Martingales and Stochastic Calculus

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Certificate of Examination

This is to certify that the dissertation titled **Martingales and Stochastic Calculus** submitted by **Neha Sharma** (Reg. No. MS11014) for the partial fulfillment of BS-MS dual degree program of the Institute, has been examined by the thesis committee duly appointed by the Institute. The committee finds the work done by the candidate satisfactory and recommends that the report be accepted.

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Declaration

The work presented in this dissertation has been carried out by me under the guidance of Dr. Lingaraj Sahu at the Indian Institute of Science Education and Research Mohali. I have also taken help from Prof. Abhay G. Bhatt at Indian Statistical Institute, Delhi.

This work has not been submitted in part or in full for a degree, a diploma, or a fellowship to any other university or institute. Whenever contributions of others are involved, every effort is made to indicate this clearly, with due acknowledgement of collaborative research and discussions. This thesis is a bonafide record of work done by me and all sources listed within have been detailed in the bibliography.

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In my capacity as the supervisor of the candidates project work, I certify that the above statements by the candidate are true to the best of my knowledge.

Dr. Lingaraj Sahu
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Neha Sharma

Notation

$\mathbb{I}$	Identity Matrix(in appropriate dimensions)
$\mathbb{C}^n$	Complex Hilbert space of dimension n
$\mathcal{M}_2^c$	Space of square integrable continuous martingale
$\mathcal{M}_2$	Space of square integrable martingale
$\mathcal{M}^{c,loc}$	Space of continuous local martingale
$\langle M \rangle_t$	Angle bracket process defined in doob's decomposition for square of a martingale M
$[X]_T^2$	$E\int_0^T X_t^2 d\langle M \rangle_t$
$\mathcal{L}(M)$	measurable with $[X]_T < \infty$
$\mathcal{L}^*(M)$	Progressively measurable with $[X]_T < \infty$
$\mathcal{L}(L^2[0,T],\Omega)$	Random variables adapted and $\int_0^T E(\|X\|^2)dt < \infty$

Abstract

Martingales are stochastic processes which model the 'fair game', i.e., these are the processes where the expected value of the next term is equal to present observed term given that we have the knowledge of all past terms. The aim of the project is to understand this special class of stochastic processes with the continuous parameter time. Martingales are processes which have unbounded first variation. Due to this we cannot define the integration of a process with respect to martingales in the Lebesgue-Steiltjes sense. However, they have a bounded second variation. Using this we can show that integral of simple processes converge to the stochastic integration in $\mathcal{L}^2$ sense and this is how we define the stochastic integral with respect to continuous martingales. The construction of stochastic integral with respect to martingales has been carried out rigorously. Further I have discussed the change of variable formula (Ito's rule) which is important to understand the calculus of stochastic processes. Also in the end, there is a discussion on the existence and uniqueness of SDEs and under what conditions we can have a weak and strong solutions to the SDE with the given coefficients.

Contents

Chapter 1

Martingales, Stopping Times and Brownian Motion

The first chapter is devoted towards understanding a class of stochastic processes which are called Martingales. Martingales are processes in which expectation at any time t is equal. Brownian motion is an important

1.1 Stochastic Processes

Definition 1.1. ***Stochastic Process*** *is a collection of random variables* $X = \{X_t; 0 \leq t < \infty\}$ *on* $(\Omega, \mathcal{F})$*, taking values on the second measurable space* (S, ρ)*, which is called the state space.*

The index $t \in [0, \infty)$ of X_t is interpreted as the time index. Also for a fixed sample point $\omega \in \Omega$, the function $t \mapsto X_t(\omega)$ is the sample path of X.

Let X and Y be two stochastic processes defined on the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$. As functions of t and ω, the concept of being equal can be categorized as the following:

- Y is a modification of X if $\mathbb{P}[X_t = Y_t] = 1$ for every $t \geq 0$.
- X and Y are indistinguishable if almost all their sample paths agree, i.e.,

$$P[X_t = Y_t;\ \forall 0 \leq t < \infty] = 1.$$

Example Let T be a positive random variable with continuous distribution, and let $X_t = 0$, $Y_t = 0$ on the set $[t \neq T]$ and $Y_t = 1$ on the set $[t = T]$. Then Y is a modification of X, since for every $t \geq 0$, the probability measure of $X_t = Y_t$ is the measure of the set $\mathbb{P}[T \neq t]$ which is equal to 1, but on the other hand we have $\mathbb{P}[Y_t = X_t; \forall\ t \geq 0] = 0$.

Definition 1.2. ***Filtrations*** *are a non decreasing family of sub σ-fields of $\mathcal{F}$ such that we have $\mathcal{F}_s \subseteq \mathcal{F}_t \subseteq \mathcal{F}$ for $0 \le s < t < \infty$. We also set $\mathcal{F}_\infty = \sigma((\bigcup_{t\ge 0})\mathcal{F}_t)$.*

Definition 1.3. *X is said to be* ***adapted*** *to the filtration $\mathcal{F}_t$ if for each $t \ge 0$, X_t is $\mathcal{F}_t$ measurable.*

Definition 1.4. *X is* ***measurable*** *if, for every $A \in \mathfrak{B}(\mathbb{R}^d)$, the set $\{(t,\omega); X_t(\omega) \in A\}$ belongs to the product σ-field $\mathfrak{B}([0,\infty]) \otimes \mathcal{F}$.*

Definition 1.5. *X is* ***progressively measurable*** *if, for each $t \ge 0$ and $A \in \mathfrak{B}(\mathbb{R}^d)$, the set $\{(s,\omega); 0 \le s \le t, \omega \in \Omega, X_s(\omega) \in A\}$ belongs to the product σ-field $\mathfrak{B}([0,t]) \otimes \mathcal{F}_t$.*

1.2 Stopping Time

The parameter t is interpreted as time and the σ-field $\mathcal{F}_t$ associated is the information accumulated upto that time t. To study natural phenomenon like an earthquake of above a certain richter scale or number of customers exceeding the safety requirements, one needs to study the instant $T(\omega)$ at which the phenomenon occurs for the first time, where T is an $\mathcal{F}$ measurable function. Therefore the event $\{\omega; T(\omega) \le t\}$ is part of the accumulated information by time t.

Definition 1.6. *Consider a measurable space $(\Omega, \mathcal{F})$ equipped with a filtration $\{\mathcal{F}_t\}$. A random time is a* ***stopping time*** *of the filtration, if the event $\{T \le t\}$ belongs to the σ-field $\mathcal{F}_t$ for every $t \ge 0$. A random time is an* ***optional time****, if the event $\{T < t\}$ belongs to the σ-field $\mathcal{F}_t$, for every $t \ge 0$.*

Theorem 1. *Every random time equal to nonnegative constant is a stopping time. Every stopping time is optional, and the two concepts are equal if the filtration is right-continuous.*

Proof If $T = C$, where C is some constant, then the set $\{\omega \in \Omega; T(\omega) = C\}$ has measure 1 and so it follows trivially. To prove the second statement we observe that $\{T < t\} = \bigcup_{n=1}^{\infty} \{T \le t - (1/n) \in \mathcal{F}_t\}$, as T is a stopping time. To show that the two concepts are equal if the filtration is right-continuous, we can write $\{T \le t\} = \bigcap_{n=r}^{\infty} \{T < t+(1/n)\}$ and conclude that $\{T \le t\} \in \mathcal{F}_{t+(1/r)}$, for every positive integer r, which proves that $\{T \le t\} \in \mathcal{F}_{t+} = \mathcal{F}_t$.

Lemma 1.1. *If T and S are stopping times, then so are $T \vee S$, $T \wedge S$ and $T + S$.*

Proof The first two follow trivially. In the third one, we decompose as;

$$\{T + S > t\} = \{T = 0, S > t\} \cup \{0 < T < t, T + S > t\} \cup \{T > 0, S = 0\} \cup \{T \ge t, S > 0\}.$$

Here first, third and fourth are in $\mathcal{F}_t$ and second can be written as:

$$\bigcup_{r\in\mathbb{Q};0<r<t} \{t > T > r, S > t - r\}.$$

Theorem 2. *Let X be a progressively measurable process, and T be the stopping time of the filtration $\{\mathcal{F}\}_t$. Then the random variable X_T is $\{\mathcal{F}\}_T$ measurable and the stopped process $\{X_{T\wedge t}, \mathcal{F}_t; 0 \leq t < \infty\}$ is progressively measurable.*

Definition 1.7. *A filtration which is right continuous and contains all the $\mathbb{P}$ negligible events in $\mathcal{F}_0$ is said to satisfy the* ***usual conditions****.*

1.3 Martingales

Definition 1.8. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be the probability space with the discrete filtration $\{\mathcal{F}\}_n$. A Process X is called a* ***discrete parameter martingale*** *if the following holds true:*

- *X is adapted relative to $\{\mathcal{F}\}_n\}$,*
- $\mathbb{E}(|X_n|) < \infty$,
- $\mathbb{E}(X_n|\mathcal{F}_{n-1}) = X_{n-1}$.

The above process is a **supermartingale** the first two conditions hold true and the third is replaced by

$$\mathbb{E}(X_n|\mathcal{F}_{n-1}) \leq X_{n-1},$$

and is a **submartingale** if

$$\mathbb{E}(X_n|\mathcal{F}_{n-1}) \geq X_{n-1}.$$

A martingale is both a submartingale and a supermartingale.

Example Let $X_1, X_2, ...,$ be a sequence of independent random variables with,

$$E(X_k) = 0, \forall k.$$

Define $S_0 := 0,\ S_n := X_1 + X_2 \cdots + X_n,\ \mathcal{F}_n := \sigma(X_1 + X_2 \cdots + X_n)$. Then S_n, for $n \geq 1$ is a martingale.

Definition 1.9. *A process $\{X_t, \mathcal{F}_t; 0 \leq t < \infty\}$ is a* ***continuous parameter martingale*** *if it satisfies the following:*

- *X is adapted to the filtration $\{\mathcal{F}_t\}$,*
- $\mathbb{E}(|X_t|) < \infty$,

- $\mathbb{E}(X_t|\mathcal{F}_s) = X_s$ *forevery* $s \leq t$

The inequalities for submartingales and supermartingales in continuous case is the same as that for the discrete case.

Definition 1.10. *On* $(\Omega, \mathcal{F}, \mathbb{P})$*, a random sequence* $\{A_n\}_{n=0}^{\infty}$*, adapted to the filtration is* ***increasing previsible process*** *if for* $\mathbb{P}$ *a.e.* $\omega \in \Omega$*, we have* $0 = A_0(\omega) \leq A_1(\omega) \leq \dots$*, and* $E[A_n] < \infty$ *for* $n \geq 1$ *and* $\{A_n\}$ *is* $\mathcal{F}_{n-1}$ *measurable.*

Definition 1.11. *A continuous adapted process* A *is called* ***increasing process*** *if for* $\mathbb{P}$ *a.e.* $\omega \in \Omega$ *we have:*

- $A_0(\omega) = 0$,
- $t \mapsto A_t(\omega)$ *is a nondecreasing, right continuous integrable function.*

Lemma 1.2. *If* $X = \{X_n, \{\mathcal{F}\}_n, n \geq 0\}$ *is a martingale and* T *is a stopping time, then the stopped process defined as* $X^T := \{X_{T\wedge n}, \{\mathcal{F}\}_n, n \geq 0\}$ *is also a martingale.*

Proof $X_{T\wedge n}$ can be written as:
$X_{T\wedge n} = X_0 + \Sigma_{i=0}^{n}(X_i - X_{i-1})\chi_{i\leq T}$.
We can write the difference of stopped processes as,

$$X_{T\wedge n} - X_{T\wedge n-1} = (X_n - X_{n-1})\chi_{i\leq T}.$$

Taking the expectation of the above term we get the result.

Let X be a martingale and T be a stopping time. We assume that T is finite almost surely, then we can define the random variable $X_T : \Omega \to \mathbb{R}$ by,

$$X_T(\omega) = X_{T(\omega)}(\omega).$$

Intutively we can say that $E(X_T)$ represents the player's expected fortune when he is stopping at time T which is random. If the game is fair, then we should have,

$$E(X_T) = E(X_0).$$

Instead we know that $T \wedge n$ converges to T pointwise almost surely as n goes to infinity. Also we know that $E(X_{T\wedge n}) = E(X_0)$ for all n. If we can show that $E[X_{T\wedge n}] \to E[X_T]$ as $n \to \infty$, we can conclude that the game is fair. However, such a convergence is not guaranteed, so we need some hypotheses under which the convergence exist.

