Understanding Quantum Teleportation using NMR

Manpreet Kaler

A dissertation submitted for the partial fulfilment of BS-MS dual degree in Science

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Dedicated to my parents

Certificate of Examination

This is to certify that the dissertation titled **"Understanding Quantum Teleportation using NMR"** submitted by **Manpreet Kaler** (Reg. No. MS10108) for the partial fulfillment of BS-MS dual degree programme of the Institute, has been examined by the thesis committee duly appointed by the Institute. The committee finds the work done by the candidate satisfactory and recommends that the report be accepted.

Dr. K. S. Viswanathan   Dr. S. Rakshit   Dr. Ramesh Ramachandran

(Supervisor)

Dated: April 23, 2015

Declaration

The work presented in this dissertation has been carried out by me under the guidance of Dr. Ramesh Ramachandran at the Indian Institute of Science Education and Research Mohali.

This work has not been submitted in part or in full for a degree, a diploma, or a fellowship to any other university or institute. Whenever contributions of others are involved, every effort is made to indicate this clearly, with due acknowledgement of collaborative research and discussions. This thesis is a bonafide record of original work done by me and all sources listed within have been detailed in the bibliography.

Manpreet Kaler
(Candidate)

Dated: April 23, 2015

In my capacity as the supervisor of the candidate's project work, I certify that the above statements by the candidate are true to the best of my knowledge.

Dr. Ramesh Ramachandran
(Supervisor)

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Abstract

Teleportation has moved from the realms of science fictions to a scientific possibility. Quantum entanglement plays a key ingredient for this process and is an invaluable resource in the field of quantum communications. Apart from photons, the fundamental unit of quantum computation: the Qbit can be realized in a variety of ways such as atoms or nuclei, ion traps etc. Nuclear Magnetic Resonance utilizes the Zeeman Splitting of the degenerate energy levels of a nuclear spin, which are then employed as quantum bits. In my thesis, I have tried to understand a few possible interpretations of the Quantum Teleportation circuit and its implementation using NMR as a tool.

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Glossary of symbols

H_N	Hilbert space for system N
p_i	Probability
$\prod$	Projection Operator
ρ	Density Matrix
Tr	Trace
$CNOT$	Controlled- NOT gate
H	Hadamard Gate
X	Pauli's Matrix
Y	Pauli's Matrix
Z	Pauli's Matrix
M	Measurement
U	Unitary Transformation
$\otimes$	Tensor Product
Pr	Probability

Chapter 1

Introduction

1.1 Cbits and Qbits

In the theory of Information, the fundamental unit of information is expressed in terms of a 'bit' and is assigned the binary logic value 0 or 1. The physical realisation of a bit (also referred to as Cbit in classical computer science) can be understood in terms of the presence/absence of voltages in electronic circuits.The quantum analogue of the classical bit is referred to as Qbit (or quantum bit) and often exists in a superposition of the basis states of a classical bit. For e.g, the Qbit[1] can be defined as a vector in the two-dimensional vector space spanned by the states $\{|0\rangle, |1\rangle\}$,known as the computational basis. The general state of a qbit can be described as:-

$$|\psi\rangle = \alpha |0\rangle + \beta |1\rangle$$

where α and $\beta \in \mathbf{Z}$ are constrained by the normalization condition: $\|\psi\| = |\alpha|^2 + |\beta|^2 = 1$ The two-level system can be understood using Bloch Sphere.

In general the state of an n-qbit system is described by,

$$|\psi\rangle = \sum_{0 \leq x < 2^n} \alpha_x |x\rangle_n$$

where $|x\rangle_n$, denotes the superposition of the possible 2^n classical states with the condition,

$$\sum_{0 \leq x < 2^n} |\alpha_x|^2 = 1$$

The major properties that make qbits different are the existence of superposition states and the concept of entanglement.

1.2 Logic Gates

The basic element of Quantum Computation are the logic gates. Although the computational logic gates follow Boolean Algebra, the classical logic gates differ from quantum logic gates in a variety of ways. The major one being that the atomic computers shall represent the bit values using the quantum state of atomic system and hence, a logic gate can neither create nor destroy bits. So, the possibility of gates such as AND are ruled out.
There are two peculiar properties which contribute to the uniqueness of the Quantum logic gates[1]-

1. Reversibility:-
Reversible operations transform the initial state of the qbit to final state using only those processes whose action can be inverted. For any operator $\hat{A}$,reversibility is defined as follows:

$$\hat{A} |\psi_i\rangle \to |\psi_f\rangle$$

$$\hat{A} |\psi_f\rangle \to |\psi_i\rangle$$

2. Unitarity:-
The reversible operations which shall transform a state, are constrained by the condition that this operation must map a state from one orthonormal basis to another and both the vectors be of a unit magnitude. Such transformations are called *unitary* and satisfy the following condition:-

$$UU^\dagger = U^\dagger U = 1$$

Consider an example where the clock ticks. The hands of the clock rotate but the length remains the same. In a similar fashion, the unitary operators preserve the norm. In quantum theory, preserving magnitudes refers to keeping the same overall probabilities and hence displaying similar physical properties. Otherwise, physically difficult to justify phenomenon may occur.
Some basic logic gates[1] widely used in implementing quantum circuits are:-

1. **The NOT gate**

This operator holds for a single qbit i.e. it is a single qbit gate.

$$X : |x\rangle \to |\bar{x}\rangle$$

$$X |0\rangle \to |1\rangle$$

$$X|1\rangle \rightarrow |0\rangle$$

The matrix form of this gate is written as:-

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

2. **The Walsh-Hadamard Gate:-**

This is again a one-qbit gate whose uniqueness lies in the fact that it transforms a one qbit pure state to the superposition of the possibilities of the states.

$$H|0\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$

$$H|1\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$$

The matrix form for the computational basis $\{|0\rangle, |1\rangle\}$ can be written as:-

$$H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

3. **The SWAP gate:-**

It essentially swaps the state of a two qbit system.

$$S|xy\rangle \rightarrow |yx\rangle$$

$$|00\rangle \rightarrow |00\rangle$$

$$|01\rangle \rightarrow |10\rangle$$

$$|10\rangle \rightarrow |01\rangle$$

$$|11\rangle \rightarrow |11\rangle$$

It is written in matrix form as:-

$$S = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

4. **The CNOT gate:-**

This is the Controlled-Not gate. This is also a two qbit gate. Its action is to perform the NOT operation on the target bit provided the control bit is already defined.

$$CNOT_{AB}\,|x\rangle\,|y\rangle = |x\rangle\,|y \oplus x\rangle$$

where $\oplus$ defines the addition modulo two function. Its matrix form is written as:-

$$CNOT_{AB} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$C\,|00\rangle \rightarrow |0\rangle\,|0 \oplus 1\rangle \rightarrow |00\rangle$$

$$C\,|01\rangle \rightarrow |0\rangle\,|1 \oplus 0\rangle \rightarrow |01\rangle$$

$$C\,|10\rangle \rightarrow |1\rangle\,|0 \oplus 1\rangle \rightarrow |11\rangle$$

$$C\,|11\rangle \rightarrow |1\rangle\,|1 \oplus 1\rangle \rightarrow |10\rangle$$

If one chooses the second bit to be the control and the first one as the target, the matrix representation changes,

$$CNOT_{BA}\,|x\rangle\,|y\rangle = |x \oplus y\rangle\,|y\rangle$$

and hence,

$$CNOT_{BA} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

The representation of a gate changes even when one moves from one basis set to another. An arbitrary basis can be used to produce an orthonormal basis by application of the *Gram-Schmidt Orthogonalization process.*

1.3 The Born's Rule

The general state of a qbit is given by

$$|\psi\rangle = \sum_{0 \leq x < 2^n} \alpha_x |x\rangle_n$$

The only irreversible operation which is applied to the qbits is the *Measurment*.Any information about a state can be gathered only by making a few measurements but the very act of measurement is believed to disturb the system itself and its originality is hence lost. One cannot realize the coefficients or the respective amplitudes of any of these possible states. The measurement gates project the state of the system to the computational basis i.e any measurement will yield either a zero or one. This is also called as *wave-function collapse*. The Born's rule establishes a link between the amplitudes and the results of any measurement. It states that the probability that the zeroes and ones resulting from measurements will give the binary expansion of the integer x is given by[1]

$$p(x) = |\alpha_x|^2$$

In a simpler language, this is the probability that a given system shall be projected onto one of the possible states.

1.4 The EPR Paradox and Bell's Inequalities

It is no secret that Quantum Mechanics is plagued with conceptual difficulties. The famous 1935 EPR paper instigated a debate which continues till date.

> *If without in any way disturbing a system, we can predict with certainty the value of a physical quantity, then there exists an element of physical reality corresponding to this physical quantity.*

EPR[2] claimed the Quantum Theory to be incomplete as it did not account for such supposedly existing real properties, later called as *"hidden variables"*.Niels Bohr rebuked Einstein and replied to the EPR paper by claiming that " No physical force could connect the observations at two ends; The polarization observed on one end 'influences' what would be seen by the observer at the other end."
Einstein rejected these so called 'influences' as 'Spooky actions at a distance'.
According to the Copenhagen Interpretation[3] of Quantum Mechanics states that:-

> *"Observations not only disturb what is to be measured, but they produce it."*

This debate was answered thirty years later by John Stewart Bell. His theory asserts[4] that the objects that have ever interacted, forever do influence each other instantaneously. These influences become undetectable after either object has interacted significantly with the environment.

1.4.1 The Bell's Theorem

The Bell's Theorem[5] is based on the following two assumptions:-

- Objects(unobserved) have physically real properties that are not created by their observation i.e it is meaningful to assign a property to a system, independent of whether the measurement of this property is carried out or not. This forms the basic definition of *Realism* or a *counter-factual definite theory*.

- There exists *separability* or *non-locality* in accord with Einstein's special theory of relativity. Two objects can be separated from each other so that what happens to one cannot instantaneously affect the other. The outcome of an experiment on a system are independent of the actions performed on a different system that has no causal connection with the first.

To explain the EPR problem, Bell proposed an inequality (often referred to as the Bell's Inequality) that is consistent with the concept of locality and counter-factual definiteness. But, it incidentally turns out that Quantum Mechanics violates these Bell's Inequalities and hence one of the conditions or both need to be false.

Consider a system[6] comprising of two identical coins. Consider three properties A,B and C that can be measured on these systems and each of these properties can assume two possibilities which can be denoted as 0 and 1.

