On the Spontaneous Localization Process of Many-Particle System

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Certificate of Examination

This is to certify that the dissertation titled **On the Spontaneous Localization Process of Many-Particle System** submitted by **Rajendra Singh Bhati** (Reg. No. MS11019) for the partial fulfillment of BS-MS dual degree programme of the Institute, has been examined by the thesis committee duly appointed by the Institute. The committee finds the work done by the candidate satisfactory and recommends that the report be accepted.

Dr. Kavita Dorai	Dr. Mandip Singh	Prof. Arvind (Supervisor)

Dated: April 22, 2016

Declaration

The work presented in this dissertation has been carried out by me under the guidance of Prof. Arvind at the Indian Institute of Science Education and Research Mohali.

This work has not been submitted in part or in full for a degree, a diploma, or a fellowship to any other university or institute. Whenever contributions of others are involved, every effort is made to indicate this clearly, with due acknowledgement of collaborative research and discussions. This thesis is a bonafide record of original work done by me and all sources listed within have been detailed in the bibliography.

Rajendra Singh Bhati
(Candidate)

Dated: April 22, 2016

In my capacity as the supervisor of the candidates project work, I certify that the above statements by the candidate are true to the best of my knowledge.

Prof. Arvind
(Supervisor)

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Notation

$\mathbb{I}$	Identity Matrix(in appropriate dimensions)
$\mathscr{H}$	Hilbert space
$\|A\|$	Norm of A
$\{A, B\}$	Anti-Commutation of A and B
$[A, B]$	Commutation of A and B
$\hbar$	Plank Constant divided by 2π
c	Speed of light in vacuum
$\otimes$	Tensor product
$\langle A\rangle$	Expectation value A
$\bar{A}$	Average of A

Abstract

Quantum Mechanics has met severe difficulties in accounting for the measurement problem. Apart from the re-interpretative and decoherence approaches an attempt based on consideration of stochastic and nonlinear modification to standard Schrodinger Equation has also been made [9]. The new dynamics unifies micro and macroscopic phenomena. The formalism of the dynamics is reviewed. We re-analyze the approach taken by G.C. Ghirardi, P. Pearle, A. Rimini [10] and A. Bassi [12] to treat spontaneous collapse process for many-particle systems. A claim was made in earlier work that due to heavier center of mass, spread in center of mass position reduces very fast and behaves like a classical particle. A big flaw in their approach showing inconsistency of their claims with those of quantum mechanics is presented. We propose a legitimate method to explain wavepacket reduction of entangled particles and investigate the role of interaction in wavepacket reduction process using GRW model.

Contents

Chapter 1

Introduction

Quantum Theory has met an unprecedented success in accounting for various phenomena in atomic, sub-atomic, optical, material, molecular, nuclear and particle physics. The theory successfully unifies three of the four fundamental interactions and has not encountered any empirical contradiction. Yet the postulates and interpretation of the theory have been the subjects of a great debate. The counter-intuitive formalism of Quantum Theory apparently meets severe contradiction with the intuitive phenomena of the 'classical world'. In other words *the transition from the Quantum to Classical world* is not clear in the formalism. This aspect of the theory is mostly known as *'Schrödinger's Cat Paradox'* or *'Measurement Problem'*.

1.1 The Quantum Measurement Problem

1.1.1 The Postulates of Quantum Theory

The formalism of quantum mechanics can be summarized in following set of postulates:

1. The physical states of a system is represented by normalized vectors (known as 'state vector') in an Hilbert space $\mathscr{H}$ associated with the same. A physical observable O is represented by a Hermitian (self-adjoint) operator in Hilbert space.
$$|\psi\rangle = \sum_i c_i|\psi_i\rangle \tag{1.1}$$
$$\mathrm{O}|\psi\rangle = o_i|\psi_i\rangle$$

2. Time evolution of the state vector between two consecutive measurement is generated by Hamiltonian H of the system. The evolution is unitary, deterministic

and reversible governed by a linear differential equation; known as 'Schrödinger equation':

$$\iota\hbar\frac{\partial|\psi\rangle}{\partial t} = \mathrm{H}|\psi\rangle \tag{1.2}$$

3. The 'measurement' on the system drastically reduces the state vector to one of the eigen-states of the observable being measured. The evolution is non-unitary, irreversible and non-deterministic. This postulate is known as the *postulate of wavepacket reduction*. If $\mathrm{P_i}$ is a projection operator then during the measurement

$$|\psi\rangle \rightarrow \frac{\mathrm{P_i}|\psi\rangle}{\|\mathrm{P_i}|\psi\rangle\|} \tag{1.3}$$

4. The outcome of the measurement is purely probabilistic. The probability of state $|\psi\rangle$ being reduced in a sate $|\phi\rangle$ is given by Born rule:

$$Prob(|\phi\rangle) = \|\langle\phi|\psi\rangle\|^2 \tag{1.4}$$

From postulates 3 and 4, it is clear that 'measurement' of an observable performed on an ensemble of identical systems represented by $|\psi\rangle$ leads the ensemble to a statistical mixture of projected states $|\psi_i\rangle$. It is impractical to keep the record of each individual system among the ensemble. To simplify the study of ensembles, the formalism of statistical operator is introduced.An ensemble of systems is represented by statistical operator ρ. Suppose individual N_i systems described by state vector $|\psi_i\rangle$ belong to an ensemble consists of N systems. Then the statistical operator corresponding to the ensemble is represented as:

$$\rho = \sum_i p_i|\psi_i\rangle\langle\psi_i| \tag{1.5}$$

where $p_i = N_i/N$. For pure states $\rho^2 = \rho$ and $\rho^2 \neq \rho$ for statistical mixtures. The time evolution of ρ corresponding to (1.2) is given by

$$\iota\hbar\frac{d}{dt}\rho(t) = \left[\mathrm{H}, \rho(t)\right] \tag{1.6}$$

The effect of measurement (corresponding to (1.3)) on ρ is:

$$\rho \rightarrow \sum_i P_i\rho P_i \tag{1.7}$$

The most peculiar feature of Quantum Mechanics is that, in general, it can give only the probabilistic description about the possible outcomes of any observable's measurements. Unlike in classical statistical mechanics, these probabilities are not due to an observer's ignorance of precise knowledge about the state of the system; rather, quantum mechanics is such that the 'measurement process' itself possesses an inbuilt 'randomization process' which leads a pure ensemble to a mixed ensemble.

