CLASSIFICATION OF SURFACES:
A COMBINATORIAL VIEWPOINT

Abhishek (MS11009)

A dissertation submitted for the partial fulfillment of BS-MS dual degree in Science.

Indian Institute of Science Education and Research Mohali

April, 2016

Certificate of Examination

This is to certify that the dissertation titled “Classification of Surfaces: A Combinatorial Viewpoint” submitted by Mr. Abhishek (Reg. No. MS11009) for the partial fulfillment of BS-MS dual degree programme of the Institute, has been examined by the thesis committee duly appointed by the Institute. The committee finds the work done by the candidate satisfactory and recommends that the report be accepted.

Dr. Chanchal Kumar

Dr. Lingaraj Sahu

Dr. Krishnendu Gongopadhyay
(Supervisor)

Dated : April 28, 2016

Declaration

The work presented in this dissertation has been carried out by me under the guidance of Dr. Krishnendu Gongopadhyay at the Indian Institute of Science Education and Research Mohali.

This work has not been submitted in part or in full for a degree, a diploma, or a fellowship to any other university or institute. Whenever contributions of others are involved, every effort is made to indicate this clearly, with due acknowledgement of collaborative research and discussions. This thesis is a bonafide record of original work done by me and all sources listed within have been detailed in the bibliography.

Abhishek

April 28, 2016

In my capacity as the supervisor of the candidate's project work, I certify that the above statements made by the candidate are true to the best of my knowledge.

Dr. Krishnendu Gongopadhyay

(Supervisor)

Acknowledgement

I would like to thank:

My parents for supporting and encouraging me in every path of my life.

My Project Guide Dr. Krishnendu Gongopadhyay for guiding me in my thesis work and helping me understand the difficult concepts.

My friend Arjit Kant Gupta for helping me with the drawings of figures.

LIST OF FIGURES

Contents:

Abstract

In this thesis, we identify the compact surfaces with their ribbon graphs. We first study the notion of Covering Space Theory and Fundamental Groups of topological spaces. For understanding surfaces, we understand Manifolds and the concept of smooth Atlas on manifolds. A 2-dimensional manifold is a topological space which is locally homeomorphic to the 2-dimensional Euclidean Plane. An atlas on a manifold is the collection of pairs (U_i, φ_i) where U_i are subsets of the manifold and φ_i are smooth maps from U_i to a subset of euclidean plane $\mathbb{R}^2$. A surface is a manifold with a smooth atlas defined on it. We define the notion of graphs embedded in a surface. A graph is a set of points (called vertices) which are joined by edges. For a vertex v in the graph, the star of v, denoted by E_v, is the set of all edges originating from v. A Ribbon graph is the graph along with a cyclic ordering on the star of every vertex. Any ribbon graph can be embedded into an oriented surface with cyclic ordering induced from orientation of the surface. Moreover, the ribbon surface associated with the filling ribbon graph is unique. The main result of our thesis is that "Every compact oriented surface S is homeomorphic to one of the surfaces S_g for $g \geq 0$".

Introduction

In this thesis, our aim is to classify compact, connected, oriented surfaces using a combinatorial approach. We have gathered all the materials in this note from the references { [1], [2], [3], [4] }. I do not claim any originality in the mathematical contents of this thesis except the flow of presentation of the material.

Chapter 1: Fundamental Groups

0.1 Fundamental Groups

Definitions:

- Let X and Y be two topological spaces and f, f' be two continuous maps from X to Y. Then, f is **Homotopic** to f' if there is a continuous map $F : X \times I \to Y$ s.t.

$$F(x,0) = f(x)$$

$$F(x,1) = f'(x)$$

for each x∈X.

- A path from x_0 to x_1 is a continuous map $f : [0,1] \to X$ s.t. $f(0) = x_0$ and $f(1) = x_1$.

- Two paths f and f' in X are **Path Homotopic** (denoted by $\backsimeq$) if they have same initial point x_0 and same final point x_1, and if there is a continuous map $F : I \times I \to X$ s.t.

$$F(s,0) = f(s)$$

$$F(s,1) = f'(s)$$

$$F(0,t) = x_0$$

$$F(1,t) = x_1$$

for each s,t∈I.

Lemma 1: The relation $\simeq$ is an equivalence relation.

Proof: We prove the three properties of an equivalence relation.

1. Reflexivity: $f \simeq f$ is trivial ; the map $F(x,t) = f(x)$ is the required path homotopy function.

2. Symmetry: If $f \simeq f'$ then we need to show that $f' \simeq f$. Let F be the path homotopy between f and f'. Then $G(x,t) = F(x, 1-t)$ is the path homotopy between f' and f.

3. Transitivity: Suppose $f \simeq f'$ and $f' \simeq f''$. We show that $f \simeq f''$. Let F be a path homotopy between f and f' and G be a path homotopy between f' and f''. Define $H : I \times I \to X$ by the equation

$$\begin{cases} H(s,t) = F(s,2t), & t \in [0, \frac{1}{2}] \\ H(s,t) = G(s,2t-1), & t \in [\frac{1}{2}, 1] \end{cases}$$

If f is a path, we will denote its path-homotopy equivalence class by $[f]$.

Definitions:

- If f is a path in X from x_0 to x_1, and if g is path in X from x_1 to x_2, we define the **product** $f * g$ to be the path h from x_0 to x_2 given by

$$\begin{cases} h(s) = f(2s), & s \in [0, \frac{1}{2}] \\ h(s) = g(2s-1), & s \in [\frac{1}{2}, 1] \end{cases}$$

For paths f and g in X, we define $[f] * [g] = [f * g]$.

- For any point x in topological space X, the loop at x is the path starting and ending at x.

- For an arbitrary point x_0 in a topological space X, the collection of all equivalence classes of loops at x_0 forms a group under the product operation. This group is called the **Fundamental Group** of X relative to the **base point** x_0, and is denoted by $\Pi_1(X, x_0)$.

  Moreover, we can see from the following theorem that the fundamental group of a path connected space is independent of the choice of base point:

Theorem 2: If X is path-connected and x_0 and x_1 are two points in X, then $\Pi_1(X, x_0)$ is isomorphic to $\Pi_1(X, x_1)$.