Now, suppose a measurement of any two properties is made on both the systems. The probability of getting the same result $P_{same}(A,B)$ needs to be taken into account where A is the first coin and B is the second. This means that either one gets 0 or 1 for both the measurements. There are four total possibilities when a measurement shall be made:$(0_A, 0_B), (0_A, 1_B), (1_A, 0_B), (1_A, 1_B)$. Hence,

$$P_{same}(A,B) = \frac{1}{2}$$

Similarly,

$$P_{same}(B,C) = P_{same}(A,C) = \frac{1}{2}$$

Also, since the particles are identical in nature

$$P_{same}(A,A) = P_{same}(B,B) = P_{same}(C,C) = 1$$

Under these defined conditions, Bell's Inequality is defined as:-

$$P_{same}(A,B) + P_{same}(B,C) + P_{same}(A,C) \geq 1$$

The implication of this relation is that if a property is measured on one of the systems, then one can predict the value of the same property on the other system irrespective of the measurements on the other system.
The word 'identical' may be considered as the classical analogue of 'entanglement'. Now, consider a quantum mechanical system. Let there be a two-qbit system $|\phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$.Let the two-valued properties A, B and C be defined by the eigen-kets:-

$$A : \begin{cases} |a_o\rangle = |0\rangle \\ |a_1\rangle = |1\rangle \end{cases}$$

$$B : \begin{cases} |b_o\rangle = \frac{1}{2}|0\rangle + \frac{\sqrt{3}}{2}|1\rangle \\ |b_1\rangle = \frac{\sqrt{3}}{2}|0\rangle - \frac{1}{2}|1\rangle \end{cases}$$

$$C : \begin{cases} |c_o\rangle = \frac{1}{2}|0\rangle - \frac{\sqrt{3}}{2}|1\rangle \\ |c_1\rangle = \frac{\sqrt{3}}{2}|0\rangle + \frac{1}{2}|1\rangle \end{cases}$$

It is interesting to note that the entangled states could be expressed in terms of the eigen-kets of the properties that are to be measured.

$$|\phi^+\rangle = \frac{1}{\sqrt{2}}(|a_o a_o\rangle + |a_1 a_1\rangle)$$

$$|\phi^+\rangle = \frac{1}{\sqrt{2}}(|b_o b_o\rangle + |b_1 b_1\rangle)$$

$$|\phi^+\rangle = \frac{1}{\sqrt{2}}(|c_o c_o\rangle + |c_1 c_1\rangle)$$

Now,

$$|\phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) = \frac{1}{2\sqrt{2}}|a_0\rangle(|b_o\rangle + \sqrt{3}|b_1\rangle) + \frac{1}{2\sqrt{2}}|a_1\rangle(\sqrt{3}|b_o\rangle - |b_1\rangle)$$

$$= \frac{1}{2\sqrt{2}}|a_0\rangle|a_o\rangle + \frac{1}{2\sqrt{2}}|a_1\rangle|a_1\rangle$$

The joint probability of obtaining the same esult for A and B i.e $|a_0\rangle|b_o\rangle + |a_1\rangle|b_1\rangle$ is given as:-

$$P_{same}(A,B) = (\frac{1}{2\sqrt{2}})^2 + (\frac{1}{2\sqrt{2}})^2 = \frac{1}{4}$$

An alternate way in which the joint probabilities can be calculated is:-

$$P_{same}(A,B) = \left|\langle a_o b_o|\ \phi^+\rangle\right|^2 + \left|\langle a_1 b_1|\ \phi^+\rangle\right|^2$$

$$= \left|\frac{1}{2\sqrt{2}}\right|^2 + \left|\frac{1}{2\sqrt{2}}\right|^2 = \frac{1}{4}$$

So,

$$P_{same}(A,B) + P_{same}(B,C) + P_{same}(A,C) = \frac{3}{4}$$

$$P_{same}(A,B) + P_{same}(B,C) + P_{same}(A,C) < 1$$

Hence, the Bell's Inequality stands violated.

1.5 Spin Quantization

1.5.1 Spin Polarization and Correlations

The term polarization can be associated with alignment or order. Each spin is considered to be quantized along an arbitrary axis prior to any external perturbation.

Perfect Correlations in the two-particle state

Let

$|\hat{n},+\rangle$ and $|\hat{n},-\rangle$ represent the two states of a two-level system quantized along an arbitrary axis $\hat{n}$.

The states can be derived using the following equations:

$$\sigma.n\,|\hat{n},+\rangle = |\hat{n},+\rangle$$

$$\sigma.n\,|\hat{n},-\rangle = -\,|\hat{n},-\rangle$$

Derivation:

The Pauli's matrices are written as:

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

The polar coordinates are written as:

$$z = \cos\theta$$

$$x = \sin\theta \sin\varphi$$

$$y = \sin\theta \cos\varphi$$

Also,

$$\sigma.n = \begin{pmatrix} \cos\theta & -i\sin\theta e^{i\varphi} \\ i\sin\theta e^{-i\varphi} & -\cos\theta \end{pmatrix}$$

The next step is to find the eigen-values for this operator:

$$\det|A - \lambda I| = 0$$

$$\begin{vmatrix} \cos\theta - \lambda & -i\sin\theta e^{i\varphi} \\ i\sin\theta e^{-i\varphi} & -\cos\theta - \lambda \end{vmatrix} = 0$$

$$-(\cos^2\theta - \lambda^2) - \sin^2\theta = 0$$

This gives,

$$\lambda^2 - 1 = 0$$

Hence,

$$\lambda = \pm 1$$

further, one needs to deduce the representation for $|\hat{n}, +\rangle$ and $|\hat{n}, -\rangle$ i.e find the corresponding eigen-vectors.
For $\lambda = 1$,

$$\begin{pmatrix} \cos\theta & -i\sin\theta e^{i\varphi} \\ i\sin\theta e^{-i\varphi} & -\cos\theta \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$$

Therefore,

$$\frac{a}{b} = \frac{1}{i}\cot\frac{\theta}{2}e^{i\varphi}$$

From the normalization condition, it follows that $|a|^2 + |b|^2 = 1$ and hence

$$b = \pm\sin\frac{\theta}{2}$$

Confining the possibilities to only the positive values of b,

$$a = -i\cos\frac{\theta}{2}e^{i\varphi}$$

or

$$a = \cos\frac{\theta}{2}e^{i(\varphi-\frac{\pi}{2})}$$

.

Similarly, for $\lambda = -1$,

$$\begin{pmatrix} \cos\theta & -i\sin\theta e^{i\varphi} \\ i\sin\theta e^{-i\varphi} & -\cos\theta \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = - \begin{pmatrix} a \\ b \end{pmatrix}$$

Therefore,

$$\frac{a}{b} = i\tan\frac{\theta}{2}e^{i\varphi}$$

Hence, $b = \pm\cos\frac{\theta}{2}$. Confining again the possibilities to only the positive values,

$$a = i\sin\frac{\theta}{2}e^{i\varphi}$$

or

$$a = \sin\frac{\theta}{2}e^{i(\varphi+\frac{\pi}{2})}$$

Following this description, one may write:

$$|\hat{n},+\rangle = \cos\frac{\theta}{2}e^{i(\varphi-\frac{\pi}{2})}\,|+\rangle + \sin\frac{\theta}{2}\,|-\rangle$$

$$|\hat{n},-\rangle = \sin\frac{\theta}{2}e^{i(\varphi+\frac{\pi}{2})}\,|+\rangle + \cos\frac{\theta}{2}\,|-\rangle$$

Employing these relations, the entangled state polarized along an arbitrary direction is represented by:-

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|n,+\rangle_1|n,-\rangle_2 - |n,-\rangle_1|n,+\rangle_2)$$

$$= \begin{pmatrix} -\cos\frac{\theta_1}{2}\sin\frac{\theta_2}{2} \\ \cos\frac{\theta_1}{2}\cos\frac{\theta_2}{2}e^{i\varphi_2} \\ -\sin\frac{\theta_1}{2}\sin\frac{\theta_2}{2}e^{i\varphi_1} \\ \sin\frac{\theta_1}{2}\cos\frac{\theta_2}{2}e^{i\varphi_1}e^{i\varphi_2} \end{pmatrix} - \begin{pmatrix} -\sin\frac{\theta_1}{2}\cos\frac{\theta_2}{2} \\ -\sin\frac{\theta_1}{2}\sin\frac{\theta_2}{2}e^{i\varphi_2} \\ \cos\frac{\theta_1}{2}\cos\frac{\theta_2}{2}e^{i\varphi_1} \\ \cos\frac{\theta_1}{2}\sin\frac{\theta_2}{2}e^{i\varphi_1}e^{i\varphi_2} \end{pmatrix}$$

So, for $\theta_1 = \theta_2$ i.e both the spins are quantized along the same axis,

$$|\psi(\hat{n})\rangle = |\psi\rangle = \frac{1}{\sqrt{2}}(|+\rangle_1|-\rangle_2 - |-\rangle_1|+\rangle_2)$$

This implies that the entangled(here singlet) state for a two-spin system is independent of the axis of quantization.

1.5.2 Statistical Correlations in two-particle state

Consider the general representation of the singlet entangled state to start with:

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|n,+\rangle_1|n,-\rangle_2 - |n,-\rangle_1|n,+\rangle_2)$$

Here, both the spins are quantized along the same axis. Another axis $\hat{n}_2$ can be chosen such that the polar angle of $\hat{n}_1$ w.r.t $\hat{n}_2$ is θ. Let the second spin be quantized along this new axis $\hat{n}_2$. Now,

$$|n_1,+\rangle_2 = \cos\frac{\theta}{2}|n_2,+\rangle_2 + \sin\frac{\theta}{2}|n_2,-\rangle_2$$

$$|n_1,-\rangle_2 = -\sin\frac{\theta}{2}|n_1,-\rangle_2 + \cos\frac{\theta}{2}|n_2,-\rangle_2$$

Also, $|\psi\rangle$ can be written in the new frame of reference as:-

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|n_1,+\rangle_1(-\sin\frac{\theta}{2}|n_1,-\rangle_2 + \cos\frac{\theta}{2}|n_2,-\rangle_2))$$

$$-\frac{1}{\sqrt{2}}(|n_1,-\rangle_1(\cos\frac{\theta}{2}|n_2,+\rangle_2 + \sin\frac{\theta}{2}|n_2,-\rangle_2))$$

The amplitude for the joint outcome corresponding to each of the four possibilities can be given as following:

$$P(|+\rangle_1|+\rangle_2) = \frac{1}{2}\sin^2\frac{\theta}{2}$$

$$P(|+\rangle_1|-\rangle_2) = \frac{1}{2}\cos^2\frac{\theta}{2}$$

$$P(|-\rangle_1|+\rangle_2) = \frac{1}{2}\cos^2\frac{\theta}{2}$$

$$P(|-\rangle_1|-\rangle_2) = \frac{1}{2}\sin^2\frac{\theta}{2}$$

This result can also be obtained using an alternate method, described below:

Assume that one of the spins is quantized along the z-axis.Let the second spin be quantized along a different axis and its polar angle with the axis of first spin is θ. Again starting with the generalized form of an entangled(singlet) state,

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|+\rangle_1|-\rangle_2 - |-\rangle_1|+\rangle_2)$$

