto (x, v + tx)$$

where $(x, v, tx) \in T\mathbb{S}^n \oplus N\mathbb{S}^n$ with $x \perp v$.

3. The direct sum of the canonical line bundle $\gamma^1_{n,\mathbb{R}}$ over $\mathbb{RP}^n$ with its complement $(\gamma^1_{n,\mathbb{R}})^\perp$ is isomorphic to the trivial bundle via the map

$$(l, v, w) \longmapsto (l, v + w)$$

where $v \in l$ and $w \perp l$.

Hence the tangent bundle over $\mathbb{S}^n$ and the canonical line bundle over $\mathbb{RP}^n$ are *stably trivial*, i.e. they become trivial after taking the direct sum with the trivial bundle.

1.3.2 Inner product

An *inner product* on a vector bundle $\xi = (E, \pi, X)$ is a map

$$\langle -, - \rangle : E \oplus E \longrightarrow X$$

which restricts on each fibre to an inner product, a positive definite symmetric bilinear form.

Proposition 1.1. *An inner product exists for a vector bundle $\xi = (E, \pi, X)$ if X is a compact Hausdorff or more generally paracompact space.*

We will deduce this result again in the Section 2.1 , on *Universal Bundles.*

We know that a vector subspace is always a direct summand by taking its orthogonal complement. The corresponding result holds for vector bundles over a paracompact base space.

We define *vector subbundle* of a vector bundle $\xi = (E, \pi, X)$ as a subspace $E_0 \subset E$, intersecting each fibre of E in a vector subspace, such that the restriction $\pi|_{E_0} : E_0 \longrightarrow X$ is a vector bundle.

Lemma 1.2. *If $\xi = (E, \pi, X)$ is a vector bundle over a paracompact base X and $E_0 \subset E$ is a vector subbundle, then there is a vector subbundle $E_0^\perp \subset E$ such that $E_0 \oplus E_0^\perp \approx E$.*

We have seen that in Examples 1.3.1 the direct sum of two bundles, one or both of which may be non trivial, is the trivial bundle. Here is the general result along these lines:

Lemma 1.3. *For each vector bundle $\xi = (E, \pi, X)$ with X compact Hausdorff there exists a vector bundle $\xi' = (E', \pi, X)$ such that $E \oplus E'$ is the trivial bundle.*

1.3.3 Pullback bundles

We will denote the set of isomorphism classes of real vector bundles of rank n over X by $\mathrm{Vect}^n_{\mathbb{R}}(X)$. The complex analogue is denoted by $\mathrm{Vect}^n_{\mathbb{C}}(X)$.

Proposition 1.2. *Given a map $f : X' \longrightarrow X$ and a vector bundle $\xi = (E, \pi, X)$, then there exists a vector bundle $\xi' = (E', \pi', X')$ with a map $\tilde{f} : E' \longrightarrow E$ taking the fibre of E' over each point $x' \in X'$ isomorphically onto the fibre of E over $f(x')$, and such a vector bundle is unique upto isomorphism. Often the vector bundle E' is written as $f^*(E)$ and called the bundle induced by f, or the pullback of E by f.*

Proof First we construct an explicit pullback by setting

$$E' = \{(x', e) \in X' \times E \mid f(x') = \pi(e)\}.$$

This subspace of $X' \times E$ fits into the following diagram at the right where $\pi'(x', e) = x'$ and $\tilde{f}(x', e) = e$.

$$\begin{array}{ccc} E' & \xrightarrow{\tilde{f}} & E \\ {\scriptstyle \pi'}\downarrow & & \downarrow{\scriptstyle \pi} \\ X' & \xrightarrow[f]{} & X \end{array}$$

Notice that the two compositions $f \circ \pi'$ and $\pi \circ \tilde{f}$ are equal. Now let

$$\Gamma_f = \{(x', f(x')) \in X' \times X\}$$

denote the graph of f. Then π' factors as the composition

$$E' \longrightarrow \Gamma_f \longrightarrow X',$$

$$(x', e) \longmapsto (x', \pi(e)) = (x', f(x')) \longmapsto x'.$$

The first of these two maps is the restriction of the vector bundle

$$\mathrm{Id} \times \pi : X' \times E \longrightarrow X' \times X$$

over the Γ_f, and the second map is a homeomorphism. The map $\tilde{f}$ obviously takes the fibre $E'_{x'}$ isomorphically onto the fibre $E_{f(x')}$.

From the construction, it is clear that $\xi' = (E', \pi', X')$ is a vector bundle. And for the uniqueness, we can construct an isomorphism from an arbitrary E' which satisfies the conditions in the proposition to the particular one constructed above by sending $e' \in E'$ to the pair $(\pi'(e'), \tilde{f}(e'))$.

$\blacksquare$

Remark 1.2. *From the uniqueness statement above, it follows that we have a function $f^* : Vect^n_{\mathbb{R}}(X) \longrightarrow Vect^n_{\mathbb{R}}(X')$ taking isomorphism classes of E to the isomorphism classes of E'.*

Properties:

1. $(f \circ g)^*(E) \approx g^*(f^*(E))$
2. $\mathrm{Id}^* \approx E$. Here $\mathrm{Id} : X \longrightarrow X$ is the identity map.
3. $f^*(E_1 \oplus E_2) \approx f^*(E_1) \oplus f^*(E_2)$
4. $f^*(E_1 \otimes E_2) \approx f^*(E_1) \otimes f^*(E_2)$

Examples

1. If f is an inclusion $Y \hookrightarrow X$ then $f^*(E)$ is just the restriction $E|_Y$, where E is the vector bundle over X.
2. The pullback bundle of *Möbius* bundle by

$$f : \mathbb{S}^1 \longrightarrow \mathbb{S}^1$$

$$z \longmapsto z^2$$

is trivial line bundle.

We define M as follows:

$$M = \{(e^{\iota\theta}, te^{\iota\frac{\theta}{2}}) \mid \theta \in [0, 2\pi], t \in \mathbb{R}\}.$$

Then

$$f^*(M) = \{(e^{\iota\theta}, te^{\iota\theta}) \mid \theta \in [0, 2\pi], t \in \mathbb{R}\}$$

which is isomorphic to $\mathbb{S}^1 \times \mathbb{R}$, where isomorphism is given by the map

$$\phi : f^*(M) \longrightarrow \mathbb{S}^1 \times \mathbb{R}$$

$$(e^{\iota\theta}, te^{\iota\theta}) \longmapsto (e^{\iota\theta}, t).$$

1.3.4 Tensor product of vector bundles

In this section, we answer the following questions:

1. How tensor products of several vector bundles is built?
2. How the same technique applies to define notions of symmetric and exterior powers of a vector bundle?

Let $E, E^1, \ldots, E^n$ be real vector bundles over X. As sets, we define

$$E^1 \otimes \ldots \otimes E^n = \coprod_{x \in X} (E^1_x \otimes \ldots \otimes E^n_x)$$

$$\mathrm{Sym}^n(E) = \coprod_{x \in X} \mathrm{Sym}^n(E_x)$$

$$\wedge^n(E) = \coprod_{x \in X} \wedge^n(E_x)$$

We map each set to X by sending the subset indexed by $x \in X$ to the point $x \in X$. Thus, each of these sets is equipped with a map π to X such that each fibre $\pi^{-1}(x)$ is the usual linear-algebra tensorial operation applied to the fibre of the given bundles.

The main task is to put reasonable topologies on these sets to make them vector bundles over X. We first address the topological aspect of the construction problem. We cover X by open subsets U. For a choice of such a U, we fix trivializations of the given bundles. In the tensor product case we fix isomorphisms

$$\varphi_i : E^i|_U \approx U \times V_i$$

as vector bundles over U, with V_i a finite dimensional $\mathbb{R}$-vector space, and in other cases we fix an isomorphism

$$\varphi : E|_U \approx U \times V$$

as vector bundles over U. Let $\tau = (U; \varphi_1, \ldots, \varphi_n)$ in the tensor product case and let $\tau = (U, \varphi)$ in other cases; this is the "trivialization data" over U for the given bundle(s). For each $x \in U$, the data in τ provides linear isomorphisms of fibres. Hence, for the case of tensor products we have bijections

$$\psi_{(U,\tau)} : \pi^{-1}(U) \approx U \times (V_1 \otimes \ldots \otimes V_n)$$

and in symmetric and exterior power cases we have bijections

$$\psi_{(U,\tau)} : \pi^{-1}(U) \approx U \times \mathrm{Sym}^n(V)$$

$$\psi_{(U,\tau)} : \pi^{-1}(U) \approx U \times \wedge^n(V).$$

We declare the bijection $\psi_{(U,\tau)}$ to be homeomorphism in each case. Let $S_{(U,\tau)}$ denote the set $\pi^{-1}(U)$ with the topology induced via $\psi_{(U,\tau)}$ from the topology on its target. The first main problem is to show that these topologies glue to define topologies on the sets $E^1 \otimes \ldots \otimes E^n, \mathrm{Sym}^n(E)$, and $\wedge^n(E)$

The method of gluing topologies reduces our task to check two things: For any open set $U, U' \subseteq X$ and the trivialization data τ, τ' over these, we need

1. the overlap $S_{(U,\tau)} \cap S_{(U',\tau')}$ is an open subset in each of the topological spaces $S_{(U,\tau)}$ and $S_{(U',\tau')}$,
2. the subspace topologies on this overlap via its inclusion into each of $S_{(U,\tau)}$ and $S_{(U',\tau')}$ are the same topology.

The overlap is the subset $\pi^{-1}(U) \cap \pi^{-1}(U') = \pi^{-1}(U \cap U')$, so the first one follows from the fact that for any topological space X', the subset $(U \cap U') \times X'$ is open in $U \times X'$ and $U' \times X'$. As for the second one, this amounts to proving that the bijective transition map

$$\psi_{(U',\tau')} \circ \psi_{(U,\tau)}^{-1} : \psi_{(U,\tau)}(\pi^{-1}(U) \cap \pi^{-1}(U')) \longrightarrow \psi_{(U',\tau')}(\pi^{-1}(U) \cap \pi^{-1}(U'))$$

is a homeomorphism. In case of tensor products, the transition map is the self-map of $(U \cap U') \times (V_1 \otimes \ldots \otimes V_n)$ given by

$$(u, t) \longmapsto (u, ((\varphi_1'(u) \circ \varphi_1(u)^{-1}) \otimes \ldots \otimes (\varphi_n'(u) \circ \varphi_n(u)^{-1}))(t))$$

for $u \in U \cap U'$ and $t \in V_1 \otimes \ldots \otimes V_n$. In case of symmetric powers, the transition map is the self map of $(U \cap U') \times \mathrm{Sym}^n(V)$ is given by

$$(u, t) \longmapsto (u, (\mathrm{Sym}^n(\varphi'(u) \circ \varphi(u)^{-1}))(t))$$

for $u \in U \cap U'$ and $t \in \mathrm{Sym}^n(V)$. In case of exterior powers, the transition map is the self map of $(U \cap U') \times \wedge^n(V)$ is given by

$$(u, t) \longmapsto (u, (\wedge^n(\varphi'(u) \circ \varphi(u)^{-1}))(t))$$

for $u \in U \cap U'$ and $t \in \wedge^n(V)$.

Upon picking bases of the vector spaces V_i and V, we get bases for tensor products and symmetric or exterior powers in the usual manner, and so the problem is to prove that linear maps

$$((\varphi_1'(u) \circ \varphi_1(u)^{-1}) \otimes \ldots \otimes (\varphi_n'(u) \circ \varphi_n(u)^{-1})), (\mathrm{Sym}^n(\varphi'(u) \circ \varphi(u)^{-1})), (\wedge^n(\varphi'(u) \circ \varphi(u)^{-1}))$$

depending on u are given by matrices in the chosen bases whose entries have dependence on $u \in U \cap U'$. Since the linear maps $\varphi_i(u)$, $\varphi_i'(u)$, $\varphi(u)$ and $\varphi'(u)$ are matrix valued functions on $U \cap U'$, hence we get desired property for the transition maps.

1.3.5 Dual and hom bundles

We now wish to define a dual vector bundle E^* and a hom bundle $\mathrm{Hom}(E', E)$ for vector bundles E and E' over X.

Roughly speaking, E^* should be the bundle over X whose fibre over x is E_x^* for every $x \in X$, and similarly $\mathrm{Hom}(E', E)$ should be the bundle over X whose fibre over

x is $\text{Hom}(E'_x, E_x)$ for every $x \in X$.

Using the method of Subsection 1.3.4 we can define the dual bundle E^* and $\text{Hom}(E', E)$, which set-theoretically are

$$\text{Hom}(E', E) = \coprod_{x \in X} \text{Hom}(E'_x, E_x),$$

and

$$E^* = \{(x, v) \mid x \in X, v \in E^*_x\}.$$

Examples

1. The dual bundle E^* is $\text{Hom}(E, X \times \mathbb{R})$. This has fibres naturally identified with $\text{Hom}(E_x, \mathbb{R}) = E^*_x \; \forall \, x \in X$.

2. Let L be a line through origin in $\mathbb{R}^{n+1}$, intersecting $\mathbb{S}^n$ in the points $\{\pm x\}$, and let $L^\perp$ be the complementary n-plane. Let $f : \mathbb{S}^n \longrightarrow \mathbb{R}^n$ denote the canonical map $f(x) = \{\pm x\}$. Now the map

$$Tf : T\mathbb{S}^n \longrightarrow T\mathbb{RP}^n$$

is induced by the map f. Thus

$$T\mathbb{RP}^n = \{(x, v), (-x, -v) \mid x.x = 1, x.v = 0\}$$

But each such pair determines, and determined by, a linear mapping

$$l : L \longrightarrow L^\perp,$$

where

$$l(x) = v.$$

The tangent space of $\mathbb{R}^n$ at $\{\pm x\}$ is canonically isomorphic to the vector space $\text{Hom}(L, L^\perp)$. Hence

$$T\mathbb{RP}^n \approx \text{Hom}(\gamma^1_{n,\mathbb{R}}, (\gamma^1_{n,\mathbb{R}})^\perp).$$

Remark 1.3. *The notation Hom(E', E) for a certain vector bundle is not to be confused with the set Hom$_X(E, E')$ of bundle morphisms over X.*

1.4 Principal bundles

1.4.1 Definitions and examples

If we consider ξ as a real n-plane bundle with Euclidean metric [2] , then there are number of bundles associated with the given bundle [3] . By easy (but repetitive arguments), we can show that projection map is indeed a local product, furthermore, transition functions are always linear. Hence, it turns out that all of the data can be subsumed in a single object called *principal $O(n)$-bundle*, which is just the bundle of orthonormal n-frames.

The concept of principal bundle acts as a powerful unifying force in algebraic topology. The homotopy classification theorem for vector bundles then emerges as a special case of the homotopy classification theorem for principal bundles.

Definition 1.5. *Let X be a topological space. Suppose that P is a right G-space equipped with a G-map*

$$\pi : P \longrightarrow X,$$

*where G acts trivially on X. We say that (P, π) is a **principal G-bundle** over X if π satisfies the following local triviality condition: X has a covering by open sets U_i such that there exist G-equivariant homeomorphisms*

$$\phi_{U_i} : \pi^{-1}(U_i) \longrightarrow U_i \times G$$

commuting the diagram

$$\begin{array}{ccc} \pi^{-1}(U_i) & \xrightarrow{\phi_{U_i}} & U_i \times G \\ & \searrow^{\pi|_{\pi^{-1}(U_i)}} & \downarrow{pr_1} \\ & & U_i \end{array}$$

Here $U_i \times G$ has the right G-action [4]

$$(u_i, g)h = (u_i, gh).$$

[2]If we do not want to consider a Euclidean metric, there is an analogous notion of $GL_n(\mathbb{R})$-bundle, bundle of linearly independent n-frames.

[3]Here bundle simply means that it is locally a product with the indicated fibre.

[4]This condition implies that G acts freely on P, and that π factors through a homeomorphism $\bar{\pi} : P/G \longrightarrow X$. And hence, X is the orbit space of P.

Thus, a principal G-bundle over X consists of a locally trivial free G-space with orbit space X.

1.4.2 Principal $GL_n(\mathbb{R})$-bundles

Let $\xi = (E, \pi, X)$ be a real vector bundle of rank n. The associated principal $GL_n(\mathbb{R})$-bundle can be defined in two equivalent ways. We emphasize that the two definitions are really little more than mild paraphrases of one another, based on the following fact:

Let W be a real vector space of dimension n and let V_nW denote the space of n-frames in W. Let $\mathrm{Iso}(\mathbb{R}^n, W)$ denote the space of all linear isomorphisms

$$f : \mathbb{R}^n \longrightarrow W.$$

Then there is a natural homeomorphism

$$g : Iso(\mathbb{R}^n, W) \longrightarrow V_nW$$

given by

$$f \longmapsto (f(e_1), \ldots, f(e_n)).$$

In the first definition, we set $P = P_\xi = V_n\xi$, where $V_n\xi$ denotes the n-frame bundle, i.e, the set of pairs $(x, (v_1, \ldots, v_n))$ with $x \in X$ and $(v_1, \ldots, v_n)$ an n-frame in E_x. We topologize P as a subspace of Whitney sum

$$nE = E \oplus \ldots \oplus E.$$

Note that $GL_n(\mathbb{R})$ acts on the right of nE by the rule

$$(v_1, \ldots, v_n)A = (v_1', \ldots, v_n')$$

where $v_j' = \sum_i v_i A_{ij}$. Moreover, given action is free, and it makes P into a principal $GL_n(\mathbb{R})$-bundle over X.

In the second definition, consider the vector bundle $\mathrm{Hom}(\epsilon^n, E)$. It can be identified with n-fold Whitney sum $E \oplus \ldots \oplus E$. Now $GL_n(\mathbb{R})$ acts on the left of ϵ^n by

$$g(x, v) = (x, gv)$$

inducing a right action on $\mathrm{Hom}(\epsilon^n, E)$. As our second definition of P_ξ, we take the subbundle $\mathrm{Iso}(\epsilon^n, E)$ of $\mathrm{Hom}(\epsilon^n, E)$ for which fibre over $x \in X$ is the space of all isomorphisms

$$\mathbb{R}^n \longrightarrow E_x.$$

Clearly, $GL_n(\mathbb{R})$ acts freely on P. Using local triviality of E, we see that P is a principal $GL_n(\mathbb{R})$-bundle over X.

Moreover, the natural map

$$\mathrm{Iso}(\epsilon^n, E) \longrightarrow \mathrm{Hom}(\epsilon^n, E)$$

given by

$$(x, f) \longmapsto (x, f(e_1), \ldots, f(e_n))$$

is an equivariant homeomorphism. Hence, this definition agrees with the previous one.

Remark 1.4. *If the vector bundle $\xi = (E, \pi, X)$ has a Euclidean metric, we can define an associated principal $O(n)$-bundle* [5] *in an analogous way. The analogue of our first definition is to take $P = V_n\xi$, the bundle of orthonormal n-frames. The analogue of our second definition is to take $P = Iso(\epsilon^n, \xi)$, the bundle whose fibre over $x \in X$ is the space of isometries of $\mathbb{R}^n \approx E_x$.*

Examples

1. Let $\gamma^1_{n,\mathbb{R}}$ be the canonical line bundle over $\mathbb{RP}^n$. The associated principal $O(1)$-bundle $P_{\gamma^1_{n,\mathbb{R}}}$ is the just the usual covering map

$$\mathbb{S}^n \longrightarrow \mathbb{RP}^n,$$

with $O(1) \approx \mathbb{Z}/2$.

[5] In a similar way, for a complex n-plane bundle with Hermitian metric, we have an associated $U(n)$ bundle.

2. Let $\gamma^1_{n,\mathbb{C}}$ be the canonical line bundle over $\mathbb{CP}^n$. The associated principal $U(1)$-bundle $P_{\gamma^1_{n,\mathbb{C}}}$ is the just the usual quotient map

$$\mathbb{S}^{2n+1} \longrightarrow \mathbb{CP}^n,$$

with $U(1) \approx \mathbb{S}^1$, acting in the usual way via complex multiplication.

3. Let $T\mathbb{S}^n$ be the tangent bundle of $\mathbb{S}^n$. Then the associated principal $O(n)$-bundle $P_{T\mathbb{S}^n}$ is the space of pairs $(x, (v_1, \dots, v_n))$ with $x \in \mathbb{S}^n$ and $(v_1, \dots, v_n)$ an orthonormal n-frame perpendicular to x. But this is just $V_{n+1}(\mathbb{R}^{n+1}) = O(n+1)$. Hence the map

$$P_{T\mathbb{S}^n} \longrightarrow \mathbb{S}^n$$

can be identified with the standard quotient map

$$O(n+1) \longrightarrow \mathbb{S}^n.$$

1.5 Clutching theorems

We would like to construct vector bundles using their restrictions to suitable subsets.

Theorem 1.2 (Clutching of morphisms). *Let $\xi = (E, \pi, X)$ and $\xi' = (E', \pi', X)$ be two vector bundles on the same base space X. Let us consider also*

1. *A cover of X consisting of open subsets U_i (respectively a locally finite cover of X of closed subsets U_i).*

2. *A collection of morphisms $\alpha_i : \xi|_{U_i} \longrightarrow \xi'|_{U_i}$ such that $\alpha_i|_{U_i \cap U_j} = \alpha_j|_{U_i \cap U_j}$.*

Then there exists a unique morphism $\alpha : \xi \longrightarrow \xi'$ such that $\alpha|_{U_i} = \alpha_i$.

Theorem 1.3 (Clutching of bundles). *Let U_i be an open cover of a space X (respectively a locally finite closed cover of X). Let $\xi_i = (E_i, \pi_i, U_i)$ be a vector bundle over each U_i, and let $g_{ji} : \xi_i|_{U_i \cap U_j} \longrightarrow \xi_j|_{U_i \cap U_j}$ be isomorphisms which satisfy compatibility condition $g_{ki}|_{U_i \cap U_j \cap U_k} = g'_{kj} \circ g'_{ji}$, where $g'_{kj} = g_{kj}|_{U_i \cap U_j \cap U_k}$ and $g'_{ji} = g_{ji}|_{U_i \cap U_j \cap U_k}$. Then there exists a vector bundle ξ over X and isomorphisms $g_i : \xi_i \longrightarrow \xi_{U_i}$ such that the diagram*

$$
\begin{array}{ccc}
\xi_i|_{U_i \cap U_j} & \xrightarrow{g_{ji}} & \xi_j|_{U_i \cap U_j} \\
 & {\scriptstyle g_i|_{U_i \cap U_j}} \searrow & \downarrow {\scriptstyle g_j|_{U_i \cap U_j}} \\
 & & \xi|_{U_i \cap U_j}
\end{array}
$$

commutes.

The bundle ξ is unique in the following sense. Let ξ' be another vector bundle, and let $g'_i : \xi_i \longrightarrow \xi'|_{U_i}$, be isomorphisms which make the diagram

$$
\begin{array}{ccc}
\xi_i|_{U_i \cap U_j} & \xrightarrow{g_{ji}} & \xi_j|_{U_i \cap U_j} \\
 & {\scriptstyle g'_i|_{U_i \cap U_j}} \searrow & \downarrow {\scriptstyle g'_j|_{U_i \cap U_j}} \\
 & & \xi'|_{U_i \cap U_j}
\end{array}
$$

commutative. Then there exists a unique isomorphism $\alpha : \xi \longrightarrow \xi'$ which makes the following diagram commutative.

$$
\begin{array}{ccc}
\xi_i & & \\
{\scriptstyle g_i} \downarrow & \searrow {\scriptstyle g'_i} & \\
\xi|_{U_i} & \xrightarrow[\alpha|_{U_i}]{} & \xi'|_{U_i}
\end{array}
$$

In fact one may construct α in the following way. The morphism $\alpha_i = g'_i \circ g_i^{-1}$ is an isomorphism from $\xi|_{U_i}$ to $\xi'|_{U_i}$, and over $U_i \cap U_j$, we have identity $g_{ji} = g_j^{-1} \circ g_i = g_j'^{-1} \circ g'_i$. Therefore, over $U_i \cap U_j$ we have $\alpha_i = g'_i \circ g_i^{-1} = g'_j \circ g_j'^{-1} = \alpha_j$. The existence of α is then guaranted by Theorem 1.2 . Its uniqueness is obvious.

