Boundaries of Negatively Curved Groups and Cannon-Thurston Maps

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To my grand parents and parents

Declaration

The work presented in this thesis has been carried out by me under the guidance of Dr. Pranab Sardar at the Indian Institute of Science Education and Research Mohali. This work has not been submitted in part or in full for a degree, a diploma, or a fellowship to any other university or institute. Whenever contributions of others are involved, every effort is made to indicate this clearly, with due acknowledgement of collaborative research and discussions. This thesis is a bonafide record of original work done by me and all sources listed within have been detailed in the bibliography.

Ravi Tomar

In my capacity as the supervisor of the candidate's thesis work, I certify that the above statements by the candidate are true to the best of my knowledge.

Dr. Pranab Sardar
(Supervisor)

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Abstract

Bass-Serre introduced the notion of graphs of groups. These can be described as groups acting on simplicial trees. A natural higher dimensional analogue of graphs of groups is the notion of complexes of groups. In this thesis, we address some questions related to the boundaries of the fundamental group of certain complexes of groups. In the first part of the thesis, we study combination theorems for convergence and relatively hyperbolic groups. We prove that the fundamental group of a finite graph of convergence groups with parabolic edge groups is a convergence group. Consequently, we deduce that the fundamental group of a finite graph of convergence groups with a dynamically malnormal family of dynamically quasiconvex edge groups is a convergence group. Then we show that the fundamental group G of a graph of relatively hyperbolic groups with edge groups either parabolic or infinite cyclic is relatively hyperbolic and give an explicit construction of the Bowditch boundary of G. Next, we show that the homeomorphism type of the Bowditch boundary of the fundamental group of a finite graph of relatively hyperbolic groups with parabolic edge groups is determined by the homeomorphism types of the Bowditch boundaries of vertex groups.

In the second part, we look at the boundaries of coned-off spaces and deduce the existence of Cannon-Thurston maps for certain subcomplexes of groups of a complex of hyperbolic groups. Suppose Y is a finite simplicial complex and $(\mathscr{G},Y)$ is a developable complex of hyperbolic groups such that all the local maps are quasiisometric embeddings. Let $Y_1 \subset Y$ and let $(\mathscr{G},Y_1)$ be the complex of groups obtained by restricting $(\mathscr{G},Y)$ to Y_1. Let H,G be the fundamental groups of $(\mathscr{G},Y_1),(\mathscr{G},Y)$, respectively. Suppose H,G are hyperbolic groups. Lastly, suppose the natural map $H \to G$ is injective and all the local groups of $(\mathscr{G},Y)$ are quasiconvex in G. We prove that if the natural map from the universal cover of $(\mathscr{G},Y_1)$ to the universal cover of $(\mathscr{G},Y)$ satisfies Mitra's criterion then the inclusion $H \to G$ admits a Cannon-Thurston map. Finally, we deduce a number of applications.

Notations

$\mathbb{R}$: set of real numbers

$\mathbb{Z}$: set of integers

$\mathbb{N}$: set of natural numbers

M_k^n: the model space, where $k \in \mathbb{R}$ and $n \in \mathbb{N}$, see [11, Chapeter I.2].

For a metric space X, the metric will be denoted by d_X.

An open ball with center $x \in X$ and radius $r \geq 0$ will be denoted by $B(x,r)$.

A geodesic joining $x, y \in X$ will be denoted by $[x,y]$.

For $Y \subset X$, geodesic in Y joining $x, y \in Y$ will be denoted by $[x,y]_Y$.

For $A, B \subset X$, Hausdorff distance between A, B will be denoted by $\mathrm{Hd}(A,B)$.

For $A \subset X$ and $R \geq 0$, $N_R(A)$ will denote the set $\{x \in X \mid d(x,A) \leq R\}$.

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Chapter 1

Introduction

Geometric group theory studies groups via their actions on geometric objects. Bass and Serre introduced one such tool to study the structure of groups through their actions on simplicial trees. This gives us the notion of graphs of groups. A natural generalization (in higher dimensions) of graphs of groups is the notion of complexes of groups.

There are two parts in this thesis and we address two different problems about complexes of groups. These are discussed in the following two sections.

Remark 1.0.1. Results discussed in Section 1.1 are published, see [73]. However, the results discussed in Section 1.2 are part of an unpublished preprint [68].

1.1 Combination theorems for convergence groups

In this section, we are interested in the following question:

Question 1. *Suppose we have a developable complex of groups such that all the local groups have a property $\mathscr{P}$. Under which condition(s) does the fundamental group of the given complex of groups have the property $\mathscr{P}$?*

For us, property $\mathscr{P}$ is either hyperbolicity, relative hyperbolicity, or convergence action.

Combination theorem dates back to F. Klein [45] where he proved a combination theorem for groups which were named after him later, namely Kleinian groups. Subsequently, Maskit ([52],[53],[54],[55]) gave far reaching generalizations of *Klein combination theorem*. In [6], Bestvina-Feighn discovered its analogue in the context of hyperbolic groups. There, they proved a combination theorem for graphs of hyperbolic groups. Motivated by this, several authors proved combination theorems for graphs of hyperbolic and relatively hyperbolic groups ([40],[61],[62],[43]). Later, Martin [48] also proved a combination theorem for complexes of hyperbolic groups. However, using a completely different technique, Dahmani [18] also proved a combination theorem for an acylindrical graph of relatively hyperbolic groups. In fact, he constructed a compact metrizable space on which the fundamental group of given graph of groups acts naturally by homeomorphisms. Then he proved that this action is in fact geometrically finite to be able to invoke a topological characterization of relatively hyperbolic groups from [78]. The following is the main theorem of [18].

Theorem 1.1.1. [18] *Suppose a finitely generated group Γ splits into a finite acylindrical graph of relatively hyperbolic groups with fully quasiconvex edge groups. Then, Γ is relatively hyperbolic with respect to the collection of images of maximal parabolic subgroups of the vertex groups.*

The above theorem motivates the following natural question, which is a particular case of Question 1:

Question 2. *Let Γ be a group admitting a decomposition into a finite graph of convergence groups with dynamically quasiconvex edge groups. Under which condition(s) is Γ a convergence group?*

We answer the Question 2 in the following cases:

1. When the edge groups are parabolic.
2. When the edge groups are cyclic.
3. When the collection of edge groups forms a dynamically malnormal family in the adjacent vertex groups.

Theorem 1.1.2. [73, Theorem 1.2] *Let Γ be a group admitting a decomposition into a finite graph of countable convergence groups with parabolic edge groups. Then Γ is a convergence group.*

Remark 1.1.3. Proof of parts (2),(3),(4) of Theorem 0.1 in [18] also go through when relatively hyperbolic groups are replaced by convergence groups, and rest of the hypotheses remain the same. With a little effort, one can also prove Theorem 1.1.2 by combining parts (2),(3), and (4) of Theorem 0.1 in [18]. Note that, in part (2),(3),(4) of Dahmani's theorem, the domain (see Definition 3.1.1) of any point in the edge boundary is infinite, but it is of star-like form. On the other hand, we allow domains to be infinite subtrees of the Bass-Serre tree of the given graph of groups. For example, let $\Gamma = G_1 *_P G_2$, where G_1, G_2 are convergence groups and P is a common parabolic subgroup but not a maximal parabolic in G_1 and in G_2, respectively. Suppose P_1, P_2 are maximal parabolics containing P in G_1, G_2, respectively. Then the domain of the edge parabolic point is the Bass-Serre tree of $P_1 *_P P_2$. Thus, we generalize the technique of Dahmani [18] and give a more direct proof of Theorem 1.1.2. In fact, we explicitly construct a compact metrizable space (see Section 3.1) on which Γ acts as a convergence group. Also, we use this construction for producing the Bowditch boundary for the group Γ as in Theorem 1.1.7.

The following is a more general combination theorem for convergence groups.

Theorem 1.1.4. [73, Theorem 1.3] *Let Γ be a group admitting a decomposition into a finite graph of groups such that the following holds:*

1. *The vertex groups are countable convergence groups.*
2. *The stabilizers of the limit sets of the edge groups form a dynamically malnormal family of dynamically quasiconvex subgroups in the adjacent vertex groups.*

Then Γ is a convergence group.

In Theorem 1.1.4, by the definition of a dynamically malnormal family (Definition 2.3.12), we see that action of Γ on the Bass-Serre tree of the graph of groups is 2-acylindrical. Thus Γ is the fundamental group of an acylindrical graph of convergence groups. When the vertex groups are convergence groups and the edge groups are fully dynamically quasiconvex in Theorem 1.1.1, it is different from Theorem 1.1.4. In fact, in Theorem 1.1.1, the limit sets of edge groups in adjacent vertex groups are homeomorphic which may not be the case here. There was no identification involved inside Bowditch boundaries of vertex groups. On the other hand, to prove Theorem 1.1.4, we identify the translates of the limit sets of edge groups in adjacent vertex groups with points.

Next proposition is a consequence of Theorem 1.1.4 and it also gives an answer to Question 2 when edge groups are infinite cyclic.

Proposition 1.1.5. [73, Proposition 1.4] *Let Γ be a group admitting a decomposition into a finite graph of countable convergence groups with infinite cyclic edge groups, which are dynamically malnormal in the adjacent vertex groups. Then Γ is a convergence group.*

Note that infinite cyclic subgroups of a convergence group are dynamically quasi-convex (Lemma 2.3.18). In Proposition 1.1.5, dynamical malnormality of edge groups can be replaced by torsion-free vertex groups (see Proposition 3.3.4). Floyd boundary was introduced by W. Floyd in [22]. One is referred to [44],[79] for related results. We have the following immediate corollary:

Corollary 1.1.6. [73, Corollary 1.5] *Let* Γ *be a group that splits into a finite graph of finitely generated groups such that the following holds:*

1. *The vertex groups are torsion-free with non-trivial Floyd boundaries.*
2. *The edge groups are infinite cyclic.*

Then Γ *is a convergence group.*

Proof. A group having a non-trivial Floyd boundary acts as a convergence group on its Floyd boundary (see [44]). Thus, the corollary follows from Proposition 3.3.4. □

In the above corollary, if the vertex groups are relatively hyperbolic then Γ is relatively hyperbolic by Theorem 1.1.7, and hence, by [26], it has a non-trivial Floyd boundary. However, if the vertex groups are not relatively hyperbolic, it is unclear whether Γ has a non-trivial Floyd boundary; see Questions 3 and 4.

The following is a combination theorem for relatively hyperbolic groups which also appears in [7].

Theorem 1.1.7. [73, Theorem 1.6] *Let* Γ *be a group admitting a decomposition into a finite graph of relatively hyperbolic groups such that the edge groups are parabolic in the adjacent vertex groups. Then* Γ *is relatively hyperbolic. Moreover, the vertex groups are relatively quasiconvex in* Γ.

We explain parabolic structure of Γ after proof of Theorem 1.1.7 in Section 6. In Theorem 1.1.7, if edge groups are not maximal parabolic in adjacent vertex groups then it does not satisfy the hypotheses of [18, Theorem 0.1]. However, proof of Theorem 1.1.7 still follows from parts (2),(3),(4) of Theorem 0.1 in [18]. Here, we give a different proof by constructing a compact metrizable space on which Γ acts geometrically finitely. Therefore, we have an explicit construction of Bowditch boundary of Γ too. Note that relative hyperbolicity in Theorem 3.0.3 also follows from the work of Bigdely and Wise [7]. Since in our situation parabolic edge groups can be infinitely generated, the first condition of the theorem of Mj-Reeves [61] is not satisfied. Thus, to prove relative hyperbolicity, we cannot use the theorem of Mj-Reeves.

Theorem 1.1.8. [73, Theorem 1.7] *Let Γ be a group that splits as a finite graph of relatively hyperbolic groups with infinite cyclic edge groups. Then Γ is relatively hyperbolic. Moreover, the vertex groups are relatively quasiconvex in Γ.*

To prove Theorem 1.1.8, we use Theorem 1.1.7. In particular, by extending the parabolic structure on vertex groups, we convert the graph of relatively hyperbolic groups with infinite cyclic edge groups into a graph of relative hyperbolic groups with parabolic edge groups. Thus, we can explicitly construct Bowditch boundary for the group Γ.

Next course of action is to study the homeomorphism type of the Bowditch boundary of the fundamental group of a graph of relatively hyperbolic groups with parabolic edge groups.

Theorem 1.1.9. [73, Theorem 1.8] *Let $\mathscr{Y}$ be a finite connected graph and let $G(\mathscr{Y}), G'(\mathscr{Y})$ be two graphs of groups satisfying the following:*

1. *For each vertex $v \in V(\mathscr{Y})$, let $(G_v, \mathbb{P}_v), (G'_v, \mathbb{P}'_v)$ be relatively hyperbolic groups.*

2. *Let $e \in E(\mathscr{Y})$ be any edge. Suppose v, w are vertices connected by e. Let P_e, P'_e be parabolic edge groups in $G(\mathscr{Y}), G'(\mathscr{Y})$, respectively. Then either P_e, P'_e have infinite index in corresponding maximal parabolic subgroups in G_v, G'_v, respectively or P_e, P'_e have the same finite index in maximal parabolic subgroups in G_v, G'_v, respectively. Similarly, either P_e, P'_e have infinite index in maximal parabolic subgroups in G_w, G'_w, respectively or P_e, P'_e have the same finite index in corresponding maximal parabolic subgroups in G_w, G'_w, respectively.*

3. *For any $v \in V(\mathscr{Y})$, let B_v, B'_v be the set of translates of parabolic points corresponding to adjacent edge groups under the action of G_v, G'_v on their Bowditch boundaries respectively. Suppose we have a homeomorphism from $\partial G_v \to \partial G'_v$ that maps B_v onto B'_v.*

Let $\Gamma = \pi_1(G(\mathscr{Y}))$, $\Gamma' = \pi_1(G'(\mathscr{Y}))$ and let $\partial\Gamma, \partial\Gamma'$ be their Bowditch boundaries, respectively. Then there exists a homeomorphism from $\partial\Gamma$ to $\partial\Gamma'$ preserving edge parabolic points, i.e. taking parabolic points corresponding to edge groups of $G(\mathscr{Y})$ to parabolic points corresponding to edge groups of $G'(\mathscr{Y})$.

Remark 1.1.10. The proof of Theorem 0.1 in [18] works only for infinite edge groups. In particular, if the edge groups are finite then the space M constructed in [18] need not be compact. In all of the above theorems, we also take infinite edge groups. In [50], Martin and Świątkowski constructed the Gromov boundary for a graph of hyperbolic groups with finite edge groups, see also [48]. If we take a graph of convergence groups with finite edge groups then, by [10], the fundamental group of graph of groups is relatively hyperbolic with respect to infinite vertex groups and hence it a convergence group.

A few words on the proofs: To prove Theorem 1.1.2, we generalize Dahmani's technique [18]. In fact, we explicitly construct a compact metrizable space on which the fundamental group of the graph of groups acts as a convergence group. In [18], the primary role of fully quasiconvex edge groups and acylindrical action was to get a uniform upper bound on the diameters of the domains. In our situation, domains for points in the boundary of edge groups may be infinite (when edge groups are not maximal parabolic). Therefore the space constructed by Dahmani in [18] does not work, see Remark 3.1.9. Thus, we need to modify the space. For that, we look at the domain of a point ξ in the boundary of an edge group. If the domain of ξ is an infinite subtree then identify all the boundary points (in the visual boundary of the Bass-Serre tree) of the domain of ξ with ξ itself. Thus, we get a new set by taking the equivalence relation generated by this. Then we define a topology on this set and see that this is our candidate space. For proving Theorem 1.1.4, we use a result of Manning from [47]. Using this result, the proof of this theorem boils down to the case of parabolic edge groups, which is done by Theorem 1.1.2. To construct Bowditch boundary for Γ as in Theorem 1.1.7, in the construction of candidate space (see Section 3.1), we take Bowditch boundaries on which vertex groups act geometrically finitely and the rest of the things remain the same.

It is sufficient to consider only the amalgam and HNN extension case to prove the above theorems. For a general graph of groups, say $G(\mathscr{Y})$, we may take a maximal tree T in $\mathscr{Y}$. By proving the theorems for amalgams, we are done for graphs of groups over T. By adding the remaining edges in $\mathscr{Y} \setminus T$ one by one, we are in the HNN extension case. By proving the theorems for HNN extensions, we are done in the general case.

1.2 Boundaries of coned-off spaces and Cannon-Thurston maps

Given a hyperbolic space X and a hyperbolic subspace Y, it is natural to ask if the natural inclusion $Y \to X$ extends continuously to $\partial Y \to \partial X$ (see [5],[57]). In particular, one may ask the same question for a pair of hyperbolic groups $H < G$. Existence of such a continuous map is known as Cannon-Thurston (CT) map. The first example of this sort was given by Cannon and Thurston in [12]. Later Mitra (Mj) [57] gave a different proof of Cannon-Thurston's result and generalized it for the tree of hyperbolic spaces. Over the time, the existence of Cannon-Thurston map has been proven in many interesting cases. A few of them are [56],[60],[63],[46],[43]. For a detailed account of the history of Cannon-Thurston maps, one is referred to [59]. However, Baker and Riley [2] answer the general question for groups negatively.

Our starting point is the following question:

Question 3. *Suppose X is a hyperbolic metric space and $\{A_i\}_{i\in I}$ is a collection of uniform quasiconvex subsets of X. Let $\hat{X}$ denotes the coned-off space obtained by coning A_i's. Suppose $Y \subset X$ is a hyperbolic space, and a CT map exists for $Y \to \hat{X}$. Does there exist a CT map for $Y \to X$ and vice versa?*

In the previous question, if Y is quasiconvex in X then, by [17, Proposition 2.11], Y is also quasiconvex in $\hat{X}$. In general, the converse of this result is false. However, under some additional assumptions, Mj-Dahmani [17, Proposition 2.12] proved the converse.

Remark 1.2.1. Let X be a geodesic metric space and let $\{A_i\}_{i\in I}$ be a collection of its subsets. The collection $\{A_i\}$ is said to be *locally finite* if for all $x_0 \in X$ and $D > 0$ there exists $N = N(x_0, D)$ such that $|\{i \in I : B(x_0, D) \cap A_i \neq \phi\}| \leq N$.

We give an answer to a variation of Question 3 in the following theorem.

Theorem 1.2.2 ([68])**.** *Suppose X is a hyperbolic geodesic metric space, $Y \subset X$ which is properly embedded in X with respect to the induced length metric from X. Let $\{A_i \subset X\}$ be a locally finite collection of uniformly quasiconvex subsets. Suppose Y is hyperbolic geodesic metric space and $\{B_j \subset Y\}$ is a collection of subsets such that each B_j is contained in $A_i \cap Y$ for some i. Also, assume that B_j's are uniformly quasiconvex in Y as well as in X. Let $\hat{Y}$ denote the coned-off space obtained by coning B_j's and let $\hat{X}$ denote the coned-off space obtained by coning A_i's. If $\hat{Y} \to \hat{X}$ satisfies Mitra's criterion then the CT map $\partial Y \to \partial X$ exists.*

Moreover, the CT map $\partial Y \to \partial X$ is injective if and only if the CT map $\partial\hat{Y} \to \partial\hat{X}$ is injective.

For Mitra's criterion, one is referred to Lemma 2.2.34.

The next part studies the existence of Cannon-Thurston map for certain subcomplexes of groups of a given complex of hyperbolic groups. As the main application of Theorem 1.2.2, we prove the following.

Theorem 1.2.3 ([68]). *Let $(\mathcal{G},Y)$ be a developable complex of groups over a finite simplicial*
subcomplex of groups obtained by restricting $(\mathcal{G},Y)$ to Y_1. Suppose the following conditions hold:

1. *The natural homomorphism $G_1 = \pi_1(\mathcal{G},Y_1) \to G = \pi_1(\mathcal{G},Y)$ is injective.*

2. *Both G_1, G are hyperbolic, and all the local groups of $(\mathcal{G},Y)$ are quasiconvex in G.*

3. *The natural map $B_1 \to B$ satisfies Mitra's criterion, where B_1, B are the universal covers of $(\mathcal{G},Y_1)$ and $(\mathcal{G},Y)$, respectively.*

Then there exists a Cannon-Thurston map for the inclusion $G_1 \to G$. Moreover, G_1 is quasiconvex in G if and only if the Cannon-Thurston map for $B_1 \to B$ is injective.

The converse of Theorem 1.2.3 is false, i.e. there are examples where the CT map exists for $G_1 \to G$, but the CT map fails to exist at the development level, see Example 4.4.17.

An immediate corollary of Theorem 1.2.3 is the following:

Corollary 1.2.4. *Let $(\mathcal{G},Y)$ be a complex of groups satisfying the hypothesis of Theorem 4.2.1. Let Y_1 be a connected subcomplex of Y and let $(\mathcal{G},Y_1)$ be the subcomplex of groups obtained by restricting $(\mathcal{G},Y)$ to Y_1. Suppose the following conditions hold.*

1. *$(\mathcal{G},Y_1)$ also satisfies the hypotheses of Theorem 4.2.1.*

2. *The natural homomorphism $H = \pi_1(\mathcal{G},Y_1) \to G = \pi_1(\mathcal{G},Y)$ is injective.*

3. *Assume that the natural map $B_1 \to B$ satisfies Mitra's criterion where B_1, B are the universal covers of $(\mathcal{G},Y)$ and $(\mathcal{G},Y)$ respectively.*

Then there exists a Cannon-Thurston map for the inclusion $H \to G$. Moreover, H is quasiconvex in G if and only if the Cannon-Thurston map for $B_1 \to B$ is injective.

Remark 1.2.5. Motivated by the work of Dahmani [18], A.Martin [48] proved a combination theorem for an acylindrical complex of hyperbolic groups. There he explicitly constructed the Gromov boundary of the fundamental group of the given complex of groups. It is worth nothing that the existence of the CT map in Corollary

1.2.4 can be proved using the technique of [48] too, i.e. one can define a natural map from Gromov boundary of H to Gromov boundary of G, and one can check that this map is continuous. But it is a long proof. However, using the geometry of the electrified space, we give an elegant and short proof. Also, our technique is applicable to situations where Martin's technique failed, see for example Theorem 1.2.3 and Corollary 4.3.3.

Further, we give an application of Theorem 1.2.3 in the context of polygons of groups. More specifically, in Section 4.4, we prove the following:

Theorem 1.2.6. *Let Y be a regular Euclidean polygon with at least 4-edges and let $(\mathcal{G},Y)$ be a simple complex of groups over Y. Let e be an edge of Y and let G_1 be the amalgamated free product corresponding to e. Suppose $(\mathcal{G},Y)$ satisfies the following:*

1. *All the local groups are hyperbolic and all the local maps are qi embeddings.*
2. *In vertex groups, the intersection of the two subgroups coming from the adjacent edges is equal to the subgroup coming from the barycenter of Y.*
3. *The universal cover B is a hyperbolic space and the action of $G = \pi_1(\mathcal{G},Y)$ on B is acylindrical.*

Then G_1 is quasiconvex in G.

Using the work of Wise [77] and Theorem 1.2.6, we also prove a combination theorem for virtually compact special groups in the setting of polygons of groups, see Proposition 4.4.12. At the end of Chapter 4, we discuss a few interesting examples.

A few words on the proofs: To prove Theorem 1.2.2, firstly, as a set, we describe the Gromov boundary of X and Y. Its immediate consequence is that there exists a natural map from ∂Y to ∂X. Finally, we check the hypotheses of Lemma 2.2.37 to be able to prove Theorem 1.2.2. We also prove a converse of Theorem 1.2.2. Next, we discuss some group theoretical applications of Theorem 1.2.2 and deduce a proof of Theorem 1.2.3. To prove Theorem 1.2.6, it is sufficient to show that the natural map $B_1 \to B$ satisfies Mitra's criterion and the corresponding CT map is injective. We prove that this map is in fact an isometric embedding. Finally, we give some interesting examples.

Layout of the thesis: In Chapter 2, we recall definitions and results that are relevant to us. In Chapter 3, we prove combination theorems for graphs of convergence and relatively hyperbolic groups. In particular, for a graph of relatively hyperbolic groups with parabolic or cyclic edge groups, we give a construction of the Bowditch boundary of the fundamental group of the graph of groups. Also, we discuss the homeomorphism type of the Bowditch boundary of the fundamental group of a graph of relatively hyperbolic groups with parabolic edge groups. Cannon-Thurston maps for coned-off metric spaces are studied in Chapter 4. As the main application, we obtain the existence of Cannon-Thurston maps for certain subcomplexes of groups of a complex of hyperbolic groups. Finally, deduce some other interesting results and examples too.

Chapter 2

Preliminaries

2.1 Coarse geometric notions

Let X be a metric space. Suppose $x, y \in X$. A **geodesic (segment)** joining x and y is an isometric embedding α from a closed interval $[0, l] \subset \mathbb{R}$ to X such that $\alpha(0) = x, \alpha(l) = y$. (Most of the time, we are interested only in the image of this embedding rather than the embedding itself.) If any two points of X can be joined by a geodesic segment then X is said to be a **geodesic metric space**. In this thesis, graphs are assumed to be connected and it is assumed that each edge is assigned a unit length so that the graphs are naturally geodesic metric spaces ([11, Section 1.9, I.1]).

Now, we collect some basic notions from large scale geometry.

Definition 2.1.1. Let X, Y be metric spaces and let $L \geq 1, k \geq 1, \varepsilon \geq 0$.

1. A map $f : X \to Y$ is said to be ***L*-Lipschitz** if $d_Y(f(x), f(y)) \leq L d_X(x, y)$. The map $f : X \to Y$ is said to be Lipschitz if it is L-Lipschitz for some $L \geq 1$.

2. A map $f : X \to Y$ is said to be ***L*-coarsely Lipschitz** if for all $x, y \in X$, we have

$$d_Y(f(x), f(y)) \leq L d_X(x, y) + L$$

   The map f is said to be coarsely Lipschitz if it is L-coarsely Lipschitz for some $L \geq 1$.

3. Let $\phi : [0, \infty) \to [0, \infty)$ be a map. A map $f : X \to Y$ is said to be a **ϕ-proper embedding** if $d_Y(f(x), f(x')) \leq M$ implies $d_X(x, x') \leq \phi(M)$ for all $x, x' \in X$. A map $f : X \to Y$ is called a proper embedding if it is a ϕ-proper embedding for some $\phi : [0, \infty) \to [0, \infty)$.

In all instances in this thesis, the space X is a subspace of Y and the metric on X is the induced length metric [11, Definition 3.3, I.3] from Y and f is the inclusion which is clearly 1-Lipschitz.

4. A map $f : X \to Y$ is said to (k,ε)- **quasiisometric embedding** (qi embedding) if for all $x, y \in X$, we have

$$\frac{1}{k}d_X(x,y) - \varepsilon \leq d_Y(f(x), f(y)) \leq kd_X(x,y) + \varepsilon$$

The map f is called quasiisometric embedding if it is (k,ε)-quasiisometric embedding for some $k \geq 1, \varepsilon \geq 0$.

5. A map $f : X \to Y$ is said to be (k,ε)-**quasiisometry** if f is a (k,ε)-quasiisometric embedding and there exists $D \geq 0$ such that $N_D(f(X)) = Y$. The map f is said to be quasiisometry if it is (k,ε)-quasiisometry for some $k \geq 1, \varepsilon \geq 0$.

6. A **quasigeodesic** (resp. **quasigeodesic ray**) in X is a (k,ε)-quasiisometric embedding from an interval $I \subset \mathbb{R}$ (resp. from $[0,\infty) \subset \mathbb{R}$) to X for some $k \geq 1, \varepsilon \geq 0$.

7. Let $I \subset \mathbb{R}$ be a closed interval with endpoints in $\mathbb{Z} \cup \{\infty, -\infty\}$. Let $J = \mathbb{Z} \cap I$ with restricted metric from $\mathbb{R}$. Then a (k,ε)-qi embedding $\alpha : J \to X$ will be called a **dotted** (k,ε)-**quasigeodesic**. Moreover, if $I = [0,\infty)$ then a (k,ε)-qi embedding $\alpha : J \to X$ is called a **dotted quasigeodesic ray**.

8. Suppose $A \subset X$. Then the **nearest point projection** of X on A is a map $P_{X,A} : X \to A$ such that $d(x, P_{X,A}(x)) = inf\{d(x, P_{X,A}(y)) : y \in A\}$ for all $x \in X$.

The following lemma is standard. For the sake of completeness, we give its proof.

Lemma 2.1.2. *Suppose X,Y are any metric spaces, $f : Y \to X$ is an L-Lipschitz and ϕ-proper embedding. Suppose Z is a geodesic metric space and $g : Z \to Y$ is a map such that $f \circ g$ is a (k,ε)-qi embedding. Then g is a (k',ε')-qi embedding where k',ε' depend only on L, ϕ, k, ε.*

Proof. Let $z, z' \in Z$. Since $f \circ g$ is a (k,ε)-qi embedding, we have the following inequality:

$$\frac{1}{k}d_Z(z,z') - \varepsilon \leq d_X(f(g(z)), f(g(z'))) \leq kd_Z(z,z') + \varepsilon$$

Since f is an L-Lipschitz map, $\frac{1}{k}d_Z(z,z') - \varepsilon \leq Ld_Y(g(z),g(z'))$. This implies that $\frac{1}{kL}d_Z(z,z') - \frac{\varepsilon}{L} \leq d_Y(g(z),g(z'))$. Next, we show that g is a Lipschitz map. Note that it is sufficient to consider the case when $d_Z(z,z') \leq 1$. Then, $d_X(f(g(z)),f(g(z'))) \leq k+\varepsilon$. Since f is ϕ-proper embedding, $d_Y(g(z),g(z')) \leq \phi(k+\varepsilon)$. Hence, g is L'-Lipschitz where L' depends on ϕ,k,ε. This completes the proof. □

In particular, when Y is a subspace of X equipped with the induced length metric from X and Z is an interval then we obtain that any quasigeodesic of X contained in Y is a uniform quasigeodesic in Y too.

Next, we record the following lemma which is straight forward.

Lemma 2.1.3. *Suppose X is a Hausdorff topological space, $x \in X$ and $\{x_n\}$ is a sequence in X. If for any subsequence $\{x_{n_k}\}$ of $\{x_n\}$, there exists a further subsequence $\{x_{n_{k_l}}\}$ of $\{x_{n_k}\}$ such that $\{x_{n_{k_l}}\}$ converges to x then $\{x_n\}$ converges to x.*

We end this section by including the following which originates from [69].

Definition 2.1.4 (Acylindrical action). Let $k \in \mathbb{N}$. An action of a group G on a metric space (X,d) is said to be k-acylindrical if whenever the pointwise stabilizer $\{x,y\}$ is infinite for $x,y \in X$ then $d(x,y) \leq k$. The action of G on X is said to be acylindrical if it is k-acylindrical for some $k \in \mathbb{N}$.

2.2 Hyperbolic spaces

In his seminal work [32], Gromov introduced the notion of hyperbolic metric spaces. For a detailed account of the work, we refer the reader to some of the standard references like [30], [11].

Definition 2.2.1 (Gromov inner product). Let (X,d) be a metric space and let $p \in X$ be a fixed point. Let $x,y \in X$. Then the Gromov inner product of x,y with respect to p is denoted by $(x.y)_p$ and defined as $\frac{1}{2}(d(p,x)+d(p,y)-d(x,y))$.

Definition 2.2.2 (Gromov hyperbolicity). Let (X,d) be a metric space and let $\delta \geq 0$. Suppose $p \in X$ is fixed. Then X is said to be δ-hyperbolic in the sense of Gromov if for all $x,y,z \in X$ the following holds:

$$(x.y)_p \geq min\{(x.z)_p,(z.y)_p\} - \delta.$$

A metric space is said to be hyperbolic in the sense of Gromov if it is δ-hyperbolic in the sense of Gromov for some $\delta \geq 0$.

Note that Gromov hyperbolicity does not depend on the choice of point $p \in X$, i.e. if X is δ-hyperbolic with respect to point p then it is 2δ-hyperbolic with respect to any other point of X (see [32]).

Lemma 2.2.3. [43, Lemma 1.40] *Given $\varepsilon \geq 0, \delta \geq 0, D \geq 0$ there exists $\delta' = \delta'(\varepsilon, \delta, D)$ such that the following holds:*

Let $f : X \to Y$ be a $(1, \varepsilon)$-quasiisometry. If X is δ-hyperbolic in the sense of Gromov then Y is δ'-hyperbolic in the sense of Gromov.

Proof. Fix an element $p \in X$. Then it is easy to verify that $|(f(x).f(y))_{f(p)} - (x.y)_p| \leq \frac{3\varepsilon}{2}$ for all $x, y \in X$. Thus, for all $x, y, z \in X$, we have

$$\begin{aligned}(f(x).f(y))_{f(p)} &\geq (x.y)_p - \frac{3\varepsilon}{2} \geq min\{(x.z)_p, (z.y)_p\} - \delta - \frac{3\varepsilon}{2} \\ &\geq min\{(f(x).f(z))_{f(p)}, (f(z).f(y))_{f(p)}\} - \delta - 3\varepsilon\end{aligned}$$

Let $y_1, y_2, y_3 \in Y$. Since f is a quasiisometry, $\exists\, D \geq 0$ such that $N_D(f(X)) = Y$. Choose $x_1, x_2, x_3 \in X$ such that $d_Y(f(x_i), y_i) \leq D$ for $i = 1, 2, 3$. Again it is easy to verify that $|(f(x_1).f(x_2))_{f(p)} - (y_1.y_2)_{f(p)}| \leq 2D$. Therefore, we get

$$\begin{aligned}(y_1.y_2)_{f(p)} &\geq (f(x_1).f(x_2))_{f(p)} - 2D \\ &\geq min\{(f(x_1).f(x_3))_{f(p)}, (f(x_3).f(x_2))_{f(p)}\} - \delta - 3\varepsilon - 2D \\ &\geq min\{(y_1.y_3)_p, (y_3.y_2)_p\} - \delta - 3\varepsilon - 4D.\end{aligned}$$

Let y_1, y_2, y_3, y be arbitrary elements of Y. Let $x \in X$ such that $d(f(x), y) \leq D$. Then, we see that $|(y_i.y_j)_{f(x)} - (y_i.y_j)_y| \leq 2D$ for $i, j \in \{1, 2, 3\}$. Hence $(y_1.y_2)_y \geq min\{(y_1.y_3)_y, (y_3.y_2)_y\} - \delta - 3\varepsilon - 4D$. By taking $\delta' = \delta + 3\varepsilon + 4D$, we are done. □

The above lemma shows that Gromov hyperbolicity is invariant under $(1, \varepsilon)$-quasiisometry. In general, this is no longer true; see [43, Example 1.39]. However, Gromov hyperbolicity is known to be invariant for length spaces under quasiisometries [43]. Next, we define hyperbolic geodesic metric space (also known as hyperbolic spaces in the sense of Rips).