Theorem 3 (Doob's Optional Stopping Theorem)**.** *[Wil91] Let* $X = \{X_n, \{\mathcal{F}\}_n, n \geq 0\}$ *be a martingale and* T *be the stopping time. Suppose that one of the following holds true:*

1. *There is a positive integer K such that we have $T(\omega) \leq K$ for all $\omega \in \Omega$.*

2. *There is a positive integer N such that we have,*

$$|X_n(\omega)| < N$$

   for all n and T is finite almost surely.

3. *$E[T]$ is finite and*

$$|X_n(\omega) - X_{n-1}(\omega)| < N.$$

Then we have X_T an integrable function with,

$$E(X_T) = E(X_0).$$

Proof In all the three cases we have T a.s. a finite random variable. Suppose (1) holds, then for $n > K$ we would have $T \wedge n = T$ for all $\omega \in \Omega$ that implies, $X_{T\wedge n} = X_T$ for all $n > K$ and we have,

$$E(X_0) = X_{T\wedge n} = E(X_0)$$

as $n \to \infty$. Suppose (2) holds, using the boundedness of X_n we can write,

$$|X_{T\wedge n}| < N$$

for all $\omega \in \Omega$ and if (3) holds we get the inequality,

$$|X_{T\wedge n}| \leq |X_0| + NT(\omega).$$

Certainly we have X_0 integrable and $E(NT) = NE(T) < \infty$ by the assumption. Therefore in either cases we have $|X_{T\wedge n}|$ bounded by constant which is integrable, so by using Dominated Convergence Theorem we get the desired result.

Theorem 4 (Submartingale Convergence)**.** *Let $\{X_t, \mathcal{F}_t; 0 \leq t < \infty\}$ be a right continuous submartingale such that $sup_{t\geq 0}E((X_t)^+) < \infty$. Then $lim_{t\to\infty}X_t(\omega)$ exists for almost every $\omega \in \Omega$.*

Theorem 5 (Doob's Decomposition)**.** *Let $X = \{X_n, \mathcal{F}_n; n \in \mathbb{Z}^+\}$ be a stochastic process which is adapted to $\{F\}_n$ and $X_n \in \mathcal{L}^1, \forall n$. Then X can be decomposed as:*

$$X_n = M_n + A_n + X_0 \quad \forall n. \tag{1.1}$$

Here $M = \{M_t, \mathcal{F}_t, t \geq 0\}$ is a martingale and $A = \{A_t, \mathcal{F}_t, t \geq 0\}$ is a previsible process. This decomposition is called as doob's decomposition and it is unique modulo indistinguishability.

Proof Defining $A_n = \sum_{k=1}^{n}\big(E[X_k \,|\, \mathcal{F}_{k-1}] - X_{k-1}\big)$ and $M_n = X_0 + \sum_{k=1}^{n}\big(X_k - \mathbb{E}[X_k \,|\, \mathcal{F}_{k-1}]\big)$ we have $E(M_n - M_{n-1}) = 0$.

For uniqueness, assume that $X_n = M_n + A_n$ and $X_n = \tilde{M}_n + \tilde{A}_n$ are two different decompositions of X. Let us define $Y_n = M_n - M_{n-1} = A_{n-1} - A_n$. Taking conditional expectation we have the desired result.
To see that A_n is increasing when X_n is a submartingale, we write,

$$\begin{aligned} E(X_n - X_{n-1}|\mathcal{F}_{n-1}) &= E(M_n - M_{n-1}|\mathcal{F}_{n-1}) + E(A_{n-1} - A_n|\mathcal{F}_{n-1}) \\ &= 0 + (A_{n-1} - A_n). \end{aligned}$$

Henceforth we have,

$$A_n = \sum_{k=1}^{n} E(X_n - X_{n-1}|\mathcal{F}_{k-1}).$$

Definition 1.12. *Let $\wp(\wp_a)$ be a class of stopping time T satisfying $\mathbb{P}[T < \infty] = 1$ (resptively, $\mathbb{P}[T < a] = 1$, for $a > 0$)). The right continuous process X is said to be of class **D**, if the family $\{X_T\}_{T\in\wp}$ is uniformly integrable and of class **DL** if the the family $\{X_T\}_{T\in\wp_a}$ is uniformly integrable, for every $0 < a < \infty$.*

Theorem 6 (Doob Meyer Decomposition)**.** *[KS91] Let $\{\mathcal{F}_t\}$ satisfy usual conditions. If X is the right continuous submartingale (martingale) and is of class of DL then it has the decomposition,*

$$X_t = M_t + A_t,$$

where M is a right continuous martingale and A is an increasing process. Also the decomposition is unique modulo indistinguishability.

Theorem 7 (Martingale /Doob's Convergence Theorem)**.** *If X is supermartingale and $\sup_n E|X_n| < \infty$ then we have that almost surely $X_n \to X_\infty$ where $X_\infty = \lim_{n\to\infty} X_n$ exist and is finite.*

1.4 Continuous Square Integrable Martingales

Continuous square integrable martingales are an important class of processes which are necessary to understand Brownian Motion. Also further we will construct integration with respect to square integrable martingales.
Throughout this section we have the triplet $(\Omega, \mathcal{F}, \mathbb{P})$ and the filtration $\{\mathcal{F}_t\}$ satisfying the usual conditions.

Definition 1.13. *A right continuous martingale $X = \{X_t, \mathcal{F}_t, 0 \le t < \infty\}$ is said to be a **square integrable martingale** if $EX_t^2 < \infty \; \forall t \ge 0$. If in addition we have, $X_0 = 0$ then we write $X \in \mathcal{M}_2$ ($X \in \mathcal{M}_2^c$, if X is continuous).*

For any $X \in \mathcal{M}_2$, we can easily observe that $X^2 = \{X_t^2, \mathcal{F}_t, 0 \leq t < \infty\}$ is a nonnegative submartingale (using jensen's inequality), hence of class DL and so it has the following decomposition:

$$X_t^2 = M_t + A_t,$$

where M and A are as defined in (Theorem 6). Further if $X \in \mathcal{M}_2^c$, then A and M are continuous.

Definition 1.14. *For $X \in \mathcal{M}_2$, we define **quadratic variation** of X to be $\langle X \rangle_t = A_t$, where A is an natural increasing process, which is also the decomposition of X^2 such that $X^2 - \langle X \rangle$ is a martingale.*

Definition 1.15. *For two martingales $X, Y \in \mathcal{M}_2$, we can define the cross variation process by*

$$\langle X, Y \rangle =: \frac{1}{4}[\langle X+Y \rangle_t - \langle X-Y \rangle_t]; \qquad 0 \leq t < \infty \tag{1.2}$$

such that XY-$\langle X, Y \rangle$ is a martingale. Also,

$$\begin{aligned} E[(X_t - X_s)(Y_t - Y_s)] &= E[(X_tY_t - X_sY_s)|\mathcal{F}_s] \\ &= E[\langle X_tY_t \rangle - \langle X_sY_s \rangle|\mathcal{F}_s], \end{aligned}$$

for every $0 \leq s < t < \infty$. X and Y are orthogonal if $\langle X, Y \rangle = 0$, in which case XY is a martingale.

Remark 1.1. *It is observed that $\langle \cdot, \cdot \rangle$ is a bilinear form on $\mathcal{M}_2$ such that the following holds: .*

1. $\langle \alpha X + \beta Y, Z \rangle = \alpha \langle X, Z \rangle + \beta \langle Y, Z \rangle$.
2. $\langle X, Y \rangle = \langle Y, X \rangle$.
3. $|\langle X, Y \rangle|^2 \leq \langle X \rangle \langle Y \rangle$.

Definition 1.16. *For any stochastic process X, we define the **p-th variation** $(p > 0)$ of X over partition Π as,*

$$V_t^p(\Pi) = \sum_{k=1}^{m} |X_{t_k} - X_{t_{k-1}}|^p,$$

where $\Pi = \{t_0, t_1, ..., t_m\}$, with $0 = t_0 \leq t_1 \leq \cdots \leq t_m = t$, be a partition of $[0, t]$.

Also mesh of the partition Π is defined as $\| \Pi \| = max_{1 \leq k \leq m} |t_k - t_{k-1}|$.
We now show that V_t^2 converges in probability as $\| \Pi \| \to 0$, and the limit is the quadratic variation of the process X.

Theorem 8. *[KS91] Let $X \in \mathcal{M}_2^c$. For partition Π of $[0,t]$, we have $lim_{\|\Pi\|\to 0} V^2{}_t(\Pi) = \langle X \rangle_t$; i.e., for every $\epsilon > 0$, $\eta > 0$, there exists $\delta > 0$ such that $\|\Pi\| < \delta$ implies*

$$P[|V_t^2(\Pi) - \langle X \rangle_t| > \epsilon] < \eta,$$

where $\|\Pi\| = max_{1 \le k \le m}|t_k - t_{k-1}|$ is the mesh of Π.

The proof to the above theorem proceeds through two lemmas. The key idea is, if $X \in \mathcal{M}_2^c$ and $0 \le s < t \le u < v$, then we can write,

$$E[(X_v - X_u)(X_t - X_s)] = E\{E[(X_v - X_u)|\mathcal{F}_u]\}(X_t - X_s).$$

Also, we can write,

$$E[(X_v - X_u)^2|\mathcal{F}_s] = E[X_v{}^2 - 2X_u E[X_v|\mathcal{F}_u] + X_u{}^2|\mathcal{F}_t]$$

$$= E[X_v{}^2 - X_u{}^2|\mathcal{F}_t] = E[\langle X \rangle_v - \langle X \rangle_u|\mathcal{F}_t].$$

Proposition 1.1. *Let $X \in \mathcal{M}_2$ satisfies $|X_s| \le C < \infty$,$\forall$ $s \in [0,t]$ a.s $\mathbb{P}$. If Π be a partition on $[0,t]$ with $t_0 \le t_1 \le .... \le t_n$ then we have*

$$E[V_t^{(2)}(\Pi)]^2 \le 6K^4$$

Proof We can use the martingale property to proceed.

$$E[\sum_{i=m+1}^{n}(X_{t_i} - X_{t_{i-1}})^2|\mathcal{F}_{t_m}] = E[\sum_{i=m+1}^{n}(X_{t_i}^2 - X_{t_{i-1}}^2)|\mathcal{F}_{t_m}] \le E[X_{t_n}^2|\mathcal{F}_{t_m}] \le K^2$$

therefore, we have,

$$E[\sum_{j=1}^{n-1}\sum_{i=m+1}^{n}(X_{t_i} - X_{t_{i-1}})^2(X_{t_j} - X_{t_{j-1}})^2] \le K^2 E[\sum_{i=m+1}^{n}(X_{t_i} - X_{t_{i-1}})^2] \le K^4.$$

Further we have,

$$E[\sum_{i=1}^{n}(X_{t_i} - X_{t_{i-1}})^4] \le K^2 E[\sum_{i=1}^{n}(X_{t_i} - X_{t_{i-1}})^2] \le 4K^4.$$

Using the above inequalities we have,

$$E[V_t^{(2)}(\Pi)]^2 \le 6K^4.$$

Lemma 1.3. *If $X \in \mathcal{M}_2^c$ satisfies $|X_x| \leq K$ $\forall s \in [0,t]$, a.s. $\mathbb{P}$. Then for partitions Π of [0,t], we have,*

$$\lim_{\|\Pi \to \infty\|} EV_t^{(4)} = 0.$$

Proof We consider $m_t(X;\delta) := \sup\{|X_p - X_q|; p, q \leq t, |p-q| < \delta\}$. We can easily see that,

$$V_t^{(4)} \leq V_t^{(2)}.m_t(X,\delta).$$

Applying the Holder inequality we have the desired result.

Proof Using the above results, we would now proceed with Theorem 8.