Examples

1. Let $\mathbb{S}^k$ be the unit sphere of $\mathbb{R}^{k+1}$. Let $\mathbb{S}^k_+$ (respectively $\mathbb{S}^k_-$) be the subset of $\mathbb{S}^k$ whose points x satisfy $x_{k+1} \geq 0$ (respectively $x_{k+1} \leq 0$). Then $\mathbb{S}^k_+$ and $\mathbb{S}^k_-$ are closed subsets, and $\mathbb{S}^k_+ \cap \mathbb{S}^k_- = \mathbb{S}^{k-1}$.

   Let $f : \mathbb{S}^{k-1} \longrightarrow GL_n(\mathbb{R})$ be a continuous map. According to Theorem 1.2 , there is a bundle E_f over $\mathbb{S}^k$ which is naturally associated with f. It is obtained from the clutching of the trivial bundles $E_1 = \mathbb{S}^k_+ \times \mathbb{R}^n$ and $E_1 = \mathbb{S}^k_- \times \mathbb{R}^n$ by the transition function $g_{21} : \mathbb{S}^{k-1} \times \mathbb{R}^n \longrightarrow \mathbb{S}^{k-1} \times \mathbb{R}^n$ (g_{11} and g_{22} are the identity maps).

In general the clutching theorems are useful in the construction of the tangent bundle of a differentiable manifold.

2. Let us find a clutching function for the canonical complex line bundle $\gamma^1_{1,\mathbb{C}}$ over $\mathbb{CP}^1 = \mathbb{S}^2$. The space $\mathbb{CP}^1$ is the quotient of $\mathbb{C}^2 - \{0\}$ under the equivalence relation
$$(z_0, z_1) \sim \lambda(z_0, z_1) \text{ for } \lambda \in \mathbb{C} - \{0\}.$$
Denote the equivalence class of (z_0, z_1) by $[z_0, z_1]$. We can also write points of $\mathbb{CP}^1$ as ratios $z = z_0/z_1 \in \mathbb{C} \cup \{\infty\} = \mathbb{S}^2$. If we do this, then points in the disk D^2_0 inside the unit circle $\mathbb{S}^1$ can be expressed uniquely in the form $[z_0/z_1, 1] = [z, 1]$ with $|z| \leq 1$, and points in the disk D^2_∞ outside $\mathbb{S}^1$ can be written uniquely in the form $[1, z_1/z_0] = [1, z^{-1}]$ with $\left|z^{-1}\right| \geq 1$. Over D^2_0, a section of the canonical line bundle is given by
$$[z, 1] \longrightarrow (z, 1)$$
and over D^2_∞ a section is
$$[1, z^{-1}] \longrightarrow (1, z^{-1}).$$
These sections determine trivializations of the canonical line bundle over these two disks, and over their common boundary $\mathbb{S}^1$. We pass from the D^2_∞ trivialization to the D^2_0 trivialization by multiplying by z. Thus, if we take D^2_∞ as D^2_+ and D^2_0 as D^2_-, then the canonical line bundle has the clutching function [6]
$$f : \mathbb{S}^1 \longrightarrow GL_1(\mathbb{C})$$
defined by
$$f(z) = (z).$$

Remark 1.5. *Theorem 1.2 is related to the problem of classification of principal G-bundles, where G is a topological subgroup of* $GL_n(k)$. *To be more precise, let us consider an arbitrary topological group G and a topological space X. A G-cocycle on X is given by an open cover* $\{U_i\}$ *of X, and continuous maps* $g_{ji} : U_i \cap U_j \longrightarrow G$ *such that* $g_{kj}(x) \circ g_{ji}(x) = g_{ki}(x)$ *for* $x \in U_i \cap U_j \cap U_k$.

[6] If we had taken D^2_+ to be D^2_0 rather than D^2_∞, then the clutching function would have been $f(z) = z^{-1}$.

Two cocycles (U_i, g_{ji}) *and* (V_r, h_{sr}) *are equivalent if there exist continuous maps* $g_i^r : U_i \cap V_r \longrightarrow G$ *such that* $g_j^s(x) \circ g_{ji}(x) \circ g_i^r(x)^{-1} = h_{sr}(x)$ *for* $x \in U_i \cap U_j \cap V_r \cap V_s$. *Equivalence relation is well defined.*

The quotient set will be denoted by $H^1(X; G)$.

Theorem 1.4. *Let* (U_i, g_{ji}) *and* (U_i, h_{ji}) *be two cocycles relative to the same open cover of a space* X. *Then the associated vector bundles* E *and* E' *are isomorphic iff there exist continuous functions* $\lambda_i : U_i \longrightarrow G = GL_n(k)$, *such that* $h_{ji}(x) = \lambda_j(x) \circ g_{ji}(x) \circ (\lambda_i(x))^{-1}$ *for* $x \in U_i \cap U_j$.

1.6 Hopf bundle

The Hopf bundle is the morphism

$$h : \mathbb{S}^3 \longrightarrow \mathbb{S}^2$$

defined by

$$h(a, b, c, d) = (a^2 + b^2 - c^2 - d^2, 2(ad + bc), 2(bd - ac)). \tag{1.1}$$

We give the important descriptions of the Hopf bundle.

The Hopf bundle via Riemann sphere

To begin with this description of the Hopf bundle, we look at $\mathbb{S}^3$, described as sitting in $\mathbb{C}^2$ as follows:

$$\mathbb{S}^3 = \{(z_1, z_2) \in \mathbb{C}^2 \mid |z_1|^2 + |z_2|^2 = 1\}.$$

Now consider the ratio z_2/z_1. This is a complex number, unless $z_1 = 0$. But in this case, since we know about $\mathbb{P}^1_{\mathbb{C}}$, the Riemann sphere, by setting $z_2/0 = \infty$, we get a map

$$h : \mathbb{S}^3 \longrightarrow \mathbb{S}^2$$

$$(z_1, z_2) \longmapsto z_2/z_1.$$

This is the Hopf bundle.

We would like to understand the fibres of this map. For this, it helps to represent z_1 and z_2 in polar coordinates, so write $z_1 = r_1 e^{\iota\theta_1}$ and $z_2 = r_2 e^{\iota\theta_2}$. Now observe that for any complex number $\lambda \in \mathbb{C}$ of unit norm, and any point (z_1, z_2) in $\mathbb{S}^3$, not only is $(\lambda z_1, \lambda z_2)$ is still in $\mathbb{S}^3$ (since $|\lambda z| = |\lambda||z| = |z|$ for any complex number z) but in fact, these points are all on the same fibre of the Hopf bundle; we have $h(z_1, z_2) = h(\lambda z_1, \lambda z_2)$, as follows:

$$\begin{aligned} h(\lambda z_1, \lambda z_2) &= \frac{\lambda z_2}{\lambda z_1} \\ &= \frac{z_2}{z_1} \\ &= h(z_1, z_2) \end{aligned}$$

Now suppose

$$h(z_1, z_2) = h(w_1, w_2)$$

As above we write these points in polar coordinates as $z_1 = r_1 e^{\iota\theta_1}$ and $z_2 = r_2 e^{\iota\theta_2}$, $w_1 = s_1 e^{\iota\phi_1}$ and $w_2 = s_2 e^{\iota\phi_2}$. From the given condition we have

$$\big|h(z_1, z_2)\big| = \big|h(w_1, w_2)\big|$$

i.e,

$$\left|\frac{z_2}{z_1}\right| = \left|\frac{w_2}{w_1}\right|$$

and hence

$$\frac{r_2}{r_1} = \frac{s_2}{s_1}.$$

Since (z_1, z_2) and (w_1, w_2) are on $\mathbb{S}^3$ and we get $s_1^2 = r_1^2$, similarly $s_2^2 = r_2^2$. In other words, there exists a unit complex number λ such that

$$(w_1, w_2) = (\lambda z_1, \lambda z_2).$$

Since (w_1, w_2) and $(\lambda z_1, \lambda z_2)$ are distinct points for $|\lambda| \neq 1$, the fibres are topological circles.

The Hopf bundle via quaternions

We now give a reformulation of the Hopf map h in terms of quaternions. First fix a distinguished point, say $P = (1, 0, 0)$ on $\mathbb{S}^2$. Given a point on $\mathbb{S}^3$, let $r = a+bi+cj+dk$ be the corresponding unit quaternion. The quaternion r then defines a rotation R_r of the 3-space and the Hopf bundle is defined by

$$r \longmapsto R_r(P) = ri\bar{r}. \tag{1.2}$$

Now we verify that both these formulas given in (1.1) and (1.2), for the Hopf bundle are equivalent.

$$\begin{aligned} ri\bar{r} &= (a + bi + cj + dk)(i)(a - bi - cj - dk) \\ &= (ai - b - ck + dj)(a - bi - cj - dk) \\ &= a^2 i + ab - ack + adj - ab + b^2 i + bcj + bdk - ack + bcj - c^2 i - cd + adj + bdk + cd - d^2 i \\ &= (a^2 + b^2 - c^2 - d^2)i + 2(ad + bc)j + 2(bd - ac)k \end{aligned}$$

Consider the point $(1, 0, 0)$ in $\mathbb{S}^2$. The set of points

$$C = \{(\cos t, \sin t, 0, 0) \in \mathbb{S}^3 \mid t \in \mathbb{R}\}$$

map to $(1, 0, 0)$ via the Hopf map h. In fact, this set C is the entire set of points that map to $(1, 0, 0)$ via h. In other words, C is the preimage set $h^{-1}(1, 0, 0)$. Clearly C is the unit circle in a plane in $\mathbb{R}^4$.

Next aim is to study configuration of fibres in $\mathbb{S}^3$. The main idea is to use stereographic projection to get an elegant decomposition of 3-space into a union of disjoint circles and a single straight line.

Now we demonstrate a method that allows us to see a little of what is going on with the Hopf bundle. Consider the projection map $\mathbb{S}^3/(1, 0, 0, 0) \longrightarrow \mathbb{R}^3$ given by

$$(w, x, y, z) \longmapsto \left(\frac{x}{1 - w}, \frac{y}{1 - w}, \frac{z}{1 - w} \right). \tag{1.3}$$

Now a circle on $\mathbb{S}^3$ that does not pass through the point $(1, 0, 0, 0)$ is mapped to a circle in $\mathbb{R}^3$. We know that fibres of the Hopf map h are circles in $\mathbb{S}^3$. It follows that stereographic projection sends them to circles (or a line, if the fibre contains the point $(1, 0, 0, 0)$ in $\mathbb{R}^3$ [7] . We conclude with the following observations that the Hopf fibres are linked.

[7] The property that stereographic projection preserves circles holds in all dimensions.

Observation 1: Let s denote the stereographic projection of $\mathbb{S}^3$ given by (1.3). Consider the point $P = (1, 0, 0)$ on $\mathbb{S}^2$,

$$s(h^{-1})(P) = s(\cos t, \sin t, 0, 0) = \frac{\sin t}{1 - \cos t}$$

Clearly, $s(h^{-1})(P)$ is the x-axis.

Observation 2: Let $C_1 = \{(0, 0, \cos t, \sin t) \mid t \in \mathbb{R}\}$ be the set of points on $\mathbb{S}^3$. We see that via Hopf map h, these points map to $(-1, 0, 0)$, say Q.

$$\begin{aligned}
ri\bar{r} &= ((\cos t)j + (\sin t)k)(i)(-(\cos t)j - (\sin t)k) \\
&= (-(\cos t)k + (\sin t)j)(-(\cos t)j - (\sin t)k) \\
&= ((\cos^2 t)(-i) + \cos t \sin t - \sin t \cos t + (\sin^2 t)(-i)) \\
&= -i
\end{aligned}$$

Hence $s(0, 0, \cos t, \sin t)$ is the unit circle in the yz-plane.

Observation 3: Now for any point $R = (x_1, x_2, x_3)$ on $\mathbb{S}^2$ not equal to P and Q, we notice that $s(h^{-1})(R)$ is a circle in $\mathbb{R}^3$ that intersects the yz-plane in exactly two points A and B, one inside and one outside the unit circle in the yz-plane.

Consider the point $R = (\cos t, 0, \sin t) \in \mathbb{S}^2$. We will show that $h^{-1}(R) = \{(0, \cos \frac{t}{2}, 0, \sin \frac{t}{2}) \in \mathbb{S}^3, t \in \mathbb{R}\}$. Apply (1.2) when $r = (0, \cos \frac{t}{2}, 0, \sin \frac{t}{2})$, we get

$$\begin{aligned}
(\cos \frac{t}{2} i + \sin \frac{t}{2} k)(i)(-\cos \frac{t}{2} i - \sin \frac{t}{2} k) &= (-\cos \frac{t}{2} + \sin \frac{t}{2} j)(-\cos \frac{t}{2} i - \sin \frac{t}{2} k) \\
&= (\cos^2 \frac{t}{2} i + \cos \frac{t}{2} \sin \frac{t}{2} k + \sin \frac{t}{2} \cos \frac{t}{2} k - \sin^2 \frac{t}{2} i) \\
&= (\cos t i + \sin t k)
\end{aligned}$$

Now $s(0, \cos \frac{t}{2}, 0, \sin \frac{t}{2}) = (\cos \frac{t}{2}, 0, \sin \frac{t}{2})$ which is clearly a circle in $\mathbb{R}^3$ as desired [8].

[8] Calculation can be done for any general point $R \in \mathbb{S}^2$. We did for the special choice of R to avoid complexity in calculations.

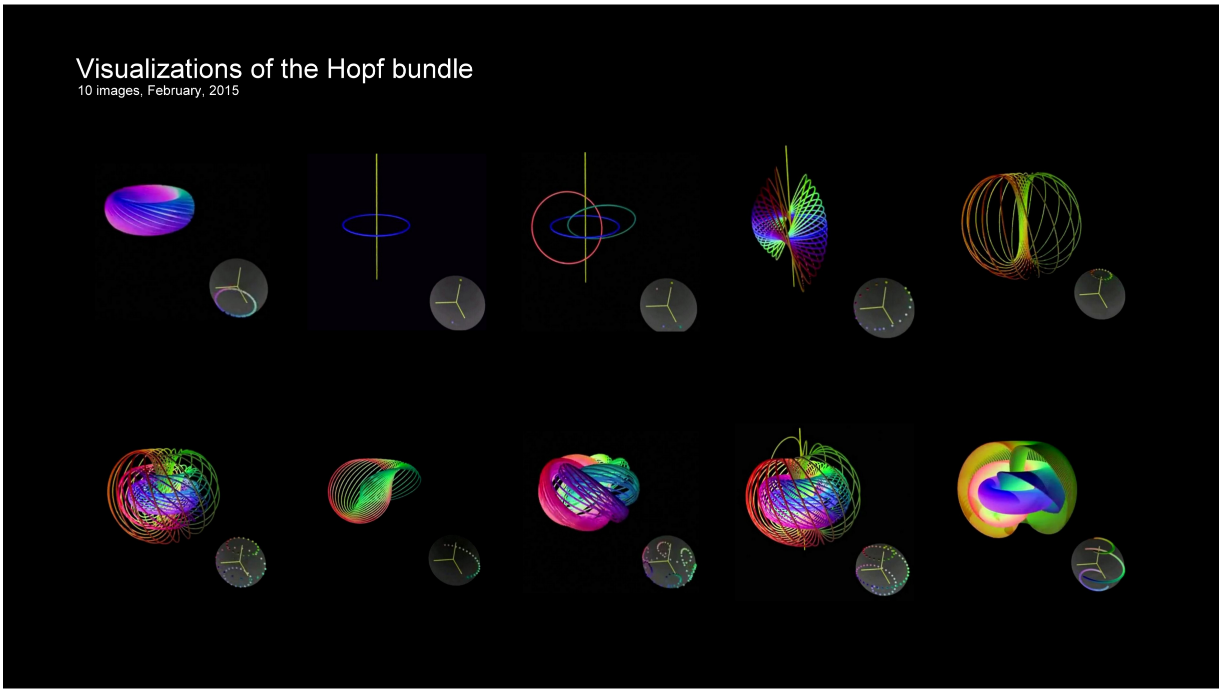


Figure 1.1: More visualizations of the Hopf bundle

Note that the plane of the circle $s(h^{-1})(R)$ cannot contain the x-axis, if it did, $s(h^{-1})(R)$ would intersect $s(h^{-1})(P)$, but fibres are disjoint. From these observations we can conclude that x-axis passes through the interior of the circle $s(h^{-1})(R)$.

Observation 4: To show that any two projected fibre circles C and D are linked, we exhibit a continuous one to one map $\phi : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ that takes C to the unit circle in the yz-plane, and takes D to some other projected fibre circle E. Since E is linked with the unit circle in the yz-plane, C and D must also be linked.

To construct the map ϕ, let P be any point on the circle C, and let $r = s^{-1}(P)$. Define $f : \mathbb{R}^4 \longrightarrow \mathbb{R}^4$ by $f(x) = xr^{-1}x$ (quaternion multiplication). The map ϕ is the composition $s \circ f \circ s^{-1}$.

For better visualizations to the Hopf bundle refer to [20] and [22] in the references.

Chapter 2

Homotopy properties of vector bundles

The sources of this chapter are primarily [1], [2], [3], [4], [12],[13], [15].

2.1 Classifying spaces and universal bundles

2.1.1 Grassmann manifolds and Grassmann bundles

Fix integers k and n so that $1 \leq k \leq n$. Let V be a vector space of dimension n. Let

$$G_k(V) = \{V' \subset V \mid \dim(V') = k\}.$$

$G_k(V)$ is called Grassmann manifold of k-planes of V, or simply a Grassmannian. In special case $V = \mathbb{R}^n$, the Grassmannian $G_k(\mathbb{R}^n)$ is often denoted by $G_{k,n}$ [1] .

Interpretation of Grassmann manifold through Lie groups

Consider the action of $O(n)$ on $\mathbb{R}^n$ which maps k-spaces to k-spaces and thus induces action of $O(n)$ on $G_{k,n}$. This action is transitive since any k-space can be transformed into any other k-space by some $A \in O(n)$. Let $V' \in G_{k,n}$ be a k-space spanned by $\{e_1, e_2, \ldots, e_k\}$, where $\{e_i\}_{1\leq i\leq k}$ is the standard basis of $\mathbb{R}^n$. Then the corresponding

[1] $G_k(\mathbb{C}^n)$ is defined similarly.

isotropy group is the subgroup $O(k) \times O(n-k)$ of $O(n)$ consisting of all matrices

$$\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}, \; A \in O(k), \; B \in O(n-k)$$

Since the action is transitive, we get

$$G_{k,n} \approx O(n)/(O(k) \times O(n-k)).$$

Recall that the *Stiefel manifold* $V_{k,n}$ is given as follows:

$$V_{k,n} = \{(v_1, \dots, v_k) \mid v_i.v_j = \delta_{i,j}\}$$

Observe that

$$O(n)/O(n-k) \approx V_{k,n}.$$

Hence

$$G_{k,n} \approx V_{k,n}/O(k).$$

Examples

1. $G_{1,3}$ is the spaces of lines through origin in $\mathbb{R}^3$, hence it is same as $\mathbb{RP}^2$.

2. $G_{2,3}$ is the space of all planes through origin in $\mathbb{R}^3$. And a plane containing the origin is completely characterized by the one and only line through the origin perpendicular to that plane (and vice versa). Hence

$$G_{2,3} \approx G_{1,3} \approx \mathbb{RP}^2.$$

   In general,

$$G_{1,n+1} \approx \mathbb{RP}^n.$$

2.1.2 Grassmann bundles

Here we take $G_{k,n}$ to be the complex Grassmann manifold.

The tautological k-plane bundle $(\xi_n^k, \pi, G_k(\mathbb{C}^n))$ is defined by

$$\xi_n^k = \{(\alpha, v) \in G_k(\mathbb{C}^n) \times \mathbb{C}^n : v \in \alpha\} \quad \text{and} \quad \pi(\alpha, v) = \alpha.$$

The direct limit of the sequence of complex Grassmann manifolds is the infinite complex Grassmann manifold $G_k(\mathbb{C}^\infty) = \cup_{n\geq k} G_{k,n}$, with the direct limit topology and is called the '*classifying space*' of k-plane bundles, and is also denoted as $BU(k)$.

The universal k-plane bundle [2] $\pi : \xi^k \longrightarrow BU(k)$ is defined by

$$\xi^k = \{(\alpha, v) \in BU(k) \times \mathbb{C}^\infty : v \in \alpha\} \quad \text{and} \quad \pi(\alpha, v) = \alpha$$

In fact,

$$\xi^k = \cup_n \xi_n^k.$$

2.1.3 Constructions of classifying spaces and universal bundles

Let us start with $G = U(k)$. Consider the group $U(N+k)$. The two subgroups

$$U(k) \subset U(N+k)$$

and

$$U(N) \subset U(N+k)$$

are included as matrices

$$\begin{pmatrix} X & 0 \\ 0 & 1 \end{pmatrix}, X \in U(k)$$

and

$$\begin{pmatrix} 1 & 0 \\ 0 & Y \end{pmatrix}, Y \in U(N)$$

These subgroups commute and hence the group $U(k)$ acts on the quotient space $U(N+k)/U(N)$. This action is free and the corresponding quotient space is homeomorphic to $U(N+k)/(U(N) \oplus U(k))$. And this space is homeomorphic to complex Grassmann manifold $G_{k,N+k}$. Thus one has principal $U(k)$-bundle, i.e.

$$\pi : U(N+k)/U(N) \longrightarrow U(N+k)/(U(N) \oplus U(k))$$

[2]The preceding constructions and results holds equally well for real vector bundles, with $G_k(\mathbb{R}^n)$.

total space of the bundle is homeomorphic to $V_{k,N+k}$. Now consider the inclusion

$$j : U(N+k) \longrightarrow U(N+k+1)$$

defined by

$$Z \longmapsto \begin{pmatrix} Z & 0 \\ 0 & 1 \end{pmatrix}, Z \in U(N+k).$$

The subgroup $U(N)$ maps to the subgroup $U(N+1)$:

$$A = \begin{pmatrix} 1 & 0 \\ 0 & Y \end{pmatrix} \longmapsto \begin{pmatrix} A & & 0 \\ & & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & & \\ 0 & A & \end{pmatrix}, Y \in U(N),$$

and the subgroup $U(k)$ maps to itself. Hence the inclusion j generates an inclusion of principal $U(k)$-bundles.

$$\begin{array}{ccc} U(N+k)/U(N) & \xrightarrow{\ j\ } & U(N+k+1)/U(N+1) \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\ U(N+k)/(U(N) \oplus U(k)) & \xrightarrow[\ j\ ]{} & U(N+k+1)/(U(N) \oplus U(k+1)) \end{array}$$

Equivalently

$$\begin{array}{ccc} V_{k,N+k} & \xrightarrow{\ j\ } & V_{k,N+k+1} \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\ G_{k,N+k} & \xrightarrow[\ j\ ]{} & G_{k,N+k+1} \end{array}$$

Let $V_{k,\infty}$ denote the direct limit

$$V_{k,\infty} = \varinjlim V_{k,N+k},$$

and $G_{k,\infty}$ the direct limit which is also called the classifying space for principal $U(k)$-bundle, denoted by $BU(k)$.

$$G_{k,\infty} = \varinjlim G_{k,N+k}.$$

The commutative diagram above induces the mapping

$$\pi : V_{k,\infty} \longrightarrow G_{k,\infty}. \tag{2.1}$$

which is a principal $U(k)$-bundle.