Definition 2.2.4. Let X be a geodesic metric space and let $\delta \geq 0$.

1. A geodesic triangle Δ in X is said to be **δ-slim** if any side of Δ is contained in the union of the δ-neighborhood of the remaining two sides.

2. The space X is said to be **δ-hyperbolic** if every geodesic triangle in X is δ-slim.

A geodesic metric space is said to be **hyperbolic** if it is δ-hyperbolic for some $\delta \geq 0$.

By [11, Proposition 1.22, III.H], Definition 2.2.2 and Definition 2.2.4 are equivalent for geodesic metric spaces. Next, we state one of the fundamental properties of hyperbolic spaces.

Lemma 2.2.5 (Stability of quasigeodesics). [11, Theorem 1.7, Chapter III.H] *Given $\delta \geq 0, k \geq 1, \varepsilon \geq 0$ there exists a constant $D = D_{2.2.5}(\delta, k, \varepsilon)$ with the following property:*

Let X be a δ-hyperbolic geodesic metric space. Then the Hausdorff distance between a (k, ε)-quasigeodesic and a geodesic joining the same pair of end points is less than or equal to D.

One of the consequences of the above lemma is that hyperbolicity is an invariant of quasiisometry.

Lemma 2.2.6. [11, Theorem 1.9, Chapter III.H] *Given $k \geq 1, \varepsilon \geq 0, \delta \geq 0$ there exists $\delta' = \delta'(k, \varepsilon, \delta)$ such that the following holds:*

Let X and Y be geodesic metric spaces and let $f : X \to Y$ be a (k, ε)-quasiisometric embedding. If Y is δ-hyperbolic then X is δ'-hyperbolic.

It follows from the Milnor-Švarc lemma [11, Proposition 8.19, I.8] that the Cayley graphs of any finitely generated group G with respect to any two finite generating sets are quasiisometric. Since hyperbolicity is invariant under quasiisometry, the following definition does not depend on the choice of finite generating sets of G.

Definition 2.2.7 (Hyperbolic groups). Let $\delta \geq 0$. A finitely generated group G is said to be δ-hyperbolic if the Cayley graph of G with respect to a finite generating set is δ-hyperbolic.

A finitely generated group is said to be hyperbolic if it is δ-hyperbolic for some $\delta \geq 0$.

Recall that a subset C of a geodesic metric space X is **convex** if for all $x, y \in C$, each geodesic joining x to y is contained in C. In [32], Gromov quasified this notion and introduced the notion of a quasiconvex subset of a geodesic metric space.

Definition 2.2.8 (Quasiconvexity). Let X be a geodesic metric space and let $Q \subset X$. Let $K \geq 0$. Then Q is said to be K-**quasiconvex** if every geodesic with end points in Q is contained in $N_K(Q)$. A subset of X is said to quasiconvex if it is K-quasiconvex for some $K \geq 0$.

A collection $\{A_i\}_{i \in I}$ of subsets of X is said to be **uniformly K-quasiconvex** if A_i is K-quasiconvex in X for all i. The collection $\{A_i\}$ is said to be uniformly quasiconvex if it is uniformly K-quasiconvex for some $K \geq 0$.

Definition 2.2.9. Let G be a finitely generated group. A subgroup H of G is said to be quasiconvex if it is a quasiconvex subset of the Cayley graph of G with respect to a finite generating set.

The following lemma shows the persistence of quasiconvexity under qi embedding of hyperbolic spaces. Thus it shows that quasiconvexity of any subset of a hyperbolic group is well-defined, i.e. independent of the Cayley graphs.

Lemma 2.2.10. *Given $K \geq 0, k \geq 0, \delta \geq 0$ there exists $K' = K'(K,k,\delta)$ such that the following holds:*

Suppose $f : X \to Y$ is a k-qi embedding of δ-hyperbolic geodesic metric spaces. If $A \subset X$ is K-quasiconvex then $f(A) \subset Y$ is K'-quasiconvex.

Proof. Let γ be a geodesic in X joining x_1, x_2. Since f is a qi embedding, $f(\gamma)$ is a quasigeodesic joining $y_i = f(x_i)$ for $i = 1,2$. The lemma is then immediate by using stability of quasigeodesics (Theorem 2.2.5). □

Next, we record the following basic lemma.

Lemma 2.2.11. [46, Corollary 2.29] *Given $\delta \geq 0, k \geq 0$ there exists $K = K(\delta,k)$ such that the following holds:*

Let X be a δ-hyperbolic geodesic metric space and let γ be a k-quasigeodesic in X with an end point y. Suppose $x \in X$ and y is a nearest point projection of x on γ. Let β be a geodesic in X joining x and y. Then the concatenation of β and γ is a K-quasigeodesic in X.

The following lemma is standard. For the sake of completeness, we give its proof.

Lemma 2.2.12. *Let X be a geodesic metric space and let Q be a K-quasiconvex subset of X. Then, given two points $x,y \in Q$, there exists a uniform dotted quasigeodesic path joining x and y whose image is contained in Q. Moreover, we have a uniform quasigeodesic in X joining x and y.*

Proof. We give a sketch of proof. Let $\alpha : [0,l] \to X$ be a geodesic joining x and y, where l is the length of α. If $l \leq 1$ then lemma follows. Suppose $l > 1$. Choose points $x_0, x_1, ..., x_n$ on α such that x_i, x_{i+1} are sufficiently separated (depending on K) for $0 \leq i \leq n-1$. Since Q is K-quasiconvex, there exists $q_i \in Q$ such that $d(x_i, q_i) \leq K$ for $i = 1,2,...,(n-1)$. We take $q_0 = x_0, q_n = x_n$. Define a map $\beta : [0,l) \cap \mathbb{Z} \cup \{l\} \to X$ such that $\beta(0) = q_0, \beta(l) = q_n$ and $\beta(i) = q_i$ for $i = 1,2,...,(n-1)$. It is immediate that β is a uniform dotted quasigeodesic. Now, join q_i and q_{i+1} by a geodesic α_i in

X. Then, it is easy to verify that the concatenation $\alpha_0 * \alpha_1 * ... * \alpha_{n-1}$ is a uniform quasigeodesic joining x, y in X. □

The following lemma shows that the finite union of quasiconvex sets is quasiconvex. The proof is clear by induction and hence we skip it.

Lemma 2.2.13. *Given $\delta \geq 0, k \geq 0$ and $n \in \mathbb{N}$ there exists $D = D(\delta, k, n)$ such that the following holds:*

Suppose X is a δ-hyperbolic geodesic metric space and $\{A_i\}_{1 \leq i \leq n}$ is a collection of k-quasiconvex subsets in X such that $A_i \cap A_{i+1} \neq \emptyset$ for all $1 \leq i \leq n-1$. Then $\cup_i A_i$ is a D-quasiconvex subset of X.

In general, an arbitrary union of quasiconvex sets need not be quasiconvex. However the following is true.

Lemma 2.2.14. *Given $\delta \geq 0, K \geq 0$ there is $D = D(\delta, K)$ such that the following holds:*

Suppose X is a δ-hyperbolic geodesic metric space, $\{A_i\}$ is any sequence of K-quasiconvex sets in X and $\gamma \subset X$ is a geodesic such that $A_i \cap A_{i+1} \cap \gamma \neq \emptyset$ for all $i \geq 1$. Then $\cup_i A_i$ is a D-quasiconvex set in X.

Proof. Let $x_i \in A_i \cap A_{i+1} \cap \gamma$ for all i. For all $i \leq j$, let $[x_i, x_j]$ denote the segment of γ from x_i to x_j. Clearly $[x_i, x_{i+1}] \subset N_K(A_{i+1})$ for all i and hence $[x_i, x_j] \subset N_K(\cup_{i+1 \leq k \leq j+1} A_k) \subset N_K(\cup_k A_k)$ for all $i \leq j$. Note that quadrilaterals in X are 2δ-slim. Now, given $x \in A_i, y \in A_j$, $i \leq j$, we have $[x, y] \subset N_{2\delta}([x, x_i] \cup [x_i, x_j] \cup [x_j, y]$. Hence, $[x, y] \subset N_{2\delta + K}(\cup_k A_k)$. Hence, we may choose $D(\delta, K) = 2\delta + K$. □

2.2.1 Boundaries of hyperbolic spaces and Cannon-Thurston maps

Definition 2.2.15 (Geodesic boundary). Let X be a geodesic hyperbolic metric space and let $x_0 \in X$. Then the geodesic boundary ∂X is defined in the following way:

$$\partial X := \{\gamma : \gamma : [0, \infty) \to X \text{ is a geodesic ray such that } \gamma(0) = x_0\} / \sim$$

where two geodesic rays $\gamma_1 \sim \gamma_2$ if and only if $Hd(\gamma_1, \gamma_2) < \infty$.

The equivalence class of a geodesic ray γ is denoted by $\gamma(\infty)$. If X is a proper geodesic hyperbolic space then the geodesic boundary of X does not depend on the choice of base point x_0, [41, Proposition 2.10].

Definition 2.2.16. Let X be a geodesic metric space. A **generalised geodesic ray** is a geodesic $\gamma : I \to X$, where either $I = [0,R]$ for some $R \geq 0$ or else $I = [0,\infty)$. In the case $I = [0,R]$ it is convenient to define $\gamma(t) = \gamma(R)$ for $t \in [R,\infty]$.

Thus for a hyperbolic geodesic metric space X, $\bar{X} := X \cup \partial X = \{\gamma(\infty) : \gamma$ is a generalised geodesic ray$\}$.

Definition 2.2.17 (Topology on $\bar{X}$). Let X be a hyperbolic geodesic metric space and let $x_0 \in X$. We define convergence in $\bar{X}$ by : a sequence $\{x_n\}$ in $\bar{X}$ converges to $x \in \bar{X}$ if and only if there exist generalised geodesic rays γ_n with $\gamma_n(0) = x_0$ and $\gamma_n(\infty) = x_n$ such that every subsequence of $\{\gamma_n\}$ contains a subsequence that converges (uniformly on compact sets) to a generalised ray γ with $\gamma(\infty) = x$. This defines a topology on $\bar{X}$: the closed subsets $C \subset \bar{X}$ are those which satisfy the condition $[x_n \in C, \forall n \geq 1$ and $x_n \to x] \implies x \in C$.

Let X be a δ-hyperbolic geodesic metric space and let γ be a geodesic ray in X such that $\gamma(0) = x_0$. Let $k > 2\delta$. For $n \in \mathbb{N}$, we define the following set:

$$V(\gamma,n) := \{\alpha : \alpha \text{ is a generalised ray such that } \alpha(0) = x_0 \text{ and } d(\gamma(n),\alpha(n)) < k\}.$$

Lemma 2.2.18. [11, Lemma 3.6, III.H] *The collection $\{V(\gamma,n)\}_{n\in\mathbb{N}}$ is a fundamental system of neighborhoods around $\gamma(\infty)$ in $\bar{X}$.*

Next lemma shows that the topology on $\bar{X}$ does not depend on the choice of base point x_0.

Lemma 2.2.19. [11, Proposition 3.7,III.H] *Let X be a proper δ-hyperbolic geodesic metric space. Then*

1. *The topology on $\bar{X}$ is independent of the choice of base point.*
2. *$X \hookrightarrow \bar{X}$ is a homeomorphism onto its image and $\partial X \subset \bar{X}$ is closed.*
3. *$\bar{X}$ is compact.*

Lemma 2.2.20. [11, Theorem 3.9,III.H] *Let X and Y be proper hyperbolic geodesic spaces. If $f : X \to Y$ is a quasiisometric embedding, then $\alpha(\infty) \to (f \circ \alpha)(\infty)$ defines a topological embedding $f_\partial : \partial X \to \partial Y$. If f is a quasiisometry then f_∂ is a homeomorphism.*

Next, we define Gromov boundary of a hyperbolic group.

Definition 2.2.21. Let G be a hyperbolic group. Then, the Gromov boundary of G is defined as the Gromov boundary of a Cayley graph of G with respect to a finite generating set.

The previous definition is independent of the choice of the Cayley graphs as different Cayley graphs with respect to different finite generating sets are quasiisometric. Thus, by Lemma 2.2.20, their Gromov boundaries are homeomorphic.

Definition 2.2.22 (Quasigeodesic boundary). Suppose X is a hyperbolic metric space. Then the quasigeodesic boundary $\partial_q X$ is defined in the following way:

$$\partial_q X = \{\alpha : \alpha \text{ is a quasigeodesic ray}\}/\sim$$

where two quasigeodesics rays $\alpha \sim \beta$ if and only if $Hd(\alpha,\beta) < \infty$.

Definition 2.2.23 (Sequential boundary). Let X be a hyperbolic metric space and let $p \in X$. Let $\mathscr{S}$ denote the set of all the sequences $\{x_n\}$ such that $\lim_{i,j\to\infty}(x_i.x_j)_p = \infty$. Define an equivalence relation $\sim$ on $\mathscr{S}$ by declaring two sequences $\{x_n\}, \{y_n\}$ are equivalent if and only if $\lim_{i,j\to\infty}(x_i.y_j)_p = \infty$. As a set, the Gromov boundary or the sequential boundary $\partial_s X$ of X is defined as $\mathscr{S}/\sim$.

The above definition does not depend on the base point p. The following is a very basic lemma.

Lemma 2.2.24. *(1) If $\{x_n\} \in \mathscr{S}$ and $\{x_{n_k}\}$ is a subsequence of $\{x_n\}$ then $\{x_{n_k}\} \in \mathscr{S}$ and $\{x_n\} \sim \{x_{n_k}\}$.*

(2) If $\{x_n\}, \{y_n\} \in \mathscr{S}$ then $\{x_n\} \sim \{y_n\}$ if and only if $\lim_{n\to\infty}(x_n.y_n)_p = \infty$.

If $\{x_n\} \in \mathscr{S}$ then the equivalence class of $\{x_n\}$ will be denoted by $[\{x_n\}]$. If $\xi = [\{x_n\}] \in \partial_s X$ then we say x_n converges to ξ. Next, we record some of the basic facts about boundaries of hyperbolic spaces.

Lemma 2.2.25. *Suppose X is a δ-hyperbolic metric space.*

1. *Given a quasigeodesic $\alpha : [0,\infty) \to X$, the sequence $\{\alpha(n)\}$ converges to infinity. This gives rise to an injective map $\partial_q X \to \partial_s X$.*

2. *There is a constant k_0 depending only on δ such that for any $x_0 \in X$ and any $\xi \in \partial_s X$ there is a k_0-quasigeodesic in X joining x_0 to ξ. In particular, the map $\partial_q X \to \partial_s X$ mentioned above is surjective.*

   Also for all $\xi_1 \neq \xi_2 \in \partial_s X$ there is a k_0-quasigeodesic line γ in X joining ξ_1, ξ_2, i.e. such that ξ_1 is the equivalence class of $\{\gamma(-n)\}$ and ξ_2 is the equivalence class of $\{\gamma(n)\}$.

3. *If Y is another hyperbolic metric space and $f : Y \to X$ is a qi embedding then f induces an injective map $\partial f : \partial_s Y \to \partial_s X$. This map is functorial:*

   (a) If $I : X \to X$ is the identity map then ∂I is the identity map on $\partial_s X$.

   (b) If $g : Z \to Y$ and $f : Y \to X$ are qi embeddings of hyperbolic metric spaces then $\partial f \circ \partial g = \partial f \circ g$.

Topology on $X \cup \partial_s X$.
There is a natural topology on $\partial_s X$ defined in the following way:

Let $\xi_n = [\{x^n_k\}_k]$ be a sequence of points in $\partial_s X$ and let $\xi = [\{x_k\}]$. Then $\xi_n \to \xi$ if and only if $\lim_{n\to\infty}(\liminf_{i,j\to\infty}(x^n_i.x_j)_p) = \infty$. Next, we include the following basic facts that we are going to need later.

If $\{x_n\}$ is a sequence in X and $\xi \in \partial_s X$ then $x_n \to \xi$ if and only if $\{x_n\}$ converges to infinity and $\xi = [\{x_n\}]$.

Remark 2.2.26. If X is a proper hyperbolic geodesic metric space then the geodesic boundary and the sequential boundary of X are naturally homeomorphic [11, Lemma 3.13, III.H]. When X is a hyperbolic metric space, we interchangeably write ∂X and $\partial_s X$ for the sequential boundary. We also write $\bar{X} = X \cup \partial_s X$.

Next lemma gives a geometric criteria for convergence of a sequence in a hyperbolic metric space.

Lemma 2.2.27. ([46, Lemma 2.41]) *For all $\delta \geq 0$ and $k \geq 1$ the following holds: Suppose $\{x_n\}$, $\{y_n\}$ are sequences in a δ-hyperbolic metric space X and $\xi \in \partial_s X$. Let $\alpha_{m,n}$ be a k-quasigeodesic joining x_m, x_n, and let $\beta_{m,n}$ be a k-quasigeodesic joining x_m, y_n for all $m, n \in \mathbb{N}$. Let γ_n be a k-quasigeodesic ray joining x_n to ξ for all $n \in \mathbb{N}$. Let $x_0 \in X$ be an arbitrary fixed point. Then*

(1) $\{x_n\}$ converges to infinity if and only if $\lim_{m,n\to\infty} d(x_0, \alpha_{m,n}) = \infty$.

(2) If both the sequences $\{x_n\}, \{y_n\}$ are converging to infinity then $\{x_n\} \sim \{y_n\}$ if and only if $\lim_{m,n\to\infty} d(x_0, \beta_{m,n}) = \infty$.

(3) $x_n \to \xi$ if and only if $\lim_{n\to\infty} d(x_0, \gamma_n) = \infty$.

The following lemma gives yet another criteria for convergence.

Lemma 2.2.28. *Given $\delta \geq 0, k \geq 1$ there exists $D = D(\delta, k)$ such that following holds:*

Let X be a δ-hyperbolic metric space and let $x_0 \in X$. Suppose α is a k-quasigeodesic ray converging to $\xi \in \partial_s X$ and $\{x_n\}$ is a sequence in X. Let α_n be a k-quasigeodesic segment joining x_0 to x_n. Then $x_n \to \xi$ if and only if for all $R \geq 0$ there is $N \in \mathbb{N}$ such that $Hd(\alpha \cap B(x_0; R), \alpha_n \cap B(x_0; R)) \leq D + d(x_0, \alpha(0))$ for all $n \geq N$.

In reference to this lemma we shall informally say that α_n**'s fellow travel** α **for a longer and longer time.** The idea of the proof is very similar to that of Lemma 1.15 and also Lemma 3.3 of [11, Chapter III.H]. Since this is very standard we skip its proof. One is also referred to [46, Lemma 2.41].

Lemma 2.2.29. *Suppose X is a δ-hyperbolic metric space and $\xi \in \partial_s X$. Suppose $R > 0$ and for all integer $n \geq R$ there is a sequence $\{x^n_k\}$ in X with $x^n_k \to \xi$ as $k \to \infty$. There is an unbounded sequence of integers $\{c_k\}$ and a subsequence $\{n_k\}$ of the sequence of natural numbers such that $\{x^{c_k}_{n_k}\} \to \xi$.*

Proof. Let $x \in X$. A sequence $\{x_n\}$ in X converges to $\xi \in \partial_s X$ if and only if $\lim_{n\to\infty}(x_n.\xi)_x \to \infty$. Thus, given $r_1 > 0$ there exists $n_1 \in \mathbb{N}$ such that $(x^{R+1}_{n_1}.\xi)_x > r_1$. Now, choose r_2 sufficiently larger than r_1 such that there exists $n_2 > n_1$ and $(x^{R+2}_{n_2}.\xi)_x > r_2$. By continuing in this way, we have increasing unbounded sequences of numbers $\{r_k\}$ and $\{n_k\}$ such that $(x^{R+k}_{n_k}.\xi)_x > r_k$. Now, by defining $c_k = R+k$, it is immediate that $\{x^{c_k}_{n_k}\} \to \xi$. □

Definition 2.2.30 (Limit set of a subset). Suppose X is a hyperbolic metric space. The *limit set of a subset Y* of X is the set $\{\xi \in \partial_s X : \exists \{x_n\} \subset Y \text{ with } \lim_{n\to\infty} x_n = \xi\}$.

Lemma 2.2.31. *Let $\delta \geq 0, k \geq 0$ be any constants. Suppose X is a δ-hyperbolic geodesic metric space and A is a k-quasiconvex subset of X. Suppose γ is a geodesic ray in X. Let $P_{X,A}$ be the nearest point projection of X onto A. If the $diam_X(P_{X,A}(\gamma))$ is infinite then $\gamma(\infty) \in \Lambda(A)$.*

Proof. We give a sketch of proof. Let $D = 2k+7\delta$ and $R = k+5\delta$. Suppose $\gamma(0) = x$. Since $diam_X(P_{X,A}(\gamma))$ is infinite, choose a point $x_1 = \gamma(t_1)$ for some $t_1 \in [0,\infty)$ such that $d(P_{X,A}(x), P_{X,A}(x_1)) \geq D$. Then, by [43, Lemma 1.120], $a_1 := P_{X,A}(x_1) \in N_R(\gamma|_{[0,t_1]})$. Thus, there exists $y_1 \in [x,x_1]$ such that $d(a_1,y_1) \leq R$. Note that $diam_X(P_{X,A}(\gamma([t_1,\infty))))$ is still infinite. Again choose $x_2 = \gamma(t_2)$ for some $t_2 \in [t_1,\infty)$ such that $d(P_{X,A}(x_2), P_{X,A}(x_1)) \geq D$. Let $a_2 := P_{X,A}(x_2)$. By the same reason, there exists $y_2 \in \gamma|_{[t_1,t_2]}$ such that $d(a_2,y_2) \leq R$. By continuing in this way, we obtain sequences $\{a_n \in A\}$ and $\{y_n \in \gamma([0,\infty))\}$ such that $d(a_n,y_n) \leq R$. Note that $\lim_{n\to\infty} y_n = \gamma(\infty)$. Now, it is easy to check that $\lim_{n\to\infty} a_n = \gamma(\infty)$. □

The following lemma gives the existence of a dotted quasigeodesic ray in a quasiconvex subset A corresponding to each limit point of A.

Lemma 2.2.32. *Given $\delta \geq 0$ and $k \geq 0$ there is $K = K(\delta,k)$ such that the following holds:*
Suppose X is a δ-hyperbolic geodesic metric space and $A \subset X$ is a k-quasiconvex

subset. Then for all $\xi \in \Lambda(A)$ and $x \in A$ there is a dotted K-quasigeodesic ray γ of X contained in A where γ joins x to ξ.

Proof. By Lemma 2.2.25(2), there is a $k_0 = k_0(\delta)$- quasigeodesic ray $\alpha : [0,\infty) \to X$ joining x to ξ. Since $\xi \in \Lambda(A)$, there is a sequence $\{x_n\}$ in A such that $\lim_{n\to\infty} x_n = \xi$. Let y_1 be a nearest point projection of x_1 on α. Then, by Lemma 2.2.11, the concatenation of $[x_1, y_1]$ and portion of α from y_1 to ξ is a uniform quasigeodesic, call it α_1. Since $x_n \to \infty$, choose n to be sufficiently large such that nearest point projection y_n of x_n on α_1 lie on the portion of α from y_1 to ξ. Again, by Lemma 2.2.11, the concatenation of $[x_1, y_1]$, the portion of α_1 between y_1, y_n, and $[x_n, y_n]$ is a uniform quasigeodesic joining x_1, x_n. Thus, by quasiconvexity of A, the portion of α_1 between y_1 and y_n lies in a D-neighborhood of A where D depends on k, δ. Hence $\alpha(n) \in N_D(A)$ for all large n. Thus, there exists $a_n \in A$ such that $d_X(\gamma(n), a_n) \leq D$ for all large n. Now, it is straight forward to check that $n \to a_n$ is a uniform dotted $(k_0 + 2D)$-quasigeodesic ray in X. □

Now, we are ready to define Cannon-Thurston map. For the history of CT maps and its offshoots, one is referred to [59],[46],[43].

Definition 2.2.33 (Cannon-Thurston map). Let $f : X \to Y$ be a map between two hyperbolic metric spaces. We say that the Cannon-Thurston (CT) map exists for f or f admits the CT map if f gives rise to a well-defined continuous map $\partial f : \partial Y \to \partial X$ in the following sense:

(1) Given any $\xi \in \partial X$ and a sequence $\{x_n\}$ in X converging to ξ, the sequence $\{f(x_n)\}$ converges to a point in ∂Y independent of $\{x_n\}$. This gives ∂f.

(2) We also require that ∂f is continuous.

However, the Lemma 2.2.36 shows that condition (2) in the definition of CT map follows from condition (1). In [57], Mj (Mitra) proved the following lemma that gives a sufficient condition for the existence of Cannon-Thurston map:

Lemma 2.2.34. ([57, Lemma 2.1]) *Suppose X, Y are hyperbolic geodesic metric spaces and $f : Y \to X$ is a coarsely Lipschitz map. Then f admits a Cannon-Thurston map if the following holds:*

Given $y_0 \in Y$ there exists a non-negative function $M(N)$ such that $M(N) \to \infty$ as $N \to \infty$ and such that for all geodesic segments λ lying outside $B(y_0, N)$ in Y, any geodesic segment in X joining the end points of $f(\lambda)$ lies outside $B(f(y_0), M(N))$ in X.

Above lemma gives a sufficient condition for the existence of a Cannon-Thurston map, it is not necessary unless the spaces involved are proper metric spaces (see [46, Subsection 2.4]).

Remark 2.2.35. In what follows, we refer to Lemma 2.2.34 as **Mitra's criterion** for the existence of Cannon-Thurston map.

The lemma below gives a criterion for the existence of CT maps that will be useful to us later.

Lemma 2.2.36. *Suppose $f : X \to Y$ is any map between hyperbolic metric spaces which satisfies the condition (1) of Definition 2.2.33. Then $\partial f : \partial X \to \partial Y$ is continuous too, i.e. ∂f is the CT map.*

Proof. Fix $x_0 \in X, y_0 \in Y$. Suppose $\{\xi_n\}$ is a sequence in $\partial_s X$ and $\xi_n \to \xi \in \partial_s X$. We want to show that $\partial f(\xi_n) \to \partial f(\xi)$. Suppose this is not the case. Let $\xi_k = [\{x^k_n\}]$ and $\xi = [\{x_n\}]$. Then there is $R \geq 0$ such that up to passing to a subsequence of $\{\xi_n\}$ we may assume that $\liminf_{m,n\to\infty}(f(x^k_m).f(x_n))_{y_0} \leq R$ and $\liminf_{m,n\to\infty}(x^k_m.x_n)_{x_0} \geq k$ for all k. This implies that for all $k \in \mathbb{N}$ there is $m_k \in \mathbb{N}$ such that $(x^k_{m_k}.x_{m_k})_{x_0} \geq k$ but $(f(x^k_{m_k}).f(x_{m_k}))_{y_0} \leq R$. Consequently, by Lemma 2.2.24 $x^k_{m_k} \to \xi$ but $f(x^k_{m_k}) \not\to \partial f(\xi)$. This gives us a contradiction. □

Corollary 2.2.37. *Suppose X,Y are hyperbolic metric spaces, and $f : X \to Y$ and $g : \partial_s X \to \partial_s Y$ are any maps which satisfy the following property:*
For any $\xi \in \partial_s X$ and any sequence $x_n \to \xi$ where $x_n \in X$ for all $n \in \mathbb{N}$ there is a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $f(x_{n_k}) \to g(\xi)$ as $k \to \infty$. Then g is the CT map induced by f.

Proof. Suppose $\xi \in \partial_s X$ and $\{x_n\}$ is a sequence in X converging to ξ. Then any subsequence of $\{f(x_n)\}$ has a subsequence which converges to $g(\xi)$. Since $\bar{Y}$ is a Hausdorff space, it clearly follows that $f(x_n) \to g(\xi)$. Then we are done by Lemma 2.2.36. □

The existence of CT maps gives the following useful criteria for quasiconvexity and a sort of converse to Lemma 2.2.20 in the setting of groups.

Lemma 2.2.38. *Suppose G is a hyperbolic group and H is a hyperbolic subgroup of G such that the inclusion $H \to G$ admits a CT map $\partial i : \partial H \to \partial G$. If ∂i is injective then H is quasiconvex in G.*

One may refer to [58, Lemma 2.5],[38, Proposition 2.13] for a proof.

2.3 Convergence groups

The study of a convergence group was introduced by Gehring and Martin [24] in order to describe the dynamical properties of Kleinian groups acting on the ideal sphere of (real) hyperbolic space. Later, this notion was generalised for groups acting on compact Hausdorff spaces by several people such as Tukia, Freden and Bowditch [74],[23],[9].

Definition 2.3.1. Let X be a compact Hausdorff space. A group G acting by homeomorphisms on X is said to be **convergence** if, whenever $\{g_n\}_n$ is a sequence of distinct points in G there is a subsequence $\{g_{n_k}\}_k$ and two points $a,b \in X$ such that $g_{n_k}|X \setminus \{b\} \to a$ and $g_{n_k}^{-1}|X \setminus \{a\} \to b$ uniformly on compact subsets of $X \setminus \{b\}, X \setminus \{a\}$, respectively.

The following is the natural instance of convergence groups. Let X be a proper hyperbolic geodesic metric space and let a group G that acts properly discontinuously and isometrically on X. Then, we get an induced action by homeomorphism on $\bar{X} = X \cup \partial X$.

Lemma 2.3.2. [9, Proposition 1.12] *G acts as a convergence group on $\bar{X}$.*

In particular, a hyperbolic group acts as a convergence group on its Gromov boundary. Let G be a group acting as a convergence group on a compactum (compact Hausdorff space) X. Given $g \in G$, we define $fix(g) = \{x \in X : gx = x\}$.

Definition 2.3.3. If the order of g is finite then g is called an elliptic element. An infinite order element $g \in G$ is said to be **parabolic** if $|fix(g)| = 1$. An infinite order element $g \in G$ is said to be **loxodromic** if $|fix(g)| = 2$.

Next lemma gives a classification of the elements of a convergence group.

Lemma 2.3.4. [9, Lemma 2.1] *Suppose G is a group that acts as a convergence group on a compactum X. Then every element of G is either elliptic, parabolic, or loxodromic.*

The following lemma shows that a parabolic element cannot be loxodromic and vice versa.

Lemma 2.3.5. [74, Theorem 2G] *Let G be a convergence group and let $g,h \in G$. Assume that g,h are either loxodromic or parabolic elements. Then $fix(g)$ and $fix(h)$ are either coincide or disjoint.*

Definition 2.3.6. Suppose G is a group that acts as a convergence group on a compactum X. A point $p \in X$ is called **parabolic** if there exists an infinite subgroup H of G such that H does not have a loxodromic element and H fixes p. The subgroup H is called a **parabolic subgroup** and $Stab_G(p)$ is called a **maximal parabolic subgroup**. The point p is called a **bounded parabolic point** if $Stab_G(p)$ acts co-compactly on $X \setminus \{p\}$.

Definition 2.3.7. Let G be a group acting as a convergence group on a compactum X. A point $z \in X$ is said to be a **conical limit point** if there exists a sequence $\{g_n\}_n$ and two distinct points $a, b \in X$ such that $g_n z$ converges to a and for all $y \in X \setminus \{z\}$, $g_n y$ converges to b.

Next we record the following lemma proved by Bowditch in [9].

Lemma 2.3.8. [9, Proposition 3.2] *A conical limit point cannot be a parabolic fixed point.*

Let G be a group that acts as a convergence group on a compact metrizable space X. If every point of X is a conical limit point then G is said to be a **uniform convergence group**, see [9, 8]. In [8], Bowditch proved the following topological characterization of hyperbolic groups .

Theorem 2.3.9. [8, Theorem 0.1] *Let X be a perfect metrizable compactum and let G be a group acting as a convergence group on X. Suppose that G is a uniform convergence group. Then, G is hyperbolic. Moreover, there is a G-equivariant homeomorphism from X onto ∂G.*

Now, we are ready to define geometrically finite convergence groups. For more details, one is referred to [10],[78],[25].

Definition 2.3.10. Let G be a group that acts as a convergence group on a compact metrizable space X. The group G is said to be **geometrically finite** if every point of X is either conical or bounded parabolic.

Next, we define the notion of the limit set of a subgroup of a convergence group.

Definition 2.3.11. Let G be a group acting as a convergence group on a compactum X and let H be a subgroup of G. The **limit set** $\Lambda(H)$ of H is the set of limit points, where a **limit point** is an accumulation point of some H-orbit in X.

It is well known that if $|\Lambda(H)| \geq 2$ then the limit set can be characterized as unique minimal non-empty closed H-invariant subset of X. The limit set of a finite subgroup is empty.

Definition 2.3.12. Let G be a group that acts as a convergence group on a compactum X. Let H be a subgroup of G.

(1) H is said to be **dynamically quasiconvex** if

$$|\{gH \in G/H : g\Lambda(H) \cap K \neq \emptyset, g\Lambda(H) \cap L \neq \emptyset\}| < \infty$$

whenever K and L are closed disjoint subsets of X.

