QUANTUM FIELD THEORY IN ACCELERATED FRAMES AND PROTON DECAY

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Certificate of Examination

This is to certify that the dissertation titled **"Quantum Field Theory in Accelerated Frames and Proton Decay"** submitted by **Abhinav Kala** (Reg. No. MS11026) for the partial fulfillment of BS-MS dual degree programme of the Institute, has been examined by the thesis committee duly appointed by the Institute. The committee finds the work done by the candidate satisfactory and recommends that the report be accepted.

Dr. H. K. Jassal	Prof. J. S. Bagla	Prof. C. S. Aulakh
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Dated: April 22, 2016

Declaration

The work presented in this dissertation has been carried out by me under the guidance of Prof. Charanjit S. Aulakh at the Indian Institute of Science Education and Research Mohali.

This work has not been submitted in part or in full for a degree, a diploma, or a fellowship to any other university or institute. Whenever contributions of others are involved, every effort is made to indicate this clearly, with due acknowledgement of collaborative research and discussions. This thesis is a bonafide record of original work done by me and all sources listed within have been detailed in the bibliography.

Abhinav Kala
(Candidate)

Dated: April 22, 2016

In my capacity as the supervisor of the candidate's project work, I certify that the above statements by the candidate are true to the best of my knowledge.

Prof. Charanjit S. Aulakh
(Supervisor)

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List of Figures

Notations & Conventions

- $g_{\mu\nu}$ = Metric tensor
- $g = det(g_{\mu\nu})$
- **Kronecker Delta** - $\delta_{ij} = 1$, iff $i = j$. Otherwise, $\delta_{ij} = 0$.
- **Re** or **Im** outside a number means real or imaginary part of that number respectively.
- γ^{μ} represent Dirac gamma matrices.
- **Feynman slash notation** - $\not{k} = k_{\mu}\gamma^{\mu}$
- **Barred fermion field** is defined as $\bar{\psi} = \psi^{\dagger}\gamma^{0}$
- **Minkowski metric** is represented by $\eta_{\mu\nu} = diag(1,-1,-1,-1)$
- **Raising/lowering of indices** - Any tensor index is lowered/raised using $g_{\mu\nu}$ and $g^{\mu\nu}$ respectively. Where, $g^{\mu\nu}g_{\nu\rho} = \delta^{\mu}_{\rho}$
- Subscript (or superscript) $,\mu$ with a quantity represents covariant derivative with respect to contravariant (or covariant) 'μ'th coordinate . eg.- $\phi_{,\mu}$ for scalar field represents partial derivative in the direction of x^{μ}.

- **Einstein summation convention**- Repeated indices in a term(one lower and one upper) are summed over. eg. - $x_a x^a = x_0 x^0 + x_1 x^1 + x_2 x^2 + x_3 x^3$

- **Units** - We have used natural units, in which $\hbar = c = 1$

- Any integral sign ($\int$) with no limits/contours mentioned with it is assumed to be same as $\int_{-\infty}^{\infty}$.

Contents

Abstract

Quantum field theory on curved spacetime is an approach to calculate the lowest order effects of gravity on interactions of quantum fields. Although, far from being the complete theory of quantum gravity it does predict some astonishing phenomena. Possibility of particle creation in an expanding spacetime is one such important result for cosmology.[Dav82] The most celebrated result is Hawking radiation emission by a black hole.[@Wik16b]

Not only non-trivial spacetime geometry, but even acceleration in Minkowski spacetime can have effects ranging from alteration of standard model's reaction rates to making some forbidden processes possible. Proton decay is one of those forbidden processes and is of much interest for physicists, since recent experiments put a lower bound of $\approx 10^{34}$ years on proton's half-life [@Wik16e]. The possibility of these non-standard model processes are a consequence of the inability to find a definition of 'particle' for a general background spacetime, which has one implication of change in the definition of particle for uniformly accelerated observers compared to inertial observers, famously known as **Fulling-Davies-Unruh effect** or just **Unruh effect**[Dav82]. This difference in particle concepts in inertial and non-inertial frame also means difference in concept of 'vacuum', which is in general defined as the state annihilated by annihilation operators $\hat{a}_k|vacuum\rangle = 0$.

Decay rate of proton(due to Fermi interaction) has been calculated and verified to be matching for both inertial and non-inertial frame calculations both in the case of

massless[Mat01a] and massive neutrinos[Yam03]. A recent claim has been made about mismatch in the decay rates calculated in both frames in presence of neutrino oscillations.[Tor15]

In this material, all the necessary background for performing decay rate calculation along with the calculation of proton decay rate is presented. Also the validity of claims in [Tor15] is questioned. The complications caused by neutrino oscillations in the evaluation of decay rate is discussed too.

1

Elements from Quantum Field Theory on Curved Spacetime

Introduction

In Quantum field theory on curved spacetime, also known as semi-classical gravity, matter is treated as interacting quantum fields, while background spacetime metric is kept fixed.

Next is presented the case of real scalar field, since this thesis contains mostly the examples related to real scalar field. In appendix B, we discuss about formulation of quantum field theory for fermions.

Formulation of Real Scalar Field Theory on Curved Spacetime

We will be working with globally hyperbolic spacetimes (Appendix A) in this text. Metric is represented by $g_{\mu\nu}$ and $g = det(g_{\mu\nu})$ The Lagrangian density L for a real scalar field ϕ with mass m, which will give scalar action when integrated over whole spacetime volume is

$$\mathcal{L}(x) = \frac{1}{2}\sqrt{-g(x)}[g^{\mu\nu}(x)\phi_{,\mu}\phi_{,\nu} - m^2\phi^2(x) + \xi R(x)\phi^2(x)] \tag{1.1}$$

Here $R(x)$ is Ricci scalar curvature at x, and the term $\xi R(x)\phi^2(x)$ is added as only possible local interaction term of scalar field and gravitational field [Dav82], with ξ being the coupling constant. The pre-factor $\sqrt{-g(x)}$ is included because action includes integration over spacetime volume and $\sqrt{-g(x)}d^4x$ is an invariant infinitesimal quantity [Wei72].

Euler-Lagrange field equation for ϕ is then

$$[\sqrt{-g(x)}\partial_\mu(\sqrt{-g(x)}g^{\mu\nu}(x)\partial_\nu) + m^2 + \xi R(x)]\phi(x) = 0 \tag{1.2}$$

Inner Product

For two solutions of eq. (1.2) $u_1(x)$ and $u_2(x)$, we define inner product by following formula

$$(u_1, u_2) = -i\int_\Sigma \{u_1(x)\partial_0 u_2^*(x) - u_2^*(x)\partial_0 u_1(x)\}[\sqrt{-g_\Sigma(x)}]d\Sigma \tag{1.3}$$

Here Σ is a spacelike hypersurface with time direction orthogonal to it. $\sqrt{-g_\Sigma(x)}$ is the determinant of the metric defined on Σ.

Now we define

$$j_{\mu_{(u_1,u_2)}}(x) = u_1(x)\partial_\mu u_2^*(x) - u_2^*(x)\partial_\mu u_1(x) \tag{1.4}$$

Using eq. 1.2, it can be shown that $j^{;\mu}_{\mu_{(u_1,u_2)}}(x) = 0$.[Mat07] Using it we get

$$(u_1,u_2)^{;0} = \frac{d(u_1,u_2)}{dt} = -i\int_\Sigma [\sqrt{-g_\Sigma(x)}]\, j^{;0}_{0_{(u_1,u_2)}}(x)d\Sigma = i\int_\Sigma [\sqrt{-g_\Sigma(x)}]\, j^{;a}_{a_{(u_1,u_2)}}(x)d\Sigma$$

Here index a takes values 1,2 and 3.

Using divergence theorem

$$\frac{d(u_1,u_2)}{dt} = i\int_\rho [\sqrt{-g_\Sigma(x)}]\, j_{a_{(u_1,u_2)}}(x).n^a(x)d\rho$$

ρ is the boundary of spacelike hypersurface Σ. n^a is the unit normal on ρ . Now if we assume j_a to be zero at spatial boundaries then inner product (1.3) is time independent.

Mode Expansion of ϕ

Suppose we have a complete set of solutions $\{u_i\}$ satisfying following properties

$(u_i,u_j) = \delta_{ij}$, $(u_i^*,u_j^*) = -\delta_{ij})$ and $(u_i,u_j^*) = 0$ Then set $\{u_i\}$ is called an orthonormal basis and the field can be expanded as

$$\phi(x) = \sum_i [a_i u_i(x) + a_i^\dagger u_i^*(x)] \tag{1.5}$$

a_i and $a_i^\dagger$ are called *creation* and *annihilation operators* for modes $\{u_i\}$. Following canonical commutation relation is imposed on the operators

$$[a_i,a_j^\dagger] = \delta_{ij}$$

Vacuum state $|0\rangle$ of the field is defined as the state annihilated by all a_i operators, ie

-

$$a_i|0\rangle = 0$$

The states created by the operation of operators $a_i^\dagger$ on vacuum are referred as particle states, which need not be typical particle states with definite momentum and energy like those of Minkowski spacetime. [3.0]

Concepts of Vacuum and Particle

In Minkowski space plane wave modes $e^{-ip.x}$ are considered modes corresponding to vacuum states. These are the preferred modes because of the Poincaré symmetry group of Minkowski space. This group includes space-time translations, boosts and rotations, whose generators are Killing vectors (A) of Minkowski space. These modes are eigenfunctions of Killing vector field $\frac{\partial}{\partial t}$, with eigenvalue $-ip^0$. Under Poincaré transformation, vacuum state corresponding to plane wave modes remains invariant. Hence, for inertial observers we can have one unambiguous vacuum.

Bogoliubov Transformations and Nonequivalent Vacuum States

The eigenmode solutions $u_i(x)$ ((1.5)) of field equation aren't the only possible set of solutions. The field operator can be expanded as a linear combination of a different complete set of modes $\{v_j\}$. The completeness implies the equivalence of both expansions

$$\phi(x) = \sum_i [a_i u_i(x) + a_i^\dagger u_i^*(x)] = \sum_j [b_j v_j(x) + b_j^\dagger v_j^*(x)] \tag{1.6}$$

The set of solutions of field equation form a vector space, because equation (1.2) is a linear differential equation. The equation (1.6) represents same vector written in two different basis $\{u_i\}$ and $\{v_j\}$. So any two vectors from these two sets should be related by a unitary transformation.

$$v_j = \sum_i [\alpha_{ji} u_i + \beta_{ji} u_i^*] \tag{1.7}$$

This transformation is called Bogoliubov transformation named after *Nikolay Bogoliubov*. The operators then can be related using (1.6) as

$$a_i = \sum_j [\alpha_{ji} b_j + \beta_{ji}^* b_j^\dagger] \tag{1.8}$$

Corresponding to two different sets of modes, one will have two different vacuum states(i.e. states annihilated by annihilation operators) $|0\rangle$ and $|0'\rangle$.
The particles of mode v_j in vacuum corresponding to modes $\{u_i\}$ will then be

$$\langle 0| b_j b_j^\dagger |0\rangle = \sum_i |\beta_{ji}|^2 \tag{1.9}$$

Non-zero β bogoliubov coefficients represent difference in two vacuum states and we'll further see in section (4.0) how it could lead to particle creation in cosmology.