Proof: We give the sketch of elementary proof here.

Since X is path connected, let f be a path from x_0 to x_1 .Now, For any loop g at x_0 in X, we define a loop h at x_1 by : $h = f \circ g$. It is easy to see that this map is indeed an isomorphism.

Hence, $\Pi_1(X, x_0)$ is isomorphic to $\Pi_1(X, x_1)$.

□

0.2 Covering Spaces

Definitions:

- Let $p : E \to B$ be a continuous surjective map. The open set U of B is said to be ***evenly covered*** by p if the inverse image $p^{-1}(U)$ can be written as the union of disjoint open sets V_α in E such that for each α , the restriction of p to V_α is a homeomorphism of V_α onto U. The collection $\{V_\alpha\}$ will be called a partition of $p^{-1}(U)$ into **slices.**

- Let $p : E \to B$ be continuous and surjective. If every point b of B has a neighbourhood U that is evenly covered by p , then p is called a ***covering map***, and E is said to be a ***covering space*** of B .

0.3 Manifolds

Definition:

- An n-dimensional **manifold** M is a topological space s.t. :

1. M is a *Hausdorff* space: For every pair of points $p,q \in M$, there are disjoint open subsets $U,V \subset M$ s.t. $p \in U$ and $q \in V$.

2. M is *Second Countable* : There exists a countable basis for the topology of M .

3. M is locally Euclidean of dimension n: Every point of M has a neighbourhood that is homeomorphic to the n-dimensional euclidean space $\mathbb{R}^n$.

- **Surface :**

  Let S be a 2-dimensional metrisable(Hausdorff) topological space. A two-dimensional *chart* for such a space is a pair (U,φ) where $U \subset S$ is an open subset and $\varphi : U \to V$ is a homeomorphism onto an open subset $V \subset \mathbb{R}^2$.

  The map φ is called the *coordinate* of the chart.

  A collection of charts $\{(U_i,\varphi_i)|i\varepsilon I\}$ is called an *atlas* for S if $S = \bigcup U_i$. An atlas is called *smooth* or C^∞ if the coordinate change maps

$$\varphi_i \circ \varphi_j^{-1} : \varphi_j(U_i \cap U_j) \to \varphi_i(U_i \cap U_j)$$

  are smooth functions $\forall i, j\varepsilon I$.

  A *surface* is a metrisable topological manifold S together with a smooth atlas.

Note: By defining smoothness of atlas for S , we can also define smoothness of functions on S.

For a function $\varphi : S \to \mathbb{R}$, the *function seen in the chart* is the composition

$$\varphi \circ \varphi_i^{-1} : \varphi_i(U_i) \to \mathbb{R}$$

A function φ is smooth if and only if the function seen in the chart is smooth.

- Let S be a surface with smooth atlas $\{(U_i, \varphi_i) | i \varepsilon I\}$. The atlas is called **oriented** if the Jacobians

$$Jac(\varphi_i, \varphi_j) := det(D(\varphi_i \circ \varphi_j^{-1}))$$

are positive for any $i, j \varepsilon I$. (Here, $D(\varphi_i \circ \varphi_j^{-1})$ is the Jacobian matrix of $\varphi_i \circ \varphi_j^{-1}$.)

A surface is called *oriented* if its atlas is oriented.

- **Surface with Boundary:**

Let

$$H^+ := \{(x,y) \in \mathbb{R}^2 | y \geq 0\}$$

be the closed upper half plane. It has a boundary

$$\partial H^+ := \{(x,y) \in \mathbb{R}^2 | y = 0\}$$

when considered as a subset of $\mathbb{R}^2$.

For a 2-dimensional metrisable(Hausdorff) topological space S, a two-dimensional *chart with boundary* is a pair (U, φ) where $U \subset S$ is an open subset and φ is a map from U to H^+ , where $V = \varphi(U)$ is open in H^+ and φ is a homeomorphism onto its image.

The map φ is called the *coordinate* of the chart.

We define the boundary of U as follows:

$$\partial U := \varphi^{-1}(\varphi(U) \cap \partial H^{+}) \subset U$$

A collection of charts with boundary $\{(U_i, \varphi_i) | i \varepsilon I\}$ is called an *atlas* for S if $S = \bigcup U_i$. An atlas is called *smooth* or C^{∞} if the coordinate change maps

$$\varphi_i \circ \varphi_j^{-1} : \varphi_j(U_i \cap U_j) \to \varphi_i(U_i \cap U_j)$$

are smooth functions from H^{+} to H^{+} $\forall i, j \varepsilon I$. [A function $f : U \to V$ for U, V open subsets of H^{+}, is called smooth if $\exists$ an open subset $\tilde{U}$ of $\mathbb{R}^2$ with $\tilde{U} \cap H^{+} = U$ and a smooth function $\widetilde{f} : \tilde{U} \to \mathbb{R}^2$ s.t. $f = \widetilde{f}|_U$.]

A *surface with boundary* is a metrisable topological manifold S together with a smooth atlas of charts with boundary.

Lemma 3: Let S be a surface with boundary and $(U_1, \varphi_1), (U_2, \varphi_2)$ be charts with boundary of S. Suppose $x \in \partial U_1 \cap U_2$ be a boundary point of U_1. Then $x \in \partial U_2$ is also a boundary point of U_2.

Proof: Since $x \in \partial U_1 = \varphi_1^{-1}(\varphi_1(U_1) \cap \partial H^{+})$,

$\Rightarrow x \in \varphi_1^{-1}(\partial H^{+})$

Also, we have the smooth coordinate change map $\varphi_2 \circ \varphi_1^{-1} : \varphi_1(U_1 \cap U_2) \to \varphi_2(U_1 \cap U_2)$.

So, we see that $\varphi_2(x) \in \partial H^{+}$ i.e. $x \in \varphi_2^{-1}(\partial H^{+})$

And since $x \in U_2$ also,

$\Rightarrow x \in \varphi_2^{-1}(\varphi_2(U_2) \cap \partial H^+) = \partial U_2$.

$\square$

- **Definition:** Let S be a surface with boundary. A point $x \in S$ is called a boundary point of S if $x \in \partial U$ for some (and from above lemma, any) chart (U, φ) containing it. The set of boundary points of S is denoted by ∂S .