(2) H is said to be **dynamically malnormal** if for all $g \in G \setminus H$, $g\Lambda(H) \cap \Lambda(H) = \emptyset$. A collection of subgroups $\{H_i < G\}_{i \in I}$ is said to form a **dynamically malnormal family** if all H_i are dynamically malnormal and for all $g \in G$, $g\Lambda(H_i) \cap \Lambda(H_j) = \emptyset$ unless $i = j$ and $g \in H_i$.

Lemma 2.3.13. [79, Lemma 2.6] *Let G be a group acting as a convergence group on a compactum X. Let H and J be two subgroups of G satisfying the following:*

1. *H is dynamically quasiconvex with $|\Lambda(H)| \geq 2$.*
2. *$H \subset J \subset G$ and $\Lambda(H) = \Lambda(J)$.*

Then, the index of H in J is finite and J is also dynamically quasiconvex.

A subgroup H of a group G is said to be **weakly malnormal** if $|gHg^{-1} \cap H| < \infty$ for $g \in G \setminus H$. The subgroup H is said to be **malnormal** if $|gHg^{-1} \cap H| = \{1\}$ for $g \in G \setminus H$.

Lemma 2.3.14. *Let G be a group acting as a convergence group on a compactum X and let $H < G$ be a subgroup. If H is dynamically malnormal then H is malnormal.*

Proof. Let $g \in G \setminus H$. Let if possible $|gHg^{-1} \cap H| = \infty$. Then, $\Lambda(gHg^{-1} \cap H) \neq \phi$. Since $\Lambda(gHg^{-1} \cap H) \subset \Lambda(gHg^{-1}) \cap \Lambda(H) = g\Lambda(H) \cap \Lambda(H)$, we get a contradiction as H is dynamically malnormal. □

Example 2.3.15. Let G be a group acting as a convergence group on a compactum X. Let P be a parabolic subgroup of G. Then $\Lambda(P)$ is a singleton. It follows from the definition that P is a dynamically quasiconvex subgroup of G. Also, if P is a maximal parabolic subgroup then P is dynamically malnormal.

Let G be a hyperbolic group. Then, quasiconvex subgroups of G are dynamically quasiconvex, see [9]. It is well known that cyclic subgroups of a hyperbolic group are quasiconvex (see [11, III.H]). Now, we prove an analogue of this in the setting of convergence groups.

Lemma 2.3.16. [73, Lemma 2.5] *Suppose G is a group acting as a convergence group on a compactum X. Then infinite cyclic subgroups of G are dynamically quasiconvex.*

First of all, we record the following proposition:

Proposition 2.3.17. *Let G be a group having a convergence action on a compact metrizable space X. Let H be a subgroup of G and let ΛH be its limit set. Then H is dynamically quasiconvex if and only if for any sequence $\{g_n\}$ in distinct left cosets of H in G there exists a subsequence $\{g_{\sigma(n)}\}$ of $\{g_n\}$ such that $g_{\sigma(n)}\Lambda H$ uniformly converges to a point.*

For relatively hyperbolic groups, the above proposition also appears in [18, Proposition 1.8]. We skip proof of the above proposition as it follows directly from the definition of a dynamically quasiconvex subgroup. Before proving Lemma 2.3.16, we prove the following:

Lemma 2.3.18. [73, Lemma 2.7] *Let X be a proper geodesic hyperbolic space and let G be a group acting by isometries on X. Suppose G acts as a convergence group on ∂X (Gromov boundary of X). Then, infinite cyclic subgroups of G are dynamically quasiconvex.*

Proof. Let $g \in G$ be an infinite order element. If g is a parabolic element for $G \curvearrowright \partial X$. Then, clearly $\langle g \rangle$ is dynamically quasiconvex. Suppose $\Lambda(\langle g \rangle) = \{x, y\}$. Suppose $\langle g \rangle$ is not dynamically quasiconvex. Then, by the above proposition, there exists a sequence $\{g_n\}_n \subset G \setminus \langle g \rangle$ such that $g_n\Lambda(\langle g \rangle)$ converges to two distinct points, say, ξ and η. Without loss of generality, we can assume that $g_n x \to \xi$ and $g_n y \to \eta$. Since the action of G on ∂X is convergence, there exists a subsequence $\{g_{n_k}\}_k$ of $\{g_n\}$ and two points $a, b \subset \partial X$ such that for all $z \in \partial X \setminus \{b\}$, $g_{n_k} z$ converges to a. Note that G also acts on $X \cup \partial X$ as a convergence group with the same attracting and repelling points. If $x \neq b$ and $y \neq b$ then both $g_{n_k} x, g_{n_k} y$ converge to a. This implies $\xi = \eta$, a contradiction. Now, suppose $x \neq b$ and $y = b$. Then $g_{n_k} x \to a$ and $g_{n_k} y \to \eta$. Thus, $a = \xi$. Now, all the points, except y, on a bi-infinite geodesic ray joining x and y converge to a under the action of g_{n_k}. Now, choose a point p on a bi-infinite geodesic ray joining ξ and η close enough to η and consider a ball around p of radius R for some $R > 0$. Then this ball does not intersect the bi-infinite geodesic rays joining $g_{n_k} x$ to $g_{n_k} y$ for sufficiently large k. This is a contradiction as $g_n x$ and $g_n y$ converging to two different points in the boundary of a proper hyperbolic space. We get a similar contradiction when $x = b$ and $y \neq b$. Hence $\langle g \rangle$ is dynamically quasiconvex for $G \curvearrowright \partial X$. □

Proof of Lemma 2.3.16: If G acts on X as an elementary convergence group, then every infinite cyclic subgroup is dynamically quasiconvex. Now, suppose G acts on X as a non-elementary convergence group. Let Q be the set of all distinct triples. Then, by [8],[72, Proposition 6.4], there is a G-invariant hyperbolic path quasimetric on Q. Thus, we can define the boundary, ∂Q, of Q as in [8]. By [8, Proposition 4.7], ∂Q is G-equivariantly homeomorphic to X. Now ∂Q is compact as X is compact. Then, by [8, Proposition 4.8], there is a locally finite hyperbolic path quasi-metric space Q' (quasiisometric to Q) and X is G-equivariantly homeomorphic to $\partial Q'$. As Q' is locally finite path quasimetric space, by [8, Section 3], Q' is G-equivariantly quasiisometric to a locally finite graph $G_r(Q')$ for some $r \geq 0$. As Q' is a locally finite hyperbolic path quasimetric space, $G_r(Q')$ is a proper hyperbolic geodesic metric space. Also, $\partial Q'$ is G-equivariantly homeomorphic to $\partial G_r(Q')$. Finally, by the above discussion, X is G-equivariantly homeomorphic to $\partial G_r(Q')$. Let ϕ be the homeomorphism induced by quasiisometry from X to $G_r(Q')$. Since G acts on X as a convergence group, G also acts on $\partial G_r(Q')$ as a convergence group [9]. Also, G acts as a convergence group on $G_r(Q') \cup \partial G_r(Q')$.

Suppose $g \in G$ such that order of g is infinite. If g is a parabolic element. Then, clearly $\langle g \rangle$ is dynamically quasiconvex for $G \curvearrowright X$. Now, suppose g is loxodromic for the action of G on M. Then, g is loxodromic for the action of G on $\partial G_r(Q')$. By Lemma 2.3.18, $\langle g \rangle$ is dynamically quasiconvex subgroup of G for $G \curvearrowright \partial G_r(Q')$. Now, let if possible $\langle g \rangle$ is not dynamically quasiconvex for $G \curvearrowright X$. Then there exist disjoint closed subsets K, L of X such that the set $\{g \in G \setminus \langle g \rangle | g\Lambda(\langle g \rangle) \cap K \neq \emptyset, g\Lambda(\langle g \rangle) \cap L \neq \emptyset\}$ is infinite. Since ϕ is a homeomorphism, $\phi(K), \phi(L)$ are disjoint closed subsets of $\partial G_r(Q')$. Then, by G-equivariance of ϕ, we see that $\langle g \rangle$ is not dynamically quasiconvex for $G \curvearrowright \partial G_r(Q')$. This gives us a contradiction. □

2.4 Relatively hyperbolic groups

The notion of relative hyperbolic groups was introduced by Gromov in [32] and has been elaborated on by several authors, for example [21],[10],[66],[37],[25]. When the group is countable and the collection of subgroups is finite then all definitions of relative hyperbolicity are equivalent, see [37],[28]. Here, we use the following two equivalent definition of relative hyperbolicity, see [10].

Definition 2.4.1. A group G is **hyperbolic relative to** a family $\mathscr{H}$ of subgroups if G admits a properly discontinuous isometric action on a proper hyperbolic space X

such that the induced action of G on ∂X is geometrically finite, and the subgroups in the family $\mathscr{H}$ are precisely the maximal parabolic subgroups.

In the above definition, the Gromov boundary of X is canonical and is called the **Bowditch boundary** of the relatively hyperbolic group G.

Definition 2.4.2. A connected graph K is said to be **fine** if each edge of K is contained in finitely many circuits of length n for each $n \in \mathbb{N}$.

Definition 2.4.3. A group G is said to be **hyperbolic relative to** a family $\mathscr{H}$ of subgroups if G admits an action on a fine hyperbolic graph with finite edge stabilizers, finitely many orbits of edges, and elements of $\mathscr{H}$ are precisely infinite vertex stabilizers.

Suppose a group G is finitely generated and that G is hyperbolic relative to a collection $\mathscr{H}$ of subgroups. Then, each element of $\mathscr{H}$ is finitely generated and the collection $\mathscr{H}$ is also finite, see [66]. Also, in [16], the author proved the equivalence of Definition 2.4.3 and the one given in [21].

Examples and non-examples:

1. Hyperbolic groups are relatively hyperbolic with respect to trivial subgroup.

2. The fundamental group of non-compact, complete, finite volume Riemannian manifold with pinched negative sectional curvature is relatively hyperbolic with respect to cusp subgroups, see [21].

3. The fundamental group of a finite graph of groups with finite edge groups is relative hyperbolic with respect to infinite vertex groups.

4. Thick groups (or spaces) are not relatively hyperbolic, see [4].

The following theorem of Yaman shows that relatively hyperbolic groups can be characterized in terms of geometrically finite action.

Theorem 2.4.4. [78, Theorem 0.1] *Let G be a group acting geometrically finitely on a non-empty perfect metrizable compactum X. Assume that the quotient of bounded parabolic points is finite under the action of G and the corresponding maximal parabolic subgroups are finitely generated. Let $\mathscr{H}$ be the family of maximal parabolic subgroups. Then $(G, \mathscr{H})$ is a relatively hyperbolic group and X is equivariantly homeomorphic to Bowditch boundary of G.*

Remark 2.4.5. The assumption that maximal parabolic subgroups are finitely generated does not play any role in proving the above theorem, but it is there merely to satisfy the hypothesis in Bowditch's definition of a relatively hyperbolic group. Also, by a result of Tukia [75, Theorem 1B], one can remove the assumption of finiteness of the set of orbits of bounded parabolic points.

Definition 2.4.6. Let G be a relatively hyperbolic group with Bowditch boundary ∂G. Let H be a group acting as a geometrically finite convergence group on a compact metrizable space ∂H. We assume that H embeds in G as a subgroup. Then, H is said to be **relatively quasiconvex** if $\Lambda(H) \subset \partial G$ is H-equivariantly homeomorphic to ∂H.

Example 2.4.7. It is immediate that the parabolic subgroups of relatively hyperbolic groups are relatively quasiconvex. Every cyclic subgroup of a relatively hyperbolic group is relatively quasiconvex, see [66]. By [29], one also sees that a subgroup of a relatively hyperbolic group is relative quasiconvex if and only if it is dynamically quasiconvex.

In the following definition K^g denotes the conjugate of K by g, i.e. gKg^{-1}.

Definition 2.4.8. Let $(G, \mathscr{H})$ be a relatively hyperbolic group and let K be a subgroup of G. We say that K is **full** in G if, for all $H \in \mathscr{H}$, $g \in G$, either $|K \cap H^g| < \infty$ or $[H^g : K \cap H^g] < \infty$.

A **full relatively quasiconvex** subgroup of G is a relatively quasiconvex subgroup that is full.

From definition, it is clear that maximal parabolic subgroups of a relatively hyperbolic group are full relatively quasiconvex. Next, we recall some basic definitions and results from [65] which are relevant to us.

Definition 2.4.9. Let G be a group hyperbolic relative to a collection of subgroups $\mathscr{H}$. A subgroup Q of G is said to be **hyperbolically embedded** in G if G is hyperbolic relative to $\mathscr{H} \cup \{Q\}$.

Suppose $(G, \mathscr{H})$ is a relatively hyperbolic group. We say that an element g of G is **parabolic** if it is conjugate to an element of $H \in \mathscr{H}$. Otherwise, it is called **hyperbolic**. For any hyperbolic element g of infinite order, we set $E(g) = \{f \in G : f^{-1}g^n f = g^{\pm}n\}$. For any hyperbolic element of infinite order, Osin proved the following theorem:

Theorem 2.4.10. [65, Theorem 4.3] *Every hyperbolic element g of infinite order in G is contained in a unique maximal elementary subgroup $E(g)$.*

From the proof of the above theorem, it follows that $[E(g):\langle g\rangle] < \infty$ and therefore $E(g)$ is elementary. Also, for an infinite order hyperbolic element, the unique maximal elementary subgroup is hyperbolically embedded in the relatively hyperbolic group G, see [65, Corollary 1.7].

2.5 Farb's electrified space

In [21], Farb introduced the notion of a coned-off Cayley graph in order to define relatively hyperbolic groups. Using the same idea, one can define coned-off space or electrified space.

Definition 2.5.1. Let X be a geodesic metric space and let $\{A_i\}$ be a collection of subsets of X. The **coned-off space** $\hat{X}$ of X with respect to $\{A_i\}$ is a new set defined as follows.

$$\hat{X} := X \bigsqcup (\sqcup_i A_i \times [0,1]) / \sim$$

where the equivalence relation $\sim$ is generated by $(x_i,1) \sim (y_i,1)$ and $(x_i,0) \sim x_i$ for all $x_i, y_i \in A_i$, and for all i. Let the equivalence class of $(x_i,1)$ be denoted by e_i. This is called the **cone point** corresponding to the set A_i. One defines a length metric on $\hat{X}$ in the usual way and it turns out that $\hat{X}$ is a metric space.

Remark 2.5.2. In this thesis, we always assume that the coned-off space of a geodesic metric space is a geodesic metric space which is not true in general. For groups, the coned-off Cayley graphs are always geodesic metric spaces.

Next, we record the following basic lemma that will be useful to us later.

Lemma 2.5.3. *Given $D \geq 0$ there exists $K = K(D)$ such that the following holds:*

Suppose X is a geodesic metric space and $\{A_i\}_{i\in I}$ and $\{B_i\}_{i\in I}$ are two collection of subsets of X. Suppose $\hat{X}_A, \hat{X}_B$ are the coned off spaces obtained from coning the A_i's and B_i's respectively. Let $\phi : \hat{X}_A \to \hat{X}_B$ be the extension of the identity map $X \to X$ obtained by mapping the open ball of radius 1 about the cone point for A_i to the cone point for B_i. If, for all i, $Hd(A_i, B_i) \leq D$ then ϕ is a K-quasiisometry.

Now, we recall some terminology in a coned-off space $\hat{X}$ with respect to $\{A_i\}$.

A path γ in $\hat{X}$ *penetrates* a subset A_i if γ passes through the cone point corresponding to A_i. A path γ in $\hat{X}$ is said to be *without backtracking* if, for any i, γ does not return to A_i after leaving it. Let γ be a geodesic (quasigeodesic) without backtracking in $\hat{X}$. Suppose γ passes through the cone points $e_1, e_2, ..., e_k$. For each $i = 1, 2, ..., k$, join the entry and exit points of γ in A_i by a geodesic, say γ_i, in A_i.

Now take the concatenation of portions of γ outside A_i's and γ_i's and call it $\hat{\gamma}$. The path $\hat{\gamma} \subset X$ is called a *de-electrification* or *enlargement* of γ.

The following result is motivated by Farb's ([21]) weak relative hyperbolicity. The statement was known to be true to the specialists for a long time. However, the first rigorous proof of it appears in [17].

Proposition 2.5.4. ([17, Proposition 2.10],[42, Proposition 2.6]) *Given $\delta \geq 0, C \geq 0$ there exists $\delta' = \delta'(\delta, C)$ such that the following holds:*

If X is a δ-hyperbolic geodesic metric space and $\{A_i\}$ is a collection of C-quasiconvex subsets of X, then the coned-off space $\hat{X}$ with respect to the collection $\{A_i\}$ is δ'-hyperbolic.

The above proposition does not assume that A_i's are "sufficiently separated". Such a requirement is present in many versions of the above proposition available in the literature, although this assumption is not in fact necessary and, in particular, Proposition 7.12 in [10] does not impose the "sufficiently separated" requirement.

In addition to the above proposition the following nice result also appears in [17].

Proposition 2.5.5. [17, Proposition 2.11] *Suppose we have the hypotheses of Proposition 2.5.4. Then for any $C \geq 0$ there is a $C' \geq 0$ such that any C-quasiconvex subset Q of X is C'-quasiconvex in $\hat{X}$.*

A form of the converse of Proposition 2.5.4 also appears in [17, Proposition 2.12]. However, this leads us to the following question. Suppose 'quasiconvexity' in the above proposition is replaced by the conditions that the inclusion $Q \to X$ is a proper embedding, Q is hyperbolic, and it admits the CT map. Can one then ask about the existence of CT map for the inclusion $Q \to \hat{X}$ and vice versa? The main technical results of Chapter 4 will try to formalize these questions and obtain answers to variations of them.

2.6 Complexes of groups

In [70], Bass and Serre described completely the groups acting without inversion on trees. It follows from their work that groups acting without inversion on trees arise as the fundamental group of graphs of groups. In the same spirit, one has the following question:

Given a simplicial action of a group G on a simply connected simplicial complex $\widetilde{X}$, how can one reconstruct the action with the given data on the quotient $X = \widetilde{X}/G$?

This leads to the theory of complexes of groups. These are natural generalization of graphs of groups introduced by Bass and Serre. When X is a 2-simplex, we have the notion of triangles of groups studied by Gerstein and Stallings [71]. Later general complexes of groups was defined and studied by Haefliger [34] and Corson [14] (in dimension 2 independently). Now, we briefly recall some definitions and results for complexes of groups. For details, one is referred to [11].

Definition 2.6.1. A **small category without loops (scwol)** $\mathscr{X}$ is a disjoint union of a set $V(\mathscr{X})$, the vertex set, and a set $E(\mathscr{X})$, the edge set, endowed with maps

$$i : E(\mathscr{X}) \to V(\mathscr{X}) \text{ and } t : E(\mathscr{X}) \to V(\mathscr{X})$$

and, if $E^2(\mathscr{X})$ denotes the set of pairs (a,b) such that $t(b) = i(a)$, with a map

$$E^2(\mathscr{X}) \to E(\mathscr{X})$$

that associates to each pair (a,b) an edge ab called their *composition* such that:

1. for all $(a,b) \in E^2(\mathscr{X})$, we have $i(ab) = i(b), t(ab) = t(a)$,

2. *associativity:* for all $a,b,c \in E(\mathscr{X})$, if $i(a) = t(b)$ and $i(b) = t(c)$, then $(ab)c = a(bc)$,

3. *no loop:* for each $a \in E(\mathscr{X})$, $i(a) \neq t(a)$.

Example 2.6.2. Let Q be a poset. Then, we can associate a scwol to Q in the following way. The set of vertices is Q and the edges are pairs $(\tau, \sigma) \in Q \times Q$ such that $\tau \subset \sigma$. Define $i((\tau, \sigma)) := \sigma$ and $t((\tau, \sigma)) := \tau$. The composition $(\tau, \rho)(\rho, \sigma) := (\tau, \sigma)$.

For each integer $k > 0$, let $E^k(\mathscr{X})$ be the set of sequences $(a_1, a_2, ..., a_k)$ of composable edges $(a_i, a_{i+1}) \in E^2(\mathscr{X})$ for $i = 1, 2, ..., (k-1)$. By convention $E^0(\mathscr{X}) = V(\mathscr{X})$. The *dimension* of $\mathscr{X}$ is the supremum of all k such that $E^k(\mathscr{X}) \neq \phi$. For each scwol $\mathscr{X}$, one can define *the geometric realisation* of $\mathscr{X}$. It is denoted by $|\mathscr{X}|$. It is a piecewise Euclidean complex. Roughly speaking it is a disjoint union of k-simplices corresponding to each sequence $(a_1, a_2, ..., a_k) \in E^k(\mathscr{X})$ with a natural relation. For more detail, one is referred to [11, pp. 522-523, III.C]. In general, $|\mathscr{X}|$ need not be a simplicial complex but its first barycentric subdivision is a simplicial complex.

Example 2.6.3. Naturally associated to each M_k-polyhedral complex (see, [11, Definition 7.37, I.7]) K there is a scwol $\mathscr{X}$ such that $|\mathscr{X}|$ is the first barycentric subdivision of K. The set of vertices of $\mathscr{X}$ is the set of cells of K. The set of edges of $\mathscr{X}$ are the set of 1-simplices of the first barycentric subdivision K' of K: each 1-simplex a of K' corresponds to a pair of cells $T \subset S$. Define $i(a)$ to be the barycenter of S and $t(a)$ to be the barycenter of T.

Definition 2.6.4 (Morphisms of scwols). Let $\mathscr{X}$ and $\mathscr{Y}$ be two scwols. A morphism $f : \mathscr{X} \to \mathscr{Y}$ is a map that sends $V(\mathscr{X})$ to $V(\mathscr{Y})$ and $E(\mathscr{X})$ to $E(\mathscr{Y})$ such that the following conditions hold:

1. For each $a \in E(\mathscr{X})$, we have $i(f(a)) = f(i(a))$ and $t(f(a)) = f(t(a))$.
2. For each $(a,b) \in E^2(\mathscr{X})$, we have $f(ab) = f(a)f(b)$.

A morphism f is said to be non-degenerate if for each $\sigma \in V(\mathscr{X})$, the restriction of f to the set of edges with initial vertex σ is a bijection onto the set of edges of $\mathscr{Y}$ with initial vertex $f(\sigma)$.

An *automorphism* of a scwol $\mathscr{X}$ is a morphism from $\mathscr{X}$ to $\mathscr{X}$ that has an inverse.

For a scwol $\mathscr{X}$, let $E^{\pm}(\mathscr{X})$ be the set of oriented edges of $\mathscr{X}$, i.e. the set of symbols a^+, a^- for $a \in E(\mathscr{X})$. For $e = a^+$, we define $i(e) = t(a), t(e) = i(a)$. For $e = a^-$, we define $i(e) = i(a), t(e) = t(a)$. An *edge path* joining the vertex τ to the vertex σ is a sequence $(e_1, e_2, ..., e_n)$ such that $i(e_1) = \tau, i(e_{j+1}) = t(e_j), t(e_n) = \sigma$. A scwol $\mathscr{X}$ is *connected* if any two vertices of $\mathscr{X}$ are joined by an edge path. Equivalently, $\mathscr{X}$ is connected if and only if $|\mathscr{X}|$ is connected. A scwol $\mathscr{X}$ is called *simply connected* if $|\mathscr{X}|$ is simply connected as a topological space.

Definition 2.6.5 (Group action on a scwol). An action of a group G on a scwol $\mathscr{X}$ is a homomorphism from G to the group of automorphisms of $\mathscr{X}$ such that the following holds:

1. For all $a \in E(\mathscr{X})$ and $g \in G$, $g.i(a) \neq t(a)$.
2. For all $g \in G$ and $a \in E(\mathscr{X})$, if $g.i(a) = i(a)$ then $g.a = a$.

Remark 2.6.6. Let K be a M_k-simplicial complex. Suppose G is a group that acts simplicially on K. We say that G acts *without inversion* on K whenever $g \in G$ leaves a cell of K invariant, then its restriction to that cell is the identity. Let $\mathscr{X}$ be the scwol associated to K. Then G acts on $\mathscr{X}$ as in Definition 2.6.5 if and only if G acts on K without inversion.

Now, we are ready to define complexes of groups.

Definition 2.6.7 (Complex of groups). Let $\mathscr{Y}$ be a scwol. A complex of groups $G(\mathscr{Y}) = (G_\sigma, \psi_a, g_{a,b})$ over $\mathscr{Y}$ is given by the following data:

1. For each $\sigma \in V(\mathscr{Y})$, a group G_σ called the *local group at* σ.
2. For each $a \in E(\mathscr{Y})$, an injective homomorphism $\psi_a : G_{i(a)} \to G_{t(a)}$. The maps ψ_a's are called the local maps.
3. For each pair of composable edges $(a,b) \in E^2(\mathscr{Y})$, a twisting element $g_{a,b} \in G_{t(a)}$, with the following compatibility conditions:

   (i) $Ad(g_{a,b})\psi_{ab} = \psi_a\psi_b$, where $Ad(g_{a,b})$ is the conjugation by $g_{a,b}$ in $G_{t(a)}$.

   (ii) For each triple $(a,b,c) \in E^3(\mathscr{Y})$ of composable edges we have the *cocycle condition*

$$\psi_a(g_{b,c})g_{a,bc} = g_{a,b}g_{ab,c}.$$

Remark 2.6.8. 1. The condition (i) is empty if $\mathscr{Y}$ is 1-dimensional. The cocycle condition is empty if dimension of $\mathscr{Y} \leq 2$.

2. A *simple* complex of groups is a complex of groups such that all the twisting elements are trivial.
3. Let Y be a M_k-simplicial complex. Then, the complex of groups over Y is the complex of groups over the scwol associated to Y. Throughout the thesis, a complex of groups over Y is denoted by $(\mathscr{G}, Y)$.
4. Let Y be a graph. Then, the complex of groups over Y is same as the graph of groups (see [70]) over Y.

Definition 2.6.9. Let $G(\mathscr{Y}') = (G_{\sigma'}, \psi_{a'}, g_{a',b'})$ be a complex of groups over a scwol $\mathscr{Y}'$. Let $f : \mathscr{Y} \to \mathscr{Y}'$ be a morphism of scwols. A morphism $\phi = (\phi_a, \phi(a))$ from $G(\mathscr{Y})$ to $G(\mathscr{Y}')$ over f consists of the following:

1. For each $\sigma \in V(\mathscr{Y})$, a homomorphism $\phi_\sigma : G_\sigma \to G_{f(\sigma)}$ of groups.
2. An element $\phi(a) \in G_{t(f(a))}$ for each $a \in E(\mathscr{Y})$ such that

   (i) $Ad(\phi(a))\psi_{f(a)}\phi_{i(a)} = \phi_{t(a)}\psi_a$.

   (ii) $\phi_{t(a)}(g_{a,b})\phi(ab) = \phi(a)\psi_{f(a)}(\phi(b))g_{f(a),f(b)}$, for all $(a,b) \in E^2(\mathscr{Y})$.

If f is an isomorphism of scwols and ϕ_σ is an isomorphism for every $\sigma \in V(\mathscr{Y})$ then ϕ is called an *isomorphism*.

When $\mathcal{Y}'$ is a vertex then the complex of groups $G(\mathcal{Y}')$ is simply a group, call it G. Then, in this special case the definition of morphism will come in the following form.

Definition 2.6.10. A morphism $\phi = (\phi_\sigma, \phi(a))$ from a complex of groups $G(\mathcal{Y})$ to a group G consists of a homomorphism $\phi_\sigma : G_\sigma \to G$ for each $\sigma \in V(\mathcal{Y})$ and an element $\phi(a) \in G$ for each $a \in E(\mathcal{Y})$ such that

$$\phi_{t(a)}\psi_a = Ad(\phi(a))\phi_{i(a)} \text{ and } \phi_{t(a)}(g_{a,b})\phi(ab) = \phi(a)\phi(b).$$

We say that ϕ is *injective on local groups* if each homomorphism ϕ_σ is injective.

2.6.1 The complex of groups associated to an action

Let G be a group acting on a scwol $\widetilde{\mathcal{Y}}$. Let $\mathcal{Y} = \widetilde{\mathcal{Y}}/G$ be the quotient scwol (see [11, 1.13, p.529]) and let $p : \mathcal{X} \to \mathcal{Y}$ be the natural projection. The *complex of groups $G(\mathcal{Y})$ associated to the action of* G on $\widetilde{\mathcal{Y}}$ is defined as follows:

For each vertex $\sigma \in \mathcal{Y}$, choose a vertex $\widetilde{\sigma} \in \widetilde{\mathcal{Y}}$ such that $p(\widetilde{\sigma}) = \sigma$. For each edge $a \in E(\mathcal{Y})$ with $i(a) = \sigma$, there exists a unique edge $\widetilde{a} \in E(\widetilde{\mathcal{Y}})$ such that $p(\widetilde{a}) = a$ and $i(\widetilde{a}) = \widetilde{\sigma}$. Choose $h_a \in G$ such that $h_a.t(\widetilde{a}) = \widetilde{t(a)}$. For each $\sigma \in \mathcal{Y}$, let G_σ be the stabilizer in G of $\widetilde{\sigma}$. For each $a \in E(\mathcal{Y})$, let $\psi_a : G_{i(a)} \to G_{t(a)}$ be the conjugation by h_a, that is, $\psi_a : g \longmapsto h_a g h_a^{-1}$. For every pair of composable edges $(a,b) \in E^2(\mathcal{Y})$, define $g_{a,b} = h_a h_b h_{ab}^{-1}$. Now, it is easy to verify that $G(\mathcal{Y}) = (G_\sigma, \psi_a, g_{a,b})$ is a complex of groups.

Another choice for the representatives of a vertex of $\mathcal{Y}$ in $\widetilde{\mathcal{Y}}$ and for the h_a's will give a complex of groups $G'(\mathcal{Y})$ with an isomorphism of $G(\mathcal{Y})$ on $G'(\mathcal{Y})$ over the identity morphism of $\mathcal{Y}$ (see [11],[33] for details).

Definition 2.6.11 (Developability). A complex of groups $G(\mathcal{Y})$ is called *developable* if it is isomorphic to a complex of groups that is associated to an action of a group G on a scwol $\widetilde{\mathcal{Y}}$ with $\mathcal{Y} = \widetilde{\mathcal{Y}}/G$.

Theorem 2.6.12 (The Basic Construction). [11, Theorem 2.13, III.C] *Suppose $G(\mathcal{Y})$ is a complex of groups over a scwol $\mathcal{Y}$.*

1. *Let G be a group. Canonically associated to each morphism $\phi : G(\mathcal{Y}) \to G$ there is an action of G on a scwol $D(\mathcal{Y}, \phi)$ with quotient $\mathcal{Y}$. If ϕ is injective on local groups then $G(\mathcal{Y})$ is the complex of groups associated to this action and $G(\mathcal{Y}) \to G$ is the associated morphism.*

2. *If $G(\mathscr{Y})$ is the complex of groups associated to an action of a group G on a scwol $\mathscr{X}$, and if $\phi : G(\mathscr{Y}) \to G$ is the associated morphism then there is a G-equivariant isomorphism $D(\mathscr{Y}, \phi) \to \mathscr{X}$ that projects to the identity of $\mathscr{Y}$.*

One immediately observes the following corollary that gives an algebraic condition for being a developable complex of groups.

Corollary 2.6.13. [11, Corollary 2.15, III.C] *A complex of groups $G(\mathscr{Y})$ is developable if and only if there exists a morphism ϕ from $G(\mathscr{Y})$ to a group G which is injective on the local groups.*

Note that not every complex of groups is developable, see [11, Example 2.17(5), II.12].

2.6.2 Fundamental group of a complex of groups

Suppose $G(\mathscr{Y})$ is a developable complex of groups. Let T be a maximal tree in the 1-skeleton of $|\mathscr{Y}|$. Such a maximal tree is not unique in general. Let $E^{\pm}(Y)$ denotes the set of oriented edges of $\mathscr{Y}$. The fundamental group of $G(\mathscr{Y})$ is a group, denoted by $\pi_1(G(\mathscr{Y}))$, generated by a set

$$\bigsqcup_{\sigma \in V(\mathscr{Y})} G_\sigma \bigsqcup E^{\pm}(\mathscr{Y})$$

with the following relations:

(1) The relations in the local groups G_σ.

(2) For each $a \in E(\mathscr{Y})$, $(a^+)^{-1} = a^-$ and $(a^-)^{-1} = a^+$.

(3) $a^+b^+ = g_{a,b}(ab)^+$ for every composable edges $a, b \in E(\mathscr{Y})$.

(4) $\psi_a(g) = a^+ g a^-$ for every $a \in E(\mathscr{Y})$.

(5) $a^+ = 1$ for every $a \in T$.

Next, we describe universal cover of a developable complex of groups. We describe it in the case when the complex of groups is over a simplicial complex. For the general construction, one is referred to the proof of [11, Theorem 2.13, III.C].

Definition 2.6.14 (Universal cover of a developable complex of groups). Let $(\mathscr{G}, Y)$ be a developable complex of groups over a simplicial complex Y and let $G = \pi_1(\mathscr{G}, Y)$. Let $i_T : (\mathscr{G}, Y) \to G$ be a morphism mapping local groups G_σ to its image in G, and the edge a to a^+ where each edge corresponds to an inclusion $\sigma' \subset \sigma$ of simplices of Y. The universal cover B associated to i_T is defined as follows:

$$B := (G \times (\bigsqcup_{\sigma \subset Y} \sigma)) / \sim$$

where $(gi_T(g'),x) \sim (g,x)$ for $g \in G, g' \in G_\sigma$, $x \in \sigma$ and $(g, i_{\sigma',\sigma}(x)) \sim (gi_T(a),x)$ for $g \in G, x \in \sigma$, $i_{\sigma',\sigma} : \sigma' \hookrightarrow \sigma$ is an inclusion.

The group G acts naturally by left multiplication on the first factor of B.

Theorem 2.6.15. [11, Theorem 3.13, III.C] *The universal cover of a developable complex of groups $G(\mathscr{Y})$ is connected and simply connected.*

Remark 2.6.16. We do not work with the CW topology on B. We rather view B as the quotient metric space obtained by gluing standard Euclidean simplices [11, Chapter I.7]. This metric space is required to be a CAT(0), hyperbolic metric space in the setting of Martin's theorem (see Theorem 4.2.1). Moreover, we note that G-action on B is through isometries.