Concept of particle state

To have a useful concept of particle, spacetime should have a timelike Killing vector field $X^\mu(x)$. Then the observers with trajectory along the flow of such field can be called special observers(like inertial observers in Minkowski case) with particle modes for them being the positive frequency eigenfunctions(f_ω) of the Killing vector field (i.e. $\mathcal{L}_X f_\omega = -i\omega f_\omega; \omega \geq 0$). Here $\mathcal{L}_X$ represents Lie derivative along field $X^\mu(x)$.Such a mode under the isometry(coordinate translation along the flow of the Killing field) remains an eigenfunction of field $X^\mu(x)$ with unchanged eigenvalue.

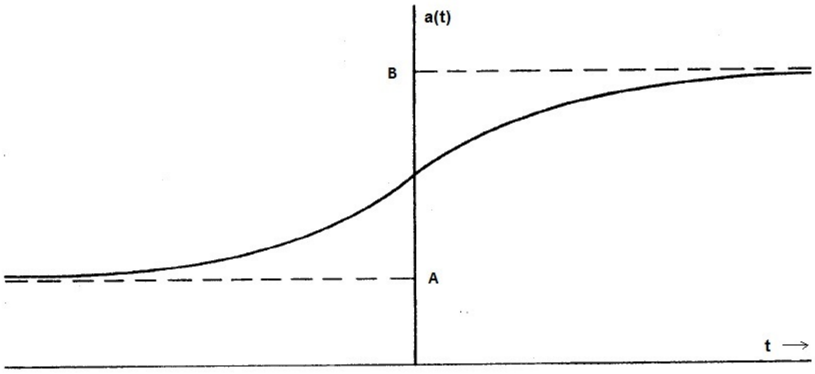

Fig. 1.1.: Qualitative evolution of scale factor

If the vacuum(before applying isometry) with respect to eigenmodes of field $X^{\mu}(x)$ was $|0_i\rangle$, then since the frequencies of modes are unchanged, hence β Bogulibov coefficients for transformation between modes before and after the application of isometry will be zero and hence vacuum after isometry $|0_f\rangle = |0_i\rangle$ (see 3.1).

Cosmological Particle Creation

We consider a FRW spacetime (App. A), which is asymptotically static in far past and far future, i.e. in these regions scale factor $a(t)$ asymptotically approaches to a constant value.

$$\lim_{t\to-\infty} a(t) \to A$$
$$\lim_{t\to\infty} a(t) \to B \tag{1.10}$$

If such is the case, then one has timelike Killing vector field in both regions ($A\frac{\partial}{\partial t}$ in far past and $B\frac{\partial}{\partial t}$ in far future) . Positive frequency plane wave modes in those two regions of spacetime can be defined. Suppose u_k are modes in far past and v_k are modes in far future. Because of the spatial homogeneity of R-W spacetime, these

plane wave modes will have the spatial part proportional to $e^{-i\vec{k}.\vec{x}}$.
So Bogulibov transformation between two sets of modes will be

$$u_k(x) = \alpha_k v_k(x) + \beta_k v^*_{-k}(x) \tag{1.11}$$

Also, we'll denote vacuum in far past as $|0_p\rangle$ and in far future as $|0_f\rangle$.

What a Detector Measures

We model our detector moving on a trajectory $x = x(\tau)$, where τ is detector's proper time. It's assumed to be interacting with scalar field via monopole interaction (similar to Birrell & Davies [Dav82]), with interaction Lagrangian density

$$\mathcal{L} = g\mu(\tau)\phi[x(\tau)]\delta(\vec{x} - \vec{x}(\tau)) \tag{1.12}$$

Here g represents the strength of coupling and $\mu(\tau)$ is monopole moment operator of the detector, which evolves due to Hamiltonian $\mathcal{H}$ of the detector

$$\mu(\tau) = e^{i\mathcal{H}\tau}\mu(0)e^{-i\mathcal{H}\tau} \tag{1.13}$$

Suppose the initial state of the field was $|0_p>$. A monopole detector in the far future region will detect the presence of $u_f(k)$ modes, if the two vacuums aren't the same. To ensure that coupling of detector doesn't cause field excitations in the region of expansion, one assumes adiabatically switching off the detector before entering the expansion region and then adiabatically switching it on after exiting that region of spacetime.
We'll take the case of an inertial detector(detectors that is carried by comoving observers) moving on a trajectory with comoving coordinates

$$\vec{x} = \vec{v}t \tag{1.14}$$

Suppose initially detector was in energy state $|E_0\rangle$ with energy E_0. It will then make a transition to a state $|E_i\rangle$ with energy E_i. If we work with first term in perturbation expansion, then the amplitude for the transition of detector to state $|E_i\rangle$ and field to state $|\phi_j\rangle$ will be

$$A = ig\langle E_i|\mu(0)|E_0\rangle \int d\tau e^{i(E_i-E_0)\tau}\langle\phi_j|\phi[x(\tau)]|0_p\rangle \tag{1.15}$$

Then the total probability for detector transition to any state $|E_i\rangle$ and field transition to any state $|\phi_j\rangle$ will be

$$P_{particle} = g^2\sum_{ij}|\langle E_i|\mu(0)|E_0\rangle|^2 \int d\tau \int d\tau' e^{i(E_i-E_0)(\tau-\tau')}\langle 0_p|\phi[x(\tau')]|\phi_j\rangle\langle\phi_j|\phi[x(\tau)]|0_p\rangle$$

Since $\{|\phi_j\rangle\}$ include all possible states of field so $\sum_j |\phi_j\rangle\langle\phi_j|$ is identity and the probability is

$$P_{particle} = g^2\sum_{i}|\langle E_i|\mu(0)|E_0\rangle|^2 \int d\tau \int d\tau' e^{i(E_i-E_0)(\tau-\tau')}\langle 0_p|\phi[x(\tau')]\phi[x(\tau)]|0_p\rangle \tag{1.16}$$

Using Bogoliubov transformation between two sets of modes (Eq. 1.11) one can write the matrix element in eq. (1.16) as

$$\begin{aligned}\langle 0_p|\phi[x(\tau')]\phi[x(\tau)]|0_p\rangle = \int d^3k[|\alpha_k|^2 v_k(x)v_k^*(x') + \alpha_k\beta_k^* v_k(x)v_{-k}^*(x') \\ +\alpha_k^*\beta_k v_{-k}^*(x)v_k^*(x') + |\beta_k|^2 v_{-k}^*(x)v_{-k}^{(}x')]\end{aligned} \tag{1.17}$$

where field expansion $\phi(x) = \int d^3k[a_k u_k(x) + a_k^\dagger u_k^*(x)]$ is used along with canonical commutation relations and orthonormality of modes (2.2). Due to spatial isotropy α and β will depend on $|\vec{k}|$ only. Using the spatial dependence of $e^{-i\vec{k}.\vec{x}}$ and trajectory on eq. 1.14 in the eq. 1.16, we can see that the integration over τ and τ' will give two delta functions for each of the four terms in eq. 1.17. These delta functions will have arguments for all four terms respectively

$$[E_i - E_0 + \gamma(\omega - \vec{k}.\vec{v})]\ and\ [-(E_i - E_0) - \gamma(\omega + \vec{k}.\vec{v})]$$
$$[E_i - E_0 + \gamma(\omega - \vec{k}.\vec{v})]\ and\ [-(E_i - E_0) + \gamma(\omega + \vec{k}.\vec{v})]$$
$$[E_i - E_0 - \gamma(\omega - \vec{k}.\vec{v})]\ and\ [-(E_i - E_0) - \gamma(\omega + \vec{k}.\vec{v})]$$
$$[E_i - E_0 - \gamma(\omega - \vec{k}.\vec{v})]\ and\ [-(E_i - E_0) + \gamma(\omega + \vec{k}.\vec{v})]$$

where $\gamma = (1-v^2)^{-1/2}$ and frequency $\omega = \sqrt{k^2+m^2}$. Using the fact that $E_i > E_0$, $v < 1$ and for massive scalar field $\omega > |\vec{k}|$, we can see that only in the last term its possible for delta functions to take non-zero values. Since r.h.s. of equation (1.16) will then depend only on $\tau - \tau'$, we can make following change of coordinates

$$\tau_1 = \tau - \tau', \quad \tau_2 = \frac{1}{2}(\tau + \tau') \tag{1.18}$$

Also we denote $\langle 0_p|\phi[x(\tau')]\phi[x(\tau)]|0_p\rangle$ by $M(\tau_1)$. Then the probability will be

$$\begin{aligned} P_{particle} &= g^2 \sum_i |\langle E_i|\mu(0)|E_0\rangle|^2 \int d\tau_2 \int d\tau_1 e^{i(E_i-E_0)\tau_1} M(\tau_1) \\ &= g^2 \sum_i |\langle E_i|\mu(0)|E_0\rangle|^2 \int d^3k \int d\tau_2 \int d\tau_1 \frac{e^{i(E_i-E_0)\tau_1}}{2\omega(2\pi)^3} |\beta_k|^2 e^{i(-\gamma(\omega+\vec{k}.\vec{v})\tau_1)} \\ &= g^2 \sum_i |\langle E_i|\mu(0)|E_0\rangle|^2 \int d^3k \int d\tau_2 |\beta_k|^2 2\pi \frac{\delta[E_i - E_0 - \gamma(\omega+\vec{k}.\vec{v})]}{2\omega(2\pi)^3} \end{aligned} \tag{1.19}$$

we've used plane wave modes $v_k = \frac{e^{i(\omega t - \vec{k}.\vec{x})}}{\sqrt{2\omega(2\pi)^3}}$.
Now for the simple case of zero velocity by going to polar coordinates and denoting $(E_i - E_0)$ by ΔE_i we get

$$\begin{aligned} P_{particle} &= g^2 \sum_i |\langle E_i|\mu(0)|E_0\rangle|^2 \int d\tau_2 \int 4\pi|k|^2 d|k| |\beta_k|^2 2\pi \frac{\delta(|k| - \sqrt{E^2-m^2})}{(2\pi)^3 2\omega} \\ &\times \frac{E}{\sqrt{E^2-m^2}} = \frac{g^2}{2\pi} \sum_i |\langle E_i|\mu(0)|E_0\rangle|^2 \int d\tau_2 \sqrt{E^2-m^2} |\beta_{\sqrt{E^2-m^2}}|^2 \end{aligned} \tag{1.20}$$

To get the detection probability per unit time we can divide both sides by infinite time integral $\int d\tau_2 \equiv T$

$$\frac{P_{particle}}{T} = \frac{g^2}{2\pi} \sum_i |\langle E_i|\mu(0)|E_0\rangle|^2 \sqrt{E^2 - m^2} |\beta_{\sqrt{E^2-m^2}}|^2 \tag{1.21}$$

So, non-zero β Bogulibov coefficients will imply detection of particles in far future region.

2

The Unruh Effect

Unruh effect is one of the most famous counter-intuitive results of non-uniqueness of particle modes(and vacuum state) in quantum field theory. It is sometimes also known as Fulling-Davies-Unruh effect. It's a flat spacetime result which is a consequence of difference in 'preferred' particle and vacuum states for an inertial observer and a uniformly accelerating observer.