1.1.2 The Measurement Scheme

An ideal measurement scheme for quantum description of measurement was first presented by John von Neumann [1].In such a scheme a microscopic system S with one of its observable O consisting eigenvalues $\{o_i\}$ is considered. An apparatus A is devised to measure the observable O of system S. There exists a set of mutually orthogonal states $\{|A_i\rangle\}$ representing different macroscopic configurations of the measuring apparatus. The macroscopic state $|A_i\rangle$ of apparatus corresponds to microscopic state $|o_i\rangle$ of the system i.e. initial state of system $|o_i\rangle$ drives apparatus to $|A_i\rangle$ configuration. If $|A_0\rangle$ is the initial configuration of apparatus then due to measurement:

$$|o_i\rangle \otimes |A_0\rangle \rightarrow |o_i\rangle \otimes |A_i\rangle \tag{1.8}$$

The interaction between apparatus and system is governed by interaction Hamiltonian of the combined system. Accordingly, the evolution of the state $(\sum_i c_i|o_i\rangle \otimes |A_0\rangle)$ in combined Hilbert space $\mathscr{H}_S \otimes \mathscr{H}_A$ is linear and can be written as:

$$\sum_i c_i|o_i\rangle \otimes |A_0\rangle \rightarrow \sum_i c_i|o_i\rangle \otimes |A_i\rangle \tag{1.9}$$

This is an entangled state of apparatus and system representing the superposition of macroscopic configurations of apparatus which is not a legitimate situation. To avoid apparent contradiction with observed classical phenomenon i.e. lack of macroscopic superposition, the *postulate of wavepacket reduction* (1.3) was taken into consideration.

1.1.3 The Problem

According to the standard quantum mechanics axioms, Hamiltonian of the system is the only operator that generates the time evolution and for any form of Hamiltonian (1.2) and (1.6) must be valid. Therefore, the system-apparatus interaction must be governed by an unitary and deterministic evolution no matter how large the combined system is. Thus the postulate of wavepacket reduction ((1.3) and (1.7)) is in direct conflict with time evolution postulate (Schrödinger equation, (1.2) and (1.6)). This contradiction among the fundamental axioms of standard quantum mechanics (known as *measurement problem of quantum mechanics*) can be summarized in following questions:

- What generates the nonlinear, non-unitary, irreversible and probabilistic time evolution during the measurement process while the time evolution in all dynamical situations in standard quantum mechanics are linear, unitary, reversible and deterministic?
- Where to draw the line between normal quantum mechanical interaction (governed by (1.2)) and measurement type interaction (governed by (1.3))?
- Where to draw the line between quantum system and classical system?

1.2 Major Approaches to Solve the Measurement Problem

Soon after the formalism of standard quantum mechanics, Schrödinger presented this paradoxical situation known as *'Cat Paradox'*. The problem has been highly debated and it has touched various aspect of theory e.g. interpretations, completeness of theory and realism. Various approaches taken to solve the problem can be summarized in following categories:

1. **Realism**: Contrary to the superposition postulate of the quantum theory (1.1), a system is always considered to be in one of the basis of observable and the non-classical behavior of system is due to existence of *'hidden variables'*. The existence of local hidden variable theories has been falsified by Bell's inequality tests[2],[3]. The non-local hidden variable theories can reproduce all quantum mechanical predictions and do not differ in predictions. Bohmian Mechanics[4] is such an example. But non-falsifiable and non-local features of the theory prevent us to accept it.

2. **Interpretation**: This approach deals with re-interpretations of measurement processes. *Copenhagen interpretation* [5] considers a sharp divide between quantum and classical world following different sets of laws of nature. The interpretation bypasses the question -'where should the divide be put?' or 'at what mass

scale quantum theory breaks down?' *von Neumann interpretation*[1] is another one which considers ideal measurement situation of §1.1.2 and measurement is assumed to take place only when a *'conscious observer'* observes it. The interpretation involves complexity of neurology and not so satisfactory. The *many-world interpretation*[7] does not consider reduction of wavepacket. According to this interpretation there exist many parallel worlds those are different branches of the universe and orthogonal outcomes of (1.9) lie in different branches of the universe, hence, the reduction is mere an appearance. This approach cannot explain why there is probabilistic outcome (Born rule) in measurement. There are other interpretations but they are also not satisfactory in explaining all aspects of the quantum theory.

3. **Decoherence**: The measurement scheme presented by von Neumann[1] (see §1.1.2) is highly criticized because of its over simplistic model. von Neumann did not took role of environment-apparatus interaction into consideration. The environment can be consider as a collection of all particles or systems present in a sphere of radius cT centered at the apparatus location, where c is speed of light and T is the time interval during which measurement process takes place. These particles can causally affect the apparatus-system interaction. This interaction is understood as decoherence process. The decoherence process and its role in localization of state vector is analyzed in [14], [15] and [16]. Its is explicitly shown that due to environment-apparatus interaction, the statistical operator of combined system becomes diagonal representing the statistical mixture. But a statistical operator can be constructed in such a way that the individual particle is not localized but the statistical operator is diagonal. It is due to ambiguity of basis for statistical operator.

4. **Dynamical Reduction Models**: In the dynamical reduction approach, standard quantum mechanics is considered to be approximation of more general theory. It is assumed that an unified time evolution equation reproduces all classical and quantum mechanical phenomena in same formalism. The models consider role of mass in reduction of wave packet. First attempt to construct such a model was made by P. Pearle (1976) [17]. The most consistent model was proposed by G.C. Ghirardi, A. Rimini and T. Weber (known as GRW model) in 1986 [8] and later modified version (continuous spontaneous localization model) was proposed in [9] and [10]. We will discourse formalism and consequences of the both through out the thesis.

Chapter 2

Spontaneous Collapse Model

First consistent collapse model was presented by Ghirardi, Rimini and Weber[8]. The model is also known as 'Quantum Mechanics with Spontaneous Localization'. According to the model, spontaneous discrete jumps cause the localization of wavepacket.

2.1 Assumptions

The model is based on following assumptions:

1. Each particle of many-particle system experiences a sudden jump to position basis with a mean rate λ. In the time interval between two successive spontaneous localization processes, the system evolves according to the Schr$\ddot{o}$dinger time evolution equation (1.2).

$$|\psi\rangle \to \frac{|\psi_x^i\rangle}{\||\psi_x^i\rangle\|} \tag{2.1}$$

$$|\psi_x^i\rangle = L_x^i|\psi\rangle$$

   where $|\psi\rangle$ is state vector in n-particle Hilbert space $\mathscr{H}^n$ and $|\psi_x^i\rangle$ is state vector of system in same Hilbert space after localization of particle i around point $\mathbf{x}$.

2. The probability density for localization to be occurred at point $\mathbf{x}$ is

$$P_i(x) = \||\psi^i(x)\rangle\|^2 \tag{2.2}$$

3. The localization operator L_x^i is given by

$$L_x^i = \left(\frac{\alpha}{\pi}\right)^{3/4} e^{-(\alpha/2)(\mathbf{q}_i - \mathbf{x})^2} \tag{2.3}$$

where α is a new parameter which sets the width of localization and $\mathbf{q}_i$ is the position operator for particle *i*.