We first consider the case when X is bounded by a constant $K < \infty$ and $\langle X \rangle_s \leq K$ holds $\forall s \in [0,t]$, a.s. $\mathbb{P}$. Then,

$$\begin{aligned} E[V_t^{(2)}(\Pi) - \langle X \rangle_t]^2 &= \sum_{i=1}^{n} E[\{(X_{t_i} - X_{t_{i-1}})^2 - (\langle X \rangle_{t_i} - \langle X \rangle_{t_{i-1}})\}]^2 \\ &\leq 2\sum_{i=1}^{n} E[(X_{t_i} - X_{t_{i-1}})^4 + (\langle X \rangle_{t_i} - \langle X \rangle_{t_{i-1}})^2] \\ &\leq 2EV_t^{(4)}(\Pi) + 2E[\langle X \rangle_t m_t(\langle X \rangle, \Pi)]. \end{aligned}$$

As mesh approaches 0 we get the desired result using [Lemma 1.3] .
The second case when X is unbounded , we use localization and proceed as the above case discussed. For the same we define sequence of stopping times :

$$T_n = \inf\{t \geq 0; |X| > n \ \ or \ \ \langle X \rangle_t \geq n\},$$

with this $X_t^{(n)} = X_{t\wedge T_n}$ and $X_{t\wedge T_n}^2 - \langle X \rangle_{t\wedge T_n}$ are bounded martingales. As $T_n \to \infty$ we get the desired result.

Proposition 1.2. *Let $X = \{X_t, \mathcal{F}_t, 0 \leq t < \infty\}$ be a continuous process having the property that for some $p > 0$ and each fixed $t > 0$,*

$$\lim_{\|\Pi\| \to 0} V_t^{(p)} = L_t \quad (\textit{in probability}).$$

Where L_t takes values in $[0,\infty)$ and is a random variable. Then for $q > p$,

$$\lim_{\|\Pi\| \to 0} V_t^{(q)} = 0 \quad (\textit{in probability}),$$

and for $0 < q < p$,

$$\lim_{\|\Pi\| \to 0} V_t^{(q)} = \infty \quad (\textit{in probability}).$$

From the above proposition we can thus conclude that martingales have finite quadratic variation but have unbounded first variation. Variations of higher order are zero. Being of unbounded first variation, these processes cannot be differentiated nor we can make sense of integration in the Lebesgue-Steielges sense. In Chapter 2, we will talk about the construction of stochastic integration using the idea of boundedness of quadratic variation.

Definition 1.17. *Let $X = \{X_t, \mathcal{F}_t, 0 \le t < \infty\}$ be a (continuous) process such that if there exist a non decreasing sequence $\{T_n\}_{n=1}^{\infty}$ of stopping times where $T_0 = 0$ and $\lim_{n\to\infty} T_n = \infty$ such that $\{X_t^{(n)} := X_{t\wedge T_n}, \mathcal{F}_t, 0 \le t < \infty\}$ is a martingale for each $n \ge 1$, then we say that X is a (continuous)* ***local martingale****. If in addition $X_0 = 0$ a.s, we write $X \in \mathcal{M}^{loc}(\mathcal{M}^{c,loc})$.*

Remark 1.2. *Every martingale is a local martingale as $\{X_{T\wedge t}, \mathcal{F}_t, 0 \le t < \infty\}$ is a sub-martingale and by optional sampling theorem, we have the result.*

Lemma 1.4. *Let $X, Y \in \mathcal{M}^{c,loc}$. Then there is a unique adapted, continuous process of bounded variation $\langle X, Y\rangle$ satisfying $\langle X, Y\rangle_0 = 0$ a.s $\mathbb{P}$, such that $XY - \langle X, Y\rangle \in \mathcal{M}^{c,loc}$.*

Definition 1.18. *For any $X \in \mathcal{M}_2$ and $0 \le t < \infty$ we define,*

$$\| X \| = \sum_{n=1}^{\infty} \frac{\| X \|_n \wedge 1}{2^n}.$$

$$\| X \|_t =: \sqrt{E(X_t^2)}.$$

The above defines a pseudo metric, $\| X \|$ on $\mathcal{M}_2$ which becomes a metric modulo indistinguishable processes.

Proposition 1.3. *Under the preceeding metric, $\mathcal{M}_2$ is a complete metric space and $\mathcal{M}_2^c$ is a closed subspace of $\mathcal{M}_2$.*

Proof Let $\{X^{(n)}\}_{n=1}^{\infty} \subseteq \mathcal{M}_2$ be a cauchy sequence in $\mathcal{M}_2$ such that $lim_{n,m\to\infty} \| X^{(n)} - X^{(m)} \| = 0$. $\{X^{(n)}\}^{\infty}{}_{n=1}$ is Cauchy in $\mathcal{L}^2$ for each fixed t and has a limit.
From $\mathcal{L}^2$ convergence and Cauchy-Schwarz inequality we have, for $A \in \mathcal{F}_s$,
$\lim_{n\to\infty} E[1_A(X_s^{(n)} - X_t)] = 0, \lim_{n\to\infty} E[1_A(X_t^{(n)} - X_t)] = 0$. From this we have $E[1_A X_t^{(n)}] = E[1_A X_s^{(n)}]$ which implies $E[1_A X_t] = E[1_A X_s]$. We then take the right continuous modification of X and conclude the result.

1.5 Brownian Motion

Definition 1.19. *A continuous process $B = \{B_t\}$ adapted to the filtration $\mathcal{F}_t$ on $(\Omega, \mathcal{F}, \mathbb{P})$ is a standard* ***Brownian Motion*** *if the following holds:*

- *$B_0 = 0$ a.s.,*
- *For $0 \leq s < t$, increments are independent, i.e $B_t - B_s$ is independent of $\mathcal{F}_s$,*
- *The increments are normally distributed with mean 0 and variance $t - s$ of $B_t - B_s$.*

Remark 1.3. *B is a square integrable martingale with quadratic variation process $\langle B \rangle_t = t, t \geq 0$.*

1.5.1 Properties

Let $B = \{B_t, \mathcal{F}_t; 0 \leq t < \infty\}$ be a standard Brownian Motion, then the following properties hold:

1. **Markov Property:** Brownian motion has stationary and independent increments which makes it a Markov Process.
2. **Martingale Property:** Brownian Motion is a continuous martingale as :

$$\begin{aligned} E[W_t|\mathcal{F}_s] &= E[W_t - W_s + W_s|\mathcal{F}_s] \\ &= E[W_t - W_s|\mathcal{F}_s] + E[W_s|\mathcal{F}_s] \\ &= E[W_t - W_s] + W_s = W_s. \end{aligned}$$

3. **Scaling and Time inversion** :If W is a Brownian Motion, then the process $X = \{X_t, \mathcal{F}_{c^2t}; 0 \leq t < \infty\}$ for $c > 0$ defined by

$$X_t = \frac{1}{c} W_{c^2t} \qquad 0 \leq t < \infty.$$

is a Brownian Motion.

Scaling is an equivalence transformation as the continuity and stationary increments property is preserved. We note that

$$\begin{aligned} Var[X_t - X_s] = Var[c^{(-1)}(W(c^2t) - W(c^2s))] &= c^{(-2)}(c^2t - c^2s) \\ &= t - s. \end{aligned}$$

The expectation is 0. Also $X_t - X_s = c^{(-1)}(W(c^2t) - W(c^2s))$ is distributed as $cN(0, c^2(t-s)) \sim N(0, (t-s))$.

Also the process $Y = \{Y_t, \mathcal{F}_t^Y; 0 \leq t < \infty\}$ defined by

$$Y_t = \begin{cases} tW_{1/t} & 0 < t < \infty \\ 0 & t = 0 \end{cases}$$

is an equivalence transformation.

The new process is continuous and 0 at origin. We have $E[Y_t] = tE[W_{1/t}] = 0$ and Y_t is a Gaussian process also we have the covariance function $E[Y_sY_t] = st(\frac{1}{s} \wedge \frac{1}{t}) = s \wedge t$.

4. **Symmetry:**
   Proof If W is a Brownian motion so is -W as continuity and stationary increments are preserved. mean and variance are not affected by the negative sign. The distribution (can be seen with the help of probability law) does not change.

5. **Finite Quadratic variation** Let $\{\Pi_n\}_{n=1}^{\infty}$ be a sequence of partitions of the interval [0,t] with $\lim_{n\to\infty} \|\Pi_n\| = 0$.

$$\sum_{i=1}^{n}[(W_{t_i} - W_{t_{i-1}})^2 - (t_i - t_{i-1})] = \sum_{i=1}^{n} X_i$$

with

$$X_i = (W_{t_i} - W_{t_{i-1}})^2 - (t_i - t_{i-1})$$

We have $E(X_iX_j) = 0$ for $i \neq j$ since the increments are independent; also $E[(W_{t_i} - W_{t_{i-1}})^2] = t_i - t_{i-1}$. We also see using computations that $E[(W_t - W_s)^4] = 3(t-s)^2$.

$$\begin{aligned} E(X_i^2) &= E\{(W_{t_i} - W_{t_{i-1}})^4 - 2t(W_{t_i} - W_{t_{i-1}})^2 - t^2\} \\ = 3\sum_{k=1}^{n}(t_j - t_{j-1})^2 &+ 2\sum_{1\leq j<k\leq n}(t_j - t_{j-1})(t_k - t_{k-1}) - t^2 \\ &= 2\sum_{k=1}^{n}(t_k - t_{k-1})^2 \\ &\leq 2t \mid \Delta_n[0,t] \mid \to 0. \end{aligned}$$

6. For almost every $\omega \in \Omega$, the sample path $W(\omega)$ is monotone in no interval

7. For almost every $\omega \in \Omega$, the Brownian sample path $W(\omega)$ **is nowhere differentiable.**

Proof $DW_t = \lim_{h\to 0} \frac{W_{t+h} - W_t}{h} \sim \lim_{h\to 0} \frac{N(0,h)}{h} \sim N(0, h^{-1})$. By this we can say that variance goes to infinity, hence the brownian motion is nowhere differentiable.

Chapter 2

Stochastic Integration

To give meaning to the ordinary differential equations that involves continuous stochastic processes, the theory of Stochastic Calculus emerged. Since many important processes that we observe in real life such as Brownian Motion, cannot be differentiated, stochastic calculus assigned meaning to the integration of such processes; through the construction of Stochastic Integration.

2.1 Construction of Stochastic Integration

Consider continuous square integrable martingales $M = \{M_t, \mathcal{F}_t; 0 \leq t < \infty\}$, on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$,equipped with the filtration $\{\mathcal{F}_t\}$. We assume $M_0 = 0$ a.s $\mathbb{P}$. $M \in \mathcal{M}_2^c$ is of unbounded first variation on any finite interval $[0, T]$, and as a consequence, the integral,

$$I_T(X) = \int_0^T X_t(\omega) dM_t(\omega)$$

cannot be defined in the Lebesgue-Stieljes sense.

Since martingale M has a bounded second variation, this allows us to proceed with the construction of stochastic integration with respect to continuous, square integrable matingales for an appropriate class of integrands. The construction was given by Ito(1942) for the case when martingale M is a Brownian Motion and later was given by Kunita and Watanabe(1967)[KS91] for the general case $M \in \mathcal{M}_2$.

We are going to talk about the case where $M \in \mathcal{M}_2^c$ and will then extend to the general, continuous, local martingales M.

We define a measure on $([0,\infty)\times\Omega, \mathcal{B}([0,\infty)\otimes\mathcal{F})$ by,

$$\mu_M(A) = E\int_0^\infty 1_A(t,\omega)d\langle M\rangle_t(\omega).$$

Two processes X and Y are **equivalent** if,

$$X_t(\omega) = Y_t(\omega);\ \mu_M - a.e., (t,\omega).$$

We also define,

$$[X]_T^2 =: E\int_0^T X_t^2 d\langle M\rangle_t.$$

Then $[X]_T^2$ is the $\mathcal{L}^2$-norm for X, as a funtion of (t,ω) on $[0,T]\times\Omega$, under the measure μ_M. $[X-Y]=0\ \forall\ T>0$ iff X and Y are equivalent.

Definition 2.1. *Let $\mathcal{L}(M)$ be the set of equivalence classes of all measurable $\{\mathcal{F}_t\}$ adapted process X, such that $[X]_T<\infty\ \forall\ T>0$. We define a metric on $\mathcal{L}$ by $[X-Y]$, where*

$$[X] = \sum_{n=1}^{\infty}\frac{[X]_n\wedge 1}{2^n}.$$

Let $\mathcal{L}^*(M)$ be the set of equivalence classes of all progressively measurable processes X, such that $[X]_T<\infty\ \forall T>0$. We define a metric on $\mathcal{L}^*$ by $[X-Y]$ in the same way.