Milnor's construction of universal bundle

There is another construction due to J. Milnor which can be used for any topological group. Let

$$E_n = G \star \ldots \star G$$

be the join of $(n+1)$ copies of G in the strong topology. We define right action of G

$$R : E_n \times G \longrightarrow E_n$$

given by

$$R(t_0 g_0 \oplus \ldots \oplus t_n g_n, g) = t_0(g_0 g) \oplus \ldots \oplus t_n(g_n g).$$

Let $X_n = E_n/G$ be the orbit space and $\pi : E_n \longrightarrow X_n$ be the quotient map.

Theorem 2.1. *G is the group of an n-universal bundle having total space E_n and base space X_n.*

Proof The space E_n is $(n-1)$ connected [3]. The bundle structure is defined as follows. Let

$$U_j = \{\pi(t_0 g_0 \oplus \ldots \oplus t_n g_n) \in X_n \mid t_j \neq 0\}.$$

Define coordinate functions

$$\phi_j : U_j \times G \longrightarrow \pi^{-1}(U_j)$$

by

$$\phi_j(\pi(t_0 g_0 \oplus \ldots \oplus t_n g_n), g) = t_0(g_0 g_j^{-1} g) \oplus \ldots \oplus t_n(g_n g_j^{-1} g).$$

Define

$$\pi_j : \pi^{-1}(U_j) \longrightarrow G$$

by

$$\pi_j(t_0 g_0 \oplus \ldots \oplus t_n g_n) = g_j.$$

[3] See Lemma 6.1 in Appendix.

The identities

$$\pi(\phi_j(x,g)) = x,$$

$$\pi_j(\phi_j(x,g)) = x,$$

$$\phi_j(\pi(e), \pi_j(e)) = e$$

show that (π, π_j) is an inverse of π_j. The coordinate transformation

$$g_{ij} : U_i \cap U_j \longrightarrow G$$

are defined by

$$g_{ij}(\pi(t_0 g_0 \oplus \ldots \oplus t_n g_n) = g_i \circ g_j^{-1},$$

and satisfy the identity

$$\pi_i(\phi_j(x,g)) = g_{ij}(x)g.$$

It is now necessary to prove that all of these functions are continuous. Starting directly from the definition of strong topology on the join, it can be proved that R and π_j are continuous. The identification map π is certainly continuous.

Let ε be the identity element of G. The identity

$$\phi_j(\pi(e), \varepsilon) = R(e, \pi_j(e)^{-1})$$

shows that $\pi_j(\pi(e), \varepsilon)$ is continuous function of e. By the definition of identification topology, this means that $\pi_j(x, \varepsilon)$ is a continuous function of x. Now

$$\phi_j(x, \varepsilon) = R(\phi_j(x, \varepsilon), g)$$

implies that ϕ_j is continuous as a function of two variables. Finally the identity

$$g_{ij}(x) = \pi_i(\phi_j(x, \varepsilon))$$

shows that g_{ij} is continuous.

■

2.2 Classifications theorems

Now we discuss technical results about pullback bundles described in Subsection 1.3.3.

Theorem 2.2. *Given a vector bundle $\xi = (E, \pi, X)$ and homotopic maps $f_0, f_1 : X' \longrightarrow X$, then the pullback bundles $f_0^*(E)$ and $f_1^*(E)$ are isomorphic if X' is a compact Hausdorff space or more generally paracompact space.*

The theorem is an immediate consequence of the following result:

Proposition 2.1. *The restrictions of a vector bundle $\xi = (E, \pi, X \times I)$ over $X \times \{0\}$ and $X \times \{1\}$ are isomorphic if X is a paracompact space.*

A basic property of the construction of bundles E_f over $\mathbb{S}^k$ described in Examples 1.5 , via clutching functions $f : \mathbb{S}^{k-1} \longrightarrow GL_n(\mathbb{R})$ is that E_f and E_g are isomorphic if f and g are homotopic. For if we have a homotopy $F : \mathbb{S}^{k-1} \times I \longrightarrow GL_n(\mathbb{R})$ from f to g, then the same sort of clutching construction can be used to produce a vector bundle E_F over $\mathbb{S}^k \times I$ that restricts to E_f over $\mathbb{S}^k \times \{0\}$ and E_g over $\mathbb{S}^k \times \{1\}$. Hence E_f and E_g are isomorphic by Proposition 2.1 . Thus if we denote by $[X, Y]$ the set of homotopy classes of maps $X \longrightarrow Y$, then the association

$$f \longmapsto E_f$$

gives a well defined map

$$\Phi : [\mathbb{S}^{k-1}, GL_n(\mathbb{R})] \longrightarrow \mathrm{Vect}^n_{\mathbb{R}}(\mathbb{S}^k).$$

We describe the following basic result about complex form of Φ before dealing with real case.

Theorem 2.3. *The map $\Phi : [\mathbb{S}^{k-1}, GL_n(\mathbb{C})] \longrightarrow Vect^n_{\mathbb{C}}(\mathbb{S}^k)$ which sends a clutching function f to the vector bundle E_f is a bijection.*

Above Theorem does not quite work for real vector bundles since $\mathrm{GL}_n(\mathbb{R})$ is not path-connected. The closest analogy with the complex case is obtained by considering oriented real vector bundles. An *orientation* of a real vector bundle $\xi = (E, \pi, X)$ is a function assigning an orientation to each fibre and the orientations of fibres of E

can be defined by ordered n tuples of independent local sections. Since $\mathrm{GL}_n^+(\mathbb{C})$ is path-connected. Here is the result along these lines:

Proposition 2.2. *The map* $\Phi : [\mathbb{S}^{k-1}, GL_n^+(\mathbb{R})] \longrightarrow Vect_+^n(\mathbb{S}^k)$ *is a bijection.*

Let $\mathbb{K}$ be $\mathbb{R}$ or $\mathbb{C}$. If we restrict our attention to maps $f : \mathbb{S}^{k-1} \longrightarrow \mathrm{GL}_n(\mathbb{K})$ such that $f(e) = 1$, where $e = (1, 0, \ldots, 0)$ is the base point of $\mathbb{S}^{k-1}$, the above discussion shows that the correspondence $f \longmapsto E_f$ defines a map from $\pi_{k-1}(\mathrm{GL}_n(\mathbb{K}))$ to $\mathrm{Vect}_{\mathbb{K}}^n(\mathbb{S}^k)$. On the other hand $\pi_0(\mathrm{GL}_n(\mathbb{K}))$ acts on $\pi_{k-1}(\mathrm{GL}_n(\mathbb{K}))$ by the map defined on representatives by $(a, f) \longmapsto a \circ f \circ a^{-1}$. Since the vector bundles E_f and $E_{a \circ f \circ a^{-1}}$ are isomorphic by Theorem 1.4 , we now have the following result:

Theorem 2.4. *The map*

$$\pi_{k-1}(GL_n(\mathbb{K}))/\pi_0(GL_n(\mathbb{K})) \longrightarrow Vect_{\mathbb{K}}^n(\mathbb{S}^k)$$

is injective.

If $\mathbb{K} = \mathbb{C}$, then the group $\mathrm{GL}_n(\mathbb{K}) = \mathrm{GL}_n(\mathbb{C})$. By matrix polar decomposition $\mathrm{GL}_n(\mathbb{C}) = U(n) \times H(n)$ where $U(n)$ = the group of unitary matrices and $H(n)$ = the set of positive definite Hermitian matrices. Since $U(n)$ is path connected, $\pi_0(\mathrm{GL}_n(\mathbb{C})) = \pi_0(U(n)) = 0$. Hence

$$\mathrm{Vect}_{\mathbb{C}}^n(\mathbb{S}^k) \approx \pi_{k-1}(U(n)).$$

Theorem 2.5. *If X is paracompact Hausdorff space, then the correspondence $[f] \mapsto [f^*\xi^k]$ sets up a natural bijection*

$$\Psi : [X, BU(k)] \longrightarrow Vect_{\mathbb{C}}^n(X).$$

For the proof, we require that if $\pi : E \longrightarrow X$ is a k-plane bundle, then $E \approx f^*(\xi^k)$ for some $f : X \longrightarrow BU(k)$ iff there is a map $g : E \longrightarrow \mathbb{C}^\infty$ which is the linear monomorphism on each fibre of E. The map g is called **Gauss map**.

It is easy to deduce that vector bundles over a paracompact base have inner products, since the bundle $\pi : \xi^k \longrightarrow G_k(\mathbb{R}^\infty)$ has an obvious inner product obtained by restricting the standard inner product in $\mathbb{R}^\infty$ to each k-plane, and this inner product on ξ^k induces an inner product on every pullback $f^*(\xi^k)$.

Next we assume that the base space X of a vector bundle $\xi = (E, \pi, X)$ is a CW-complex.

2.3 Exact homotopy sequence

Theorem 2.6. *Let $\xi = (E, \pi, X)$ be a vector bundle. Then π satisfies the homotopy lifting axiom.*

Theorem 2.7. *Let $\xi = (E, \pi, X)$ be a vector bundle, $x_0 \in X$, $y_0 \in \pi^{-1}(x_0) = F$ and let*

$$j : F \longrightarrow E$$

be natural inclusion of the fibre in the total space. Then there is homomorphism of homotopy groups

$$\partial : \pi_k(X, x_0) \longrightarrow \pi_{k-1}(F, y_0) \tag{2.2}$$

such that the sequence of homomorphisms

$$\begin{aligned} \ldots \longrightarrow \pi_k(F, y_0) \xrightarrow{j_*} \pi_k(E, y_0) \xrightarrow{\pi_*} \pi_k(X, x_0) \xrightarrow{\partial} \\ \xrightarrow{\partial} \pi_{k-1}(F, y_0) \xrightarrow{j_*} \pi_{k-1}(E, y_0) \xrightarrow{\pi_*} \pi_{k-1}(X, x_0) \xrightarrow{\partial} \ldots \\ \ldots \longrightarrow \pi_1(F, y_0) \xrightarrow{j_*} \pi_1(E, y_0) \xrightarrow{\pi_*} \pi_1(X, x_0) \end{aligned} \tag{2.3}$$

is an exact sequence of groups.

Proof To construct the map ∂ let

$$\varphi : I^k \longrightarrow X$$

be a representative of an element of the group $\pi_k(X, x_0)$ and note that

$$I^k = I^{k-1} \times I.$$

Hence

$$\varphi(I^{k-1} \times \{0\}) = \varphi(I^{k-1} \times \{1\}) = x_0.$$

The map φ can be extended as a homotopy of the cube I^{k-1}. Put

$$\psi(u, 0) = y_0, \ \forall \ u \in I^{k-1}.$$

Then the map ψ is a lifting of the map $\varphi(u, 0)$. By the HLP [4] , there is a homotopy

$$\psi : I^{k-1} \times T \longrightarrow E,$$

such that

$$\pi\psi(u, t) = \varphi(u, t).$$

In particular,

$$\pi\psi(u, 1) = x_0.$$

This means that $\psi(u, 1)$ maps the cube I^{k-1} into the fibre F. Moreover, since

$$\varphi(\partial I^{k-1} \times I) = x_0,$$

the lifting ψ can be chosen such that

$$\psi(\partial I^{k-1} \times I) = y_0.$$

Hence, the map $\psi(u, 1)$ maps ∂I^{k-1} to the point y_0. This means that the map $\psi(u, 1)$ defines an element of the group $\pi_{k-1}(F, y_0)$. This homomorphism ∂ is well defined. Indeed, if the map φ is homotopic to the map φ', then the corresponding lifting maps ψ and ψ' are homotopic and a homotopy Ψ between ψ and ψ' can be constructed as a lifting of the homotopy Φ between φ and φ'. Next, we need to prove the exactness of the sequence (2.3).

Exactness at the term $\pi_k(E, y_0)$. If

$$\varphi : \mathbb{S}^k \longrightarrow F$$

represents the element of the group $\pi_k(F, y_0)$, then the map

$$\pi \circ j \circ \varphi : \mathbb{S}^k \longrightarrow X$$

represents the image of this element with respect to homomorphism $\pi_* \circ j_*$. Since

$$\pi(F) = x_0,$$

[4]See Definition 6.1 in Appendix.

we have

$$\pi \circ j \circ \varphi(\mathbb{S}^k) = x_0.$$

Hence

$$\mathbf{Im}(j_*) \subset \mathbf{Ker}(\pi_*).$$

Conversely, let

$$\varphi : \mathbb{S}^k \longrightarrow E$$

be a map such that $\pi \circ \varphi$ is homotopic to a constant, that is,

$$[\varphi] \in \mathbf{Ker}(\pi_*).$$

Let Φ be a homotopy between $\pi \circ \varphi$ and a constant map. By the HLP, there is a lifting homotopy Ψ between φ and a map φ', which is lifting of the constant map. Thus

$$\pi \circ \varphi'(\mathbb{S}^k) = x_0,$$

that is,

$$\varphi'(\mathbb{S}^k) \subset F.$$

Hence

$$[\varphi'] = [\varphi] \in \mathbf{Im}(j_*),$$

that is,

$$\mathbf{Ker}(\pi_*) \subset \mathbf{Im}(j_*).$$

Exactness at the term $\pi_k(X, x_0)$. Let

$$\varphi : I^k \longrightarrow B$$

represent an element of the group $\pi_k(X, x_0)$. The composition

$$\pi \circ \varphi : I^k = I^k \times I \longrightarrow X$$

is lifted by φ. According to the definition, the restriction

$$\varphi(u, 1) : I^{k-1} \longrightarrow F \subset E$$

represents the element $\partial[\pi \circ \varphi]$. Since

$$\varphi(u,1) = y_0,$$

we have

$$\partial \circ \pi_*[\varphi] = 0.$$

Now, let

$$\psi : I^k \times I \longrightarrow E$$

be a lifting of the map φ. Then the restriction $\psi(u,1)$ represents the element $\partial[\varphi]$. If the map

$$\psi : I^{k-1} \longrightarrow F$$

is homotopic to the constant map, then there is a homotopy

$$\psi' : I^{k-1} \times I \longrightarrow F$$

between $\psi(u,1)$ and constant map. Let us now construct a new map

$$\psi'' : I^{k-1} \times I \longrightarrow E,$$

$$\psi''(u,1) = \begin{cases} \psi(u,2t), & 0 \le t \le \frac{1}{2} \\ \psi'(u,2t-1), & \frac{1}{2} \le t \le 1. \end{cases}$$

Then

$$\pi \circ \psi'(u,1) = \begin{cases} \varphi(u,2t), & 0 \le t \le \frac{1}{2} \\ x_0, & \frac{1}{2} \le t \le 1. \end{cases}$$

Hence, the map $\pi \circ \psi''$ is homotopic to the map φ, that is,

$$\pi_*[\psi''] = [\varphi].$$

or

$$\mathbf{Ker}(\partial) \subset \mathbf{Im}(\pi_*).$$

Exactness at the term $\pi_k(F, y_0)$. Let the map

$$\varphi : I^{k+1} = I^k \times I \longrightarrow X$$

represents an element of the group $\pi_{k+1}(X, x_0)$, and let

$$\psi : I^k \times I \longrightarrow E$$

lifts the map φ. Then the restriction $\psi(u, 1)$ represents the element

$$\partial([\varphi]) \in \pi_k(F, y_0).$$

Hence, the element $j_* \circ \partial([\varphi])$ is represented by the map $\psi(u, 1)$. The latter is homotopic to constant map $\psi(u, 0)$, that is,

$$j_* \circ \partial([\varphi]) = 0.$$

Conversely, if

$$\varphi : I^k \times I \longrightarrow E$$

is a homotopy between $\varphi(u, 1) \subset F$ and constant map $\varphi(u, 0)$, then according to the definition, the map $\varphi(u, 1)$ represents the element $\partial([\pi \circ \varphi])$. Hence

$$\mathbf{Ker}(j_*) \subset \mathbf{Im}(\partial).$$

■

Remark 2.1. *Let $\pi_0(X, x_0)$ denote the set of connected components of the space X. Then the exact sequence (2.3) can be extended as follows*

$$\begin{aligned} \ldots \longrightarrow \pi_1(F, y_0) \xrightarrow{j_*} \pi_1(E, y_0) \xrightarrow{\pi_*} \pi_1(X, x_0) \xrightarrow{\partial} \\ \xrightarrow{\partial} \pi_0(F, y_0) \xrightarrow{j_*} \pi_0(E, y_0) \xrightarrow{\pi_*} \pi_0(X, x_0). \end{aligned}$$

This sequence is the exact in the sense that the image coincides with the inverse image of the fixed element.

Examples

1. Consider the covering map

$$\mathbb{R}^1 \longrightarrow \mathbb{S}^1$$

in which the fibre $\mathbb{Z}$ is the discrete space of integers. The exact homotopy sequence

$$\begin{aligned} \ldots \longrightarrow \pi_k(\mathbb{Z}) \xrightarrow{j_*} \pi_k(\mathbb{R}^1) \xrightarrow{\pi_*} \pi_k(\mathbb{S}^1) \xrightarrow{\partial} \ldots \\ \ldots \xrightarrow{\partial} \pi_1(\mathbb{Z}) \xrightarrow{j_*} \pi_1(\mathbb{R}^1) \xrightarrow{\pi_*} \pi_1(\mathbb{S}^1) \xrightarrow{\partial} \\ \longrightarrow \pi_0(\mathbb{Z}) \xrightarrow{j_*} \pi_0(\mathbb{R}^1) \xrightarrow{\pi_*} \pi_0(\mathbb{S}^1) \end{aligned}$$

has the following form

$$\begin{aligned} \ldots \longrightarrow 0 \xrightarrow{j_*} 0 \xrightarrow{\pi_*} \pi_k(\mathbb{S}^1) \xrightarrow{\partial} 0 \longrightarrow \ldots \\ \ldots \xrightarrow{\partial} 0 \xrightarrow{j_*} 0 \xrightarrow{\pi_*} \pi_1(\mathbb{S}^1) \xrightarrow{\partial} \\ \longrightarrow \mathbb{Z} \xrightarrow{j_*} 0 \xrightarrow{\pi_*} 0 \end{aligned}$$

Since

$$\pi_k(\mathbb{R}^1) = 0, \quad k \geq 0,$$

and

$$\pi_k(\mathbb{Z}) = 0, \quad k \geq 1.$$

It follows that

$$\pi_1(\mathbb{S}^1) = \mathbb{Z},$$

$$\pi_k(\mathbb{S}^1) = 0, \quad k \geq 2.$$

2. Consider the Hopf bundle

$$h : \mathbb{S}^3 \longrightarrow \mathbb{S}^2.$$

The corresponding exact homotopy sequence is

$$\begin{aligned}
\ldots \longrightarrow \pi_k(\mathbb{S}^1) \xrightarrow{j_*} \pi_k(\mathbb{S}^3) \xrightarrow{\pi_*} \pi_k(\mathbb{S}^2) \xrightarrow{\partial} \ldots \\
\ldots \xrightarrow{\partial} \pi_3(\mathbb{S}^1) \xrightarrow{j_*} \pi_3(\mathbb{S}^3) \xrightarrow{\pi_*} \pi_3(\mathbb{S}^2) \xrightarrow{\partial} \\
\xrightarrow{\partial} \pi_2(\mathbb{S}^1) \xrightarrow{j_*} \pi_2(\mathbb{S}^3) \xrightarrow{\pi_*} \pi_2(\mathbb{S}^2) \xrightarrow{\partial} \\
\xrightarrow{\partial} \pi_1(\mathbb{S}^1) \xrightarrow{j_*} \pi_1(\mathbb{S}^3) \xrightarrow{\pi_*} \pi_1(\mathbb{S}^2) \longrightarrow 0
\end{aligned}$$

It takes the following form

$$\begin{aligned}
\ldots \longrightarrow 0 \xrightarrow{j_*} \pi_k(\mathbb{S}^3) \xrightarrow{\pi_*} \pi_k(\mathbb{S}^2) \xrightarrow{\partial} 0 \ldots \\
\ldots \xrightarrow{\partial} 0 \xrightarrow{j_*} \mathbb{Z} \xrightarrow{\pi_*} \pi_3(\mathbb{S}^2) \xrightarrow{\partial} 0 \\
\overset{\partial}{0} \xrightarrow{j_*} 0 \xrightarrow{\pi_*} \pi_2(\mathbb{S}^2) \xrightarrow{\partial} \\
\xrightarrow{\partial} \mathbb{Z} \xrightarrow{j_*} 0 \xrightarrow{\pi_*} 0
\end{aligned}$$

since

$$\pi_3(\mathbb{S}^1) = \pi_2(\mathbb{S}^1) = \pi_2(\mathbb{S}^3) =$$

$$= \pi_1(\mathbb{S}^3) = \pi_1(\mathbb{S}^2) = 0.$$

Hence

$$\pi_2(\mathbb{S}^2) = \mathbb{Z} = \pi_1(\mathbb{S}^1) = \mathbb{Z},$$

$$\pi_3(\mathbb{S}^2) = \mathbb{Z}.$$

The space $\mathbb{S}^2$ is the simplest example of a space where there are nontrivial homotopy groups in degrees greater than the dimension of the space. Notice also that the Hopf bundle $\xi = (\mathbb{S}^3, h, \mathbb{S}^2)$ gives a generator of the group $\pi_3(\mathbb{S}^2) = \mathbb{Z}$. Indeed, from the exact homotopy sequence, we see that a generator of the group $\pi_3(\mathbb{S}^3)$ is mapped to a generator of the group $\pi_3(\mathbb{S}^2)$ by the homomorphism π_*. A generator of the group $\pi_3(\mathbb{S}^3)$ is represented by the identity $\varphi : \mathbb{S}^3 \longrightarrow \mathbb{S}^3$ and hence

$$\pi_*[\varphi] = [\pi \circ \varphi] = [\pi].$$

3. In the general case of the above example, there are two bundles: the real Hopf

bundle

$$\mathbb{S}^n \longrightarrow \mathbb{RP}^n$$

with fibre $\mathbb{Z}_2$, and the complex Hopf bundle

$$\mathbb{S}^{2n-1} \longrightarrow \mathbb{CP}^n$$

with fibre $\mathbb{S}^1$. In the first case, the exact homotopy sequence shows that

$$\pi_1(\mathbb{S}^n) = 0 \longrightarrow \pi_1(\mathbb{RP}^n) \longrightarrow \pi_0(\mathbb{Z}_2) = \mathbb{Z}_2 \longrightarrow 0, \quad \pi_1(\mathbb{RP}^n) = \mathbb{Z}_2,$$
$$\pi_k(\mathbb{S}^n) = 0 \longrightarrow \pi_k(\mathbb{RP}^n) \longrightarrow \pi_{k-1}(\mathbb{Z}_2) = 0, \quad 2 \leq k \leq n.$$

Hence

$$\pi_k(\mathbb{RP}^n) = 0, \quad 2 \leq k \leq n-1.$$

In the complex case, we have

$$\pi_2(\mathbb{S}^{2n-1}) = 0 \longrightarrow \pi_2(\mathbb{CP}^n) \longrightarrow \pi_1(\mathbb{S}^1) = \mathbb{Z} \longrightarrow$$
$$\longrightarrow \pi_1(\mathbb{S}^{2n-1}) = 0 \longrightarrow \pi_1(\mathbb{CP}^n) \longrightarrow \pi_0(\mathbb{S}^1) = 0$$

$$\pi_k(\mathbb{S}^{2n-1}) = 0 \longrightarrow \pi_k(\mathbb{CP}^n) \longrightarrow \pi_{k-1}(\mathbb{S}^1) = 0, \quad 2n-1 > k \leq 2.$$

Hence

$$\begin{aligned} &\pi_1(\mathbb{CP}^n) = 0, \\ &\pi_2(\mathbb{CP}^n) = \mathbb{Z}, \\ &\pi_k(\mathbb{CP}^n) = 0 \quad \text{for} \quad 3 \leq k \leq 2n-1. \end{aligned}$$

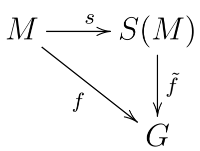
Chapter 3

K-Theory

The sources of this chapter are primarily [1], [3], [6], [11], [14], [15], [19].