The generalized form of the state for spin-2 is represented by:

$$|n,+\rangle_2 = \begin{pmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2}e^{i\varphi} \end{pmatrix}, |n,-\rangle = \begin{pmatrix} -\sin\frac{\theta}{2} \\ \cos\frac{\theta}{2}e^{i\varphi} \end{pmatrix}$$

So, the joint probabilities can be calculated as follows:- The probability of getting the $|n,+\rangle_2$ state for spin-2 when spin-1 is measured as $|+\rangle_1$ is given by:

$$P(|+\rangle_1 |n,+\rangle_2) = |\langle n,+|_2 \, |\psi\rangle|^2 = \frac{1}{2}\text{sin}^2\frac{\theta}{2}$$

EPR correlations imply that if spin-1 is measured to be $|+\rangle_1$, spin-2 is automatically forced or 'influenced' to be in the $|-\rangle_2$

$$P(|n,+\rangle_2) = |\langle n,+| \, \beta\rangle|^2 = \left|(\cos\frac{\theta}{2}\,\langle\alpha| + \sin\frac{\theta}{2}e^{i\varphi}\,\langle\beta|)\,|\beta\rangle\right|^2$$

Also, the amplitude is $\sin\frac{\theta}{2}e^{i\varphi}$ and,

$$Probability = |Amplitude|^2 = \frac{1}{2}(\sin\frac{\theta}{2}e^{i\varphi}.\sin\frac{\theta}{2}e^{-i\varphi})$$

$$\frac{1}{2}\text{sin}^2\frac{\theta}{2}$$

On similar grounds, the probability of obtaining spin-1 in $|-\rangle_1$ is 0.5. Once this happens, the state of spin-2 is $|+\rangle_1$. As explained earlier, the probability is calculated as:

$$P(|n,+\rangle_2) = |\langle n,+| \, \alpha\rangle|^2 = \left|(\cos\frac{\theta}{2}\,\langle\alpha| + \sin\frac{\theta}{2}e^{i\varphi}\,\langle\beta|)\,|\alpha\rangle\right|^2$$

So,

$$Probability = \left|\cos\frac{\theta}{2}\right|^2 = \cos^2\frac{\theta}{2}$$

1.5.3 Magnetization of spins at Thermal Equlibrium

Each spin can be considered possess its own arbitrary quantization axis prior to any external perturbation.Hence, it shall have its own state represented by a ket or bra.

$$\overrightarrow{S} = S_x\hat{i} + S_y\hat{j} + S_z\hat{k}$$

$$\overrightarrow{n} = \sin\theta\cos\phi\hat{i} + \sin\theta\sin\phi\hat{j} + \cos\theta\hat{k}$$

$$\overrightarrow{S}.\overrightarrow{n} = S_x n_x + S_y n_y + S_z n_z$$

Therefore,

$$\overrightarrow{S}.\overrightarrow{n} = \frac{\hbar}{2}\begin{pmatrix} \cos\theta & \sin\theta e^{-i\phi} \\ \sin\theta e^{i\phi} & -\cos\theta \end{pmatrix}$$

Also,

$$|n,+\rangle = \begin{pmatrix} \cos\frac{\theta}{2} \\ e^{i\phi}\sin\frac{\theta}{2} \end{pmatrix}, |n,-\rangle = \begin{pmatrix} -\sin\frac{\theta}{2} \\ e^{i\phi}\cos\frac{\theta}{2} \end{pmatrix}$$

With respect to this arbitrary axis, the above states result in $\frac{\hbar}{2}$ and $-\frac{\hbar}{2}$ as the eigen-values.

$$S.n\,|n,+\rangle = \frac{\hbar}{2}\,|n,+\rangle$$

$$S.n\,|n,-\rangle = -\frac{\hbar}{2}\,|n,-\rangle$$

This implies that the spins are also quantized along there own axis. Now,

$$|\psi\rangle = \begin{pmatrix} \cos\frac{\theta}{2} \\ e^{i\phi}\sin\frac{\theta}{2} \end{pmatrix}$$

The net magnetization along the z-axis is given as,

$$\langle I_z\rangle = \langle\psi \mid I_z\,|\psi\rangle \tag{1.1}$$

$$= \begin{pmatrix} \cos\frac{\theta}{2} & e^{i\phi}\sin\frac{\theta}{2} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \cos\frac{\theta}{2} \\ e^{i\phi}\sin\frac{\theta}{2} \end{pmatrix} \tag{1.2}$$

$$= \cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2} \tag{1.3}$$

$$= \cos\theta \tag{1.4}$$

Taking an average over all the possible orientations,

$$\int_0^\pi \sin\theta d\theta.\cos\theta = \frac{1}{2}\int_0^\pi \sin 2\theta d\theta = 0$$

Hence, the expectation value along the z-direction is zero. The same is true for the x and y axes. Under the influence of applied magnetic field, the spins tend to align themselves along the direction of this applied field. The spins shall then be quantized along this new direction.

Case-1: The vector is quantized along the x-axis:

$$|n,+\rangle_x = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}, |n,-\rangle_x = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$S_x|n,+\rangle_x = \frac{\hbar}{2\sqrt{2}}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$
$$= \frac{\hbar}{2\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$S_x|n,-\rangle_x = \frac{\hbar}{2\sqrt{2}}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} -1 \\ 1 \end{pmatrix}$$
$$= -\frac{\hbar}{2\sqrt{2}}\begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

Case-2: The vector is quantized along the y-axis:

$$|n,+\rangle_y = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix}, |n,-\rangle_y = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ i \end{pmatrix}$$

$$S_y|n,+\rangle_y = \frac{\hbar}{2\sqrt{2}}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 1 \\ i \end{pmatrix}$$
$$= \frac{\hbar}{2\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix}$$

$$S_y|n,-\rangle_y = \frac{\hbar}{2\sqrt{2}}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} -1 \\ i \end{pmatrix}$$
$$= \frac{-\hbar}{2\sqrt{2}}\begin{pmatrix} -1 \\ i \end{pmatrix}$$

Case-3: If the vector is quantized along the z-axis:

$$|n,+\rangle_z = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, |n,-\rangle_z = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$S_z|n,+\rangle_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$
$$= \frac{\hbar}{2}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$S_z|n,-\rangle_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$
$$= -\frac{\hbar}{2}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

These show that irrespective of the quantization axis, the eigen-values remain the same.

1.6 Singlet State

The singlet state can be described as:

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|n,+\rangle_1|n,-\rangle_2 - |n,-\rangle_1|n,+\rangle_2)$$

Now,

$$|n,+\rangle = \cos\frac{\theta}{2}|\alpha\rangle + e^{i\phi}\sin\frac{\theta}{2}|\beta\rangle$$
$$|n,-\rangle = -\sin\frac{\theta}{2}|\alpha\rangle + e^{i\phi}\cos\frac{\theta}{2}|\beta\rangle$$

So,

$$|\psi\rangle = \frac{1}{\sqrt{2}}(\sin\frac{\theta_1}{2}\cos\frac{\theta_2}{2} - \cos\frac{\theta_1}{2}\sin\frac{\theta_2}{2})|\alpha\rangle_1|\alpha\rangle_2 + \frac{1}{\sqrt{2}}e^{i\phi_2}(\cos\frac{\theta_1}{2}\cos\frac{\theta_2}{2} + \sin\frac{\theta_1}{2}\sin\frac{\theta_2}{2})|\alpha\rangle_1|\beta\rangle_2$$
$$-\frac{1}{\sqrt{2}}e^{i\phi_1}(\sin\frac{\theta_1}{2}\sin\frac{\theta_2}{2} + \cos\frac{\theta_1}{2}\cos\frac{\theta_2}{2})|\beta\rangle_1|\alpha\rangle_2 + \frac{1}{\sqrt{2}}e^{i(\phi_1+\phi_2)}(\sin\frac{\theta}{2}\cos\frac{\theta}{2} - \cos\frac{\theta}{2}\sin\frac{\theta}{2})|\beta\rangle_1|\beta\rangle_2$$

If both the spins are quantized along x-axis,

$$|\alpha\rangle_x = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}, |\beta\rangle_x = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$|\psi\rangle_x = \frac{1}{\sqrt{2}}\left\{\frac{1}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix}\otimes\begin{pmatrix} -1 \\ 1 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} -1 \\ 1 \end{pmatrix}\otimes\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right\}$$

Hence,

$$|\psi\rangle_x = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}$$

If both the spins are quantized along z-axis:

$$|\alpha\rangle_z = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, |\beta\rangle_z = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$|\psi\rangle_z = \left\{\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \end{pmatrix}\otimes\begin{pmatrix} 0 \\ 1 \end{pmatrix} - \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \end{pmatrix}\otimes\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right\}$$

$$|\psi\rangle_z = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}$$

Therefore, irrespective of the quantization axis, the representation of the singlet state remains invariant.

Bell's Inequalities

Consider an experiment[7] where the singlet state is represented as:-

$$|\psi\rangle = \frac{1}{\sqrt{2}}|\alpha\rangle_1|\beta\rangle_2 - \frac{1}{\sqrt{2}}|\beta\rangle_1|\alpha\rangle_2$$

The *Local Hidden Variables Theory* makes following two assumptions:-
1. The particles have a definite spin.
2. The particles were previously entangled.
If detector 1 measures a particle of type (x+, y-, z-) then detector 2 should measure a particle of type (x-, y+, z+) in order to conserve the total angular momentum.

Population	**Partcle1**	**Particle2**
N_1	$(\alpha\alpha\alpha)$	$(\beta\beta\beta)$
N_2	$(\alpha\alpha\beta)$	$(\beta\beta\alpha)$
N_3	$(\alpha\beta\alpha)$	$(\beta\alpha\beta)$
N_4	$(\alpha\beta\beta)$	$(\beta\alpha\alpha)$
N_5	$(\beta\alpha\alpha)$	$(\alpha\beta\beta)$
N_6	$(\beta\alpha\beta)$	$(\alpha\beta\alpha)$
N_7	$(\beta\beta\alpha)$	$(\alpha\alpha\beta)$
N_8	$(\beta\beta\beta)$	$(\alpha\alpha\alpha)$

Table 1.1: Populations corresponding to respective polarizations

$$N_3 + N_4 \leq (N_2 + N_4) + (N_3 + N_7)$$

If the detector 1 measures $(S_1.x)$ to be + and the detector 2 measures $(S_2.y)$ to be +, then the probability $(S_1.x = +, S_2.y = +)$ can be given as :

$$P(x+, y+) = \frac{N_3 + N_4}{\sum_{i=1}^{8} N_i}$$

Therefore,

$$P(x+, y+) \leq P(x+, z+) + P(z+, y+)$$

Hence, the Hidden variables theory satisfy the Bell's Inequality.
Now, moving on to the Orthodox Interpretation of Quantum Mechanics, again the following two important assumptions play a crucial role:-
1. Each particle emitted from a source has no definite spin.
2. Each particle is in a superposition of states.
The quantum mechanical projection operator can be written as:

$$\prod(a\pm) = \frac{1}{2}(1 \pm \sigma.a)$$

The quantum mechanical probability of finding particle 1 in the + state (along x , y or z) can be given by $\frac{1}{2} Tr \prod(x) = \frac{1}{2}$
Thus, the joint probability can be given as:

$$P(x+, y+) = \frac{1}{2} Tr\{\prod(x)\prod(y)\} = \frac{1}{4}(1 + x.y)$$

If for simplicity, x , y and z are chosen such that $S_x.S_z = S_y.S_z = \cos 2\alpha$ and $S_x.S_y = \cos 4\alpha$,then, for some cases the following condition crops up:

$$\sin^2 2\alpha \leq 2\sin^2\alpha$$

This is not true in general.
Hence, Quantum Mechanics violates Bell's Inequalities. This leads to an unavoidable consequence as opposed to the Einstein's special theory of relativity, that Quantum Mechanics is *'non-local'*.