Definition 2.2. *X is simple if there exists an increasing sequence in $\mathbb{R}$ with $t_0=0$ and $\lim_{n\to\infty}t_n=\infty$, sequence of r.v. $\{\xi_n\}_{n=0}^\infty$ and nonrandom constant $C\geq 0$ with $\sup_{n\geq 0}|\xi_n(\omega)|\leq C$, such that for every $\omega\in\Omega$, ξ_n is $\mathcal{F}_{t_n}$ measurable and*

$$X_t = \xi_0(\omega)1_{\{0\}}(t) + \sum_{i=0}^{\infty}\xi_i(\omega)1_{(t_i,t_{i+1}]}(t).$$

The class of simple processes is denoted by $\mathcal{L}_0$.

As members of $\mathcal{L}_0$ are progressively measurable and bounded, we have,

$$\mathcal{L}_0\subseteq\mathcal{L}^*\subseteq\mathcal{L}.$$

For X in $\mathcal{L}_0$ we define the **martingale transform** as,

$$I_t(X) := \sum_{i=1}^{n-1} \xi_i(M_{t_{i+1}} - M_{t_i}) + \xi_n(M_t - M_{t_n}) \tag{2.1}$$

$$= \sum_{i=1}^{\infty} \xi_i(M_{t\wedge t_{i+1}} - M_{t\wedge t_i}) \qquad ; 0 \le t < \infty. \tag{2.2}$$

This definition is then extended to the class of integrands in $\mathcal{L}$ and $\mathcal{L}^*$ by approximations by simple processes. To get there we require a couple of results.

Lemma 2.1. *Let X be bounded, measurable, $\{\mathcal{F}_t\}$ adapted process. Then there exist a sequence $\{X^{(m)}\}_{m=1}^{\infty}$ of simple process such that*

$$\sup_{T>0} \lim_{m\to\infty} \mathbb{E} \int_0^T |X_t^{(m)} - X_t|^2 dt = 0.$$

Proof [Idea of the proof]We divide the proof in three cases where the process is continuous, progressively measurable and the third case when the process is measurable and adapted.

Case 1 When X is continuous, we construct a sequence of simple processes:

$$X_t{}^{(n)}(\omega) = X_0(\omega)1_{\{0\}}(t) + \sum_{i=0}^{2n-1} X_{kT/2^n}(\omega)1_{(kT/2^n,(k+1)T/2^n]}(t); \; n \ge 1$$

that satisfies $\lim_{m\to\infty} \mathbb{E} \int_0^T |X_t^{(n)} - X_t|^2 dt = 0$ by bounded convergence theorem.

Case 2 When X is progressively measurable, consider continuous progressively measurable processes,

$$F_t(\omega) := \int_0^{t\wedge T} X_s(\omega)ds; \; \tilde{X_t}^{(m)}(\omega) := m[F_t(\omega) - F_{t-1/m\vee 0}(\omega)]; \; m \ge 1,$$

By virtue of case 1, we again define a sequence of simple processes. $\{\tilde{X_t}^{(m,n)}\}_{n=0}^{\infty}$ and use bounded convergence theorem to proceed.

Case 3 When X is measurable, we cannot quarantee that the continuous process F is progressively measurable, because we do not know if F is adapted. However, we can have a progressively measurable modification Y of X. Using this modification we can proceed as case 2.

Proposition 2.1. *If the function $t \mapsto \langle M\rangle_t(\omega)$ is absolutely continuous with respect to Lebesgue measure for $\mathbb{P}$ a.e ω, then $\mathcal{L}_0$ is dense in $\mathcal{L}$ with respect to the defined metric.*

Theorem 9. *Let $\{A_t; 0 \leq t < \infty\}$ be a continuous increasing process adapted to the filtration of the martingale $M = \{M_t; \mathcal{F}_t,\ 0 \leq < t < \infty\}$. If $X = \{X_t; 0 \leq t < \infty\}$ is a progressively measurable process and $\mathbb{E}\int_0^T X_t^2 dA_t < \infty$ for each $T > 0$ then there exist $\{X^{(n)}\}_{n=1}^{\infty}$ of simple process such that,*

$$\sup_{T>0} \lim_{m\to\infty} \mathbb{E} \int_0^T |X_t^{(m)} - X_t|^2 dA_t = 0.$$

Proof [Basic Idea] WLOG we can assume that X is bounded (in case of unbounded we can define a sequence of bounded processes converging to X and use dominated convergence theorem) i.e.,

$$|X_t(\omega)| \leq C < \infty;\ \forall t \geq), \omega \in \Omega.$$

It suffices to show for each fixed T, a sequence of simple processes $\{X^{(n)}\}_{n=1}^{\infty}$ such that

$$\lim_{n\to\infty} \mathbb{E} \int_0^T |X_t^{(n)} - X_t|^2 dA_t = 0.$$

Again, WLOG we have,

$$X_t(\omega) = 0;\ \forall t > T, \omega \in \Omega$$

and define strictly increasing inverse function $T_s(\omega)$ such that,

$$A_{T_s(\omega)}(\omega) + T_s(\omega) = s;\ \forall s \geq 0.$$

This $T_s(\omega)$ is a stopping time w.r.t the filtration $\{F\}_t$. We also define the new filtration with s as the new time variable such that

$$\mathcal{G}_s = \mathcal{F}_{T_s},$$

and define the new time changed process as,

$$Y_s(\omega) = X_{T_s(\omega)}(\omega).$$

This process is adapted to $\mathcal{G}_s$ because of the progressively measurability of X. Also,

$$E\int_0^\infty {Y_s}^2 ds = E\int_0^\infty 1_{T_s \leq T} {X_{T_s}}^2 ds \tag{2.3}$$

$$= E\int_0^{A_T+T} {X_{T_s}}^2 ds \leq C^2(EA_T + T) < \infty. \tag{2.4}$$

We use **Lemma 2.1** to construct a family of simple processes $\{Y_s\}^\epsilon$ and show that these processes converge to Y w.r.t the defined metric.

Proposition 2.2. *The set $\mathcal{L}_0$ of simple processes is dense in $\mathcal{L}^*$ with respect to the metric defined.*

we take $A = \langle M \rangle$ in Theorem 9 and conclude.

2.2 Construction and Properties of the Integral

We have defined the stochastic integral of simple processes $X \in \mathcal{L}_0$. For $X, Y \in \mathcal{L}_0$ we have the following elementary properties:

$$I_0(X) = 0, \qquad a.s\ \mathbb{P} \tag{2.5}$$

$$I(\alpha X + \beta Y) = \alpha I(X) + \beta I(Y), \qquad \alpha, \beta \in \mathbb{R} \tag{2.6}$$

$$E[I_t(X)|\mathcal{F}_s] = I_s(X), \tag{2.7}$$

$$E(I_t(X))^2 = E\int_0^t X_u^2 d\langle M \rangle_u, \tag{2.8}$$

$$\|I(X)\| = [X], \tag{2.9}$$

$$E[(I_t(X) - I_s(X))^2|\mathcal{F}_s] = E[\int_s^t X_u^2 d\langle M \rangle_u |\mathcal{F}_s], \tag{2.10}$$

Properties (2.5) and (2.6) are obvious. Property (2.7) follows from,

$$E[\xi_i(M_{t\wedge t_{i+1}} - M_t \wedge t_i)|\mathcal{F}_s] = \xi_i(M_{s\wedge t_{i+1}} - M_s \wedge t_i).$$

Thus we see that $I(X)$ defined as :

$$I_t(X) = \sum_{i=1}^{i=n-1} \xi_i(M_{t_{i+1}} - M_{t_i}) + \xi_n(M_t - M_{t_n})$$

is a continuous martingale. For $s < t$ and $t_{k-1} < s < t_k$ and $t_n < t < t_{n+1}$ we have,

$$\begin{aligned}
E[(I_t(X) - I_s(X))^2|\mathcal{F}_s] &= E[\{\xi_{k-1}(M_{t_k} - M_s) + \sum_{i=k}^{n-1} \xi_i(M_{t_{i+1}} - M_{t_i}) + \xi_n(M_t - M_{t_n})\}^2|\mathcal{F}_s] \\
&= E[\xi_{k-1}^2(M_{t_k} - M_s)^2 + \Sigma_{i=k}^{n-1}\xi_i^2(M_{t_{i+1}} - M_{t_i})^2 + \xi_n^2(M_t - M_{t_n})^2|\mathcal{F}_s] \\
&= E[\xi_{k-1}^2(\langle M \rangle_{t_k} - \langle M \rangle_s)^2 + \sum_{i=k}^{n-1} \xi_i^2(\langle M \rangle_{t_{i+1}} - \langle M \rangle_{t_i})^2 + \xi_n^2(\langle M \rangle_t - \langle M \rangle_{t_n})^2|\mathcal{F}_s] \\
&= E[\int_s^t X_u^2 d\langle M \rangle_u |\mathcal{F}_s].
\end{aligned}$$

This proves (2.10) and shows that $I(X)$ is square integrable i.e $I(X) \in \mathcal{M}_2^c$, having the quadratic variation as,

$$\langle I(X) \rangle_t = \int_0^t X_u^2 d\langle M \rangle_u.$$

With $s = 0$ and taking expectations in (2.10) we get (2.8). (2.9) also follows from the metric defined.

For $X \in \mathcal{L}^*$, Proposition 2.2 implies that we can construct a sequence $\{X^{(n)}\}_{n=1}^{\infty}$ in $\mathcal{L}^0$ such that $[X^{(n)} - X] \longrightarrow 0$ as $n \to \infty$. By (2.6) and (2.9) we say that,

$$\|I(X^{(n)}) - I(X^{(k)})\| = \|I(X^{(n)} - X^{(k)})\| = [X^{(n)} - X^{(k)}] \to 0$$

as $n, k \to \infty$. This helps us understanding that $\{I(X^{(n)})\}$ forms a cauchy sequence in $\mathcal{M}_2^c$. We know that $\mathcal{M}_2^c$ is closed in $\mathcal{M}_2$ and therefore the limit $I(X) = \{I_t(X); 0 \leq t < \infty\}$ exists and is in $\mathcal{M}_2^c$ such that $\|I(X^{(n)}) - I(X)\| \to 0$ as $n \to \infty$. This shows that $I_t(X) = \{I_t(X), \mathcal{F}_t; 0 \leq t < \infty\}$ is a continuous martingale that follows properties (2.5) and (2.7). We also have, for $0 \leq s < t$, processes $\{I(X_s^{(n)})\}$, $\{I(X_t^{(n)})\}$ converging to $I_s(X)$ and $I_t(X)$ respectively in $\mathcal{L}^2$ sense. Therefore for $A \in \mathcal{F}_s$ we have,

$$\begin{aligned} E[1_A(I_t(X) - I_s(X))^2] &= \lim_{n\to\infty} E[1_A(I_t(X^{(n)}) - I_s(X^{(n)}))^2] \\ &= \lim_{n\to\infty} E[1_A \int_s^t ((X_u^{(n)}))^2 d\langle M\rangle_u] \\ &= E[1_A \int_s^t (X_u^2 d\langle M\rangle_u)]. \end{aligned}$$

Showing that $I(X)$ also satisfies the above properties mentioned.

As mentioned, we know that X and M are progressively measurable, implying that $\int_s^t ((X_u^{(n)}))^2 d\langle M\rangle_u$ is also $\mathcal{F}_t$ measurable for fixed $s < t$. Also the process $I(X)$ for $X \in \mathcal{L}^*$ is well defined and we have,

$$\langle I(X)\rangle = \int_0^t X_u^2 d\langle M\rangle_u.$$

Definition 2.3. *For $X \in \mathcal{L}^*$, the* ***stochastic integral*** *of X with respect to the martingale M in $\mathcal{M}_2^c$ is $I(X) = \{I_t(X), \mathcal{F}_t; 0 \leq t < \infty\}$, which is a unique square integrable martingale that satisfies $\|I(X^{(n)}) - I(X)\| = 0$ for every sequence $\{X^{(n)}\}_{n=1}^{\infty} \subseteq \mathcal{L}^0$ with $\lim_{n\to\infty}[X^{(n)} - X] = 0$. We write*

$$I_t(X) = \int_0^t X_s dM_s.; \qquad 0 \leq t < \infty \tag{2.11}$$

Remark 2.1. *If for every $\omega \in \Omega$ the map $t \mapsto \langle M\rangle_t(\omega)$ of the quadratic variation process $\langle M\rangle$ are absolutely continuous functions of t for $\mathbb{P}$ a.e ω, then Proposition 2.1 could be used directly to define stochastic integral $I(X)$ for every $X \in \mathcal{M}_2^c$.*

In the case when M is a standard Brownian Motion with $\langle M\rangle = t$, we can use Proposition 2.1 again to come to the conclusion without defining the time inversion process and using the results of Theorem 9.