3.1 The Grothendieck group $K(X)$

Let us first consider an abelian monoid M. We can associate an abelian group $S(M)$ with M and a homomorphism $s : M \longrightarrow S(M)$, having the following universal property: for any abelian group G, and any homomorphism of the underlying monoids $f : M \longrightarrow G$, there is unique group homomorphism $\tilde{f} : S(M) \longrightarrow G$ which makes the following diagram commutative.



The following constructions of s and $S(M)$, give the same result upto isomorphism.

1. Consider the product $M \times M$ and form the quotient under the equivalence relation.

$$(m, n) \sim (m', n') \Longleftrightarrow \exists\, p \in M \quad \text{such that} \quad m + n' + p = n + m' + p.$$

   The quotient monoid is a group and $s(m)$ is the class of the pair $(m, 0)$.

2. Another construction is to consider the quotient of $M \times M$ by the equivalence relation

$$(m, n) \sim (m', n') \Longleftrightarrow \exists\, p, q \in M \quad \text{such that} \quad (m, n) + (p, p) = (m', n') + (q, q),$$

with $s(m)$ the class of $(m, 0)$ once again.

Remark 3.1. *In the above constructions, we notice that every element of $S(M)$ can be written as $s(m) - s(n)$ where $m, n \in \mathbb{N}$.*

Examples

1. The most natural example is to consider $M = \mathbb{N} \cup \{0\}$. Then, $S(M) \approx \mathbb{Z}$.

2. Take $M = \mathbb{Z} - \{0\}$, an abelian monoid w.r.t multiplication. Then $S(M) \approx \mathbb{Q} - \{0\}$.

Remark 3.2. *The group $S(M)$ depends functorially on M in an obvious way: if $f : M \longrightarrow N$, the universal property enables us to define a unique homomorphism $S(f) : S(M) \longrightarrow S(N)$ which makes the diagram*

$$\begin{array}{ccc} M & \xrightarrow{f} & N \\ \downarrow & & \downarrow \\ S(M) & \xrightarrow[s(f)]{} & S(N) \end{array}$$

commutative. Moreover, $S(g \circ f) = S(g) \circ S(f)$ and $S(Id_M) = Id_{S(M)}$. The group $S(M)$ is called the symmetrization of the abelain monoid M.

Let us take 'vector bundle' to mean 'complex vector bundle' unless otherwise specified. Base spaces are always assumed to be compact Hausdorff.

Consider the vector bundles over a fixed base space X. We write trivial vector bundle of rank n over X as ϵ^n. We define the following equivalence relations on vector bundles over X:

1. Two vector bundles E_1 and E_2 over X are said to be *stably isomorphic*, written as $E_1 \approx_s E_2$, if $E_1 \oplus \epsilon^n \approx E_2 \oplus \epsilon^n$ for some n.

2. We define $E_1 \sim E_2$, if $E_1 \oplus \epsilon^m \approx E_2 \oplus \epsilon^n$ for some m and n.

A zero element is the class of ϵ^0.

Proposition 3.1. *If X is a compact Hausdorff space, then the set of $\sim$-equivalence classes of vector bundles over X forms an abelian group with respect to $\oplus$. This group is called $\tilde{K}(X)$.*

For the direct sum operation on $\approx_s$-equivalence classes, only the zero element, the class of ϵ^0, can have inverse since $E \oplus E' \approx_s \epsilon^0$ implies $E \oplus E' \oplus \epsilon^n \approx \epsilon^n$ for some n, which can only happen if E and E' has rank 0. However, even though inverses do not exist, we have cancellation property that

$$E_1 \oplus E_2 \approx_s E_1 \oplus E_3 \tag{3.1}$$

implies

$$E_2 \approx_s E_3,$$

over a compact base space X, since we can add to both sides of (3.1) a bundle E_1' such that $E_1 \oplus E_1' \approx \epsilon^n$ for some n.

We can form for a compact space X, an abelian group $K(X)$ consisting of formal differences $E - E'$ of vector bundles E and E' over X, with the equivalence relation

$$E_1 - E_1' = E_2 - E_2' \quad \text{iff} \quad E_1 \oplus E_2' \approx_s E_2 \oplus E_1'.$$

The following addition rule

$$(E_1 - E_1') + (E_2 - E_2') = (E_1 \oplus E_2) - (E_1' \oplus E_2'),$$

makes $K(X)$ into an abelian group. The zero element is the equivalence class of $E - E$ for any E, and the inverse of $E - E'$ is $E' - E$.

Proposition 3.2. *Let ϵ^n denote the trivial bundle of rank n over a compact base X. Then every element of $K(X)$ can be written as $E - \epsilon^n$ for some n, and some vector bundle E over X. Moreover, $E - \epsilon^n = E' - \epsilon^p$ if and only if there exists an integer q such that $E \oplus \epsilon^{p+q} \approx E' \oplus \epsilon^{n+q}$.*

Proof We already know that each element of $K(X)$ can be written as $E_1 - E_1'$, where E_1 and E_1' are vector bundles over X. According to Lemma 1.3 , there is a vector bundle E_2 such that $E_1' \oplus E_2$ is a trivial bundle, say ϵ^n. Then

$$\begin{aligned} E_1 - E_1' &= (E_1 - E_1') + (E_2 - E_2) \\ &= (E_1 \oplus E_2) - (E_1' \oplus E_2) \\ &= E - \epsilon^n \end{aligned}$$

where $E = E_1 \oplus E_2$.

Now suppose that

$$E - \epsilon^n = E' - \epsilon^p$$

We can find a vector bundle G such that

$$E \oplus \epsilon^p \oplus G \approx E' \oplus \epsilon^n \oplus G.$$

Let G_1 be a vector bundle such that

$$G \oplus G_1 \approx \epsilon^q.$$

Then we have

$$\begin{aligned} E \oplus \epsilon^{p+q} &\approx E \oplus \epsilon^p \oplus G \oplus G_1 \\ &\approx E' \oplus \epsilon^n \oplus G \oplus G_1 \\ &\approx E' \oplus \epsilon^{n+q} \end{aligned}$$

The converse is obvious.

■

Corollary 3.1. *Let E and E' be vector bundles over X. Then $E = E'$ in $K(X)$ iff $E \oplus \epsilon^n \approx F \oplus \epsilon^n$ for some n.*

Since, the functor K is contravariant on the category of compact spaces, the projection of X onto a point P induces a homomorphism

$$\alpha : \mathbb{Z} \approx K(P) \longrightarrow K(X),$$

whose cokernel is denoted by $\tilde{K}(X)$ and is called the reduced K-group of X.

Proposition 3.3. *If $X \neq \emptyset$, then we have an exact sequence*

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\alpha} K(X) \xrightarrow{\beta} \tilde{K}(X) \longrightarrow 0$$

The choice of a point x_0 in X defines a canonical splitting so that

$$\tilde{K}(X) \approx Ker\ [K(X) \longrightarrow K(\{x_0\}) \approx \mathbb{Z}]$$

and

$$K(X) \approx \mathbb{Z} \oplus \tilde{K}(X).$$

Proof Let us choose a point $P = \{x_0\}$. Then the inclusion of $\{x_0\}$ in X induces a homomorphism from $K(X)$ to $K(\{x_0\}) \approx \mathbb{Z}$ which is a left inverse for α.

■

Let $\text{Vect}^n_{\mathbb{C}}(X)$ denote the abelian monoid of isomorphism classes of vector bundles over X, and let γ denote the composition

$$\text{Vect}^n_{\mathbb{C}}(X) \xrightarrow{s} K(X) \xrightarrow{\beta} \tilde{K}(X).$$

Proposition 3.4. *The homomorphism γ is surjective. Moreover, $\gamma(\dot{E}) = \gamma(\dot{E}')$ if and only if $E \oplus \epsilon^n \approx E' \oplus \epsilon^p$ for some trivial bundles ϵ^n and ϵ^p. Here $\dot{E}$ and $\dot{E}'$ denote the isomorphism classes of vector bundles E and E' over X.*

Proof Since the class of ϵ^n in $\tilde{K}(X)$ is zero, and since any element of $K(X)$ can be written as $E - \epsilon^n$, we have

$$\begin{aligned} \beta(E - \epsilon^n) &= \beta(E) \\ &= \gamma(\dot{E}) \end{aligned}$$

Thus proving the first part. On the other hand, identity

$$\gamma(\dot{E}) = \gamma(\dot{E}')$$

is equivalent to

$$E - E' = \epsilon^q - \epsilon^r$$

for some q and r. Hence from Corollary 3.1 , we have

$$E \oplus \epsilon^r \oplus \epsilon^t \approx E' \oplus \epsilon^q \oplus \epsilon^t$$

for some t. Hence

$$E \oplus \epsilon^n \approx E' \oplus \epsilon^p$$

where $n = r + t$ and $p = q + t$. The converse is obvious.

■

Proposition 3.5. *Suppose X is the disjoint union of open subspaces $X_1 \cup \ldots \cup X_n$. Then the inclusions of X_i in X induce a decomposition of $K(X)$ as a direct product $K(X_1) \times \ldots \times K(X_n) = K(X_1) \oplus \ldots \oplus K(X_n)$.*

Remark 3.3. *This last proposition is false for functor $\tilde{K}(X)$. For example, if X is the disjoint union of two points P_1 and P_2, then $\tilde{K}(X) \approx \mathbb{Z}$, but $\tilde{K}(P_i) = 0$ for $i = 1, 2$.*

Recall that every vector bundle E defines a locally constant function

$$r : X \longrightarrow \mathbb{N}$$

given by

$$r(x) = \mathrm{Dim}(E_x).$$

If we let $H^0(X; \mathbb{N})$ denote the abelian monoid of locally constant functions on X with values in $\mathbb{N}$, we see that r defines a monoid homomorphism, also denoted by r as:

$$r : \mathrm{Vect}^n_{\mathbb{C}}(X) \longrightarrow H^0(X; \mathbb{N}).$$

It is clear that symmetrization of $H^0(X; \mathbb{N})$ [1] is the abelian group $H^0(X; \mathbb{Z})$ of locally constant functions on X with values in $\mathbb{Z}$. Therefore, we have

$$r : K(X) \longrightarrow H^0(X; \mathbb{Z}).$$

Proposition 3.6. *Letting $K'(X) = Ker\ [K(X) \longrightarrow H^0(X; \mathbb{Z})]$, we have an exact sequence*

$$0 \longrightarrow K'(X) \longrightarrow K(X) \xrightarrow{r} H^0(X; \mathbb{Z}) \longrightarrow 0$$

Proof Let

$$f : X \longrightarrow \mathbb{N}$$

be a locally constant function. Since X is compact, f only takes on finite number of values $n_1, \ldots, n_p$, and $X = X_1 \cup \ldots \cup X_p$ where $X_i = f^{-1}(\{n_i\})$. Let E be the

[1] $H^0(X; \mathbb{N})$ is the first Čech cohomology group of X.

vector bundle over X_i defined by $X_i \times \mathbb{C}^{n_i}$. Then the correspondence $f \longmapsto E$ defines a monoid homomorphism

$$t : H^0(X; \mathbb{N}) \longrightarrow \mathrm{Vect}^n_{\mathbb{C}}(X)$$

such that

$$r \circ t = \mathrm{Id}.$$

By symmetrization, t induces a group map

$$H^0(X; \mathbb{Z}) \longrightarrow K(X)$$

which is right inverse to

$$r : K(X) \longrightarrow H^0(X; \mathbb{Z}).$$

If X is connected, then we have $H^0(X; \mathbb{Z}) \approx \mathbb{Z}$, and $K'(X) \approx \mathrm{Coker}[\mathbb{Z} \longrightarrow K(X)]$. Now the map $\mathbb{Z} \longrightarrow K(X)$ is identical to the one induced by the projection of X to a point. Hence $K'(X) \approx \tilde{K}(X)$.

■

Again, let $\mathrm{Vect}^n_{\mathbb{C}}(X)$ be the set of isomorphism classes of complex vector bundles of rank n over X. Taking the Whitney sum by trivial bundles enables us to define an inductive system of sets

$$\mathrm{Vect}^0_{\mathbb{C}}(X) \longrightarrow \mathrm{Vect}^1_{\mathbb{C}}(X) \longrightarrow \ldots \longrightarrow \mathrm{Vect}^n_{\mathbb{C}}(X) \longrightarrow \ldots$$

The direct sum of this system, $\mathrm{Vect}'_{\mathbb{C}}(X)$, can be provided with an abelian monoid structure, using the maps

$$\mathrm{Vect}^n_{\mathbb{C}}(X) \times \mathrm{Vect}^p_{\mathbb{C}}(X) \longrightarrow \mathrm{Vect}^{n+p}_{\mathbb{C}}(X)$$

induced by the Whitney sum of vector bundles.

If $\dot{E}$, the class of vector bundle E, is an element of $\mathrm{Vect}^n_{\mathbb{C}}(X)$, we have $E - \epsilon^n \in Ker\ (r)$. The correspondence

$$\dot{E} \longmapsto E - \epsilon^n$$

for $\dot{E} \in \mathrm{Vect}^n_{\mathbb{C}}(X)$ induces a monoid homomorphism

$$\mathrm{Vect}'_{\mathbb{C}}(X) \longrightarrow K'(X).$$

Proposition 3.7. *The homomorphism*

$$Vect'_{\mathbb{C}}(X) \longrightarrow K'(X)$$

defined above, is an isomorphism. Hence $Vect'_{\mathbb{C}}(X)$ is an abelian group.

Proof If $E - \epsilon^n = E' - \epsilon^p$ in $K(X)$, by Proposition 3.2 , there exists an integer q such that

$$E \oplus \epsilon^{p+q} \approx E' \oplus \epsilon^{n+q}.$$

Hence the map

$$\mathrm{Vect}'_{\mathbb{C}}(X) \longrightarrow K'(X) \subset K(X)$$

is injective. Let u be an element of $K'(X)$. Again by Proposition 3.2 , u can be written as $E - \epsilon^n$ with $r(E - \epsilon^n) = 0$, i.e. $\mathrm{Dim}(E_x) = n$ for every point $x \in X$; hence the map is surjective.

■

Proposition 3.8. *Let BO be the inductive limit of the system of topological spaces*

$$BO(1) \longrightarrow \ldots \longrightarrow BO(n) \longrightarrow \ldots$$

where the map

$$BO(n) \longrightarrow BO(n+1)$$

is induced by the map between Grassmann manifolds

$$G_n(\mathbb{R}^N) \longrightarrow G_{n+1}(\mathbb{R}^{N+1})$$

which consists of adding subspaces generated by last vector $e_{N+1} = (0, \ldots, 1)$. Then we have a natural isomorphism of functors

$$K'_{\mathbb{R}}(X) \approx [X, BO].$$

In the same way, let BU be the inductive limit of the system of topological spaces

$$BU(1) \longrightarrow \ldots \longrightarrow BU(n) \longrightarrow \ldots$$

Then we have a natural isomorphism of functors

$$K'_{\mathbb{C}}(X) \approx [X, BU].$$

Proof Since the spaces $BO(n)$ are paracompact and hence normal, and since $BO(n)$ is closed in $BO(n+1)$, we have

$$[X, BO] \approx \varinjlim [X, BO(n)]$$

because X is compact.

According to Theorem 2.5 ,

$$\mathrm{Vect}^n_{\mathbb{R}}(X) \approx [X, BO(n)]$$

and the map

$$\mathrm{Vect}^n_{\mathbb{R}}(X) \longrightarrow \mathrm{Vect}^{n+1}_{\mathbb{R}}(X)$$

induced by addition of trivial vector bundle of rank one, coincides with the map induced by the inclusion of $BO(n)$ in $BO(n+1)$. Therefore

$$\begin{aligned} K'_{\mathbb{R}}(X) &\approx \mathrm{Vect}'_{\mathbb{R}}(X) \\ &\approx \varinjlim \mathrm{Vect}^n_{\mathbb{R}}(X) \\ &\approx \varinjlim [X, BO(n)] \\ &\approx [X, BO] \end{aligned}$$

For $k = \mathbb{C}$, the proof is analogous.

■

Theorem 3.1. *For every compact space X, we have natural isomorphisms*

$$K_{\mathbb{R}}(X) \approx [X, \mathbb{Z} \times BO]$$

and

$$K_{\mathbb{C}}(X) \approx [X, \mathbb{Z} \times BU]$$

where $\mathbb{Z}$ *is equipped with discrete topology.*

Proof We will only give proof for the real case, since the complex case is similar. From Proposition 3.6 , we have

$$K_{\mathbb{R}}(X) \approx H^0(X;\mathbb{Z}) \oplus K'_{\mathbb{R}}(X).$$

Since

$$H^0(X;\mathbb{Z}) \approx [X, \mathbb{Z}],$$

it follows that

$$\begin{aligned} K_{\mathbb{R}}(X) &\approx [X, \mathbb{Z}] \times [X, BO] \\ &\approx [X, \mathbb{Z} \times BO]. \end{aligned}$$

■

3.2 Ring structure of $K(X)$

There is also a natural multiplication in $K(X)$ coming from tensor product of vector bundles, hence providing the ring structure to $K(X)$. So for arbitrary elements of $K(X)$, their product in $K(X)$ is defined as follows:

$$(E_1 - E'_1)(E_2 - E'_2) = E_1 \otimes E_2 - E_1 \otimes E'_2 - E'_1 \otimes E_2 + E'_1 \otimes E'_2 \tag{3.2}$$

This operation makes $K(X)$ into a commutative ring with identity ϵ^1. And the commutativity of the ring is follows as:

$$\begin{aligned} (E'_1 - E_1)(E'_2 - E_2) &= E'_1 \otimes E'_2 - E'_1 \otimes E_2 - E_1 \otimes E'_2 + E_1 \otimes E_2 \\ &= (-(E_1 - E'_1))(-(E_2 - E'_2)) \\ &= (E_1 - E'_1)(E_2 - E'_2) \end{aligned}$$

If we choose a base point x_0, then the map

$$K(X) \longrightarrow K(\{x_0\})$$

obtained by restricting vector bundles to their fibres over x_0 is a ring homomorphism. Its kernel, which can be identified with $\tilde{K}(X)$, is an ideal, hence also a ring in its own right, though not necessarily a ring with identity.

3.2.1 External product in $K(X)$

An external product

$$\mu : K(X) \otimes K(Y) \longrightarrow K(X \times Y)$$

can be defined by

$$\mu(a \otimes b) = p_1^*(a)p_2^*(b)$$

where

$$p_1 : X \times Y \longrightarrow X$$

and

$$p_2 : X \times Y \longrightarrow Y$$

are natural projections. The tensor product of rings is a ring, with multiplication defined by

$$(a \otimes b)(c \otimes d) = ac \otimes bd,$$

and μ is a ring homomorphism since

$$\begin{aligned}
\mu((a \otimes b)(c \otimes d)) &= \mu(ac \otimes bd) \\
&= p_1^*(ac)p_2^*(bd) \\
&= p_1^*(a)p_1^*(c)p_2^*(b)p_2^*(d) \\
&= p_1^*(a)p_2^*(b)p_1^*(c)p_2^*(d) \\
&= \mu(a \otimes b)\mu(c \otimes d).
\end{aligned}$$

Taking Y to be $\mathbb{S}^2$, we have an external product [2]

$$\mu : K(X) \otimes K(\mathbb{S}^2) \longrightarrow K(X \times \mathbb{S}^2)$$

Now, our goal is to show that the map μ (being an essential core of the proof of Bott periodicity) is an isomorphism, which will be proved in Section 3.3.

[2] The external product in ordinary cohomology is called ‘cross product’. We will sometimes use the notation $a * b$ as shorthand for $\mu(a \otimes b)$.

3.3 Bott Periodicity

3.3.1 Generalized clutching construction

Given a vector bundle

$$\pi : E \longrightarrow X,$$

let

$$f : E \times \mathbb{S}^1 \longrightarrow E \times \mathbb{S}^1$$

be an automorphism of the product vector bundle

$$\pi \times \mathrm{Id} : E \times \mathbb{S}^1 \longrightarrow X \times \mathbb{S}^1.$$

Thus for each $x \in X$ and $z \in \mathbb{S}^1$, f specifies an isomorphism

$$f(x, z) : \pi^{-1}(x) \longrightarrow \pi^{-1}(x).$$

From E and f, we construct a vector bundle over $X \times \mathbb{S}^2$. We write this bundle as $[E, f]$, and call f a clutching function for $[E, f]$.

If

$$f_t : E \times \mathbb{S}^1 \longrightarrow E \times \mathbb{S}^1$$

is a homotopy of clutching functions, then

$$[E, f_0] \approx [E, f_1].$$

And from the definitions, we have

$$[E_1, f_1] \oplus [E_2, f_2] \approx [E_1 \oplus E_2, f_1 \oplus f_2].$$

Here are some examples:

1. $[E, \mathrm{Id}]$ is the pullback of E via projection $X \times \mathbb{S}^2 \longrightarrow X$.

2. Take X to be a point, we know that from Example 1.5 that

$$[\epsilon^1, z] \approx \gamma^1_{1,\mathbb{C}}.$$

where ϵ^1 is the trivial line bundle over X, z means scalar multiplication by

$z \in \mathbb{S}^1 \subset \mathbb{C}^1$ and $\gamma^1_{1,\mathbb{C}}$ is the canonical line bundle over $\mathbb{S}^2$. For simplicity, we denote the canonical line bundle over $\mathbb{S}^2$ by γ. More generally we have

$$[\epsilon^1, z^n] \approx \gamma^n.$$

where γ^n denotes the n-fold tensor product with itself.

Proposition 3.9. *Every vector bundle*

$$E' \longrightarrow X \times \mathbb{S}^2$$

is isomorphic to $[E, f]$ for some E and f.

Proof   Let the unit circle $\mathbb{S}^1 \subset \mathbb{S}^2$ decomposes $\mathbb{S}^2$ into two disks D_0 and D_∞. Let

$$E_\alpha = E'|_{X \times D_\alpha} \quad \text{for} \quad \alpha = 0, \infty$$

and

$$E = E'|_{X \times \{1\}}.$$

Now the maps

$$g : X \times D_\alpha \longrightarrow X \times \{1\},$$

$$\mathrm{Id} : X \times D_\alpha \longrightarrow X \times D_\alpha.$$

are homotopic. So

$$E_\alpha \approx g^*(E).$$

And hence we have an isomorphism

$$h_\alpha : E_\alpha \longrightarrow E \times D_\alpha.$$

So $f = h_0 \circ h_\infty^{-1}$ is the clutching function for E'.

■

We want to reduce the form of arbitrary clutching function to the simpler clutching function's.

Step 1: To reduce to Laurent polynomial clutching functions.