Chapter 2

Quantum Information Processing

2.1 The Density Matrix

It is impractical in the real world to think of a system where a spin can be completely isolated from the other spins or the environment in general. These systems generally exist in an ensemble of mixed states. The wave-function of such systems is described using density matrices. The density operator for a pure state $|\psi\rangle$ is defined as:-

$$\rho = |\psi\rangle \langle\psi|$$

The trace of a density operator is always 1 due to the conservation of probabilities.

$$Tr(\rho) = \sum_j \langle u_j| \rho |u_j\rangle = \sum_j \langle u_j| \psi\rangle \langle\psi| u_j\rangle$$

$$= \sum_j c_j {c_j}^* = \sum_j |c_j|^2 = 1$$

Also, the density operator is Hermitian. i.e $\rho = \rho^\dagger$. If a system is in pure state, then

$$Tr(\rho^2) = 1$$

while this condition stands violated for the mixed states.
For a state $|\psi\rangle$ defined in an n-dimensional basis vector space, let the probability that a member of the ensemble exist in the state $|\psi_i\rangle$ be p_i. The density operator for the entire system(mixed state) is given as,

$$\rho = \sum_{i=1}^{n} p_i \rho_i = \sum_{i=1}^{n} p_i |\psi_i\rangle \langle\psi_i|$$

The Schrödinger Equation describes the time-evolution of a state under an applied Hamiltonian:-

$$i\hbar\frac{\partial\psi}{\partial t} = \hat{H}\psi$$

The Liouville-von Neumann equation is used to describe how a density operator evolves in time.

$$i\hbar\frac{\partial\rho}{\partial t} = [H, \rho]$$

where,

$$\rho(t) = e^{-iHt/\hbar}\rho(0)e^{iHt/\hbar}$$

$$\rho(t) = U\rho(0)U^{\dagger}$$

where, $U = e^{-iHt/\hbar}$ The evolution of an ensemble for under applied logic gate is given in this way, where the Hamiltonian is the logic gate itself.

2.1.1 The Reduced Density Matrix

These come into play when one thinks of the constituent subsystems of a composite system, where one system is held by Bob while the other by Alice, for e.g the entangled state. The reduced density matrices are the distilled form of a system or density matrix of the system from the perspective of only one of the observers, i.e either Alice or Bob in this case.

Consider an example where Alice and Bob previously share an entangled pair of states and then each one of them flies off to a distant location.

$$\rho_{AB} = \left|\phi_{AB}{}^{-}\right\rangle \left\langle\phi_{AB}{}^{-}\right|$$

$$= \frac{1}{\sqrt{2}}(|0\rangle_A|0\rangle_B - |1\rangle_A|1\rangle_B)\frac{1}{\sqrt{2}}(\langle 0|_A\langle 0|_B - \langle 1|_A\langle 1|_B)$$

$$= \frac{1}{2}(|0\rangle_A|0\rangle_B\langle 0|_A\langle 0|_B - |0\rangle_A|0\rangle_B\langle 1|_A\langle 1|_B - |1\rangle_A|1\rangle_B\langle 0|_A\langle 0|_B + |1\rangle_A|1\rangle_B\langle 1|_A\langle 1|_B)$$

The one point perspective can be observed by computing the[8] trace by summing over the basis states of one observer alone. If one looks at the system from Bob's frame of reference,then

$$\rho_B = Tr_A(\rho) = Tr_A(\left|\phi_{AB}{}^{-}\right\rangle \left\langle\phi_{AB}{}^{-}\right|)$$

$$= \langle 0_A| \left(\left|\phi_{AB}{}^{-}\right\rangle \left\langle\phi_{AB}{}^{-}\right|\right) |0_A\rangle + \langle 1_A| \left(\left|\phi_{AB}{}^{-}\right\rangle \left\langle\phi_{AB}{}^{-}\right|\right) |1_A\rangle$$

Now,

$$= \frac{|0\rangle_B\langle 0|_B}{2} + \frac{|1\rangle_B\langle 1|_B}{2}$$

So, the density matrix for Bob is,

$$\rho_B = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

The subscripts can be dropped as this state now belongs only to Bob. Also,

$${\rho_B}^2 = \frac{I^2}{4}$$

$$Tr({\rho_B}^2) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2} < 1$$

Hence, a completely mixed state lies in the possession of Bob. In this particular case, Alice shall also hold the same state in her hand.

2.2 Entanglement

Einstein's 'spooky actions' and Bohr's 'influences' are what we today call as Quantum Entanglement. The phenomenon of Entanglement is an indispensable resource in the realms of Quantum Communications. Entanglement is the name given to the correlations which exist among two systems which had ever interacted in the past irrespective of the spatial difference. There are certain claims that the entire Universe can be explained using a single wave-function and hence such correlations exist but there is no substantial proof behind such bizarre theories. Entanglement is considered to be an example of superluminal communications i.e transfer of information at a pace greater than the speed of light.

Consider the systems A and B described such that $|\psi\rangle \in H_A$ and $|\phi\rangle \in H_B$; thus, $|\psi\rangle \otimes |\phi\rangle \in H_A \otimes H_B$. If the state of the system is described as:-

$$|\psi\rangle \otimes |\phi\rangle = \begin{pmatrix} m \\ n \\ o \\ p \end{pmatrix}$$

then the systems are separable only if $mn \neq op$ The EPR states are well known examples of the entangled systems.

$$|\phi\rangle = \frac{|1\rangle|0\rangle \pm |0\rangle|1\rangle}{\sqrt{2}}$$

$$|\psi\rangle = \frac{|0\rangle|0\rangle \pm |1\rangle|1\rangle}{\sqrt{2}}$$

Entanglement Fidelity

The amount of statistical[8] overlap between two systems, known as *fidelity* is given by,

$$F(\rho, \sigma) = Tr\left(\sqrt{\sqrt{\rho}\sigma\sqrt{\rho}}\right)$$

where, ρ and σ are the corresponding density matrices.
Fidelity is a number such that,

$$0 \leq F(\rho, \sigma) \leq 1$$

Since, no two arbitrary states can be exact equals w.r.t the no cloning theorem, fidelity measure is required to know the amount of correspondence among the two systems.

Quantifying Entanglement

To create entanglement, there is an immediate need to measure the enatnglement and calculate the *'entanglement of formation'*. Concurrence is a means to characterize the amount by which a system is entangled.The enatnglement needs to be quantified. This can be done as follows:-
Concurrence is defined as:

$$C(|\psi\rangle) = \left|\left\langle \psi \,\middle|\, \tilde{\psi} \right\rangle\right|$$

where,

$$\left|\tilde{\psi}\right\rangle = Y \otimes Y \left|\psi^*\right\rangle$$

and Y is the Pauli's matrix:-

$$Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

An alternate way to look at concurrence is by evaluating the eigenvalues$\lambda_1, \lambda_2, \lambda_3.......\lambda_n$ of the matrix defined by, :-

$$M = \rho(Y \otimes Y)\rho^{\dagger}(Y \otimes Y)$$

or,

$$M = \sqrt{\sqrt{\rho}\tilde{\rho}\sqrt{\rho}}$$

The concurrence[8] is then defined as,

$$C(\rho) = \max\{0, \lambda_1 - \lambda_2 - \lambda_3...\}$$

The *entanglement of formation* is given by,

$$E(\rho) = h\left(\frac{1+\sqrt{1-C(\rho)^2}}{2}\right)$$

Consider the singlet state,

$$|\psi\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle|1\rangle - |1\rangle|0\rangle\right)$$

$$\rho = \frac{1}{2}\left(|0\rangle|1\rangle\langle 0|\langle 1| - |0\rangle|1\rangle\langle 1|\langle 0| - |1\rangle|0\rangle\langle 0|\langle 1| + |1\rangle|0\rangle\langle 1|\langle 0|\right)$$

$$\rho = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and,

$$Y\otimes Y = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

Therefore,

$$\rho(Y\otimes Y)\rho^{\dagger}(Y\otimes Y) = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Now,det $|A-\lambda I| = 0$ So, the eigen-values of this matrix are:$\lambda_1 = 1, \lambda_2 = 0$
Therefore,

$$C(\rho) = \max\{0,1\} = 1$$

Also, the entanglement of formation is,

$$E(\rho) = \frac{h}{2}$$

2.3 The Measurement Theory

The process of measurement is an obligatory and inevitable condition fundamental to the Quantum Information Processing. To extract any information about the state of a system, one must employ 'measurements.' Realistic quantum systems are always a subject to interaction with the surrounding environment and can never be completely isolated. A quantum system is always supposed to exist in arbitrary superpositions; the measurement or the observer is the one responsible for the apparent classical behaviour.

2.3.1 The Von-Neumann Measurement Scheme

If S is a microscopic or quantum system whose basis vector space$\{|s_n\rangle\} \in H_s$ subjected to a measurement[9] apparatus A with basis vector space $\{|a_n\rangle\} \in H_A$, where H denotes the corresponding *Hilbert Space*. The basis vectors $\{|a_n\rangle\}$ correspond to the outcome of a measurement if S is in the state $|s_n\rangle$. for e.g a superposition of states is only projected onto the computational basis (with values 0 or 1). The system S is supposed to exist in a superposition of states; $S : \sum_n c_n |s_n\rangle$ and let A be in initial ready state $|a_a\rangle$. So, the total system $SA \in H_S \otimes H_A$ shall evolve as:-

$$(\sum_n c_n |s_n\rangle) |a_a\rangle \rightarrow \sum_n c_n |s_n\rangle |a_n\rangle$$

This is the pre-measurement state of the total system. The post-measurement state is one of the possibilities of the apparatus basis states i.e $\{|a_n\rangle\}$ which is called as the *system collapse* or the *wave-function collapse*. The inability to measure or observe the *'actual reality'* are held responsible for such discrepancies. The dual nature of electron and light waves are a well known example of the dependence of the result of their nature on the phenomenon of observation or measurement.