2.3 Characterization of the Integral

Let $M = \{M_t, \mathcal{F}_t; 0 \le t < \infty\}$ and $N = \{N_t, \mathcal{F}_t; 0 \le t < \infty\}$ are in $\mathcal{M}_2^c$ and we have $X \in \mathcal{L}^*(\mathcal{M})$ and $Y \in \mathcal{L}^*(\mathcal{N})$. Then ${I^M}_t(X) = \int_0^t X_s dM_s$ and ${I^N}_t(X) = \int_0^t Y_s dM_s$ are also in $\mathcal{M}_2^c$ with all the defined properties (2.5)-(2.10).

We can extend this result from simple processes to $X \in \mathcal{L}^*(\mathcal{M})$ and $Y \in \mathcal{L}^*(\mathcal{N})$.

Proposition 2.3 (An inequality of Kunita and Watanabe(1967)). *[KW67] If $M, N \in \mathcal{M}_2^c$ and $X \in \mathcal{L}^*(\mathcal{M})$ and $Y \in \mathcal{L}^*(\mathcal{N})$, then a.s,*

$$\int_0^t |X_s Y_s| d\tilde{\xi} \le (\int_0^t {X_s}^2 d\langle M\rangle_s)(\int_0^t {Y_s}^2 d\langle N\rangle_s), \tag{2.12}$$

where $\tilde{\xi}$ denotes the total variation of $\xi := \langle M, N\rangle$ on $[0, s]$

Lemma 2.2. *If $M, N \in \mathcal{M}_2^c$ and $X \in \mathcal{L}^*(\mathcal{M})$ and $\{X^{(n)}\}_{n=1}^{\infty} \subseteq \mathcal{L}^*(\mathcal{M})$ such that for $T > 0$,*

$$\lim_{n\to\infty} \int_0^T |{X_u}^{(n)} - X_u|^2 d\langle M\rangle_u = 0;\ a.s\ \mathbb{P}, \tag{2.13}$$

then,

$$\lim_{n\to\infty} \langle \tilde{I}(X^{(n)}), N\rangle_t = \langle I(X), N\rangle;\ 0 \le t \le T\ a.s\ \mathbb{P}. \tag{2.14}$$

Proof Using the inequality,

$$|\langle I(X)^{(n)} - \langle I(X), N\rangle, N\rangle_t|^2 \le \langle I(X)^{(n)} - X\rangle_t \langle N\rangle_t,$$

$$\le \int_0^T |X_u^{(n)} - X_u|^2 d\langle M\rangle_u \langle N\rangle_T$$

we can conclude the result.

Lemma 2.3. *If $M, N \in \mathcal{M}_2^c$ and $X \in \mathcal{L}^*(\mathcal{M})$, then,*

$$\langle I^M(X), N\rangle_t = \int_0^t X_u \langle M, N\rangle_u. \tag{2.15}$$

Proposition 2.4. *If $M, N \in \mathcal{M}_2^c$ and $X \in \mathcal{L}^*(\mathcal{M})$ and $Y \in \mathcal{L}^*(\mathcal{N})$, then,*

$$\langle I^M(X), I^N(Y)\rangle = \int_0^t X_u Y_u d\langle M, N\rangle_u. \tag{2.16}$$

$$E[(I^M{}_t(X) - I^M{}_s(X))(I^N{}_t(Y) - I^N{}_s(Y))|\mathcal{F}_s] \tag{2.17}$$

$$= E[\int_s^t X_u Y_u d\langle M, N\rangle_u|\mathcal{F}_s]$$

holds.

Proof From Lemma 2.3 we know that $d\langle M, I^N(Y)\rangle_t = Y_u d\langle M, N\rangle_u$. Swaping N with $I^N(Y)$, we get,

$$\langle I^M(X), I^N(Y)\rangle_t = \int_0^t X_u Y_u d\langle M, N\rangle_u;\ t \geq 0,\ \mathbb{P} - a.s. \tag{2.18}$$

2.4 Integration with respect to Continuous, Local Martingales

Let $M \in \mathcal{M}^{c,loc}$ i.e a continuous local martingale. Then for a certain class of processes we can define the stochastic integration w.r.t to continuous, local martingales.

Definition 2.4. *Let $\mathcal{P}$ be the class of measurable, adapted process $X = \{X_t, \mathcal{F}_t; 0 \leq t < \infty\}$ satisfying*

$$\mathbb{P}([\int_0^T X_t^2 d\langle M\rangle_t < \infty]) = 1 \qquad \forall\ T \in [0, \infty).$$

Let $\mathcal{P}^*$ be the class of progressively measurable process agreeing to above condition.

We can observe that $\mathcal{P} \subseteq \mathcal{L}$ and $\mathcal{P}^* \subseteq \mathcal{L}^*$. We will continue our discussion when the class of integrands belong to $\mathcal{P}^*$. If for a.e. path, the quadratic variation process is absolutely continuous w.r.t to the time parameter, we can talk about a larger class of integrands, mainly $\mathcal{P}$ for integration w.r.t continuous, local martingales.

As taken $M \in \mathcal{M}^{c,loc}$; hence there exist a sequence of stopping time $\{T_n\}_{n=1}^{\infty}$ such that $\lim_{n\to\infty} T_n = \infty$ and we have $M_{t\wedge T_n}$ belongs to $\mathcal{M}_{\in}{}^c$. For $X \in \mathcal{P}^*$ we construct another sequence of stopping time,

$$R_n = n \wedge \inf\{0 \leq t < \infty; \int_0^t X_t^2 d\langle M\rangle_t \geq n\}.$$

We can see that $\lim_{n\to\infty} R_n = \infty$, and is bounded for each n. For $n \geq 1,\ \omega \in \Omega$, we set,

$$S_n = R_n \wedge T_n,$$

$$M_t^n(\omega) = M_{t\wedge S_n}(\omega),$$

$$X_t^n(\omega) = X_t(\omega) 1_{\{T_n(\omega)\geq n\}} \qquad 0 \leq t < \infty.$$

Then we have, $M^{(n)} \in \mathcal{M}_2^c$ and $X^{(n)} \in \mathcal{L}^*(M^{(n)})$ such that we can define for $X \in \mathcal{P}^*$,

$$I_t(X) = I_t^{(M^{(n)})}(X^{(n)}), \tag{2.19}$$

which is a local martingale.

Definition 2.5. *For $M \in \mathcal{M}^{c,loc}$ and $X \in \mathcal{P}^*$ we define the stochastic integral of X with respect to $M \in \mathcal{M}^{c,loc}$ as $I_t(X) = \{I_t(X), \mathcal{F}_t; 0 \leq t < \infty\} \in \mathcal{M}^{c,loc}$ defined by (3.19).*

Proposition 2.5. *Let $M \in \mathcal{M}^{c,loc}$ and $X \in \mathcal{P}^*(M)$. Then there exists a sequence of simple process $\{X^{(n)}\}_{n=1}^{\infty}$ such that, for every $T > 0$,*

$$\lim_{n\to\infty} \int_0^T |X_s^{(n)} - X_s| d\langle M\rangle_s = 0.$$

Also

$$\lim_{n\to\infty} \sup_{0\leq t<T} |I_s(X^{(n)}) - I(X)| = 0.$$

Example Let $M = W$ a standard brownian motion and $X \in \mathcal{P}$. We define

$$\zeta_t^s(X) := \int_s^t X_u dW_u - \frac{1}{2}\int_s^t X_u^2 du.$$

The process $\{\exp(\zeta_t(X)), \mathcal{F}_t; 0 \leq t < \infty\}$ is a super martingale because $\int_s^t X_u dW_u \in \mathcal{M}^{c,loc}$. It is a martingale if $X \in \mathcal{L}_0$.

2.5 The Change of Variable Formula

To study the integral-differential calculus of stochastic processes we should study the change of variable formula, or the **Ito's Rule**. We start with the following definition.

Definition 2.6. *A continuous **semimartingale** $X = \{X_t, \mathcal{F}_t; 0 \leq t < \infty\}$ is an adapted process which has the decomposition a.s*

$$X_t = X_0 + M_t + B_t; \quad 0 \leq t < \infty,$$

where ($M \in \mathcal{M}^{c,loc}$) and $B = \{B_t, \mathcal{F}_t; 0 \le t < \infty\}$ is the difference of continuous, nondecreasing, adapted processes

Theorem 10 (Ito's Rule)**.** *[KS91] Let $f: \mathbb{R} \to \mathbb{R}$ belong to the class of $C^2(\mathbb{R})$ function. Let $X = \{X_t, \mathcal{F}_t; 0 \le t < \infty\}$ be a continuous semi-martingale with above decomposition. Then, a.s,*

$$f(X_t) = f(X_0) + \int_0^t f^{'}(X_s)dM_s + \int_0^t f^{'}(X_s)dB_s + \frac{1}{2}\int_0^t f^{''}(X_s)d\langle M\rangle_s. \tag{2.20}$$

Remark 2.2. *$X_s(\omega)$, for a fixed ω, is a bounded function for $0 \le s \le t$ due to which $f^{'}(X_s)$ is bounded as well on this interval. It follows that integral $\int_0^t f^{'}(X_s)dM_s$ exists and it is a continuous, local martingale(from the last section). The other two intergals can be evaluated in the Lebesgue-Stieljes sense, as the process $\langle M\rangle$ is of bounded variations and $f^{'}(X_s)$ is also a bounded function and so, are functions of bounded variations. It follows that $\{f(X_t), \mathcal{F}_t; 0 \le t < \infty\}$ is a continuous semimartingale.*

Proof We divide the proof into several steps.
Step 1: We define the stopping time,

$$T_n = \begin{cases} 0 & \text{if } |X_0| \ge n, \\ \inf\{t \ge 0; |M_t| \ge n \text{ or } B_t \ge n \text{ or } \langle M\rangle_t \ge n\} & \text{if } |X_0| < n, \\ \infty & \text{if } |X_0| < n \text{ and } \inf\{t \ge 0; |M_t| \ge n \text{ or } B_t \ge n \text{ or } \langle M\rangle_t \ge n\} = \varnothing. \end{cases}$$

This stopping time is defined for localization. We have $\{T_n\}_{n=1}^{\infty}$ which is an increasing sequence with $\lim_{n\to\infty} T_n = \infty$. If we can show (2.20) for $X_{t\wedge T_n}(\omega), M_{t\wedge T_n}(\omega), \langle M\rangle_t(\omega)$, then we have the result. We can assume that the processes are bounded by a common constant K such that M is a bounded martingale. With this assumption, we have $|X_t(\omega)| \le 3K$ and f has a compact support implying that both $f^{'}$ and $f^{"}$ are bounded.
Step 2: For fix $t > 0$ and partition $\Pi = \{t_0, t_1, \cdots, t_m\}$ with $\{t_0 = 0 < t_1 < \cdots < t_m = t\}$. We use the **Taylor's Expansion** and get

$$f(X_t) - f(X_0) = \sum_{k=1}^{m}\{f(X_{t_k}) - f(X_{t_{k-1}})\} \tag{2.21}$$

$$= \sum_{k=1}^{m} f'(X_{t_k})\{X_{t_k} - X_{t_{k-1}}\} + \frac{1}{2}\sum_{k=1}^{m} f''(\eta_k)\{X_{t_k} - X_{t_{k-1}}\}^2 \tag{2.22}$$

where $\eta_k = X_{t_{k-1}}(\omega) + \theta_k(\omega)\{X_{t_k} - X_{t_{k-1}}\}$. We choose θ_k such that $f''(\eta_k)$ is measurable and $0 \le \theta_k \le 1$.