Definition 3.1. *Laurent polynomial clutching functions have the form*

$$l(x,z) = \sum_{|i|\leq n} a_i(x) z^i$$

where a_i restricts to the linear transformation $a_i(x)$ in each fibre $\pi^{-1}(x)$.

For a compact space X, we want to approximate a continuous function by Laurent polynomial function

$$l(x,z) = \sum_{|n|\leq N} a_n(x) z^n = \sum_{|n|\leq N} a_n(x) e^{\iota n\theta}$$

where $z = e^{\iota\theta}$ and

$$a_n : X \longrightarrow \mathbb{C}$$

is a continuous function.

Set

$$a_n(x) = \frac{1}{2\pi}\int_0^{2\pi} f(x, e^{\iota\theta}) e^{-\iota n\theta} d\theta$$

For positive real r, let

$$u(x,r,\theta) = \sum_{n\in\mathbb{Z}} a_n(x) r^{|n|} e^{\iota n\theta}$$

For $r < 1$, the above series converges absolutely and uniformly as (x,θ) ranges over $X \times [0, 2\pi]$. And by compactness of $X \times \mathbb{S}^1$, we have $\left|f(x, e^{\iota\theta})\right|$ is bounded and hence $\left|a_n(x)\right|$.

Step 1.1: To show: As $r \longrightarrow 1$, $u(x,r,\theta) \longrightarrow f(x, e^{\iota\theta})$ uniformly in x and θ.

For $r < 1$, we have

$$\begin{aligned} u(x,r,\theta) &= \sum_{n=-\infty}^{\infty} \frac{1}{2\pi}\int_0^{2\pi} r^{|n|} e^{\iota n(\theta - t)} f(x, e^{\iota t}) dt \\ &= \int_0^{2\pi} \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} r^{|n|} e^{\iota n(\theta - t)} f(x, e^{\iota t}) dt \end{aligned}$$

We can write above expression for $u(x, r, \theta)$ as

$$u(x, r, \theta) = \int_0^{2\pi} P(r, (\theta - t)) f(x, e^{\iota t}) dt$$

where $P(r, (\theta - t))$ is the Poisson kernel [3] .

Now to show uniform convergence of $u(x, r, \theta)$ to $f(x, e^{\iota\theta})$ we first observe that,

$$\begin{aligned} \left| u(x, r, \theta) - f(x, e^{\iota\theta}) \right| &= \left| \int_0^{2\pi} P(r, (\theta - t)) f(x, e^{\iota t}) dt - \int_0^{2\pi} P(r, (\theta - t)) f(x, e^{\iota\theta}) dt \right| \\ &\leq \int_0^{2\pi} P(r, (\theta - t)) \left| f(x, e^{\iota t}) - f(x, e^{\iota\theta}) \right| dt \end{aligned}$$

Given $\epsilon > 0$, there exists a $\delta > 0$ such that $\left| f(x, e^{\iota t}) - f(x, e^{\iota\theta}) \right| < \epsilon$ for $|t - \theta| < \delta$ and all x, since f is uniformly continuous on the compact space $X \times \mathbb{S}^1$. Let

$$I_\delta = \int P(r, (\theta - t)) \left| f(x, e^{\iota t}) - f(x, e^{\iota\theta}) \right| dt$$

over the interval $|t - \theta| < \delta$ and I'_δ denote the above the integral over the complement of the interval $|t - \theta| < \delta$ in an interval of length 2π. Then we have

$$I_\delta \leq \int_{|t-\theta|<\delta} P(r, (\theta - t)) \epsilon \, dt \leq \epsilon \int_0^{2\pi} P(r, (\theta - t)) dt = \epsilon$$

Now the maximum value of $P(r, (\theta - t))$ on $|t - \theta| \geq \delta$ is $P(r, \delta)$. So

$$I'_\delta \leq P(r, \delta) \int_0^{2\pi} \left| f(x, e^{\iota t}) - f(x, e^{\iota\theta}) \right| dt$$

The integral here has a uniform bound for all x and θ since f is bounded. Thus we can make $I'_\delta \leq \epsilon$ by taking r closer enough to 1. Therefore,

$$\left| u(x, r, \theta) - f(x, e^{\iota\theta}) \right| \leq I_\delta + I'_\delta \leq 2\epsilon.$$

[3] We define Poisson Kernel as $P(r, \varphi) = \frac{1}{2\pi} \sum_{-\infty}^{\infty} r^{|n|} e^{\iota n \varphi}$ for $0 \leq r < 1$ and $\varepsilon \in \mathbb{R}$.

Now the following proposition will provide an answer to what we had asked in **Step 1.**

Proposition 3.10. *Every vector bundle $[E, f]$ is isomorphic to $[E, l]$ for some Laurent polynomial clutching function l. Laurent polynomial clutching functions l_0 and l_1 which are homotopic through clutching functions are homotopic by a Laurent polynomial clutching function homotopy $l_t(x, z) = \sum_i a_i(x, t)z^i$.*

Proof To show that $f \in \mathrm{End}(X \times \mathbb{S}^1)$ can be approximated by Laurent polynomial endomorphisms, choose open cover $\{U_i\}_{i \in I}$ of X. From the trivialization property, we have following isomorphisms

$$h_i : \pi^{-1}(U_i) \longrightarrow U_i \times \mathbb{C}^{n_i}$$

Further assume that h_i takes the chosen inner product in $\pi^{-1}(U_i)$ to the standard inner product on $\mathbb{C}^{n_i}$, by applying Gram-Schmidt process to h_i^{-1} of the standard basis vectors. Let $\{\phi_i\}_{i \in I'}$ be a partition of unity subordinate to $\{U_i\}_{i \in I}$ and let X_i be the support of ϕ_i, a compact set in U_i. Via h_i, the linear maps $f(x, z)$ for $x \in X_i$ can be viewed as matrices , where entries define functions

$$f_i : X_i \times \mathbb{S}^1 \longrightarrow \mathbb{C}$$

From the **Step 1.1**, we can find Laurent polynomial matrices $l_i(x, z)$ whose entries uniformly approximate those of $f(x, z)$ for $x \in X_i$. Hence from these approximations we obtain a convex linear combination $l = \sum_i \phi_i l_i$, a Laurent polynomial approximation of f over all of $X \times \mathbb{S}^1$.

■

A Laurent polynomial clutching function can be written as $l = z^{-m}q$ for a polynomial clutching function q and hence we have

$$[E, l] \approx [E, q] \otimes \hat{\gamma}^{-m}.$$

Step 2: To reduce polynomial clutching functions to linear clutching functions

Proposition 3.11. *If q is a polynomial clutching function of degree atmost n, then*

$$[E, q] \oplus [nE, Id] \approx [(n+1)E, L^n q]$$

where $L^n q$ is a linear clutching function.

Proof Let

$$q(x, z) = a_n(x)z^n + \ldots + a_0(x)$$

Now the matrices

$$A = \begin{pmatrix} 1 & -z & 0 & \ldots & 0 & 0 \\ 0 & 1 & -z & \ldots & 0 & 0 \\ 0 & 0 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 1 & -z \\ a_n & a_{n-1} & a_{n-2} & \ldots & a_1 & a_0 \end{pmatrix}$$

and

$$B = \begin{pmatrix} 1 & 0 & 0 & \ldots & 0 & 0 \\ 0 & 1 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 1 & 0 \\ 0 & 0 & 0 & \ldots & 0 & q \end{pmatrix}$$

defines an endomorphism of $(n+1)E$. Hence

$$A = \{a_{ij} \mid a_{ij} : E_j \longrightarrow E_i\}$$

where E_j and E_i are the j-th and i-th summand of $(n+1)E$.

We define column operations [4] by $c_{j+1} = c_{j+1} + zc_j$, where $0 \leq j \leq n$ and we denote the first column of matrix A by c_0. And for $0 \leq i \leq n-1$, $r_n = r_n -$ (appropriate multiple of r_i). Here also we denote first row by r_0.

[4]These row and column operations are not quite elementary in the traditional sense since entries of the matrices are linear maps. However, by restricting to a fibre of E and suitable choice of basis, each entry in A becomes a matrix itself.

The matrix B is a clutching function for

$$[nE, \mathrm{Id}] \oplus [E, q],$$

Hence in each fibre B is non-singular, hence invertible. Similarly A is also invertible. This means that A is an automorphism of $(n+1)E$ for each $z \in \mathbb{S}^1$, and therefore determines a clutching function which we denoted by $L^n q$. Since $L^n q$ is a linear clutching function. The matrices A and B define homotopic clutching functions since the elementary row and column operations can be achieved by continuous one-parameter families of such operations. Since homotopic clutching functions produce isomorphic bundles, so we obtain an isomorphism

$$[nE, \mathrm{Id}] \oplus [E, q] \approx [(n+1)E, L^n q].$$

■

Proposition 3.12. *Given a bundle* $[E, a(x)z + b(x)]$*, there is a splitting* $E \approx E_+ \oplus E_-$ *with*

$$[E, a(x)z + b(x)] \approx [E_+, Id] \oplus [E_-, z].$$

3.3.2 The fundamental product theorem

Let $\gamma^1_{1,\mathbb{C}} = \gamma$ be the canonical line bundle over $\mathbb{CP}^1 = \mathbb{S}^2$. We have

$$(\gamma \otimes \gamma) \oplus \epsilon^1 \approx \gamma \oplus \gamma.$$

In $K(\mathbb{S}^2)$, this is the formula $\gamma^2 + \epsilon^1 = 2\gamma$. We can also write this as $(\gamma - \epsilon^1)^2$, so we have a natural ring homomorphism

$$\mathbb{Z}[\gamma]/(\gamma - \epsilon^1)^2 \longrightarrow K(\mathbb{S}^2)$$

whose domain is the quotient ring of the polynomial ring $\mathbb{Z}[\gamma]$ by the ideal generated by $(\gamma - \epsilon^1)^2$.

We define a homomorphism μ as the composition

$$\mu : K(X) \otimes \mathbb{Z}[\gamma]/(\gamma - \epsilon^1)^2 \longrightarrow K(X) \otimes K(\mathbb{S}^2) \longrightarrow K(X \times \mathbb{S}^2),$$

where the second map is the external product.

Theorem 3.2. *The homomorphism*

$$\mu : K(X) \otimes \mathbb{Z}[\gamma]/(\gamma - \epsilon^1)^2 \longrightarrow K(X \times \mathbb{S}^2)$$

is an isomorphism of rings for all compact Hausdorff spaces X.

Proof **μ is surjective:**

The preceding steps imply that in $K(X \times \mathbb{S}^2)$ we have

$$\begin{aligned}
[E, F] &= [E, z^{-m}q] \\
&= [E, q] \otimes \hat{\gamma}^{-m} \\
&= [(n+1)E, L^n q] \otimes \hat{\gamma}^{-m} - [nE, \mathrm{Id}] \otimes \hat{\gamma}^{-m} \\
&= [(n+1)E_+, L^n q] \otimes \hat{\gamma}^{-m} + [(n+1)E_-, L^n q] \otimes \hat{\gamma}^{-m} - [nE, \mathrm{Id}] \otimes \hat{\gamma}^{-m} \\
&= (n+1)E_+ * \gamma^{-m} + (n+1)E_- * \gamma^{1-m} - nE * \gamma^{-m}
\end{aligned}$$

This last expression is in the image of μ. Since every vector bundle over $X \times \mathbb{S}^2$ has the form $[E, f]$, it follows that μ is surjective.

μ is injective: To show this, we construct

$$\nu : K(X \times \mathbb{S}^2) \longrightarrow K(X) \otimes \mathbb{Z}[\gamma]/(\gamma - \epsilon^1)^2$$

such that $\nu \circ \mu = \mathrm{Id}$.

The main idea is to define $\nu([E, f])$ as linear combination of terms $E \otimes \gamma^k$ and $(n+1)E_\pm \otimes \gamma^k$ that is independent of all the choices.

We define the following formulas [5]

1. $[(n+2)E, L^{n+1}q] \approx [(n+1)E, L^n q] \oplus [E, \mathrm{Id}]$.
2. $[(n+2)E, L^{n+1}(zq)] \approx [(n+1)E, L^n q] \oplus [E, z]$.
3. For $[E, \mathrm{Id}]$ the summand E_- is 0 and $E_+ = E$.
4. For $[E, z]$ the summand E_+ is 0 and $E_- = E$.

[5] These formulas are to investigate the dependence of the terms in the formula for $[E, f]$ displayed above on m and n.

Suppose we define

$$\nu([E, z^{-m}q]) = (n+1)E_- \otimes (\gamma - \epsilon^1) + E \otimes \gamma^{-m} \in K(X) \otimes \mathbb{Z}[\gamma]/(\gamma - \epsilon^1)^2$$

for $n \geq \deg(q)$.

We claim that ν is well defined. $\nu([E, z^{-m}q])$ does not depend on n. To see this is independent of m we must see that it is unchanged when $z^{-m}q$ is replaced by $z^{-m-1}(zq)$. By (2) and (4) we have the following equalities:

$$\begin{aligned}
\nu([E, z^{-m-1}(zq)]) &= (n+1)E_- \otimes (\gamma - \epsilon^1) + E \otimes (\gamma - \epsilon^1) + E \otimes \gamma^{-m-1} \\
&= (n+1)E_- \otimes (\gamma - \epsilon^1) + E \otimes (\gamma^{-m} - \gamma^{-m-1}) + E \otimes \gamma^{-m-1} \\
&= (n+1)E_- \otimes (\gamma - \epsilon^1) + E \otimes \gamma^{-m} \\
&= \nu([E, z^{-m}q])
\end{aligned}$$

Next, we claim that $\nu([E, z^{-m}q])$ depends only on the the bundle $[E, z^{-m}q]$, not on the clutching function $z^{-m}q$ for this bundle. Applying Propositions 3.11 and 3.14 over $X \times I$ with a Laurent polynomial homotopy as clutching function, we conclude that the two bundles $(n+1)E_-$ over $X \times \{0\}$ and $X \times \{1\}$ are isomorphic.

Now ν takes sums to sums since

$$L^n(q_1 \oplus q_2) = L^n(q_1) \oplus L^n(q_2).$$

Also $\pm$-splitting preserves the sums. It follows that ν extends to a homomorphism

$$K(X \times \mathbb{S}^2) \longrightarrow K(X) \otimes \mathbb{Z}[\gamma]/(\gamma - \epsilon^1)^2.$$

The last thing to verify is that $\nu \circ \mu = \mathrm{Id}$. The group $\mathbb{Z}[\gamma]/(\gamma - \epsilon^1)^2$ is generated by ϵ^1 and γ, so in view of the relation

$$\gamma + \gamma^{-1} = 2$$

which follows from $(\gamma - \epsilon^1)^2 = 0$, we see that $K(\mathbb{S}^2)$ is also generated by ϵ^1 and γ^{-1}. Thus it suffices to show that $\nu \circ \mu = \mathrm{Id}$ on elements of the form $E \otimes \gamma^{-m}$ for $m \geq 0$.

We have

$$\begin{aligned}\nu(\mu(E \otimes \gamma^{-m})) &= \nu([E, z^{-m}]) \\ &= E_- \otimes (\gamma - \epsilon^1) + E \otimes \gamma^{-m}\end{aligned}$$

Since $E_- = 0$, the polynomial q being Id so that (3) applies.

■

Corollary 3.2. *Let X be a point. We have*

$$\mu : \mathbb{Z}[\gamma]/(\gamma - \epsilon^1)^2 \longrightarrow K(\mathbb{S}^2)$$

is an isomorphism of rings.

3.3.3 Bott periodicity from the product theorem

Proposition 3.13. *If X is compact Hausdorff and $A \subset X$ is a closed subspace, then the inclusion and quotient maps*

$$A \xrightarrow{i} X \xrightarrow{q} X/A$$

induce homomorphisms

$$\tilde{K}(X/A) \xrightarrow{q^*} \tilde{K}(X) \xrightarrow{i^*} \tilde{K}(A)$$

for which

$$Ker(i^*) = Im(q^*).$$

From Proposition 3.13 , we have the following exact sequence [6] of $\tilde{K}$ groups [7]

$$\ldots \longrightarrow \tilde{K}(SX) \longrightarrow \tilde{K}(SA) \longrightarrow \tilde{K}(X/A) \longrightarrow \tilde{K}(X) \longrightarrow \tilde{K}(A) \qquad (3.3)$$

[6] In the exact sequence SX denote the suspension of X. For detail description see Definition 6.4 in Appendix.

[7] We have also used the result to get (3.3) that if A is contractible, the quotient map $q : X \longrightarrow X/A$ induces a bijection $q^* : \mathrm{Vect}^n_{\mathbb{C}}(X/A) \longrightarrow \mathrm{Vect}^n_{\mathbb{C}}(X)$ for all n.

If $X = A \vee B$, then $X/A = B$ and the sequence breaks up into split exact sequences, which implies that the map

$$\tilde{K}(X) \longrightarrow \tilde{K}(A) \oplus \tilde{K}(B) \tag{3.4}$$

obtained by restriction to A and B is an isomorphism.

We can use the above exact sequence to obtain a reduced version of the external product,

$$\tilde{K}(X) \otimes \tilde{K}(Y) \longrightarrow \tilde{K}(X \wedge Y)$$

To define the reduced product, consider the long exact sequence for the pair $(X \times Y, X \vee Y)$ as follows:

$$\tilde{K}(S(X \times Y)) \longrightarrow \tilde{K}(S(X \vee Y) \longrightarrow \tilde{K}(X \wedge Y) \longrightarrow \tilde{K}(X \times Y) \longrightarrow \tilde{K}(X \vee Y),$$

where we have the following isomorphisms:

1. $\tilde{K}(S(X \vee Y) \approx \tilde{K}(SX) \oplus \tilde{K}(SY)$.
2. $\tilde{K}(X \vee Y) \approx \tilde{K}(X) \oplus \tilde{K}(Y)$.

Now for $a \in \tilde{K}(X) = \mathrm{Ker}(K(X) \longrightarrow K(x_0))$ and $b \in \tilde{K}(Y) = \mathrm{Ker}(K(Y) \longrightarrow K(y_0))$, the external product $a * b = p_1^*(a)p_2^*(b) \in K(X \times Y)$ has $p_1^*(a)$ restricting to zero in $K(Y)$ and $p_2^*(b)$ restricting to zero in $K(X)$, so $p_1^*(a)p_2^*(b)$ restricts to zero in both $K(X)$ and $K(Y)$. In particular, $a * b$ lies in $\tilde{K}(X \times Y)$, and from the short exact sequence above, $a * b$ pulls back to a unique element of $\tilde{K}(X \wedge Y)$. This defines the reduced external product

$$\tilde{K}(X) \otimes \tilde{K}(Y) \longrightarrow \tilde{K}(X \wedge Y) \tag{3.5}$$

It is essentially the restriction of the unreduced external product, as shown below. So the reduced external product is also a ring homomorphism, and we shall use the same notation $a * b$ for both reduced and unreduced external product.

Moreover, the reduced external product gives rise to a homomorphism [8]

$$\beta : \tilde{K}(X) \longrightarrow \tilde{K}(S^2X), \quad \beta(a) = (\gamma - \epsilon^1) * a$$

where γ is the canonical line bundle over $\mathbb{S}^2 = \mathbb{CP}^1$. And finally we have the Bott Periodicity Theorem:

Theorem 3.3. *The homomorphism*

$$\beta : \tilde{K}(X) \longrightarrow \tilde{K}(S^2X)$$

given by

$$\beta(a) = (\gamma - \epsilon^1) * a$$

is an isomorphism for all compact Hausdorff spaces X.

Proof The map β is the composition

$$\tilde{K}(X) \longrightarrow \tilde{K}(\mathbb{S}^2) \otimes \tilde{K}(X) \longrightarrow \tilde{K}(S^2X)$$

where the first map

$$a \longmapsto (\gamma - \epsilon^1) \otimes a$$

is an isomorphism since $\tilde{K}(\mathbb{S}^2)$ is infinite cyclic generated by $(\gamma - \epsilon^1)$, and the second map is the reduced external product. So the fact that this is an isomorphism is equivalent to the product theorem proved in the previous section.

■

Corollary 3.3. *1.* $\tilde{K}(\mathbb{S}^{2n+1}) = 0$,

2. $\tilde{K}(\mathbb{S}^{2n}) = \mathbb{Z}$*, generated by the n-fold reduced external product* $(\gamma - \epsilon^1) * \ldots * (\gamma - \epsilon^1)$.

3.3.4 Consequences of Bott periodicity

Here we mention some important consequences of Bott periodicity which we will use to get other important results in Section 5.2 .

[8]Since $\mathbb{S}^n \wedge X$ (Smash product) is the n-fold iterated reduced suspension $\Sigma^n X$, which is the quotient of the ordinary n-fold suspension S^nX obtained by collapsing an n-disk to a point. Hence the quotient map $S^nX \longrightarrow \mathbb{S}^2 \wedge X$ induces an isomorphism on $\tilde{K}$. Refer to Appendix for more details.

1. Since

$$\tilde{K}(\mathbb{S}^n) = \begin{cases} \mathbb{Z}; & \text{if } n \text{ is even} \\ 0; & \text{if } n \text{ is odd} \end{cases}$$

   In particular, we see that a generator of $\tilde{K}(\mathbb{S}^{2k})$ is the k-fold external product $(\gamma - \epsilon^1) * \ldots * (\gamma - \epsilon^1)$. And multiplication in $\tilde{K}(\mathbb{S}^{2k})$ is trivial since this ring is the k-fold tensor product of the ring $\tilde{K}(\mathbb{S}^2)$, which has trivial multiplication by Corollary 3.2 .

2. The external product

$$\tilde{K}(\mathbb{S}^{2k}) \otimes \tilde{K}(X) \longrightarrow \tilde{K}(\mathbb{S}^{2k} \wedge X)$$

   is an isomorphism since it is an iterate of the Bott periodicity isomorphism.

3. The external product

$$K(\mathbb{S}^{2k}) \otimes K(X) \longrightarrow K(\mathbb{S}^{2k} \times X)$$

   is an isomorphism. This follows from (2) by the same reasoning which showed the equivalence of the reduced and unreduced forms of Bott periodicity. Since the external product is a ring homomorphism, the isomorphism

$$\tilde{K}(\mathbb{S}^{2k} \wedge X) \approx \tilde{K}(\mathbb{S}^{2k}) \otimes \tilde{K}(X)$$

   is a ring isomorphism. For example, since $\tilde{K}(\mathbb{S}^{2k})$ can be described as the quotient ring $\mathbb{Z}[\alpha]/(\alpha^2)$, we can deduce that $K(\mathbb{S}^{2k} \times \mathbb{S}^{2l})$ is $\mathbb{Z}[\alpha, \beta]/(\alpha^2, \beta^2)$, where α and β are the pullback of generators of $\tilde{K}(\mathbb{S}^{2k})$ and $\tilde{K}(\mathbb{S}^{2l})$ under the projections map [9]

$$\pi_1 : \mathbb{S}^{2k} \times \mathbb{S}^{2l} \longrightarrow \mathbb{S}^{2k}$$

$$\pi_2 : \mathbb{S}^{2k} \times \mathbb{S}^{2l} \longrightarrow \mathbb{S}^{2l}$$

   An additive basis for $K(\mathbb{S}^{2k} \times \mathbb{S}^{2l})$ is $\{\epsilon^1, \alpha, \beta, \alpha\beta\}$.

3.3.5 An extension to cohomology theory

If we set

[9]These projections maps give us maps between $\tilde{K}$-groups.