2.3.2 Projective Measurements

The potential to read a register of qbits is an important aspect of quantum communications. The type of measurements usually encountered in quantum mechanics are the projective measurements which project the system onto one of its eigen-values. For example, To answer the question so as to which of the states, a system of superpositions exist in, projective measurements come into play.
Given an n-dimensional vector space with a basis given by $|1\rangle, |2\rangle, |3\rangle .... |n\rangle$ such that m<n. Then, the operator given by:-

$$P = \sum_{i=1}^{m} |i\rangle \langle i|$$

projects onto the subspace spanned by the set $|1\rangle, |2\rangle, |3\rangle .... |m\rangle$. Also, the probability of finding the i^{th} measurement may be given as:

$$\Pr(i) = |P_i |\psi\rangle|^2 = (P_i |\psi\rangle)^\dagger (P_i |\psi\rangle) \tag{2.1}$$

$$= \langle\psi| P_i{}^2 |\psi\rangle = \langle\psi| P_i |\psi\rangle \tag{2.2}$$

These operators follow two important properties:-
1. Projection operators are *Hermitian*. i.e

$$P^\dagger = P$$

2. They satisfy the completeness relation:-

$$\sum_i P_i = I$$

For the spectral decomposition given by the operator

$$A = \sum_{i=1}^{n} a_i |u_i\rangle \langle u_i|$$

, the projection operator$P_i = |u_i\rangle \langle u_i|$ projects onto the subspace defined by the eigen-value a_i. This eigen-value represents a given measurement result for the operator A. The state of a system

$$\psi = \sum_{i=1}^{n} (\langle u_i | \psi\rangle) |u_i\rangle \tag{2.3}$$

$$= \sum_{i=1}^{n} c_i |u_i\rangle \tag{2.4}$$

$c_i = \langle u_i | \psi\rangle$ is the probability amplitude for obtaining the measurement result result a_i when the system is in the state $|\psi\rangle$. The probability of obtaining a measurement result is given by:-

$$\Pr(i) = \frac{|\langle u_i | \psi\rangle|^2}{\langle\psi | \psi\rangle}$$

The denominator emphasizes on the normalization of the states.

2.3.3 Generalized Measurements

Given a measurement operator M_r where r is a possible outcome of the measurement, and a state $|\psi\rangle$, the probability of finding the measurement result r is given by:-

$$\Pr(r) = \langle\psi| {M^\dagger}_r M_r |\psi\rangle$$

After the measurement, the stateof te system can be described by:-

$$\psi^{'} = \frac{M_r |\psi\rangle}{\sqrt{\langle\psi| {M^\dagger}_r M_r |\psi\rangle}}$$

For the quantum systems whose state is described by density matrices, the probability of obtaining the result r is given by:-

$$\Pr(r) = Tr({M^\dagger}_r M_r \rho)$$

The state of the system post measurement can be then described by:-

$$\rho^{'} = \frac{M_r \rho {M^\dagger}_r}{Tr({M^\dagger}_r M_r \rho)}$$

2.3.4 Measurement in Bell Basis

The Bell Basis is given by the following four states which are said to be maximally entangled:-

$$\left|\phi^+\right\rangle = \frac{1}{\sqrt{2}}(|0\rangle|0\rangle + |1\rangle|1\rangle)$$

$$\left|\phi^-\right\rangle = \frac{1}{\sqrt{2}}(|0\rangle|0\rangle - |1\rangle|1\rangle)$$

$$\left|\psi^+\right\rangle = \frac{1}{\sqrt{2}}(|0\rangle|1\rangle + |1\rangle|0\rangle)$$

$$\left|\psi^-\right\rangle = \frac{1}{\sqrt{2}}(|0\rangle|1\rangle - |1\rangle|0\rangle)$$

The measurement basis for most of the experiments in generally the computational basis $\{|0\rangle, |1\rangle\}$. The basis states are mapped to the computational basis. So, they shall collapse to a value out of the four different possibilities i.e $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$. This[1] is essentially done by applying CNOT gate where control is on the first bit and second is the target bit, succeeded by a Hadamard operation only on the first bit.

$$CNOT_{AB}(\frac{1}{\sqrt{2}}(|0\rangle|1\rangle + |1\rangle|0\rangle)) \rightarrow \frac{1}{\sqrt{2}}(|0\rangle|1\rangle + |1\rangle|1\rangle)$$

$$H_A(\frac{1}{\sqrt{2}}(|0\rangle|1\rangle + |1\rangle|1\rangle)) \rightarrow |01\rangle$$

One may proceed similarly for the other states as well.

2.4 Entropy

Entropy is always associated with degree of randomness and is a measure of the information content in a signal. The amount of uncertainty involved in the measurement is entropy. This concept was introduced by Shannon.

For a random variable$X$ which[8] can assume values x_i with probability p_i, the entropy is given as:-

$$H(X) = -\sum_i p_i \log_2 p_i$$

Consider an example such that

a)Signal:-1111111111111........

$$H = -\sum 1 \times \log_2 1 = 0$$

b)Signal:-11001100111000......

$$p(1) = p(0) = 0.5$$

$$H = -\frac{1}{2}\log_2\frac{1}{2} - \frac{1}{2}\log_2\frac{1}{2} = \frac{1}{2} + \frac{1}{2} = 1$$

c)1001110001000111000....... i.e if there are three equally likely possibilities,

$$H = -\frac{1}{3}\log_2\frac{1}{3} - \frac{1}{3}\log_2\frac{1}{3} - \frac{1}{3}\log_2\frac{1}{3} = 1.585$$

The greater the amount of randomness, greater is the entropy and hence, more information can be extracted from the system per measurement.

The quantum analogue of Shannon Entropy is given by the Von Neumann Entropy:-

$$S(\rho) = -Tr(\rho \log_2 \rho)$$

For a density matrix with eigen-values λ_i,

$$S(\rho) = -\sum_i \lambda_i \log_2 \lambda_i$$

The entropy of a pure state is 0 while that of a maximally entangled state is 1. Entropy remains invariant under a change of basis. The entropy of mixed states $\in (0, 1)$

2.5 The NMR Theory

The physical implementation of qbits for quantum information processing must be free of doubts. An innate example could be an isolated spin $\frac{1}{2}$ subjected to an external magnetic field. The state of such a particle is generally described as:-

$$|\psi\rangle = a\,|\alpha\rangle + b\,|\beta\rangle$$

where,$|\alpha\rangle$ and $|\beta\rangle$ are the states corresponding to the I_z basis. The dominant NMR interactions among the spins are the Zeeman, chemical shift, scalar coupling and the dipolar coupling.
The states for a spin- $\dfrac{1}{2}$ particle are given as follows,

$$|\alpha\rangle = \left|m = +\frac{1}{2}\right\rangle = |0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$|\beta\rangle = \left|m = -\frac{1}{2}\right\rangle = |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

The interaction between two systems or spins is given by the tensor product.

$$|\alpha\rangle \otimes |\beta\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

The logic gates are implemented in[10] NMR using Radiofrequency pulse sequences. These operations denote the rotations of the spins.

$$R_{\hat{n}}(\theta) = \exp(-i\theta n.I)$$

$$I = I_x\hat{i} + I_y\hat{j} + I_x\hat{k}$$

The action of an on-resonance pulse applied for a duration t_p, with a phase ϕ is described by:-

$$(\theta)^I_\phi = \exp(-i\omega t_p I_\phi) = \exp(-i\theta I_\phi)$$

where,

$$I_\phi = I_x\cos(\phi) + I_y\sin(\phi)$$

The Hadamard Gate can be described as:-

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$H = I_x(\pi)I_y(\pi/2)$$

The elementary CNOT gate where A is the control and B the target, is employed as follows:-

$$CNOT_{AB} = (\frac{\pi}{2})_z^{I_1}(-\frac{\pi}{2})_z^{I_2}(\frac{\pi}{2})_z^{I_2}U_J(\frac{1}{2J})(\frac{\pi}{2})_y^{I_2}$$

$$= \frac{(1-i)}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

The CNOT gate when B is the control and A is the target bit can be implemented as follows:-

$$CNOT_{BA} = (\frac{\pi}{2})_z^{I_2}(-\frac{\pi}{2})_z^{I_1}(\frac{\pi}{2})_z^{I_1}U_J(\frac{1}{2J})(\frac{\pi}{2})_y^{I_1}$$

$$CNOT_{BA} = \frac{(1-i)}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

Chapter 3

Quantum Teleportation

In compliance with, or in contrary to the name, Teleportation refers to the transfer of information instead of the object itself. In 1993, Bennett et al.[11] moved the concept of Quantum Teleportation from the jungles of fiction to the realms of scientific possibility. The concept of transfer of the state of a particle onto anoher particle via an EPR channel is what they termed as *"Teleportation."* The No-Cloning Theorem and the Quantum Entanglement play a key ingredient in this process.

3.1 The No-Cloning Theorem

Quantum Mechanics prohibits the possibility of cloning the quantum state of any particle. There ceases to exist[1] a transformation such that

$$U|\alpha\rangle_n|0\rangle_n \to |\alpha\rangle_n|\alpha\rangle_n$$

Proof:
Consider a unitary transformation such that:-

$$U\left(|\alpha\rangle\,|0\rangle\right) = |\alpha\rangle\,|\alpha\rangle$$

Hence,

$$U\left(|\beta\rangle\,|0\rangle\right) = |\beta\rangle\,|\beta\rangle$$

By linearity,

$$\begin{aligned} U(a\,|\alpha\rangle + b\,|\beta\rangle)\,|0\rangle &= aU\left(|\alpha\rangle\,|0\rangle\right) + bU\left(|\beta\rangle\,|0\rangle\right) \\ &= a\,|\alpha\rangle\,|\alpha\rangle + b\,|\beta\rangle\,|\beta\rangle \end{aligned}$$

Also, by the definition of the transformation itself,

$$U(a|\alpha\rangle + b|\beta\rangle)|0\rangle = (a|\alpha\rangle + b|\beta\rangle)(a|\alpha\rangle + b|\beta\rangle)$$
$$= a^2|\alpha\rangle|\alpha\rangle + ab|\alpha\rangle|\beta\rangle + ba|\beta\rangle|\alpha\rangle + b^2|\beta\rangle|\beta\rangle$$

The above two equations can not be equal in general. Hence, proved.

3.2 Various Interpretations of the Circuit

The circuit diagram[12] for teleportation is generally given as:-

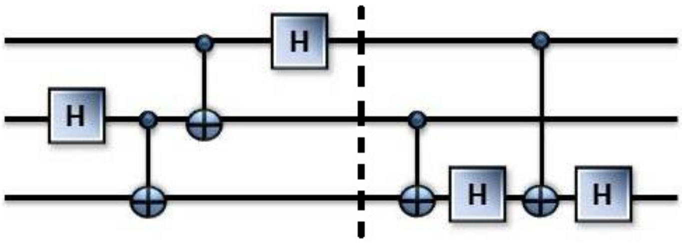

Figure 3.1: The Teleportation Circuit

This process occurs in four steps essentially,

1. Creation of an EPR pair. i.e Alice and Bob share an entangled pair of qbits.

$$|\psi^-\rangle_{23} = |\alpha\rangle_2|\beta\rangle_3 - |\beta\rangle_2|\alpha\rangle_3$$

This is an example of the singlet state which is also one of the Bell states, also known as the EPR state.