Rindler Coordinates

We'll consider two dimensional Minkowski spacetime with rectangular coordinates (t, z) and with metric

$$ds^2 = dt^2 - dz^2 \tag{2.1}$$

Rindler coordinates (u, v) are defined by following coordinate transformation

$$\begin{aligned} z &= a^{-1} e^{av} \cosh au \\ t &= a^{-1} e^{av} \sinh au \end{aligned} \tag{2.2}$$

Here a is a constant. These coordinates cover region 1 of the spacetime (fig. 2.1). Metric in these coordinates becomes

$$ds^2 = e^{2av}[du^2 - dv^2] \tag{2.3}$$

This metric is conformal to Minkowski metric. A uniformly accelerated observer(Rindler oberver) with positive proper acceleration α moves on following hyperbola

$$z^2 - t^2 = \alpha^{-2} \tag{2.4}$$

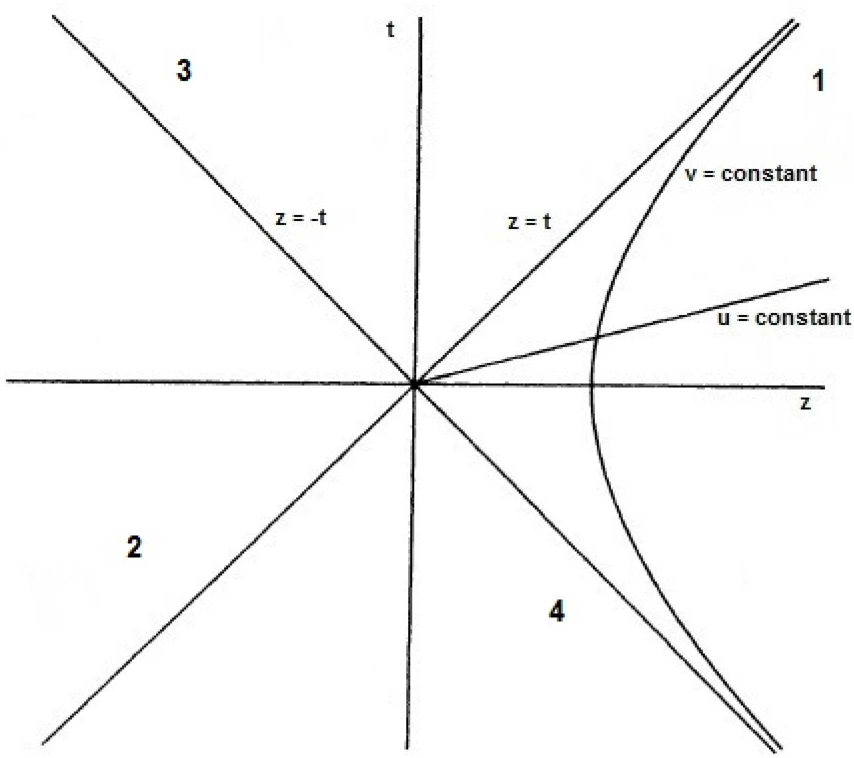

Fig. 2.1.: 2-d Minkowski space - Rindler and rectangular coordinates

Where α is proper acceleration of the Rindler observer. In Rindler coordinates this trajectory is

$$a^{-2}e^{2av} = \alpha^{-2} \tag{2.5}$$

So, for uniformly trajectory v has constant value. Also proper acceleration $\alpha = ae^{-av}$. Proper time(τ) for such an observer will be

$$\tau = e^{av}u \tag{2.6}$$

So time experienced by observer is proportional to coordinate u. This fact will help to choose a particular mode as particle for Rindler observer(2.1).
Lines $z = t$ and $z = -t$ divide spacetime into four regions. The regions 1 and 2 are causally disconnected with each other. Since, to go from region 1 to 2 (or vice versa), a trajectory has to make an angle of more than 45° with t-axis.
One can similarly provide Rindler coordinates to region 2.

$$\begin{aligned} z &= -a^{-1}e^{av}\cosh au \\ t &= -a^{-1}e^{av}\sinh au \end{aligned} \tag{2.7}$$

For a Rindler observer moving with negative proper acceleration, the proper time will be $\tau = -e^{av}u$.

Scalar field quantization

We'll consider massless scalar field $\phi(x)$ in 2-d Minkowski spacetime, where x refers to coordinate pair. Following are the Lagrangian($\mathcal{L}$) and equation of motion for such a field

$$\begin{aligned} \mathcal{L}(x) = \frac{1}{2}[(\partial_t\phi(x))(\partial_t\phi(x)) - (\partial_z\phi(x))(\partial_z\phi)(x)] \\ (\partial_t^2 - \partial_z^2)\phi(x) = 0 \end{aligned} \tag{2.8}$$

The plane wave modes with positive frequency with respect to timelike Killing vector ∂_t are

$$f_k(x) = \frac{1}{\sqrt{4\pi\omega}}e^{i(kz-\omega t)} \quad , \omega = |k| \tag{2.9}$$

Quantization in Rindler coordinates

Since metric in equation(2.3) is conformal(see appendix [A] and [Dav82]) to Minkowski metric, so for 2 dimensional spacetime the field equation in Rindler coordinates will have same form [Dav82]

$$(\partial_u^2 - \partial_v^2)\phi(x) = 0 \tag{2.10}$$

We can see that $K^\mu = (1,0)$ is a Killing vector field of Rindler metric. To see this we first calculate following Christoffel symbols/ affine connections(see [Wei72] for details) for Rindler metric

$$\begin{aligned} \Gamma^u_{uu} = \Gamma^u_{vv} = 0 \\ \Gamma^u_{uv} = \Gamma^u_{vu} = 2a \end{aligned} \tag{2.11}$$

Where $\Gamma^\lambda_{\mu\nu} == \frac{1}{2} g^{\rho\lambda} [\frac{\partial g_{\nu\rho}}{\partial x^\mu} + \frac{\partial g_{\mu\rho}}{\partial x^\nu} - \frac{\partial g_{\nu\mu}}{\partial x^\rho}]$. Now, the following equation holds for all μ and ν.

$$K_{\mu;\nu} + K_{\nu;\mu} = 0 \tag{2.12}$$

Hence, K^μ is a Killing vector field of Rindler metric, which is also timelike(see A). The modes $g^1_k(x) = \frac{1}{\sqrt{4\pi\omega}} e^{i(kv-\omega u)}$ are positive frequency eigenfunctions of timelike Killing vector ∂_u. Hence we can select them as particle modes for Rindler observer in region 1. Similarly, we can prove that $-\partial_u$ is a timelike Killing vector in region 2. And, we can use modes $g^2_k(x) = \frac{1}{\sqrt{4\pi\omega}} e^{i(kv+\omega u)}$ as particle modes for Rindler observer in region 2. And again for both $g^1_k(x)$ and $g^2_k(x)$, $\omega = |k|$.

Set of $g^1_k(x)$ and $g^2_k(x)$, is a complete set of modes in region 1 and 2 respectively. However the lines $z = t$ or $z = -t$ are Cauchy surfaces(Appendix A) for the whole spacetime. So a set comprised of both $g_k^1(x)$'s and $g_k^2(x)$'s will be a complete set of modes for field expansion. So, we finally define these modes in both regions

$$\begin{aligned} g_k^1(x) &= \frac{1}{\sqrt{4\pi\omega}} e^{i(kv-\omega u)};\ \text{in region 1} \\ &= 0;\quad \text{in region 2} \\ g_k^2(x) &= \frac{1}{\sqrt{4\pi\omega}} e^{i(kv+\omega u)};\ \text{in region 2} \\ &= 0;\quad \text{in region 1} \end{aligned} \tag{2.13}$$

These are called Rindler modes and these can be extended to regions 3 and 4 also.[Mat07]

None of these modes can be linear combination of only positive frequency Minkowski modes $f_k^*(x)$(eq. 2.9), because these modes are not analytic at origin $x = z = 0$, because of discontinuity at that point. So, they should also contain negative frequency modes $f_k^*(x)$. Now if we consider the following combinations of modes

$$\begin{aligned} h_k^1(x) &= g_k^1(x) + e^{-\pi\omega/a} g_{-k}^{2*}(x) \\ h_k^2(x) &= e^{\pi\omega/a} g_k^2(x) + g_{-k}^{1*}(x) \end{aligned} \tag{2.14}$$

We can see that these have analytic structure similar to $f_k(x)$ in equation (2.9). Since for $k > 0$, $f_k(x) = \frac{1}{\sqrt{4\pi\omega}} e^{i(kz-\omega t)}$ is bounded and analytic in region of $(t-z)$ complex plane, where imaginary part of $(t-z)$, $\mathbf{Im}(t-z)$ is less than zero.

Now for $k > 0$

$$\begin{aligned} h_k^1(x) &= \alpha[e^{-i\omega(u-v)}] = \alpha[e^{-a(u-v)(i\omega/a}] = \alpha(-a(t-z))^{i\omega/a};\ \text{in region 1} \\ &= e^{i\pi i\omega/a}\alpha(-a(t-z))^{i\omega/a} = (-1)^{-i\omega/a}\alpha(-a(t-z))^{i\omega/a} \\ &= \alpha(a(t-z))^{i\omega/a};\ \text{in region 2} \end{aligned} \tag{2.15}$$

Here we've used equations(2.2) and (2.7). $\alpha = \frac{1}{\sqrt{4\pi\omega}}$ is normalization constant. This is analytic and bounded for region $\mathbf{Im}(t-z) < 0$ in complex $(t-z)$ plane. Also, it's analytic at $z = t = 0$. Similarly we can show for $k < 0$,

$h_k^1(x) \propto (a(t+z))^{-i\omega/a}$. This is analytic and bounded for region $\mathbf{Im}(t+z) < 0$ similar to $f_k(x)$ with $k < 0$. And for $h_k^2(x)$ we can similarly have

$$
\begin{aligned}
h_k^1(x) &\propto ((a(t+z))^{i\omega/a}; \text{ for } k > 0 \\
&\propto ((a(t-z))^{-i\omega/a}; \text{ for } k < 0
\end{aligned}
$$

These two also share property of being analytic and bounded in regions $\mathbf{Im}(t+z) < 0$ and $\mathbf{Im}(t-z) < 0$. So these modes are only composed of positive frequency Minkowski modes.