1. $\tilde{K}^{-n}(X) = \tilde{K}(S^n X)$ and

2. $\tilde{K}^{-n}(X, A) = \tilde{K}(S^n(X/A))$,

then for a pair (X, A) of compact Hausdorff spaces, we have the following exact sequence [10] of $\tilde{K}$ groups [11] :

$$\begin{array}{ccccccccccccccc}
\tilde{K}(S^2X) & \to & \tilde{K}(S^2A) & \to & \tilde{K}(S(X/A)) & \to & \tilde{K}(SX) & \to & \tilde{K}(SA) & \to & \tilde{K}(X/A)) & \to & \tilde{K}(X) & \to & \tilde{K}(A) \\
\| & & \| & & \| & & \| & & \| & & \| & & \| & & \| \\
\tilde{K}^{-2}(X) & \to & \tilde{K}^{-2}(A) & \to & \tilde{K}^{-1}(X,A) & \to & \tilde{K}^{-1}(X) & \to & \tilde{K}^{-1}(A) & \to & \tilde{K}^{0}(X,A) & \to & \tilde{K}^{0}(X) & \to & \tilde{K}^{0}(A) \\
\beta\uparrow & & \beta\uparrow & & & & & & & & & & & & \\
\tilde{K}^{0}(X) & \longrightarrow & \tilde{K}^{0}(A) & & & & & & & & & & & &
\end{array}$$

The lower left corner of the diagram containing the Bott periodicity isomorphisms β commutes since external tensor product with $(\gamma - \epsilon^1)$ commutes with maps between spaces. So, we now have the six-tem periodic exact sequence obtained from long exact sequence above, given as follows:

$$\begin{array}{ccccc}
\tilde{K}^0(X,A) & \longrightarrow & \tilde{K}^0(X) & \longrightarrow & \tilde{K}^0(A) \\
\uparrow & & & & \downarrow \\
\tilde{K}^1(A) & \longleftarrow & \tilde{K}^1(X) & \longleftarrow & \tilde{K}^1(X,A)
\end{array}$$

Now from Equation (3.5) we obtain the following product, by replacing X and Y by S^iX and S^jY

$$\tilde{K}^i(X) \otimes \tilde{K}^i(Y) \longrightarrow \tilde{K}^{i+j}(X \wedge Y) \tag{3.6}$$

Define

$$\tilde{K}^*(X) = \tilde{K}^0(X) \oplus \tilde{K}^1(X).$$

Then this gives a product

$$\tilde{K}^*(X) \otimes \tilde{K}^*(Y) \longrightarrow \tilde{K}^*(X \wedge Y) \tag{3.7}$$

[10] The sequence is called Puppe sequence.

[11] Negative indices are chosen here, so that the coboundary maps in this sequence increase dimension, as in ordinary cohomology.

The relative form of (3.7) is gives by

$$\tilde{K}^*(X,A)\otimes\tilde{K}^*(Y,B)\longrightarrow\tilde{K}^*(X\times Y, X\times B\cup A\times Y)$$

Composing the external product

$$\tilde{K}^*(X)\otimes\tilde{K}^*(X)\longrightarrow\tilde{K}^*(X\wedge X)$$

with the map

$$\tilde{K}^*(X\wedge X)\longrightarrow\tilde{K}^*(X)$$

induced by the diagonal map

$$X\longrightarrow X\wedge X,$$

we obtain a mutiplication in $\tilde{K}^*(X)$ making it into a ring [12] .

General relative form of this product on $\tilde{K}^*(X)$ is a product

$$\tilde{K}^*(X,A)\otimes\tilde{K}^*(X,B)\longrightarrow\tilde{K}^*(X,A\cup B)$$

which is induced by the relativized diagonal map

$$X/(A\cup B)\longrightarrow X/A\wedge X/B.$$

Examples

1. The product in $\tilde{K}^*(\mathbb{S}^n)$ is trivial for $n>0$. For $n=0$, the multiplication is the usual multiplication of integers since $\mathbb{R}^m\otimes\mathbb{R}^n\approx\mathbb{R}^{mn}$.

2. Suppose that $X=A\cup B$, where A and B are compact contractible spaces of X containing the base point [13] . Then the product

$$\tilde{K}^*(X)\otimes\tilde{K}^*(X)\longrightarrow\tilde{K}^*(X)$$

is identically zero since it is equivalent to the composition

$$\tilde{K}^*(X,A)\otimes\tilde{K}^*(X,B)\longrightarrow\tilde{K}^*(X,A\cup B)\longrightarrow\tilde{K}^*(X)$$

[12]Multiplication in $\tilde{K}^*(A)$ comes with a sign, i.e. $\alpha\beta=(-1)^{ij}\beta\alpha$ for $\alpha\in\tilde{K}^i(A)$ and $\beta\in\tilde{K}^j(A)$.

[13]Above example and the given condition illustrates the necessity of the condition that A and B both contain the basepoint of X, since without this condition we could take A and B to be the two points of $\mathbb{S}^0$.

and $\tilde{K}^*(X, A \cup B) = 0$ since $X = A \cup B$.

3. We can generalize the above example by taking X as the union of compact contractible subspaces $A_1, \ldots, A_n$ containing the base point then the n-fold product

$$\tilde{K}^*(X, A_1) \otimes \ldots \otimes \tilde{K}^*(X, A_n) \longrightarrow \tilde{K}^*(A_1 \cup \ldots \cup A_n)$$

is trivial, so all n-fold products in $\tilde{K}^*(X)$ are trivial [14] .

Proposition 3.14. *The exact sequence*

$$\begin{array}{ccc} \tilde{K}^*(X, A) & \longrightarrow & \tilde{K}^*(X) \\ & \nwarrow \quad \swarrow & \\ & \tilde{K}^*(A) & \end{array}$$

is an exact sequences of $\tilde{K}^*(X)$-modules [15] *, with the maps being homomorphisms of $\tilde{K}^*(X)$-modules.*

Proof Consider the following diagram

$$\begin{array}{ccccccc} \tilde{K}(S^jSA) & \longrightarrow & \tilde{K}(S^j(X/A)) & \longrightarrow & \tilde{K}(S^jX) & \longrightarrow & \tilde{K}(S^jA) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \tilde{K}(S^jX \wedge S^jSA) & \longrightarrow & \tilde{K}(S^jX \wedge S^j(X/A)) & \longrightarrow & \tilde{K}(S^jX \wedge S^jX) & \longrightarrow & \tilde{K}(S^jX \wedge S^jA) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \tilde{K}(S^{i+j}SA) & \longrightarrow & \tilde{K}(S^{i+j}(X/A)) & \longrightarrow & \tilde{K}(S^{i+j}X) & \longrightarrow & \tilde{K}(S^{i+j}A) \end{array}$$

where the vertical maps between the first two rows are external product with a fixed element of $\tilde{K}(S^iX)$ and the vertical maps between second and third rows are induced by diagonal maps.

Next we show that the diagram commutes. For the upper two rows it follows from external product since the horizontal maps are induced by maps between spaces. The lower two rows are induced from the suspensions of maps between spaces,

[14]In particular all elements of $\tilde{K}^*(X)$ are nilpotent.

[15]$\tilde{K}^*(X)$-module structure on $\tilde{K}^*(A)$ is defined by $\xi.\alpha = i^*(\xi)\alpha$, where i is the inclusion map and the product on RHS is the multiplication in the ring $\tilde{K}^*(A)$. To define the module structure on $\tilde{K}^*(X, A)$, key observation is that diagonal map $X \longrightarrow X \wedge X$ induces a quotient map $X/A \longrightarrow X \wedge (X/A)$, and this leads to a product $\tilde{K}^*(X) \otimes \tilde{K}^*(X, A) \longrightarrow \tilde{K}^*(X, A)$.

$$
\begin{array}{ccccccc}
A & \longrightarrow & X & \longrightarrow & X/A & \longrightarrow & SA \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
X \wedge A & \longrightarrow & X \wedge X & \longrightarrow & X \wedge (X/A) & \longrightarrow & X \wedge SA
\end{array}
$$

So it suffices to show that this diagram commutes upto homotopy. We only need to show for the right square and we have:

$$
\begin{array}{ccc}
X \cup CA & \xrightarrow{1} & SA \\
\downarrow{\scriptstyle 2} & & \downarrow{\scriptstyle 2'} \\
X \wedge (X \cup CA) & \xrightarrow{1'} & X \wedge SA
\end{array}
$$

where the horizontal maps 1 and $1'$ collapse the copy of X in $X \cup CA$ to a point. From the vertical maps we have

$$x \in X \overset{2}{\longmapsto} (x, x) \in X \cup CA$$

and

$$(a, s) \in CA \overset{2'}{\longmapsto} (a, a, s) \in X \wedge CA$$

Hence the commutativity follows.

∎

Chapter 4

Characteristic classes

The sources of this chapter are primarily [3], [5], [7], [9], [17].

We know that any vector bundle can be obtained as an inverse image or pull back of a universal bundle by a continuous map of the base spaces. In particular, isomorphisms of vector bundles over X are characterized by homotopy classes of continuous maps of the space X to the classifying space $BO(n)$ (or $BU(n)$ for complex bundles). But it is usually difficult to describe homotopy classes of maps from X into $BO(n)$. Instead, it is usual to study certain invariants of vector bundles defined in terms of the homology or cohomology groups of the space X.

Following this idea, we use the term *characteristic class* [1] for a correspondence α which associates to each vector bundle ξ of rank n over X a cohomology class $\alpha(\xi) \in H^*(X)$ with some fixed coefficient group for the cohomology groups. In addition, we require functoriality, if

$$f : X \longrightarrow Y$$

is a continuous map, η a vector bundle of rank n over Y, and $\xi = f^*(\eta)$, the pull back vector bundle over X, then

$$\alpha(\xi) = f^*(\alpha(\eta)), \tag{4.1}$$

where f^* denotes the induced natural homomorphism of cohomology groups

$$f^* : H^*(Y) \longrightarrow H^*(X).$$

[1] We can have such a correspondence for principal G-bundles, but we are manily interested in vector bundles. And we will only consider the mod 2 cohomology of $BO(n)$ whereas we consider the integral cohomology of $BU(n)$.

Theorem 4.1. *The family of all characteristic classes of n-dimensional real (complex) vector bundle is in one-to-one correspondence with the cohomology ring $H^*(BO(n))$ (respectively, with $H^*(BU(n))$).*

Proof Let ξ^n be a universal bundle over the classifying space $BO(n)$ and α a characteristic class. Then $\alpha(\xi^n) \in H^*(BO(n))$ is the associated cohomology class.

Conversely, if $x \in H^*(BO(n))$ is arbitrary cohomology class then a characteristic class α is defined by the following rule: If

$$f : X \longrightarrow BO(n)$$

is a continuous map and $\xi = f^*(\xi^n)$ [2] . Put

$$\alpha(\xi) = f^*(x) \in H^*(X). \tag{4.2}$$

We have to check that the correspondence (4.2) gives a characteristic class. If

$$g : X \longrightarrow Y$$

is a continuous map and

$$h : Y \longrightarrow BO(n)$$

is a map such that

$$\eta = h^*(\xi^n), \xi = g^*(\eta),$$

then

$$\begin{aligned}
\alpha(\xi) &= \alpha((hg)^*(\xi^n)) \\
&= (hg)^*(x) \\
&= g^*(h^*(x)) \\
&= g^*(\alpha(h^*(\xi^n))) \\
&= g^*(\alpha(\eta))
\end{aligned}$$

If

$$f : BO(n) \longrightarrow BO(n)$$

[2]Note that here $f^*(\xi^n)$ denotes the pull back of the universal bundle ξ^n.

is the identity map then

$$\alpha(\xi^n) = f^*(x) = x.$$

Hence the class α corresponds to the cohomology class x.

■

Theorem 4.2. *1. The integral cohomology algebra of $BU(n)$ is isomorphic to polynomial algebra over integers in generators $C_i \in H^{2i}(BU(n);\mathbb{Z}), 1 \leq i \leq n$.*

2. The mod 2 cohomology algebra of $BO(n)$ is isomorphic to polynomial algebra over integers in generators $W_i \in H^i(BO(n);\mathbb{Z}/2\mathbb{Z}), 1 \leq i \leq n$.

Instead of proving the theorem, we will only describe the the classes $C_i \in H^{2i}(BU(n);\mathbb{Z})$ and $W_i \in H^i(BO(n);\mathbb{Z}/2\mathbb{Z})$.

Consider $n = 1$ case, we have

$$BU(1) = \mathbb{CP}^\infty$$

and

$$BO(1) = \mathbb{RP}^\infty.$$

Let $x \in H^2(\mathbb{CP}^\infty;\mathbb{Z}) \approx \mathbb{Z}$ denote a generator. Similarly, let $t \in H^1(\mathbb{RP}^\infty;\mathbb{Z}/2\mathbb{Z}) \approx \mathbb{Z}/2\mathbb{Z}$ denote the generator. Apply Künneth formula 6.1 , we see that

$$H^*(B(U(1))^n);\mathbb{Z}) = \mathbb{Z}[x_1,\dots,x_n].$$

The inclusion

$$U(1)^n \longrightarrow U(n)$$

induces a map

$$\rho : B(U(1))^n \longrightarrow BU(n).$$

The symmetric group S_n acts on $U(1)^n$ [3] by permuting the coordinates and

$$\rho \circ \sigma = \rho, \forall\ \sigma \in S_n.$$

[3] Regard $U(1)^n$ as a diagonal subgroup of $U(n)$.

The map ρ induces a monomorphism in cohomology. Since $\rho \circ \sigma = \rho$, we see that the image of the map

$$\rho^* : H^*(BU(n); \mathbb{Z}) \longrightarrow H^*(B(U(1)^n); \mathbb{Z})$$

is contained in subalgebra of $H^*(B(U(1)^n); \mathbb{Z})$ which is invariant under the action of S_n. It is evident that S_n acts on $H^*(B(U(1)^n); \mathbb{Z}) = \mathbb{Z}[x_1, \ldots, x_n]$ by permuting x_i. Therefore, defining $C_i \in H^{2i}(BU(n); \mathbb{Z})$ to be the element such that $\rho^*(C_i)$ is the ith elementary symmetric polynomial in $x_1, \ldots, x_n$. C_i's are algebraically independent and that they generate integral cohomology algebra of $BU(n)$ [4] .

4.1 Chern and Stiefel-Whitney classes

Definition 4.1. *If $1 \leq i \leq n$, let $C_i \in H^{2i}(BU(n); \mathbb{Z})$ and $W_i \in H^i(BO(n); \mathbb{Z}/2\mathbb{Z})$ be as defined above. Set $C_0 = 1 \in H^0(BU(n); \mathbb{Z})$ and $W_0 = 1 \in H^0(BO(n); \mathbb{Z}/2\mathbb{Z})$. Assume X is paracompact.*

1. *If*

$$f : X \longrightarrow BU(n)$$

*is a classifying map of a complex vector bundle η of rank n over X, then $c_i(\eta) \in H^{2i}(X; \mathbb{Z}), 1 \leq i \leq n$, ith **Chern class** of η defined to be the element $f_\eta^*(C_i)$. The **total Chern class** of η is the element*

$$c(\eta) := 1 + c_1(\eta) + \ldots + c_n(\eta) \in H^*(X, \mathbb{Z}).$$

2. *If*

$$g : X \longrightarrow BO(n)$$

*is a classifying map of a real vector bundle ξ of rank n over X, then $w_i(\xi) \in H^i(X; \mathbb{Z}/2\mathbb{Z}), 1 \leq i \leq n$, ith **Stiefel-Whitney class** of ξ defined to be the element $g_\xi^*(W_i)$. The **total Stiefel-Whitney class** of ξ is the element*

$$w(\xi) := 1 + w_1(\xi) + \ldots + w_n(\xi) \in H^*(X, \mathbb{Z}/2\mathbb{Z}).$$

Remark 4.1. *1. In the view of above definitions, the $W_i, 1 \leq i \leq n$, and $C_i, 1 \leq$*

[4] The above statements hold when $U(n)$ and $U(1)$ are replaced by $O(n)$ and $O(1)$ respectively provided $H^{2i}(\ \ ; \mathbb{Z})$ is replaced by $H^i(\ \ ; \mathbb{Z}/2)$ throughout.

$i \leq n$, are called respectively ***Universal Stiefel-Whitney classes*** *and* ***Universal Chern classes****.*

2. *Note that we have defined Stiefel-Whitney classes and Chern classes only under the assumption that the base space is paracompact.*

3. *Here after we supress the coefficient ring of the cohomology. It is understood that, unless explicitly mentioned otherwise, integer coefficients are used in the context of Chern classes and mod 2 coefficients are used in case of Stiefel-Whitney classes.*

Theorem 4.3. *The Stiefel-Whitney classes and Chern classes satisfy the following axioms: Let ξ and η be, respectively, real and complex vector bundles over X.*

1. *Naturality: If*

$$f : Y \longrightarrow X$$

is a continuous map, then

$$c_i(f^*(\eta)) = f^*(c_i(\eta)) \in H^{2i}(Y)$$

and

$$w_i(f^*(\xi)) = f^*(w_i(\xi)) \in H^i(Y).$$

2. *Whitney Product Formula: If ξ_1 and ξ_2 are real vector bundles over X, then*

$$w_k(\xi_1 \oplus \xi_2) = \sum_{i+j=k} w_i(\xi_1) w_j(\xi_2).$$

Equivalently

$$w(\xi_1 \oplus \xi_2) = w(\xi_1) w(\xi_2).$$

Similarly for complex vector bundles η_1 and η_2 over X one has

$$c(\eta_1 \oplus \eta_2) = c(\eta_1) c(\eta_2).$$

3. *Non-triviality: If ξ_1 is the Hopf line bundle over $\mathbb{RP}^1$, then $w_1(\xi_1)$ is the generator of $H^1(\mathbb{RP}^1) \approx \mathbb{Z}/2\mathbb{Z}$. If η_1 is the Hopf line bundle over $\mathbb{CP}^1 \approx \mathbb{S}^2$, then $c_1(\eta_1) \in H^2(\mathbb{S}^2) \approx \mathbb{Z}$ is the negative of positive generator.*

Remark 4.2. *Note that a constant map*

$$f : X \longrightarrow BU(n)$$

classifies the trivial bundle ϵ^n. It follows that

$$c_i(\epsilon^n) = 0 \textit{ for } i > 0.$$

By Whitney product formula, we have

$$c(\eta \oplus \epsilon^n) = c(\eta).$$

An entirely analogous statement holds for real vector bundles.

Examples

1. For the tangent bundle $T\mathbb{S}^n$ of the unit sphere $\mathbb{S}^n$, the class

$$w(\mathbb{S}^n) = w(T\mathbb{S}^n).$$

In other words, $T\mathbb{S}^n$ cannot be distinguished from trivial bundle over $\mathbb{S}^n$ by means of Stiefel-Whitney classes.

2. The total Stiefel-Whitney class of canonical line bundle over $\mathbb{RP}^n$ is given by

$$w(\gamma_n^1) = 1 + a,$$

where $a \in H^1(\mathbb{RP}^n; \mathbb{Z}/2)$.

3. By its definition, the line bundle γ_n^1 over $\mathbb{RP}^n$ is contained as a subbundle in the trivial bundle ϵ^{n+1}. Let $\gamma^\perp$ denote the orthogonal complement of γ_n^1 in ϵ^{n+1}. Then

$$w(\gamma^\perp) = 1 + a + \ldots + a^n.$$

Theorem 4.4. *The Whitney sum $T\mathbb{RP}^n \oplus \epsilon^1$ is isomorphic to the $(n+1)$-fold Whitney sum $\gamma_n^1 \oplus \ldots \oplus \gamma_n^1$. Hence the total Stiefel-Whitney class of $\mathbb{RP}^{n+1}$ is given by*

$$w(\mathbb{RP}^n) = (1+a)^{n+1} = 1 + \binom{n+1}{1} a + \ldots + \binom{n+1}{n} a^n.$$

4.2 K-theory and Chern character

In the Section 4.1 we defined Chern classes for any complex vector bundle. If we consider characteristic classes in cohomology with rational coefficients then they may all expressed as polynomials in Chern classes $c_1(\eta), \ldots, c_n(\eta)$ with rational coefficients. Now we define an important invariant of a complex vector bundle η over a paracompact base X.

Consider the function

$$\varphi^k(t_1, \ldots, t_n) = t_1^k + \ldots + t_n^k$$

Let σ_i be elementary symmetric polynomials:

$$\sigma_0(t_1, \ldots, t_n) = 1,$$

$$\sigma_1 = \sigma_1(t_1, \ldots, t_n) = t_1 + \ldots + t_n,$$

$$\ldots$$

$$\sigma_n = \sigma_n(t_1, \ldots, t_n) = t_1 t_2 \ldots t_n,$$

$$\sigma_m = 0 \text{ for } m > n.$$

The functions φ^k can be expressed as polynomials of $\sigma_1, \ldots, \sigma_n$. For example:

$$\varphi^1 = \sigma_1;$$

$$\varphi^2 = \sigma_1^2 - 2\sigma_2;$$

$$\varphi^3 = \sigma_1^3 - 3\sigma_1\sigma_2 + 3\sigma_3;$$

$$\vdots$$

As a result, we can write

$$\varphi^k = s_k(\sigma_1, \ldots, \sigma_n).$$

Let η be a complex vector bundle of rank n. The k^{th} Chern character of η is defined as:

$$ch_k(\eta) = \frac{s_k(c(\eta))}{k!}$$

for $k > 0$. If $k = 0$, we set

$$ch_0(\eta) = \mathrm{rank}(\eta).$$

Definition 4.2. *Let η be a complex vector bundle of rank n over a paracompact base space X. The characteristic class*

$$ch(\eta) = rank(\eta) + \sum_{k=1}^{\infty} \frac{s_k(c(\eta))}{k!} \tag{4.3}$$

is called the Chern character of the bundle η.

We explain the above definition where we are given with the *Chern polynomial* of a complex vector bundle η.

Let $\mathrm{rank}(\eta) = n$ and consider the Chern polynomial

$$c_t(\eta) = 1 = c_1(\eta)t + c_2(\eta)t^2 + \ldots + c_n(\eta)t^n$$

where t is an indeterminate. Let $x_1, \ldots, x_n$ be the formal roots of this polynomial. That is

$$c_t(\eta) = \prod_{1 \leq k \leq n} (1 + x_k t).$$

Thus c_i equals the ith elementary symmetric polynomial in the formal variables $x_1, \ldots, x_n$. In particular any symmetric polynomial (with integer coefficients) in x_i's can be expressed as a polynomial (with integer coefficients) in c_i's. The x_i's are often reffered to as *Chern roots* of η. Hence we define Chern character of η, alternatively, as the following

Definition 4.3. *The Chern character of η is defined to be the formal sum*

$$ch(\eta) = \sum_{k \geq 0} \left(\sum_{1 \leq j \leq n} (x_j^k / k!) \right) \tag{4.4}$$

in the rational cohomology ring $H^*(X; \mathbb{Q})$ where $x_1, \ldots, x_n$ are the Chern roots of η.

Examples

1. If η is a line bundle

$$ch(\eta) = 1 + c_1 + \frac{c_1^2}{2!} + \ldots,$$

2. $ch(\epsilon^n) = n$, where ϵ^n is the trivial bundle of rank n.

Proposition 4.1. *Let η_1 and η_2 be the complex vector bundles over a paracompact base space X. Then*

1. *$ch(\eta_1 \oplus \eta_2) = ch(\eta_1) \oplus ch(\eta_2)$; in particular if $\eta_1 \oplus \eta_2 \approx \eta_1' \oplus \eta_2$, then $ch(\eta_1) = ch(\eta_1')$,*

2. *$ch(\eta_1 \otimes \eta_2) = ch(\eta_1)ch(\eta_2)$,*

3. *If $f : X' \longrightarrow X$ is any continuous map, then $ch(f^*(\eta)) = f^*(ch(\eta))$.*

From the above proposition Chern character can be extended to a homomorphism of rings

$$ch : K(X) \longrightarrow H^{2*}(X; \mathbb{Q}) \tag{4.5}$$

As a result, the Chern character contributes on investigating the algebraic properties of elements of the K-group. The following proposition gives an application of the Chern character to K-theory:

Proposition 4.2. *Suppose η_1 and η_2 are two complex vector bundles over a common paracompact base space X. If*

$$ch(\eta_1) \neq ch(\eta_2),$$

then η_1 and η_2 give different elements in the K-group.