Using the semimartingale decomposition of X and taylor's expansion we divide $f(X_t) - f(X_0)$ as sum of three decompositions. They are,

$$f(X_t) - f(X_0) = J_1(\Pi) + J_2(\Pi) + J_3(\Pi), \tag{2.23}$$

where ,

$$J_1(\Delta) = \sum_{k=1}^{m} f'(X_{t_{k-1}})\{B_{t_k} - B_{t_{k-1}}\}.$$
$$J_2(\Delta) = \sum_{k=1}^{m} f'(X_{t_{k-1}})\{M_{t_k} - M_{t_{k-1}}\}.$$
$$J_3(\Delta) = \sum_{k=1}^{m} f''(\eta_k)\{X_{t_k} - X_{t_{k-1}}\}^2.$$

As for $J_1(\Pi)$, it can be seen that it converges to the Lebesgue-Stieltjes integral $\int_0^t f'(X_s)dB_s$ as mesh goes to 0 because $f^{'}$ is bounded and B is of bounded variation.

We also observe that $f'(X_s(\omega))$ is an adapted, bounded and continuous (in $\mathcal{L}^*$). We can approximate it be simple processes and write as,

$$Y_s^{(\Pi)} = f'(X_0(\omega)1_{\{0\}})(s) + \sum_{k=1}^{m} f'(X_{t_{k-1}}(\omega))1_{(t_{k-1},t_k]}(s).$$

We have,

$$EI_t^2(Y^{\Pi} - Y) = E\int_0^t |Y_s^{\Pi} - Y_s|^2 d\langle M\rangle_s \longrightarrow 0$$

as $\|\Pi\| \to 0$, by bounded convergence theorem. Therefore $J_2(\Pi) = \int_0^t {Y_s}^{\Pi} dM_s \longrightarrow \int_0^t Y_s dM_s$ in quadratic mean.

Step 3: We can further write $J_3(\Pi)$ as

$$J_3(\Pi) = J_4(\Pi) + J_5(\Pi) + J_6(\Pi),$$

where,

$$J_4(\Pi) = \sum_{k=1}^{m} f''(\eta_k)\{B_{t_k} - B_{t_{k-1}}\}^2. \tag{2.24}$$

$$J_5(\Pi) = \sum_{k=1}^{m} f''(\eta_k)\{B_{t_k} - B_{t_{k-1}}\}\{M_{t_k} - M_{t_{k-1}}\}. \tag{2.25}$$

$$J_6(\Pi) = \sum_{k=1}^{m} f''(\eta_k)\{M_{t_k} - M_{t_{k-1}}\}^2. \tag{2.26}$$

B is bounded variation process which is bounded by K, thus we have,

$$|J_4(\Pi)| + |J_5(\Pi)| \leq 2K\|f''\|_\infty(\max_{1\leq k\leq m} |B_{t_k} - B_{t_{k-1}}| + \max_{1\leq k\leq m} |M_{t_k} - M_{t_{k-1}}|)$$

which converges to 0 as mesh goes to zero.

For $J_6(\Pi)$, we define the following sum,

$$J_6{}^*(\Pi) = \Sigma_{k=1}^{m} f''(X_{t_{k-1}})\{M_{t_k} - M_{t_{k-1}}\}^2.$$

We observe that,

$$|J_6(\Pi) - J_6^*(\Pi)| \leq V_t^2(\Pi) \cdot \max_{1\leq k\leq m} |f''(\eta_k) - f''(X_{t_{k-1}})|,$$

where $V_t^2(\Pi)$ is the quadratic variation of martingale M over the partition. Using Cauchy-Schwarz inequality we have,

$$\begin{aligned} E|J_6(\Pi) - J_6^*(\Pi)| &\leq E(V_t^2(\Delta) \cdot \max_{1\leq k\leq m} |f''(\eta_k) - f''(X_{t_{k-1}})|) \\ &\leq \sqrt{6K^4}\sqrt{E(\max_{1\leq k\leq m} |f''(\eta_k) - f''(X_{t_{k-1}})|)^2}. \end{aligned}$$

X is continuous and using bounded convergence theorem we have the above inequality going to 0 as mesh goes to 0. To show the convergence of $J_3(\Pi)$ to $\int_0^t f''(X_s)d\langle M\rangle_s$ in $\mathcal{L}^1$ as mesh goes to 0, we need to compare $J_6^*(\Pi)$ to the sum,

$$J_a(\Pi) = \sum_{k=1}^{m} f''(X_{t_{k-1}})\{(\langle M\rangle_{t_k} - \langle M\rangle_{t_{k-1}})\}.$$

$$E|J_6^*(\Pi) - J_a(\Pi)|^2 = E|\sum_{k=1}^{m} f''(X_{t_{k-1}})\{(M_{t_k} - M_{t_{k-1}})^2 - (\langle M\rangle_{t_k} - \langle M\rangle_{t_{k-1}})\}|^2$$

$$= E|\sum_{k=1}^{m}[f''(X_{t_{k-1}})]^2\{(M_{t_k} - M_{t_{k-1}})^2 - (\langle M\rangle_{t_k} - \langle M\rangle_{t_{k-1}})\}^2|$$

$$\leq 2\|f''\|_\infty^2 \cdot E[\sum_{k=1}^{m}(M_{t_k} - M_{t_{k-1}})^4 + \sum_{k=1}^{m}(\langle M\rangle_{t_k} - \langle M\rangle_{t_{k-1}})^2]$$

$$\leq 2\|f''\|_\infty^2 \cdot E[(V_t^{(4)}(\Pi)) + \langle M\rangle_t \max_{1\leq k\leq m}(\langle M\rangle_{t_k} - \langle M\rangle_{t_{k-1}})].$$

We know that $(V_t^{(4)}(\Pi))$ goes to zero and using the bounded convergence theorem , the last term goes to zero as mesh goes to zero.
Convergence in $\mathcal{L}^2$ implies convergence in $\mathcal{L}^1$, from this we can conclude that,

$$\lim_{\|\Pi\|\to 0} J_3(\Pi) \longrightarrow \int_0^t f''(X_s)d\langle M\rangle_s.$$

Step 4: Further, if we have a sequence of partions $\{\Pi^{(n)}\}$ such that $\|\Pi^{(n)}\| \longrightarrow 0$, then for some subsequence $\{\Pi^{(n_k)}\}_{k=1}^{\infty}$ we have a.s :

$$\lim_{k\to\infty} J_1(\Pi_{(n_k)}) = \int_0^t f'(X_s)dB_s.$$
$$\lim_{k\to\infty} J_2(\Pi_{(n_k)}) = \int_0^t f'(X_s)dM_s.$$
$$\lim_{k\to\infty} J_3(\Pi_{(n_k)}) = \int_0^t f''(X_s)d\langle M\rangle_s.$$

□

Applications of Ito's Formula

The following characterization of the Weiner Process can be obtained using the Ito's Formula

Theorem 11 (Kunita-Watanabe)**.** *Let $\mathcal{M} \in \mathcal{M}_c^l oc$ in $\mathbb{R}^d$ with $M_0 = 0$ and $\langle M\rangle_t = tI$ (I is the $d \times D$ identity matrix). Then we have:*

- *(M_t) is a d dimensional Weiner Process.*
- *For $s \leq t$, $\sigma[M_v - M_u : s \leq u < v \leq t]$ is orthogonal to $\mathcal{F}_s$. Then on each interval $[0, T]$, (M_t) is continuous $\mathcal{L}^2$ martingale.*

Here we are not assuming the fitration to be right continuous.

Proof Let us first assume that $(\mathcal{F}_s)$ is right continuous. For fixed $s \leq T$ and $u = (u_1, \dots, u_d) \in \mathbb{R}^d$, with $s \leq t \leq T$, define,

$$g(t) = \int_A e^{i(u, M_t - M_s)} d\mathbb{P} \quad A \in \mathcal{F}_s.$$

Applying Ito's Formula to $F(x) = e^{i(u,x)}$ we have,

$$e^{i(u,M_t)} = 1 + i \sum_j u_j \int_0^t e^{i(u,M_v)} dM_v{}^j - \frac{1}{2}|u|^2 \int_0^t e^{i(u,M_v)} dv,$$

where $|u|^2 = \sum_j u_j{}^2$. Thus we have,

$$e^{i(u,M_t - M_s)} = 1 + i \sum_j u_j \int_0^t e^{i(u,M_v - M_s)} dM_v{}^j - \frac{1}{2}|u|^2 \int_0^t e^{i(u,M_v - M_s)} dv.$$

With s fixed and integrating with respect to $\mathbb{P}$ over A , we have,

$$E[e^{i(u,M_t - M_s)}|\mathcal{F}_s] = 0 \;\; a.s.$$

From the above we observe that,

$$g(t) = P(A) - \frac{1}{2}|u|^2 \int_s^t g(v) dv.$$

On differentiating with respect to the time parameter t we obtain,

$$g^{'}(t) = -\frac{1}{2}|u|^2 g(t).$$

Hence, $g(t) = e^{-\frac{1}{2}|u|^2(t-s)\mathbb{P}(A)}$. From this we get the desired result.

Example 1 $f(x) = x^2$ and $X_0 = 0$ and M = W = Brownian Motion. Then M as a semimartingale will have the decomposition such that $B_t = 0 \; \forall t$. Using the Ito's rule we have,

$$W_t^2 = 2 \int_0^t W_s dW_s + t.$$

Example 2 Let the martingale be the Standard Brownian Motion and let $X \in \mathcal{P}$. We define,

$$\zeta_t^s(X) := \int_s^t X_u dW_u - \frac{1}{2} \int_s^t X_u^2 du,$$

and

$$Z_t = exp(\zeta_t); \quad 0 \leq t < \infty.$$

Then the process $\{\exp(\zeta_t(X)), \mathcal{F}_t; 0 \leq t < \infty\}$ is a supermartingale. Using Ito's rule we show that the process Z satisfies the stochastic integral equation,

$$Z_t = 1 + \int_0^t Z_s X_s dW_s; \qquad 0 \leq t < \infty.$$

With $f(x) = e^x$, we have,

$$\begin{aligned} Z_t = f(\zeta_t) = f(\zeta_0) + \int_0^t f'(\zeta_s) dM_s + \int_0^t f'(\zeta_s) dB_s + \frac{1}{2}\int_0^t f''(\zeta_s) d\langle M \rangle_s \\ = 1 + \int_0^t Z_s X_s dW_s + \int_0^t Z_s(-\frac{1}{2}X_s^2) ds + \frac{1}{2}\int_0^t Z_s X_s^2 ds \\ = 1 + \int_0^t Z_s X_s dW_s. \end{aligned}$$

Chapter 3

Stochastic Differential Equations

3.1 Introduction

In this chapter we will explore the existence and uniqueness for solutions to stochastic differential equations. We would also explore under what conditions do we have a unique solution modulo indistinguishability. SDEs have a lot of applications in the world of mathematical economics as they are used to model the fluctuations of stock prices and also problems related to the consumption/investments. The stochastic differential equation is of the form,

$$dX_t = b(t, X_t) + \sigma(t, X_t)dW_t. \tag{3.1}$$

3.2 Strong Solutions

We begin with the introduction to stochastic differential equations with respect to the Brownian motion and its solution in the strong sense. We also discuss the properties and answer questions about their existence and uniqueness.

Let $\{b_i(t, x),\ \sigma_{ij}(t, x),\ 1 \leq i \leq d,\ 1 \leq j \leq r\}$ be borel measurable functions from $[0, \infty) \times \mathbb{R}^d$ to $\mathbb{R}$ and we also define $b(t, x) = \{b_i(t, x)\}_{1 \leq i \leq d}$ which is the drift vector and $\sigma(t, x) = \{\sigma_{ij}(t, x)\}_{1 \leq i \leq d\ 1 \leq j \leq r}$ which is the dispersion matrix. We want to assign meaning to the differential equation,

$$dX_t = b(t, X_t)dt + \sigma(t, X_t)dW_t, \tag{3.2}$$

which can be written componentwise as,

$$dX_t^{(i)} = b_i(t, X_t)dt + \sum_{j=1}^{r} \sigma_{ij}(t, X_t)dW_t^{(j)}; \qquad 1 \leq i \leq d. \tag{3.3}$$

Here W is an r-dimensional Brownian Motion and X is a stochastic process with continuous sample paths with values in $\mathbb{R}^d$.