Observation of thermal state by Rindler observer

We can write the expansion of field ϕ in terms of $h_k^1(x)$ and $h_k^1(x)$

$$\phi(x) = \sum_k [C_1(b_k^1 h_k^1(x) + b_k^{1\dagger} h_k^{1*}(x)) + C_2(b_k^2 h_k^2(x) + b_k^{2\dagger} h_k^{2*}(x))] \tag{2.16}$$

Where $C_1 = \frac{e^{\pi\omega/a}}{\sqrt{2sinh(\pi\omega/a)}}$ with $h_k^1(x)$ and $C_2 = \frac{e^{-\pi\omega/a}}{\sqrt{2sinh(\pi\omega/a)}}$ with $h_k^2(x)$ is inserted to make the modes normalized, since $g_k^1(x)$'s and $g_k^1(x)$'s are already normalized. This expansion should be equal to the expansion in the basis of modes $g_k^1(x)$'s and $g_k^1(x)$'s

$$\phi(x) = \sum_k [a_k^1 g_k^1(x) + a_k^{1\dagger} g_k^{1*}(x) + a_k^2 g_k^2(x) + a_k^{2\dagger} g_k^{2*}(x)] \tag{2.17}$$

Using equation (2.14), we can find the following relations between operators a_k's and b_k's

$$
\begin{aligned}
a_k^1 &= \frac{1}{\sqrt{2sinh(\pi\omega/a)}}[e^{\pi\omega/2a} b_k^2 + e^{-\pi\omega/2a} b_{-k}^{1\dagger}] \\
a_k^2 &= \frac{1}{\sqrt{2sinh(\pi\omega/a)}}[e^{\pi\omega/2a} b_k^1 + e^{-\pi\omega/2a} b_{-k}^{2\dagger}]
\end{aligned}
\tag{2.18}
$$

Since h_k are composed of positive frequency Minkowski modes only, operators d_k^1 and d_k^2 should annihilate Minkowski vacuum state $|0\rangle$, i.e. $d_k^1|0\rangle = d_k^1|0\rangle = 0$. Now using this fact and equation (2.18), we can find number of Rindler particles in region 1 with frequency ω and wave number k

$$\langle 0|a_k^{1\dagger} a_k^1|0\rangle = \frac{1}{e^{2\pi\omega/a} - 1} \tag{2.19}$$

Same number would be obtained for number of Rindler modes in region 2 with frequency ω and wave number k. The number distribution of quanta is same as in thermal state for bosons. Hence, we can say that Rindler observer witnesses thermal bath in Minkowski vacuum. The frequency in equation (2.19) will be the one observed by observers with proper time $= u$. These are the observers with coordinate $v = 0$. For an observer with general value of v, the frequency will be ωe^{av}. Then comparing r.h.s. of equation (2.19) with bosonic thermal factor $\frac{1}{e^{\omega/k_BT}}$ we'll have $\frac{\omega}{k_BT} = \frac{2\pi\omega}{e^{-av}a}$. Here, T is temperature observed and k_B is Boltzmann constant. From equation (2.5), we see that proper acceleration $= e^{-av}a$. Then the temperature(Unruh temperature) for observer with uniform proper acceleration α will be

$$T_{Unruh} = \frac{\alpha}{2\pi k_B} \tag{2.20}$$

Some Remarks

The case presented here was for massless bosons, but similar thermal state is observed by Rindler observer for massive bosons and fermions.[Mat07] Although, significant Unruh temperature can only be observed for extremely high accelerations, eg.- The acceleration of protons in LHC ring takes 20 minutes to accelerate proton from energy of 450 GeV to 4 TeV. [@CER]
This means an average proper acceleration of $\approx 10^12\ m/s^2$. This corresponds to proton observing a temperature of $\approx 10^{-8}$ K.

There have been many suggestions in literature about implications of Unruh effect on standard model processes and many experiments have been proposed too.[Mat07; Mil04] The main implications are about modification of cross sections/decay rates of certain processes and possibility of some forbidden processes of standard model in Minkowski spacetime. In next chapter one of such processes *weak decay of an accelerated proton* is described.

3

Decay of an Accelerated Proton

The possibility of decay of an accelerated proton(p) due to following *weak interaction* process was first proposed by Vanzella and Matsas.[Mat01a]

$$p \rightarrow n + e^{+} + \nu_e \tag{3.1}$$

The final products are a neutron(n), a positron(e^{+}) and an electron neutrino(ν_e). They evaluated the decay rate in both inertial frame and accelerated frame and assumed neutrino to be massless. They obtained identical rates in both frames by solving numerically the integrals in accelerated frames.[Mat01a] However it's possible to get the equality analytically and for massive neutrinos too. Here is presented the approach for proton decay rate calculation due to Suzuki and Yamada.[Yam03]

An inertial proton in standard flat space quantum field theory can't decay through this process, since the mass of a neutron is more than the mass of a proton and process has to conserve four momentum due to space-time translation invariance of theory. But, if the proton is accelerating then any process containing proton need not obey this constraint and proton decay could occur. We will see that it could also be explained by an alternative picture in the rest frame of proton where it's able to interact with Rindler particles.

Semi-classical vector current and interaction

An accelerated proton is assumed to follow a classical trajectory. The assumption is valid if the acceleration $<<$ mass of proton(m_p), so that radiation reaction could be ignored. The further restriction on the acceleration is due to internal structure of protons. The assumption of pointlike proton would break if the acceleration(a) is caused by electromagnetic force(which is always the case), which is strong enough to probe the inner structure of proton. This condition is satisfied if the force applied $<< m_{pion}^2$ and hence $a << m_{pion}^2/m_p$.[Tor15]
To use the Fermi effective theory for of weak interactions the protons should be accelerated for small enough time so that their energies are $<<$ mass of W-boson(M_W) $\approx$ 100 GeV. If τ is the proper time of proton and the trajectory is $x = x[\tau]$ with four velocity $V_\nu[\tau]$, then semi-classical vector current is

$$j^\nu(x) = \hat{g}[\tau]V^\nu[\tau]\frac{\delta^{(3)}(x - x[\tau])}{V^0} \tag{3.2}$$

Here $\hat{g}$ is a quantum mechanical operator on the vector space of proton and neutron, which would make transition between two states possible because it's matrix element $\langle n|\hat{g}[\tau]|p\rangle$ is assumed to be non-zero. If H is the Hamiltonian in the space of proton and neutron state, such that $H|p\rangle = m_p|p\rangle$ and $H|n\rangle = m_n|n\rangle$. Then

$$\hat{g}[\tau] = e^{iH\tau}\hat{g}[0]e^{-iH\tau} \tag{3.3}$$

And we define effective Fermi constant G_{eff} by

$$G_{eff} = \langle n|\hat{g}[0]|p\rangle \tag{3.4}$$

The Fermi interaction of two vector currents of following form is taken as interaction Lagrangian

$$\mathcal{L}_I = \int d^4x\sqrt{-g}\, j_\nu [\bar{\psi}_e \gamma^\nu \psi_{\nu_e} + \bar{\psi}_{\nu_e} \gamma^\nu \psi_e] \tag{3.5}$$

Fermion field quantization

Inertial frame

Fermion field $\psi(x)$ with mass m satisfies Dirac equation

$$(i\partial\!\!\!/ - m)\psi(x) = 0 \tag{3.6}$$

It can be expanded in the basis of fixed energy-momentum solutions $u_{p\sigma}(x)$ and $v_{p\sigma}(x)$, where p is the four momentum and energy $= p^0 = \omega$. σ is polarization which can take one of the two values $+$ or $-$. These (normalized) wavefunctions have following form

$$\begin{aligned} u_{p\sigma}(x) &= \frac{e^{-ip_\mu.x^\mu}}{(2\pi)^{3/2}} \frac{p\!\!\!/ + m}{\sqrt{2\omega(\omega+m)}} \alpha_\sigma \\ v_{p\sigma}(x) &= \frac{e^{+ip_\mu.x^\mu}}{(2\pi)^{3/2}} \frac{p\!\!\!/ - m}{\sqrt{2\omega(\omega-m)}} \alpha_\sigma \end{aligned} \tag{3.7}$$

where $\omega = \sqrt{k_1^2 + k_2^2 + k_3^2 + m^2} \equiv \sqrt{|\vec{k}|^2 + m^2}$ and α_σ are

$$\alpha_+ = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \alpha_- = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \tag{3.8}$$

In this text we've used Dirac representation of gamma matrices. [@Wik16b]

$\psi(x)$ can be expanded as

$$\psi(x) = \int d^3p \sum_{\sigma} [a_{p\sigma} u_{p\sigma}(x) + b^{\dagger}_{p\sigma} v_{p-\sigma}(x)] \tag{3.9}$$

Here $a_{p\sigma}$ is fermion annihilation operator, while $b^{\dagger}$ is anti-fermion creation operator. These operators satisfy canonical anti-commutation relations

$$\begin{aligned} \{a_{p\sigma}, a^{\dagger}_{p'\sigma'}\} &= \{b_{p\sigma}, b^{\dagger}_{p'\sigma'}\} = \delta^{(3)}(p-p')\delta_{\sigma\sigma'} \\ \{a_{p\sigma}, a_{p'\sigma'}\} &= \{b_{p\sigma}, b_{p'\sigma'}\} = 0 \end{aligned} \tag{3.10}$$

Accelerated Frame

We extend definition of Rindler coordinates in region 1 to 4-dimensional spacetime. First we change coordinates $a^{-1}e^{av} \to v$ and $au \to u$ in equations (2.2). The trajectory of a uniformly accelerated observer is still $v = a^{-1}$. Other two coordinates remain same as Minkowski coordinates and coordinates of a point are

$$x_{\mu} = (u, x^1, x^2, v) \tag{3.11}$$

The metric is

$$g_{\mu\nu} = v^2 du^2 - (dx^1)^2 - (dx^2)^2 - dv^2 \tag{3.12}$$

Following vierbeins (Appendix B) are selected

$$X_0^{\nu} = v^{-1}\delta_0^{\mu}\,, \;\; X_a^{\mu} = \delta_a^{\mu} \tag{3.13}$$

Using fermion covariant derivative((Appendix B), we will get following Dirac equation

$$i\frac{\partial\psi(x)}{\partial u} = \left(\gamma^0 mv - i\gamma^0\gamma^3/2 - iv\gamma^0\gamma^b\partial_b\right)\psi(x) \tag{3.14}$$

Index 'b' runs from 1 to 3.

We now define following objects

$$\begin{aligned}\mathbf{w} &= (\omega, p^1, p^2)\\ \mathbf{x} &= \vec{x} = (x^1, x^2, v)\end{aligned} \tag{3.15}$$

We'll just state the solution for Dirac equation, details about solving this Dirac equation can be found in [Yam03]. $\psi(x)$ can be expanded in terms of positive frequency particle wavefunctions $U_{\omega\sigma}(x)$ and anti-particle wavefunctions $V_{\omega\sigma}(x)$. These are eigenfunctions of Hamiltonian.[Yam03]

$$\psi(x) = \int_0^\infty d\omega \int d^2p \sum_\sigma [\hat{p}_{\mathbf{w}\sigma} U_{\mathbf{w}\sigma}(x) e^{-i(\omega u/a - p_b x^c)} + \hat{q}^\dagger_{\mathbf{w}\sigma} V_{\mathbf{w}-\sigma}(x) e^{i(\omega u/a - p_b x^b)}] \tag{3.16}$$

Here subscript 'c' takes values '1' and '2'. σ can take one of the two values '$+$' or '$-$'. Here operators $\hat{p}_{\mathbf{w}\sigma}$ and $\hat{q}^\dagger_{\mathbf{w}\sigma}$ respectively are annihilation operators for fermion and creation operators for anti-fermions. These operators satisfy following anti-commutation relations

$$\{\hat{p}_{\mathbf{w}\sigma}, \hat{p}^\dagger_{\mathbf{w}'\sigma'}\} = \{\hat{q}_{\mathbf{w}\sigma}, \hat{q}^\dagger_{\mathbf{w}'\sigma'}\} = \delta^{(3)}(\mathbf{w} - \mathbf{w}')\delta_{\sigma\sigma'} \tag{3.17}$$