Lemma 4 [COLLAR LEMMA] : Let S be a surface with boundary ∂S and $c \subset \partial S$ a connected component. Then $\exists$ a neighbourhood U (called collar neighbourhood) of c in S and a diffeomorphism $\psi : U \to V$ onto a subset $V \subset \mathbb{R}^2$ of the form $V \cong c \times [0,1)$ mapping c onto $c \times \{0\}$.

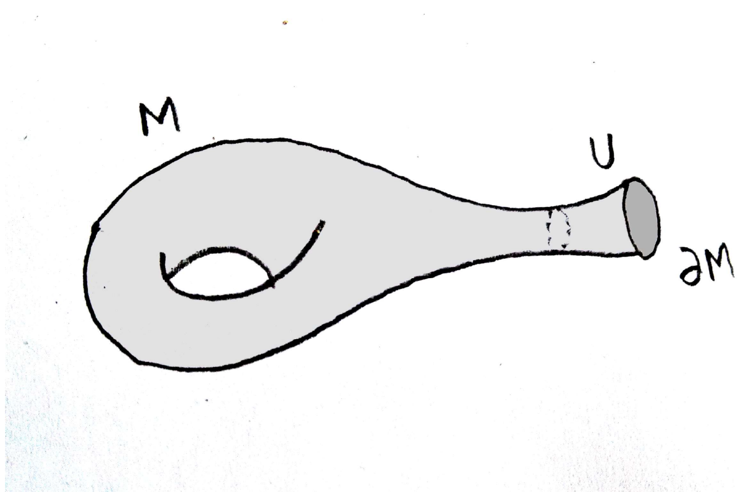

Figure 1: A Collar Neighbourhood

0.4 Gluing Surfaces

Let S_1 and S_2 be two surfaces with boundary ∂S_1 and ∂S_2 respectively. Then we can glue S_1 and S_2 together, along their collar neighboourhood (as shown in the diagram below) as follows:

Let c_1 and c_2 be collar neighbourhoods of $S_1$and S_2 resp. and let $\varphi : c_1 \to c_2$ be a diffeomorphism. Now, on the disjoint union $S_1 \sqcup S_2$, we define the equivalence relation $\sim$ by:

$$x \sim y \iff y = \varphi(x)$$

for $x \in c_1, y \in c_2$. Now, we define a topological space $S_1 \cup_\varphi S_2$ (called the *gluing* of S_1 and S_2 along φ) as the quotient of $S_1 \sqcup S_2$ by the equivalence relation $\sim$.

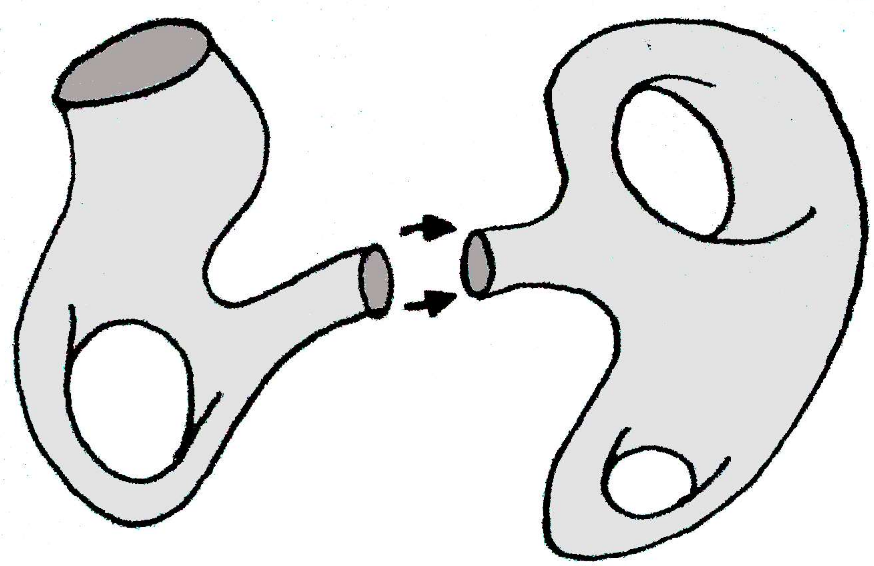

Figure 2: Gluing Two Surfaces

Remark: The space $S_1 \cup_\varphi S_2$ is metrisable.

We now construct the atlas for $S_1 \cup_\varphi S_2$ by first dividing $S_1 \cup_\varphi S_2$ into four components $S_1 \backslash c_1$, $S_2 \backslash c_2$, the complement of gluing curve in $S_1 \cup_\varphi S_2$, and the union of collar neighbourhoods of S_1 and S_2 ;

1. For $S_1 \backslash c_1$ and $S_2 \backslash c_2$, we take smooth atlas $\{(U_i, \varphi_i) | i \in I\}$ and $\{(U_j, \varphi_j) | j \in J\}$ respectively.

2. Let $i_1 : S_1 \to S_1 \cup_\varphi S_2$, $i_2 : S_2 \to S_1 \cup_\varphi S_2$ be the canonical inclusions. Then,

$$\{(i_1(U_i), \varphi_i \circ i_1^{-1}) | i \in I\} \cup \{i_2(U_j), \varphi_j \circ i_2^{-1}) | j \in J\}$$

   will be the atlas for the complement of gluing curve in $S_1 \cup_\varphi S_2$.

3. Let U_1, U_2 be collar neighbourhoods of c_1, c_2 , ψ_1 a diffeomorphism from U_1 onto $c_1 \times$(-1,0] (mapping c_1 to $c_1 \times \{0\}$) and ψ_2 a diffeomorphism from U_2 onto $c_2 \times$[0,1) (mapping c_2 to $c_2 \times \{0\}$) . Define an open subset $O := U_1 \cup U_2$ of $S_1 \cup_\varphi S_2$ and an embedding $i : c_2 \times (-1, 1) \to \mathbb{R}^2$. Then we have a chart along with the coordinates:

$$\psi : O \to \mathbb{R}^2$$

   given by:

$$\begin{cases} x \longmapsto i \circ (\varphi, id) \circ \psi_1(x), & x \in U_1 \\ x \longmapsto i \circ \psi_2(x), & x \in U_2 \end{cases}$$

Then, we have the following proposition:

Proposition 5: $S_1 \cup_\varphi S_2$ is a surface with boundary with smooth atlas given by

$$\{(i_1(U_i), \varphi_i \circ i_1^{-1})|i \in I\} \cup \{i_2(U_j), \varphi_j \circ i_2^{-1})|j \in J\} \cup \{O, \psi\}$$

Proof: We need to prove that the coordinates $\varphi_i \circ i_1^{-1}$, $\varphi_j \circ i_2^{-1}$ and ψ are indeed smooth.