To develop the concept of strong solution we require a suitable filtration. We consider a probability triple $(\Omega, \mathcal{F}, \mathbb{P})$ with r-dimensional Brownian Motion $W = \{W_t, \mathcal{F}_t^W, 0 \leq t < \infty\}$. We also accommodate a random vector ξ with values in $\mathbb{R}^d$ independent of $\mathcal{F}_\infty^W$ with the distribution defined as $\mu(\Gamma) = \mathbb{P}[\xi \in \Gamma]$; $\Gamma \in \mathcal{B}(\mathbb{R}^d)$. This vector helps us in defining the notion of strong solution.
We start with construction of the filtration. We consider a left continuous fitration,

$$\mathcal{G}_t := \sigma(\xi, W_s, 0 \leq s \leq t),$$

with a collection of null sets,

$$\mathcal{N} := \{N \subseteq \Omega; \exists G \in \mathcal{G}_\infty \text{ with } N \subseteq G \text{ and } \mathbb{P}(G) = 0\}$$

to get the augmented filtration which is defined as,

$$\mathcal{F}_t = \sigma(\mathcal{G}_t \cup \mathcal{N}). \tag{3.4}$$

Following is an example of a SDE.

Example: Let $X_t = cos(B(t))$ and $Y_t = sin(B(t))$. The column vector V_t with components X_t and Y_t represents the position at time t of an object moving in unit circle with angle governed by Brownian motion. Applying Ito's formula

$$dX_t = -sinB(t)dB(t) - \frac{1}{2}cosB(t)dt = -Y_t dB(t) - \frac{1}{2}X_t dt,$$

$$dY_t = cosB(t)dB(t) - \frac{1}{2}sinB(t)dt = X_t dB(t) - \frac{1}{2}Y_t dt.$$

Therefore, the stochastic differential equation of V_t is obtained as:

$$dV_t = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} V_t dB(t) - \frac{1}{2}V_t dt, \quad V_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Definition 3.1. *A strong solution of the stochastic differential equations, on $(\Omega, \mathcal{F}, \mathbb{P})$ with respect to the fixed Brownian motion W and having the initial condition ξ, is again a stochastic process $X = \{X_t; 0 \leq t < \infty\}$ with continuous sample paths, with the following properties :*

1. *X is adapted to the filtration $\{\mathcal{F}_t\}$.*

2. $\mathbb{P}[X_0 = \xi] = 1$.

3. $\mathbb{P}[\int_0^t \{|b_i(s, X_s)| + \sigma_{ij}^2(s, X_s)\} < \infty]$ *holds for every* $1 \leq i \leq d$, $1 \leq j \leq r$ *and* $0 \leq t < \infty$.

4. *The integral version of (3.2),*

$$X_t^i = X_0^i + \int_0^t b_i(s, X_s)ds + \sum_{j=1}^{r} \int_0^t \sigma_{ij}(s, X_s)dW_s^{(j)}.$$

holds almost surely.

Definition 3.2. *Let the drift vector* $b(t, X_t)$ *and the dispersion matrix* $\sigma(t, X_t)$ *be given. Whenever* W *is an r-dimensional Brownian Motion with initial condition* ξ*, where* ξ *is an independent, r-dimensional random vector on* $\{\mathcal{F}_t\}$ *and* X *and* $\tilde{X}$ *are the two strong solutions of (3.2), then* $\mathbb{P}[X = \tilde{X}; 0 \leq t < \infty] = 1$. *In such case strong uniqueness holds for* (b, σ).

If $\sigma(t, X_t)$, which is the dispersion matrix is identically zero, then the integral form of SDE can be written as,

$$X_t = X_0 + \int_0^t b(s, X_s)ds. \tag{3.5}$$

In the theory of such equations, we have to impose the assumption that $b(t, x)$ satisfies the local Lipchitz condition in the space variable x and is bounded on compact subsets to ensure that for sufficiently small t we can apply the Picard-Lindelof iterations where,

$$X_t{}^{(0)} = X_0; \quad X_t{}^{(n+1)} = X_0 + \int_0^t b(s, X_s)ds,$$

so that the above converges to the integral solution (3.5) such that the solution is unique. In the absence of this condition the equation might fail to be solvable.

To develop the theory of stochastic differential equations, the condition of Lipschitz continuity was imposed and shown what kind of existence and uniqueness results can be obtained. The theory was developed by K. Ito.

To begin with we would start with the Gronwall inequality.

Proposition 3.1. *Let* $g(t)$ *be a continuous function that satisfies,*

$$0 \leq g(t) \leq f(t) + b\int_0^t g(s)ds. \tag{3.6}$$

Then we have the following equality:

$$g(t) \leq f(t) + b\int_0^t f(t)\exp(b(t-s))ds.$$

Proof We define,

$$v(s) = exp(-b[s-0]) \int_0^s bg(r)dr, \quad s \in [0,T]$$

taking the derivative, we have the inequality,

$$v^{'}(s) = g(s)exp(-b[s-0]) - bexp(-b[s-0]) \int_0^s bg(r)dr$$

$$= [g(s) - \int_0^s bg(r)dr]bexp(-b[s-0]).$$

where $g(s) - \int_0^s bg(r)dr \leq f(s)$.
Since b and exponential are non negative, the above equation gives an upper bound for derivative of v.

Since $v(0) = 0$, integration of this inequality from 0 to t gives us,

$$v(t) \leq \int_0^t f(s)bexp(-b[s-0])ds.$$

Using the definition of $v(t)$ and the inquality above, we obtain

$$\int_0^t bg(r)dr \leq \int_0^t f(r)b\exp[b(t-0) - b(s-0)]dr$$

$$\leq b \int_0^t f(r)exp(-b[t-s]).$$

substituting this inequality into the assumed integral we get the desired result.

Remark 3.1. *For every $d \times r$ matrix , we define,*

$$\|\sigma\|^2 = \sum_{i=1}^{d} \sum_{j=1}^{r} {\sigma_{ij}}^2.$$

Theorem 12. *Let us suppose that the coefficients $b_i(s, X_s)$ and $\sigma_{ij}(s, X_s)$ are locally Lipschitz continuous in the space variable, that is, for every $n \geq 1$ there exist a constant A_n such that for every $t \geq 0$, $\|x\| \leq n$ and $\|y\| \leq n$ we have:*

$$\|b(t,x) - b(t,y)\| + \|\sigma(t,x) - \sigma(t,y)\| \leq A_n \|x-y\|. \tag{3.7}$$

Then strong uniqueness holds for (3.1).

Proof Let X and $\tilde{X}$ be the strong solution of (3.1) defined for all $t \geq 0$, with respect to some Brownian motion W and initial conditions ξ on the probability space. For each $n \geq 0$, we define the stopping time as $\tau_n = \inf\{t \geq 0; \|X\|_t \geq n; n \geq 1\}$ and $\tilde{\tau_n} = \inf\{t \geq 0; \|\tilde{X}\|_t \geq n; n \geq 1\}$. We also set $S_n = \tau \wedge \tilde{\tau}$. Clearly we have $\lim_{n \to \infty} S_n = \infty$ and,

$$X_{t\wedge S_n} - \tilde{X}_{t\wedge S_n} = \int_0^{t\wedge S_n} \{b(u, X_u) - b(u, (\tilde{X_u}))\}du + \int_0^{t\wedge S_n} \{\sigma(u, X_u) - \sigma(u, (\tilde{X_u}))dW_u.$$

Using the vector inequality $\|v_1 + v_2 + \cdots v_k\| \leq k^2(\|v_1\|^2 + \|v_2\|^2 \cdots \|v_k\|^2)$ and the Holder inequality for Lebesgue integrals, we have for $0 \leq t \leq T$,

$$\begin{aligned} E\|X_{t\wedge S_n} - \tilde{X}_{t\wedge S_n}\|^2 &\leq E[\int_0^{t\wedge S_n} \|\{b(u, X_u) - b(u, (\tilde{X_u}))du\}\|]^2 \\ +4E\sum_{i=1}^{d}[\sum_{i=1}^{r} \int_0^{t\wedge S_n} &\|\{\sigma_{ij}(u, X_u) - \sigma_{ij}(u, (\tilde{X_u}))dW_u^{(j)}\}\|]^2 \\ &= 4tE\int_0^{t\wedge S_n} \|\{b(u, X_u) - b(u, (\tilde{X_u}))\}\|^2 du \\ +4E\int_0^{t\wedge S_n} &\|\{\sigma(u, X_u) - \sigma(u, (\tilde{X_u}))\}\|^2 du \\ &\leq 4(T+1)A_n^2 \int_0^t E\|X_{t\wedge S_n} - \tilde{X}_{t\wedge S_n}\|^2 du. \end{aligned}$$

Using Gronwall inequality with $g(t) = E\|X_{t\wedge S_n} - \tilde{X}_{t\wedge S_n}\|^2$ we can conclude that X and $\tilde{X}$ are modifications of one another, thus are indistinguishable. Letting $n \to \infty$ we get the desired result.

Remark 3.2. *Even for the ordinary differential equations the condition of local Lipschitz is not sufficient to guarantee global existence of a solution. For example:*

$$X_t = 1 + \int_0^t X_s^2 ds,$$

is $X_t = 1/(1-t)$ which shoots to infinity as $t \uparrow 1$. We thus need a stronger condition to show the existence of the solution.

Theorem 13. *[KS91] Let the coefficients $b_i(s, X_s)$ and $\sigma_{ij}(s, X_s)$ satisfy the global Lipschitz and linear growth conditions, i.e,*

$$\|b(t,x) - b(t,y)\| + \|\sigma(t,x) - \sigma(t,y)\| \leq A\|x - y\|. \tag{3.8}$$

$$\|b(t,x)\|^2 + \|\sigma(t,x)\|^2 \leq A^2(1 + \|x\|^2). \tag{3.9}$$

for every $0 \leq t < \infty$, $x \in \mathbb{R}^d$, $y \in \mathbb{R}^d$ and $A \in [0, \infty)$. On probability triple $(\Omega, \mathcal{F}, \mathbb{P})$, let $\mathcal{F}_t$ be the filtration defined and let ξ be $\mathbb{R}^d$ valued random variable which is independent of the

Brownian motion $W = \{W_t, \mathcal{F}_t^W, 0 \le t < \infty\}$ of dimension r, such that ξ satisfies,

$$E\|\xi\|^2 < \infty. \tag{3.10}$$

Then there exist a continuous process $X = \{X_t, \mathcal{F}_t, 0 \le t < \infty\}$ adapted to the filtration such that it is a strong solution of the SDE with respect to the given Brownian motion W and initial conditions ξ. This process is also a square integrable process such that for every $T \ge 0$, we have constant C, depending on A and T such that we have,

$$E\|X_t\|^2 \le C(1 + E\|\xi\|^2)e^{Ct};\ 0 \le t \le T \tag{3.11}$$

The idea of the proof is to use iterative method and to construct recursively using Picard's iterations a sequence of successive approximations with $X_t(0) = \xi$ and,

$$X_t^{(k+1)} := \xi + \int_0^t b(s, X_s^{(k)})ds + \int_0^t \sigma(s, X_s^{(k)})dW_s; \quad 0 \le t < \infty. \tag{3.12}$$

Lemma 3.1. *For every $T > 0$, we have a positive constant C that depends only on A and T, such that the iterations in (3.12) satisfy:*

$$E\|X_t{}^{(k)}\|^2 \le C(1 + E\|\xi\|^2)e^{Ct};\ 0 \le t \le T; \quad 0 \le t \le T,\ k \ge 0. \tag{3.13}$$

Proof We need to show that for $k \ge 0$ we must have the following:

$$\int_0^t \{\|b(t, X_t{}^{(k)})\| + \|\sigma(t, X_t{}^{(k)})\|\} < \infty.$$

for $k = 0$, the above holds true. We can show by induction that,

$$\sup_{0 \le t \le T} E\|X_t^{(k)}\|^2 < \infty. \tag{3.14}$$

Let us assume that (3.14) is true for some value k. Then using the idea of Theorem 12 we get a bound for t i.e,

$$E\|X_t^{(k+1)}\|^2 \le 9E\|\xi\|^2 + 9(T+1)A^2 \int_0^t (1 + E\|X_t^{(k)}\|^2)ds.$$

which gives (3.14) for $k + 1$. Also we have,

$$E\|X_t^{(k+1)}\|^2 \le C(1 + E\|\xi\|^2) + C\int_0^t E\|X_t^{(k)}\|^2 ds.$$

Iteration of this inequality gives:

$$E\|X_t^{(k+1)}\|^2 \leq C(1+E\|\xi\|^2)[1+Ct+\frac{(Ct)^2}{2!}+\cdots+\frac{(Ct)^{k+1}}{(k+1)!}].$$

This gives us the desired results.