And all other anti-commutators vanish. The operators $\hat{p}_{\omega\sigma}$ and $\hat{q}_{\omega\sigma}$ annihilate Rindler vacuum state and not Minkowski vacuum state.
The expressions for wavefunctions are following

$$\begin{aligned}U_{\mathbf{w}\sigma}(x) &= \sqrt{\frac{\cosh\Omega\pi}{\pi L}} \frac{1}{(2\pi)^{\frac{3}{2}}} \gamma^0 [(P_d\gamma^d + M) K_{i\Omega+\frac{1}{2}}(lv) + iL\gamma^3 K_{i\Omega-\frac{1}{2}}(lv)]\alpha_\sigma\\ V_{\mathbf{w}\sigma}(x) &= \sqrt{\frac{\cosh\Omega\pi}{\pi L}} \frac{1}{(2\pi)^{\frac{3}{2}}} \gamma^0 [(P_d\gamma^d - M) K_{i\Omega+\frac{1}{2}}(lv) + iL\gamma^3 K_{i\Omega-\frac{1}{2}}(lv)]\alpha_\sigma\end{aligned} \tag{3.18}$$

$$\alpha_+ = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad \alpha_- = \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix} \tag{3.19}$$

Index 'c' takes values '1' or '2'. $l = \sqrt{(p^1)^2 + (p^2)^2 + m^2}$ Here $K_\nu(y)$ is modified Bessel function of second kind. And following quantities are defined

$$L = \frac{l}{a}; \quad P_d = \frac{p_d}{a}; \quad \Omega = \frac{\omega}{a}; \quad M_i = \frac{m_i}{a} \tag{3.20}$$

Inertial frame calculation

We consider a uniformly accelerated proton along x^3 direction with proper acceleration α in region 1 and Rindler coordinate $v = 0$.(see sec. 1.0)

$$\begin{aligned} &\text{Minkowski coordinates of proton} = x^\mu = (a^{-1}\sinh au, 0, 0, a^{-1}\cosh au) \\ &\text{Proper time of proton} = \tau = u = a^{-1}\sinh^{-1} ax^0 \\ &\text{Four velocity of proton} = V^\mu = (\cosh a\tau, 0, 0, \sinh a\tau) \end{aligned} \tag{3.21}$$

At tree level, the non-zero contribution in proton decay is only due to second term in Lagrangian and the process has following amplitude

$$\mathcal{A}_{in} = \int d^4x\sqrt{-g}\langle n| \otimes \langle \nu_e, e^+| j_\nu [\bar{\psi}_{\nu_e}\gamma^\nu \psi_e]|p\rangle \tag{3.22}$$

With Lagrangian in equation (3.5) we'll have

$$\begin{aligned} \mathcal{A}_{in} &= G_{eff}\int dx^0 dx^3 e^{i\Delta m\tau} V_\nu \frac{\delta(x^3 - \sqrt{a^{-2} + x^{02}})}{V^0} \langle \nu_e, e^+|[\bar{\psi}_{\nu_e}\gamma^\nu \psi_e]|0\rangle \\ &= G_{eff}\int d\tau V^0 \frac{e^{i\Delta m\tau}}{(2\pi)^3} \frac{V_\nu}{V^0}[\bar{u}_{p\nu_e}\gamma^\nu v_{ke^+}] e^{\{i(\omega_e + \omega_{\nu_e})a^{-1}\sinh a\tau - (k^3 + p^3)\cosh a\tau\}} \end{aligned} \tag{3.23}$$

k and p are momentum of final state positron and neutrino respectively and fermion polarization indices and coordinate dependence implicit is left in last expression. The differential decay probability will be= $|\mathcal{A}_{in}|^2$. To get unpolarized/total decay probability(P) we will sum over all the possible final state polarizations(σ).

$$P = \frac{G_{eff}}{(2\pi)^6} \times \int d^3k d^3p \int d\tau d\tau' V_\mu V_\nu^* e^{\{i(\omega_e+\omega_{\nu_e})a^{-1}(\sinh a\tau - \sinh a\tau') - (k^3+p^3)(\cosh a\tau - \cosh a\tau')\}} \sum_\sigma [\bar{u}_{p\nu_e}\gamma^\nu v_{ke^+}][\bar{u}_{p\nu_e}\gamma^\mu v_{ke^+}]^* \tag{3.24}$$

We will use following fermion spin sum rules to evaluate spin sum in the last expression

$$\begin{aligned} \sum_\sigma u_{p\sigma}\bar{u}_{p\sigma} &= \frac{\not{p}+m}{2\omega} \\ \sum_\sigma v_{p\sigma}\bar{v}_{p\sigma} &= \frac{\not{p}-m}{2\omega} \end{aligned} \tag{3.25}$$

Then spin sum in equation (3.24) will be

$$\begin{aligned} &\frac{1}{4\omega_{e^+}\omega_{\nu_e}} \text{Trace}[(\not{p}+m_{\nu_e})\gamma^\nu(\not{k}-m_e)\gamma^\mu] \\ =&\frac{1}{\omega_{e^+}\omega_{\nu_e}} p_\alpha k_\beta [g^{\alpha\nu}g^{\beta\mu} - g^{\alpha\beta}g^{\nu\mu} + g^{\alpha\mu}g^{\beta\nu} - m_{\nu_e}m_e g^{\nu\mu}] \end{aligned}$$

Where formulas for trace of product of gamma matrices are used (for reference see section 5.1 of [V.S05]). We can leave terms linear in k^1,k^2,p^1 and p^2, since they will contribute zero due to anti-symmetry. And we take product of this spin sum with $V_\nu[\tau]V_\mu^*[\tau']$ to get

$$\begin{aligned} \frac{1}{\omega_{e^+}\omega_{\nu_e}}[(\omega_{e^+}\omega_{\nu_e} + k^3p^3)\cosh a(\tau+\tau') - (\omega_{e^+}p^3 + \omega_{\nu_e}k^3)\sinh a(\tau+\tau') \\ - m_{\nu_e}m_e \cosh a(\tau-\tau')] \end{aligned}$$

Now we make following transformations

$$\tau = r + \frac{l}{2}$$
$$\tau' = r - \frac{l}{2} \tag{3.26}$$

$$k^{3'} = -\omega_{e^+}\sinh ar + k^3\cosh ar \quad \text{and} \quad p^{3'} = -\omega_{\nu_e}\sinh ar + p^3\cosh ar$$
$$\omega^{'}_{e^+} = \omega_{e^+}\cosh ar - k^3\sinh ar \quad \text{and} \quad \omega^{'}_{\nu_e} = \omega_{\nu_e}\cosh ar - p^3\sinh ar \tag{3.27}$$

These substitutions will lead us to following expression

$$P = \int dr \frac{G_{eff}}{(2\pi)^6}\int d^3k' d^3p' \int dl e^{i\{\Delta m r + 2(\omega^{'}_{e^+} + \omega^{'}_{\nu_e})a^{-1})\sinh\frac{al}{2}\}}$$
$$\times\,(\omega^{'}_{e^+}\omega^{'}_{\nu_e} + k^{3'}p^{3'} - m_{\nu_e}m_e\cosh al) \tag{3.28}$$

Since integrand does not depend on r, we can integrate $\int dr$ to get an infinite factor of total proper time T. Dividing left hand side of equation (3.28) with T, we will get decay rate($\Gamma_{\mathtt{in}}$) , i.e. decay probability per unit time. We also introduce following dimensionless quantities

$$\Delta M = \frac{\Delta m}{a}; \quad M_e = \frac{m_e}{a}; \quad M_{\nu_e} = \frac{M_{\nu_e}}{a}$$
$$K = \frac{k}{a}; \quad P = \frac{p}{a}; \quad \Omega_e = \frac{\omega_e}{a}; \quad \Omega_{\nu_e} = \frac{\omega_{\nu_e}}{a} \tag{3.29}$$

And we introduce new integration variable $\alpha \equiv e^{ar/2}$, which results in

$$\Gamma_{\mathtt{in}} = \frac{a^5 G^2_{eff}}{32\pi^6}\int_0^\infty d\alpha \int d^3K d^3P \frac{e^{i(\Omega_e + \Omega_{\nu_e})(\alpha - 1/\alpha)}}{\Omega_e\Omega_{\nu_e}}\alpha^{2i\Delta M - 1}$$
$$\times[\Omega_e + \Omega_{\nu_e} - \frac{1}{2}M_e M_{\nu_e}(\alpha^2 + \frac{1}{\alpha^2})] \tag{3.30}$$

We can use following integral representation of modified Bessel function of second kind in last equation [Str12]

$$K_\nu(z) = \frac{z^\nu}{2^{\nu+1}a^\nu}\int_0^\infty w^{\nu-1}e^{-\frac{a}{w} - \frac{z^2 w}{4a}}dw \tag{3.31}$$

We can then transform equation (3.30) to

$$\Gamma_{\texttt{in}} = \frac{a^5 G_{eff}^2}{16\pi^6 e^{\pi\Delta M}} \int d^3K d^3P \, [K_{2i\Delta M}(2(\Omega_e + \Omega_{\nu_e})) + \frac{M_e M_{\nu_e}}{2\Omega_e \Omega_{\nu_e}} \{K_{2i\Delta M+2}(2(\Omega_e + \Omega_{\nu_e})) + K_{2i\Delta M-2}(2(\Omega_e + \Omega_{\nu_e}))\}] \tag{3.32}$$

We can further use following contour integral representation of $K_\nu(z)$ in last equation [Yam03; @Wol16]

$$K_\nu(x) = \frac{1}{4\pi i} \int_{C_y} dy \, \Gamma(-y)\Gamma(-y-\mu) \left(\frac{x}{2}\right)^{2y+\mu} \tag{3.33}$$

Closed contour C_y is chosen such that it encircles all the poles of $\Gamma(-y)$ and $\Gamma(-y-\mu)$. We further substitute $y + i\Delta M$ for y and obtain

$$\begin{aligned} \Gamma_{\texttt{in}} = \frac{a^5 G_{eff}^2}{32\pi^6 e^{\pi\Delta M}} \int d^3K d^3P \int_{C_y} \frac{dy}{2\pi i} (\sqrt{|\vec{K}|^2 + M_e^2} + \sqrt{|\vec{P}|^2 + M_{\nu_e}^2})^{2y} \\ \times \left[\Gamma(-y + i\Delta M)\Gamma(-y - i\Delta M) + \frac{M_e M_{\nu_e}}{2\Omega_e \Omega_{\nu_e}} \{\Gamma(-y + i\Delta M + 1) \right. \\ \times \Gamma(-y - i\Delta M - 1) + \Gamma(-y + i\Delta M - 1)\Gamma(-y - i\Delta M + 1)\}] \end{aligned} \tag{3.34}$$

In the following formula[Yam03] we substitute $A = \sqrt{|\vec{K}|^2 + M_e^2}$ and $B = \sqrt{|\vec{P}|^2 + M_{\nu_e}^2}$

$$(A+B)^x = \int_{C_z} \frac{dz}{2\pi i} \frac{\Gamma(-z)\Gamma(z-x)}{\Gamma(-x)} A^{-z+x} B^z \tag{3.35}$$

The contour C_z is chosen to enclose all poles of $\Gamma(-z)$ if $A > B$ and to enclose all the poles of $\Gamma(z-x)$ if $A < B$. Then we can integrate over d^3K and d^3P in polar coordinates. Only writing the relevant part for integration of equation (3.34)

$$\int d^3K d^3P \int_{C_y} (\sqrt{|\vec{K}|^2 + M_e^2} + \sqrt{|\vec{P}|^2 + M_{\nu_e}^2})^{2y} = 8\pi^2 \int_0^\infty d|\vec{K}| d|\vec{P}| |\vec{K}|^2 |\vec{P}|^2 \int_{C_z} \frac{dz}{2\pi i} \frac{\Gamma(-z)\Gamma(z-2y)}{\Gamma(-2y)} (\sqrt{|\vec{K}|^2 + M_e^2})^{-z+2y} (\sqrt{|\vec{P}|^2 + M_{\nu_e}^2})^z$$