2.2 Formalism and Consequences

Let us assume a single particle system experiences a sudden jump from state $|\psi\rangle$ to state $|\psi_x\rangle = L_x^i|\psi\rangle$ i.e. localization happens around point $\mathbf{x}$ with probability $P(x)$. The process transforms a pure ensemble into mixed ensemble. Transformation of statistical operator can be written as:

$$\rho \rightarrow \rho_T$$

$$|\psi\rangle\langle\psi| \rightarrow \int d^3x P(x) \frac{|\psi_x\rangle\langle\psi_x|}{\||\psi_x\|^2} = \int d^3x L_x^i |\psi\rangle\langle\psi| L_x^i \tag{2.4}$$

here we have used (2.2). Since, the localization probability distribution is Poissonian, λdt is the probability for a jump to occur in dt time interval and $(1 - \lambda dt)$ is the probability that state evolves according to Schrödinger equation (1.2). If an ensemble contains total N identically prepared quantum systems in state $|\psi\rangle$ at time t, after time $(t+dt)$, the ensemble will be divided into two parts - one contains the $(1-\lambda dt)N$ systems which have evolved according to standard quantum mechanical evolution (1.6) and another contains $(\lambda dt)N$ systems which have experienced spontaneous localization processes(2.4). Statistical operator after time $t + dt$ can be written using (1.5).

$$\rho(t + dt) = (1 - \lambda dt)\rho'(t + dt) + \lambda dt \rho_T \tag{2.5}$$

according to (1.6)

$$\rho'(t + dt) = \rho(t) - \frac{\iota}{\hbar}\big[\mathrm{H}, \rho(t)\big] dt$$

and ρ_T given by (2.4). Now,

$$\rho(t + dt) = (1 - \lambda dt)\left(\rho(t) - \frac{\iota}{\hbar}\big[\mathrm{H}, \rho(t)\big] dt\right) + \lambda dt \rho_T$$

$$\Rightarrow \frac{\mathrm{d}}{\mathrm{d}t}\rho(t) = -\frac{\iota}{\hbar}\big[\mathrm{H}, \rho(t)\big] - \lambda\left(\rho(t) - \rho_T(t)\right) \tag{2.6}$$

This is the master master equation of GRW model. Using (2.3) and (2.4) one can write expression for $\langle \mathbf{x}'|\rho_T|\mathbf{x}''\rangle$ as:

$$\begin{aligned}\langle \mathbf{x}'|\rho_T|\mathbf{x}''\rangle =&\left(\frac{\alpha}{\pi}\right)^{3/2}\int d^3x\langle \mathbf{x}'|e^{-(\alpha/2)(\mathbf{q}-\mathbf{x})^2}|\psi\rangle\langle\psi|e^{-(\alpha/2)(\mathbf{q}-\mathbf{x})^2}|\mathbf{x}''\rangle\\ =&\langle \mathbf{x}'|\rho|\mathbf{x}''\rangle\left(\frac{\alpha}{\pi}\right)^{3/2}\int d^3xe^{-(\alpha/2)\{(\mathbf{x}'-\mathbf{x})^2+(\mathbf{x}''-\mathbf{x})^2\}}\\ =&\langle \mathbf{x}'|\rho|\mathbf{x}''\rangle e^{-(\alpha/4)\{(\mathbf{x}'-\mathbf{x}'')^2\}}\end{aligned} \tag{2.7}$$

Feeding (2.7) into (2.6) we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle \mathbf{x}'|\rho(t)|\mathbf{x}''\rangle = -\frac{\iota}{\hbar}\langle \mathbf{x}'|\Big[\mathrm{H},\rho(t)\Big]|\mathbf{x}''\rangle - \lambda\left(1-e^{-(\alpha/4)\{(\mathbf{x}'-\mathbf{x}'')^2\}}\right)\langle \mathbf{x}'|\rho(t)|\mathbf{x}''\rangle \tag{2.8}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle \mathbf{x}'|\rho(t)|\mathbf{x}''\rangle = \frac{\mathrm{d}}{\mathrm{d}t}\langle \mathbf{x}'|\rho(t)|\mathbf{x}''\rangle\Big|_{unitary} + \frac{\mathrm{d}}{\mathrm{d}t}\langle \mathbf{x}'|\rho(t)|\mathbf{x}''\rangle\Big|_{spont.collapse}$$

Here

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle \mathbf{x}'|\rho(t)|\mathbf{x}''\rangle\Big|_{spont.collapse} = -\lambda\left(1-e^{-(\alpha/4)\{(\mathbf{x}'-\mathbf{x}'')^2\}}\right)\langle \mathbf{x}'|\rho(t)|\mathbf{x}''\rangle \tag{2.9}$$

For analytic study of dynamics of spontaneous collapse processes, unitary evolution in (2.8) can be dropped. The rate of change in statistical operator due to collapse is given by (2.9). It is clear from above results that diagonal elements of statistical operator remains unaffected and only off-diagonal elements reduces. Since the evolution of density matrix elements due to (1.5) is unitary and norm-conserving, we can integrate (2.9) to get $\|\langle \mathbf{x}'|\rho(t)|\mathbf{x}''\rangle\|$.

$$\|\langle \mathbf{x}'|\rho(t)|\mathbf{x}''\rangle\| = \|\langle \mathbf{x}'|\rho_0|\mathbf{x}''\rangle\|\exp\Big\{-\lambda\left(1-e^{-(\alpha/4)\{(\mathbf{x}'-\mathbf{x}'')^2\}}\right)t\Big\} \tag{2.10}$$

where α is taken such that particle is sharply localized around point $\mathbf{x}$. $\alpha \approx 10^{-12}m^{-2}$ is the value suggested in [8]. For $(\mathbf{x}'-\mathbf{x}'') \gg \frac{1}{\sqrt{\alpha}} \sim 10^{-6}$, $e^{-(\alpha/4)\{(\mathbf{x}'-\mathbf{x}'')^2\}} \ll 1$. In this case (2.10) can be rewritten as:

$$\|\langle \mathbf{x}'|\rho(t)|\mathbf{x}''\rangle\| = \|\langle \mathbf{x}'|\rho_0|\mathbf{x}''\rangle\|\exp\{-\lambda t\} \tag{2.11}$$

It is clear that for $(x'-x'') \gg \frac{1}{\sqrt{\alpha}}$ off-diagonal elements of density matrix reduce exponentially where diagonal elements remain constant.