Now we prove that the universal cover of a developable complex of groups over a finite simplicial complex is a tree-graded space. A subset A of a geodesic metric space X is said to be a *geodesic subset* if any two points of A can be joined by a geodesic contained in A.

Definition 2.6.17 (Tree-graded space). [20, Definition 2.1] Let X be a complete geodesic metric space and let $\mathscr{S}$ be a collection of closed geodesic subsets of X (called pieces). The space X is said to be tree-graded with respect to $\mathscr{S}$ if the following holds:

1. Every two different pieces have at most one common point.

2. Every simple geodesic triangle (simple loop composed of three geodesics) in X is contained in one piece.

Let $(\mathscr{G}, Y)$ be a developable complex of groups over a finite simplicial complex Y. Let G be the fundamental group of the complex of groups $(\mathscr{G}, Y)$ and let B be the universal cover of $(\mathscr{G}, Y)$. By Theorem 2.6.15, we see that B is simply connected simplicial complex. As the isometry type of simplices of B is finite, the universal cover B is a complete geodesic metric space (see [11, I.7]).

Define an equivalence relation on B generated by the following:

$\sigma_1 \sim \sigma_2$ if and only if σ_1 and σ_2 have a common face of dimension at least 1. Let $\mathscr{S}$ be the collection of equivalence classes. We prove the following.

Proposition 2.6.18. *B is a tree-graded space with respect to $\mathscr{S}$.*

Proof. Suppose B is not a tree-graded space. We need to consider the following two cases.

Case 1. Let $S_1, S_2 \in \mathscr{S}$ such that S_1, S_2 intersect at more than one vertex. Let v, w be two vertices in $S_1 \cap S_2$. Form a new space B' by adding an edge e at vertex v so that B and B' are homotopically equivalent (for adding e, we are just stretching the vertex v). Let v_1, v_2 be end points of edge e. Note that $v_1 \in S_1$ and $v_2 \in S_2$. Now, remove the interior of edge e from B'. Let B'' be the remaining space.

Claim. B'' is path connected.

Fix a point $s_1 \in S_1$ and let z be any point in B''. If we already have a path in B' joining z and s_1 which does not pass through v then we have a path in B'' joining z and s_1. If $z \in S_1$ and we have a path joining s_1, z passing through the vertex v in B, then we get a path α in S_1 joining s_1 to v_1. Let β be a path joining v_1, z in S_1. Concatenation of α, β gives a path from s_1 to z in $S_1 \subset B''$. If $z \in S_2$ and we have a path from s_1 to z passing through v in B. Again, we get a path α joining s_1 and v_1 in S_1. Join v_1 and w by a path β(say) in S_1. Let γ be a path connecting w and z in S_2. Then concatenation of α, β, γ gives a path joining s_1 and z in B''. Now, let $z \in B'' \setminus \{S_1 \cup S_2\}$. Clearly, we have a path α in B'' either from z to v_1 or v_2 or from z to some common vertex w of S_1 and S_2. By the above discussion, we have path β either from s_1 to v_1 or v_2 or from s_1 to w in $S_1 \cup S_2 \setminus \{v\}$. Concatenation of α and β gives a path from s_1 to z in B''. Hence the space B'' is a path connected CW-complex. Join vertices v_1, v_2 by a path β in 1-skeleton of B''. Since the path β is a contractible subcomplex of B'', the space B' and hence B is homotopically equivalent to the wedge sum of a circle and a CW-complex (homotopically equivalent to B''). This is a contradiction as B is simply connected.

Case 2. Let $\{S_1, S_2, ..., S_n\} \subset \mathscr{S}$ be a collection of pieces such that they form a cycle and consecutive pieces are intersecting at a vertex, i.e. $S_i \cap S_{i+1} \neq \phi$ for $i = 1, 2..., (n-1)$ and $S_n \cap S_1 \neq \phi$. Choose any two consecutive pieces S_i, S_{i+1} and let $v_i = S_i \cap S_{i+1}$. As in the previous case, form a new space B' by adding a new edge e at any of the vertex v_i so that B and B' are homotopically equivalent. Let B'' be the space obtained from B' by removing the interior of edge e. Similar to the previous case, we see that B'' is path connected and B is homotopically equivalent to the wedge sum of a circle and a CW-complex. Again it is a contradiction as B is simply connected. □

Next, we define complexes of spaces over simplicial complexes (see [48, Definition 1.3]).

Definition 2.6.19 (Complexes of spaces). A complex of spaces $C(Y)$ over a simplicial complex Y consists of the following data:

1. For every simplex σ of Y, a CW-complex C_σ, called local space.

$$\begin{array}{ccc} \tilde{\mathbb{Y}} & \longrightarrow & \mathbb{Y} \\ \downarrow & & \downarrow \\ B & \longrightarrow & Y \end{array}$$

Figure 2.1:

2. For every pair of simplices $\sigma \subset \sigma'$, an embedding $\phi_{\sigma,\sigma'} : C_{\sigma'} \to C_\sigma$ called a gluing map such that for every $\sigma \subset \sigma' \subset \sigma''$, we have $\phi_{\sigma,\sigma''} = \phi_{\sigma,\sigma'} \circ \phi_{\sigma',\sigma''}$.

The *realization of a complex of spaces* $C(Y)$ is the quotient space

$$|C(Y)| = \Big(\bigsqcup_{\sigma \subset Y} \sigma \times C_\sigma \Big) / \sim,$$

where $(i_{\sigma,\sigma'}(x), s) \sim (x, \phi_{\sigma,\sigma'(s)})$ for $x \in \sigma \subset \sigma'$, $s \in C_{\sigma'}$ and $i_{\sigma,\sigma'} : \sigma \hookrightarrow \sigma'$ is the inclusion.

Given a complex of groups $(\mathscr{G}, Y)$ over a finite simplicial complex Y, for each simplex σ of Y one takes any simplicial complex (generally a $K(G_\sigma, 1)$-space), say Y_σ with a base point such that

(1) $\pi_1(Y_\sigma) \simeq G_\sigma$.

(2) For every pair of simplices $\tau \subset \sigma$ one has a base point preserving continuous map $Y_\sigma \to Y_\tau$ which induces the local map $\psi_{(\sigma,\tau)} : G_\sigma \to G_\tau$.

This defines a complex of spaces over Y. By abuse of terminology we also call the realization, $\mathbb{Y}$ say, of this to be a complex of spaces as well. Note that we have a natural simplicial map $\mathbb{Y} \to Y$.

The fundamental group $\pi_1(\mathscr{G}, Y)$ of $(\mathscr{G}, Y)$ is the same as $\pi_1(\mathbb{Y})$. It is a standard consequence of Seifert-Van Kampen theorem that this is independent of the complex of spaces thus chosen. Suppose the natural homomorphism $\pi_1(Y_\sigma) \to \pi_1(\mathbb{Y})$ is injective for all simplex $\sigma \subset Y$. Suppose $\tilde{\mathbb{Y}} \to \mathbb{Y}$ is a universal cover of $\mathbb{Y}$. Then, one considers the composition $\tilde{\mathbb{Y}} \to \mathbb{Y} \to Y$, say f, and collapses the connected components of $f^{-1}(y)$ for all $y \in Y$. Then, the resulting simplicial complex is isomorphic to the universal cover B of $(\mathscr{G}, Y)$ as in Definition 2.6.14 (see figure 2.1). For the details of this paragraph, one is referred to [15],[14],[33].

2.7 Model space for G-action

We start here by fixing some notation. Let Y be a finite connected simplicial complex. Let $\mathscr{B}(Y)$ denote the directed graph whose vertex set is the set of simplices of Y and given two simplices $\tau \subset \sigma$ we have directed edge e from σ to τ. In this case we write $e = (\sigma, \tau)$, $o(e) = \sigma$ and $t(e) = \tau$. Two directed edges e, e' are said to be *composable* if $t(e) = o(e')$.

In this section, we assume that $(\mathscr{G}, Y)$ is a developable complex of hyperbolic groups such that all the local maps are qi embeddings. Let $G = \pi_1(\mathscr{G}, Y)$. Our aim here is to recall the construction of a complex of spaces $p : X \to B$ where G acts on X by isometries such that the action is simplicial, proper and cocompact and p is G-equivariant. Any geodesic metric space Z on which a group H acts by isometries properly and cocompactly will be referred to as a *model space* for H in this section since any orbit map $H \to Z$ is a quasiisometry by Švarc-Milnor lemma.

Construction of X: It is a standard fact that for a hyperbolic group Γ there is a finite dimensional Rips complex $P_n(\Gamma)$ on which Γ acts properly and cocompactly, see [11, Theorem 3.21, III.Γ]. Now, it is not hard to see that there are model spaces X_σ for G_σ's, $\sigma \subset Y$ and qi embeddings $\phi_e : X_\sigma \to X_\tau$, for all $e = (\sigma, \tau) \in E(\mathscr{B}(Y))$ equivariant with respect to the local maps $G_\sigma \to G_\tau$. For a rigorous treatment of this, one is referred to [49, Theorem 2].

Then as in [48, Definition 2.3, Theorem 2.4], we have a finite dimensional, locally finite CW complex with a proper, cocompact simplicial G-action:

$$X := (G \times \bigsqcup_{\sigma \subset Y} (\sigma \times X_\sigma))/\sim,$$

where (1) $(g, i_{\sigma,\sigma'}(x), s) \sim (ge^{-1}, x, \phi_e(s))$, where $s \in X_{\sigma'}$, $x \in \sigma$, $g \in G$, $i_{\sigma,\sigma'} : \sigma \hookrightarrow \sigma'$ and $\phi_e : X_{\sigma'} \to X_\sigma$ and $e = (\sigma', \sigma) \in E(\mathscr{B}(Y))$. (2) $(gg', x, s) \sim (g, x, g's)$ if $x \in \sigma, s \in X_\sigma$, $g' \in G_\sigma$, $g \in G$.

We note that there is a natural G-equivariant projection $p : X \to B$. Thus, the space X can be seen as a complex of spaces over the universal cover B. Now, by [11, I.7], X (take the barycentric subdivision of X if necessary) admits a geodesic metric so that the G-action is through isometries and the metric topology is the same as the CW topology on X. Therefore, X is a model space for G with the metric.

Let ϕ be a quasiisometry from G to X. Note that the local group G_σ is quasiisometric to X_σ for $\sigma \subset Y$. Also observe that for $\sigma \subset Y$, Hausdorff distance between $\phi(G_\sigma)$ and X_σ is uniformly bounded. Thus, the local space gX_σ is uniformly close to $\phi(gG_\sigma)$ for $g \in G$ and $\sigma \subset Y$. Finally, it follows that the inclusion $gX_\sigma \hookrightarrow X$ is a uniform quasiisometric embedding for $g \in G, \sigma \subset Y$, see figure 2.2.

$$\begin{array}{ccc} gG_\sigma & \longrightarrow & gX_\sigma \\ \downarrow & & \downarrow \\ G & \xrightarrow{\phi} & X \end{array}$$

Figure 2.2:

Remark 2.7.1. From the above discussion, we have a quasiisometry $\phi : G \to X$ such that $Hd(\phi(gG_\sigma), gX_\sigma)$ is uniformly bounded for $g \in G, \sigma \subset Y$. Also, we have quasiinverse $\psi : X \to G$ such that $Hd(\psi(gX_\sigma), gG_\sigma)$ is uniformly bounded for $g \in G, \sigma \subset Y$. Finally, if $(\mathscr{G}, Y)$ is a complex of hyperbolic groups such that G is a hyperbolic group, all the local maps are qi embeddings, and all the local groups are uniformly qi embedded in G then X is a hyperbolic geodesic metric space and all the local spaces are uniformly qi embedded in X. Suppose $\hat{G}$ is the coned-off space obtained by coning gG_σ's and $\hat{X_\phi}$ is the coned-off space obtained by coning $\phi(gG_\sigma)$'s where X_σ's are model spaces for local groups G_σ's and $g \in G$. Then, by [67, Lemma 1.2.31], $\hat{G}$ is quasiisometric to $\hat{X_\phi}$. Let $\hat{X}$ denote the coned-off space obtained by coning gX_σ. Then, according to Lemma 2.5.3, $\hat{X}$ is quasiisometric to $\hat{X_\phi}$.

Chapter 3

Combination theorems for convergence groups

In this chapter, we prove the following results.

Theorem 3.0.1. *Let Γ be a group admitting a decomposition into a finite graph of countable convergence groups with parabolic edge groups. Then Γ is a convergence group.*

Theorem 3.0.2. *Let Γ be a group admitting a decomposition into a finite graph of groups such that the following holds:*

1. *The vertex groups are countable convergence groups.*
2. *The stabilizers of the limit sets of the edge groups form a dynamically malnormal family of dynamically quasiconvex subgroups in the adjacent vertex groups.*

Then Γ is a convergence group.

In the setting of parabolic edge groups, we have the following combination theorem for relatively hyperbolic groups.

Theorem 3.0.3. *Let Γ be a group admitting a decomposition into a finite graph of relatively hyperbolic groups such that the edge groups are parabolic in the adjacent vertex groups. Then Γ is relatively hyperbolic. Moreover, the vertex groups are relatively quasiconvex in Γ.*

For cyclic edge groups, we prove the following.

Theorem 3.0.4. *Let Γ be a group that splits as a finite graph of relatively hyperbolic groups with infinite cyclic edge groups. Then Γ is relatively hyperbolic. Moreover, the vertex groups are relatively quasiconvex in Γ.*

For the homeomorphism type of Bowditch boundary of the fundamental group of a graph of relatively hyperbolic groups with parabolic edge groups, we obtain the following result.

Theorem 3.0.5. *Let $\mathcal{Y}$ be a finite connected graph and let $G(\mathcal{Y}), G'(\mathcal{Y})$ be two graphs of groups satisfying the following:*

1. *For each vertex $v \in V(\mathcal{Y})$, let $(G_v, \mathbb{P}_v), (G'_v, \mathbb{P}'_v)$ be relatively hyperbolic groups.*

2. *Let $e \in E(\mathcal{Y})$ be any edge. Suppose v, w are vertices connected by e. Let P_e, P'_e be parabolic edge groups in $G(\mathcal{Y}), G'(\mathcal{Y})$, respectively. Then either P_e, P'_e have infinite index in corresponding maximal parabolic subgroups in G_v, G'_v, respectively or P_e, P'_e have the same finite index in maximal parabolic subgroups in G_v, G'_v, respectively. Similarly, either P_e, P'_e have infinite index in maximal parabolic subgroups in G_w, G'_w, respectively or P_e, P'_e have the same finite index in corresponding maximal parabolic subgroups in G_w, G'_w, respectively.*

3. *For any $v \in V(\mathcal{Y})$, let B_v, B'_v be the set of translates of parabolic points corresponding to adjacent edge groups under the action of G_v, G'_v on their Bowditch boundaries respectively. Suppose we have a homeomorphism from $\partial G_v \to \partial G'_v$ that maps B_v onto B'_v.*

Let $\Gamma = \pi_1(G(\mathcal{Y}))$, $\Gamma' = \pi_1(G'(\mathcal{Y}))$ and let $\partial\Gamma, \partial\Gamma'$ be their Bowditch boundaries, respectively. Then there exists a homeomorphism from $\partial\Gamma$ to $\partial\Gamma'$ preserving edge parabolic points, i.e. taking parabolic points corresponding to edge groups of $G(\mathcal{Y})$ to parabolic points corresponding to edge groups of $G'(\mathcal{Y})$.

Layout of the chapter: For proving Theorem 3.0.1, we construct a compact metrizable on which Γ acts naturally as a convergence group. Section 3.1 contains this construction and Section 3.2 contains a proof of Theorem 3.0.1. In Section 3.3, we prove Theorem 3.0.2 and give one of its consequence. In Section 3.4, we prove combination theorems for relatively hyperbolic groups, Theorem 3.0.3, 3.0.4. Section 3.5 is devoted to prove Theorem 3.5.1. We deduce some interesting applications and examples in Section 3.6.

3.1 Construction of compact metrizable space M

In this section, we construct our candidate space M on which the fundamental group of a finite graph of convergence groups with parabolic edge groups naturally acts as a convergence group. As mentioned in the introduction, it is sufficient to consider the amalgam and the HNN extension case for proving our theorems. This will be our standing assumption for this chapter. Let Γ be an amalgamated free product or an HNN extension of convergence groups along with parabolic edge group. Let T be the Bass-Serre tree of this splitting and let τ be a subtree of T, an edge in case of amalgam, and a vertex in case of HNN extension.

Notation: For a vertex v of T, we write Γ_v for the stabilizer of v in Γ. Similarly, for an edge e, we write Γ_e for its stabilizer. For each vertex v and each edge e incident on v, Γ_v is a convergence group, and Γ_e is a parabolic subgroup in Γ_v. We denote X_v and X_e as compact metrizable spaces on which the vertex group Γ_v and the edge group Γ_e act as convergence groups. In our situation, X_e is a singleton.

3.1.1 Definition of M as a set

Contribution of the vertices of T

Let $\mathscr{V}(\tau)$ be the set of vertices of τ. Set Ω to be $\Gamma \times (\bigsqcup_{v \in \mathscr{V}(\tau)} X_v)$ divided by the natural relation

$$(\gamma_1, x_1) = (\gamma_2, x_2) \text{ if } \exists v \in \mathscr{V}(\tau), x_1, x_2 \in X_v, \gamma_2^{-1}\gamma_1 \in \Gamma_v, (\gamma_2^{-1}\gamma_1)x_1 = x_2.$$

In this way, Ω is the disjoint union of compactums corresponding to the stabilizers of the vertices of T. Also, for each $v \in \mathscr{V}(\tau)$, the space X_v naturally embeds in Ω as the image of $\{1\} \times X_v$. We identify it with its image. The group Γ naturally acts on the left on Ω. For $\gamma \in \Gamma$, γX_v is the compactum corresponding to the vertex stabilizer $\Gamma_{\gamma v}$.

Contribution of the edges of T

Each edge allows us to glue together compactums corresponding to the stabilizers of the vertices along with the limit set of the stabilizer of the edge. Since each edge group embeds as a parabolic subgroup in adjacent vertex groups, its limit set is a singleton. Let $e = (v_1, v_2)$ be the edge in τ, there exist equivariant maps $f_{e,v_i} : X_e \to X_{v_i}$ for $i = 1, 2$. Similar maps are defined by translation for edges in $T \setminus \tau$.

The equivalence relation $\sim$ on Ω is generated by the following (see figure 3.1). Let v and v' be vertices of T. The points $x \in X_v$ and $x' \in X_{v'}$ are equivalent in Ω

if there is an edge e between v and v' and a point $x_e \in X_e$ such that $x = f_{e,v}(x_e)$ and $x' = f_{e,v'}(x_e)$ simultaneously. Let $\Omega/_\sim$ be the quotient under this relation and let $\pi' : \Omega \to \Omega/_\sim$ be the corresponding projection. An equivalence class $[x]$ of an element $x \in \Omega$ is denoted by x itself.

Let ∂T be the (visual) boundary of the Bass-Serre tree T. We define M' as following:

$$M' = \partial T \sqcup (\Omega/_\sim)$$

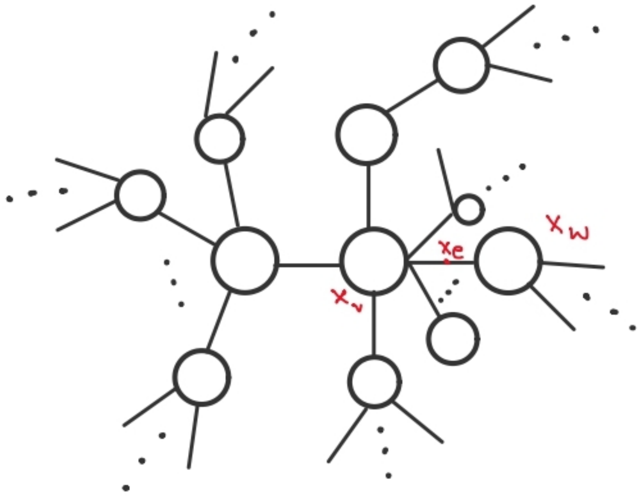

Figure 3.1: Circles denote the Bowditch boundaries.

Definition 3.1.1 (Domains). Let $x \in \Omega/_\sim$. We define the *domain* of x as

$$D(x) = \{v \in \mathcal{V}(T) | x \in \pi'(X_v)\}$$

We also say that domain of a point $\xi \in \partial T$ is $\{\xi\}$ itself.

3.1.2 Final construction of M

The construction of M' is exactly the same as the construction of Dahmani in [18]. However, suppose M' is equipped with the topology defined in [18] then it turns out that M' is not a Hausdorff space (see 3.1.9). Therefore we need to modify M'. For getting the desired space, we further define an equivalence relation on the set M'. Firstly observe that for the domain of each element in $\Omega/_\sim$ there are following three

possibilities:
(1) a singleton (2) a finite non-singleton subtree of T (3) an infinite subtree of T.

Suppose, for some $x \in \Omega/_\sim$, $D(x)$ is an infinite subtree of T. We identify the boundary points of $D(x)$ in ∂T to x itself. By considering the equivalence relation generated by these relations, we denote the quotient of M' by M. Again, M can be written as a disjoint union of two sets of equivalence classes :

$$M = \Omega' \sqcup (\partial T)'$$

where Ω' (as a set it is same as $\Omega/_\sim$) is the set of equivalence classes of elements in $\Omega/_\sim$ and $(\partial T)'$ is the equivalence classes of the remaining elements in ∂T as some elements of ∂T are identified with parabolic points of edge groups. Note that the equivalence class of each remaining element in ∂T is a singleton.

Definition 3.1.2 (Domains of points in M). We define the domain of each element in M as previously defined.

Remark 3.1.3. Note that, if $\eta \in \partial T$ is identified with parabolic a point, say x, of an edge group then we take domain of η same as that of x. Also, we write p for an element of M, if $p \in \Omega'$ then $p = x, y, z, ...$ and if $p \in (\partial T)'$ then $p = \xi, \eta, \zeta, ...$

From the construction of M, we observe the following.

Lemma 3.1.4. *For distinct $p, q \in M$, either $D(p) \cap D(q) = \emptyset$ or $D(p) \cap D(q)$ is a singleton.*

Proof. If either p or q is in $(\partial T)'$ then, by definition of the domain, $D(p) \cap D(q) = \emptyset$. Now, assume that both p and q are in Ω'. Let $p \in X_v$ for some $v \in \mathcal{V}(T)$.

Case 1. Suppose $p \notin X_e$ for any edge e adjacent to v. Then $D(p) = \{v\}$. Since for any $q \in \Omega'$ the domain $D(q)$ is a subtree of T, $|D(p) \cap D(q)| \leq 1$.

Case 2. Suppose $p \in X_e$ for some edge e adjacent to v. Since for any edge $e \in T$ the space X_e is a singleton and domains are subtrees of T, it follows that for any $p \neq q \in \Omega'$, $|D(p) \cap D(q)| \leq 1$. □

It is clear that for each v in T, the restriction of projection map π' from X_v to $\Omega/_\sim$ is injective. Let π'' be the projection map from M' to M. Let π be the composition of the restriction of π'' to $\Omega/_\sim$ and π'. Now, the following lemma is immediate.

Lemma 3.1.5. *For each $v \in T$, the restriction of π to X_v is injective.*

Proof. Since the restriction of π'' to $\pi'(X_v)$ is injective and the composition of two injective map is again injective, π'' is injective. □

3.1.3 Definition of neighborhoods in M

In this subsection, we closely follow the definition of neighborhoods given by Dahmani in [18, Section 2.3] and define a family $(W_n(p))_{n\in\mathbb{N},p\in M}$ of subsets of M that generates a topology on M. For a vertex v and an open subset U of X_v, we define the subtree $T_{v,U}$ of T as

$$\{w \in \mathscr{V}(T) : X_e \cap U \neq \emptyset\}$$

where e is the first edge of the geodesic $[v,w]$ joining v and w. For each vertex v in T, let us choose $\mathscr{U}(v)$, a countable basis of open neighborhoods of X_v. Without loss of generality, we can assume that for all v, the collection of open subsets $\mathscr{U}(v)$ contains X_v. Let x be in Ω' and $D(x) = \{v_1,...,v_n,...\} = (v_i)_{i\in I}$. Here I is a subset of $\mathbb{N}$. For each $i \in I$, let $U_i \subset X_{v_i}$ be an element of $\mathscr{U}(v_i)$ containing x such that for all but finitely many indices $i \in I, U_i = X_{v_i}$. We define the set $W_{(U_i)_{i\in I}}(x)$ as the disjoint union of three subsets

$$W_{(U_i)_{i\in I}}(x) = A \cup B \cup C$$

Description of A,B,C in words: The set A is nothing but the collection of all boundary points of subtrees T_{v_i,U_i} which are not identified with parabolic points corresponding to the edge groups. The set B is a collection of all points y outside $\bigcup_{i\in I} X_{v_i}$ in Ω' whose domains lie inside $\bigcap_{i\in I} T_{v_i,U_i}$. C is simply the union of all neighborhoods U_i around x in X_v for each v in $D(x)$.

In notation A,B,C are defined as follows:

$$A = (\bigcap_{i\in I} \partial T_{v_i,U_i}) \cap (\partial T)'$$
$$B = \{y \in \Omega' \setminus (\bigcup_{i\in I} X_{v_i}) | D(y) \subset \bigcap_{i\in I} T_{v_i,U_i}\}$$
$$C = \bigcup_{i\in I} U_i$$

As $A \subset (\partial T)'$, the remaining elements of $\bigcap_{i\in I} \partial T_{v_i,U_i}$ are in B. In this way, A, B, C are disjoint subsets of M.

Remark 3.1.6. (1) The set $W_{(U_i)_{i\in I}}(x)$ is completely defined by the data of the domain of x, the data of a finite subset J of I, and the data of an element of $\mathscr{U}(v_j)$ for each index $j \in J$. Therefore there are only countably many different sets $W_{(U_i)_{i\in I}}(x)$, for $x \in \Omega'$, and $U_i \in \mathscr{U}(v_i), v_i \in D(x)$. For each x, we choose an arbitrary order and denote them by $W_m(x)$.

(2) The main difference between the neighborhoods defined above and the neighborhoods defined by Dahmani in [18, Section 2.3] lies in the description of A. In [18], the author defines $A = \bigcap_{i\in I} \partial T_{v_i,U_i}$. However, as some elements of $\bigcap_{i\in I} \partial T_{v_i,U_i}$ are identified with the boundaries of some edge groups, we exclude those identified points in order to hold all the required things (see also 3.1.9).

Next, we define neighborhoods around the points of $(\partial T)'$. Let $\xi \in (\partial T)'$ and choose a base point v_0 in T. Firstly, we define the subtree $T_m(\xi) = \{w \in V(T) : |[v_0, w] \cap [v_0, \xi)| > m\}$. We set

$$W_m(\xi) = \{\zeta \in M | D(\zeta) \subset T_m(\xi)\}$$

This definition does not depend on the choice of base point v_0, up to shifting the indices.

3.1.4 Topology of M

Consider the smallest topology $\mathscr{T}$ on M such that the family of sets $\{W_n(p) : n \in \mathbb{N}, p \in M\}$ are open subsets of M. In this subsection, we prove that M with this topology is a compact metrizable space. This is proved with the help of following lemmata. We start with a simple observation.

Note: The statements and proof's ideas of the lemmas in this subsection are adapted from [18, Subsection 2.4] with appropriate changes.

Lemma 3.1.7 (Avoiding an edge). *Let p be a point in M and let e be an edge in T with at least one vertex not in $D(p)$. Then there exists an integer m such that $W_m(p) \cap X_e = \emptyset$.*

Proof. If $p \in (\partial T)'$ then both of the vertices of e are not in $D(p)$. The lemma follows by taking one of the vertex as the base point and $m \geq 1$. If $p \in \Omega'$ then a unique segment exists from $D(p)$ to the edge e. Let v be the vertex from where this segment starts and e_0 be the first edge. Then X_{e_0} does not contain p. So to find a $W_m(p)$ such that it does not intersect X_e, it is sufficient to find a neighborhood around p in X_v which does not intersect X_{e_0}. But it is evident as X_{e_0} is just a point. □

Lemma 3.1.8. *$(M, \mathscr{T})$ is a Hausdorff space*

Proof. Let p and q be two distinct points in M. There are two cases to be consider.

Case 1. Suppose $D(p) \cap D(q) = \emptyset$. Then there is a unique geodesic segment from a vertex of $D(p)$ to a vertex of $D(q)$ in T having an edge e on this segment such that both vertices of e are neither contained in $D(p)$ nor in $D(q)$. Then, by Lemma 3.1.7, we can find disjoint neighborhoods around p and q.

Case 2. Suppose $D(p) \cap D(q) \neq \emptyset$. Then, according to Lemma 3.1.4, there is only one vertex in this intersection. Let $D(p) \cap D(q) = \{v\}$. Since X_v is Hausdorff space, we can find disjoint open subsets of X_v around p and q, respectively. Using these neighborhoods in X_v, it is clear that we have disjoint neighborhoods around p and q in M. □

Remark 3.1.9. The reason why we need to define a further equivalence relation on the set M' is the following:

In M', if we consider a parabolic point p corresponding to an edge group (which is not a maximal parabolic) then $D(p)$ is an infinite subtree of T, and there are uncountably many boundary points of $D(p)$. If we take one boundary point η of $D(p)$ then on the geodesic ray $[v_0, \eta)$ there is no edge with at least one vertex not belonging to $D(p)$, where v_0 is some vertex in $D(p)$. Thus we can't find a disjoint neighborhood around η and p. The same kind of situation will arise when we try to prove that M' is a regular space.

Lemma 3.1.10 (Filtration). *For every $p \in M$, every integer n, and every $q \in W_n(\xi)$, there exists m such that $W_m(q) \subset W_n(p)$.*

Proof. Suppose $p \in (\partial T)'$ and $W_n(p)$ is a neighborhood around p. Let $q \in W_n(p)$. If $q \in (\partial T)'$ then choose $m = n$ and $W_m(q) = W_n(p)$. Let q be some point in Ω'. Suppose the subtree $T_n(p)$ starts at the vertex v and let e be the last edge on the geodesic segment from a base vertex to v. Then except X_e, all the points in M corresponding to subtree $T_n(p)$ is in $W_n(p)$. By this observation, it is clear that there exists a neighborhood $W_m(q)$ such that $W_m(q) \subset W_n(p)$ for sufficiently large m. Now, suppose that $p \in \Omega'$ and $q \in W_n(p)$. If $D(p) \cap D(q) = \emptyset$ then there exists an edge e on a unique geodesic segment from $D(p)$ to $D(q)$ such that both vertices of e neither lie in $D(p)$ nor in $D(q)$. Then, by Lemma 3.1.7, one can find a neighborhood $W_m(q)$ sitting inside $W_n(p)$. If $D(p) \cap D(q) \neq \emptyset$ then it is a singleton. Let this intersection be $\{v\}$. As $q \in W_n(p)$, q is in some U_i, where U_i is in neighborhood basis of X_{v_i} and $D(p) = (v_i)_{i \in I}$. Now, we find a neighborhood V_i around q sitting inside U_i. Finally, using this V_i, we see that there exists $W_m(q)$ such that $W_m(q) \subset W_n(p)$. □

Lemma 3.1.11. *The family of sets $\{W_n(p) : n \in \mathbb{N}, p \in M\}$ forms a basis for the topology $\mathscr{T}$.*

Proof. Using Lemma 3.1.10, it remains to prove that if $W_{n_1}(p_1)$ and $W_{n_2}(p_2)$ are two neighborhoods and $q \in W_{n_1}(p_1) \cap W_{n_2}(p_2)$ then there exists a neighborhood around q, namely $W_m(q)$, such that $W_m(q) \subset W_{n_1}(p_1) \cap W_{n_2}(p_2)$. Again Lemma 3.1.10, there exist m_1 and m_2 such that $W_{m_1}(q) \subset W_{n_1}(p_1)$ and $W_{m_2}(q) \subset W_{n_1}(p_2)$. Note that, $W_k(q) \subset W_{m_1}(q) \cap W_{m_2}(q)$ for some k. Hence the lemma. □

Lemma 3.1.12. *For each $v \in \mathscr{V}(T)$, $\pi_{|X_v} : X_v \to M$ is continuous.*

Proof. Let x be an element of X_v and let $\pi(x)$ be its image in M. We denote this image by x. Consider the neighborhood $W_n(x)$ around x. Now, by the definition of neighborhoods, it is clear that the inverse image of $W_n(x)$ under $\pi_{|X_v}$ is an open subset of X_v. Hence, the restriction of π to X_v is continuous. □

Lemma 3.1.13 (Regularity). *The topology $\mathscr{T}$ is regular, i.e. for all $p \in M$ and for all $W_m(p)$ there exists n such that $\overline{W_n(p)} \subset W_m(p)$.*

Proof. Let $p \in M$ and let $W_m(p)$ be a neighborhood around p.

Case 1. Let $p \in (\partial T)'$. Let v be a vertex from where the subtree $T_m(\xi)$ starts. Let e be the last edge of the geodesic segment from v_0 to v. Observe that the closure of $W_m(p)$ contains only one extra point, namely X_e. By taking n to be sufficiently large, we see $\overline{W_n(p)} \subset W_m(p)$.