To integrate over $|\vec{K}|$ and $|\vec{P}|$, we can use the following representation of Beta function[@Wik16a]

$$\beta(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} = \int_0^\infty \frac{t^{x-1}}{(1+t)^{x+y}} dt \tag{3.36}$$

We finally make following substitutions to arrive at final expression for decay rate

$$y \to y - \frac{z}{2} + \frac{3}{2}$$
$$z \to -\frac{z}{2} + \frac{3}{2}$$

$$\begin{aligned}\Gamma_{\texttt{in}} = \frac{a^5 G_{eff}^2}{32\pi^{7/2} e^{\pi\Delta M}} \int_{C_y} \frac{dy}{2\pi i} \int_{C_z} \frac{dz}{2\pi i} \frac{M_e^{2y} M_{\nu_e}^{2z}}{\Gamma(-y-z+3)\Gamma(-y-z+\frac{7}{2})} \\ \times \left[|-y-z+i\Delta M+3|^2 \Gamma(-y)\Gamma(-z)\Gamma(-y+2)\Gamma(-z+2) \right. \\ + \mathbf{Re}\{\Gamma(-y-z+i\Delta M+2)\Gamma(-y-z-i\Delta M+4)\} \\ \left. \times \Gamma(-y+\frac{1}{2})\Gamma(-z+\frac{1}{2})\Gamma(-y+\frac{3}{2})\Gamma(-z+\frac{3}{2}) \right]\end{aligned} \tag{3.37}$$

Accelerated frame calculation

We consider a uniformly accelerated proton. In Rindler coordinates

$$\begin{aligned}&\text{Proton's coordinates} = x^\mu = (u, o, o, \frac{1}{a}) \\ &\text{Proton's proper time} = \frac{u}{a} \\ &\text{Proton's four velocity} = u^\mu = (a, 0, 0, 0)\end{aligned} \tag{3.38}$$

In accelerated frame, the proton can't go through weak decay process. We can see it by calculating the amplitude(A_1), using interaction term in equation (3.5) in Rindler coordinates of this process. The Lagrangian and amplitude are

$$\begin{aligned}
\mathcal{L}_I &= \int d^4x\sqrt{-g}\, j_\nu[\bar{\psi}_e X^\nu_\mu \gamma^\mu \psi_{\nu_e} + \bar{\psi}_{\nu_e} X^\nu_\mu \gamma^\mu \psi_e] \\
A_1 &= \langle n, \nu_e, e^+|\mathcal{L}_I|o\rangle = \frac{G_{eff}}{(2\pi)^2}\int du e^{i(\Delta m+\omega_{e^+}+\omega_{\nu_e})u}\bar{u}_{\nu_e}\gamma^0 v_{e^+} \\
&= \frac{G_{eff}}{2\pi}\delta(\Delta m+\omega_{e^+}+\omega_{\nu_e})\bar{u}_{\nu_e}\gamma^0 v_{e^+}
\end{aligned} \tag{3.39}$$

Now since $\Delta m = m_n - m_p > 0$, the argument of delta function is always > 0 . Hence, the amplitude vanishes. Although, an accelerated proton will 'observe' presence of Rindler particles(*Unruh effect*) and can go through following three processes.

$$\begin{aligned}
&(a)\ p + e^- + \bar{\nu}_e \to n \\
&(b)\ p + e^- \to n + \nu \\
&(c)\ p + \bar{\nu}_e \to e^+ + n
\end{aligned} \tag{3.40}$$

The amplitudes for these processes at first order in G_{eff} are(as done in equation (3.39))

$$\begin{aligned}
A_{(a)} &= \frac{G_{eff}}{2\pi}\delta(\Delta m - \omega_{e^-} - \omega_{\bar{\nu}_e})\bar{v}_{\bar{\nu}_e}\gamma^0 u_{e^-} \\
A_{(b)} &= \frac{G_{eff}}{2\pi}\delta(\Delta m - \omega_{e^-} + \omega_{\nu_e})\bar{u}_{\nu_e}\gamma^0 v_{e^+} \\
A_{(c)} &= \frac{G_{eff}}{2\pi}\delta(\Delta m + \omega_{e^+} - \omega_{\bar{\nu}_e})\bar{u}_{e^+}\gamma^0 u_{\bar{\nu}}
\end{aligned} \tag{3.41}$$

Spin indices and dependence on 'lu' on fermion wavefunctions are left implicit. The probability of transition will be modified due to already present fermions in a thermal state. So, for each fermion in initial state with frequency ω the probability should be multiplied with a factor of $\theta(\omega) = \frac{1}{1+e^{2\pi\Omega}}$ and for each fermion in final state with frequency ω it should be multiplied by a factor of $(1-\theta(\omega))$.

The differential transition probabilities are

$$\begin{aligned}
\frac{d^6P_{(a)}}{d\omega_{e^-}d\omega_{\bar{\nu}_e}d^2p_{e^-}d^2p_{\bar{\nu}_e}} &= \beta\delta(\Delta m-\omega_{e^-}-\omega_{\bar{\nu}_e})^2|\bar{v}_{\bar{\nu}_e}\gamma^0u_{e^-}|^2\theta(\omega_{\bar{\nu}_e})\theta(\omega_{e^-})\\
\frac{d^6P_{(b)}}{d\omega_{e^-}d\omega_{\nu_e}d^2p_{e^-}d^2p_{\nu_e}} &= \beta\delta(\Delta m-\omega_{e^-}+\omega_{\nu_e})^2|\bar{u}_{\nu_e}\gamma^0v_{e^+}|^2(1-\theta(\omega_{\nu_e}))\theta(\omega_{e^-})\\
\frac{d^6P_{(c)}}{d\omega_{e^+}d\omega_{\bar{\nu}_e}d^2p_{e^+}d^2p_{\bar{\nu}_e}} &= \beta\delta(\Delta m+\omega_{e^+}-\omega_{\bar{\nu}_e})^2|\bar{u}_{e^+}\gamma^0u_{\bar{\nu}}|^2\theta(\omega_{\bar{\nu}_e})(1-\theta(\omega_{e^+}))
\end{aligned} \tag{3.42}$$

$\beta=\frac{G_{eff}^2}{4\pi^2}$. To calculate transition rates we can use $\delta(0)$ from r.h.s., which will be equal to infinite time factor($T/2\pi$) and we can divide these equations from the $\delta(0)$. We'll further do fermion spin sums to get unpolarized transitions rates.

$$\begin{aligned}
\frac{1}{T}\frac{d^6P_{(a)}}{d\omega_{e^-}d\omega_{\bar{\nu}_e}d^2p_{e^-}d^2p_{\bar{\nu}_e}} &= \frac{G_{eff}^2}{32\pi^3}\frac{\delta(\Delta m-\omega_{e^-}-\omega_{\bar{\nu}_e})\sum_\sigma|\bar{v}_{\bar{\nu}_e}\gamma^0u_{e^-}|^2}{\cosh\pi\Omega_e^-\cosh\Omega_{\bar{\nu}_e}e^{\pi(\Omega_{e^-}+\Omega_{\bar{\nu}_e})}}\\
\frac{1}{T}\frac{d^6P_{(b)}}{d\omega_{e^-}d\omega_{\nu_e}d^2p_{e^-}d^2p_{\nu_e}} &= \frac{G_{eff}^2}{32\pi^3}\frac{\delta(\Delta m-\omega_{e^-}+\omega_{\nu_e})\sum_\sigma|\bar{u}_{\nu_e}\gamma^0v_{e^+}|^2}{\cosh\pi\Omega_e^-\cosh\Omega_{\nu_e}e^{\pi(\Omega_{e^-}-\Omega_{\nu_e})}}\\
\frac{1}{T}\frac{d^6P_{(c)}}{d\omega_{e^+}d\omega_{\bar{\nu}_e}d^2p_{e^+}d^2p_{\bar{\nu}_e}} &= \frac{G_{eff}^2}{32\pi^3}\frac{\delta(\Delta m+\omega_{e^+}-\omega_{\bar{\nu}_e})\sum_\sigma|\bar{u}_{e^+}\gamma^0u_{\bar{\nu}}|^2}{\cosh\pi\Omega_e^+\cosh\Omega_{\bar{\nu}_e}e^{\pi(-\Omega_{e^+}+\Omega_{\bar{\nu}_e})}}
\end{aligned} \tag{3.43}$$

We can do spin sums using the spin sum rules given in [Yam03]. After doing spin sums and making use of δ functions, we integrate these equations get following expression for total decay rate($\Gamma_{\mathtt{acc}}$)

$$\begin{aligned}
\Gamma_{\mathtt{acc}} &= \frac{G_{eff}^2}{2^3\pi^7e^{\pi\Delta M}}\\
&\times\int_{-\infty}^{\infty}d\Omega[\int d^2P_eL_e|K_{i\Omega+\frac{1}{2}}(L_e)|^2\int d^2P_\nu L_\nu|K_{i(\Omega-\Delta M)+\frac{1}{2}}(L_e)|^2\\
&+M_eM_\nu\mathbf{Re}\left(\int d^2P_eK^2_{i\Omega+\frac{1}{2}}(L_e)\int d^2P_\nu K^2i(\Omega-\Delta M)+\frac{1}{2}(L_\nu)\right)]
\end{aligned} \tag{3.44}$$

Now we can transform this expression using equation(3.33) for functions $K_\mu(z)$ as done in section (3.0).Then we integrate over momenta K's in polar coordinates(Since $L_i^2 = (P_i^1)^2 + (P_i^2)^2 + M_i^2$) after doing another transformation like in equation(3.35). These all manipulations lead to the expression

$$\begin{aligned}
&\Gamma_{\texttt{acc}} = \\
&\frac{G_{eff}^2}{2^3\pi^7 e^{\pi\Delta M}} \int_{C_y} \frac{dy}{2\pi i} \int_{C_z} \frac{dz}{2\pi i} \int d\Omega \frac{M_e^{2z+1} M_{\nu_e}^{2y+1}}{\Gamma(2y+1)\Gamma(2z+1)\Gamma(-y+1)\Gamma(-z+1))} \\
&\times [\Gamma(-y+1)\Gamma(-z+1)|\Gamma(-y+i(\Omega-\Delta M)+1)\Gamma(-z+i\Omega+1)|^2 \\
&M_e M_\nu \Gamma(-y-1)\Gamma(-z-1)\mathbf{Re}\{\Gamma(-y+i(\Omega-\Delta M)+1)\Gamma(-y-i(\Omega-\Delta M)) \\
&\times \Gamma(-z+i\Omega+1)\Gamma(-z-i\Omega)\}]
\end{aligned} \tag{3.45}$$

The contours C_z and C_y are chosen as done in section(3.0). Then we use following formula to integrate over Ω

$$\begin{aligned}
&\int_{-i\infty}^{i\infty} d\Omega \Gamma(a_1+\Omega)\Gamma(a_2+\Omega)\Gamma(a_3-\Omega)\Gamma(a_4-\Omega) \\
&= 2\pi i \frac{\Gamma(a_1+a_3)\Gamma(a_1+a_4)\Gamma(a_2+a_3)\Gamma(a_2+a_4)}{\Gamma(a_1+a_2+a_3+a_4)}
\end{aligned} \tag{3.46}$$