Chapter 3

Continuous Spontaneous Localization Model

3.1 Formalism

Let's consider a state vector $|\psi\rangle$ in Hilbert space satisfying the It$\hat{o}$ stochastic differential equation [11]:

$$\begin{aligned} d|\psi\rangle &= [\mathrm{C}\,dt + \mathbf{A}\cdot d\mathbf{B}]|\psi\rangle \\ d\langle\psi| &= \langle\psi|[\mathrm{C}^\dagger\,dt + \mathbf{A}^\dagger\cdot d\mathbf{B}] \end{aligned} \tag{3.1}$$

where C is an operator, $\mathbf{A} \equiv \{\mathrm{A}_i\}$ is a set of operators and $\mathbf{B} \equiv \{\mathrm{B}_i\}$ is a set of real Markov processes i.e.

$$\overline{d\,\mathrm{B}_i} = 0 \tag{3.2}$$

$$\overline{d\,\mathrm{B}_i\,d\,\mathrm{B}_j} = \delta_{ij}\gamma dt \tag{3.3}$$

γ is a constant representing the strength of noise field (appa. B) and the dot product in (3.1) has usual meaning:

$$\mathbf{A}\cdot d\mathbf{B} = \sum_i \mathrm{A}_i\,d\,\mathrm{B}_i \tag{3.4}$$

To see whether (3.1) preserves norm, we can calculate infinitesimal change in norm-square of $|\psi\rangle$ in time interval dt.

$$d\||\psi_t\rangle\|^2 = \||\psi_{t+dt}\rangle\|^2 - \||\psi_t\rangle\|^2 \tag{3.5}$$

If the change in state vector in dt time is $d|\psi\rangle = |d\psi\rangle$ and initial vector (at time t) is $|\psi_t\rangle$, state vector at time $t + dt$ is:

$$|\psi_{t+dt}\rangle = |\psi_t\rangle + |d\psi\rangle \tag{3.6}$$

Similarly,

$$\langle\psi_{t+dt}| = \langle\psi_t| + \langle d\psi| \tag{3.7}$$

$$\begin{aligned} \||\psi_{t+dt}\rangle\|^2 &= \langle\psi_{t+dt}|\psi_{t+dt}\rangle \\ &= \langle\psi_t|\psi_t\rangle + \langle d\psi|\psi_t\rangle + \langle\psi_t|d\psi\rangle + \langle d\psi|d\psi\rangle \end{aligned} \tag{3.8}$$

Using (3.8), (3.1), (3.2), (3.3) and (3.5) we get:

$$d\||\psi_t\rangle\|^2 = \langle\psi_t|(\mathbf{A} + \mathbf{A}^\dagger)|\psi\rangle \cdot d\mathbf{B} + \langle\psi_t|(\mathrm{C} + \mathrm{C}^\dagger)|\psi_t\rangle dt + \langle\psi_t|\mathbf{A}^\dagger \cdot \mathbf{A}|\psi_t\rangle\gamma dt \tag{3.9}$$

$$\Rightarrow \quad \overline{d\||\psi_t\rangle\|^2} = \overline{\langle\psi_t|(\mathbf{A} + \mathbf{A}^\dagger)|\psi\rangle \cdot d\mathbf{B}} + \langle\psi_t|(\mathrm{C} + \mathrm{C}^\dagger)|\psi_t\rangle dt + \langle\psi_t|\mathbf{A}^\dagger \cdot \mathbf{A}|\psi_t\rangle\gamma dt \tag{3.10}$$

Using property (3.2):

$$\overline{d\||\psi_t\rangle\|^2} = \langle\psi_t|(\mathrm{C} + \mathrm{C}^\dagger)|\psi_t\rangle dt + \langle\psi_t|\mathbf{A}^\dagger \cdot \mathbf{A}|\psi_t\rangle\gamma dt \tag{3.11}$$

It is clear from (3.11) that for arbitrary operators $\mathbf{A}$ and C may not preserve norm of the process. For norm preserving condition we take $\overline{d\||\psi_t\rangle\|^2} = 0$.

$$\langle\psi_t|(\mathrm{C} + \mathrm{C}^\dagger)|\psi_t\rangle dt + \langle\psi_t|\mathbf{A}^\dagger \cdot \mathbf{A}|\psi_t\rangle\gamma dt = 0$$

$$\Rightarrow \quad \mathrm{C} + \mathrm{C}^\dagger = -\gamma\mathbf{A}^\dagger \cdot \mathbf{A} \tag{3.12}$$

And (3.9) reduces to

$$d\||\psi_t\rangle\|^2 = \langle\psi_t|(\mathbf{A} + \mathbf{A}^\dagger)|\psi\rangle \cdot d\mathbf{B} \tag{3.13}$$

Every operator can be written as sum of a Hermitian and an anti-Hermitian part. The anti-Hermitian part which can play role for Hamiltonian generated time evolution is considered to be $-(\iota/\hbar)\,\mathrm{H} = (\mathrm{C} - \mathrm{C}^\dagger)/2$ and Hermitian pert is $(\mathrm{C} + \mathrm{C}^\dagger)/2 = -(\gamma/2)\mathbf{A}^\dagger \cdot \mathbf{A}$. Now, (3.1) can be rewritten as:

$$d|\psi\rangle = \Big[-\frac{\iota}{\hbar}\,\mathrm{H}\,dt - \frac{\gamma}{2}\mathbf{A}^\dagger \cdot \mathbf{A} + \mathbf{A} \cdot d\mathbf{B} \Big] |\psi\rangle \tag{3.14}$$

After re-normalization one can easily get:

$$d|\psi_t\rangle = \Big[-\frac{\iota}{\hbar}\,\mathrm{H}\,dt - \frac{\gamma}{2}(\mathbf{A}-\mathbf{R})^2 dt + \sqrt{\gamma}(\mathbf{A}-\mathbf{R})\cdot d\mathbf{W}\Big]|\psi_t\rangle \tag{3.15}$$

Here we have used $\mathbf{A}^\dagger = \mathbf{A}$, $\mathbf{R} = \langle\psi_t|\mathbf{A}|\psi_t\rangle$ and $d\mathbf{B} = \sqrt{\gamma}\mathbf{dW}$. (3.15) is the universal dynamical time evolution equation of the continuous spontaneous localization (CSL) model. The equation is nonlinear, stochastic and norm-conserving. Operator $\mathbf{A}$ is the observable which is being observed. In general every measurement is made on position basis from which the eigenvalues of other correlated observables like spin, momentum, energy etc. are measured. Thus, taking this phenomena into the consideration, in CSL model localization is always assumed to be taking place in position basis. In that case A is position operator. To explain the classical behavior of macroscopic systems, the model must consider role of mass in dynamical time evolution and it must also explain all quantum mechanical predictions successfully. To fulfill all requirements, there should be proper bound on the parameter γ. In CSL model γ is assumed to be mass dependent in following way:

$$\gamma = \frac{m}{m_0}\gamma_0 \tag{3.16}$$

where m_0 is a reference mass taken to be mass of proton and γ_0 is a constant of model.