2. Alice flies off to a distant location and so does Bob. Alice has an unknown qbit whose state needs to be teleported to Bob.

$$|\psi\rangle_1 = a|\alpha\rangle_1 + b|\beta\rangle_1$$

3. She performs joint measurements on her system.

$$|\psi\rangle_1 \otimes |\psi\rangle^-{}_{23} = \frac{1}{\sqrt{2}}(a|0\rangle_1|0\rangle_2|1\rangle_3 - a|0\rangle_1|1\rangle_2|0\rangle_3 + b|1\rangle_1|0\rangle_2|1\rangle_3 - b|1\rangle_1|1\rangle_2|0\rangle_3)$$

The state of this system may be written as follows:-

$$\left|\phi^{+}\right\rangle_{12} \otimes (-b|\alpha\rangle_3 + a|\beta\rangle_3) + \left|\psi^{+}\right\rangle_{12} \otimes (-a|\alpha\rangle_3 + b|\beta\rangle_3)$$

$$+$$

$$\left|\phi^{-}\right\rangle_{12} \otimes (b|\alpha\rangle_3 + a|\beta\rangle_3) + \left|\psi^{-}\right\rangle_{12} \otimes (-a|\alpha\rangle_3 - b|\beta\rangle_3)$$

This is where the game actually starts. As can be seen in the decomposition above, the particle 2 and 3 are no longer entangled i.e the entanglement is broken. The interaction of a third particle (the unknown bit), situated at a distant location has 'influenced' the particle 3 at Bob's end without any spatial contact or exchange of information(from the observer's perspective). The Bob's particle now exists in any of the four possibilities or one may say that it exists in a superposition of these states.

4. Bob performs certain rotations or unitary transformations on his set of particles in accord with Alice's results.

Alice will now make measurements on her set of particles and shall obtain one out of four different possibilities.This is called as *'measurement in Bell Basis'*. She communicates the result to Bob through a classical channel after which he performs a set of operations on his particle.
The circuit can be explicitly explained as follows:-
Let the bits 2 and 3 be in the $|\beta\rangle$ each. The first two gates in the circuit are used to generate an entangled pair.

$$H_2|\beta\rangle_2|\beta\rangle_3 = \frac{1}{\sqrt{2}}\left(|\alpha\rangle_2 - |\beta\rangle_2\right)|\beta\rangle_3$$

$$CNOT_{AB}\frac{1}{\sqrt{2}}\left(|\alpha\rangle_2 - |\beta\rangle_2\right)|\beta\rangle_3 = \frac{1}{\sqrt{2}}\left(|\alpha\rangle_2|\beta\rangle_3 - |\beta\rangle_2|\alpha\rangle_3\right)$$

Hence, the singlet state is obtained:-

$$\left|\psi^{-}\right\rangle_{23} = \frac{1}{\sqrt{2}}\left(|\alpha\rangle_2|\beta\rangle_3 - |\beta\rangle_2|\alpha\rangle_3\right)$$

Alice and Bob share this entangled state and now move to a distant location. Alice wants to send a message to Bob which is encoded in another bit 1, but the state is unknown to her. Her systems interact in the following way:-

$$\begin{aligned}|\psi\rangle_1 \otimes \left|\psi^{-}\right\rangle_{23} = \left|\phi^{+}\right\rangle_{12} \otimes (-b|\alpha\rangle_3 + a|\beta\rangle_3) + \left|\psi^{+}\right\rangle_{12} \otimes (-a|\alpha\rangle_3 + b|\beta\rangle_3)\\ +\left|\phi^{-}\right\rangle_{12} \otimes (b|\alpha\rangle_3 + a|\beta\rangle_3) + \left|\psi^{-}\right\rangle_{12} \otimes (-a|\alpha\rangle_3 - b|\beta\rangle_3)\end{aligned}$$

She performs measurement on her system as follows:-
a)$|\phi\rangle^+{}_{1,2} \otimes (-b|\alpha\rangle_3 + a|\beta\rangle_3)$

$$CNOT_{AB}|\phi\rangle^+{}_{1,2} = CNOT_{AB}\frac{1}{\sqrt{2}}\left(|\alpha\rangle_1|\alpha\rangle_2 + |\beta\rangle_1|\beta\rangle_2\right)$$

$$= |\alpha\rangle_1|\alpha\rangle_2 + |\beta\rangle_1|\alpha\rangle_2$$

She further applies the Hadamard gate on her fist bit(the unknown),

$$H_A\left(|\alpha\rangle_1|\alpha\rangle_2 + |\beta\rangle_1|\alpha\rangle_2\right) = \frac{1}{2}\left((|\alpha\rangle_1 + |\beta\rangle_1)|\alpha\rangle_2 + (|\alpha\rangle_1 - |\beta\rangle_1)|\alpha\rangle_2\right)$$

$$= |\alpha\rangle_1|\alpha\rangle_2$$

Similarly,
b)$|\psi\rangle^+{}_{1,2} \otimes (-a|\alpha\rangle_3 + b|\beta\rangle_3)$

$$CNOT_{AB}|\psi\rangle^+{}_{1,2} = CNOT_{AB}\frac{1}{\sqrt{2}}\left(|\alpha\rangle_1|\beta\rangle_2 + |\beta\rangle_1|\alpha\rangle_2\right)$$

$$= |\alpha\rangle_1|\beta\rangle_2 + |\beta\rangle_1|\beta\rangle_2$$

$$H_A\left(|\alpha\rangle_1|\beta\rangle_2 + |\beta\rangle_1|\beta\rangle_2\right) = \frac{1}{2}\left((|\alpha\rangle_1 + |\beta\rangle_1)|\beta\rangle_2 + (|\alpha\rangle_1 - |\beta\rangle_1)|\beta\rangle_2\right)$$

$$= |\alpha\rangle_1|\beta\rangle_2$$

For,
c)$|\phi\rangle^-{}_{1,2} \otimes (-a|\alpha\rangle_3 - b|\beta\rangle_3)$

$$CNOT_{AB}|\phi\rangle^-{}_{1,2} = CNOT_{AB}\frac{1}{\sqrt{2}}\left(|\alpha\rangle_1|\alpha\rangle_2 - |\beta\rangle_1|\beta\rangle_2\right)$$

$$= |\alpha\rangle_1|\alpha\rangle_2 - |\beta\rangle_1|\alpha\rangle_2$$

$$H_A\left(|\alpha\rangle_1|\alpha\rangle_2 - |\beta\rangle_1|\alpha\rangle_2\right) = \frac{1}{2}\left((|\alpha\rangle_1 + |\beta\rangle_1)|\alpha\rangle_2 - (|\alpha\rangle_1 - |\beta\rangle_1)|\alpha\rangle_2\right)$$

$$= |\beta\rangle_1|\alpha\rangle_2$$

For,
d)

$$|\psi\rangle^-{}_{1,2} \otimes (b|\alpha\rangle_3 + a|\beta\rangle_3)$$

$$CNOT_{AB}|\psi\rangle^-{}_{1,2} = CNOT_{AB}\frac{1}{\sqrt{2}}\left(|\alpha\rangle_1|\beta\rangle_2 - |\beta\rangle_1|\alpha\rangle_2\right)$$

$$= |\alpha\rangle_1|\alpha\rangle_2 - |\beta\rangle_1|\beta\rangle_2$$

$$H_A\left(|\alpha\rangle_1|\alpha\rangle_2-|\beta\rangle_1|\beta\rangle_2\right)=\frac{1}{2}\left((|\alpha\rangle_1+|\beta\rangle_1)|\alpha\rangle_2-(|\alpha\rangle_1-|\beta\rangle_1)|\beta\rangle_2\right)$$
$$=|\beta\rangle_1|\beta\rangle_2$$

Interpretation One

The state of the system post Bell state measurements can be summarized as follows:-

$$\left|\phi^+\right\rangle_{12}\rightarrow|\alpha\rangle_1|\alpha\rangle_2\left(-b|\alpha\rangle_3+a|\beta\rangle_3\right)$$
$$\left|\psi^+\right\rangle_{12}\rightarrow|\alpha\rangle_1|\beta\rangle_2\left(-a|\alpha\rangle_3+b|\beta\rangle_3\right)$$
$$\left|\phi^-\right\rangle_{12}\rightarrow|\beta\rangle_1|\alpha\rangle_2\left(b|\alpha\rangle_3+a|\beta\rangle_3\right)$$
$$\left|\psi^-\right\rangle_{12}\rightarrow|\beta\rangle_1|\beta\rangle_2\left(-a|\alpha\rangle_3-b|\beta\rangle_3\right)$$

These show that in the fourth case, the state at Bob's end is already transformed and teleportation has successfully occurred, without any classical transmission. The probability of success hence is 0.25 or $\frac{1}{4}$.
Now, the moment Alice makes a measurement on her part of the entangled pair, the Bob's bit is forced to be in an opposite state (because of anti-correlations developed due to entanglement).
This may be understood as follows:

a)$|\alpha\rangle_1|\alpha\rangle_2|\beta\rangle_3$

$$CNOT_{23}\left(|\alpha\rangle_1|\alpha\rangle_2|\beta\rangle_3\right)=|\alpha\rangle_1|\alpha\rangle_2|\beta\rangle_3$$
$$H_3\left(|\alpha\rangle_1|\alpha\rangle_2|\beta\rangle_3\right)=\frac{1}{\sqrt{2}}|\alpha\rangle_1|\alpha\rangle_2\left(|\alpha\rangle_2-|\beta\rangle_3\right)$$
$$CNOT_{13}\frac{1}{\sqrt{2}}|\alpha\rangle_1|\alpha\rangle_2\left(|\alpha\rangle_2-|\beta\rangle_3\right)=\frac{1}{\sqrt{2}}\left(|\alpha\rangle_1|\alpha\rangle_2|\alpha\rangle_3+|\alpha\rangle_1|\alpha\rangle_2|\beta\rangle_3\right)$$
$$H_3\frac{1}{\sqrt{2}}\left(|\alpha\rangle_1|\alpha\rangle_2|\alpha\rangle_3+|\alpha\rangle_1|\alpha\rangle_2|\beta\rangle_3\right)=|\alpha\rangle_1|\alpha\rangle_2|\alpha\rangle_3$$