Case 2. Let $p \in \Omega'$ and let $D(p) = (v_i)_{i \in I}$. Again observe that in the closure of $W_m(p)$ only extra points are the points in the closure of each U_i in X_{v_i} (If some point is not in the closure of U_i then one can easily find a neighborhood around that point disjoint from $W_m(p)$). Since for each $v \in T$ the space X_v is regular, choose a neighborhood V_i of p in X_{v_i} such that $\overline{V_i} \subset U_i$. Now, it is immediate that $\overline{W_{\{V_i\}_{i \in I}}(p)} \subset W_m(p)$. □

Proposition 3.1.14. *The space M is perfect metrizable.*

Proof. By the previous lemmas, we see that the topology on M is second countable, Hausdorff, and regular. Thus, by Urysohn's metrization theorem, we see that M is a metrizable space. Since every point of M is a limit point, M is perfect. □

Convergence criterion: A sequence $\{p_n\}_n$ in M converges to a point p if and only if $\forall n$, $\exists m_0 \in \mathbb{N}$ such that $\forall m > m_0$, $p_m \in W_n(p)$.

Now we show that the space M is compact. The idea of proof of this fact is adapted from [18, Theorem 2.10].

Lemma 3.1.15. *The metrizable space M is compact.*

Proof. Since M is metrizable, it is sufficient to show that M is sequentially compact. Let $\{p_n\}_n$ be a sequence in M. Let us fix a vertex $v \in T$. For each n, choose a vertex

v_n (if $p_n \in (\partial T)'$ then take $v_n = p_n$) in $D(p_n)$. We see that, up to extraction of a subsequence, either the Gromov inner products $(v_n.v_m)_v$ remain bounded or they go to infinity. In the latter case, the sequence $\{v_n\}$ converges to a point $q \in \partial T$. If q is the point which is identified with some edge boundary point then there is a ray in the domain of that edge boundary point converging to q. Then, by the definition of the neighborhood around q, we see that p_n converges to q. If $q \in (\partial T)'$ then again there is a ray converges to q, and by the convergence criterion, p_n converges to q. Now, in the first case up to extracting a subsequence, we assume that Gromov inner products is equal to some constant N. For each n, let g_n be the geodesic from v to v_n then there exists a geodesic $g = [v, v']$ of length N such that g lies in each g_n. For $n \neq m$, g_n and g_m do not have a common prefix whose length is bigger than the length of g. We have the following two cases:

Case 1. There exists a subsequence $\{g_{n_k}\}$ such that $g_{n_k} = g$. Since $X_{v'}$ is compact, we get a subsequence of $\{p_n\}$ which converges to a point of $X_{v'}$.

Case 2. There exists a subsequence $\{g_{n_k}\}$ of $\{g_n\}$ such that each g_{n_k} strictly longer than g. Let e_{n_k} be the edges just after v'. Note that they are all distinct. As for each edge e, X_e is a singleton, so $\{X_{e_{n_k}}\}$ forms a sequence of points in $X_{v'}$. Since $X_{v'}$ is compact, there exists a subsequence which converges to a point in $X_{v'}$. Then, by the convergence criterion, we see that there exists a subsequence of $\{p_n\}$ converging to this point of $X_{v'}$. (It may be possible that all the X_{n_k} are equal to p for some p in $X_{v'}$. However, the sequence converges to p in this situation also.) □

3.2 Dynamics of Γ on *M*

In this section, we prove that the group Γ acts on M as a convergence group. For that we need the following two lemmas.

Note: By taking edge length 1 on each edge of the Bass-Serre tree T of Γ, we consider T as a metric graph. Hence, T is a geodesic metric space. In this subsection, d_T denotes the metric on T.

Remark 3.2.1. The statements and the idea of proofs of the following lemmata are adapted from [18, Section 3] with appropriate modifications.

Lemma 3.2.2 (Large Translation). *Let $\{\gamma_n\}_n$ be a sequence in Γ. Assume that, for some (hence any) vertex $v_0 \in T$, $d_T(v_0, \gamma_n v_0) \to \infty$. Then, there is a subsequence $\{\gamma_{\sigma(n)}\}_n$, two points $p \in M$ and $\zeta \in (\partial T)'$ such that for all compact $K \subset M \setminus \{\zeta\}$, $\gamma_{\sigma(n)} K$ converges uniformly to p.*

Proof. Let $p_0 \in X_{v_0}$. Since M is sequentially compact, there exists a subsequence $\{\gamma_{\sigma(n)}\}$ such that $\{\gamma_{\sigma(n)}p_0\}$ converges to a point $p \in M$. Note that we still have $d_T(v_0, \gamma_{\sigma(n)}v_0) \to \infty$. Let v_1 be a vertex in T such that $v_1 \neq v_0$. For each n, the lengths of the geodesic segment $[\gamma_n v_0, \gamma_n v_1]$ equal to the length of $[v_0, v_1]$. As the $d_T(v_0, \gamma_{\sigma(n)}v_0)$ goes to ∞, for all m there is n_m such that for all $n > n_m$, the segments $[v_0, \gamma_{\sigma(n)}v_0]$ and $[v_0, \gamma_{\sigma(n)}v_1]$ have prefix of length more than m. Then, by the convergence criterion 3.1.4, for all $v \in T$, $\gamma_{\sigma(n)}X_v$ converges uniformly to p. Let $\zeta_1, \zeta_2 \in (\partial T)'$. As triangles in $\bar{T} = T \cup \partial T$ are degenerate, the triangle with vertices v_0, ζ_1, ζ_2 has the center a vertex v in T. Therefore, for all $m \geq 0$, the segments $[v_0, \gamma_{\sigma(n)}v_0]$ and $[v_0, \gamma_{\sigma(n)}v]$ coincide on a subsegment of length more than m for sufficiently large n. Now, at least for one ζ_i, the ray $[v_0, \gamma_{\sigma(n)}\zeta_i]$ has a common prefix with $[v_0, \gamma_{\sigma(n)}v_0]$ of length at least m. Then by convergence criterion $\gamma_{\sigma(n)}\zeta_i$ converges to p. Therefore, there exists at most one ζ in $(\partial T)'$ such that, for all $\zeta' \in ((\partial T)' \setminus \{\zeta\})$, $\gamma_{\sigma(n)}\zeta' \to p$ (since for any two points in $(\partial T)'$ at least one of them converges to p under the action of γ_n).

Let K be a compact subset of $M \setminus \{\zeta\}$. Then there exists a vertex v, $x \in \Omega'$, and a neighborhood $W_n(x)$ around x containing K such that $\zeta \notin W_n(x)$. Consider the segment $[v_0, v]$. By the discussion at the beginning of the proof, for all $v \in T$, the sequence $\{\gamma_{\sigma(n)}X_v\}_n$ uniformly converges to p, and the sequence $\{\gamma_{\sigma(n)}W_m(x)\}_n$ uniformly converges to p. Hence the convergence is uniform on K. □

Lemma 3.2.3 (Small Translation). *Let $\{\gamma_n\}_n$ be a sequence of distinct elements in Γ and let for some (hence any) vertex v_0, the sequence $\{\gamma_n v_0\}_n$ is bounded in T. Then there exists a subsequence $\{\gamma_{\sigma(n)}\}_{n\in\mathbb{N}}$, a vertex v, a point $p \in X_v$, and another point $p' \in \Omega'$, such that for all compact $K \subset M \setminus \{p'\}$, one has $\{\gamma_{\sigma(n)}\}K \to p$ uniformly.*

Proof. There are two cases to be consider.

Case 1. Assume that for some vertex v and some element $\gamma \in \Gamma$ there exists a subsequence $\{h_n\}_n$ in Γ_v such that $\gamma_n = h_n\gamma$ for all n. Since Γ_v acts as a convergence group on X_v, one can further extract a subsequence of $\{\gamma_n\}$ (we shall denote it again by $\{\gamma_n\}$) and a point p' in $X_{\gamma^{-1}v}$ such that, for all compact subsets K of $X_{\gamma^{-1}v} \setminus \{p'\}$, $\gamma_n K \to p$ uniformly for some $p \in X_v$. Suppose p' is a conical limit point. Then p' is not in any X_e's contained in $X_{\gamma^{-1}v}$. Let e_n be the possible edges starting from the vertex $\gamma^{-1}v$. For any $q \in M \setminus \{X_{\gamma^{-1}v}\}$, we see that the unique edge path from $\gamma^{-1}v$ to w contains e_n for some n, and $q \in X_w$. Since $\gamma_n r \to p$ for all r in $X_{\gamma^{-1}v} \setminus \{p'\}$, by the convergence criterion, we see that $\gamma_n q$ converges to p, and the same is true for the points in $(\partial T)'$. Hence for all compact $K \subset M \setminus \{p'\}$ we have $\gamma_n K \to p$ uniformly. Suppose p' is a parabolic point, $\gamma p'$ is also a parabolic point in X_v. Then

$\gamma_n p'$ also converges to p otherwise $\gamma p'$ is a conical limit point. Thus again, by the same argument as above, we see that for all $q \in M$, $\gamma_n q \to p$.

Case 2. Suppose such a sequence $\{h_n\}_n$ and a vertex v do not exist. After possible extraction we can assume that $d_T(v_0, \gamma_n(v_0))$ is constant. Let us choose a vertex v such that there exists a subsequence $\{\gamma_{\sigma(n)}\}_{n\in\mathbb{N}}$ such that the segments $[v_0, \gamma_{\sigma(n)} v_0]$ share a common segment $[v_0, v]$ and the edges $e_{\sigma(n)}$ located just after v are all distinct. Since X_v is compact, one can extract a subsequence $\{e_{\sigma'(n)}\}_{n\in\mathbb{N}}$ such that spaces corresponding to these edges converge to a point p in X_v. Then, by the convergence criterion, $\gamma_{\sigma'(n)} X_{v_0}$ converge uniformly to p. Let $\xi \in (\partial T)'$. Then v is not on the ray $[\gamma_{\sigma'(n)} v_0, \gamma_{\sigma'(n)} \xi]$ for all sufficiently large n. For if v is there then, for infinitely many n, $\gamma^{-1}_{\sigma'(n)} v = w$ for some fixed vertex w on $[v_0, \xi)$, and we see that we are in the first case which is a contradiction. Thus, for all $\xi \in (\partial T)'$, we see that the unique rays from v to $\gamma_{\sigma'(n)} \xi$ contains the edge $e'_{\sigma(n)}$ for some n. Hence, by the convergence criterion, $\gamma_{\sigma'(n)} \xi \to p$ for all $\xi \in (\partial T)'$. Now, let $x \in \Omega'$ such that $x \notin X_{v_0}$. Suppose $x \in X_{v'}$ where $v' \neq v_0$. Again by the same reasoning, we see that $v \notin [\gamma_{\sigma'(n)} v_0, \gamma_{\sigma'(n)} v']$ for all sufficiently large n. Then the unique geodesic segment from v to $\gamma_{\sigma'(n)} v'$ contains $e'(n)$ for sufficiently large n. Hence by definition of neighborhoods $\gamma_{\sigma'(n)} x \to p$. Thus, for all compact subsets $K \subset M$, we see that $\gamma_{\sigma'(n)} K \to p$ uniformly. □

Using the previous two lemmas, we get the following:

Corollary 3.2.4. *The group Γ acts on M as a convergence group.*

Proof. Fix a vertex v_0 in T. Let $\{\gamma_n\}_n$ be a sequence in Γ. Then, up to extracting a subsequence, either the distance from v_0 to $\gamma_n v_0$ goes to infinity or the distance from v_0 to $\gamma_n v_0$ is bounded. In either case, the previous lemmas imply the corollary. □

3.3 Proof of Theorem 3.0.2 and consequences

First of all, we recall a construction from [47]. Let Γ be a hyperbolic group and let $\mathscr{G}$ be a finite malnormal family of quasiconvex subgroups of G. In order to prove that $(\Gamma, \mathscr{G})$ is relatively hyperbolic, in [47], Manning constructed a space which is the quotient of the Gromov boundary of Γ and showed that Γ acts geometrically finitely on the quotient. The quotient was obtained by collapsing all the translates of the limit set of subgroups in $\mathscr{G}$. Using Yaman's characterization (Theorem 2.4.4) of relative hyperbolicity, this quotient was turn out to be the Bowditch boundary of $(\Gamma, \mathscr{G})$. There, to prove that the action of Γ on the quotient is convergence, we do not require that Γ is hyperbolic and $\mathscr{G}$ is a malnormal family of quasiconvex subgroups.

Suppose Γ acts on a compact metrizable space X as a convergence group. Let $\mathscr{G}$ is a dynamically malnormal family of dynamically quasiconvex subgroups. Then form a quotient space $X/_\sim$ as in [47] by collapsing the translates of the limit sets of subgroups in $\mathscr{G}$. Also, assume that $|\Lambda(H)| \geq 2$, where $H \in \mathscr{G}$. To prove Proposition 2.2 in [47], we require that the collection of limit sets of the cosets of the elements in $\mathscr{G}$ forms a null sequence which follows from Proposition 2.3.17. Hence, we immediately have the following:

Lemma 3.3.1. *1. $X/_\sim$ is a compact metrizable space.*

2. The group Γ acts on $X/_\sim$ as a convergence group.

Proof. (1) As noted above the collection of the limit sets of the cosets of elements in $\mathscr{G}$ form a null sequence. Thus, by [47, Proposition 2.1], $X/\sim$ is compact metrizable.

(2) It follows from Proof of Claim 2 in [47]. □

Also, note that each subgroup in $\mathscr{G}$ becomes a parabolic subgroup for the action of Γ on $X/_\sim$.

Proof of Theorem 3.0.2. Let v be a vertex in T and let Γ_v be a vertex group that acts as a convergence group on X_v. Take the collection of edges incident to v and take a collection of those edge groups which are not parabolic in Γ_v. Consider the stabilizers of the limit sets of these edge groups in Γ_v. By assumption, they form a dynamically malnormal family of dynamically quasiconvex subgroups. Therefore, by Lemma 3.3.1, we obtain a quotient of X_v, namely $X_v/_\sim$ on which Γ_v acts as a convergence group and all edge groups incident to v become parabolic subgroups of Γ_v. Hence by following the same process at each vertex group, we are in the situation where we have a graph of convergence groups with edge groups parabolic in adjacent vertex groups. By invoking Theorem 3.0.1, we are done. □

Now, we prove the following proposition which is a consequence of the above theorem and also give an answer to Question 2 when edge groups are infinite cyclic.

Proposition 3.3.2. *Let Γ be a group that splits as a finite graph of countable convergence groups with infinite cyclic edge groups, which are dynamically malnormal in the adjacent vertex groups. Then Γ is a convergence group.*

Proof. Again it is sufficient to prove the proposition when Γ is either an amalgamated free product or an HNN extension.

Case 1. Let $\Gamma = \Gamma_1 *_{\langle\gamma_1\rangle\simeq\langle\gamma_2\rangle} \Gamma_2$. If γ_1 is parabolic in Γ_1 then clearly $\langle\gamma_1\rangle$ is dynamically quasiconvex in Γ_1. If γ_1 is loxodromic then again $\langle\gamma_1\rangle$ is dynamically quasiconvex in Γ_1 by Lemma 2.3.16. Thus, by Lemma 2.3.13, the stabilizer of the limit set of $\langle\gamma_1\rangle$ in Γ_1 is a dynamically quasiconvex subgroup of Γ_1. Also, by assumption, the stabilizer of the limit set of $\langle\gamma_1\rangle$ in Γ_1 is dynamically malnormal. Similarly, the stabilizer of the limit set of $\langle\gamma_2\rangle$ in Γ_2 is dynamically quasiconvex and dynamically malnormal. Hence, by Theorem 3.0.2, Γ is a convergence group.

Case 2. Let $\Gamma = \Gamma_1 *_{\langle\gamma_1\rangle\simeq\langle\gamma_2\rangle}$. Again, as in case 1, the stabilizers of the limit sets of $\langle\gamma_1\rangle$ and $\langle\gamma_2\rangle$ are dynamically quasiconvex subgroups of Γ_1. Since $\langle\gamma_1\rangle$ and $\langle\gamma_2\rangle$ are cyclic subgroups, their limit sets are either equal or disjoint. By assumption, $\langle\gamma_1\rangle$ and $\langle\gamma_2\rangle$ are dynamically malnormal subgroups of Γ_1. So, the intersection of their limit set is empty. Thus, the stabilizers of limit sets of $\langle\gamma_1\rangle$ and $\langle\gamma_2\rangle$ form a dynamically malnormal family. Hence, by Theorem 3.0.2, Γ is a convergence group. □

If the vertex groups in Proposition 3.3.2 are torsion-free then, by the following lemma, we do not need to assume dynamical malnormality of edge groups in the adjacent vertex groups.

Lemma 3.3.3. *Let Γ be a torsion-free group that acts on X as a convergence group. Let $\gamma \in \Gamma$ be an infinite order element and let $H = Stab_\Gamma(\Lambda(\langle\gamma\rangle))$ Then H is a dynamically quasiconvex and dynamically malnormal subgroup of Γ.*

Proof. If γ is parabolic the conclusion follows. Suppose γ is a loxodromic element. Then, $\langle\gamma\rangle$ is dynamically quasiconvex in Γ by Lemma 2.3.16. Thus, the dynamical quasiconvexity of H follows from Lemma 2.3.13. Now, we prove dynamical malnormality of H. Let $\gamma_1 \in \Gamma\setminus H$ and assume that $\gamma_1\Lambda(H)\cap\Lambda(H) \neq \emptyset$. Let $\Lambda(H) = \{x_1, x_2\}$ and let γ_1 fixes, either x_1 or x_2. Since Γ is torsion-free, γ_1 has infinite order. Thus γ_1 is either a parabolic or a loxodromic element. By Lemma 2.3.8, a parabolic point cannot be a fixed point of a loxodromic element so γ_1 can not be parabolic. By Lemma 2.3.5, if γ_1 is loxodromic then γ_1 must fix the other point. But this implies that γ_1 is in H, which is a contradiction. Now, suppose that γ_1 does not fix any of x_i and $\gamma_1 x_1 = x_2$. Consider the element $\gamma_1' = \gamma_1^{-1}\gamma\gamma_1$ which fixes x_1. Again γ_1' cannot be parabolic so it has to be loxodromic, but this implies that $\gamma_1' x_2 = x_2$. Thus $\gamma_1 x_2 = x_1$ and this implies that γ_1 is in H, which is again a contradiction. □

Hence for torsion-free groups, we have the following:

Proposition 3.3.4. *Let Γ be the fundamental group of a finite graph of torsion-free countable convergence groups with infinite cyclic edge groups. Then Γ is a convergence group.* □

Remark 3.3.5. Although we have answered Question 2 in the special cases, but the general case is still not answered. For the general case, if we try to work with Dahmani's construction, we need to identify more points in the space constructed in [18] but it is unclear which points are needed to be identified.

3.4 Proof of Theorems 3.0.3, 3.0.4

Let Γ be as in Theorem 3.0.1. Firstly, we show that if each vertex group Γ_v acts geometrically finitely on compact metrizable space X_v then the group Γ acts geometrically finitely on the space M constructed in Section 3.1. Using this, we give a proof of Theorems 3.0.3, 3.0.4. To prove that Γ acts geometrically finitely, we demonstrate that every point of M is either a conical limit point or a bounded parabolic point. So, we start proving the following lemmata:

Lemma 3.4.1. *Every point in $(\partial T)' \subset M$ is a conical limit point for Γ in M.*

Proof. Let $\eta \in (\partial T)' \subset M$ and let v_0 be a vertex of T. Then there exists a sequence $\{\gamma_n\}_n$ in Γ such that $\gamma_n v_0$ lies on the unique geodesic ray $[v_0, \eta)$ for all n. Then, by lemma 3.2.2, there exists a subsequence denoted by γ_n and a point $p \in M$ such that for all q in M except possibly a point in $(\partial T)'$, we have $\gamma_n^{-1} q$ converges to p. Now after multiplying each γ_n on the the right by an element of Γ_{v_0}, we can assume that p does not belong to X_{v_0}. To prove that η is a conical limit point of Γ in M, it is sufficient to prove that $\gamma_n \eta$ does not converge to p. Observe that the ray $[\gamma_n^{-1} v_0, \gamma_n^{-1}\eta)$ always have v_0 on this ray for all n. If the sequence $\{\gamma_n^{-1}\eta\}$ converges to p then, as the sequence $\gamma_n x$ also converges to p for any $x \in X_{v_0}$, we see that p belongs to Γ_{v_0}, which is a contradiction to our choice of p. □

Now, we prove that each conical limit point for the action of the vertex group Γ_v on X_v is conical for the action of Γ on M.

Lemma 3.4.2. *Every point in Ω' which is the image of a conical limit point in the vertex stabilizer's boundary is a conical limit point for Γ in M.*

Proof. Let $x \in X_v$ be a conical limit point for Γ_v in X_v. There exists a sequence $\{\gamma_n\}_n \subset \Gamma_v$ and two distinct points y and z in X_v such that $\gamma_n x \to y$ and $\gamma_n x' \to z$ for all $x' \neq x$. Now, we show that $\pi(x)$ is a conical limit point for Γ in M. Since the restriction of π to X_v is Γ_v-equivariant continuous from X_v to M. Therefore $\pi(\gamma_n x) = \gamma_n \pi(x) \to \pi(y)$ and $\pi(\gamma_n x') = \gamma_n \pi(x') \to \pi(z)$. Since restriction of π is injective, so $\pi(y)$ and $\pi(z)$ are distinct. Thus, $\pi(x)$ is a conical limit point for Γ_v in M. Hence $\pi(x)$ is a conical limit point for Γ in M. □

Lemma 3.4.3. *Every point in Ω' which is image by π of a bounded parabolic point in some vertex stabilizer's boundary is a bounded parabolic point for Γ in M.*

Proof. **(Amalgam Case)** We prove the lemma in two cases:

Case 1. Let p be a bounded parabolic point for a vertex group Γ_v in X_v. Suppose p is not in any edge space attached to X_v. We denote $\pi(p)$ by p. Let $D(p) = \{v\}$ and let P be the stabilizer of p in Γ. Since P fixes the vertex v, $P \leq \Gamma_v$. In fact, P is the stabilizer of p in Γ_v. As p is bounded parabolic point for Γ_v in X_v, P acts co-compactly on $X_v \setminus \{p\}$. Let K be a compact subset of $X_v \setminus \{p\}$ such that $PK = X_v \setminus \{p\}$. Suppose $\mathscr{E}$ is the set of edges whose boundaries intersect K. Let e be the edge with one vertex v, then there exists $h \in P$ such that $X_e \cap hK \neq \emptyset$. Therefore the set of edges $\cup_{h\in P} h\mathscr{E}$ contains every edge with one and only one vertex v. Let T_v be the connected subtree of T which starts from v and contains all vertices w of T such that the first edge of $[v,w]$ is in $\mathscr{E}$. Then $T_v \setminus \{v\} = \sqcup_{i\in I} T_i$ where T_i's are connected components of $T_v \setminus \{v\}$. Let p' be the natural projection from $M \to T \cup (\partial T)'$ defined as follows: for $q \in X_v \subset \Omega'$, define $p'(q) = v$ and for $\eta \in (\partial T)'$, define $p'(\eta) = \eta$. Let $K_i = p'^{-1}(T_i \cup \partial T_i)$. By definition of neighborhoods on M, it follows that K_i's are closed subsets of M. Note that $K \cap X_e \in \bigcup_{i\in I} K_i$ for all $e \in \mathscr{E}$. Define $K' = K \cup (\bigcup_{i\in I} K_i)$.

Claim: K' is closed in M

If the indexing set I is finite then K' is compact in M as finite union of compact sets is compact. Suppose I is an infinite set and $\{x_n\}$ is a sequence in K' such that $x_n \to x$. We show that $x \in K'$. If, for infinitely many n, $x_n \in K$ then $x \in K \subset K'$ as K is closed in M. If, except for finitely many n, $x_n \in K_i$ for some i then again $x \in K_i$ as K_i's are closed in M. Finally, suppose there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \in K_i$ for all i. Then, there are following two situations:

(i) $y = K \cap K_i$ for all i such that $x_{n_i} \in K_i$. Then in this case it is clear that $\{x_{n_i}\}$ converges to y and hence $x = y \in K'$

(ii) We have a sequence $\{z_i\}$ such that $z_i = K_i \cap K$ for all i such that $x_{n_i} \in K_i$. As K is compact in M, there is a subsequence of $\{z_i\}$ converging to $z \in K$. Then by definition of topology on M, it follows that $\lim x_{n_i} = z$ and thus $x = z \in K'$. Hence the claim and it is clear that $PK' = M \setminus \{p\}$.

Case 2. Suppose $p \in X_e$ for some edge $e \in T$. If $D(p)$ is a finite subtree of T then the proof follows from [18, Lemma 3.6] (or one can prove in a similar manner as in Case 1). Now, suppose that $D(p)$ is infinite. Suppose that v_1 and v_2 are vertices of e. Let P_1, P_2 be the maximal parabolic subgroups in the vertex groups $\Gamma_{v_1}, \Gamma_{v_2}$ respectively and let P be the parabolic edge group corresponding to e. Then $D(p)$ is nothing but the Bass-Serre tree of the amalgam $Q = P_1 *_P P_2$ and it is also the stabilizer of p in Γ. Under the action of $P_1 *_P P_2$ the quotient of $D(p)$ is the edge e. Since P_1, P_2 act co-compactly on $X_{v_1} \setminus \{p\}, X_{v_2} \setminus \{p\}$ respectively, there exists a compact subset K_i of $X_{v_i} \setminus \{p\}$ such that $P_i K_i = X_{v_i}$ for $i = 1,2$. Consider $\mathscr{E}_i$ the set of edges starting at v_i whose boundary intersects K_i but does not contain p. Let e' be an edge with only one vertex in $D(p)$ and let v' be this vertex. Then there exists $h \in P_i$ such that $X_e \cap hK_i \neq \emptyset$ for $i = 1,2$. Therefore the set of edges $\cup_{i=1,2} Q\mathscr{E}_i$ contains every edge with one and only one vertex in $D(p)$.

Construction of a compact subset: Again we follow the same scheme as in Case 1. Let T_{v_i} be the connected subtree T which starts from v_i and contains all vertices w of T such that the first edge of $[v,w]$ is in $\mathscr{E}_i$ for $i = 1,2$. Then $T_{v_i} \setminus \{v_i\} = \sqcup_{j \in J} T_j^i$ where T_j^i's are connected components of $T \setminus \{v_i\}$. Let p' be the natural projection as in Case 1. Then, by definitions of neighborhoods in M, $K_j^i = p'^{-1}(T_j^i \cup \partial T_j^i)$ is a closed subset of M for all j and $i = 1,2$. Define $K_i' = K_1 \cup (\bigcup_{j \in J} K_j^i)$ for $i = 1,2$. Then it follows from the claim in Case 1 that K_1' and K_2' are compact subsets of M. Hence $K'' = \cup_{i=1,2} K_i'$ is a compact set of M not containing p and $QK'' = M \setminus \{p\}$. Therefore p is a bounded parabolic point for Γ in M.

(HNN extension case) Again, there are two cases to be consider.

Case 1. Let Γ_v be vertex group in HNN extension and let P be a parabolic subgroup sitting inside a maximal parabolic subgroup, say P_1, and isomorphic to a subgroup P' of P_1. In this case, the proof is exactly the same as in the amalgam case except that the maximal parabolic subgroup corresponding to the edge boundary point is $P_1 *_{P \simeq P'}$, and maximal parabolic subgroups corresponding to parabolic points which are not in any edge spaces are maximal parabolic for Γ in M.

Case 2. Let Γ_v be same as in case 1 and suppose P is sitting inside P_1 and is isomorphic to a subgroup P' of maximal parabolic subgroup P_2, which is not conjugate to P_1 in Γ_v. Then, in this case, we can write $\Gamma_v *_{P \simeq P'} = (\Gamma_v *_P P')*_{P'}$, and we apply the amalgam and case 1 of HNN extension respectively to get the result. □

From the above lemmata, it is clear that if each Γ_v acts on X_v geometrically finitely then Γ acts on M geometrically finitely. Now, we are in the position of proving the following proposition:

Proposition 3.4.4. *Let Γ be a finitely generated group admitting a decomposition into a finite graph of convergence groups with finitely generated parabolic edge groups. Then Γ acts geometrically finitely on M (constructed in Section 3.1) if and only if each vertex group Γ_v acts geometrically finitely on compact metrizable space X_v.*

Proof. Since each edge group is finitely generated, by [7, Lemma 2.5], each vertex group Γ_v is finitely generated. Suppose each vertex group Γ_v acts geometrically finitely on X_v. Then, by Lemma 3.4.1, 3.4.2, 3.4.3, Γ acts geometrically finitely on M. Conversely, suppose that Γ acts geometrically finitely on M. Since each edge group is parabolic for the action of Γ on M, each edge group is a relatively quasiconvex subgroup of Γ. By [36, Proposition 5.2], each vertex group Γ_v is a relatively quasiconvex subgroup of Γ. In particular, each Γ_v acts geometrically finitely on X_v as X_v is the limit set of Γ_v for Γ acting on M. □

Proof of Theorem 3.0.3

Let Γ be either amalgam or HNN extension of relatively hyperbolic groups with parabolic edge groups. Since each vertex group Γ_v is relatively hyperbolic, Γ acts geometrically finitely on its Bowditch boundary. To prove that Γ is relatively hyperbolic, we use Yaman's characterization of relative hyperbolicity (Theorem 2.4.4). For constructing a space on which Γ acts geometrically finitely, we follow the same construction as in Section 3.1 by taking compactum X_v, X_e for Γ_v, Γ_e as Bowditch boundaries of these groups, respectively. Therefore, we have a compact perfect metrizable space M as in Section 3.1. Now, the proof of Theorem 3.0.1 gives that Γ acts on M as a convergence group. Since each vertex group acts geometrically on its Bowditch boundary, from the above lemmata, the groups Γ acts geometrically finitely on M. Hence, Γ is relatively hyperbolic, and M is equivariantly homeomorphic to the Bowditch boundary of Γ.

The limit set of each vertex group Γ_v for the action on M is Γ_v-equivariantly homeomorphic to its Bowditch boundary $\partial\Gamma_v$. It is clear that Γ_v acts geometrically finitely on its limit set. Hence, Γ_v is a relatively quasiconvex subgroup of Γ. This complete the proof of the theorem. □

Parabolic structure: Let Γ be as in the proof of the above theorem. Consider the collection $\mathscr{G}$ containing two types of subgroups of Γ: (1) the stabilizers of bounded parabolic points in Bowditch boundary of vertex groups which are not identified with edge parabolic point. (2) the stabilizers of edge parabolic points in Γ. Then Γ is hyperbolic relative to $\mathscr{G}$.

Next, we prove the combination theorem for graphs of relatively hyperbolic groups with cyclic edge groups. Again, it is sufficient to consider the amalgam and the HNN extension case.

Proof of Theorem 3.0.4

Case 1. Let $\Gamma = \Gamma_1 *_{Z_1 \simeq Z_2} \Gamma_2$. Suppose $Z_1 = \langle \gamma_1 \rangle$ and $Z = \langle \gamma_2 \rangle$. If both γ_1, γ_2 are parabolic elements in Γ_1, Γ_2 respectively then we are in the amalgam case of Theorem 3.0.3, and hence Γ is relatively hyperbolic. Suppose at least one of them is a hyperbolic element. Then by Theorem 2.4.10, we have a maximal elementary subgroup containing the cyclic subgroup generated by a hyperbolic element which is hyperbolically embedded. Again, we are in the amalgam case of Theorem 3.0.3 and therefore Γ is relatively hyperbolic.

Case 2. Let $\Gamma = \Gamma_1 *_{Z \simeq Z'}$ and let $Z = \langle \gamma \rangle, Z' = \langle \gamma' \rangle$ are isomorphic subgroups of Γ_1. Again if both γ, γ' are parabolic elements then we are in the HNN extension case of Theorem 3.0.3, and hence Γ is relatively hyperbolic. If at least one of them is a hyperbolic element, then by applying the Theorem 2.4.10, we get maximal elementary subgroups containing that cyclic subgroup that is hyperbolically embedded. Thus, we are in the HNN extension case of Theorem 3.0.3, and hence Γ is relatively hyperbolic.

In either case, we have proved that Γ is relatively hyperbolic and applying Theorem 3.0.3, we have a description of the Bowditch boundary. Also, each vertex groups is relatively quasiconvex in Γ. □

3.5 Homeomorphism type of Bowditch boundary

In [50], the authors proved that the homeomorphism type of Gromov boundary of the fundamental group of a graph of hyperbolic groups with finite edge groups depends only on the set of homeomorphism type of Gromov boundary of non-elementary hyperbolic vertex groups. It is not clear that the same result can be extended to the case of a graph of relatively hyperbolic groups with finite edge groups. However, under some assumptions, we prove a similar result for a graph of relatively hyperbolic groups with parabolic edge groups. For the convenience of the reader, we are again stating the following theorem:

Theorem 3.5.1. *Let $\mathscr{Y}$ be a finite connected graph and let $G(\mathscr{Y}), G'(\mathscr{Y})$ be two graphs of groups satisfying the following:*

1. *For each vertex $v \in V(\mathscr{Y})$, let $(G_v, \mathbb{P}_v), (G'_v, \mathbb{P}'_v)$ be relatively hyperbolic groups.*

2. *Let $e \in E(\mathscr{Y})$ be any edge. Suppose v, w are vertices connected by e. Let P_e, P'_e be parabolic edge groups in $G(\mathscr{Y}), G'(\mathscr{Y})$, respectively. Then either P_e, P'_e have infinite index in corresponding maximal parabolic subgroups in*

G_v, G'_v, respectively or P_e, P'_e have the same finite index in maximal parabolic subgroups in G_v, G'_v, respectively. Similarly, either P_e, P'_e have infinite index in maximal parabolic subgroups in G_w, G'_w, respectively or P_e, P'_e have the same finite index in corresponding maximal parabolic subgroups in G_w, G'_w, respectively.

3. *For any $v \in V(\mathscr{Y})$, let B_v, B'_v be the set of translates of parabolic points corresponding to adjacent edge groups under the action of G_v, G'_v on their Bowditch boundaries respectively. Suppose we have a homeomorphism from $\partial G_v \to \partial G'_v$ that maps B_v onto B'_v.*

Let $\Gamma = \pi_1(G(\mathscr{Y}))$, $\Gamma' = \pi_1(G'(\mathscr{Y}))$ and let $\partial\Gamma, \partial\Gamma'$ be their Bowditch boundaries, respectively. Then there exists a homeomorphism from $\partial\Gamma$ to $\partial\Gamma'$ preserving edge parabolic points, i.e. taking parabolic points corresponding to edge groups of $G(\mathscr{Y})$ to parabolic points corresponding to edge groups of $G'(\mathscr{Y})$.