3.2 Analysis of the Free Particle Dynamics

In this section we review analysis of the free particle dynamics using CSL model presented in [12]. Considering universal position localization (3.15) is re-written as:

$$d|\psi_t\rangle = \Big[-\frac{\iota}{\hbar}\,\mathrm{H}\,dt - \frac{\gamma}{2}(\mathbf{Q}-\mathbf{R})^2 dt + \sqrt{\gamma}(\mathbf{Q}-\mathbf{R})\cdot d\mathbf{W}\Big]|\psi_t\rangle \tag{3.17}$$

where $\mathbf{Q}$ is position operator of a free particle of mass m. A general solution of the above equation for free particle is taken to be Gaussian:

$$\psi_t(x) = N\exp\{-a_t(x-\bar{x}_t)^2 + \iota\bar{k}_t x + \phi_t + \iota\theta_t\} \tag{3.18}$$

where N is a normalization factor, a_t is supposed to be complex function of time, while $\bar{s}_t$, $\bar{k}_t$, ϕ_t and θ_t are considered to be real. θ_t is a global phase and not relevant here. Feeding (3.18) into (3.17) one finds the following stochastic equations for above mentioned variables:

$$da_t = \left[\gamma - \frac{2\iota\hbar}{m}(a_t)^2\right]dt \tag{3.19}$$

$$d\bar{x}_t = \frac{\hbar}{m}\bar{k}_t dt + \frac{\sqrt{\gamma}}{2a_t^R}dW_t \tag{3.20}$$

$$d\bar{k}_t = -\sqrt{\gamma}\frac{a_t^I}{a_t^R}dW_t \tag{3.21}$$

$$d\phi_t = \left[\gamma\bar{x}_t^2 + \frac{\hbar}{m}a_t^I\right]dt + \sqrt{\gamma}\bar{x}_t dW_t \tag{3.22}$$

Equation (3.19) is easily solvable. It is solved using residue formula of complex integral:

$$\int \frac{da_t}{\left[\gamma - \frac{2\iota\hbar}{m}(a_t)^2\right]} = \int dt$$

$$a_t = a_t^R + \iota a_t^I = \frac{1-\iota}{2}\sqrt{\frac{m\gamma}{\hbar}}\tanh\left[t(1+\iota)\sqrt{\frac{\hbar\gamma}{m}} + k\right] \tag{3.23}$$

where a_t^R and ιa_t^I represent real and imaginary parts of a_t and

$$k = \tanh^{-1}\left[\frac{a_0}{c}\right]$$

Applying normalization condition to (3.18) we get:

$$\mathrm{N}^2\int \psi_t^*\psi_t = 1$$

$$N = e^{-\phi_t}\left(\frac{2a_t^R}{\pi}\right)^{\frac{1}{4}} \tag{3.24}$$

To study the time dependence of spread in position and momentum we calculate variances:

$$\sigma_q(t) = \sqrt{\langle q^2\rangle - \langle q\rangle^2} = \frac{1}{2}\sqrt{\frac{1}{a_t^R}} \tag{3.25}$$

$$\sigma_p(t) = \sqrt{\langle p^2\rangle - \langle p\rangle^2} = \hbar\sqrt{\frac{(a_t^R)^2 + (a_t^I)^2}{a_t^R}} \tag{3.26}$$

Using expression for a_t in (3.23), one can get explicit time dependence of spread in position and momentum as

$$\sigma_q(t) = \sqrt{\frac{\hbar}{m\omega}\frac{\cosh(\omega t + \delta_1) + \cos(\omega t + \delta_2)}{\sinh(\omega t + \delta_1) + \sin(\omega t + \delta_2)}} \tag{3.27}$$

$$\sigma_p(t) = \sqrt{\frac{\hbar m\omega}{2}\frac{\cosh(\omega t + \delta_1) - \cos(\omega t + \delta_2)}{\sinh(\omega t + \delta_1) + \sin(\omega t + \delta_2)}} \tag{3.28}$$

Where δ_1 and δ_2 are function of initial conditions.

$$\omega = 2\sqrt{\frac{\hbar\gamma_0}{m_0}} \simeq 10^{-5}s^{-1}, \quad \gamma_0 \simeq 10^{-2}m^{-2}s^{-1}$$

Spread in position and momentum asymptotically reduces to stationary values $\sigma_q(\infty)$ and $\sigma_p(\infty)$ as:

$$\sigma_q(\infty) = \sqrt{\frac{\hbar}{m\omega}} \simeq \frac{10^{-15}}{\sqrt{m}} \quad m \tag{3.29}$$

$$\sigma_p(\infty) = \sqrt{\frac{\hbar m\omega}{2}} \simeq 10^{-19}\sqrt{m} \quad \frac{kg.m}{s} \tag{3.30}$$

$$\Rightarrow \quad \sigma_q(\infty)\sigma_p(\infty) = \frac{\hbar}{\sqrt{2}} \tag{3.31}$$

The Eq. (3.31) shows that spread in position and momentum takes the minimum values provided by Heisenberg uncertainty principle. Values of $\sigma_q(\infty)$ and $\sigma_p(\infty)$ are mass dependent. For macroscopic mass $m \approx 1gm$, the collapse strength γ is $\approx 10^{22}s^{-1}$ and $\sigma_q(\infty) \approx 10^{-14}m$ and $\sigma_p(\infty) \approx 10^{-20}kg.m/s$ which are very small quantities showing position and momentum of precisely localized particle i.e. classical behavior of macroscopic bodies. For a proton $\sigma_q(\infty) \approx 1cm$ and $\sigma_p(\infty) \approx 10^{-28}kg.m/s$ corresponding to $\approx 10^{-1}m/s$ uncertainty in velocity. These quantities are as per the predictions of standard quantum mechanics. So the CSL model successfully unifies the dynamics of macroscopic and microscopic dynamics.

3.3 Reduction of Statistical Operator in CSL model

Starting from (3.15) we can write:

$$|\psi_{t+dt}\rangle = |\psi_t\rangle + \Big[-\frac{\iota}{\hbar}\mathrm{H}\,dt - \frac{\gamma}{2}(\mathbf{A}-\mathbf{R})^2 dt + \sqrt{\gamma}(\mathbf{A}-\mathbf{R})\cdot d\mathbf{W}\Big]|\psi_t\rangle$$

Using above expression for $|\psi_{t+dt}\rangle$ one can derive an expression for time derivative of $\rho_t = |\psi_t\rangle\langle\psi_{t+dt}|$ as:

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho_t = -\frac{\iota}{\hbar}[\mathrm{H}, \rho_t] + \gamma\mathbf{A}\rho_t\mathbf{A} - \frac{\gamma}{2}\{\mathbf{A}^2, \rho_t\} \tag{3.32}$$

where $\{.,.\}$ represents the anti-commutator. This is the master equation for CSL model. As we have used in §3.2, $\mathbf{A} \equiv \mathbf{Q}$ the position operator, we can calculate $\langle x'|\rho_t|x''\rangle$:

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle x'|\rho_t|x''\rangle = -\frac{\iota}{\hbar}\langle x'|[\mathrm{H},\rho_t]|x''\rangle + \gamma\langle x'|\mathbf{A}\rho_t\mathbf{A}|x''\rangle - \frac{\gamma}{2}\langle x'|\{\mathbf{A}^2,\rho_t\}|x''\rangle \tag{3.33}$$

To study the effect of only localization process we can drop the term of evolution due to Schrödinger time evolution. Using the anti-commutator expression $\{\mathrm{A},\mathrm{B}\} = \mathrm{A}\,\mathrm{B}+\mathrm{B}\,\mathrm{A}$:

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle x'|\rho_t|x''\rangle = -\frac{\gamma}{2}(x'-x'')^2\langle x'|\rho_t|x''\rangle \tag{3.34}$$

$$\Rightarrow \quad \langle x'|\rho_t|x''\rangle = \langle x'|\rho_0|x''\rangle e^{-\{\frac{\gamma}{2}(x'-x'')^2\}t} \tag{3.35}$$

Analyzing Eq. (3.35) one can reach at following conclusions:

- The mode of off-diagonal terms of density matrix decays exponentially with critical time i.e. time when $\langle x'|\rho_{t_c}|x''\rangle = \frac{\langle x'|\rho_0|x''\rangle}{e}$, given by:

$$t_c = \frac{2}{\gamma(x'-x'')^2}$$

- The critical time depends on collapse strength γ and $x'-x''$ as well. It is clear that farther the element from diagonal in density matrix faster it reduces and similarly heavier the system faster it localizes. For a proton $\gamma = \gamma_0 \approx 10^{-2}s^{-1}$, taking $x'-x'' = 1m$ we get critical time 200 second. Similarly for $10^9 amu$ i.e. mass of one billion nucleons (order of mass of nano-particles) we get critical time 2 nanosecond. This explains the transition from quantum to classical world as mass increases.

- A pure ensemble transforms into statistical mixture with time.

- For $x' = x''$ i.e. the diagonal elements, critical time is infinite. So the diagonal elements remain unchanged preserving Born probability rule.

In a measurement scheme as presented in §1.1.2, a microscopic system interacts with a macroscopic apparatus and forms a combined system. The state of macroscopic apparatus is always observed to be well localized in space. This is analyzed using dynamics of center of mass of the combined system.

3.4 Many-particle Systems

We investigate the treatment for spontaneous localization effects on many-particle systems in [9], [10] and [12] and reviewed in [13].

3.4.1 Center of Mass with Decoupled Dynamics

Let's consider a many particle system with N distinguishable particles. The i^{th} particle has mass m_i and corresponding collapse strength γ_i. The equation (3.17) is generalized for such a system as following:

$$d|\psi_t(\{x\})\rangle = \Big[-\frac{\iota}{\hbar}\, \mathrm{H}_T\, dt - \frac{1}{2}\sum_{i=1}^{N} \gamma_i(\mathbf{Q_i} - \mathbf{R_i})^2 dt + \sum_{i=1}^{N} \sqrt{\gamma_i}(\mathbf{Q_i} - \mathbf{R_i}) \cdot d\mathbf{W_i}\Big] |\psi_t(\{x\})\rangle \tag{3.36}$$

where $\mathrm{H_T}$ is Hamiltonian for composite system, $\mathbf{Q}_i$ and $\mathbf{R}_i$ are position operator and expectation value of position operator for particle i, $\{\mathbf{W}_i\}$ is set of N independent Wiener processes and set $\{x\}$ represents N spatial coordinates corresponding to N particles. To simplify analysis one can switch to center of mass frame $\mathbf{X}$ and relative coordinates $\{\tilde{\mathbf{x}}_i\}$:

$$\begin{aligned} \mathbf{X} =& \frac{1}{M}\sum_{i=1}^{N} m_i x_i \\ \{\tilde{\mathbf{x}}_i\} =& \mathbf{x} - \mathbf{X} \\ M =& \sum_{i=1}^{N} m_i \end{aligned} \tag{3.37}$$

where M is the mass associated to the center of mass. Under the assumption $\mathrm{H}_T = \mathrm{H}_{cm} + \mathrm{H}_{rel}$, dynamics for the relative motion of all particle and that for the center of mass decouples i.e. wave function for whole system can be written as tensor product of center of mass-motion wave function and those of all relative motion wave functions.

$$\psi = \psi_{cm}(\mathbf{X}) \otimes \tilde{\psi_1}(\tilde{\mathbf{x}}_1) \otimes \tilde{\psi_2}(\tilde{\mathbf{x}}_2) \otimes ...... \otimes \tilde{\psi_N}(\tilde{\mathbf{x}}_N)$$

$$\psi_{rel} = \tilde{\psi_1}(\tilde{\mathbf{x}}_1) \otimes \tilde{\psi_2}(\tilde{\mathbf{x}}_2) \otimes ...... \otimes \tilde{\psi_N}(\tilde{\mathbf{x}}_N)$$

Following claims and conclusions were made in [9], [10] and [12]:

- $\psi_{cm}(\mathbf{X})$ and $\psi_{rel}(\{\mathbf{x}\})$ satisfy the following equations:

$$d\psi_{rel}(\{\tilde{\mathbf{x}}\}) = \Big[-\frac{\iota}{\hbar}\,\mathrm{H}_{rel}\,dt - \frac{1}{2}\sum_{i=1}^{N}\gamma_i(\tilde{\mathbf{Q}}_{\mathbf{i}} - \tilde{\mathbf{R}}_{\mathbf{i}})^2 dt + \sum_{i=1}^{N}\sqrt{\gamma_i}(\tilde{\mathbf{Q}}_{\mathbf{i}} - \tilde{\mathbf{R}}_{\mathbf{i}})\cdot d\mathbf{W}_{\mathbf{i}}\Big]\psi_{rel}(\{\tilde{\mathbf{x}}\}) \tag{3.38}$$

$$d\psi_{cm} = \Big[-\frac{\iota}{\hbar}\,\mathrm{H}_{cm}\,dt - \frac{\gamma_{cm}}{2}(\mathbf{Q}-\mathbf{R})^2 dt + \sqrt{\gamma_{cm}}(\mathbf{Q}-\mathbf{R})\cdot d\mathbf{W}\Big]\psi_{cm} \tag{3.39}$$

- γ_{cm} can be taken as collapse strength of a individual particle of mass $M = \sum_i m_i$:

$$\gamma_{cm} = \frac{\sum_i m_i}{m_0}\gamma_0 = \sum_{i=1}^{N}\gamma_i \tag{3.40}$$

- Let us consider a many particle system consists of $2\mu g$ carbon ($\sim 10^{17}$ atoms). Collapse strength for center of mass is very large $\sim 10^{17}$ times that of individual atom. Following analysis of §3.2 and §3.3 one can conclude -the rate of reduction in position and momentum spread of center of mass is much faster than that of an individual atom, since, the critical time for the center of mass localization is $\sim 10^{-17}$ times that for an individual atom.