1. Since the maps φ_i, φ_j, i_1, i_2 are smooth maps,

   Hence, $\varphi_i \circ i_1^{-1}$, $\varphi_j \circ i_2^{-1}$ are also smooth.

2. Similarly, since i , φ , ψ_1 , ψ_2 are smooth,

   Hence ψ is also smooth.

   Thus, we have a smooth atlas for $S_1 \cup_\varphi S_2$.

□

We can also see that the orientation of $S_1 \cup_\varphi S_2$ can be induced from the orientations of S_1 and S_2.

Proposition 6: Let S_1 and S_2 be oriented surfaces with boundary and c_1, c_2 be connected components of the respective boundaries. Then c_1, c_2 carry an induced orientation. If $\varphi : c_1 \to c_2$ is an orientation-reversing diffeomorphism, then $\exists!$ orientation of $S_1 \cup_\varphi S_2$ compatible with the orientations of S_1 and S_2 .

Chapter 2: Graph Theory

0.5 Graphs

Definitions:

- An *oriented* graph Γ is a pair of sets (V,E) , where V is a finite non-empty set of elements called *vertices* and E is a finite set of elements called *edges*(each of which has two associated vertices), along with a map

  $$\varphi : E \longrightarrow V \times V$$

  $$e \longmapsto (e_-, e_+)$$

  where the vertex e_- is called the *origin* of e and e_+ is called *terminus* of e .

- A *Graph* is a pair (Γ, I), where $\Gamma = (V, E, \varphi)$ is an oriented graph and

  $$I : E \longrightarrow E$$

  $$e \longmapsto \overline{e}$$

is an involution on E satisfying

$$\overline{e}_+ = e_-, \overline{e}_- = e_+.$$

The pair $(e, \overline{e})$ is called the *geometric edge* of (Γ, I) .

- The *geometric realisation* $\mid \Gamma \mid_I$ of a graph (Γ, I) is the topological space

$$\frac{\mid \Gamma \mid_I = E \times [0,1]}{\sim}$$

where $\sim$ is the equivalence relation given by following relations:

1. $(e,t) \sim (\overline{e}, 1-t)$.
2. If $e, f \in E$ with $e_- = f_-$ then $(e,0) \sim (f,0)$.
3. If $e, f \in E$ with $e_+ = f_+$ then $(e,1) \sim (f,1)$.

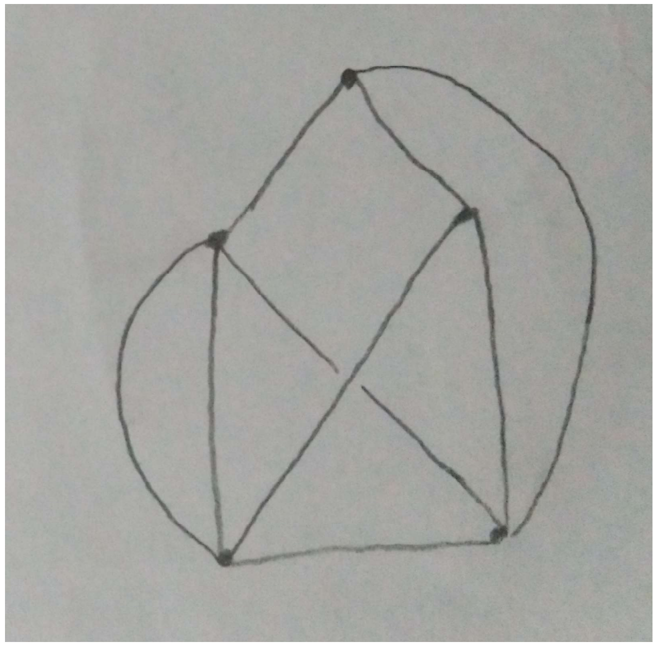

Figure 3: The geometric realisation of a graph.

- **Cyclic ordering on a finite set S** is a bijection $s : S \to S$ s.t. for any $x \in S$ the orbit $\{s^n(x)\}$ is S . For $x \in S$, we define $s(x)$ the successor of x and $s^{-1}(x)$ the predecessor of x.

We will need this notion of cyclic ordering on graphs to define some extra structure.

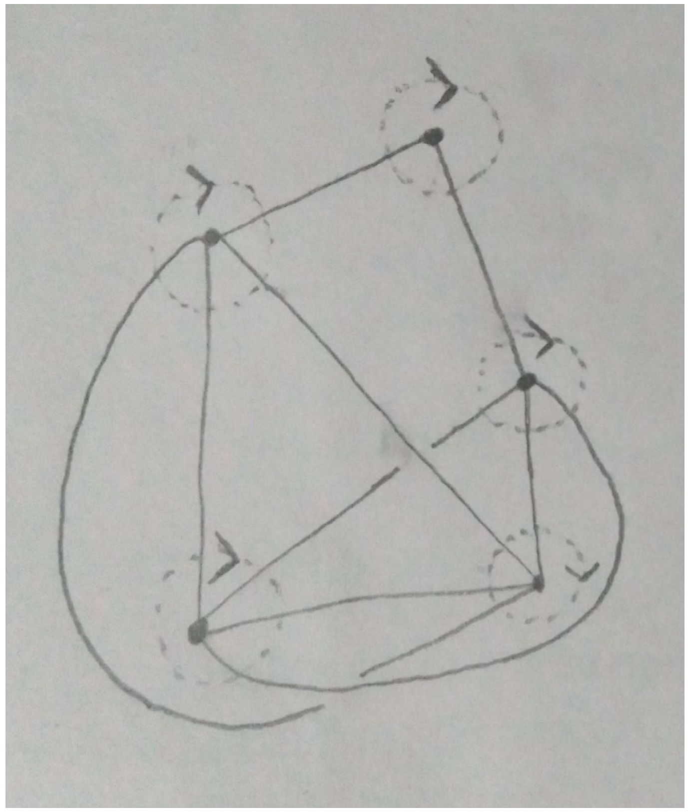

Figure 4: Cyclic ordering of edges in a planar graph

0.6 Ribbon Graphs

Definition:

- Let (Γ, I) be a graph. For $v \in V$, the *star* of v is

$$E_v := \{e \in E \mid e_- = v\},$$

the set of vertices starting from v . A **Ribbon Graph** is the data given by a graph and a cyclic ordering

$$s_v : E_v \longrightarrow E_v$$

on the star of every vertex.