Which will lead us to final expression, which is same as the expression for decay rate in inertial frame (equation(3.37))

$$\begin{aligned}
\Gamma_{\texttt{acc}} = &\frac{a^5 G_{eff}^2}{32\pi^{7/2} e^{\pi\Delta M}} \int_{C_y} \frac{dy}{2\pi i} \int_{C_z} \frac{dz}{2\pi i} \frac{M_e^{2y} M_{\nu_e}^{2z}}{\Gamma(-y-z+3)\Gamma(-y-z+\frac{7}{2})} \\
&\times \left[|-y-z+i\Delta M+3|^2 \Gamma(-y)\Gamma(-z)\Gamma(-y+2)\Gamma(-z+2)\right. \\
&+\mathbf{Re}\{\Gamma(-y-z+i\Delta M+2)\Gamma(-y-z-i\Delta M+4)\} \\
&\left.\times \Gamma(-y+\frac{1}{2})\Gamma(-z+\frac{1}{2})\Gamma(-y+\frac{3}{2})\Gamma(-z+\frac{3}{2})\right] = \Gamma_{\texttt{in}}
\end{aligned} \tag{3.47}$$

Some remarks

The analytic equivalence of ‘decay rates’ in both frames is a good theoretical hint for the necessity of the existence of Unruh effect. Although, to detect decay of accelerated protons the accelerations required are of much higher magnitude than those could be produced by humans. As estimated by *Vanzella and Matsas* in [Mat01b], in typical pulsars, in circular motion under extremely high magnetic fields ($\approx 10^1 4$ Gauss) protons can have an acceleration of $\approx$ 100 MeV. Such a proton can have proper lifetime of the order of 10^{-7} seconds.

One interesting theoretical side of such processes would be to look at energy conservation and causality. It seems counter intuitive to imagine that two seemingly completely different process in two different frames actually correspond to same physical process. For example - particle absorption in one frame might correspond to emission of particle in the other frame. Such a process of detection of a scalar Rindler particle by an accelerated detector is explored by Unruh and Wald.[Wal84] Much more complex processes are worth analyzing , like proton decay process we discussed.

4

Neutrino Oscillations and Proton Decay

Neutrino Mixing

Neutrinos in standard model come in three flavors - electron neutrino(ν_e), muon neutrino(ν_μ) and tau neutrino(ν_τ). Eigenstates of these neutrino flavors are produced in standard model interactions. Neutrinos are now observed to have masses. But, these flavor eigenstates need not be and are not the mass eigenstates.[@Wik16d] This phenomenon is called neutrino mixing which is believed to be the resolution for solar neutrino problem. Observations of solar electron-neutrino flux deficit led to suggest that neutrinos can change flavor as they travel from sun to earth.[@Wik16f]

We denote neutrino mass eigenstates and fields with Roman subscripts, while neutrino flavor eigenstates and fields with Greek subscripts. Then neutrino mixing is represented by following equations for fields and states respectively

$$\begin{aligned} \nu_\mu &= \sum_a U^\dagger_{\mu a} \nu_a \\ |\nu_\mu\rangle &= \sum_a U_{a\mu} |\nu_a\rangle \end{aligned} \tag{4.1}$$

Here $U_{a\mu}$ is a unitary matrix called PMNS matrix or neutrino-mixing matrix. Phenomenon of neutrinos oscillating between different flavor eigenstates as they travel is called *neutrino oscillations*. In next two sections mechanism of neutrino oscillations is described - first it's described using a typical mechanism and next it's validity is questioned and one other possibility is discovered as done by [Lig09]

Standard Mechanism for Neutrino Oscillations

Here we will consider two flavor mixing. Neutrinos are assumed to have two mass eigenstates with masses m_a and m_b, and energies E_a and E_b. Two neutrino flavors are α and β.
In a typical explanation of neutrino oscillation at time $t = 0$ neutrino is assumed to be in a flavor(say α) eigenstate.

$$|\nu_\alpha(0)\rangle = U_{a\alpha}|\nu_a(0)\rangle + U_{b\alpha}|\nu_b(0)\rangle \tag{4.2}$$

Both of the mass eigenstates are assumed to have same three momentum p. The states after time t and at location x will be then

$$|\nu_\alpha(t)\rangle = e^{ip.x}(e^{-iE_a t}U_{a\alpha}|\nu_a(0)\rangle + e^{-iE_b t}U_{b\alpha}|\nu_b(0)\rangle) \tag{4.3}$$

Also, since

$$|\nu_\beta\rangle = U_{a\beta}|\nu_a(0)\rangle + U_{b\beta}|\nu_b(0)\rangle \tag{4.4}$$

We can calculate transition probability at time t from $|\nu_\alpha\rangle$ to $|\nu_\beta\rangle$

$$|\langle\nu_\beta|\nu_\alpha(t)\rangle|^2 = |U^*_{a\beta}U_{a\alpha}e^{-iE_a t} + U^*_{b\beta}U_{b\alpha}e^{-iE_b t}|^2 \tag{4.5}$$

This expression contains terms with time dependent phase with sum proportional to

$$cos((Ea - Eb)t + \theta)$$

θ is some real angle, which depends on PMNS matrix elements. The frequency of oscillations is then $\propto (E_a - E_b)$. For very high neutrino velocities $E_i \approx |p_i| + \frac{m_i^2}{2|p_i|}$. Assuming $|p_i|1 \approx E_i \equiv E$, oscillation frequency is then

$$\omega_{ab} = \frac{m_a^2 - m_b^2}{2E} \tag{4.6}$$

Neutrino oscillations with same dependence of oscillation frequency on difference in neutrino mass squared can be obtained even when one assumes both mass eigenstates to have same energy but different three momenta.[Smi09]

Argument against 'Standard' Mechanism for Neutrino Oscillations

A process in flat space quantum field theory must conserve four momentum. So, is initial state is energy-momentum eigenstate, then final one should be that too and with same eigenvalue. For a final state with flavor eigenstate neutrino, if we assume common momentum or common energy for mass eigenstates in final state, then energy-momentum conservation can't be obeyed, because neutrino will not have definite energy or momenta.

Entanglement of Final state particles

One possible way out of the problem we faced in last section is due to [Lig09]. They noted that a neutrino in final state in standard model is always accompanied by some other particle/particles (eg.- In weak decay there is always one more lepton with neutrino). So, if the energy-momentum of the process has to be conserved, then one can do so by entangling neutrino four momentum with the other particle in final state. Let us take an example of one particle (X) decaying into particle e and neutrino with flavor $\alpha(\nu_\alpha)$. We assume two neutrino flavors with following PMNS matrix

$$U = \begin{pmatrix} cos\theta & -sin\theta \\ sin\theta & cos\theta \end{pmatrix} \tag{4.7}$$

Also

$$|\nu_\alpha\rangle = cos\theta|\nu_1\rangle + sin\theta|\nu_2\rangle \tag{4.8}$$

Where $|\nu_1\rangle$ and $|\nu_2\rangle$ are two masss eigenstates. If initial particle has four momentum Q,$|\nu_1\rangle$ and $|\nu_2\rangle$ has four momenta p_1 and p_2. Then following final state is a momentum eigenstate

$$|\psi\rangle = N[\int d\Gamma(p_1,k_1)cos\theta|\nu_1(p_1),b(k_1)\rangle + \int d\Gamma(p_2,k_2)sin\theta|\nu_2(p_2),b(k_2)\rangle] \tag{4.9}$$

N is normalization factor and $d\Gamma(p,k)$ is final state phase space

$$d\Gamma(p,k) = \frac{d^3pd^3k}{(2\pi)^3 2\omega_p (2\pi)^3 2\omega_k}(2\pi)^4\delta^{(4)}(Q-p-k) \tag{4.10}$$

where $\omega_p = p^0$ and $\omega_k = k^0$. Delta function keeps track of four momentum conservation.

Neutrino oscillation Mechanism

For final state $|\psi\rangle$, if we take partial trace with respect to particle b then we will find density matrix for neutrino (ρ_ν)

$$\rho_\nu = \int d^3k\langle b(k)|\psi\rangle\langle\psi|b(k)\rangle \tag{4.11}$$

Where we made use of orthonormality relation $\langle b(k_i)|b(k_j)\rangle = (2\pi)^3 2\omega_{k_i}\delta^{(3)}(k_i - k_j)$ and the fact that k_1 and k_2 can't take the same value. Then density matrix of neutrino

$$\rho_\nu \propto \big(d\Gamma(p_1,k_1)cos^2\theta|\nu_1(p_1)\rangle\langle\nu_1(p_1)| + d\Gamma(p_2,k_2)sin^2\theta|\nu_2(p_2)\rangle\langle\nu_2(p_2)|\big) \tag{4.12}$$

This density matrix is that of an ensemble of neutrinos with fraction $cos^2\theta$ of them being ν_1 and fraction $sin^2\theta$ of them being ν_2. If measurement of neutrino flavor is performed over this final state, then probability of getting flavor α will be $(cos^4\theta + sin^4\theta)$ without any spacetime dependence and there will be no neutrino oscillations.

However, neutrino oscillations can still be observed due to non-ideal experimental conditions, eg -

- Due to localization(due to detection) of particle b in the final state.

- Due to the spread in the momentum space of initial particle.

We discuss here the first case, for second case reader is referred to [Lig09]. Suppose particle b is detected in a localized state $|b\rangle$ such that the matrix elements $\langle b|b(k_1)\rangle$ and $\langle b|b(k_2)\rangle$ are almost equal then the neutrino states $|b(k_1)\rangle$ and $|b(k_2)\rangle$ can get disentangled from the state of particle b. Then the neutrino state can be written by taking projection by $|b\rangle\langle b|$ over $|\psi\rangle$

$$|\psi\rangle_\nu \propto cos\theta|\nu_1(p_1)\rangle + sin\theta|\nu_2(p_2)\rangle \tag{4.13}$$

Now amplitude for detection of ν_α will be interference between two terms with different spacetime dependence in their phases

$$\langle\nu_\alpha(x)|\psi\rangle_\nu \propto cos^2\theta\, e^{ip_1.x} + sin^2\theta\, e^{ip_2.x} \tag{4.14}$$

Probability of detection of ν_α will have interference terms with phase $\propto (p_1 - p_2).x$ Suppose only neutrinos are detected are moving along x^1-direction with a detector located at a trajectory $x^1 = vt$, with 't' being the time of detection and 'v' the velocity of detector. In real world experiments the neutrinos do have some spread in momentum space and hence some localization in position space. The velocities(group velocities) of both mass eigenstates along x_1-direction will have different values in general . To have the interference the separation between peak of the wavepackets should be much smaller than the size of neutrino wave packets(D)

$$|v_1 - v_2|t << D \tag{4.15}$$

The interference phase when both neutrinos have group velocities d/t is

$$\begin{aligned}(p_1 - p_2).x = (p_1 - p_2).(t, vt, 0, 0) &\approx t(p_1 - p_2.)\frac{(p_1 + p_2)}{(\omega_1 + \omega_2)} \\ = t\frac{m_1^2 - m_2^2}{(\omega_1 + \omega_2)}\end{aligned} \tag{4.16}$$

Where group velocity,$v \approx \frac{(p_1+p_2)}{(\omega_1+\omega_2)}$ is used, which is because neutrinos are relativistic with negligible masses.