Although the Chern roots, as we have defined them, are formal symbols, they can be given concrete meaning as being first Chern class of certain line bundles via the *Splitting principle* which is explained as follows:

Theorem 4.5. *Let ξ be a $\mathbb{K}$-vector bundle of rank $n \geq 2$ over a paracompact base space X. Then there exists a topological space X' and a continuous map*

$$f : X' \longrightarrow X$$

such that

1. *$f^*(\xi)$ is isomorphic to a Whitney sum of $\mathbb{K}$-line bundles $\xi_1, \ldots, \xi_n$, and,*

2. *$f^* : H^*(X; R) \longrightarrow H^*(X'; R)$ is a monomorphism where $R = \mathbb{Z}/2\mathbb{Z}$ or $\mathbb{Z}$ according to $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ respectively.*

The cohomology of sphere bundles can be examined by relatively simpler methods via the *Gysin sequence*.

4.3 The Gysin sequence of a vector bundle

Suppose η is complex vector bundle of rank n, with total space E and the base space X. We construct a canonical bundle η_0 of rank $(n-1)$ over the deleted total space E_0, which is obtained by deleting the zero section of E. A point in E_0 is determined by fibre F of η, and a non-zero vector $v \in F$. We define fibre of η_0 over v to be the quotient of $F/(\mathbb{C}v)$, where $\mathbb{C}v$ is the 1-dimensional subspace spanned by the non-zero vector v. From this construction, $F/(\mathbb{C}v)$ is a complex vector space of dimension $(n-1)$, and clearly can be considered as fibre of η_0. So η_0 is indeed a vector bundle over E_0.

Theorem 4.6. *Any real oriented $2n$-plane bundle*

$$\pi : E \longrightarrow X$$

possesses a long exact sequence of the form

$$\longrightarrow H^{i-2n}(X) \xrightarrow{\gamma} H^i(X;R) \xrightarrow{\pi^*} H^i(E_0;R) \xrightarrow{g} H^{i-2n+1}(X;R) \longrightarrow$$

Let E be $(n-1)$-sphere bundle over X. If the bundle is a product bundle, i.e.

$$E = \mathbb{S}^{n-1} \times X,$$

then there is a section

$$s : X \longrightarrow E$$

such that

$$\pi \circ s = \mathrm{Id}_X.$$

From the Gysin sequence [5] above injectivity of π^* and exactness implies that Gysin sequence breaks into split short exact sequences:

$$0 \longrightarrow H^i(X) \xrightarrow{\pi^*} H^i(E) \xrightarrow{g} H^{i-n+1}(X) \longrightarrow 0$$

[5] Note that here we obtain Gysin sequence for $(n-1)$-sphere bundle.

This is true as long as a section exists, regardless of whether or not the bundle is product bundle.

Example

Let $T\mathbb{S}^n$ be the unit tangent bundle to $\mathbb{S}^n$. This is sometimes called the real Stiefel manifold $V_{2,n+1}$, or the space of 2-frames in $\mathbb{R}^{n+1}$. A section of this bundle is the same as a field of unit tangent vectors on $\mathbb{S}^n$, which exists if and only if n is odd. Hence the Gysin sequence tells us that $T\mathbb{S}^n$ has the same cohomology groups as $\mathbb{S}^{n-1} \times \mathbb{S}^n$ when n is odd.

4.3.1 Gysin sequence and Chern classes

We now give an alternate definition of Chern classes.

Definition 4.4. *Let η be the complex n-plane bundle, with total space E and the base space X. Denote η_0 be the $(n-1)$-bundle over the deleted space E_0. For $i < n$, the i^{th} Chern class of η is defined to be:*

$$c_i(\eta) = \begin{cases} (\pi^*)^{-1}(c_i(\eta_0)) & \text{if } 0 < i < n; \\ 1 & \text{if } i = 0. \end{cases}$$

where $(\pi^*)^{-1}$ [6] is defined in the Gysin sequence:

$$\ldots \longrightarrow H^{2i-2n}(X) \xrightarrow{\gamma} H^{2i}(X) \xrightarrow{\pi^*} H^{2i}(E_0) \xrightarrow{g} H^{2i-2n+1}(X) \longrightarrow \ldots$$

Note that $c_i(\eta) \in H^{2i}(X;\mathbb{Z})$. For $i > n$ we just set $c_i(\eta) = 0$.

4.3.2 Chern character of Hopf bundle

Let η be the complex Hopf bundle. Here we compute Chern character of the complex Hopf bundle.

Lemma 4.1. *For $2 \le i \le n$, $c_i(\eta) = 0$.*

Proof Suppose $n = 2$. Since η has rank 1, and $n = 2$, so by Definition 4.4 , we immediately have $c_2(\eta) = 0$. Therefore, claim holds for $n = 2$.

[6] $(\pi^*)^{-1} : H^{2i}(X) \longrightarrow H^{2i}(E_0)$ is an isomorphism for $i < n$. So i^{th} Chern class is well defined and unique.

Assume that the claim holds for $n \in \{2, \dots, k-1\} \subseteq \mathbb{N}$. If $n = k$, we have

$$c_i(\eta) = (\pi^*)^{-1}(c_i(\eta_0)).$$

By assumption, $c_i(\eta_0) = 0$ for every $2 \leq i \leq n$. Since $(\pi^*)^{-1}$ is an isomorphism, we must have

$$c_i(\eta) = (\pi^*)^{-1}(c_i(\eta_0)) = (\pi^*)^{-1}(0) = 0.$$

Therefore by induction, the claim holds as desired.

■

Proposition 4.3. *Let x be generator of $H^2(\mathbb{CP}^n; \mathbb{Z})$. Then $ch(\eta) = e^x$.*

Proof Using equation (4.3), we have:

$$\begin{aligned} ch(\eta) &= 1 + c_1(\eta) + \frac{c_1^2(\eta) - 2c_2(\eta)}{2!} + \frac{c_1^3(\eta) - 3c_1(\eta)c_2(\eta) + 3c_3(\eta)}{3!} + \dots \\ &= 1 + c_1(\eta) + \frac{c_1^2(\eta)}{2!} + \frac{c_1^3(\eta)}{3!} + \dots \\ &= e^x \end{aligned}$$

■

Chapter 5

Some calculations in K-theory

The sources of this chapter are primarily [1], [3], [7], [8], [15].

5.1 Some computations of K-groups

5.1.1 K-theory of $\mathbb{R}^n$ and $\mathbb{S}^n$

For each natural number n, we have

$$K^0(\mathbb{R}^n) \approx \begin{cases} \mathbb{Z} & if\ n\ is\ even \\ 0 & if\ n\ is\ odd. \end{cases}$$

This in turn yields

$$K^{-1}(\mathbb{R}^n) \approx K^0(\mathbb{R}^{n+1}) \approx \begin{cases} 0 & if\ n\ is\ even \\ \mathbb{Z} & if\ n\ is\ odd. \end{cases}$$

The n-sphere $\mathbb{S}^n$ is the one-point compactification of $\mathbb{R}^n$. Thus

$$K^0(\mathbb{S}^n) \approx \begin{cases} \mathbb{Z} \oplus \mathbb{Z} & if\ n\ is\ even \\ \mathbb{Z} & if\ n\ is\ odd. \end{cases}$$

Also

$$K^{-1}(\mathbb{S}^n) \approx \begin{cases} 0 & if\ n\ is\ even \\ \mathbb{Z} & if\ n\ is\ odd. \end{cases}$$

5.1.2 K-theory of figure eight

We know that figure eight is $\mathbb{S}^1 \vee \mathbb{S}^1$. Let

$$i : \mathbb{S}^1 \longrightarrow \mathbb{S}^1 \vee \mathbb{S}^1$$

be the inclusion map. Define a continuous function

$$\psi : \mathbb{S}^1 \vee \mathbb{S}^1 \longrightarrow \mathbb{S}^1$$

by identifying the two copies of $\mathbb{S}^1$ to a single circle. Then $\psi \circ i$ is the identity map on $\mathbb{S}^1$. The topological space $(\mathbb{S}^1 \vee \mathbb{S}^1)/\mathbb{S}^1$ is homeomorphic to $\mathbb{R}$. Hence we have

$$K^0(\mathbb{S}^1 \vee \mathbb{S}^1) \approx K^0(\mathbb{S}^1) \oplus K^0(\mathbb{R}) \approx \mathbb{Z}$$

$$K^{-1}(\mathbb{S}^1 \vee \mathbb{S}^1) \approx K^{-1}(\mathbb{S}^1) \oplus K^{-1}(\mathbb{R}) \approx \mathbb{Z} \oplus \mathbb{Z}.$$

The group $K^0(\mathbb{S}^1 \vee \mathbb{S}^1)$ is generated by trivial bundles. For each pair of integers (m, n), define $f_{(m,n)}$ on $\mathbb{S}^1 \vee \mathbb{S}^1$ to be the function that is z^m on one circle and z^n on the other circle. Then the map

$$(m, n) \longmapsto [f_{(m,n)}]$$

is an isomorphism from

$$\mathbb{Z} \oplus \mathbb{Z} \longrightarrow K^{-1}(\mathbb{S}^1 \vee \mathbb{S}^1).$$

5.1.3 K-theory of torus

Let

$$i : \mathbb{S}^1 \times \{1\} \longrightarrow \mathbb{S}^1 \times \mathbb{S}^1$$

and let

$$\psi : \mathbb{S}^1 \times \mathbb{S}^1 \longrightarrow \mathbb{S}^1 \times \{1\}$$

be the projection map. Then $\psi \circ i$ is the identity map. The complement of $\mathbb{S}^1 \times \{1\}$ in $\mathbb{S}^1 \times \mathbb{S}^1$ is homeomorphic to $\mathbb{S}^1 \times \mathbb{R}$, whence

$$K^0(\mathbb{S}^1 \times \mathbb{S}^1) \approx K^0(\mathbb{S}^1) \oplus K^0(\mathbb{S}^1 \times \mathbb{R}) \approx K^0(\mathbb{S}^1) \oplus K^{-1}(\mathbb{S}^1) \approx \mathbb{Z} \oplus \mathbb{Z}$$

$$K^{-1}(\mathbb{S}^1 \times \mathbb{S}^1) \approx K^{-1}(\mathbb{S}^1) \oplus K^{-1}(\mathbb{S}^1 \times \mathbb{R}) \approx K^{-1}(\mathbb{S}^1) \oplus K^{-2}(\mathbb{S}^1) \approx \mathbb{Z} \oplus \mathbb{Z}.$$

5.1.4 K-theory of complex projective spaces

Theorem 5.1.

$$K^q(\mathbb{CP}^n) = \begin{cases} \mathbb{Z}^{\oplus n+1} \text{ if } q \text{ is even;} \\ 0 \text{ if } q \text{ is odd.} \end{cases}$$

Moreover, as a ring,

$$K^0(\mathbb{CP}^n) = \mathbb{Z}[\Gamma]/\langle \Gamma^{n+1} \rangle,$$

where $\Gamma = \eta - \epsilon^1$, η is the Hopf bundle, and ϵ^1 is the trivial bundle of rank 1.

Proof By Bott periodicity theorem, we can focus on the cases $q = 0$ and $q = 1$.

By CW structure of $\mathbb{CP}^n$, we have $\mathbb{CP}^n/\mathbb{CP}^{n-1}$ is homeomorphic to $\mathbb{S}^{2n}$.

Step 1: We compute $K^0(\mathbb{CP}^n)$ by induction on n.

If $n = 1$, we have

$$\begin{aligned} K^0(\mathbb{CP}^n) &= K^0(\mathbb{S}^2) \\ &= \tilde{K}(\mathbb{S}^2) \oplus \mathbb{Z} \\ &= \mathbb{Z} \oplus \mathbb{Z} \end{aligned}$$

and

$$
\begin{aligned}
K^1(\mathbb{CP}^n) &= K^1(\mathbb{S}^2) \\
&= \tilde{K}(\mathbb{S}^3) \\
&= \pi_3(BU) \\
&= 0
\end{aligned}
$$

Hence the claim holds for $n = 1$ [1]. Assume that the claim holds for $n \in \{1, \ldots, k\} \subseteq \mathbb{N}$. If $n = k+1$, by the CW pair $(\mathbb{CP}^{k+1}, \mathbb{CP}^k)$, the part of exact sequence of $\tilde{K}$-groups yields:

$$
0 \longleftarrow \tilde{K}(\mathbb{CP}^k) \longleftarrow \tilde{K}(\mathbb{CP}^{k+1}) \longleftarrow \tilde{K}(\mathbb{CP}^{k+1}/\mathbb{CP}^k) \longleftarrow \tilde{K}(S\mathbb{CP}^k)
$$

which reduced to the following diagram:

$$
\begin{array}{ccccccccc}
0 & \longleftarrow & \tilde{K}(\mathbb{CP}^k) & \longleftarrow & \tilde{K}(\mathbb{CP}^{k+1}) & \longleftarrow & \tilde{K}(\mathbb{CP}^{k+1}/\mathbb{CP}^k) & \longleftarrow & \tilde{K}(S\mathbb{CP}^k) \\
\Vert & & \Vert & & \Vert & & \Vert & & \Vert \\
0 & \longleftarrow & \mathbb{Z}^{\oplus k} & \longleftarrow & \tilde{K}(\mathbb{CP}^{k+1}) & \longleftarrow & \tilde{K}(\mathbb{S}^{2k}) & \longleftarrow & \tilde{K}(S\mathbb{CP}^k) \\
\Vert & & \Vert & & \Vert & & \Vert & & \Vert \\
0 & \longleftarrow & \mathbb{Z}^{\oplus k} & \longleftarrow & \tilde{K}(\mathbb{CP}^{k+1}) & \longleftarrow & \mathbb{Z} & \longleftarrow & \tilde{K}(S\mathbb{CP}^k)
\end{array}
$$

Now, apply Five lemma [2] to the diagram:

$$
\begin{array}{ccccccccc}
0 & \longleftarrow & \mathbb{Z}^{\oplus k} & \longleftarrow & \tilde{K}(\mathbb{CP}^{k+1}) & \longleftarrow & \mathbb{Z} & \longleftarrow & \tilde{K}(S\mathbb{CP}^k) \\
\Vert & & \Vert & & \downarrow & & \Vert & & \Vert \\
0 & \longleftarrow & \mathbb{Z}^{\oplus k} & \longleftarrow & \mathbb{Z} \oplus \mathbb{Z}^{\oplus k} & \longleftarrow & \mathbb{Z} & \longleftarrow & \tilde{K}(S\mathbb{CP}^k)
\end{array}
$$

We have $\tilde{K}(\mathbb{CP}^{k+1}) = \mathbb{Z} \oplus \mathbb{Z}^{\oplus k} = \mathbb{Z}^{\oplus k+1}$. Since $K(X) = \tilde{K}(X) \oplus \mathbb{Z}$, we have

$$
K^0(\mathbb{CP}^{k+1}) = \mathbb{Z}^{\oplus k+1} \oplus \mathbb{Z} = \mathbb{Z}^{\oplus k+2}.
$$

Step 2: We follow the above procedure to get $K^1(\mathbb{CP}^n)$

[1]Last equality follows from Theorem 6.2 in Appendix.
[2]See Lemma 6.2 in Appendix.

By assuming $K^1(\mathbb{CP}^n) = 0$ for $n \in \{1, \dots, k\} \subseteq \mathbb{N}$, when $n = k+1$, we have the exact sequence:

$$\begin{array}{ccccccccc}
K^0(\mathbb{CP}^{k+1}, \mathbb{CP}^k) & \longleftarrow & K^{-1}(\mathbb{CP}^k) & \longleftarrow & K^{-1}(\mathbb{CP}^{k+1}) & \longleftarrow & K^{-1}(\mathbb{CP}^{k+1}, \mathbb{CP}^k) & \longleftarrow & K^{-2}(\mathbb{CP}^k) \\
\Vert & & \Vert & & \Vert & & \Vert & & \Vert \\
K^0(\mathbb{CP}^{k+1}, \mathbb{CP}^k) & \longleftarrow & K^{1}(\mathbb{CP}^k) & \longleftarrow & K^{1}(\mathbb{CP}^{k+1}) & \longleftarrow & K^{1}(\mathbb{CP}^{k+1}, \mathbb{CP}^k) & \longleftarrow & K(\mathbb{CP}^k) \\
\Vert & & \Vert & & \Vert & & \Vert & & \Vert \\
\mathbb{Z} & \longleftarrow & 0 & \longleftarrow & K^{1}(\mathbb{CP}^{k+1}) & \longleftarrow & 0 & \longleftarrow & \mathbb{Z}^{\oplus k}
\end{array}$$

By Bott periodicity theorem, we have

$$K^{-1}(X) = K^1(X),$$

so we get second row from the first row. By assumption, we have $K^1(\mathbb{CP}^k) = 0$. Moreover, we have

$$K^1(\mathbb{CP}^{k+1}, \mathbb{CP}^k) = 0.$$

Combining the result we get for $K^0(\mathbb{CP}^n)$, we get third row from the second row. Now again apply Five lemma to the diagram:

$$\begin{array}{ccccccccc}
\mathbb{Z} & \longleftarrow & 0 & \longleftarrow & K^1(\mathbb{CP}^{k+1}) & \longleftarrow & 0 & \longleftarrow & \mathbb{Z}^{\oplus k+1} \\
\Vert & & \Vert & & \downarrow & & \Vert & & \Vert \\
\mathbb{Z} & \longleftarrow & 0 & \longleftarrow & 0 & \longleftarrow & 0 & \longleftarrow & \mathbb{Z}^{\oplus k+1}
\end{array}$$

We have $K^1(\mathbb{CP}^{k+1}) = 0$. So by induction, the claim holds as desired.

Step 3: Ring structure of $K(\mathbb{CP}^n)$

We first compute the Chern character of Γ. By Proposition 4.3 , we have:

$$ch(\eta) = e^x,$$

where x is a generator of $H^2(\mathbb{CP}^n; \mathbb{Z})$. Using Proposition 4.1 , we get:

$$ch(\Gamma) = ch(\eta - \epsilon^1) = e^x - 1,$$

and

$$
\begin{aligned}
ch(\Gamma^k) &= ch(\Gamma^{\otimes k}) \\
&= (ch(\Gamma))^k \\
&= x^k + \frac{k}{2}x^{k+1} + \dots
\end{aligned}
$$

for $1 < k < n$. Now, when $k = n+1$, by Proposition 6.2 , we have $x^{n+1} = 0$. So we get

$$ch(\Gamma^{n+1}) = 0$$

and

$$ch(\Gamma^n) = x^n. \tag{5.1}$$

By Proposition 4.2 , $\epsilon^1, \Gamma, \dots, \Gamma^n$ represent different elements in $K(\mathbb{CP}^n)$.

Now we show that $\epsilon^1, \Gamma, \dots, \Gamma^n$ generate $K^*(\mathbb{CP}^n)$ over $\mathbb{Z}$.

We prove by induction on n. When $n = 1$, the claim follows immediately. So assume that the claim holds for $n \in \{1, \dots, k-1\} \subseteq \mathbb{N}$. If $n = k$, the part of exact sequence of K-groups yields:

$$K(\mathbb{CP}^n, \mathbb{CP}^{n-1}) \xrightarrow{p^*} K(\mathbb{CP}^n) \xrightarrow{i^*} K(\mathbb{CP}^{n-1}) \longrightarrow 0$$

By assumption, $K(\mathbb{CP}^{n-1})$ is generated by $\epsilon^1, \Gamma, \dots, \Gamma^{n-1}$. Consider the map

$$g : K(\mathbb{CP}^{n-1}) \longrightarrow K(\mathbb{CP}^n)$$

such that

$$g(\Gamma) = \Gamma.$$

It is clear that $i^* \circ g$ is identity map on $K(\mathbb{CP}^{n-1})$. So the previous exact sequence of K-groups splits:

$$K(\mathbb{CP}^n) = \mathrm{Im}(p^*) \oplus \mathrm{Im}(g).$$

Thus, if $\alpha \in K(\mathbb{CP}^n)$, then α can be written as:

$$\begin{aligned} \alpha &= g(i^*(\alpha)) + p^*(\beta) \\ &= r_0 + r_1\Gamma + \ldots + r_{n-1}\Gamma^{n-1} + p^*(\beta), \end{aligned} \tag{5.2}$$

where $r_i \in \mathbb{Z}$, and $\beta \in \tilde{K}(\mathbb{S}^{2n})$.

We wish to show that

$$p^*(\beta) = r\Gamma^n$$

for some $r \in \mathbb{Z}$, which can be done by computing Chern character of $p^*(\beta)$. We have the following commutative diagram:

$$\begin{array}{ccc} \tilde{K}(\mathbb{S}^{2n}) & \xrightarrow{p^*} & K(\mathbb{CP}^n) \\ {\scriptstyle ch}\downarrow & & \downarrow{\scriptstyle ch} \\ H^{2n}(\mathbb{S}^{2n}) & \xrightarrow[p^*]{} & H^{2n}(\mathbb{CP}^n) \end{array}$$

Now by Proposition 6.1 , we know that

$$H^{2n}(\mathbb{CP}^n; \mathbb{Z}) = \mathbb{Z} = \mathbb{Z}x^n$$

where $x^n = ch(\Gamma^n)$. So we must have $ch(p^*(\beta)) = rx^n$ for some $r \in \mathbb{Z}$. Now, since $H^{2n}(\mathbb{CP}^n; \mathbb{Z})$ is torsion free. So the map

$$ch : K(\mathbb{CP}^n) \longrightarrow H^{2n}(\mathbb{CP}^n)$$

has trivial kernel. Combining this with the (5.1), we must have:

$$p^*(\beta) = r\Gamma^n.$$

And hence Equation (5.2), becomes:

$$\begin{aligned} \alpha &= g(i^*(\alpha)) + p^*(\beta) \\ &= r_0 + r_1\Gamma + \ldots + r_{n-1}\Gamma^{n-1} + p^*(\beta) \\ &= r_0 + r_1\Gamma + \ldots + r_{n-1}\Gamma^{n-1} + r_n\Gamma^n. \end{aligned}$$

As a result by induction we conclude that $\epsilon^1, \Gamma, \ldots, \Gamma^n$ generate $K(\mathbb{CP}^n)$ over $\mathbb{Z}$ as a ring.

■

5.2 Division algebras and the Hopf invariant

Definition 5.1. *A division algebra, also called a "division ring" or "skew field," is a ring in which every nonzero element has a multiplicative inverse, but multiplication is not necessarily commutative. An identity element is also not assumed.*

Examples

1. Every field is therefore a division algebra. And moreover, $\mathbb{H}$, the quaternions, is well known example of a skew field.

2. On $\mathbb{R}^8$ one has a multiplication which makes it non-associative division algebra over the reals and is called *Cayley numbers* or *octonians*, denoted $\mathbb{O}$ [3] .