Remark: Any planar graph is a ribbon graph.

Justification: Given any embedding of a planar graph[1] into $\mathbb{R}^2$, the orientation of $\mathbb{R}^2$ induces a cyclic ordering on the star of each vertex as follows: Consider a circle around each vertex intersecting each edge in the star of given vertex only once. Then the orientation of circle defines the ordering on the graph.

We can generalise the above remark as the following proposition:

Proposition 7: Any embedding of a graph into an oriented surface gives rise to a cyclic ordering on each of the sets E_v for each vertex $v \in V$.

Now, we shall use this notion of ribbon graphs for the construction of oriented surfaces. First, we embed a ribbon graph into an open oriented surface.

Lemma 8: Every ribbon graph can be embedded into an oriented surface such that its cyclic orderings are induced from the orientation of the surface.

PROOF: We construct a surface S by constructing neighbourhoods around the star of each vertex as follows:

For every vertex $v \in V$ define subset $U_v \subset \mathbb{R}^2$ with $| E_v |$ boundary components labelled by the elements of E_v in the following way: Start with an arbitrary edge

[1] A planar graph is the projection of the geometric realisation of the graph.

$e \in E_v$ and the next boundary component in the counterclockwise sense is labelled by $s_v(e)$ and so on. [As shown in figure 5] Similarly, around each edge e , we construct a strip V_e whose two boundary components (as given in figure 6) are labeled by $e_-$and e_+.

Now, for each $e \in E$, we glue the connected component of $\partial\{V_e\}$ labelled by v with the connected component of $\partial\{U_v\}$ labelled by e (if v is the origin of e) or $\overline{e}$ (if v is the terminus). Now, from Proposition 6, we obtain an oriented surface which will be compatible with s_v since the glueing process preserves the orientation.

$\square$

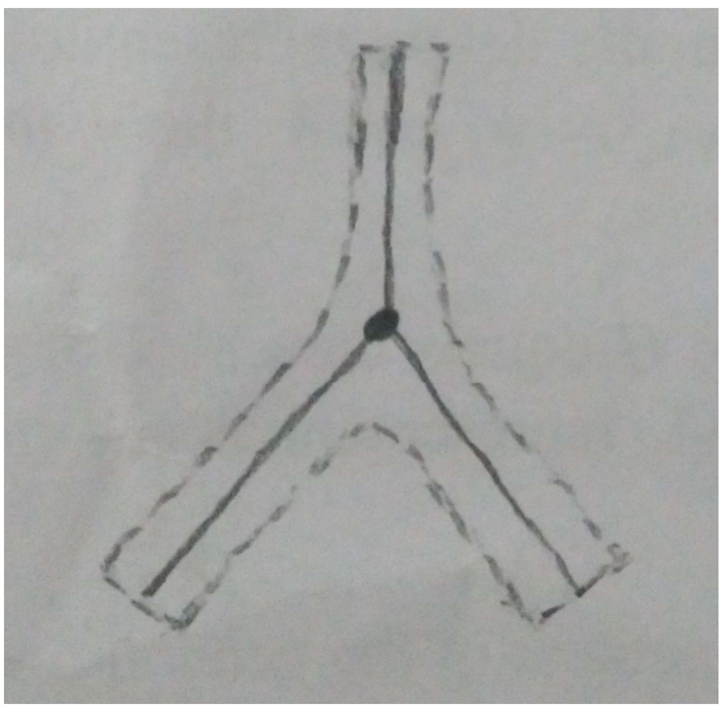

Figure 5: A surface associated with a vertex.

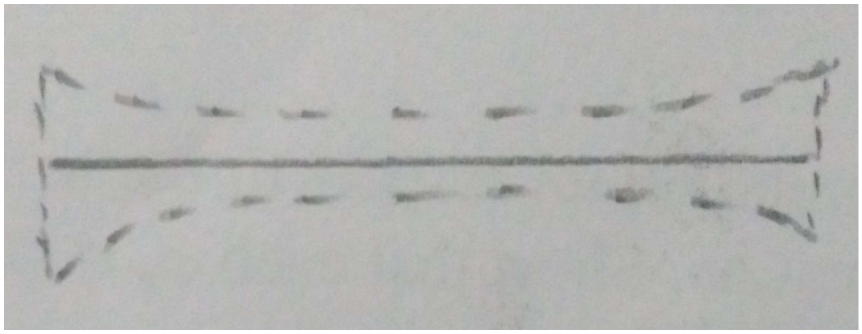

Figure 6: A surface associated with an edge.

The *open* oriented surface constructed in the above-mentioned way is called the *Associated* Ribbon Surface of that graph.

Now we need to construct closed oriented surfaces. For this, we need to "close the *holes*" in the open ribbon surface by gluing the discs inside the holes. A disc can be constructed as follows: Start with an arbitrary edge e and follow this edge to the next vertex. Then we take the successor of $\overline{e}$. We repeat this procedure untill we get a closed curve. The curve constructed this way will be the boundary of a disc in the surface we want to construct.

We formalise this procedure using this definition:

Definition: Let Γ be a ribbon graph. A face is an equivalence class (upto cyclic permutation) of n-tuples $(e_1, \dots, e_n)$ of edges such that $(e_p)_+ = (e_{p+1})_-$ and $s_{(e_p)_+}(\overline{e}_p) = e_{p+1}$ $\forall$ $1 \leq p \leq n$, where the addition is modulo n .
Below are given few examples of graphs with different number of faces.