The state at Bob's end has been transformed to $|\alpha\rangle_3$
b)$|\alpha\rangle_1|\beta\rangle_2|\alpha\rangle_3$

$$CNOT_{23}\left(|\alpha\rangle_1|\beta\rangle_2|\alpha\rangle_3\right)=|\alpha\rangle_1|\beta\rangle_2|\beta\rangle_3$$
$$H_3\left(|\alpha\rangle_1|\alpha\rangle_2|\beta\rangle_3\right)=\frac{1}{\sqrt{2}}|\alpha\rangle_1|\beta\rangle_2\left(|\alpha\rangle_3-|\beta\rangle_3\right)$$

$$CNOT_{13}\frac{1}{\sqrt{2}}|\alpha\rangle_1|\beta\rangle_2\left(|\alpha\rangle_3-|\beta\rangle_3\right)=\frac{1}{\sqrt{2}}\left(|\alpha\rangle_1|\beta\rangle_2|\alpha\rangle_3-|\alpha\rangle_1|\beta\rangle_2|\beta\rangle_3\right)$$

$$H_3\frac{1}{\sqrt{2}}\left(|\alpha\rangle_1|\beta\rangle_2|\alpha\rangle_3-|\alpha\rangle_1|\beta\rangle_2|\beta\rangle_3\right)=|\alpha\rangle_1|\beta\rangle_2|\beta\rangle_3$$

c)$|\beta\rangle_1|\beta\rangle_2|\alpha\rangle_3$

$$CNOT_{23}\left(|\beta\rangle_1|\beta\rangle_2|\alpha\rangle_3\right)=|\beta\rangle_1|\beta\rangle_2|\beta\rangle_3$$

$$H_3\left(|\beta\rangle_1|\beta\rangle_2|\beta\rangle_3\right)=\frac{1}{\sqrt{2}}|\beta\rangle_1|\beta\rangle_2\left(|\alpha\rangle_3-|\beta\rangle_3\right)$$

$$CNOT_{13}\frac{1}{\sqrt{2}}|\beta\rangle_1|\beta\rangle_2\left(|\alpha\rangle_3-|\beta\rangle_3\right)=\frac{1}{\sqrt{2}}\left(|\beta\rangle_1|\beta\rangle_2|\beta\rangle_3-|\beta\rangle_1|\beta\rangle_2|\alpha\rangle_3\right)$$

$$H_3\frac{1}{\sqrt{2}}\left(|\beta\rangle_1|\beta\rangle_2|\beta\rangle_3-|\beta\rangle_1|\beta\rangle_2|\alpha\rangle_3\right)=-|\beta\rangle_1|\beta\rangle_2|\beta\rangle_3$$

The particle at Bob's end is the same as what the measurement on the unknown particle yields.

d)$|\beta\rangle_1|\alpha\rangle_2|\beta\rangle_3$

$$CNOT_{23}\left(|\beta\rangle_1|\alpha\rangle_2|\beta\rangle_3\right)=|\beta\rangle_1|\alpha\rangle_2|\beta\rangle_3$$

$$H_3\left(|\beta\rangle_1|\alpha\rangle_2|\beta\rangle_3\right)=\frac{1}{\sqrt{2}}|\beta\rangle_1|\alpha\rangle_2\left(|\alpha\rangle_3-|\beta\rangle_3\right)$$

$$CNOT_{13}\frac{1}{\sqrt{2}}|\beta\rangle_1|\alpha\rangle_2\left(|\alpha\rangle_3-|\beta\rangle_3\right)=\frac{1}{\sqrt{2}}\left(|\beta\rangle_1|\alpha\rangle_2|\beta\rangle_3-|\beta\rangle_1|\alpha\rangle_2|\alpha\rangle_3\right)$$

$$H_3\frac{1}{\sqrt{2}}\left(|\beta\rangle_1|\alpha\rangle_2|\beta\rangle_3-|\beta\rangle_1|\alpha\rangle_2|\alpha\rangle_3\right)=-|\beta\rangle_1|\alpha\rangle_2|\beta\rangle_3$$

If the circuit is looked at in this manner, then the teleportation efficiency is only 0.75 i.e only 3 out of 4 times on an average.

Now, one can also question the correspondence of the coefficients in both the cases. The coefficients are unknown for the state of particle 1 and so for particle 3. The confirmation of this equality stands debatable.

Interpretation Two

Another way of looking at it from Bob's shoes may be understood as follows:

If to begin with, consider the fact that Bob's state has not succumbed to one of the values in the computational basis. Bob, then has a linear superposition of the states $|\alpha\rangle$ and $|\beta\rangle$ i.e Bob's particle is in the following state:-

$$(c|\alpha\rangle_3 + d|\beta\rangle_3)$$

Then,

a)$|\alpha\rangle_1|\alpha\rangle_2\,(c|\alpha\rangle_3 + d|\beta\rangle_3)$

$$CNOT_{23}|\alpha\rangle_1|\alpha\rangle_2\,(c|0\rangle_3 + d|1\rangle_3) = |\alpha\rangle_1|\alpha\rangle_2\,(c|\alpha\rangle_3 + d|\beta\rangle_3)$$

$$H_3|\alpha\rangle_1|\alpha\rangle_2\,(c|\alpha\rangle_3 + d|\beta\rangle_3) = |\alpha\rangle_1|\alpha\rangle_2\,(c(|\alpha\rangle_3 + |\beta\rangle_3) + d(|\alpha\rangle_3 - |\beta\rangle_3))$$

$$CNOT_{13}|\alpha\rangle_1|\alpha\rangle_2\,(c(|\alpha\rangle_3 + |\beta\rangle_3) + d(|\alpha\rangle_3 - |\beta\rangle_3)) = |\alpha\rangle_1|\alpha\rangle_2\,(c(|\alpha\rangle_3 + |\beta\rangle_3) + d(|\alpha\rangle_3 - |\beta\rangle_3))$$

$$H_3|\alpha\rangle_1|\alpha\rangle_2\,(c(|\alpha\rangle_3 + |\beta\rangle_3) + d(|\alpha\rangle_3 - |\beta\rangle_3)) = |\alpha\rangle_1|\alpha\rangle_2\,(c|\alpha\rangle_3 + d|\beta\rangle_3)$$

b)$|\alpha\rangle_1|\beta\rangle_2\,(c|\alpha\rangle_3 + d|\beta\rangle_3)$

$$CNOT_{23}|\alpha\rangle_1|\beta\rangle_2\,(c|\alpha\rangle_3 + d|\beta\rangle_3) = |\alpha\rangle_1|\beta\rangle_2\,(c|\beta\rangle_3 + d|\alpha\rangle_3)$$

$$H_3|\alpha\rangle_1|\beta\rangle_2\,(c|\beta\rangle_3 + d|\alpha\rangle_3) = |\alpha\rangle_1|\beta\rangle_2\,(c(|\alpha\rangle_3 - |\beta\rangle_3) + d(|\alpha\rangle_3 + |\beta\rangle_3))$$

$$CNOT_{13}|\alpha\rangle_1|\beta\rangle_2\,(c(|\alpha\rangle_3 - |\beta\rangle_3) + d(|\alpha\rangle_3 + |\beta\rangle_3)) = |\alpha\rangle_1|\beta\rangle_2\,(c(|\alpha\rangle_3 - |\beta\rangle_3) + d(|\alpha\rangle_3 + |\beta\rangle_3))$$

$$H_3|\alpha\rangle_1|\beta\rangle_2\,(c(|\alpha\rangle_3 - |\beta\rangle_3) + d(|\alpha\rangle_3 + |\beta\rangle_3)) = |\alpha\rangle_1|\beta\rangle_2\,(c|\beta\rangle_3 + d|\alpha\rangle_3)$$

c)$|\beta\rangle_1|\alpha\rangle_2\,(c|\alpha\rangle_3 + d|\beta\rangle_3)$

$$CNOT_{23}|\beta\rangle_1|\alpha\rangle_2\,(c|\alpha\rangle_3 + d|\beta\rangle_3) = |\beta\rangle_1|\alpha\rangle_2\,(c|\alpha\rangle_3 + d|\beta\rangle_3)$$

$$H_3|\beta\rangle_1|\alpha\rangle_2\,(c|\alpha\rangle_3 + d|\beta\rangle_3) = |\beta\rangle_1|\alpha\rangle_2\,(c(|\alpha\rangle_3 + |\beta\rangle_3) + d(|\alpha\rangle_3 - |\beta\rangle_3))$$

$$CNOT_{13}|\beta\rangle_1|\alpha\rangle_2\,(c(|\alpha\rangle_3 + |\beta\rangle_3) + d(|\alpha\rangle_3 - |\beta\rangle_3)) = |\beta\rangle_1|\alpha\rangle_2\,(c(|\beta\rangle_3 + |\alpha\rangle_3) + d(|\beta\rangle_3 - |\alpha\rangle_3))$$

$$H_3|\beta\rangle_1|\alpha\rangle_2\,(c(|\alpha\rangle_3 + |\beta\rangle_3) + d(|\alpha\rangle_3 - |\beta\rangle_3)) = |\beta\rangle_1|\alpha\rangle_2\,(c|\alpha\rangle_3 - d|\beta\rangle_3)$$

d)$|\beta\rangle_1|\beta\rangle_2\,(c|\alpha\rangle_3 + d|\beta\rangle_3)$

$$CNOT_{23}|\beta\rangle_1|\beta\rangle_2\,(c|\alpha\rangle_3 + d|\beta\rangle_3) = |\beta\rangle_1|\beta\rangle_2\,(c|\beta\rangle_3 + d|\alpha\rangle_3)$$

$$H_3|\beta\rangle_1|\beta\rangle_2\,(c|\beta\rangle_3 + d|\alpha\rangle_3) = |\beta\rangle_1|\beta\rangle_2\,(c(|\alpha\rangle_3 - |\beta\rangle_3) + d(|\alpha\rangle_3 + |\beta\rangle_3))$$

$$CNOT_{13}|\beta\rangle_1|\beta\rangle_2\,(c(|\alpha\rangle_3 - |\beta\rangle_3) + d(|\alpha\rangle_3 + |\beta\rangle_3)) = |\beta\rangle_1|\beta\rangle_2\,(c(|\beta\rangle_3 - |\alpha\rangle_3) + d(|\beta\rangle_3 + |\alpha\rangle_3))$$

$$H_3|\beta\rangle_1|\beta\rangle_2\left(c(|\beta\rangle_3 - |\alpha\rangle_3) + d(|\beta\rangle_3 + |\alpha\rangle_3)\right) = |\beta\rangle_1|\beta\rangle_2\left(-c|\beta\rangle_3 + d|\alpha\rangle_3\right)$$