3.4.2 Spread in Center of Mass

In this section we present a complete analysis of spread in center of mass coordinates and its reduction rate affected by reductions in that of individual particle. Let us consider a system of N distinguishable particles. Particle i has mass m_i and coordinates x_i. In case the particles are not entangled, the wave function of composite system can be written as

$$\begin{aligned} |\psi\rangle &= |\psi_1\rangle \otimes |\psi_2\rangle \otimes |\psi_3\rangle \ldots\ldots \otimes |\psi_N\rangle \\ \psi(\{x_i\}) &= \prod_{i=1}^{N}\psi(x_i) \end{aligned} \tag{3.41}$$

Let us define an operator Q as:

$$\mathrm{Q} = \sum_{i=1}^{N}\alpha_i\,\mathrm{q}_i \tag{3.42}$$

where q_i is the position operator of particle i in N-particle Hilbert space.

$$\mathrm{q}_i \equiv \mathrm{I}_1 \otimes .... \, \mathrm{q}_{i-1} \otimes \mathrm{q}_i \otimes \mathrm{I}_{i+1} \, ... \otimes \mathrm{I}_N$$

and

$$\alpha_i = \frac{m_i}{\sum_{i=1}^{N} m_i}$$

If we consider $|\psi_i\rangle$ as eigen state of position operator for particle-i i.e. particle is well localized in space, $|\psi\rangle$ becomes eigen state of operator Q with eigen value $\sum_{i=1}^{N} \alpha_i x_i$. This is the center of mass coordinates for classical system i.e. all particles are well localized. Therefore, operator Q can be interpreted as position operator of center of mass. It is just another self-adjoint operator in N-particle Hilbert space representing an physical observable and need not to be associated with a particle like virtual system, as it was done in previous section §3.4.1. The expectation value of Q can be calculated easily:

$$\langle \mathrm{Q} \rangle = \int \prod_{i=1}^{N} \psi^*(\{x_i\}) \, \mathrm{Q} \, \psi(\{x_i\})$$

Because the system consists non-interacting particles, the dynamics of motions of all particles are decoupled from each others. Then

$$\langle \mathrm{Q} \rangle = \sum_{i=1}^{N} \alpha_i \langle \mathrm{q}_i \rangle \tag{3.43}$$

$$\langle \mathrm{Q} \rangle^2 = \sum_{i=1}^{N} \alpha_i^2 \langle \mathrm{q}_i \rangle^2 + 2 \sum_{i \neq j}^{N} \alpha_i \alpha_j \langle \mathrm{q}_i \rangle \langle \mathrm{q}_j \rangle \tag{3.44}$$

Similarly $\langle \mathrm{Q}^2 \rangle$ can be calculated:

$$\mathrm{Q}^2 = \sum_{i=1}^{N} \alpha_i^2 \, \mathrm{q}_i^2 + \sum_{i \neq j}^{N} \alpha_i \alpha_j \{\mathrm{q}_i, \mathrm{q}_j\} \tag{3.45}$$

$$\langle \mathrm{Q}^2 \rangle = \sum_{i=1}^{N} \alpha_i^2 \langle \mathrm{q}_i^2 \rangle + 2 \sum_{i \neq j}^{N} \alpha_i \alpha_j \langle \mathrm{q}_i \rangle \langle \mathrm{q}_j \rangle \tag{3.46}$$

From 3.44 and 3.46, variance in Q can be calculated:

$$\sigma_Q^2 = \langle \mathrm{Q}^2 \rangle - \langle \mathrm{Q} \rangle^2$$

$$\sigma_Q^2 = \sum_{i=1}^{N} \alpha_i^2 (\langle \mathrm{q}_i^2 \rangle - \langle \mathrm{q}_i \rangle^2)$$
$$\sigma_Q^2 = \sum_{i=1}^{N} \alpha_i^2 \sigma_{q_i}^2 \tag{3.47}$$

where $\sigma_{q_i}^2$ is the variance in position of particle-i. Now we consider an change in variance of individual particle position $\Delta\sigma_{q_i}^2$ in an infinitesimal time interval Δt. If variance in center of mass coordinate at time t is $\sigma_Q^2(t)$

$$\sigma_Q^2(t+dt) = \sum_{i=1}^{N} \alpha_i^2 (\sigma_{q_i}^2(t) + \Delta\sigma_{q_i}^2)$$
$$\Rightarrow \quad \sigma_Q^2(t+dt) - \sigma_Q^2(t) = \sum_{i=1}^{N} \alpha_i^2 \Delta\sigma_{q_i}^2$$

$$\Rightarrow \quad \frac{\partial}{\partial t}\big(\sigma_Q^2\big) = \sum_{i=1}^{N} \alpha_i^2 \frac{\partial}{\partial t}\big(\sigma_{q_i}^2\big) \tag{3.48}$$

To visualize (3.46) we simplify the dynamics taking all particle of same mass m. Now $\alpha_i = \frac{1}{N}$, and (3.46) becomes:

$$\Rightarrow \quad \frac{\partial}{\partial t}\big(\sigma_Q^2\big) = \frac{1}{N^2} \sum_{i=1}^{N} \frac{\partial}{\partial t}\big(\sigma_{q_i}^2\big) \tag{3.49}$$

For large N, reduction rate for center of mass is very small. This result contradicts with claims made in [9], [10] and [12]. Possible explanation for this is that center of mass was treated as a separate particle and an independent Wiener process and collapse strength was considered without paying attention to appropriate justifications. So the treatment provided in above references for many-particle systems fails. A different approach within the framework of continuous spontaneous localization model is needed to treat macroscopic systems which are composed of many microscopic systems. We present such a scheme in next section.

3.5 Spontaneous Localization for Entangled State

As we have shown in §3.4.2 that approach taken in §3.4.1 not useful to explain micro to macro transition. Here we propose a new approach to treat many particle systems. We suggest that the entanglement among particles can play a role in rapid reduction.

A simple spontaneous localization treatment has been presented for entangled pair particles state. Let us consider states of two particles are spatially correlated as following:

$$\psi(x_1, x_2) = \frac{\psi_1(x_1)\psi_2(x_2) - \psi_2(x_1)\psi_1(x_2)}{\sqrt{2}} \tag{3.50}$$

where ψ_1 and ψ_1 are two different non-overlapping spatial wave-functions. x_1 and x_2 are coordinates of particle-1 and 2. In discrete spontaneous localization process, suppose, particle-1 experiences a jump to position x according to (2.1) and (2.3). Now wave-function for particle-2 can be calculated tracing over the wave-function of particle-1 in (3.48). Since the particle-1 is well localized in space at a random position x, one can see, particle-2 takes one of the wave-functions ψ_1 and ψ_1 with probability half. Accordingly, in an ensemble of N identically prepared systems (consist of total $2N$ particles) of considered entangled pair particles, after the realization of localization process λdtN particles will be in ρ_T ensemble state and λdtN particles will be in ρ'_T ensemble state corresponding to every particle-2 of all entangled pairs. Rest of the ensemble $(1 - 2\lambda dt)N$ are evolved with standard quantum mechanics. Statistical operator at time $(t + dt)$ can be written as:

$$\rho(t + dt) = (1 - 2\lambda dt)\rho'(t + dt) + \lambda dt\rho_T + \lambda dt\rho'_T \tag{3.51}$$