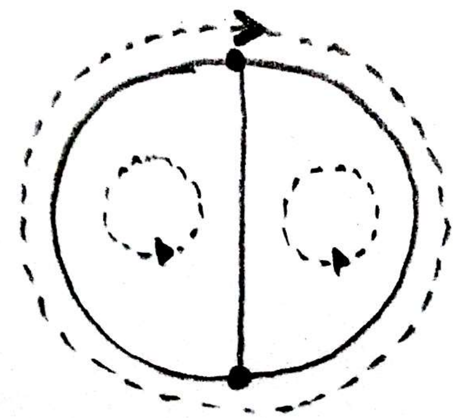

Figure 7: A graph with three faces.

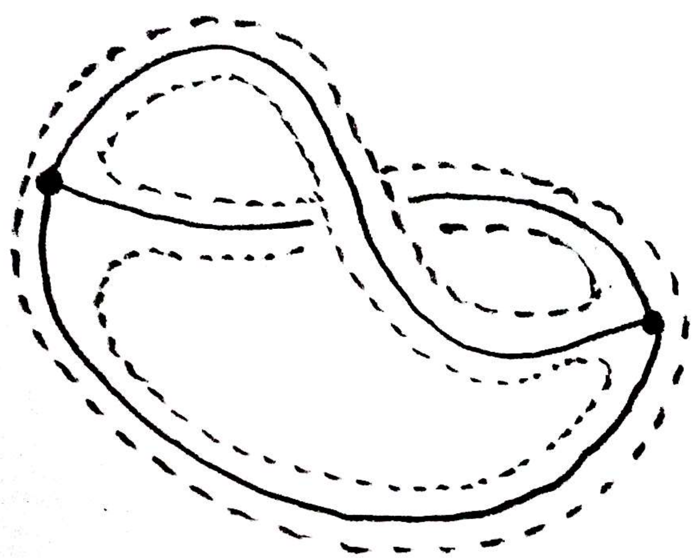

Figure 8: A graph with one face

Note that every oriented edge is contained in a unique face. Now, on each face, we can construct the boundary of the disc in the way described above. Consequently, we can fill the holes with the discs.

Definition: A graph Γ embedded in a surface S is said to be *filling* if each connected component of $S \setminus \mid \Gamma \mid$ is diffeomorphic to the disc. i.e.

$$S \setminus \mid \Gamma \mid = \sqcup D_f.$$

Then, we have the following proposition:

Proposition 9: Every ribbon graph Γ has a filling embedding into a compact oriented surface S and the connected components of $S \setminus \mid \Gamma \mid$ are in bijection with the faces of Γ.

Proof: Each face in the ribbon graph defines a circle in the ribbon surface. So, by gluing a disc for each of these faces, we obtain a closed surface.

$\square$

In fact, the surface obtained in the above proposition is unique (up to a homeomorphism). We prove this statement using the following lemma:

Lemma 10: [CLUCTHING LEMMA] : Let $X = U \cup V$ be a decomposition of a topological space X into closed subsets U, V. If $f_1 : U \to Y$ and $f_2 : V \to Y$ are continuous maps, where Y is some topological space, with $f_1 \mid_{U \cap V} = f_2 \mid_{U \cap V}$ then the induced map $f : X \to Y$ is also continuous.

Proof: The proof is elementary as the induced map f is:

$$f(x) = \begin{cases} f_1(x), & x \in U \\ f_2(x) & x \in V \end{cases}.$$

and since for $x \in U \cap V$, $f_1(x) = f_2(x)$,

$\Rightarrow f$ is continuous.

$\square$

Now, we have the following weak uniqueness result:

Proposition 11: Let $\Gamma \subset S, \Gamma' \subset S'$ be filling ribbon graphs of compact oriented surfaces and let $\varphi : \Gamma \to \Gamma'$ be an isomorphism of ribbon graphs. Then φ induces an orientation-preserving homeomorphism $\varphi : \mid \Gamma \mid \to \mid \Gamma' \mid$ of geometric realisations which extends to a homeomorphism $S \to S'$.

PROOF: Since any isomorphism of graphs sends edges to edges and vertices to vertices, the geometric realisation of the graph (which is just the equivalence class of the space $E \times [0,1]$ defined by relations as given in section 0.5) will be sent homeomorphically to the geometric realisation of the isomorphic graph in such a way that the orientation is preserved. Now, let

$$S \setminus \mid \Gamma \mid = \sqcup D_f$$

$$S' \setminus \mid \Gamma' \mid = \sqcup D'_f.$$

and let S_Γ and $S_{\Gamma'}$ be the ribbon surfaces associated with $\mid \Gamma \mid$ and $\mid \Gamma' \mid$. Then,

$$S = \overline{S_\Gamma} \cup (\sqcup \overline{d_f}),$$

where d_f is a disc slightly smaller than D_f and

$$\overline{S_\Gamma} \cap (\sqcup \overline{d_f}) = \sqcup \partial \{\overline{d_f}\}$$

is a union of circles. Similarly, for S',

$$S' = \overline{S_{\Gamma'}} \cup (\sqcup \overline{d'_f}),$$

and

$$\overline{S_{\Gamma'}} \cap (\sqcup \overline{d'_f}) = \sqcup \partial \{\overline{d'_f}\}.$$

Thus, by Lemma 10, it is enough to show that for each face f, $\exists$ a homeomorphism $\overline{d_f} \to \overline{d'_f}$ which agrees with the extension of φ to S_Γ on the boundary $\partial\{\overline{d_f}\}$:

Let ψ be any homeomorphism of circles. Then it can be extended to corresponding discs in the following manner:

For $x \in D$ an element of disc,

$$x = re^{i\theta}$$

in polar coordinates, for $r \in [0,1]$ and $e^{i\theta}$ a point on the circle. Then we can define

$$\varphi(re^{i\theta}) := r\psi(e^{i\theta})$$

to obtain the desired homeomorphism of surfaces.

□

Using Proposition 9 and 11, we get the desired uniqueness result:

Corollary 12: For any ribbon graph Γ, $\exists$ a unique (upto homeomorphism) compact oriented surface S_Γ such that Γ can be embedded as a filling graph into S_Γ.