Remark 3.5.2. In the above theorem, it is not possible that for $G(\mathscr{Y})$ the edge groups are maximal parabolic, and for $G'(\mathscr{Y})$ the edge groups are parabolic (Not maximal parabolic). Thus in both graphs of groups either edge groups are maximal parabolic or edge groups are parabolic. The example below justifies this situation.

Example 3.5.3. Let $\mathscr{Y}$ be an edge and let $F(a,b)$ be a free group of rank 2. Let $G(\mathscr{Y})$ be a double of the free group $F(a,b)$ along $\langle [a,b] \rangle$ and $G'(\mathscr{Y})$ be a double of free group $F(a,b)$ along $\langle [a,b]^2 \rangle$. Thus $\Gamma = F(a,b) *_{\langle [a,b] \rangle \simeq \langle [\bar{a},\bar{b}] \rangle} F(\bar{a},\bar{b})$ and $\Gamma' = F(a,b) *_{\langle [a,b]^2 \rangle \simeq \langle [\bar{a},\bar{b}]^2 \rangle} F(\bar{a},\bar{b})$. Since $\langle [a,b] \rangle$ is a maximal cyclic subgroup of $F(a,b)$, $\langle [a,b] \rangle$ is a malnormal quasiconvex subgroup of $F(a,b)$. Thus, $F(a,b)$ is relatively hyperbolic with respect to $\langle [a,b] \rangle$. By Theorem 3.0.3, Γ and Γ' are relatively hyperbolic with respect to $\langle [a,b] \rangle$ and $\langle [a,b] \rangle *_{\langle [a,b]^2 \rangle \simeq \langle [\bar{a},\bar{b}]^2 \rangle} \langle [\bar{a},\bar{b}] \rangle$ respectively. Here, we just take the identity map between Bowditch boundaries of vertex groups. From the construction of Bowditch boundaries, it is clear that if we remove parabolic points from Bowditch boundaries $\partial\Gamma, \partial\Gamma'$ respectively, we have two connected components, infinitely many connected components, respectively. Therefore there is no homeomorphism from $\partial\Gamma$ to $\partial\Gamma'$ preserving edge parabolic points.

Also, the above theorem does not deal with the case when parabolic edge groups have different finite indexes in corresponding maximal parabolic subgroups. Here, we give a specific example in this direction.

Example 3.5.4. Consider the two groups $\Gamma = F(a,b) *_{\langle [a,b] \rangle \simeq \langle [\bar{a},\bar{b}] \rangle} F(\bar{a},\bar{b})$ and $\Gamma' = F(a,b) *_{\langle [a,b] \rangle \simeq \langle [a,b]^2 \rangle} F(\bar{a},\bar{b})$. Since $\langle [a,b] \rangle$ is a maximal cyclic subgroup of

$F(a,b)$, $\langle [a,b] \rangle$ is a malnormal quasiconvex subgroup of $F(a,b)$. Thus, $F(a,b)$ is relatively hyperbolic with respect to $\langle [a,b] \rangle$. Both the groups Γ, Γ' are relatively hyperbolic with Bowditch boundary $\partial\Gamma$, $\partial\Gamma'$ respectively. Again, from the construction of Bowditch boundary, removing a parabolic point from $\partial\Gamma$ gives two connected components but removing a parabolic point from $\partial\Gamma'$ gives three connected components. Thus there is no homeomorphism from $\partial\Gamma$ to $\partial\Gamma'$ preserving edge parabolic points. Similarly, if we take $\Gamma = F(a,b) *_{\langle [a,b] \rangle \simeq \langle [a,b]^3 \rangle} F(\bar{a},\bar{b})$ and $\Gamma' = F(a,b) *_{\langle [a,b] \rangle \simeq \langle [a,b]^2 \rangle} F(\bar{a},\bar{b})$ then there is no homeomorphism from $\partial\Gamma$ to $\partial\Gamma'$ preserving edge parabolic points.

To prove Theorem 3.5.1, it is sufficient to consider amalgam and HNN extension case.

Proof of Theorem 3.5.1 in the amalgam case:

Let $\mathscr{Y}$ be an edge. Then $G(\mathscr{Y})$ and $G'(\mathscr{Y})$ are amalgams of two relatively hyperbolic groups with parabolic edge groups. Let $\Gamma = \pi_1(G(\mathscr{Y}))$ and let $\Gamma' = \pi_1(G'(\mathscr{Y}))$. By Theorem 3.0.3, both Γ, Γ' are relatively hyperbolic groups. Let T, T' be the Bass-Serre trees for $G(\mathscr{Y}), G'(\mathscr{Y})$, respectively. For each edge $e \in T$ and $e' \in T'$, let P_e, P'_e be parabolic edge groups in Γ, Γ' respectively. Also, in adjacent vertices of e and e', let P_v, P_w and let P'_v, P'_w be maximal parabolic subgroups corresponding to P_e, P'_e respectively. Let $\partial\Gamma, \partial\Gamma'$ denotes Bowditch boundaries of Γ, Γ' respectively. Keeping the construction of Bowditch boundaries in mind, we define a map f from $\partial\Gamma$ to $\partial\Gamma'$.

Definition of f: Let e, e' be two edges of T, T' with vertices v, w and v', w' respectively. Suppose we have homeomorphisms $\partial G_v \to \partial G'_v$ and $\partial G_w \to \partial G'_w$ as in Theorem 3.5.1(3). By definition of these homeomorphisms, we have bijections between cosets of P_v in G_v and cosets of P'_v in G'_v. Also, there is a bijection between cosets of P_e in P_v and cosets of P'_e in P'_v. Combining these two, we get a bijection between cosets of P_e in G_v and cosets of P'_e in G'_v. Similarly, we have a bijection between cosets of P_e in G_w and cosets of P'_e in G'_w. By following this process inductively, we have an isomorphism ϕ from T to T'. Let $\xi \in \partial G_v$ for some vertex $v \in V(T)$. Define $f(\xi) := f_v(\xi)$, where f_v is a homeomorphism from ∂G_v to $\partial G_{\phi(v)}$. Note that if ξ is a parabolic point in ∂G_v and let $D(\xi)$ be its domain then $\phi|_{D(\xi)} = D(f(\xi))$. Since ϕ is an isomorphism, we have a homeomorphism $\partial\phi$ from ∂T to $\partial T'$. Observe that if some point of ∂T is identified with some parabolic point, then its image under $\partial\phi$ is also identified with some parabolic point. If $\eta \in \partial T$ such that it is not identified with some edge parabolic point, define $f(\eta) := \phi(\eta)$. Thus, we have a map f from $\partial\Gamma$ to $\partial\Gamma'$.

f **is a homeomorphism:** Clearly, f is a bijection. To prove that f is a homeomorphism, it is sufficient to prove that f is continuous as Bowditch boundaries $\partial\Gamma, \partial\Gamma'$ are compact Hausdorff. Let $\xi \in \partial G_v$ for some $v \in V(T)$ and let U be a neighborhood of $f(\xi)$ in $\partial\Gamma'$. Note that $\phi(D(\xi)) = D(f(\xi))$. For each vertex $u \in D(\xi)$, we can choose a neighborhood V_u such that $f_u(V_u) \subset U_{\phi(u)}$ as f_u is a homeomorphism, where $U_{\phi(u)}$ is a neighborhood around $f(\xi)$ in $\partial G_{\phi(u)}$. Now, it is clear from the definition of neighborhoods in $\partial\Gamma$ and definition of maps f, ϕ that we can find a neighborhood V of ξ in $\partial\Gamma$ such that $f(V) \subset U$. Now, let $\eta \in \partial T$ such that it is not identified with a parabolic point. Let U be a neighborhood of $f(\eta)$ in $\partial\Gamma'$. From the construction of map ϕ, it is clear that ϕ takes subtree $W_m(\eta)$ (see section 3.1) onto the subtree $W_m(f(\eta))$. Then again, by definition of neighborhoods, we can find a neighborhood V of η in $\partial\Gamma$ such that $f(V) \subset U$. The map f is continuous, and hence f is a homeomorphism. □

Proof of Theorem 3.5.1 in HNN extension case: There are the following two cases:

Case 1. Let $\Gamma = G*_{P_1 \simeq P_2}, \Gamma' = G'*_{P_1' \simeq P_2'}$, where $(G, \mathbb{P}), (G', \mathbb{P}')$ are relatively hyperbolic groups. Assume that P_1, P_2, and P_1', P_2' are both sitting inside the same maximal parabolic subgroups in G, G', respectively. Let T, T' be the Bass-Serre trees of Γ, Γ' respectively. Also, we have a homeomorphism between Bowditch boundaries $\partial G, \partial G'$ satisfying (3) in Theorem 3.5.1. We get an isomorphism ϕ from T to T' in a similar manner as in the amalgam case. Also, we can define a map from $\partial\Gamma$ to $\partial\Gamma'$ in the same way as we define in the amalgam case. Note that f is a bijection. To prove that f is a homeomorphism, it is sufficient to prove that f is continuous as $\partial\Gamma, \partial\Gamma'$ are compact Hausdorff. Again continuity is clear from the definition of map f and definition of neighborhoods in $\partial\Gamma, \partial\Gamma'$ respectively.

Case 2. Let $\Gamma = G*_{P_1 \simeq P_2}, \Gamma' = G'*_{P_1' \simeq P_2'}$, where $(G, \mathbb{P}), (G', \mathbb{P}')$ are relatively hyperbolic groups. In this case, P_1, P_2, and P_1', P_2' are sitting inside in different (not conjugate) maximal parabolic subgroups in G, G', respectively. Now, we can write $\Gamma = (G *_{P_1} P_2)*_{P_2}$ and $\Gamma' = (G' *_{P_1'} P_2')*_{P_2'}$, respectively. By applying amalgam and case 1 of the HNN extension respectively, we get the desired homeomorphism from $\partial\Gamma$ to $\partial\Gamma'$. □

3.6 Applications and examples

3.6.1 Example of a subgroup of a relatively hyperbolic group with exotic limit set

In this subsection, we closely follow the construction of space given in Section 3.1 and give an example of a relatively hyperbolic group having a non-relatively quasiconvex subgroup whose limit set is not equal to the limit of any relatively quasiconvex subgroup. This is motivated by the work of I.Kapovich [39], where he gave such an example in the setting of hyperbolic group.

Consider the torus with one puncture and let ψ be a pseudo-Anosov homeomorphism fixing the puncture. Suppose M_ψ is the mapping torus for the homeomorphism ψ. Let G, F be the fundamental groups of M_ψ, puncture torus respectively. Then it is well known that G is relatively hyperbolic with respect to a subgroup isomorphic to $Z \oplus Z$, and subgroup F is not relatively quasiconvex in G. Let $\Gamma = G *_{\langle z \rangle} \overline{G}$ and let $H = F *_{\langle z \rangle} \overline{F}$, where $z \in F$ is primitive, be doubles of G and H respectively along cyclic subgroup $\langle z \rangle$. The groups Γ, H are relatively hyperbolic, by Theorem 3.0.4. We have the following:

Lemma 3.6.1. *H is not a relatively quasiconvex subgroup of Γ.*

Proof. Suppose H is relatively quasiconvex in Γ. Since F is relatively quasiconvex in H and H is relatively quasiconvex in Γ, F is relatively quasiconvex in Γ by [7, Lemma 2.3]. Since F is a normal subgroup of G, the limit set of F in G is the same as the Bowditch boundary of G that is homeomorphic to the limit set of G in Γ. Also the limit set of F in Γ is the same as the Bowditch boundary of G. Hence F acts geometrically finitely on its limit in G. Thus, F is relatively quasiconvex in G which is a contradiction. □

As we observe in the proof of Theorem 3.0.4, the edge group in Γ is parabolic, or we change the parabolic structure in vertex group G so that edge group become parabolic. Therefore, following Section 3.1, we can give a construction of Bowditch boundary of relatively hyperbolic group Γ. Now, we prove the following:

Lemma 3.6.2. *$Stab_\Gamma(\Lambda(H)) = H$, i.e. H is maximal in its limit set.*

Proof. Let T, T' be the Bass-Serre trees of the groups Γ, H respectively. Note that the tree T' embeds in T, and each vertex group in H is normal in the corresponding vertex group of Γ. Also, there is topological embedding between Gromov boundaries of T' and T. Let M be Bowditch boundary of Γ. Here we explicitly know the construction of M (see Section 3.1). Let p be a map from $M \to T \cup \partial T$ defined as follows: for $\xi \in \partial G_v$, define $p(\xi) = v$ and for $\eta \in \partial T$, define $p(\eta) = \eta$. Consider N, a subset of M, the inverse image of $T' \cup \partial T'$ under the map p. It is clear from the definition of topology on M that N is the minimal closed H-invariant set. Thus $\Lambda(H) = N$. Now, the lemma follows immediately from the construction of Bowditch boundary M. □

Now, we prove that the limit set of the subgroup H is exotic, i.e. there is no relatively quasiconvex subgroup of Γ whose limit set is equal to the limit set of H. Recall that a subgroup of a relatively hyperbolic group is dynamically quasiconvex if and only if it is relatively quasiconvex (see [29]).

Lemma 3.6.3. *There is no relatively quasiconvex subgroup of Γ whose limit set is equal to the limit set of H.*

Proof. If possible, there is a relatively quasiconvex subgroup Q of Γ such that $\Lambda(Q) = \Lambda(H)$. Since $Stab_\Gamma(\Lambda(H)) = Stab_\Gamma(\Lambda(Q)) = H$, we see that $Q \subset H$. Thus, by Lemma 2.3.13, we see that H is a dynamically quasiconvex subgroup of Γ. Therefore H is relatively quasiconvex too and hence we get a contradiction as H is not relatively quasiconvex. □

Remark 3.6.4. In [18], Dahmani gave a construction of Bowditch boundary for the fundamental group of an acylindrical graph of relatively hyperbolic groups with fully quasiconvex edge groups. In particular, we can construct Gromov boundary of the fundamental group of an acylindrical graph of hyperbolic groups with quasiconvex edge groups. Let $S_g, g \geq 2$ be a closed orientable surface of genus g and let ϕ be a pseudo-Anosov homeomorphism of S_g. Let M_ϕ be the mapping torus corresponding to ϕ and let G be the fundamental group of M_ϕ. Then, it is well known that G is a hyperbolic group and $F = \pi_1(S_g)$ is a non-quasiconvex subgroup of G. Let $z \in F$ be such that z is not a proper power in F and hence it is not a proper power in G. Consider the double Γ of group G along $\langle z \rangle$, i.e. $\Gamma = G *_{\langle z \rangle} \overline{G}$. Note that the group Γ is hyperbolic by [6] and the subgroups $G, \overline{G}$ of Γ are quasiconvex. Consider the group $H = F *_{\langle z \rangle} \overline{F}$. Again, by [6], H is hyperbolic. Now, using the construction of Gromov boundary of Γ from [18], we can explicitly construct the limit set of subgroup H (as we did in Lemma 3.6.2). Then, we have $Stab_\Gamma \Lambda(H) = H$. Then, using the same idea as above, H is not quasiconvex, and there is no quasiconvex subgroup of Γ whose limit set equal $\Lambda(H)$. Thus, we have a different proof of I.Kapovich's result from [39].

3.6.2 Example of a family of non-convergence groups

The class of convergence groups contains the class of hyperbolic and relatively hyperbolic groups. So, it is natural to ask whether there is a non-elementary convergence group which is not relatively hyperbolic. In general constructing non-convergence groups is not easy. However, in this subsection, we give an example of a family of groups that do not act on a compact metrizable space as a non-elementary convergence group.

Proposition 3.6.5. *Let G be a torsion-free group and let H be a subgroup of G satisfying the following:*

1. *H is malnormally closed in G, i.e. there is no proper subgroup of G containing H, which is malnormal in G*
2. *$[Comm_G(H) : H] > 1$, where $Comm_G(H)$ denotes the commensurator of H in G.*

Consider the double Γ of the group G along H, i.e. $\Gamma = G *_{H\simeq\overline{H}} \overline{G}$. Then, Γ does not act on a compact metrizable space as a non-elementary convergence group.

First of all, we give an example satisfying the hypotheses of the above theorem.

Example 3.6.6. Let $G = F(a,b)$ be a free group of rank 2 and let $H = \langle a, b^2, bab^{-1}\rangle$ be a subgroup of G. Note that H is a normal subgroup of G as $[G : H] = 2$. Since $b^2 \in bHb^{-1} \cap H$, b lies in any malnormal subgroup of G containing H. Thus, H is malnormally closed in G. As H is a normal subgroup of G, $Comm_G(H) = G$ and therefore $[Comm_G(H) : H] > 1$. Now, consider the double $\Gamma = G *_{H\simeq\overline{H}} \overline{G}$. By the above proposition, we see that Γ does not act on a compact metrizable space as a non-elementary convergence group.

Now, we collect some basic facts about subgroups of a convergence group.

Lemma 3.6.7. *Suppose a group G acts on a compact metrizable space as a convergence group. Then any subgroup P of G, which is isomorphic to $Z \oplus Z$, is parabolic.*

Proof. Since P is Abelian, $|\Lambda(P)| \leq 2$. Let if possible $|\Lambda(P)| = 2$. Then P contains a loxodromic element p (say). The fixed point set of p, $\mathrm{Fix}(p) = \Lambda(P)$ and $Stab_G(\Lambda(P))$ contains P. By [74, Theorem 2I], $\langle p\rangle$ has finite index in $Stab_G(\Lambda(P))$. In particular, $\langle p\rangle$ has finite index in P, which is impossible. Hence $|\Lambda(P)| = 1$ and P is a parabolic subgroup. □

Now, we observe the following:

Lemma 3.6.8. *Let G be a group that acts on a compact metrizable space as a convergence group. Then maximal parabolic subgroups are weakly malnormal.*

Proof. Let P be a maximal parabolic subgroup with $\Lambda(P) = \{p\}$. Suppose for some $g \in G$, $|P \cap gPg^{-1}| = \infty$. As $P \cap gPg^{-1} \subset P$, $\Lambda(P \cap gPg^{-1}) = \Lambda(P) = \{p\}$. Similarly, $\Lambda(P \cap gPg^{-1}) = \Lambda(gPg^{-1}) = g\Lambda(P)$. Hence g fixes the parabolic point p and therefore $g \in P$. □

Note that if G is torsion-free in the above lemma, then maximal parabolic subgroups are malnormal. Now, we obtain the following:

Lemma 3.6.9. *Let Γ be the double of G along H as above. If $H \cap gHg^{-1} \neq \{1\}$ then $H \cap gHg^{-1}$ is a subgroup of a parabolic subgroup.*

Proof. Let $w \in H \cap gHg^{-1}$. Then $w = ghg^{-1}$ for some $h \in H$. This implies $h = g^{-1}wg$. Since Γ is double of group G along H, $g^{-1}wg = \bar{g}^{-1}w\bar{g}$. Thus $\bar{g}g^{-1}$ commute with w. As Γ is torsion-free, $\langle \bar{g}g^{-1}, w\rangle \simeq Z \oplus Z$. By the above lemma, $\langle \bar{g}g^{-1}, w\rangle$ is a parabolic subgroup. Since w is an arbitrary element of $H \cap gHg^{-1}$, $H \cap gHg^{-1}$ is a subgroup of a parabolic subgroup. □

Proof of Proposition 3.6.5. Since $[Comm_G(H) : H] > 1$, let $g \in Comm_G(H)$ such that $g \notin H$. By Lemma 3.6.9, $H \cap gHg^{-1}$ is a subgroup of a parabolic subgroup of Γ. Since $[H : H \cap gHg^{-1}] < \infty$, H and $H \cap gHg^{-1}$ sit inside the same maximal parabolic subgroup P (say). Now, by Lemma 3.6.8, the subgroup P is malnormal in Γ. As H is malnormally closed in G, $G \subset P$. Similarly, we can show that $\overline{G} \subset P$. Hence $\Gamma \subset P$ and therefore Γ is not a non-elementary convergence group. □

Note: Let H, K be two subgroups of G such that $H \subset K \subset G$ and $1 < [K : H] < \infty$. Then one can check that $K \subset Comm_G(H)$. Also, assume that H is malnormally closed in G. Then the double $\Gamma = G *_{H \simeq \overline{H}} \overline{G}$ is not a non-elementary convergence group.

Chapter 4

Boundaries of coned-off spaces and Cannon-Thurston maps

This chapter aims to study the geometry of electric spaces and deduce the existence of CT maps for certain subcomplexes of groups of a complex of hyperbolic groups.

Following is our setup for Theorem 4.0.1.

1. Suppose X is a δ-hyperbolic geodesic metric space and $\{A_i\}$ is a locally finite collection of uniformly quasiconvex sets in X.
2. Suppose $Y \subset X$ is a subspace such that with respect to the induced length metric from X, the inclusion $Y \to X$ is a proper embedding.
3. Suppose that Y is also a δ-hyperbolic metric space.
4. Lastly, suppose that there is a collection of subsets $\{B_j\}$ in Y such that each B_j is contained in $A_i \cap Y$ for some i and each B_j is uniformly quasiconvex in X as well as in Y.
5. Let $\hat{X}$ be the space obtained from X by coning the sets A_i and let $\hat{Y}$ be the space obtained from Y by coning the sets $\{B_j\}$.

We note that the coned-off spaces $\hat{Y}$ and $\hat{X}$ are (uniformly) hyperbolic by Proposition 2.5.4. The following is the main theorem in this chapter.

Theorem 4.0.1. *Suppose the inclusion $\hat{Y} \to \hat{X}$ satisfies Mitra's criterion. Then the inclusion $Y \to X$ admits a CT map. Moreover, the CT map $\partial Y \to \partial X$ is injective if and only if the CT map $\partial \hat{Y} \to \partial \hat{X}$ is injective.*

In the setting of complexes of groups, we prove the following.

Theorem 4.0.2. *Let $(\mathscr{G},Y)$ be a developable complex of groups over a finite simplicial complex Y. Let Y_1 be a connected subcomplex of Y and let $(\mathscr{G},Y_1)$ be the subcomplex of groups obtained by restricting $(\mathscr{G},Y)$ to Y_1. Suppose the following conditions hold.*

1. *The natural homomorphism $G_1 = \pi_1(\mathscr{G},Y_1) \to G = \pi_1(\mathscr{G},Y)$ is injective.*
2. *Both G_1, G are hyperbolic, and all the local groups of $(\mathscr{G},Y)$ are quasiconvex in G.*
3. *The natural map $B_1 \to B$ satisfies Mitra's criterion, where B_1, B are the universal covers of $(\mathscr{G},Y_1)$ and $(\mathscr{G},Y)$, respectively.*

Then there exists a Cannon-Thurston map for the inclusion $G_1 \to G$. Moreover, G_1 is quasiconvex in G if and only if the Cannon-Thurston map for $B_1 \to B$ is injective.

Layout of the chapter: In Section 4.1, we give a proof of Theorem 4.0.1. There we give a set-theoretic description of the Gromov boundary of X too. In Section 4.2, as an application of Theorem 4.0.1, we prove Theorem 4.0.2. Section 4.3 is devoted to complexes of groups with finite edge groups. Some further applications and examples are discussed in Section 4.4.

4.1 Electric geometry and Cannon-Thurston maps

Throughout the section, X, Y are geodesic metric spaces satisfying the hypotheses of Theorem 4.0.1. This section is devoted to a proof of Theorem 4.0.1. Our starting point is to give a description, as a set, of the Gromov boundary of X, see Theorem 4.1.9. Then, using this there is a natural map from $\bar{Y}$ to $\bar{X}$ and we show that this map is continuous.

We start here by proving some basic lemmas about geodesics in $\hat{X}$.

Lemma 4.1.1. *Let γ is a geodesic of finite length in $\hat{X}$ and let $A_1, A_2, ..., A_k$ be the quasiconvex subsets penetrated by γ. Then $(\gamma \cap X) \cup (\bigcup_{i=1}^{k} A_i)$ is a uniform quasiconvex subset of X.*

Proof. Since γ is of finite length, say l, γ penetrates at most l quasiconvex subsets. Hence, $k \le l$. Thus, $\tilde{\gamma} = (\gamma \cap X) \cup (\bigcup_{i=1}^{k} A_i)$ is the concatenation of finitely many geodesic segment of γ in X and at most l uniform quasiconvex subsets in X. Then it is straight forward to check that $\tilde{\gamma}$ is a quasiconvex subset of X, see Lemma 2.2.13. □

Lemma 4.1.2. *Suppose $x,y \in X$. There is a uniform dotted quasigeodesic γ in X and a uniform quasigeodesic β in $\hat{X}$ both joining x,y such that $Hd_{\hat{X}}(\gamma,\beta)$ is uniformly bounded.*

Proof. Let $[x,y]$ be any geodesic in X joining x,y. Then, by Proposition 2.5.5, $[x,y]$ is a uniformly quasiconvex subset of $\hat{X}$. Hence, by Lemma 2.2.12 there is a uniform dotted quasigeodesic, say

$$\alpha : x_0 = x, x_1, \dots, x_n = y,$$

in $\hat{X}$ joining x,y contained in $[x,y]$. Now $d_{\hat{X}}(x_i,x_{i+1})$ being uniformly small, if α_i is a geodesic in $\hat{X}$ joining x_i,x_{i+1} then, by Lemma 4.1.1, there is a uniformly quasiconvex subset, call it α_i^{de}, in X containing x_i,x_{i+1}. Hence, there is a uniform dotted quasigeodesic in X, say

$$\gamma_i : x_i = y_{i0}, y_{i1}, \dots, y_{im_i},$$

joining x_i,x_{i+1} contained in α_i^{de}. Since, x_i's were on a geodesic it follows that the concatenation of the various γ_i's is a uniform dotted quasigeodesic in X. Similarly, the concatenation of the α_i's is a uniform quasigeodesic in $\hat{X}$. Note that $Hd_{\hat{X}}(\gamma_i,\alpha_i)$ is uniformly bounded. Hence, defining γ to be the concatenation of γ_i's and β to be the concatenation of the α_i's finishes the proof. □

We then immediately have the following which is also proved by Kapovich and Rafi [42, Corollary 2.4].

Corollary 4.1.3. *There exists $D_0 \geq 0$ such that the following holds:*

Suppose $x,y \in X$ and γ is a geodesic segment of X joining x,y. Let β be a geodesic segment of $\hat{X}$ joining x,y. Then $Hd_{\hat{X}}(\gamma,\beta) \leq D_0$.

Proof. There exists a uniform dotted quasigeodesic γ_1 in X and a uniform quasigeodesic β_1 in $\hat{X}$, both joining x,y such that $Hd_{\hat{X}}(\gamma_1,\beta_1)$ is uniformly bounded, according to Lemma 4.1.2. Now, using the stability of quasigeodesics in X and $\hat{X}$, there exists $D_0 \geq 0$ such that $Hd_{\hat{X}}(\gamma,\beta) \leq D_0$. □

Corollary 4.1.4. *For all $D > 0$ there is $D' > 0$ such that $D' \to \infty$ as $D \to \infty$ and the following holds.*
Suppose $x_0,x,y \in X$ and $d_{\hat{X}}(x_0,[x,y]_{\hat{X}}) \geq D$. Then $d_X(x_0,[x,y]_X) \geq D'$.

Proof. Proof is immediate from the previous corollary. □

4.1.1 ∂X vs $\partial \hat{X}$

In this subsection, we give a description of ∂X in terms of $\Lambda(A_i)$ and $\partial \hat{X}$. Such a description also appears in [1]. There, the authors additionally required "acylindrical action of a group along a subgroups" on X. However, we just need a mild assumption (see Remark 1.2.1) on X which is always true for groups.

The following is a consequence of Lemma 4.1.3. However, we provide a sketch of proof for completeness.

Proposition 4.1.5. *Suppose γ is a geodesic ray in X such that its image in $\hat{X}$ is unbounded. Then it is uniformly close to a uniform quasigeodesic ray of $\hat{X}$.*

Proof. Let γ be a geodesic ray in X such that its image in $\hat{X}$ is unbounded. Let $\gamma(0) = x$ and let α be a geodesic in $\hat{X}$ joining x to any point of γ, say $\gamma(t)$ for some $t \in [0,\infty)$. Then, according to Lemma 4.1.3, $Hd_{\hat{X}}(\gamma|_{[0,t]}, \alpha) \leq D_0$. Since image of γ is unbounded in $\hat{X}$, choose a sequence $\{x_n\}_n$ of points on γ which are sufficiently far away from x such that if α_n is a geodesic in $\hat{X}$ joining x and x_n then the projection y_i of x_i on α_n for $1 \leq i \leq (n-1)$ are such that y_{i+1} comes after y_i. Note that the Hausdorff distance in $\hat{X}$ between α_n and portion of γ joining x and x_n is bounded above by D_0. Now, join the consecutive points x_i and x_{i+1} by a geodesic in $\hat{X}$ and take the concatenation of all such geodesics, call it β. Then it is easy to verify that β is a uniform quasigeodesic ray in $\hat{X}$. Again by invoking Lemma 4.1.3, we see that β is uniformly close to γ. □

Remark 4.1.6. From the proof of the above corollary, we see that if the image of a geodesic ray in X is unbounded in $\hat{X}$ then it has a unique limit point in $\partial \hat{X}$ (see also [1, Corollary 6.4]).

Proposition 4.1.7. *Suppose γ is a geodesic ray in X such that its image in $\hat{X}$ is a bounded set. Then, $\gamma(\infty) \in \Lambda(A_i)$ for some i.*

Proof. Let $D_0 = Diam_{\hat{X}}(\gamma)$ for some $D_0 > 0$. Let $\gamma(0) = x_0$ and let α_n be a geodesic in $\hat{X}$ joining x_0 and $\gamma(n)$.

Step 1. Since $l(\alpha_n) \leq D_0$, each α_n penetrates at most D_0 quasiconvex subsets. For each n, let $A_1^n, A_2^n, ..., A_k^n$ be the quasiconvex subsets penetrated by α_n and $k \leq D_0$. Note that $B(x_0, D_0) \cap A_1^n \neq \phi$ for all n. But, by local finiteness of A_i's, there can be only finitely many such quasiconvex subsets. Thus, up to passing to a subsequence, we can assume that the first penetrated quasiconvex subset by each α_n is fixed, say A_1.

Step 2. If diameter of nearest point projection of γ on A_1 is infinite then $\gamma(\infty) \in \Lambda(A_1)$ by Lemma 2.2.31 and therefore we are done. Thus, we assume that the diameter of the nearest point projection of γ on A_1 is finite. By Lemma 4.1.1, for each n, $(\alpha_n \cap X) \cup (\bigcup_{i=1}^{k} A_i^n)$ is a uniform K-quasiconvex subset of X for some $K \geq 0$. Let x_n be the entry of point of α_n in A_1 and denote the portion of α_n from x_0 to x_n by $[x_0, x_n]$. Let $\gamma_{|_{[0,t_n]}}$, for some $t_n > 0$, be the maximal subsegment of γ which is K-close to $A_1 \cup [x_0, x_n]$. Then $\gamma(t_n + 1)$ is $(K + D_0)$-close to a point of A_i^n for some $1 < i \leq D_0$, say a_i^n. By our choice of t_n, there exists $y_n \in A_1 \cup [x_0, x_n]$ such that $d_X(\gamma(t_n), y_n) \leq K$. Suppose $y_n \in A_1$. Note that, for all n, $d_X(x_n, y_n) \leq K_1$ for some $K_1 \geq 0$. Choose a X-geodesic of length at most K_1 connecting x_n and y_n, call it β_n. Take a X-geodesic of length at most K joining y_n and $\gamma(t_n)$, call it α_n'. Now take the concatenation of $[x_0, x_n], \beta_n, \alpha_n', \gamma_{|_{[t_n, t_n+1]}}$ and X-geodesic of length at most $(K + D_0)$ joining $\gamma(t_n + 1), a_i^n$. If $y_n \in [x_0, x_n]$ then we just take concatenation of $[x_0, y_n] \subset [x_0, x_n]$, X-geodesic of length at most K joining $\gamma(t_n)$ and y_n, $\gamma_{|_{[t_n, t_n+1]}}$, and X-geodesic of length at most $(K + D_0)$ joining $\gamma(t_n + 1)$ to a_i^n. In either case, let us denote such concatenation by γ_n. Note that the X-length of each γ_n is bounded by some fixed constant. Thus all the quasiconvex subsets in which a_i^n lies belong to a finite radius X-ball centered at x_0. Again by local finiteness of A_i's, up to extracting a subsequence, all a_i^n lie in a fixed quasiconvex subset, say A_2, and each α_n penetrate A_2.

Step 3. Again, if the diameter of the nearest point projection of γ on A_2 is infinite then we are done by Lemma 2.2.31. Otherwise, repeat Step 2 for the union of $\gamma_n \cup A_2$'s and portions of α_n's after A_2. Since each α_n penetrates at most D_0 quasiconvex subsets, we can repeat the above procedure finitely many times. Finally, up to extracting finitely many subsequences, we see that each α_n penetrates a quasiconvex subset, call it A_j and after that α_n's does not penetrate any quasiconvex subset. Hence, for some $D \geq 0$, $\gamma(n)$ lies in the D-neighborhood of A_j for all large n. Therefore $\gamma(\infty) \in \Lambda(A_j)$. □

Notation. We let $\partial_h X$ the set of points $\xi \in \partial X$ such that there is a (quasi)geodesic ray γ in X with $\gamma(\infty) = \xi$ and such that γ is unbounded in $\hat{X}$. Also we denote $\cup_{i \in I} \Lambda(A_i)$ by $\partial_v X$. Intuitively we think of the rays converging to points of $\partial_h X$ as *horizontal* ones relative to the map $X \to \hat{X}$ and the those to converging to points of $\partial_v X$ as *vertical.*

However, Proposition 4.1.5 and Corollary 4.1.4 immediately imply the following result which was proved first in [19, Theorem 3.2]. We include a sketch of proof for the sake of completeness.