Making use of (1.6):

$$\rho(t + dt) = (1 - 2\lambda dt)\left(\rho(t) - \frac{\iota}{\hbar}\big[\mathrm{H}, \rho(t)\big]dt\right) + \lambda dt\rho_T + \lambda dt\rho'_T \tag{3.52}$$

$$\Rightarrow \frac{\mathrm{d}}{\mathrm{d}t}\rho(t) = -\frac{\iota}{\hbar}\big[\mathrm{H}, \rho(t)\big] - 2\lambda\left(\rho(t) - \frac{\rho_T(t) + \rho_T(t)'}{2}\right) \tag{3.53}$$

$$\Rightarrow \quad \frac{\mathrm{d}}{\mathrm{d}t}\langle \mathbf{x}'|\rho(t)|\mathbf{x}''\rangle = -\frac{\iota}{\hbar}\langle \mathbf{x}'|\Big[\mathrm{H}, \rho(t)\Big]|\mathbf{x}''\rangle - 2\lambda\langle \mathbf{x}'|\left(\rho(t) - \frac{\rho_T(t) + \rho_T(t)'}{2}\right)|\mathbf{x}''\rangle \tag{3.54}$$

where $\rho(t)$ is the reduced density matrix for particle-1 and $\rho_T(t)'$ is given by:

$$\rho_T(t)' = \frac{1}{2}[|\psi_1\rangle\langle\psi_1| + |\psi_2\rangle\langle\psi_2|]$$

$$\begin{aligned}\langle \mathbf{x}'|\rho_T(t)'|\mathbf{x}''\rangle =& \frac{1}{2}\langle \mathbf{x}'|[|\psi_1\rangle\langle\psi_1| + |\psi_2\rangle\langle\psi_2|]\mathbf{x}''\rangle \\ =& \frac{1}{2}[\psi_1(x')\psi_1^*(x'') + \psi_2(x')\psi_2^*(x'')]\end{aligned} \tag{3.55}$$

From (2.7):

$$\langle \mathbf{x}'|\rho_T(t)|\mathbf{x}''\rangle = \langle \mathbf{x}'|\rho|\mathbf{x}''\rangle e^{-(\alpha/4)\{(\mathbf{x}'-\mathbf{x}'')^2\}}$$

Now,

$$\langle \mathbf{x}'|\left(\frac{\rho_T(t)+\rho_T(t)'}{2}\right)|\mathbf{x}''\rangle = \frac{1}{2}\langle \mathbf{x}'|\rho|\mathbf{x}''\rangle e^{-(\alpha/4)\{(\mathbf{x}'-\mathbf{x}'')^2\}}+\frac{1}{4}[\psi_1(x')\psi_1^*(x'')+\psi_2(x')\psi_2^*(x'')] \tag{3.56}$$

Let us now consider ψ_1 and ψ_2 two Gaussian wave packets centers at x_0 and $-x_0$ respectively with equal variance $\sigma^2 \approx 10^{-12}m^2$ i.e. wavepacket is sharply localized in space around x_0 or $-x_0$ respectively. It can be easily concluded using (3.50) −smaller the variances larger the correlation between particles. For example we consider $\sigma \to 0$, the state corresponding to (3.50) becomes:

$$|\psi\rangle = \frac{|x_0\rangle|-x_0\rangle - |-x_0\rangle|x_0\rangle}{\sqrt{2}} \tag{3.57}$$

It is clear from arguments provided above that for $(x'-x'') \gg 10^{-6}m$ RHS of (3.56) is negligible. Equation (3.54) now takes the form:

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle \mathbf{x}'|\rho(t)|\mathbf{x}''\rangle = -\frac{\iota}{\hbar}\langle \mathbf{x}'|\Big[\mathrm{H},\rho(t)\Big]|\mathbf{x}''\rangle + 2\lambda\langle \mathbf{x}'|\rho(t)|\mathbf{x}''\rangle \tag{3.58}$$

To study the effects of spontaneous collapse we drop the evolution term governed by Hamiltonian in (3.58).

$$\|\langle \mathbf{x}'|\rho(t)|\mathbf{x}''\rangle\| = \|\langle \mathbf{x}'|\rho_0|\mathbf{x}''\rangle\| \exp\{-2\lambda t\} \tag{3.59}$$

From the above result we can see that the collapse strength for spontaneous reduction of a maximally entangled particle system is approximately two times that of a non-entangled one. If the particles are not maximally entangled i.e entangled state differs from that provided in (3.57), $\langle \mathbf{x}'|\rho_T(t)'|\mathbf{x}''\rangle$ becomes significantly large and reduces the collapse strength in (3.54). Following the above analysis we make following conclusion and claims:

- Considering center of mass as an independent particle of large collapse strength, to explain the localization process of many-particle systems, is not a legitimate scheme.

- Reduction of wavepacket of entangled systems is more rapid than that of non entangled system. Therefore entanglement can play a vital role in explaining the reduction mechanism in many-particle interacting systems. The proposed treatment, therefore, can be able to explain the quantum-to-classical transition phenomena.

- Role of localization process in transforming pure state into a statistical mixture in case of entangled systems, it should be detectable in decoherence processes for such ensembles.

Chapter 4

Summary

- A review of GRW and CSL model along with the analysis for a free particle reduction dynamics is presented.
- It was proved that treatment for many non-interacting particle system presented by G. C. Ghirardi, P. Pearle, Weber (1990) and A. Bassi (2005) is not legitimate (§3.4).
- The role of interaction in wavepacket reduction was investigated and a new treatment for localization process of interacting particle was formulated. It is shown that for maximally correlated particles the reduction rate is double of that for non-entangled one (§3.5).

Bibliography

[11] *Stochastic differential equations: Theory and applications*, Wiley-Interscience Publication, New York, 1937.

[1] *Mathematical foundation of quantum mechanics*, Princeton University Press, Princeton, NJ, 1955.

[13] Seema Satin T.P. Singh H. Ulbricht A. Bassi, K. Lochan, J. Phys. A: Math. Gen. **85** (2013), 471.

[16] A. J. Leggett A. O.Caldeira, Phys. Rev. Lett. **46** (1981), 211.

[12] A. Bassi, J. Phys. A: Math. Gen. **38** (2005), 3173.

[2] J.S. Bell, Physics **1** (1964), 195.

[5] N. Bohr, Nature(London) **121** (1928), 580.

[4] D. Bohm, Phys. Rev. **85** (1952), 166.

[8] T. Weber G.C. Ghirardi, A. Rimini, Phys. Rev. D **34** (1986), 470.

[9] A. Rimini G.C. Ghirardi, P. Pearle, Phys. Rev. A **42** (1990), 78.

[7] III H. Evertt, Rev. Mod. Phys. **29** (1957), 454.

[3] Nature **526** (2015), 682.

[17] P. Pearle, Phys. Rev. D **13** (1976), 857.

[10] ———, Phys. Rev. A **39** (1989), 2277.

[14] L. Stodolsky R. A. Harris, J. Chem. Phys. **74** (1981), 2145.

[15] W. H. Zurek W. G. Unruh, Phys. Rev. D **40** (1989), 1071.