The converse of this corollary is also true:

Proposition 13: Every compact oriented surface admits a filling ribbon graph.

However, the proof of this proposition is not straight-forward and requires some basic results from Riemannian Geometry and Morse Theory which we will not discuss here.

In the next section, we state and prove the main result of our discussion which classifies every compact oriented surface.

Chapter 3: Classification Theorem

0.7 Graph with g petals

First, we consider the graph Γ_1 shown in figure 9 and glue together g copies of it. The resultant graph will be called the *graph with g petals* , denoted by Γ_g.

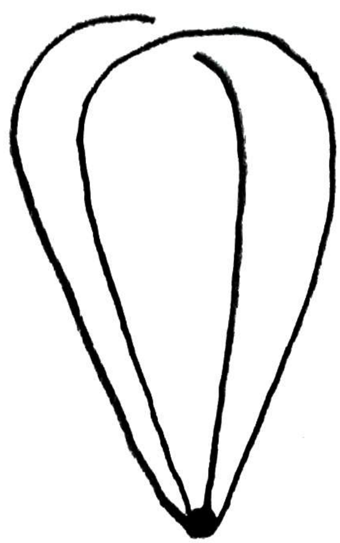

Figure 9: The graph Γ_1.

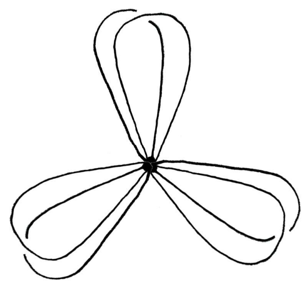

Figure 10: The graph Γ_g for $g = 3$.

Observation:

- The graph Γ_1 has one face and the associated surface $S_1 := S_{\Gamma_1}$ is a torus.
- $S_g := S_{\Gamma_g}$ is the surface of a handlebody with g handles. [Each copy of Γ_1 in Γ_g corresponds to a torus with a hole, and g copies are glued together by gluing the consecutive holes.]

The surface S_3 is shown in figure 11.

There is another description of S_g too: [As shown in figure 12]; Let D_g be the disc in S_g corresponding to the unique face of Γ_g. Then, we obtain S_g by gluing the boundary of D_g. Each oriented edge of Γ_g occurs exactly once in the boundary of D_g; Let a_i, b_i be the two edges of the i-th copy of Γ_1 in Γ_g. Then the boundary of D_g is given by the series of edges

$$a_1, b_1, a_1^{-1}, b_1^{-1}, \dots, a_g, b_g, a_g^{-1}, b_g^{-1}.$$

For convenience we define $S_0 := S^2$, the two-sphere. Now, we state the statement of the classification of surfaces:

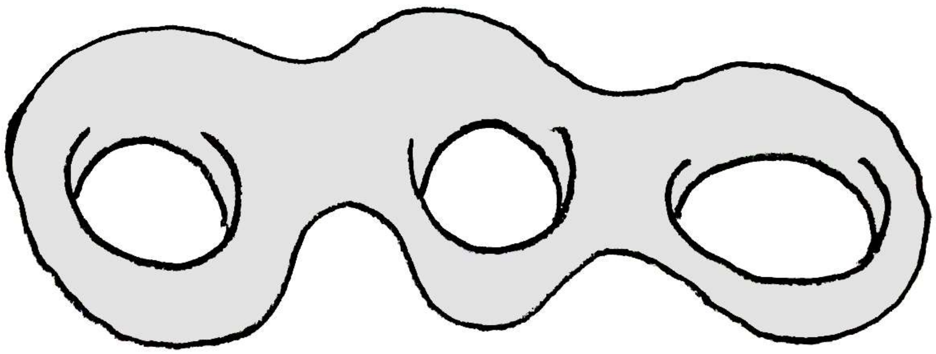

Figure 11: The surface S_3 .

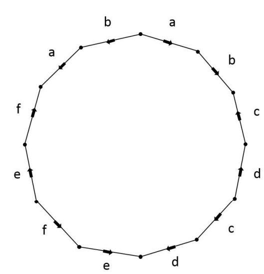

Figure 12: Constructing S_3 by gluing a disc.

Theorem 14: Every compact oriented surface S is homeomorphic to one of the surfaces S_g for $g \geq 0$.

PROOF: We know from Proposition 13 that every compact oriented surface admits a filling ribbon graph. Let $\Gamma = (V, E, \varphi)$ be the filling ribbon graph for S .

If the graph has no edges, then S is the 2-sphere S^2. So, we assume for the rest of the proof that Γ has at least one edge. Since (from Corollary 12) the filling graph is unique for a surface, it is now enough to transform Γ into Γ_g, for some $g \geq 0$, without changing its filling property. Let us assume that Γ has more than one face. We reduce the number of faces without changing its filling property as follows:

Consider the geometric edge $(e, \overline{e})$ such that e and $\overline{e}$ are in different faces. Then, we delete the edge e and $\overline{e}$ (as shown in fig. 13) from Γ and obtain a new filling ribbon graph Γ' for S . Continuing in this manner, we can get a filling ribbon graph with only one face. Thus, WLOG, we assume that Γ has only one face.

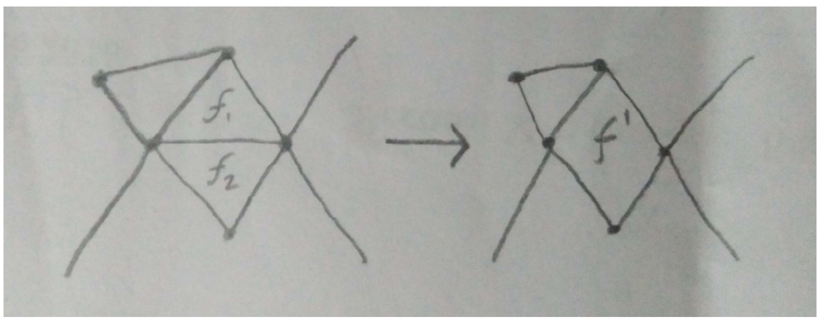

Figure 13: Reducing the no. of faces.