Theorem 4.1.8 ([19]). *(1) Suppose x_n is a sequence in X. Then x_n converges to a point of $\partial_h X$ if and only if x_n converges to a point of $\partial\hat{X}$. Consequently we have a map $\phi_X : \partial_h X \to \partial\hat{X}$.*

(2) The map ϕ_X is a homeomorphism.

Sketch of proof: (1) Suppose $x_n \to \xi \in \partial_h X$. Let γ be a quasigeodesic ray in X with $\gamma(\infty) = \xi$ and $\gamma(0) = x_1$. Then, by Lemma 2.2.28, $[x_1, x_n]_X$ fellow travels γ in X for longer and longer time as $n \to \infty$. It then follows from Corollary 4.1.3 that $[x_1, x_n]_{\hat{X}}$ fellow travels γ in $\hat{X}$ for longer and longer time as $n \to \infty$. By Lemma 4.1.5 γ is within a finite Hausdorff distance of a quasigeodesic ray, say β, in $\hat{X}$. It follows that $x_n \to \beta(\infty)$ in $\hat{X}$.

Conversely, suppose that $x_n \to \xi \in \partial\hat{X}$. Then, $\{x_n\}_n$ is an unbounded sequence in X. Fix $x_0 \in X$. For $n \neq m$, let $\alpha_{n,m}$ be a quasigeodesic in $\hat{X}$ joining x_n, x_m. Then $d_{\hat{X}}(x_0, \alpha_{n,m}) \to \infty$ as $n, m \to \infty$ by Lemma 2.2.25(1). By invoking Corollary 4.1.4, $d_X(x_0, [x_n, x_m]_X) \to \infty$ as $n, m \to \infty$. Hence, $\{x_n\}$ is converging to a point of ∂X, say η. Let γ be a quasigeodesic joining x_0 to η in X. Note that γ is unbounded in $\hat{X}$. Hence, $\eta \in \partial_h X$ and $x_n \to \eta$ as $n \to \infty$.

(2) The proof of this is exactly similar to the proof of (1) and uses Corollary 4.1.4 and Lemma 4.1.5. □

From Theorem 4.1.8, we can realize $\partial_h X$ as a subset of ∂X. Finally, Proposition 4.1.7 and Lemma 4.1.5 immediately give the following:

Theorem 4.1.9. *$\partial X = \partial_h X \cup \partial_v X$.*

Proof. Clearly, $\partial_h X \cup \partial_v X \subset \partial X$. Suppose γ is a quasigeodesic ray in X and $i : X \to \hat{X}$ is the natural inclusion. If $i(\gamma)$ is bounded in $\hat{X}$ then $\gamma(\infty) \in \partial_v X$ by Proposition 4.1.7. If $i(\gamma)$ is unbounded in $\hat{X}$ then $\gamma(\infty) \in \partial_h X$ by Proposition 4.1.5. Thus, $\partial X \subset \partial_h X \cup \partial_v X$. Hence the theorem. □

We note that this theorem in the context of groups was proved by Abbott and Manning. See [1, Theorem 6.7, Remark 6.8, and Theorem 1.6]. However, the proof was in the context of a group action with certain properties.

4.1.2 Main Theorem

Suppose X, Y are geodesic metric spaces satisfying the hypotheses of Theorem 4.0.1. We shall assume that $x_0 \in Y$ is a fixed base point once and for all.

Proof of Theorem 4.0.1: Since the inclusion $\hat{Y} \to \hat{X}$ satisfies Mitra's criterion we have a CT map, say $g : \partial\hat{Y} \to \partial\hat{X}$. Now we consider the following map $h : \partial Y \to \partial X$:

$$h(\xi) = \begin{cases} \xi & \text{if } \xi \in \partial_v Y, \\ \phi_X^{-1} \circ g \circ \phi_Y(\xi) & \text{if } \xi \in \partial_h Y \end{cases}$$

We shall show that h is the CT map $\partial Y \to \partial X$ by verifying the hypotheses of Corollary 2.2.37. Suppose $\{x_n\}$ is a sequence in Y and $x_n \to \xi \in \partial Y$ in $\bar{Y}$. We need to verify that $x_n \to h(\xi)$ in $\bar{X}$. The proof is divided into two cases.

Case 1. Suppose $\xi \in \partial_h Y$. Then $x_n \to \phi_Y(\xi)$ in $\hat{Y}$ by Theorem 4.1.8. However the inclusion $\hat{Y} \to \hat{X}$ admits a CT map by hypothesis. Hence, $x_n \to g \circ \phi_Y(\xi)$ in $\hat{X}$. Hence, again by Theorem 4.1.8 we have $x_n \to \phi_X^{-1} \circ g \circ \phi_Y(\xi) = h(\xi)$ in $\bar{X}$.

Case 2. Suppose $\xi \in \Lambda(B_j)$ for some j. Let $x \in B_j$. By Lemma 2.2.32, there is a uniform dotted quasigeodesic ray γ of X contained in B_j which joins x to ξ. Then γ is a uniform dotted quasigeodesic in Y as well by Lemma 2.1.2. There are two subcases to consider:

Subcase 1. Suppose that $\{x_n\}$ is bounded in $\hat{Y}$. Let α_n be geodesic in $\hat{Y}$ joining x to x_n. Let γ_n be the union of penetrated quasiconvex subsets by α_n and the portion of α_n outside quasiconvex subsets penetrated by γ_n. By Corollary 4.1.1, γ_n is a uniformly quasiconvex set in Y as well as in X. Let β_n be the union of γ and γ_n for all $n \in \mathbb{N}$. Then β_n is a uniformly quasiconvex in both X and Y. We note that $\xi \in \Lambda_X(\beta_n)$. Hence, we can choose, by Lemma 2.2.32, a uniform dotted quasigeodesic of X, say $\beta_n' \subset \beta_n$, joining x_n to ξ. Then by Lemma 2.1.2 it is a uniform quasigeodesic in Y as well. Since $x_n \to \xi$ in $\bar{Y}$ we have $d_Y(x_0, \beta_n') \to \infty$ by Lemma 2.2.27(2). Since Y is properly embedded in X, $d_X(x_0, \beta_n') \to \infty$. That in turn implies that $x_n \to \xi$ in $\bar{X}$ again by Lemma 2.2.27(2).

Subcase 2. Suppose that $\{x_n\}$ is unbounded in $\hat{Y}$. Passing to a subsequence, if needed, we may assume that $d_{\hat{Y}}(x_0, x_n) > n$. Now, for all $R \in \mathbb{N}$ and $n \geq R$, let $x_n^R \in Y$ be the farthest point of $[x_0, x_n]_Y$ such that $d_{\hat{Y}}(x_0, x_n^R) = R$. We note that the sequence of geodesics $[x_0, x_n]_Y$ fellow travel γ for longer and longer time as $n \to \infty$ by Lemma 2.2.28. This implies that $d_Y(x_0, [x_n^R, x_n]_Y) \to \infty$ as $n \to \infty$ for all large R since the inclusion map $Y \to \hat{Y}$ is Lipschitz. Hence, $x_n^R \to \xi$ in $\bar{Y}$ for large enough R. By the Subcase 1, for any such R we have $x_n^R \to \xi$ in $\bar{X}$ too.

By choice of the points, x_n^R we see that $d_{\hat{Y}}(x_0, [x_n^R, x_n]_Y) \geq R$. Hence, by Corollary 4.1.3 $d_{\hat{Y}}(x_0, [x_n^R, x_n]_{\hat{Y}}) \geq R_1$ where $|R_1 - R|$ is uniformly small. Since the inclusion $\hat{Y} \to \hat{X}$ satisfies Mitra's criterion, there exists $R_2 \geq 0$ depending on R_1 such that $d_{\hat{X}}(x_0, [x_n^R, x_n]_{\hat{X}}) \geq R_2$. Hence, by Corollary 4.1.4 $d_X(x_0, [x_n^R, x_n]_X) \geq R_3$. We note that $R_3 \to \infty$ as $R \to \infty$. Now since $x_n^R \to \xi$ for all R large enough, by Lemma 2.2.29, one may find an unbounded sequence of integers $\{C_k\}$ and a subsequence $\{n_k\}$ of the sequence of natural numbers such that $x^{C_k}_{n_k} \to \xi$ too. On the other hand, this means $d_X(x_0, [x^{C_k}_{n_k}, x_{n_k}]_X) \to \infty$. It follows that $x_{n_k} \to \xi$ in $\bar{X}$ as $k \to \infty$ by Lemma 2.2.27. Finally, by invoking Corollary 2.2.37, we are done.

Suppose the CT map $h : \partial Y \to \partial X$ is injective. By definition of h, g is nothing but the restriction of h to $\partial \hat{Y}$ and hence g is injective. Conversely, if g is injective then clearly h is an injective CT map. □

Proposition 4.1.10 (Converse to Theorem 4.0.1). *Suppose we have the hypothesis Theorem 4.0.1 and that there is a CT map $\partial i : \partial Y \to \partial X$. Then there is a CT map $\partial i : \partial \hat{Y} \to \partial \hat{X}$ if and only if for any A_i and any $\xi \in \Lambda_X(A_i)$, $(\partial i)^{-1}(\xi) = \emptyset$ or $\{\xi\}$.*

Proof. Suppose the CT map $f : \partial \hat{Y} \to \partial \hat{X}$ exists. Then for any $\eta \in \partial \hat{Y} \subset \partial Y$ and any sequence $y_n \to \eta$ in Y, we have $y_n \to \eta$ in $\hat{Y}$ and hence $y_n \to f(\eta)$ in $\hat{X}$. That further implies that $y_n \to f(\eta) \in \partial \hat{X} \subset \partial X$ in X. But, $\{\partial i(y_n)\}$ also converges to $\partial i(\eta)$ as ∂i is a CT map. Therefore, $\partial i(\eta) = f(\eta) \in \partial \hat{X}$. Hence, for any A_i, no point of $\Lambda_X(A_i)$ will be in $\partial i(\partial \hat{Y})$ and f is nothing but the restriction of ∂i to $\partial \hat{Y}$. However, the map ∂i restricted to $\partial Y \setminus \partial \hat{Y}$ is clearly injective. Thus for any A_i, and any $\xi \in \Lambda_X(A_i)$, $(\partial i)^{-1}(\xi) = \emptyset$ or $\{\xi\}$. The converse is also similar and hence we skip repeating the proof. □

Example 4.1.11. Consider an exact sequence of hyperbolic groups $1 \to \pi_1(\Sigma) \to G \stackrel{\pi}{\to} \mathbb{Z} \to 1$ where Σ is a closed orientable surface of genus at least 2. Let $g \in G$ such that $\mathbb{Z} =< \pi(g) >$ and let $K =< g >$. Let H be the image of $\pi_1(\Sigma)$ in G. Then for any $x \in G$, $xK \cap H = \emptyset$ or (1). However, the CT map $\partial H \to \partial G$ is surjective. It follows that the CT map $\partial \hat{H} = \partial H \to \partial \hat{G}$ does not exist.

The following is a special case of Theorem 4.0.1 that will be useful in proving Theorem 4.0.2.

Theorem 4.1.12. *Suppose X is a hyperbolic geodesic metric space and let $\{A_i \subset X\}$ be a locally finite collection of uniform quasiconvex subsets. Suppose $Y \subset X$ and with respect to induced length metric from X, the inclusion $Y \to X$ is a proper embedding. Suppose Y is also a hyperbolic geodesic metric space and $\{B_j \subset Y\}$ is a collection of subsets which is uniformly quasiconvex in X as well as in Y. Lastly, assume that $\{B_j\} \subset \{A_i\}$. Let $\hat{Y}$ denote the coned-off space obtained by coning B_i's and let $\hat{X}$ denote the coned-off space obtained by coning A_i's. If $\hat{Y} \to \hat{X}$ satisfy Mitra's criterion then the CT map $\partial Y \to \partial X$ exists. Moreover, the CT map $\partial Y \to \partial X$ is injective if and only if the CT map $\partial \hat{Y} \to \partial \hat{X}$ is injective.*

Proof. We give a sketch of the proof. First of all note that by Theorem 4.1.9, we can write ∂X as $\partial_h X \cup \partial_v X$. Similarly, $\partial Y = \partial_h Y \cup \partial_v Y$. Since $\hat{Y} \to \hat{X}$ satisfies Mitra's criterion, we have a CT map $g : \partial\hat{Y} \to \partial\hat{X}$. Then we define a map $h : \partial Y \to \partial X$ in the following manner:

$$h(\xi) = \begin{cases} \xi & \text{if } \xi \in \Lambda(B_j), \\ \phi_X^{-1} \circ g \circ \phi_Y(\xi) & \text{if } \xi \in \partial_h Y \end{cases}$$

Now it remains to verify Lemma 2.2.36. This follows exactly in the same way as the proof of Theorem 4.0.1. □

Application to relatively hyperbolic spaces: Let X be a geodesic metric space and let $\{A_i\}$ be a collection of uniformly properly embedded subsets of X. Let X^h denote the cusp space [10],[61],[43] with respect to $\{A_i\}$. Then X is said to be hyperbolic relative to $\{A_i\}$ if X^h is a Gromov hyperbolic space. Also, the Gromov boundary of X^h is defined to be the Bowditch boundary of relatively hyperbolic space X. The following lemma is pretty standard, for example see [67, Lemma 1.2.28].

Lemma 4.1.13. *Suppose X is a geodesic metric space and $\{A_i\}$ is a collection of subsets of X. Let $\hat{X^h}$ denote the coned-off space obtained by coning off $\{A_i^h\}$'s and let $\hat{X}$ denote the coned-off space obtained by coning off $\{A_i\}$'s. Then the natural inclusion $\hat{X} \to \hat{X^h}$ is a quasiisometry.*

We also record the following basic lemma about cusp spaces similar to Lemma 2.5.3.

Lemma 4.1.14. *Given $D \geq 0$ there exists $K = K(D)$ such that the following holds:*

Suppose X is a geodesic metric space and $\{A_i\}_{i\in I}, \{B_i\}_{i\in I}$ are two collections of subsets of X. Let X_A^h and X_B^h denote the cusp spaces with respect to A_i's and B_i's, respectively. Let ψ be the extension of the identity map $X \to X$ obtained by sending $A_i \times (0,\infty)$ to $B_i \times (0,\infty)$. If, for all i, $Hd(A_i, B_i) \leq D$ then ψ is a K-quasiisometry.

Lemma 4.1.15. [43, Lemma 9.2] *Let X be a geodesic metric space that is hyperbolic relative to $\{A_i \subset X : i \in I\}$. Then, the hyperbolic cone A_i^h is uniformly qi embedded in X^h for all $i \in I$.*

Theorem 4.1.16. *Let X be a geodesic metric space hyperbolic relative to a locally finite collection $\{A_i \subset X\}$ of uniformly properly embedded subsets. Let $Y \subset X$. Suppose, with respect to the induced length metric, Y is a geodesic metric space and hyperbolic relative to $\{B_j \subset Y\}$ and each B_j is contained in $A_i \cap Y$ for some i. Let $\hat{X}$ denote the coned-off space obtained by coning A_i's and $\hat{Y}$ denote the coned-off space obtained by coning B_j's. Suppose that $\hat{Y} \to \hat{X}$ satisfy Mitra's criterion. Then the CT*

$$\begin{array}{ccc} Y^h & \longrightarrow & X^h \\ \big\downarrow & & \big\downarrow \\ \hat{Y^h} & \longrightarrow & \hat{X^h} \\ {\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle g} \\ \hat{Y} & \longrightarrow & \hat{X} \end{array}$$

Figure 4.1: Horizontal maps are inclusion, f, g are natural quasiisometries

map $\partial Y^h \to \partial X^h$ exists. Moreover, $\partial Y^h \to \partial X^h$ is injective if and only if $\partial \hat{Y} \to \partial \hat{X}$ is injective.

Proof. We verify the hypotheses of Theorem 4.0.1 for the spaces Y^h and X^h. Since Y, X are hyperbolic relative to $\{A_i\}, \{B_j\}$ respectively, Y^h, X^h are proper hyperbolic geodesic metric spaces. By Lemma 4.1.15, $A_i^h \to X^h$ is uniform qi embedding for all $i \in I$. Hence, by Lemma 2.2.10, A_i^h is uniformly quasiconvex in X^h for all i. Similarly, B_j^h's are uniformly quasiconvex in Y^h as well as in X^h. Note that X^h is locally finite with respect to $\{A_i^h\}$. Let $\hat{X^h}$ be the coned-off space obtained by coning A_i^h's and let $\hat{Y^h}$ be coned-off space obtained by coning B_j^h's. Then, by Lemma 4.1.13, $\hat{X^h}, \hat{Y^h}$ are quasiisometric to $\hat{X}, \hat{Y}$ respectively. Since $\hat{Y} \to \hat{X}$ satisfies Mitra's criterion, $\hat{Y^h} \to \hat{X^h}$ also satisfies Mitra's criterion (see figure 4.1). Finally, we are done by invoking Theorem 4.0.1. □

4.1.3 Group theoretical analogue of Theorem 4.0.1

Theorem 4.0.1 has the following immediate group theoretic consequence.

Theorem 4.1.17. *1. Suppose G is a hyperbolic group and $\{K_i : 1 \leq i \leq n\}$ is a set of quasiconvex subgroups of G.*

2. Suppose $H < G$ is hyperbolic subgroup.

3. Suppose $K'_1, K'_2, \ldots, K'_m$ are subgroups of H such that each $K'_j \subset H \cap K_i$ for some i, and is quasiconvex in G.

4. *Let $\hat{G}$ be the coned-off space obtained by coning the cosets of K_i's in G and let $\hat{H}$ denote the coned-off spaces obtained by coning the various cosets of K_i''s in H.*

If the inclusion $\hat{H} \to \hat{G}$ satisfies Mitra's criterion then there is a CT map $\partial H \to \partial G$. Moreover, if the CT map $\partial \hat{H} \to \partial \hat{G}$ is injective then H is quasiconvex in G.

Proof. Note that $\hat{H}$ and $\hat{G}$ are hyperbolic geodesic spaces by Proposition 2.5.4. By Theorem 4.0.1 the CT map $\partial H \to \partial G$ exists and it is injective if and only if so is the CT map $\partial \hat{H} \to \partial \hat{G}$. In the latter case, by Lemma 2.2.38, H is quasiconvex in G. □

In the context of relatively hyperbolic groups, one has the following.

Theorem 4.1.18. *1. Suppose G is a finitely generated group hyperbolic relative to a finite collection of (qi embedded) subgroups $\{K_i\}$.*

2. *Suppose $H < G$ which is hyperbolic relative to $K_1', K_2', \ldots, K_m'$ where for all $1 \leq i \leq m$ there is an element $g_i \in G$ and a subgroup $K_{n_i} \in \{K_i\}$ with $K_i' < g_i K_{n_i} g_i^{-1} \cap H$.*

3. *For all $g \in G$, and K_i, $H \cap gK_ig^{-1}$ is contained in the conjugate in H of some K_j'.*

4. *Let $\hat{G}$ be the coned-off space obtained by coning the cosets of K_i's in G and let $\hat{H}$ denote the coned-off spaces obtained by coning the various cosets of K_i''s in H.*

If the inclusion $\hat{H} \to \hat{G}$ satisfies Mitra's criterion then there is a CT map $\partial H \to \partial G$ in level of the Bowditch boundaries of the groups.

Moreover, if the inclusion $\hat{H} \to \hat{G}$ is a qi embedding then H is relatively quasiconvex in G.

Proof. Since the number of subgroups K_i''s are finite, there exists $D \geq 0$ be such that $K_i' \subset N_D(g_i K_{n_i}) \cap H$ for all $1 \leq i \leq m$. As G is hyperbolic relative to $\{K_i\}$, after attaching hyperbolic cusps to the various cosets of K_i's in G we get a hyperbolic space, call it Z. Let X be a space obtained by attaching hyperbolic cusps to the D-neighborhoods of all the cosets of the subgroups $\{K_i\}$ in G. Since by Lemma 4.1.14 the spaces Z and X are quasiisometric, X is also a hyperbolic space. Let Y be the space obtained from H by attaching hyperbolic cusps to the cosets in H of the subgroups in $\{K_i'\}$. The resulting space, say Y, is also hyperbolic and $Y \subset X$. We note that by coning-off the hyperbolic cusps we get spaces, say $\hat{X}$ and $\hat{Y}$, which are naturally quasiisometric to the spaces obtained from the groups G, H by coning off the cosets of the various subgroups in $\{K_i\}$ and $\{K_i'\}$ respectively (see Lemma 4.1.13). Since the inclusion $\hat{H} \to \hat{G}$ satisfies the Mitra's criterion, the inclusion

$\hat{Y} \to \hat{X}$ satisfies Mitra's criterion. Clearly the hyperbolic cusps of Y are uniformly quasiconvex in X. Hence, all the hypotheses of Theorem 4.0.1 checks out. It follows that CT map exists for the inclusion $Y \to X$.

For the second part, we note that ∂H is equivariantly homeomorphic to the image of the CT map. Hence, H-action on the image of the CT map is geometrically finite whence H is relatively quasiconvex by the results of [37, Section 7]. □

4.2 Application to complexes of groups

In this section, we prove Theorem 4.0.2. Following is the set up for that.

- Suppose $(\mathscr{G}, Y)$ be a developable complex of hyperbolic groups with local maps qi embeddings, $\pi_1(\mathscr{G}, Y) = G$ and development B.
- Suppose G is hyperbolic and all the face groups are quasiconvex in G. Then it follows that B is a hyperbolic metric space (Proposition 2.5.4).
- Suppose Y_1 is a connected subcomplex of Y and $(\mathscr{G}, Y_1)$ is the subcomplex of groups obtained by restricting $(\mathscr{G}, Y)$ to Y_1. Then $(\mathscr{G}, Y_1)$ is a complex of hyperbolic groups with local maps qi embeddings. Let $G_1 = \pi_1(\mathscr{G}, Y_1)$.
- Suppose the natural homomorphism $G_1 \to G$ is injective. Then, by Corollary 2.6.13, $(\mathscr{G}, Y_1)$ is a developable complex of hyperbolic groups. Let B_1 be the development of $(\mathscr{G}, Y_1)$.
- Suppose G_1 is also hyperbolic. Then it follows that all the face groups of $(\mathscr{G}, Y_1)$ are quasiconvex in G_1 and, by Proposition 2.5.4, then B_1 is also a hyperbolic metric space.

Proof of Theorem 4.0.2:

Without loss of generality we may assume the maps $G_1 \to G, B_1 \to B$ to be inclusion and that $G_1 < G$. Now, we have a commutative diagram, see figure 4.2. In the diagram, $\hat{G}_1$ denotes the coned-off Cayley graph of G obtained by coning various cosets of local groups of $(\mathscr{G}, Y_1)$ in G_1 and $\hat{G}$ denotes the coned-off Cayley graph of G obtained by coning cosets of local groups of $(\mathscr{G}, Y)$ in G. By [13, Theorem 5.1], $\hat{G}_1, \hat{G}$ are quasiisometric to B_1, B, respectively. We denote these quasiisometries by f, g. Then Mitra's criterion for $B_1 \to B$ implies that Mitra's criterion holds for the natural map $\hat{G}_1 \to \hat{G}$. Hence, the first part of the theorem follows by Theorem 4.1.12.

$$\begin{array}{ccc} G_1 & \longrightarrow & G \\ \big\downarrow & & \big\downarrow \\ \hat{G}_1 & \longrightarrow & \hat{G} \\ {\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle g} \\ B_1 & \longrightarrow & B \end{array}$$

Figure 4.2: Horizontal maps are inclusion

Clearly, the CT map $\partial G_1 \to \partial G$ is injective if and only if the CT map $\partial \hat{G}_1 \to \partial \hat{G}$ is injective. Hence, the second part follows from the first part and by Lemma 2.2.38. □

In [48] (along with [49, Corollary, p. 805]) A. Martin proved the following combination theorem for complexes of hyperbolic groups.

Theorem 4.2.1. [48, p. 34] *Let $(\mathcal{G}, Y)$ be a developable complex of groups such that the following holds:*

1. *Y is a finite connected simplicial complex.*
2. *The local groups are hyperbolic and local maps are quasiisometric embeddings.*
3. *The universal cover of $(\mathcal{G}, Y)$ is a CAT(0) hyperbolic space, and*
4. *The action of the fundamental group $\pi_1(\mathcal{G}, Y)$ of $(\mathcal{G}, Y)$ on the universal cover is acylindrical.*

Then $\pi_1(\mathcal{G}, Y)$ is a hyperbolic group and the local groups are quasiconvex in $\pi_1(\mathcal{G}, Y)$.

Now, as a corollary of Theorem 4.0.2, we have the following.

Corollary 4.2.2. *Let $(\mathcal{G}, Y)$ be a complex of groups satisfying the hypothesis of Theorem 4.2.1. Let Y_1 be a connected subcomplex of Y and let $(\mathcal{G}, Y_1)$ be the subcomplex of groups obtained by restricting $(\mathcal{G}, Y)$ to Y_1. Suppose the following conditions hold.*

1. *$(\mathcal{G}, Y_1)$ also satisfies the hypotheses of Theorem 4.2.1.*

2. *The natural homomorphism* $H = \pi_1(\mathcal{G}, Y_1) \to G = \pi_1(\mathcal{G}, Y)$ *is injective.*

3. *Assume that the natural map* $B_1 \to B$ *satisfies Mitra's criterion where* B_1, B *are the universal covers of* $(\mathcal{G}, Y)$ *and* $(\mathcal{G}, Y)$ *respectively.*

Then there exists a Cannon-Thurston map for the inclusion $H \to G$*. Moreover,* H *is quasiconvex in* G *if and only if the Cannon-Thurston map for* $B_1 \to B$ *is injective.*

4.3 A remark on complexes of groups with finite edge groups

In this section, we consider developable complexes of groups whose edge groups are finite and whose developments are hyperbolic, which may or may not be CAT(0). Then the following theorem is immediate from the work of [10] and [51].

Theorem 4.3.1. *Suppose* $(\mathcal{G}, Y)$ *is a developable complex of groups such that the edge groups are finite and the universal cover is a hyperbolic metric space. Then the fundamental group of* $(\mathcal{G}, Y)$*, say* G*, is hyperbolic relative to* $\{G_v : v \in V(Y)\}$.

A word about the proof: In [10], Bowditch showed that a finitely generated group G is hyperbolic relative to a collection of finitely generated infinite subgroups $\{H_i\}$ if and only if there is a fine hyperbolic graph X on which G has a co-finite action and such that the edge stabilizers are finite and the infinite vertex stabilizers are precisely the conjugates of H_i's in G. Using this criterion, one only needs to check that the 1-skeleton of the universal cover of $(\mathcal{G}, Y)$ is a fine graph. This follows from [51, Corollary 2.11, Theorem 1.3].

Now, in Theorem 4.3.1 if one assumes in addition that the vertex groups are hyperbolic then one has the following.

Theorem 4.3.2. *Suppose* $(\mathcal{G}, Y)$ *is a developable complex of groups such that the vertex groups are hyperbolic and the edge groups are finite. Suppose the universal cover of* $(\mathcal{G}, Y)$ *is hyperbolic metric space. Then the fundamental group* G *of* $(\mathcal{G}, Y)$ *is a hyperbolic and the vertex groups are quasiconvex in* G.

In fact one uses Theorem 4.3.1 along with [66, Corollary 2.41] or [35, Theorem 2.4] for the proof of Theorem 4.3.2. Next we prove an analogue of Corollary 4.2.2 in the setting of Theorem 4.3.2.

Next, we deduce existence of the CT map for certain subcomplexes of groups of a complex of hyperbolic groups with finite edge groups.

Set up: Suppose $(\mathcal{G},Y)$ is a complex of groups as in Theorem 4.3.2. Suppose $Y_1 \subset Y$ is a finite connected subcomplex of Y and $(\mathcal{G},Y_1)$ is the complex of groups obtained by restricting $(\mathcal{G},Y)$ to Y_1. Let G,H be the fundamental groups of $(\mathcal{G},Y),(\mathcal{G},Y_1)$, respectively. Then G is a hyperbolic group and all the local groups of $(\mathcal{G},Y)$ are quasiconvex in G. Suppose H is hyperbolic and the natural map $H \to G$ is injective. Then, it is clear that all the local groups of $(\mathcal{G},Y_1)$ are quasiconvex in H. Let B_1,B be the universal covers of $(\mathcal{G},Y_1),(\mathcal{G},Y)$, respectively. Now, by Proposition 2.5.4, B_1 is also a hyperbolic metric space.

The following is immediate from Theorem 4.0.2.

Corollary 4.3.3. *If the inclusion $B_1 \to B$ satisfies Mitra's criterion then the CT map $\partial H \to \partial G$ exists. Moreover, the H is quasiconvex in G if and only if the CT map $\partial B_1 \to \partial B$ is injective.*

4.3.1 Description of uniform quasigeodesics.

In this subsection, we give a description of uniform quasigeodesics in a complex of spaces corresponding to a complex of groups satisfying the hypotheses of Theorem 4.3.2. In that direction, first of all, we recall some concepts that are relevant to us. Trees of spaces were defined by Bestvina and Feighn [6]. Then several authors used a number of equivalent definitions. We shall use the following modified version.

Definition 4.3.4. ([57, Section 3],[43, Definition 2.12]) A *tree of metric spaces* is a geodesic metric space X equipped with a 1-Lipschitz surjective map $\pi : X \to T$ onto a simplicial tree T satisfying the following:
(1) For each vertex $v \in V(T)$, the corresponding vertex space $X_v := \pi^{-1}(v) \subset X$ is rectifiably connected.
(2) For every edge $e \in E(T)$, the edge space $X_e := \pi^{-1}(m(e))$ is rectifiably connected.
(3) All the vertex spaces and the edge spaces are geodesic metric spaces with respect to the induced length metric from X.
(4) Every oriented edge $e = [v,w]$ comes equipped with a uniformly Lipschitz, uniformly proper map $f_e : X_e \times [v,w] \to X$, such that $f_e(X_e \times \{v\}) \subset X_v$.

One uses the notation f_{ev} for the natural composition $X_e \to X_e \times \{v\} \xrightarrow{f_e} X_v$. In this definition it is implicitly assumed that the length of each edge of T is 1.

Remark 4.3.5. If we have additional hypotheses on the vertex and edge spaces and on the maps f_{ev}'s, then as in [57, Section 3] one can define trees of space with those properties, like trees of hyperbolic spaces, or trees of (hyperbolic) spaces with qi embedding condition etc.

Construction of candidate paths.

Let $\pi : X \to T$ be a tree of hyperbolic spaces with qi embedding condition. Let $x \in X_u, y \in X_v$ for $u, v \in V(T)$. Let $t_0 = u, t_1, ..., t_m = v$ and $e_1, e_2, ..., e_{m-1}$ denote the consecutive vertices and edges on $[u, v]$. Suppose $f_{e_i t_{i-1}}, f_{e_i t_i}$ are incidence maps from X_{e_i} to $X_{t_{i-1}}, X_{t_i}$ for $i = 1, 2, ..., m-1$, respectively. Define $A_{i-1} := f_{e_i t_{i-1}}(X_{e_i})$ and $B_i := f_{e_i t_i}(X_{e_i})$ for $i = 1, 2, ..., m-1$. One inductively constructs the points x_i^-, x_i^+ for $i = 0, 1, ..., m$ as follows.

Set $x_0^- = x$ and $x_0^+ = P_{X_{t_0}, A_0}(x_0^-)$ where $P_{X_{t_0}, A_0}$ is the nearest point projection of X_{t_0} onto A_0. Suppose we have defined x_i^-, x_i^+ for $i < m$. Then we define x_{i+1}^- to be an arbitrary point of $f_{e_{i+1} t_{i+1}}(f_{e_i t_i}^{-1}(x_i^+)$. Define $x_{i+1}^+ := P_{X_{t_{i+1}}, A_{i+1}}(x_{i+1}^-)$. Lastly, define $x_m^+ := y$. Consider the path $\gamma(x, y)$ as concatenations of geodesic segments $[x_i^-, x_i^+]_{X_{t_i}}$ for $i = 0, 1, ..., m$, and a segment of length 1 joining x_i^+ and x_{i+1}^- for $i = 0, 1, ..., m-1$.

Remark 4.3.6. We shall refer to the paths $\gamma(x, y)$ as the **standard paths** joining x and y in what follows.

Let $(\mathcal{G}, Y)$ be a developable complex of hyperbolic groups such that local maps are qi embeddings. Let B be the universal cover of $(\mathcal{G}, Y)$. Let X be the complex of spaces as constructed in Section 2.7. The lemma below hints at the relevance of trees of spaces for us. Since the proof is obvious, we skip it.

Lemma 4.3.7. *Let $B^{(1)}$ be the 1-skeleton of B with the induced length metric from B. Let $\alpha \subset B^{(1)}$ be a geodesic. Let $X_\alpha := p^{-1}(\alpha)$ be endowed with the induced length metric from X. Then $p : X_\alpha \to \alpha$ is naturally a tree of hyperbolic metric spaces whose incidence maps are quasiisometric embeddings.*

Let $(\mathcal{G}, Y)$ be a developable complex of groups as in Theorem 4.3.2. Let X be a complex of spaces corresponding to $(\mathcal{G}, Y)$ as constructed in Section 2.7. Above, we have seen that G is a hyperbolic group. Since G is quasiisometric to X, X is also a hyperbolic space. Let ϕ be the quasiisometry from G to X. Let G^h be the cusp space with respect to gG_σ's and X_ϕ^h be the cusp space with respect to $\phi(gG_\sigma)$ where G_σ's are the local groups and $g \in G$. Then, by [67, Lemma 1.2.31], G^h is quasiisometric to X_ϕ^h. Let X^h be the cusp space with respect to gX_σ's. Now, according to Lemma 4.1.14, X^h is quasiisometric to X_ϕ^h. Hence, X is relatively hyperbolic with respect to its hyperbolic vertex spaces. We use the following theorem from [35] for describing uniform quasigeodesics in X.