   The multiplication rule is:

   $$(q_1, q_2).(q_1', q_2') := (q_1 q_1' - \bar{q_2'} q_2, q_2' q_1 + q_2 \bar{q_1'})$$

   where a Cayley number is being thought of as an ordered pair of quaternions. Denoting $\|.\|$ the usual Euclidean norm on $\mathbb{R}^8$, the multiplication is $\mathbb{R}$-bilinear and norm preserving, i.e.

   $$\|cc'\| = \|c\|.\|c'\| \tag{5.3}$$

   In particular $cc' = 0$ iff either $c = 0$ or $c' = 0$ for any two Cayley numbers c, c'.

Remark 5.1. *Squaring both sides of equation(5.3), we obtain*

$$\|c\|^2 \|c'\|^2 = \|cc'\|^2.$$

For example, taking $c = x_1 + \iota y_1, c' = x_1 + \iota y_2 \in \mathbb{C}$ we obtain the 'two-square formula'

[3] Although the Cayley numbers are non-associative, they form an *alternate algebra*, i.e. for any $x, y \in \mathbb{O}$, we have $(x^2)y = x(xy)$ and $x(y^2) = (xy)y$.

$$(x_1^2+y_1^2)(x_2^2+y_2^2) = (x_1x_2 - y_1y_2)^2 + (x_1y_2 + x_2y_1)^2.$$

Using the multiplication rules for quaternions and the octonions, one obtains 'four-square' and 'eight-square' formulae. If there exist bilinear forms $q_1, \ldots, q_n$ *on* $\mathbb{R}^n$ *such that*

$$\|x\|^2\|y\|^2 = q_1(x,y)^2 + \ldots + q_n(x,y)^2,$$

then the linear map

$$q : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$$

is defined as

$$(x,y) \longmapsto (q_1(x,y), \ldots, q_n(x,y))$$

would be norm preserving [4] .

Example

A division algebra structure on $\mathbb{R}^n$ is a multiplication map

$$d : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$$

such that the maps

$$x \longmapsto ax \tag{5.4}$$

$$x \longmapsto xa \tag{5.5}$$

are linear for each $a \in \mathbb{R}^n$ and invertible if $a \neq 0$. Since we are dealing with linear maps, invertibility is equivalent to having trivial kernel, and hence translates the statement that the multiplication has no zero divisors.

Choose a unit vector $e \in \mathbb{R}^n$. After composing the multiplication with an invertible linear map

$$\mathbb{R}^n \longrightarrow \mathbb{R}^n$$

taking

$$e^2 \longmapsto e$$

we may assume that

$$e^2 = e.$$

[4]Hurwitz showed that n-square formulae exist only for $n = 1, 2, 4$ or 8.

Let

$$x \xrightarrow{\alpha} xe$$

and

$$x \xrightarrow{\beta} ex.$$

Now the map

$$(x, y) \longmapsto \alpha^{-1}(x)\beta^{-1}(y)$$

sends (x, e) to $\alpha^{-1}(x)\beta^{-1}(e) = \alpha^{-1}(x)e = x$, and similarly it sends (e, y) to y. Since the maps (5.4) and (5.5) are surjective for each $a \neq 0$ and hence the equations

$$ax = e,$$

$$xa = e$$

are solvable, so nonzero elements of the division algebra have multiplicative inverses on the left and the right.

Theorem 5.2. $\mathbb{R}^n$ *is a division algebra only for* $n = 1, 2, 4,$ *and* 8.

Question 5.1. 1. *For what value of* n *does there exist a norm preserving division algebra structure on* $\mathbb{R}^n$*?*

2. *More generally, given* $n \geq 1$*, find the maximum integer* k *such that there exists a norm preserving bilinear map*

$$\mu : \mathbb{R}^k \times \mathbb{R}^n \longrightarrow \mathbb{R}^n.$$

Remarks

1. By fixing a non-zero $v \in \mathbb{R}^n$, the map

$$\mu(-, v) : \mathbb{R}^k \longrightarrow \mathbb{R}^n$$

is a monomorphism since for any $u \in \mathbb{R}^k$, $\mu(u, v) = 0$ implies

$$\begin{aligned} 0 &= \|\mu(u, v)\| \\ &= \|u\|\|v\|. \end{aligned}$$

Since $v \neq 0$, it follows that $u = 0$. Therefore $k \leq n$ [5] .

2. When $\mathbb{R}^n$ is a division algebra with a norm preserving multiplication then one can restrict the multiplication μ to obtain a map

$$\mu : \mathbb{S}^{n-1} \times \mathbb{S}^{n-1} \longrightarrow \mathbb{S}^{n-1}.$$

Hence the following discussion give the topological interpretation of the questions posed before.

Definition 5.2. *An H-space* [6] *, is a topological space X(generally assumed to be connected) together with a continuous map*

$$\mu : X \times X \longrightarrow X$$

with an identity element e so that

$$\mu(e, x) = \mu(x, e) = x \; \forall \; x \in X.$$

Proposition 5.1. $\mathbb{S}^{2k}$ *is not an H-space if $k > 0$.*

Proof Suppose

$$\mu : \mathbb{S}^{2k} \times \mathbb{S}^{2k} \longrightarrow \mathbb{S}^{2k}$$

is an H-space multiplication. The induced homomorphism of K-rings then has the form

$$\mu^* : \mathbb{Z}[\lambda]/(\lambda)^2 \longrightarrow \mathbb{Z}[\alpha, \beta]/(\alpha^2, \beta^2).$$

We claim that

$$\mu^*(\lambda) = \alpha + \beta + m\alpha\beta$$

for some integer m.

The composition

$$\mathbb{S}^{2k} \xrightarrow{i} \mathbb{S}^{2k} \times \mathbb{S}^{2k} \xrightarrow{\mu} \mathbb{S}^{2k}$$

[5]Question 5.1 was solved by A. Hurwitz who posed part (2) of Question 5.1 as an open problem. He, and independentely R. Radon, solved part (2).

[6]Every topological group is an H-space, however, in the general case, as compared to a topological group, H-spaces may lack associativity and inverses.

is identity, where i is the inclusion onto $\mathbb{S}^{2k} \times \{e\}$ or $\{e\} \times \mathbb{S}^{2k}$, with e the identity element of the H-space structure. The map i^* for i the inclusion onto first factor sends

$$\alpha \longmapsto \lambda,$$

$$\beta \longmapsto 0.$$

Hence the coefficient of α in $\mu^*(\lambda)$ must be 1. Similarly coefficient of β must be 1, proving the claim. But

$$\begin{aligned}\mu^*(\lambda^2) &= (\alpha + \beta + m\alpha\beta)^2 \\ &= 2\alpha\beta \\ &\neq 0\end{aligned}$$

which is impossible since $\lambda^2 = 0$.

■

Now the difficult problem is to show that $\mathbb{S}^{n-1}$ is not an H-space when n is even and different from 2, 4 and 8.

Step 1: We associate a map

$$\hat{\mu} : \mathbb{S}^{2n-1} \longrightarrow \mathbb{S}^n$$

to a map

$$\mu : \mathbb{S}^{n-1} \times \mathbb{S}^{n-1} \longrightarrow \mathbb{S}^{n-1}.$$

We write $\mathbb{S}^{2n-1}$ as

$$\partial(D^n \times D^n) = \partial D^n \times D^n \cup D^n \times \partial D^n,$$

and $\mathbb{S}^n$ as a union of two disks D^n_+ and D^n_- with their boundaries identified. Now we define $\hat{\mu}$ as:

$$\hat{\mu}(x,y) = \begin{cases} |y|\,\mu\left(x, \frac{y}{|y|}\right) \in D^n_+ \,; & \text{where } (x,y) \in \partial D^n \times D^n, \\ |x|\,\mu\left(\frac{x}{|x|}, y\right) \in D^n_- \,; & \text{where } (x,y) \in D^n \times \partial D^n. \end{cases}$$

Note that $\hat{\mu}$ is well defined and continuous, even when $|x|$ or $|y|$ is zero, and $\hat{\mu}$ agrees with μ on $\mathbb{S}^{n-1} \times \mathbb{S}^{n-1}$.

Step 2: We specialize to the case where n is even by replacing n by $2n$. For a map

$$f : \mathbb{S}^{4n-1} \longrightarrow \mathbb{S}^{2n}$$

let

$$C_f := e^{4n} \cup_f \mathbb{S}^{2n}$$

obtained by identifying $x \in \mathbb{S}^{4n-1}$ with $f(x) \in \mathbb{S}^{2n}$. Thus C_f is a CW complex with one cell in each dimension $0, 2n$ and $4n$. Therefore

$$H^i(C_f; \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{when } i = 2n, 4n, \\ 0 & \text{otherwise.} \end{cases}$$

Since $\tilde{K}^1(\mathbb{S}^{2n}) = \tilde{K}^1(\mathbb{S}^{4n}) = 0$, the exact sequence of the pair $(C_f, \mathbb{S}^{2n})$ becomes a short exact sequence

$$0 \longrightarrow \tilde{K}(\mathbb{S}^{4n}) \xrightarrow{g_1} \tilde{K}(C_f) \xrightarrow{g_2} \tilde{K}(\mathbb{S}^{2n}) \longrightarrow 0$$

Let

$$g_1((\gamma - \epsilon^1) * \ldots * (\gamma - \epsilon^1)) = \alpha, \tag{5.6}$$

where $x_1 = (\gamma - \epsilon^1) * \ldots * (\gamma - \epsilon^1)$ is the generator of the group $\tilde{K}(\mathbb{S}^{4n})$. And

$$g_2(\beta) = (\gamma - \epsilon^1) * \ldots * (\gamma - \epsilon^1) = x_2,$$

where x_2 is the generator of the group $\tilde{K}(\mathbb{S}^{2n})$.

From above we have

$$\beta \longmapsto 0 \in \tilde{K}(\mathbb{S}^{2n}).$$

By exactness, we have

$$\beta^2 = H(f)\alpha, \tag{5.7}$$

for some integer $H(f)$ [7] called the *Hopf invariant* of f.

[7] Notice that $H(f)$ is well defined.

Now we are ready to state the following important result:

Lemma 5.1. *If*

$$\mu : \mathbb{S}^{2n-1} \times \mathbb{S}^{2n-1} \longrightarrow \mathbb{S}^{2n-1}$$

is an H-space multiplication, then the associated map

$$\hat{\mu} : \mathbb{S}^{4n-1} \longrightarrow \mathbb{S}^{2n}$$

has Hopf invariant ± 1.

Proof Let $e \in \mathbb{S}^{2n-1}$ be the identity element for H-space multiplication, and let $\hat{\mu} = f$. In the view of definition of f, it is natural to view characteristic map Φ of the $4n$-cell of C_f as a map

$$(D^{2n} \times D^{2n}, \partial(D^{2n} \times D^{2n})) \longrightarrow (C_f, \mathbb{S}^{2n}).$$

Consider the following commutative diagram

$$\begin{array}{ccc}
\tilde{K}(C_f) \otimes \tilde{K}(C_f) & \longrightarrow & \tilde{K}(C_f) \\
\uparrow \approx & & \\
\tilde{K}(C_f, D^{2n}_-) \otimes \tilde{K}(C_f, D^{2n}_+) & \longrightarrow & \tilde{K}(C_f, \mathbb{S}^{2n}) \\
\downarrow \Phi^* \otimes \Phi^* & & \downarrow \Phi^* \\
\tilde{K}(D^{2n} \times D^{2n}, \partial D^{2n} \times D^{2n}) \otimes \tilde{K}(D^{2n} \times D^{2n}, D^{2n} \times \partial D^{2n}) & \longrightarrow & \tilde{K}(D^{2n} \times D^{2n}, \partial(D^{2n} \times D^{2n})) \\
\downarrow \approx & \nearrow \approx & \\
\tilde{K}(D^{2n} \times \{e\}, \partial D^{2n} \times \{e\}) \otimes \tilde{K}(\{e\} \times D^{2n}, \{e\} \times \partial D^{2n}) & &
\end{array}$$

Here horizontal maps are the product maps. The diagonal map is external product

$$\tilde{K}(\mathbb{S}^{2n}) \otimes \tilde{K}(\mathbb{S}^{2n}) \longrightarrow \tilde{K}(\mathbb{S}^{4n}),$$

which is an isomorphism since it is an iterate of the Bott periodicity isomorphism. The map Φ restricts to a homeomorphism from $D^{2n} \times \{e\}$ onto D^{2n}_+ and from $\{e\} \times D^{2n}$ to D^{2n}_-. It follows that the element $\beta \otimes \beta \in \tilde{K}(C_f) \otimes \tilde{K}(C_f)$ maps to a generator of the group in the bottom row of the diagram, since β restricts to a generator of $\tilde{K}(\mathbb{S}^{2n})$ by definition. Hence from the commutativity of the diagram, the product map in the top row sends $\beta \otimes \beta$ to $\pm\alpha$ since α as defined above in (5.6). Thus we have $\beta^2 = \pm\alpha$, which says that the Hopf invariant of f is ± 1.

■

In the view of the above Lemma, Theorem 5.2 becomes a consequence of the following deep result due to Adams:

Theorem 5.3. *There exists a map*

$$f : \mathbb{S}^{4n-1} \longrightarrow \mathbb{S}^{2n}$$

of the Hopf invariant ± 1 *if and only if when* $n = 1, 2,$ *or* 4.

5.3 Vector fields on spheres

Recall that a continuous (respectively smooth) vector field s on a smooth manifold M is cross section

$$s : M \longrightarrow TM$$

of the tangent bundle of M. Thus $s(x)$ is a tangent vector at x to M for every $x \in M$ which varies continuously (respectively smoothly) w.r.t x.

Example

A vector field on $\mathbb{S}^{n-1}$ is just a map

$$s : \mathbb{S}^{n-1} \longrightarrow \mathbb{S}^{n-1}$$

such that $x \perp s(x) \; \forall \; x \in \mathbb{S}^{n-1}$.

Definition 5.3. 1. *Span of a smooth manifold M is the maximum number k such that there exists k vector fields $s_1, \ldots, s_k$ on M such that $s_1(x), \ldots, s_k(x) \in T_xM$ are linearly independent for every $x \in M$. We say that $(s_1, \ldots, s_k)$ is a k-frame. If M is Riemannian manifold*[8] *, we say that a k-frame $s_1, \ldots, s_k$ is orthonormal if $\langle s_i(x), s_j(x) \rangle = \delta_{ij}$ for all $x \in M$.*

2. *We say M is parallelizable if M admits a d-frame where $d = dim(M)$.*

[8]If $M \subset \mathbb{R}^N$ then M inherits a Riemannian metric from the usual metric. Starting with any k-frame on a Riemannian manifold M, one can apply the Gram-Schmidt process and obtain an orthonormal k-frame.

Example

An orthonormal 1-frame on $\mathbb{S}^{n-1}$ is just a self-map

$$s : \mathbb{S}^{n-1} \longrightarrow \mathbb{S}^{n-1}$$

such that $x \perp s(x) \ \forall \ x \in \mathbb{S}^{n-1}$.

Question 5.2. *For what values of n, is the sphere $\mathbb{S}^{n-1}$ is parallelizable?*

Remark 5.2. *If*

$$s : \mathbb{S}^{n-1} \longrightarrow V_{k,n}$$

where $V_{k,n}$ is the Stiefel manifold, is a cross section of the bundle [9]

$$\pi : V_{k,n} \longrightarrow \mathbb{S}^{n-1},$$

i.e. if $\pi \circ s$ equals the identity map of $\mathbb{S}^{n-1}$, then $s(x) = (s_1(x), \dots, s_{k-1}(x))$ is a $(k-1)$-frame on $\mathbb{S}^{n-1}$. Conversely starting with a $(k-1)$-frame on $\mathbb{S}^{n-1}$, we obtain a cross section of the above bundle. Thus we have translated Question 5.2 in terms of existence of a certain continuous map.

Proposition 5.2. *If there is norm preserving bilinear map*

$$\mu : \mathbb{R}^k \times \mathbb{R}^n \longrightarrow \mathbb{R}^n,$$

then the span of $\mathbb{S}^{n-1} \geq k-1$. In particular, if $\mathbb{R}^n$ is a division algebra, then $\mathbb{S}^{n-1}$ is parallelizable.

Corollary 5.1. *If $\mathbb{R}^n$ is a division algebra, then $\mathbb{RP}^n$ is parallelizable.*

Example

The spheres $\mathbb{S}^1, \mathbb{S}^3$ and $\mathbb{S}^7$ are parallelizable. In fact

$$s(x, y) = (-y, x)$$

defines a 1-field on $\mathbb{S}^1$. To see parallelizability of $\mathbb{S}^3$, we exhibit explicitly an orthonormal 3-field s_1, s_2, s_3 as follows:

$$s_1(x, y, z, w) = (-y, x, -w, z),$$

[9]Moreover, the bundle $\pi : V_{k,n} \longrightarrow \mathbb{S}^{n-1}$ is a fibre bundle with fibre $V_{k-1,n-1}$.

$$s_2(x, y, z, w) = (-z, w, x, -y),$$

$$s_3(x, y, z, w) = (-w, -z, y, x).$$

Observe that identifying (x, y) with $x + \iota y \in \mathbb{C}$ the 1-frame on $\mathbb{S}^1$ is obtained by multiplication by i. Likewise the 3-frame is obtained by regarding elements of $\mathbb{S}^3$ as unit quaternions so that $s_1(q), s_2(q), s_3(q)$ are just iq, jq and kq respectively.

Chapter 6

Appendix

6.1 Recollections from Algebraic topology

Definition 6.1. *A mapping $\pi : E \longrightarrow B$ has the homotopy lifting property (HLP), with respect to the space Y if, given a homotopy $G : Y \times I \longrightarrow B$ and mapping $g : Y \times 0 \longrightarrow E$ such that $\pi g(y, 0) = G(y, 0)$, then there is homopoty lifting $\tilde{G}$: $Y \times I \longrightarrow E$ such that $\tilde{G}(y, 0) = g(y, 0)$ and $\pi\tilde{G} = G$.*

$$\begin{array}{ccc} Y \times 0 & \xrightarrow{g} & E \\ \downarrow & \nearrow^{\tilde{G}} & \downarrow \pi \\ Y \times I & \xrightarrow[G]{} & B \end{array}$$

*A mapping with the HLP with respect to all spaces is called a **Hurewicz fibration**. By a fibration we mean Hurewicz fibration.*

We can think of fibration as twisted cartesian product. Fibrations are generalizations of fibre bundles.

Definition 6.2. *The cone CX of a topological space X is the quotient space*

$$CX = (X \times I)/(X \times \{0\})$$

of the product of X with the unit interval $I = [0, 1]$.

Examples

1. The cone over a circle is the curved surface of the solid cone:

$$\{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 = z^2 \text{ and } 0 \leq z \leq 1\}.$$

   This in turn is homeomorphic to the closed disc.

2. In general, the cone over an n-sphere is homeomorphic to the closed $(n+1)$-ball.

Definition 6.3. *The join of two topological spaces X and Y, often denoted by $X \star Y$, is defined to be the quotient space*

$$X \star Y = (X \times Y \times I)/ \sim,$$

where I is the interval $[0, 1]$ and $\sim$ is the equivalence relation generated by

$$(x, y_1, 0) \sim (x, y_2, 0) \quad \text{for all } x \in X \text{ and } y_1, y_2 \in Y,$$
$$(x_1, y, 1) \sim (x_2, y, 1) \quad \text{for all } x_1, x_2 \in X \text{ and } y \in Y.$$

Example

The join of the spheres $\mathbb{S}^n$ and $\mathbb{S}^m$ is the sphere $\mathbb{S}^{n+m+1}$.

Lemma 6.1. *The join of $(n + 1)$ non-empty spaces is always $(n - 1)$-connected. In fact if each space X_i is $(c_i - 1)$-connected, then $X_0 \star \ldots \star X_n$ is $(c_0 + c_1 + \ldots + c_n + n - 1)$-connected.*

Definition 6.4. *The suspension* [1] *SX of a topological space X is the quotient space*

$$SX = (X \times I)/\{(x_1, 0) \sim (x_2, 0) \text{ and } (x_1, 1) \sim (x_2, 1) \text{ for all } x_1, x_2 \in X\}.$$

Definition 6.5. *If X is a pointed space (with basepoint x_0), then the reduced suspension or based suspension ΣX of X is the quotient space*

$$\Sigma X = (X \times I)/(X \times \{0\} \cup X \times \{1\} \cup \{x_0\} \times I).$$

[1] The space SX is sometimes called the unreduced suspension of X, to distinguish it from the reduced suspension.

Definition 6.6. *The smash product of two pointed spaces (i.e. topological spaces with distinguished basepoints, say x_0 and y_0) X and Y is the quotient of the product space $X \times Y$ under the identifications*

$$(x, y_0) \sim (x_0, y) \; \forall \, x \in X \text{ and } y \in Y.$$

Equivalently, the smash product is the quotient

$$X \wedge Y = (X \times Y)/(X \vee Y).$$

Examples

1. The smash product of any pointed space X with a 0-sphere is homeomorphic to X.

2. The smash product of two circles is a quotient of the torus homeomorphic to the 2-sphere.

3. The k-fold iterated reduced suspension of X is homeomorphic to the smash product of X and a k-sphere

$$\Sigma^k X \approx X \wedge \mathbb{S}^k.$$

Künneth formulas describe the homology or cohomology of a product space in terms of the homology or cohomology of the factors. In nice cases these formulas take the form

$$H_*(X \times Y; R) \approx H_*(X; R) \otimes H_*(Y; R)$$

or

$$H^*(X \times Y; R) \approx H^*(X; R) \otimes H^*(Y; R)$$

for a coefficient ring R. For the case of cohomology, such a formula with the hypotheses of finite generation and freeness on the cohomology of one factor, is given as follows:

Theorem 6.1 (Künneth formula)**.** *The cross product*

$$H^*(X; R) \otimes_R H^*(Y; R) \longrightarrow H^*(X \times Y; R)$$

is an isomorphism of rings if X and Y are CW complexes and $H^k(Y; R)$ is a finitely generated free R-modules for all k.

Lemma 6.2 (Five lemma)**.** *In a commutative diagram of abelian groups*

$$\begin{array}{ccccccccc} A & \xrightarrow{i} & B & \xrightarrow{j} & C & \xrightarrow{k} & D & \xrightarrow{l} & E \\ \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} & & \downarrow{\scriptstyle\delta} & & \downarrow{\scriptstyle\epsilon} \\ A' & \xrightarrow{i'} & B' & \xrightarrow{j'} & C' & \xrightarrow{k'} & D' & \xrightarrow{l'} & E' \end{array}$$

if the two rows are exact and α, β, δ and ϵ are isomorphisms, then γ is an isomorphism also.

Proposition 6.1. *The cohomology of $\mathbb{CP}^n$ is given as follows*

$$H^{2k}(\mathbb{CP}^n;\mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if} \quad 0 \le k \le n; \\ 0 & \text{if else.} \end{cases}$$

Proposition 6.2. *The cohomology ring of $\mathbb{CP}^n$ is given as*

$$H^*(\mathbb{CP}^n;\mathbb{Z}) = \mathbb{Z}[\alpha]/\langle \alpha^{n+1} \rangle$$

where α is a generator of $H^2(\mathbb{CP}^n;\mathbb{Z})$.

Theorem 6.2 (The Bott periodicity theorem for unitary groups)**.** *For $i \ge 2$,*

$$\pi_{i-2}(U) \approx \pi_i(U) = \begin{cases} 0 & if\ i \text{ is even} \\ \mathbb{Z} & if\ i \text{ is odd.} \end{cases}$$

Here U denotes the infinite unitary group. From the exact homotopy sequence (2.3), we have

$$\pi_{i-2}(BU) \approx \pi_i(BU) = \begin{cases} \mathbb{Z} & if\ i \text{ is even} \\ 0 & if\ i \text{ is odd.} \end{cases}$$

where BU is the classifying space of U.

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