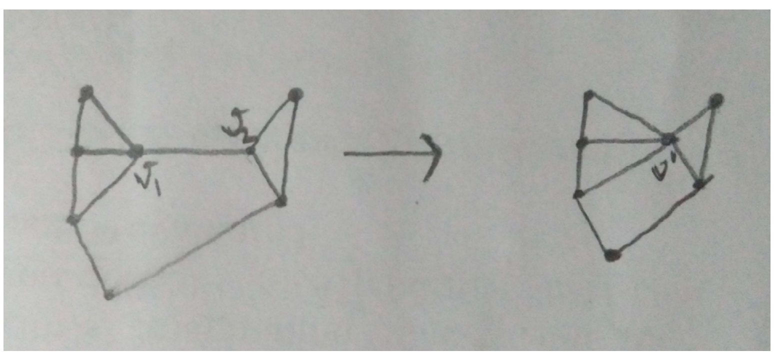

Figure 14: Reducing the no. of vertices.

Now, we reduce the number of vertices. Since Γ has at least one edge, the number of vertices will be at least two. If Γ has more than two vertices, then $\exists$ an edge joining two different vertices e_- and e_+. Now, we construct a new graph $\Gamma' = (V', E', \varphi')$, where

- $V' = (V \setminus \{e_+, e_-\}) \cup \{e_0\}$: We merge the two vertices e_- and e_+ into a new vertex e_0 ,
- $E' = E \setminus \{e, \overline{e}\}$,
- φ' sends e_0 to an arbitrary point in the geometric edge $\varphi(e)$ and extend it to the edges which previously originated from e_- or e_+.

This process reduces the number of vertices by one without increasing the number of faces. Hence, now we can also assume WLOG that $\mathit{\Gamma}$ has only two vertices and one face, which means that S is obtained by gluing the boundary of a polygon. The sides of polygon are labelled by edges of $\mathit{\Gamma}$ and the gluing is given by identifying e and $\overline{e}$ with reversed orientation.

Definition: A pair of geometric edges $\big((a,\overline{a}),(b,\overline{b})\big)$ are said to be *linked* if their relative position is as shown in figure 15.

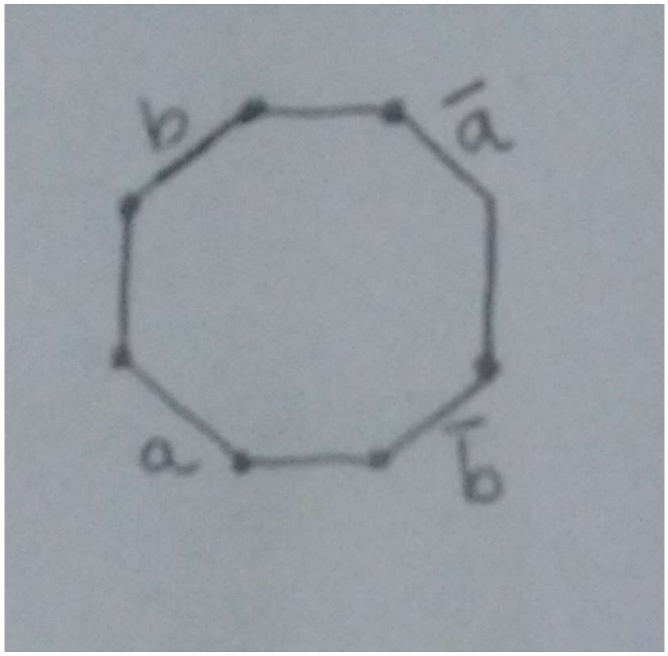

Figure 15: Linked Edges

Claim 1: Any geometric edge is linked to at least one other geometric edge.

Proof: If a geometric edge is not linked to any other geometric edge, then it would produce an extra face which contradicts our assumption that Γ has only one face.

Claim 2: Let $\big((a,\overline{a}),(b,\overline{b})\big)$ be a *linked* pair of geometric edges. Then there is a way to rearrange the labelling of the polygon without changing the resulting space such that:

- $a, b, \overline{a}, \overline{b}$ appears as a subsequence,
- and no subsequence of type $c, d, \overline{c}, \overline{d}$ is destroyed during this process.

  Proof: We formulate a process of adding and deleting certain edges of the graph (as shown in figure 16). First, we add an edge which is shown dotted in the figure. It divides the graph in two faces (shown in different colors in the picture). Now, we erase the green colored edge in the graph. In the polygon, this step glues together the two green sides into the red one.

Repeating this procedure two more times, we obtain the desired surface in which all the edges lie on the boundary and no subsequence of the form $c, d, \overline{c}, \overline{d}$ is destroyed. [This new surface is obviously homeomorphic to the original one.]

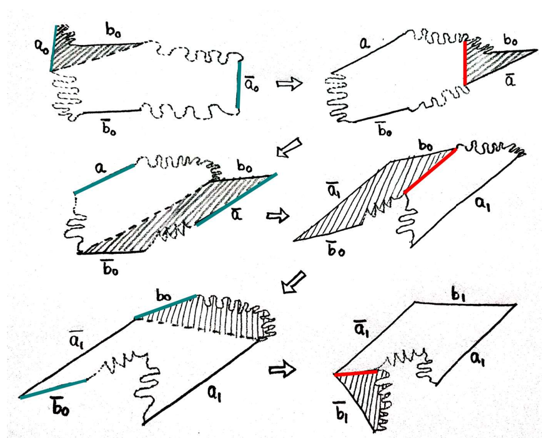

Figure 16: Moving linked edges into generic position.

This claim enables us to relabel the edges of the polygon such that the resultant graph will be Γ_g for some $g \geq 0$. Hence, the corresponding ribbon surface will be S_g for some $g \geq 0$.

Now, from Corollary 12, we get the result that the surface S is homeomorphic to S_g for some $g \geq 0$.

□

Bibliography

[1] Francois Labourie, Lectures on Respresentation of Surface Groups, Notes of a course given in ETH-Zurich Fall 2006, and Orsay Spring 2007-2008, August 2012.

[2] John M. Lee, Introduction to Smooth Manifolds, 1950, SPRINGER.

[3] Lowell W. Beineke, Robin J. Wilson, Topics in Topologocal Graph Theory, 2009, CAMBRIDGE UNIVERSITY PRESS.

[4] James R. Munkres, Topology, 2000, PEARSON EDUCATION, Inc..