Theorem 4.3.8. [35, Theorem 2.4] *Let $\mathcal{G}$ be a metric graph that is hyperbolic relative to a family $\mathcal{H} = \{H_c | c\}$ of complete connected subgraphs. Suppose each of the graph H_c is δ-hyperbolic for some $\delta \geq 0$. Then $\mathcal{G}$ is hyperbolic and enlargements of geodesics in coned-off space $\hat{\mathcal{G}}$ are uniform quasigeodesics in $\mathcal{G}$.*

Note that, in Theorem 4.3.8, if we take quasigeodesic in place of geodesic then the same proof gives us that enlargements of quasigeodesics without backtracking are also uniform quasigeodesics in $\mathcal{G}$.

Theorem 4.3.9. *Suppose $(\mathcal{G}, Y)$ is a complex of groups satisfying the hypotheses of Theorem 4.3.2 and $p : X \to B$ is the complex of spaces associated to $(\mathcal{G}, Y)$. Let $u, v \in B$ be any two points and let $[u, v]$ be a geodesic joining them in $B^{(1)}$. Then $X_{[u,v]} := p^{-1}([u, v])$ is uniformly qi embedded in X.*

In particular, uniform quasigeodesics in $X_{[u,v]}$ are uniform quasigeodesics in X.

Proof. Let $\hat{X}$ denotes coned-off space with respect to its vertex spaces X_v's. Let x, y be two elements of X and let $p(x), p(y)$ be their projections respectively. Without loss of generality, we can assume that x, y are in the vertex spaces of X and thus $p(x), p(y)$ are vertices of B. Join $p(x)$ and $p(y)$ by a geodesic, say γ, in $B^{(1)}$. Let $v_0 = p(x), v_1, ..., v_n = p(y)$ be the vertices on γ and let c_i be the cone points in $\hat{X}$ corresponding to vertex spaces X_{v_i} for $i = 0, 1, ..., n$. Note that, by Lemma 4.3.7, $X_\gamma = p^{-1}[v_0, v_n]$ is a tree of hyperbolic spaces with qi embedding conditions. Also we have standard paths in X_γ. Suppose v_0, v_1 are joined by an edge e_1. From the construction of standard path, choose points $x^-_{e_1}, x^+_{e_1}$ in the images of X_e in X_{v_0}, X_{v_1}, respectively. Join $x, x^+_{e_1}$ by a concatenation of paths of length 1 connecting $x, x^-_{e_1}$ through the cone point c_0 and an edge joining $x^-_{e_1}, x^+_{e_2}$. Now, following the same procedure for other vertices, we obtain a path α in $\hat{X}$ passing through cone points $c_0, c_1, ..., c_n$. Let ϕ be the natural quasiisometry from $\hat{X}$ to B. Note that $\phi \circ \alpha = \gamma$ and thus α is a quasigeodesic in $\hat{X}$ without backtracking. Now, by Theorem 4.3.8, we see that enlargement of α (which is same as standard path) is a uniform quasigeodesic in X joining x and y and hence X_γ is uniformly qi embedded in X. $\square$

4.4 Further applications and examples

In this section, we discuss several applications of Theorem 4.0.2 in the context of polygons of groups. At the end, we also give some interesting examples.

4.4.1 Polygons of groups

In this section, we discuss several applications of Theorem 4.0.2 in the context of polygons of groups. At the end, we also give some interesting examples.

4.4.2 Polygons of groups

Standing assumption: Let Y be a regular Euclidean polygon with at least 4 edges. Let $(\mathscr{G},Y)$ be a simple (i.e. all twisting elements are trivial) polygon of groups over Y. Also, assume that in vertex groups, the intersection of the two subgroups coming from the adjacent edges is equal to the subgroup coming from the barycenter of Y. Let $G=\pi_1(\mathscr{G},Y)$. Unless stated otherwise, this will be our standing assumption till subsection 4.4.3.

By [11, 12.29, p. 390, II.12], $(\mathscr{G},Y)$ is non-positively curved and hence it is developable. Let B be the universal cover of $(\mathscr{G},Y)$ which is a piecewise Euclidean polygon complex. Then, by [11, p. 562, III.C], B is a CAT(0) space.

Next, we adapt the following definition from [11, I.7, Definition 7.8] in our context.

Definition 4.4.1. Suppose $x\in B$. Define

$$\varepsilon(x)=\inf\{\varepsilon(x,P) : P \text{ is a polygon in } B \text{ containing } x\}$$

where $\varepsilon(x,P):=\{d_P(x,e) : e \text{ is an edge of } P \text{ not containing } x\}$.

We record the following lemma that can be proved in the same way as [11, Lemma 7.9]. For completeness, we give a sketch of proof. For the definition of an m-string in B, one is referred to [11, I.7, Definition 7.4]. Since shapes of B is finite, B is a complete geodesic metric space by [11, Chapter I.7].

Lemma 4.4.2. *Fix $x\in B$. If $y\in B$ is such that $d(x,y)<\varepsilon(x)$, then any polygon P which contains y also contains x, and $d(x,y)=d_P(x,y)$.*

Sketch of proof: To prove the lemma, it is sufficient to show that if $\Sigma=(x=x_0,x_1,...,x_n=y)$ is an m-string of $l(\Sigma)<\varepsilon(x)$, with $m\geq 2$, then $\Sigma'=(x_0,x_2,...,x_n)$ is an $(m-1)$-string with $l(\Sigma')\leq l(\Sigma)$. Now, by definition of m-string, there is a polygon P_1 such that $x_1,x_2\in P_1$. Since $l(\Sigma)<\varepsilon(x)$, $x_0\in P_1$. By triangle inequality, $d_{P_1}(x_0,x_2)\leq d_{P_1}(x_0,x_1)+d_{P_1}(x_1,x_2)$. Thus, $(x_0,x_2,...,x_n)$ is an $(m-1)$-string of length less that or equal to $l(\Sigma)$.

Now we are ready to prove the following:

Lemma 4.4.3. *The natural inclusion of each polygon in B is an isometric embedding.*

Proof. Let P be a polygon in B and let x be an interior point of P. Then, by Lemma 4.4.2, an open P-ball of radius $\varepsilon(x)$ around x embeds isometrically in B. Firstly, we show that the interior of P embeds isometrically in B. Let x,y be any two points in the interior of P and let α be the P-geodesic connecting x and y. To show that

α is a geodesic in B, it is sufficient to show that it is a local geodesic in B since in a CAT(0) space a local geodesic is a geodesic. Let $z \in Im(\alpha)$ be any point. Then $B_P(z, \varepsilon(z)) \cap \alpha$ embeds isometrically in B. Thus α is a local geodesic in B. Now, let x be an interior point of P and let y be a point on a side of P. Let β be the P-geodesic joining x and y. Then there is a sequence $\{x_n\}$ in the interior of P such that $x_n \in \beta$ and $x_n \to y$ in P. Since the inclusion $P \to B$ is 1-Lipschitz, $x_n \to y$ in B too. Now, $\lim_{n\to\infty} d_B(x,x_n) = d_B(x,y)$ and $\lim_{n\to\infty} d_P(x,x_n) = d_P(x,y)$. But $d_P(x,x_n) = d_B(x,x_n)$. Thus, $d_P(x,y) = d_B(x,y)$. Similarly, it can be shown that if x and y lie on distinct sides of P then $d_P(x,y) = d_B(x,y)$. Finally, suppose $x \neq y$ are two vertices on a side, say e, of P. Let $e' \neq e$ be a side of P having y as a vertex. Suppose $\{x_n \in e'\}$ is a sequence such that $x_n \to y$ in P. This implies that $x_n \to y$ in B too. As in the previous case, $d_P(x,x_n) = d_B(x,x_n)$. Also, $\lim_{n\to\infty} d_P(x,x_n) = d_P(x,y)$ and $\lim_{n\to\infty} d_B(x,x_n) = d_B(x,y)$. Hence $d_P(x,y) = d_B(x,y)$. This completes the proof. □

Lemma 4.4.4. *Let P_1 and P_2 be two polygons in B such that either $P_1 \cap P_2$ is a vertex or $P_1 \cap P_2$ is an edge. Let e_1 and e_2 be two edges in P_1 and P_2 respectively such that $e_1 \cap e_2$ is a vertex of $P_1 \cap P_2$. Then the concatenation α of e_1 and e_2 is a geodesic in B.*

Proof. Let α be the concatenation of e_1 and e_2. By Lemma 4.4.3, e_1, e_2 are geodesics in B. Let $v := e_1 \cap e_2$. To prove the lemma, it is sufficient to show that a small neighborhood of v in α embeds isometrically in B. Let I be a ball of radius $\frac{1}{4}$ around v in α. Suppose x, y are the endpoints of I lying in e_1, e_2 respectively. Note that $\Sigma_0 = (x, v, y)$ is a 2-string in B and $l(\Sigma_0) = \frac{1}{2}$. If, except Σ_0, there is no string in B connecting x and y then we are done. Therefore, we show that if $\Sigma = (x_0 = x, x_1, ..., x_n = y)$ is any other n-string in B then $l(\Sigma) > l(\Sigma_0)$. By definition of an n-string, there exists a sequence of polygons $(P_0, P_1, ..., P_{n-1})$ such that $x_i, x_{i+1} \in P_i$ for $i = 0, 1, ..., (n-1)$. Without loss of generality, we can assume that x_i, x_{i+1} lie on different sides of P_i' for $0 \leq i \leq (n-1)$. Note that if x_1 belongs to a side of P_0 not containing v then $d_{P_0}(x, x_1) > \frac{1}{2}$ as the angle at each vertex of P_0 is at least $\frac{\pi}{2}$ and hence $l(\Sigma) > \frac{1}{2}$. Thus we assume that x_1 belongs to a side of P_0, say e_3, containing v. Since the angle between e_1 and e_3 is at least $\frac{\pi}{2}$, $d_{P_0}(x, x_1) > \frac{1}{4}$. By the same reasoning, x_2 belongs to a side of P_1 containing v. By continuing in this way either x_{n-1} belongs to a side of P_{n-1} containing v or x_{n-1} lies on a side of P_{n-1} not containing v. In either case $d_{P_{n-1}}(x_{n-1}, x_n) > \frac{1}{4}$. Hence, $l(\Sigma) \geq d_{P_0}(x, x_1) + d_{P_{n-1}}(x_{n-1}, x_n) > \frac{1}{2}$. This completes the proof. □

Now, assume that all the local groups of $(\mathcal{G}, Y)$ are hyperbolic and all the local maps are qi embeddings. Suppose B is a hyperbolic space and the action of G on B is acylindrical. Then, by Theorem 4.2.1, G is a hyperbolic group. Let Y_1 be an edge of Y and let $(\mathcal{G}, Y_1)$ be the restriction of $(\mathcal{G}, Y)$ to Y_1. Let $H = \pi_1(\mathcal{G}, Y_1)$. Then, as an application of Theorem 4.0.2, we obtain the following:

Theorem 4.4.5. *H is a quasiconvex subgroup of G.*

Proof. Let $Y_1 = e$ be the edge with vertices v, w. Then $H = G_v *_{G_e} G_w$. Let B_1 denote the Bass-Serre tree of H. Then B_1 is a CAT(0) space. By Theorem 4.0.2, it is sufficient to prove that the natural map $B_1 \to B$ is a quasiisometric embedding. We prove that it is in fact an isometric embedding. Note that, if $B_1 \to B$ is a qi embedding then one can check that the action of H on B_1 is acylindrical and hence H is a hyperbolic group by [6]. To prove that $B_1 \to B$ is an isometric embedding, it is enough to show that the restriction of the inclusion $B_1 \to B$ to any geodesic of B_1 is a local isometry since B is CAT(0) a space. Let α be any geodesic of B_1. We note that the inclusion $\alpha \to B$ is simplicial and the edges of B are geodesics by Lemma 4.4.3. Thus, the map $\alpha \to B$ is locally isometric at all points other than possibly the vertices. Hence it is enough to prove the following:

Claim. If b is a vertex of α then the inclusion in B of a small neighborhood of b in α is an isometric map.

We know that vertices and edges of B_1 and B correspond to cosets of vertex and edge groups in H and G respectively, and the 2-dimensional faces of B correspond to cosets of G_τ in G, where τ is the 2-dimensional face of Y (see Theorem 2.6.12). Without loss of generality, we can assume that the vertex b of α corresponds to local group G_v and $e \in \alpha$ (e is the edge corresponding to G_e). Let $g_v G_e$ be an edge of α adjacent to b where $g_v \in G_v \setminus G_e$. Note that $g_v \notin G_\tau$. Then, $g_v G_\tau$ denotes a 2-dimensional face of B having G_v as common vertex. Let $e_1 := g_v G_e$. Then, by Lemma 4.4.4, the concatenation of e and e_1 is a geodesic in B. This proves the claim. □

Remark 4.4.6. Theorem 4.4.5 is not true in the case of a triangle of groups, i.e. there are examples of developable triangles of groups such that development is a CAT(0) hyperbolic space, the fundamental group G of the triangle of groups is hyperbolic, and amalgamated free product corresponding to an edge is not quasiconvex in G, see Example 4.4.18.

An immediate consequence of Theorem 4.4.5 is the following:

Corollary 4.4.7. *Let Y be an Euclidean polygon with at least 4-edges. Suppose $(\mathscr{G},Y)$ is a developable simple polygon of groups satisfying the conditions $(1),(2),(3)$ of Theorem 4.2.1. Let e be an edge of Y and let H be the amalgamated free product corresponding to e. Then H is quasiconvex in $\pi_1(\mathscr{G},Y)$.*

The next lemma is pretty standard. For completeness, we give its proof.

Lemma 4.4.8. *The group G splits as amalgamated free product.*

Proof. Let J be an internal line segment in Y joining the midpoints of two edges, say e_1,e_2 such that it divides the set of vertices into two sets each having at least 2 vertices. Let V_1,V_2 be these two sets of vertices. Note that J divides the 1-skeleton of Y into parts, say Y_1,Y_2, each having V_1,V_2 as vertex sets. Let $(\mathscr{G},Y_1),(\mathscr{G},Y_2)$ denote the complexes of groups restricted to Y_1,Y_2 and let G_1,G_2 be their fundamental groups respectively. Let G_{e_1},G_{e_2} be edge groups corresponding to e_1,e_2 and let G_τ be group corresponding to the barycenter of Y. Suppose $H = G_{e_1} *_{G_\tau} G_{e_1}$ is the natural amalgamated free product. Note that we can realise H as a subgraph of groups of $(\mathscr{G},Y_1),(\mathscr{G},Y_2)$ respectively. Then, the natural maps from H to G_1 and G_2 are injective, see [3, 2.15, p.25]. Then, using the Eilenberg-Maclane spaces for local groups in $(\mathscr{G},Y)$, one defines a complex of spaces whose fundamental group is G (see Subsection 2.6.2). Finally, by Seifert-Van Kampen theorem, one can see that $G = G_1 *_H G_2$. □

In [77], Wise proved a celebrated combination theorem for virtually compact special groups (see [76],[77]). It says that if a hyperbolic group G splits as $A *_C B$ or $A*_C$, where A,B are virtually compact special groups and C is quasiconvex in G then G is virtually compact special. We prove a similar result in the case of a polygon of virtually compact special groups. Before that, we recall the notion of the height of a subgroup of a group.

In [31], Gitik et al. generalized the concept of malnormality and introduced the following notion of *height* of a subgroup of a group.

Definition 4.4.9 (Height)**.** The height of an infinite subgroup H in G is the maximal $n \in \mathbb{N}$ such that if there exists distinct cosets $g_1H, g_2H, ..., g_nH$ such that $g_1Hg_1^{-1} \cap g_2Hg_2^{-1} \cap ... \cap g_nHg_n^{-1}$ is infinite. The height of a finite subgroup define to be 0.

In [31], the authors proved the following:

Theorem 4.4.10. [31, p. 322] *Quasiconvex subgroups of hyperbolic groups have finite height.*

Next, we note the following simple lemma.

Lemma 4.4.11. *Suppose $(\mathcal{G},Y)$ is a finite graph of groups with fundamental group G. If all the edge groups have finite height in G then the action of G on the Bass-Serre tree of $(\mathcal{G},Y)$ is acylindrical.*

Proof. Since $(\mathcal{G},Y)$ is a finite graph of groups, there exists $l \in \mathbb{N}$ such that the height of each edge group is at most l. Let T be the Bass-Serre tree of $(\mathcal{G},Y)$. Note that the G-stabilizer of a geodesic α in T is the intersection of the G-stabilizers of edges on α. Let if possible the action of G on T is not acylindrical. Then, given $k \in \mathbb{N}$, there is a geodesic in T with length bigger than k and the G-stabilizer of the geodesic is infinite. Now, choose k to be sufficiently larger than l such that there is a geodesic segment β of length bigger than k and G-stabilizer of β is infinite. Since there are finitely many edge groups, $Stab_G(\beta)$ is contained in the intersection of more than l-conjugates of an edge group. This gives a contradiction as the height of each edge group is at most l. □

Now, we are ready to prove the following result:

Proposition 4.4.12. *Suppose $(\mathcal{G},Y)$ satisfies the following conditions:*

1. *All the vertex groups are hyperbolic and virtually compact special.*
2. *All the edge groups are quasiconvex in adjacent vertex groups.*
3. *The universal cover B is a hyperbolic space and the action of G on B is acylindrical.*

Then, G is virtually compact special.

Proof. By our assumptions, $(\mathcal{G},Y)$ is a complex of groups satisfying all the hypotheses of Theorem 4.2.1. Hence, G is a hyperbolic group and all vertex groups are uniformly quasiconvex in G. Now, by Lemma 4.4.8, write $G = G_1 *_H G_2$, where G_1, G_2 are the fundamental groups of trees of hyperbolic groups with qi embedding conditions, say $(\mathcal{G},Y_1),(\mathcal{G},Y_2)$ respectively. Assume that Y_2 contain only two vertices. Since all the vertex groups of $(\mathcal{G},Y)$ are quasiconvex in G and intersection of two quasiconvex subgroups is again quasiconvex, the edge groups in $(\mathcal{G},Y_1),(\mathcal{G},Y_2)$ are quasiconvex in G. Thus, by Lemma 4.4.10, the edge groups of $(\mathcal{G},Y_1),(\mathcal{G},Y_2)$ have finite height in G and hence they have finite height in G_1, G_2 also. According to Lemma 4.4.11, $(\mathcal{G},Y_1),(\mathcal{G},Y_2)$ are acylindrical graphs of hyperbolic groups with qi embedding condition and hence G_1, G_2 are hyperbolic (see [6]). By Theorem 4.4.5, G_2 is a quasiconvex subgroup of G. By [43, Theorem 8.73], the group H is quasiconvex in G_2 and therefore H is quasiconvex in G. Now, by [77, Theorem

13.1], G_1, G_2 are virtually compact special. As H is quasiconvex in G, G is virtually compact special by [77, Theorem 13.1]. □

Remark 4.4.13. From the proof of Proposition 4.4.12, one can deduce the following:

Suppose $(\mathcal{G}, Y)$ is a polygon of groups satisfying the hypotheses of Theorem 4.4.5. Let Y_1 be a connected subcomplex of Y such that $|Y \setminus Y_1| \geq 3$. Let $(\mathcal{G}, Y_1)$ be the restriction of $(\mathcal{G}, Y)$ to Y_1 with fundamental group G_1. Then G_1 is quasiconvex in G.

Now assume that the vertex groups of $(\mathcal{G}, Y)$ are hyperbolic and the edge groups are finite. If B is a hyperbolic space then, by Theorem 4.3.1, G is hyperbolic. This also follows from Theorem 4.2.1. Now we show that, in the case of polygons of groups, its converse is also true.

Lemma 4.4.14. *The following are equivalent:*

1. *G is a hyperbolic group.*

2. *The universal cover B is a hyperbolic space.*

Proof. (1) $\implies$ (2) By Lemma 4.4.8, we can write $G = G_1 *_F G_2$, where G_1, G_2 are the fundamental group of trees of hyperbolic groups with finite edge groups, F is an amalgamated free product of finite groups. Clearly, G_1 and G_2 are hyperbolic groups. Since vertex groups of F are finite, their actions on the Bass-Serre trees of G_1, G_2 are acylindrical. Thus, by [43, Theorem 8.73], F is quasiconvex in G_1 as well as in G_2. By splitting Y along a different internal line segment, we can again write G as amalgamated free product $G'_1 *_{F'} G'_2$, where G'_1, G'_2 are hyperbolic. Again, by the same reason, F' is quasiconvex in G'_1 as well as in G'_2. Now, regard F as subgraph of groups of this new amalgamated free product. Note that, by [3, 2.15, p.25], the natural map from the Bass-Serre tree of F to Bass-Serre tree of $G'_1 *_{F'} G'_2$ is injective. Thus, by [43, Theorem 8.73], we see that F is quasiconvex in G. Hence, G_1, G_2 are also quasiconvex in G. Since all the vertex groups of G_1 and G_2 are quasiconvex in G_1 and G_2 respectively, all the vertex groups of $(\mathcal{G}, Y)$ are quasiconvex in G. Now, the coned-off Cayley graph $\hat{G}$ of G with respect to the local groups of $(\mathcal{G}, Y)$ is quasiisometric to B (see [13, Theorem 5.1]). But $\hat{G}$ is a hyperbolic geodesic metric space by Proposition 2.5.4. Hence, B is a hyperbolic geodesic metric space.

(2) $\implies$ (1) Since universal cover is a hyperbolic space, by Proposition 4.3.1, G is a hyperbolic group. □

It is not clear that the Lemma 4.4.14 is true for general complex of hyperbolic groups with finite edge groups. Also, there are example of developable complex of hyperbolic groups with finite edge groups such that the universal cover is not a hyperbolic space (see below). However, we have the following:

Lemma 4.4.15. *Let $(\mathcal{G},Y)$ be a developable complex of hyperbolic groups with finite edge groups over a finite simplicial complex Y. Assume that $G=\pi_1(\mathcal{G},Y)$ is hyperbolic. Then the universal cover of $(\mathcal{G},Y)$ is a hyperbolic space if and only if vertex groups are quasiconvex in G.*

Proof. If the universal cover is a hyperbolic space then by Theorem 4.3.1, G is hyperbolic and vertex groups are quasiconvex in G. Converse is obvious as coned-off Cayley graph of G with respect to the collection of cosets of vertex groups in G is a hyperbolic space, see Proposition 2.5.4, and it is quasiisometric to the universal cover. □

4.4.3 Examples

In this subsection, we discuss various types of examples which show that some of the hypotheses in the theorems of this chapter are necessary. First of all note that in Theorem 4.3.1, we cannot drop the hypothesis that the universal cover of complex of hyperbolic groups with finite edge groups is a hyperbolic space, see [11, Example 12.17(3), II.12]. In that example, we have a developable triangle of finite groups $(\mathcal{G},Y)$ whose universal cover is Euclidean space $\mathbb{E}^2$. Since, $\pi_1(\mathcal{G},Y)$ is quasiisometric to $\mathbb{E}^2$, $\pi_1(\mathcal{G},Y)$ is not a hyperbolic group.

Let $(\mathcal{G},Y)$ be a developable complex of groups over a finite simplicial complex and let $Y_1\subset Y$ be a connected subcomplex of Y. Consider the subcomplex of groups $(\mathcal{G},Y_1)$ obtained by restricting $(\mathcal{G},Y)$ to Y_1. The following example shows that the natural homomorphism from $\pi_1(\mathcal{G},Y_1)\to\pi_1(\mathcal{G},Y)$ need not be injective.

Example 4.4.16. Let Y be a triangle as in figure 4.3. Define a triangle of groups $(\mathcal{G},Y)$ as follows:

Suppose $G_{v_1}=\langle a,b|ab=ba\rangle, G_{v_2}=\langle c,d|cd=dc\rangle, G_{v_3}=\langle e,f|ef=fe\rangle$. Suppose all the edge groups G_{e_1},G_{e_2},G_{e_3} are cyclic and the images of monomorphisms from G_{e_3} to G_{v_1},G_{v_2} are $\langle a\rangle,\langle c\rangle$ respectively. Similarly, images of monomorphisms for e_2 are $\langle e\rangle,\langle b\rangle$ and for e_1 are $\langle d\rangle,\langle f\rangle$. Since the face group G_τ is a subgroup of the intersection of edge groups, G_τ is trivial. Note that $\pi_1(\mathcal{G},Y)=\langle a,b,d|ab=ba,ad=da,bd=db\rangle$. Let $Y_1=e_3$. Then, $\pi_1(\mathcal{G},Y_1)=\langle a,b,d|ab=ba,ad=da\rangle$. Clearly, the natural map $\pi_1(\mathcal{G},Y_1)\to\pi_1(\mathcal{G},Y)$ is not injective.

However, in contrast to triangles of groups, consider polygons of groups $G(Y)$ with at least 4 edges. Also, assume that in vertex groups, the intersection of the two

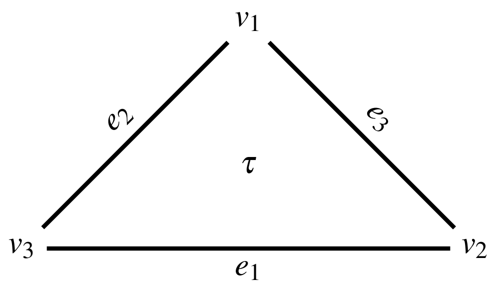

Figure 4.3:

subgroups coming from adjacent edges is equal to the subgroup coming from the barycenter of Y. If Y_1 is an edge of Y then the natural map $\pi_1(\mathscr{G}, Y_1) \to \pi_1(\mathscr{G}, Y)$ is always injective. Also, by Lemma 4.4.8, we see that such a polygon of groups is always developable. Note that one can easily cook up an example of a triangle of groups which is not developable. Now, we give an example of a developable complex of hyperbolic groups such that there exists a subcomplex of groups with the property that the natural inclusion from the universal cover of the subcomplex of groups to the universal cover of the complex of groups is not a proper embedding. In particular, the following example shows that the converse of Theorem 4.0.2 is not true in general.

Example 4.4.17. Let Y be a triangle as in figure 4.3. Define a triangle of groups $(\mathscr{G}, Y)$ in the following manner:

Let $G_{v_1} = \langle a, b | a^2, b^2 \rangle$, $G_{v_2} = \langle c, d | c^2, d^2 \rangle$, $G_{v_3} = \langle e, f, g | e^2, f^2, g^2 \rangle$. Assume that all the edge groups $G_{e_1}, G_{e_2}, G_{e_3}$ are of order 2. Let the images of monomorphisms from G_{e_3} to G_{v_1}, G_{v_2} be $\langle a \rangle, \langle c \rangle$. Similarly, the images of monomorphisms for edge e_1 are $\langle d \rangle, \langle f \rangle$ and for e_2 are $\langle e \rangle, \langle b \rangle$. Since the intersection of edge groups is trivial, the face group G_τ is trivial. Note that $\pi_1(\mathscr{G}, Y) = \langle a, b, d, g | a^2, b^2, d^2, g^2 \rangle$. Now, it is clear that it is a developable complex of groups and its fundamental group is hyperbolic. Also, all the vertex groups are quasiconvex in $\pi_1(\mathscr{G}, Y)$. Thus, the universal cover B of $(\mathscr{G}, Y)$ is a hyperbolic space. Let $Y_1 = e_3$ and let $(\mathscr{G}, Y_1)$ be the restriction of $(\mathscr{G}, Y)$ to Y_1. One can also check that $\pi_1(\mathscr{G}, Y_1) = \langle a, b \rangle *_{\langle a \rangle} \langle c, d \rangle$ is quasiconvex in $\pi_1(\mathscr{G}, Y)$. The universal cover B_1 of $(\mathscr{G}, Y_1)$ is the Bass-Serre tree of $\pi_1(\mathscr{G}, Y_1)$. Consider the two vertices G_{v_1} and $dbdb...db\ (n - times) G_{v_2}$ of B_1. The distance between these two vertices in B_1 is $n+1$. On the other hand, by construction of B [11, Theorem 2.13,III.C], one can see that the distance between these two vertices in B is always 2. Hence, the natural map $B_1 \to B$ is not a proper embedding. Moreover, there is no Cannon-Thurston map for the inclusion $B_1 \to B$.

It is worth noting that, in the above example, we can also use Theorem 4.1.10 to show that there is no CT map from B_1 to B. In the following example, for the definition of hyperbolic automorphism of a free group one is referred to [6].

Example 4.4.18. Let Y be a triangle as in figure 4.3 and let $(\mathcal{G},Y)$ be a complex of groups defined as follows:

Suppose $G_{v_1} = \langle f,g,h,t|tft^{-1} = \phi(f),tgt^{-1} = \phi(g),tht^{-1} = \phi(h)\rangle$ where ϕ is a hyperbolic automorphism of the free group generated by f,g and h. Hence G_{v_1} is a hyperbolic group by [6]. Suppose $G_{v_3} = \langle a,b,c\rangle$ and $G_{v_2} = \langle d,e\rangle$. Suppose the edge group G_{e_2} is a free group on 2 generators. Suppose the images of the generators under the monomorphisms from G_{e_2} to G_{v_3} and G_{v_1} are a,b and h,f. Suppose that the edge groups G_{e_1},G_{e_3} are cyclic and the images of the monomorphisms from G_{e_1} to G_{v_3},G_{v_2} are $\langle c\rangle$,$\langle d\rangle$ respectively. Similarly, the images of the monomorphisms for e_3 are $\langle e\rangle,\langle g\rangle$. Clearly, the face group G_τ is trivial. Note that $\pi_1(\mathcal{G},Y) = \langle a,b,e,t|tat^{-1} = \phi(a),tbt^{-1} = \phi(b),tet^{-1} = \phi(e)\rangle * \langle c\rangle$. Let $Y_1 = e_1$ and let $(\mathcal{G},Y_1)$ be the restriction of $(\mathcal{G},Y)$. Now, $\pi_1(\mathcal{G},Y_1) = \langle a,b,e\rangle * \langle c\rangle$. It is well known that $\langle f,g,h\rangle$ is not quasiconvex in G_{v_1}. Thus, one sees that $\pi_1(\mathcal{G},Y_1)$ is not quasiconvex in $\pi_1(\mathcal{G},Y)$.

Next, we give an example of a developable triangle of groups such that the universal cover is not a CAT(0) space.

Example 4.4.19. Let Y be a Euclidean equilateral triangle as in figure 4.3. Define a triangle of groups $(\mathcal{G},Y)$ in the following way:

Assume that $G_{v_1} = \langle a,b,g|a^2,b^2,g^2,ab=ba,ag=ga,bg=gb\rangle$, $G_{v_2} = \langle c,d|c^2,d^2,cd=dc\rangle$, $G_{v_3} = \langle e,f,h|e^2,f^2,h^2,ef=fe,eh=he,fh=hf\rangle$. Assume that all the edge groups are of order 2. Let the images of monomorphisms from G_{e_3} to G_{v_1},G_{v_2} be $\langle a\rangle,\langle c\rangle$. Similarly, images of monomorphisms for edge e_1 are $\langle d\rangle,\langle f\rangle$ and for e_2 are $\langle e\rangle,\langle b\rangle$. The face group G_τ is trivial. Note that Gersten-Stallings's angle [71] at each vertex is $\frac{\pi}{2}$. Thus, this triangle of groups is not non-positively curved in the sense of Gersten-Stallings [71]. Now, the universal cover B of $(\mathcal{G},Y)$ is a M_κ^2 complex for $\kappa = 0$. By our choice of local groups, we have an injective loop of length 4 in the link of a vertex of B . Thus, the development does not satisfy the link condition by [11, Lemma 5.6, II.5]. Hence, the development is not a CAT(0) space.

Chapter 5

Further questions

In this section we include some questions naturally inspired by this thesis. The author would like to pursue them as future projects.

Question 1. *Suppose $(\mathscr{G}, Y)$ is a developable complex of groups over a finite simplicial complex Y such that the following holds:*

1. *All the vertex groups are convergence.*
2. *All the edge groups are parabolic.*
3. *The universal cover of $(\mathscr{G}, Y)$ is a CAT(0) hyperbolic.*

Is the fundamental group G of $(\mathscr{G}, Y)$ convergence?

For the above question, following [48], one needs to construct a compact metrizable space X out of compact metrizable spaces for local groups such that G acts on X as a convergence group.

In [48], Martin proved a combination theorem for an acylindrical complex of hyperbolic groups, say, $(\mathscr{G}, Y)$. There, he assumed that the universal cover of $(\mathscr{G}, Y)$ is a CAT(0) space. It will be interesting to remove this hypothesis from his theorem. More precisely, we have the following question:

Question 2. *Suppose $(\mathscr{G}, Y)$ is a developable complex of hyperbolic groups over a finite simplicial complex Y such that all the local maps are qi embeddings. Let G be the fundamental group of $(\mathscr{G}, Y)$ and let B be its universal cover. Suppose the action of G on B is acylindrical and B is a hyperbolic metric space. Is G a hyperbolic group?*

We would like to answer the above questions for complexes of relatively hyperbolic groups too.

By Floyd mapping theorem in [27], we see that relatively hyperbolic groups have non-trivial Floyd boundary. The converse of this fact is a question of Olshanskii-Osin-Sapir [64, Problem 7.11], i.e. if a finitely generated group has non-trivial Floyd boundary then it is hyperbolic relative to a collection of proper subgroups. We end this chapter with the following questions.

Question 3. *Give an example of a non-elementary convergence group that is not relatively hyperbolic?*

Answer to the above question does not give a counterexample to Olshanskii-Osin-Sapir conjecture because of the following:

Question 4. *Give an example of a non-elementary convergence group with a trivial Floyd boundary?*

Answer to Question 4 will also answer Question 3. Given a finitely presented group Q, Rips constructed a hyperbolic group G and a surjection from G to Q. Let N be the kernel of this surjection. Then it is not known whether N is relatively hyperbolic with respect to a collection of proper subgroups of N or if it has trivial Floyd boundary.

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