Interpretation Three

Now, consider the EPR state with Alice and Bob to be as follows:-

$$\left|\phi^+\right\rangle_{23} = \frac{1}{\sqrt{2}}\left(|\alpha\rangle_2|\alpha\rangle_3 + |\beta\rangle_2|\beta\rangle_3\right)$$

$$|\psi\rangle_1 = a|\alpha\rangle_1 + b|\beta\rangle_3$$

$$|\psi\rangle_1 \otimes \left|\phi^+\right\rangle_{23} = \frac{1}{\sqrt{2}}\left(a((|\alpha\rangle_1|\alpha\rangle_2|\alpha\rangle_3) + (|\alpha\rangle_1|\beta\rangle_2|\beta\rangle_3)) + b((|\beta\rangle_1|\alpha\rangle_2|\alpha\rangle_3) + (|\beta\rangle_1|\beta\rangle_2|\beta\rangle_3))\right)$$

$$= \frac{1}{2}\left(\left|\phi^+\right\rangle_{12} \otimes (a|\alpha\rangle_3 + b|\beta\rangle_3)\right) + \frac{1}{2}\left(\left|\psi^+\right\rangle_{12} \otimes (b|\alpha\rangle_3 + a|\beta\rangle_3)\right)$$

$$+$$

$$\frac{1}{2}\left(\left|\phi^-\right\rangle_{12} \otimes (a|\alpha\rangle_3 - b|\beta\rangle_3)\right) + \frac{1}{2}\left(\left|\psi^-\right\rangle_{12} \otimes (-b|\alpha\rangle_3 + a|\beta\rangle_3)\right)$$

Now,

$$CNOT_{AB}\left|\phi^+\right\rangle_{12} = \frac{1}{\sqrt{2}}\left(|\alpha\rangle_1|\alpha\rangle_2 + |\beta\rangle_1|\alpha\rangle_2\right)$$

$$H_1\frac{1}{\sqrt{2}}\left(|\alpha\rangle_1|\alpha\rangle_2 + |\beta\rangle_1|\alpha\rangle_2\right) = |\alpha\rangle_1|\alpha\rangle_2$$

$$CNOT_{AB}\left|\psi^+\right\rangle_{12} = \frac{1}{\sqrt{2}}\left(|\alpha\rangle_1|\beta\rangle_2 + |\beta\rangle_1|\beta\rangle_2\right)$$

$$H_1\frac{1}{\sqrt{2}}\left(|\alpha\rangle_1|\beta\rangle_2 + |\beta\rangle_1|\beta\rangle_2\right) = |\alpha\rangle_1|\beta\rangle_2$$

$$CNOT_{AB}\left|\phi^-\right\rangle_{12} = \frac{1}{\sqrt{2}}\left(|\alpha\rangle_1|\alpha\rangle_2 - |\beta\rangle_1|\alpha\rangle_2\right)$$

$$H_1\frac{1}{\sqrt{2}}\left(|\alpha\rangle_1|\alpha\rangle_2 - |\beta\rangle_1|\alpha\rangle_2\right) = |\beta\rangle_1|\alpha\rangle_2$$

$$CNOT_{AB}\left|\psi^-\right\rangle_{12} = \frac{1}{\sqrt{2}}\left(|\alpha\rangle_1|\beta\rangle_2 - |\beta\rangle_1|\beta\rangle_2\right)$$

$$H_1\frac{1}{\sqrt{2}}\left(|\alpha\rangle_1|\beta\rangle_2 - |\beta\rangle_1|\beta\rangle_2\right) = |\beta\rangle_1|\beta\rangle_2$$

Now, the entire circuit can be written as,

$$\frac{1}{2}|\alpha\rangle_1|\alpha\rangle_2 \otimes (a|\alpha\rangle_3 + b|\beta\rangle_3)$$

$$\frac{1}{2}|\alpha\rangle_1|\beta\rangle_2 \otimes (b|\alpha\rangle_3 + a|\beta\rangle_3)$$

$$\frac{1}{2}|\beta\rangle_1|\alpha\rangle_2 \otimes (a|\alpha\rangle_3 - b|\beta\rangle_3)$$

$$\frac{1}{2}|\beta\rangle_1|\beta\rangle_2 \otimes (-b|\alpha\rangle_3 + a|\beta\rangle_3)$$

Here, it is considered that making a measurement on particle 2 at Alice's end shall no longer effect the state of particle 3 at Bob's end i.e Bob's particle is not projected onto the computational basis.

Now, Bob shall perform the rotations on his end. This must be noticed that prior to any rotations, Bob's particle is in the same state as the ancillary bit for $\frac{1}{4}$ times. The efficiency of the process at this instant is hence only 25 percent.

Now, consider the unitary tranformations at Bob's end.

The sequence of rotations is:-

$$CNOT_{23} \rightarrow H_3 \rightarrow CNOT_{13} \rightarrow H_3$$

a)$|\alpha\rangle_1|\alpha\rangle_2 \otimes (a|\alpha\rangle_3 + b|\beta\rangle_3)$

The CNOT gates will not produce any change since both have control bits as $|\alpha\rangle$. Also, $HH = I$, therefore there is no net change on Bob's particle and it remains in the state

$$a|\alpha\rangle_1 + b|\beta\rangle_3$$

which is the same as the unknown bit with which the circuit was started.

b)$|\alpha\rangle_1|\beta\rangle_2 \otimes (b|\alpha\rangle_3 + a|\beta\rangle_3)$

$$CNOT_{23}(|\alpha\rangle_1|\beta\rangle_2 \otimes (b|\alpha\rangle_3 + a|\beta\rangle_3)) = |\alpha\rangle_1|\beta\rangle_2 \otimes (b|\beta\rangle_3 + a|\alpha\rangle_3)$$

$$H_3(|\alpha\rangle_1|\beta\rangle_2 \otimes (b|\beta\rangle_3 + a|\alpha\rangle_3)) = |\alpha\rangle_1|\beta\rangle_2 \otimes (b\frac{(|\alpha\rangle_3 - |\beta\rangle_3)}{\sqrt{2}} + a\frac{(|\alpha\rangle_3 + |\beta\rangle_3)}{\sqrt{2}})$$

$CNOT_{13}$ produces no chance as the control bit here is $|\alpha\rangle_1$,

$$H_3|\alpha\rangle_1|\beta\rangle_2 \otimes (b\frac{(|\alpha\rangle_3 - |\beta\rangle_3)}{\sqrt{2}} + a\frac{(|\alpha\rangle_3 + |\beta\rangle_3)}{\sqrt{2}}) = |\alpha\rangle_1|\beta\rangle_2(a|\alpha\rangle_3 + b|\beta\rangle_3)$$

c)$|\beta\rangle_1|\alpha\rangle_2 \otimes (a|\alpha\rangle_3 - b|\beta\rangle_3)$

$CNOT_{23}$ produces no change.

$$H_3|\beta\rangle_1|\alpha\rangle_2 \otimes (a|\alpha\rangle_3 - b|\beta\rangle_3) = |\beta\rangle_1|\alpha\rangle_2 \otimes (a\frac{(|\alpha\rangle_1 + |\beta\rangle_2)}{\sqrt{2}} - b\frac{(|\alpha\rangle_1 - |\beta\rangle_2)}{\sqrt{2}})$$

$$CNOT_{13}|\beta\rangle_1|\alpha\rangle_2 \otimes (a\frac{(|\alpha\rangle_3 + |\beta\rangle_3)}{\sqrt{2}} - b\frac{(|\alpha\rangle_1 - |\beta\rangle_2)}{\sqrt{2}})$$

$$= |\beta\rangle_1|\alpha\rangle_2 \otimes (a\frac{(|\beta\rangle_3 + |\alpha\rangle_3)}{\sqrt{2}} - b\frac{(|\beta\rangle_3 - |\alpha\rangle_3)}{\sqrt{2}})$$

$$H_3|\beta\rangle_1|\alpha\rangle_2 \otimes (a\frac{(|\beta\rangle_1 + |\alpha\rangle_2)}{\sqrt{2}} - b\frac{(|\beta\rangle_1 - |\alpha\rangle_2)}{\sqrt{2}})$$

$$= |\beta\rangle_1|\alpha\rangle_2(a|\alpha\rangle_3 + b|\beta\rangle_3)$$

d)$|\beta\rangle_1|\beta\rangle_2 \otimes (-b|\alpha\rangle_3 + a|\beta\rangle_3)$

$$CNOT_{23}|\beta\rangle_1|\beta\rangle_2 \otimes (-b|\alpha\rangle_3 + a|\beta\rangle_3) = |\beta\rangle_1|\beta\rangle_2 \otimes (-b|\beta\rangle_3 + a|\alpha\rangle_3)$$

$$H_3|\beta\rangle_1|\beta\rangle_2 \otimes (-b|\beta\rangle_3 + a|\alpha\rangle_3) = |\beta\rangle_1|\beta\rangle_2 \otimes (-b\frac{(|\alpha\rangle_3 - |\beta\rangle_3)}{\sqrt{2}} + a\frac{(|\alpha\rangle_3 + |\beta\rangle_3)}{\sqrt{2}})$$

$$CNOT_{13}|\beta\rangle_1|\beta\rangle_2 \otimes (-b\frac{(|\alpha\rangle_3 - |\beta\rangle_3)}{\sqrt{2}} + a\frac{(|\alpha\rangle_3 + |\beta\rangle_3)}{\sqrt{2}})$$

$$= |\beta\rangle_1|\beta\rangle_2 \otimes (-b\frac{(|\beta\rangle_3 - |\alpha\rangle_3)}{\sqrt{2}} + a\frac{(|\beta\rangle_3 + |\alpha\rangle_3)}{\sqrt{2}})$$

$$H_3|\beta\rangle_1|\beta\rangle_2 \otimes (-b\frac{(|\beta\rangle_3 - |\alpha\rangle_3)}{\sqrt{2}} + a\frac{(|\beta\rangle_3 + |\alpha\rangle_3)}{\sqrt{2}}) = |\beta\rangle_1|\beta\rangle_2(a|\alpha\rangle_3 + b|\beta\rangle_3)$$

The efficiency post transformations at Bob's end is 100 percent for this case.

Proceeding in the same fashion for the singlet state, Bob shall perform the transformations as follows:-

$$CNOT_{23} \rightarrow H_3 \rightarrow CNOT_{13} \rightarrow H_3$$

$a)|\alpha\rangle_1|\alpha\rangle_2 \otimes (-b|\alpha\rangle_3 + a|\beta\rangle_3)$
The resultant state at Bob's end is

$$(-b|\alpha\rangle_3 + a|\beta\rangle_3)$$

$b)|\alpha\rangle_1|\beta\rangle_2 \otimes (-a|\alpha\rangle_3 + b|\beta\rangle_3)$
This gives the resultant state as:

$$(b|\alpha\rangle_3 - a|\beta\rangle_3)$$

$c)|\beta\rangle_1|\alpha\rangle_2 \otimes (-b|\alpha\rangle_3 + a|\beta\rangle_3)$
The resultant state in this case is,

$$(-b|\alpha\rangle_3 - a|\beta\rangle_3)$$

$d)|\beta\rangle_1|\beta\rangle_2 \otimes (-a|\alpha\rangle_3 - b|\beta\rangle_3)$

The final state at Bob's end is now given by,

$$(-b|\alpha\rangle_3 + a|\beta\rangle_3)$$

The efficiency in this case is only 25 percent even after application of the required transformations.

The interpretations hereby stand debatable.

"The 'paradox' is only a conflict between reality and your feeling of what reality 'ought to be'.

-Richard Feynmann

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