Proposition 3.2 (Martingale Moment Inequalities)**.** *For a continuous martingale M which is bounded along with its quadratic variation process $\langle M\rangle$, we have, for every stopping time T,*

$$E|M_T|^{(2m)} \leq A_m E\langle M_T\rangle^{(m)}. \tag{3.15}$$

$$B_m E\langle M_T\rangle^{(m)} \leq E|M_T|^{(m)}. \tag{3.16}$$

$$B_m E\langle M_T\rangle^{(m)} \leq E(M_T^*)^{2m} \leq A_m E\langle M_T\rangle^{(m)}. \tag{3.17}$$

where $M_t^* = \max_{s\leq t}|M_s|$

Proposition 3.3. *Let M be a d-dimensional continuous local martingale, i.e., $M^{(i)} \in \mathcal{M}_2^c$. Also*

$$\|M\|_t^* = \max_{s\leq t}\|M_s\| \; ; \qquad A_t = \sum_{i=1}^{d}\langle M^{(i)}\rangle_t. \tag{3.18}$$

Then for every stopping time T we have constants λ_m and Λ_m such that :

$$\lambda_m E(A_T^m) \leq E(\|M\|_T^*)^{2m} \leq \Lambda_m E(A_T^m). \tag{3.19}$$

Remark 3.3. *If $M_t^{(i)}$ in the above Proposition is given by,*

$$M_t^{(i)} = \sum_{i=1}^{d}\int_0^t X_s^{i,j}dW_s^{(j)},$$

where W is a r dimensional Brownian motion and

$$X = \{X_t = X_t^{i,j};\ 1\leq i\leq d;\ 1\leq j\leq r\},$$

such that X_t is $\mathcal{F}_t$ measurable and,

$$\|X_t\|^2 = \sum_{i=1}^{d}\sum_{j=1}^{r}\left(X_t^{i,j}\right)^2.$$

Then the above proposition holds with,

$$A_T = \int_0^T \|X_t\|^2 dt$$

Proof of Theorem 13 We can decompose $X_t(k+1) - X_t(k) = B_t + M_t$ where,

$$B_t = \int_0^t \{b_i(s, X_s^k) - b_i(s, X_s^{k-1})\}ds \quad M_t = \int_0^t \{\sigma_i(s, X_s^k) - \sigma_i(s, X_s^{k-1})\}dW_s.$$

Using the global Lipschitz condition and growth condition along with the martingale moment inequalities we have,

$$E[\max_{s\le t}\|M_s\|^2] \le \Lambda_1 E\int_0^t \|\sigma(s, X_s^k) - \sigma(s, X_s^{k-1})\|ds \le \Lambda_1 A^2 E\int_0^t \|X_s^k - X_s^{k-1}\|^2 ds.$$

For the square integrable continuous martingale M, we also have,

$$E\|B_t\|^2 \le A^2 tE\int_0^t \|X_s^k - X_s^{k-1}\|^2 ds.$$

Therefore,

$$E[\max_{s\le t}\|X_s^{k+1} - X_s^k\|^2] \le 4A^2(\Lambda + T)E\int_0^t \|X_s^k - X_s^{k-1}\|^2 ds.$$

The above inequality can be iterated successively to get,

$$E[\max_{s\le t}\|X_s^{k+1} - X_s^k\|^2] \le \max_{s\le T} E\|X_t^1 - \xi\|^2 \frac{(4A^2(\Lambda+T)t)^k}{k!}.$$

Using the Chebyshev's inequality we now get,

$$\mathbb{P}\Big[\max_{s\le T}\|X_s^{k+1} - X_s^k\|^2 \ge \frac{1}{2^k}\Big] \le 4\max_{s\le t}\|X_t^1 - \xi\|^2 \cdot \frac{(4\cdot 4A^2(\Lambda+T)t)^k}{k!}.$$

A general term of the convergent series with above terms has an upper bound. Using Borel Cantelli lemma we can say that $\exists\ \Omega^* \in \mathcal{F}$ such that, we have $\mathbb{P}(\Omega^*) = 1$ and corresponding to each $\omega \in \Omega^*$, we have an integer valued random variable such that $\forall\ \omega \in \Omega^*$,

$$\max_{s\le T}\|X_s^{k+1} - X_s^k\| \le 1/2^{k+1} \quad \forall\ k \ge N(\omega).$$
$$\max_{s\le T}\|X_s^{k+m} - X_s^k\| \le 1/2^{k+1} \quad \forall\ k \ge N(\omega)\ \forall\ m \ge 1.$$

On the space of continuous functions, we see that that the sequence of sample paths $\{N_t^{(k)}(\omega); s \ge T\}$ are convergent in sup norm. Using this convergence, we can say that the continuous limit $\{X_t; t \le T\}$ is exists. T is arbitrary and the process is continuous, the sample paths converges uniformly on the compact sets. Finally we show that $X_t = \lim X_t^k \ \ t \ge 0$ satisfies the 4th condition of the definition. Since the process is square integrable and satisfy linear growth condition we have property 3 being satisfied of the definition.

Remark 3.4. *On the one dimensional case, we can considerably relax the Lipschitz conditions on the dispersion coefficients*

Proposition 3.4 (Yamade and Watanabe(1971))**.** *[KS91] Let us suppose that the coefficients of the one dimensional equation where $d = r = 1$ can be written as,*

$$dX_t = b(t, X_t)dt + \sigma(t, X_t)dW_t,$$

which satisfies

$$|b(t,x) - b(t,y)| \leq A|x - y|, \tag{3.20}$$

$$|\sigma(t,x) - \sigma(t,y)| \leq h(|x - y|), \tag{3.21}$$

$\forall\ t \in [0, \infty)$ *and* $x, y \in \mathbb{R}$, *where* A *is a positive constant and* h *is a strictly increasing function with* $h(0) = 0$ *and*

$$\int_{(0,\epsilon)} h^{-2}(u)du = \infty; \quad \forall \epsilon > 0.$$

Then we have strong uniqueness for (3.1).

Proof There exists a decreasing sequence $\{a_n\}_{n=0}^{\infty}$ in [0,1) such that $\lim_{n\to\infty} a_n = 0$ and $\int_{a_n}^{a_{n-1}} h^{-2}(u)du = n\ \forall\ n \geq 1$. For each n there exist a continuous function ρ_n on $\mathbb{R}$ with support in (a_{n-1}, a_n) such that $0 \leq \rho_n(x) \leq (\dfrac{2}{nh^2(x)})$; $\forall x \in (0, \infty)$ and $\int_{a_n}^{a_{n-1}} \rho_n(x) = 1$. Then we have,

$$\psi_n(x) := \int_0^{|x|} \int_0^y \rho_n(u)dudy; \quad x \in \mathbb{R}. \tag{3.22}$$

This is a $C^2(\mathbb{R})$ function such that $|\psi_n'(x)| \leq 1$ and the sequence $\{\psi_n\}_{n=1}^{\infty}$ is non decreasing. Let us take two solutions of (3.1) $X^{(1)}, X^{(2)}$. By definition we take,

$$E\int_0^t |\sigma(s, X_s)|^2 ds < \infty; \quad 0 \leq t < \infty.$$

$$\Xi_t = X_t^{(1)} - X_t^{(2)} = \int_0^t \{b(s, X_s^{(1)}) - b(s, X_s^{(2)})\}ds + \int_0^t \{\sigma(s, X_s^{(1)}) - \sigma(s, X_s^{(2)})\}ds$$

We further apply the Ito's Formula to get,

$$\psi_n(\Xi_t) = \int_0^t \psi_n'(\Xi_s)[b(s,X_s^{(1)}) - b(s,X_s^{(2)})]ds + \frac{1}{2}\int_0^t \psi_n''(\Xi_s)[\sigma(s,X_s^{(1)}) - \sigma(s,X_s^{(2)})]^2 ds$$
$$+ \int_0^t \psi_n'(\Xi_s)[\sigma(s,X_s^{(1)}) - \sigma(s,X_s^{(2)})]^2 dW_s.$$

We now have $E[\int_0^t \psi_n'(\Xi_s)[\sigma(s,X_s^{(1)}) - \sigma(s,X_s^{(2)})]^2 dW_s] = 0$ as W is a brownian motion and, $E[\int_0^t |\sigma(s,X_s)|^2 ds] < \infty; 0 \leq t < \infty$. Further we have for the second term

$$E[\int_0^t \psi_n''(\Xi_s)[\sigma(s,X_s^{(1)}) - \sigma(s,X_s^{(2)})]^2 ds] \leq E[\int_0^t \psi_n''(\Xi_s)[h|\Xi_s|]^2 ds] \leq 2t/n.$$

Hence forth,

$$E\psi_n(\Xi_t) = E\int_0^t \psi_n'(\Xi_s)[b(s,X_s^{(1)}) - b(s,X_s^{(2)})]ds + t/n$$
$$\leq A\int_0^t E|\Xi_s|ds + t/n$$

as $n \to \infty$ gives us $E|\Xi_t| \leq A\int_0^t E|\Xi_s|ds$. By applying Gronwall Inequality we get the desired results.

3.3 Weak Solutions

There are two concepts of uniqueness that can be associated with the existence of the weak solution to stochastic differential equations. The first talks about the pathwise uniqueness which is a generalization of strong solution and the other is uniqueness in law which is more weaker sense of equality. Also pathwise uniqueness implies uniqueness in law.

Definition 3.3. *Whenever* $\{(X,W),\ (\Omega,\mathcal{F},\mathbb{P})\}$, $\{\mathcal{F}_t\}$ *and* $(\tilde{X},W)$, $\{(\Omega,\mathcal{F},\mathbb{P})\}$, $\{\tilde{\mathcal{F}}_t\}$ *are weak solutions to SDE with common Brownian motion and on the same probability space, with the same initial value, then the two solutions* X *and* $\tilde{X}$ *are indistinguishable i.e.,* $\mathbb{P}[X = \tilde{X}; \forall 0 \leq t < \infty] = 1$. *Then* X *and* $\tilde{X}$ *are said to have pathwise uniqueness.*

Definition 3.4. *Uniqueness in the sense of distribution is said to hold if for any two weak solutions* (X,W), $(\Omega,\mathcal{F},\mathbb{P})$,$\{F_t\}$ *and* $(\tilde{X},\tilde{W})$, $(\Omega,\mathcal{F},\mathbb{P})$, $\{\tilde{\mathcal{F}}_t\}$ *with the same initial distribution, i.e.*

$$\mathbb{P}[X_0 \in \Gamma] = \tilde{\mathbb{P}}[\tilde{\mathbb{X}}_0 \in \Gamma]; \quad \Gamma \in \mathcal{B}(\mathbb{R}^d),$$

have the same law.

To state the existence of a weak solution we need to state the Girsanov's theorem. Consider for the process

$$Y_t = exp(\int_0^t X_s dB_s - \frac{1}{2}\int_0^t (X)_s{}^2 ds.$$

This process is a martingale with,

$$E[Y_T] = E[Y_0] = 1.$$

Theorem 14. *Let the process X be defined on the probability space $\{\Omega, \mathcal{F}, \mathbb{P}\}$ such that $\{B_t, \mathcal{F}_t; t \geq 0\}$ is a standard Brownian motion. Let Y be the process defined by,*

$$Y_t = exp(\int_0^t X_s dB_s - \frac{1}{2}\int_0^t (X)_s{}^2 ds)$$

with,

$$E[Y_T] = 1.$$

Then on the same probability space we define a measure by,

$$\tilde{\mathbb{P}}(d\omega) = Y_T(\omega)\mathbb{P}(\omega),$$

such that the process $\{\tilde{B}, \mathcal{F}; t \geq 0\}$ on the probability space $\{\Omega, \mathcal{F}, \tilde{\mathbb{P}}\}$ is a Brownian motion with

$$\tilde{B}_t = \int_0^t X_s ds + B_t.$$

We can now talk about the existence of weak solution to the SDE

Theorem 15. *Consider the stochastic differential equation*

$$dX_t = b(t, X_t) + \sigma(t, X_t)dB_t; \quad 0 \leq t \leq T$$

with the initial conditions $X_0 = x$, such that X, b and σ are adapted to the filtration, then the weak solution exists.

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