So, we see that the oscillation frequency of equation(4.6) can still be reproduced.

Decay of an Inertial Particle

We consider weak decay of an inertial particle a into three particles

$$a \to b + e^+ + \nu_e \tag{4.17}$$

This is similar to proton decay considered in chapter 3, but it's assumed to be allowed in inertial frame, i.e. the mass difference $\Delta m = m_a - m_b > m_e + m_{\nu_e}$ We work in the rest frame of particle 'a' (3.0)

$$\begin{aligned} &\text{Coordinates of a} = x^\mu = (t,0,0,0) \\ &\text{Proper time of a} = \tau = t \\ &\text{Four velocity of a} = V^\mu = (1,0,0,0) \end{aligned} \tag{4.18}$$

Interaction term is

$$\begin{aligned} &\mathcal{L}_I = \int d^4x j_\nu [\bar{\psi}_e \gamma^\nu \psi_{\nu_e} + \bar{\psi}_{\nu_e} \gamma^\nu \psi_e] \\ &\text{where semiclassical vector current} = j^\nu(x) = \hat{g}[\tau] V^\nu[\tau] \frac{\delta^{(3)}(x - x[\tau])}{V^0} \\ &\text{Effective Fermi constant } G = \langle b|\hat{g}[0]|a\rangle \end{aligned} \tag{4.19}$$

We consider neutrino oscillation in this calculation

$$|\nu_e\rangle = cos\theta|\nu_1\rangle + sin\theta|\nu_2\rangle \tag{4.20}$$

Final state $|\psi\rangle$ is considered with particle'b' in four momentum eigenstate and ν_e entangled with e^+.(see below equation (4.9) for meaning of symbols)

$$\begin{aligned} |\psi\rangle = |b(p_b)\rangle \otimes N[\int d\Gamma(p_1,k_1) cos\theta |\nu_1(p_1), b(k_1)\rangle \\ + \int d\Gamma(p_2,k_2) sin\theta |\nu_2(p_2), b(k_2)\rangle] \end{aligned} \tag{4.21}$$

The transition amplitude at first order in G is then

$$\begin{aligned} &\mathcal{A} = \langle\psi|\mathcal{L}_I|a(p_a)\rangle \\ &= \frac{G}{(2\pi)^3} \int dt V_0 N e^{-i\Delta m t} \left[\int d\Gamma(p_1,k_1) cos^2\theta e^{i(\omega_{1e}+\omega_{1\nu})t} \right. \\ &\left. \times (\bar{u}_\nu(p_1)\gamma^0 v_e(k_2)) + \int d\Gamma(p_2,k_2) sin^2\theta e^{i(\omega_{2e}+\omega_{2\nu})t} (\bar{u}_\nu(p_2)\gamma^0 v_e(k_2)) \right] \end{aligned} \tag{4.22}$$

We'll start by simplifying the first term

$$\begin{aligned}
&\frac{G}{(2\pi)^3}\int dtV_0Ne^{-i\Delta mt}\left[\int d\Gamma(p_1,k_1)cos^2\theta e^{i(\omega_{1e}+\omega_{1\nu})t}\right](\bar{u}_\nu(p_1)\gamma^0 v_e(k_1))\\
&=\frac{GN}{(2\pi)^3}\int d\Gamma(p_1,k_1)cos^2\theta(2\pi)\delta(\omega_{1e}+\omega_{1\nu}-\Delta m)(\bar{u}_\nu(p_1)\gamma^0 v_e(k_1))\\
&=\frac{GN}{(2\pi)^2}\int\frac{d^3p_1d^3k_1\,cos^2\theta}{(2\pi)^3 2\omega_{1e}(2\pi)^3 2\omega_{1\nu}}(2\pi)^4\delta^{(4)}(p_1+k_1+p_b-p_a)\\
&(2\pi)\delta(\omega_{1e}+\omega_{1\nu}-\Delta m)\times(\bar{u}_\nu(p_1)\gamma^0 v_e(k_1))
\end{aligned} \tag{4.23}$$

Now by using three delta functions from $\delta^{(4)}(p_1+k_1+p_b-p_a)$ we can integrate over $\int d^3k_1$ and use $\vec{k}_1=-\vec{p}_1$ in the $v_e(k_1)$ implicitly.

$$\frac{GN}{(2\pi)^3}\int\frac{d^3p_1\,cos^2\theta}{2\omega_{1e}2\omega_{1\nu}}(\delta(\omega_{1e}+\omega_{1\nu}-\Delta m))^2(\bar{u}_\nu(p_1)\gamma^0 v_e(k_1)) \tag{4.24}$$

Now following will be the expression for $(\bar{u}_\nu(p_1)\gamma^0 v_e(k_1))=(u_\nu^\dagger(p_1)v_e(k_1))$ using equation (3.7)

$$\begin{aligned}
\sqrt{2\omega_{1\nu}(\omega_{1\nu}+m_{1\nu})}u_\nu^\dagger(p_1)_+ &= \begin{pmatrix}\omega_{1\nu}+m_{1\nu} & 0 & -p_1^3 & -p_1^1+ip_1^2\end{pmatrix}\\
\sqrt{2\omega_{1\nu}(\omega_{1\nu}+m_{1\nu})}u_\nu^\dagger(p_1)_- &= \begin{pmatrix}0 & \omega_{1\nu}+m_{1\nu} & -p_1^1-ip_1^2 & p_1^3\end{pmatrix}
\end{aligned} \tag{4.25}$$

$$\sqrt{2\omega_{1e}(\omega_{1e}-m_e)v_e(k_1)}_-=\begin{pmatrix}\omega_{1e}-m_e\\0\\p_1^3\\p_1^1+ip_1^2\end{pmatrix}\quad\sqrt{2\omega_{1e}(\omega_{1e}-m_e)}v_e(k_1)_+=\begin{pmatrix}0\\\omega_{1e}-m_e\\p_1^1-ip_1^2\\-p_1^3\end{pmatrix} \tag{4.26}$$

'$+$' or '$-$' represents fermion polarization. Upper index represents component of momentum.

$$\begin{aligned}
u_\nu^\dagger(p_1)_+v_e(k_1)_- &= \frac{(\omega_{1\nu}+m_{1\nu})(\omega_{1e}-m_e)-|\vec{p}_1|^2}{2\sqrt{\omega_{1e}(\omega_{1e}-m_e)}\sqrt{\omega_{1\nu}(\omega_{1\nu}+m_{1\nu})}}\\
u_\nu^\dagger(p_1)_+v_e(k_1)_+ &= 0\\
u_\nu^\dagger(p_1)_-v_e(k_1)_- &= 0\\
u_\nu^\dagger(p_1)_-v_e(k_1)_+ &= \frac{(\omega_{1\nu}+m_{1\nu})(\omega_{1e}-m_e)-|\vec{p}_1|^2}{2\sqrt{\omega_{1e}(\omega_{1e}-m_e)}\sqrt{\omega_{1\nu}(\omega_{1\nu}+m_{1\nu})}}
\end{aligned} \tag{4.27}$$

Now we evaluate expression (4.24) in polar coordinates for polarization of positron and neutrino are '$-$' and '$+$' respecively.

$$
\begin{aligned}
&\frac{GN}{2(\pi)^2}\int_0^\infty d|\vec{p}_1||\vec{p}_1|^2\, cos^2\theta \frac{[(\omega_{1\nu}+m_{1\nu})(\omega_{1e}-m_e)-|\vec{p}_1|^2](\delta(\omega_{1e}+\omega_{1\nu}-\Delta m))^2}{8\omega_{1e}\omega_{1\nu}\sqrt{\omega_{1e}(\omega_{1e}-m_e)}\sqrt{\omega_{1\nu}(\omega_{1\nu}+m_{1\nu})}} \\
&= \frac{GN}{2(\pi)^2}\int_0^\infty d|\vec{p}_1||\vec{p}_1|^2\, cos^2\theta \frac{(\omega_{1\nu}+m_{1\nu})(\omega_{1e}-m_e)-|\vec{p}_1|^2}{8\omega_{1e}\omega_{1\nu}\sqrt{\omega_{1e}(\omega_{1e}-m_e)}\sqrt{\omega_{1\nu}(\omega_{1\nu}+m_{1\nu})}} \\
&\times(\delta(\omega_{1e}+\omega_{1\nu}-\Delta m))(\delta(|\vec{p}_1|-p_{01}))\frac{\sqrt{m_e^2+p_{01}^2}\sqrt{m_{1\nu}^2+p_{01}^2}}{p_{01}(\sqrt{m_e^2+p_{01}^2}+\sqrt{m_{1\nu}^2+p_{01}^2})} \\
&= \frac{GN}{16(\pi)^2} cos^2\theta\,\delta(0)p_{01}\left(\frac{(\omega_{1\nu}+m_{1\nu})(\omega_{1e}-m_e)-p_{01}^2}{\sqrt{\omega_{1e}(\omega_{1e}-m_e)}\sqrt{\omega_{1\nu}(\omega_{1\nu}+m_{1\nu})}(\omega_{1e}+\omega_{1\nu})}\right) \\
&\equiv \frac{GN}{16(\pi)^2} cos^2\theta\,\delta(0)Q(p_{01})
\end{aligned}
\tag{4.28}
$$

Here p_{01} is the value of $|\vec{p}_1|$ at which $\omega_{1e}+\omega_{1\nu}-\Delta m = 0$. And

$$
p_{01} = \sqrt{\frac{(m_e^2-m_{1\nu}^2-\Delta m^2)^2-4\Delta m^2 m_{1\nu}^2}{4\Delta m^2}} \tag{4.29}
$$

In equation (4.28) inside function $Q(p_{01})$ all the ω's are evaluated at three momentum amplitude $= p_{01}$. Now in equation(4.22), the second term will give a similar expression for polarization of positron and neutrino are '$-$' and '$+$' respecively.

$$
\frac{GN}{16(\pi)^2} sin^2\theta\,\delta(0)Q(p_{02}) \tag{4.30}
$$

p_{02} has the same functional form as p_{01}, but with all neutrino subscripts '1' replaced by '2'.

$$
p_{02} = \sqrt{\frac{(m_e^2-m_{2\nu}^2-\Delta m^2)^2-4\Delta m^2 m_{2\nu}^2}{4\Delta m^2}} \tag{4.31}
$$

And from equation (4.27) we can see that if polarizations of positron and neutrino are '+' and '−' respecively, then also both terms will have same value in equation (4.22) for amplitude. So, total probability of transition will be

$$\mathbf{P} = 2\left|\left(\frac{GN}{16(\pi)^2}cos^2\theta\delta(0)Q(p_{01}) + \frac{GN}{16(\pi)^2}sin^2\theta\delta(0)Q(p_{02})\right)\right|^2 \tag{4.32}$$

The decay rate (Γ) can be calculated by dividing l.h.s. by $\delta(0)$ from r.h.s.

$$\Gamma = \frac{g^2N^2}{2^7\pi^4}\delta(0)\left|\left(cos^2\theta Q(p_{01} + sin^2\theta Q(p_{02})\right|^2 \tag{4.33}$$

Proton decay and Neutrino Oscillations

It was claimed in [Tor15] that if neutrino oscillations are considered in the calculation of decay rate of an accelerated proton, then one will have disagreement in decay rates calculated in an inertial frame and in accelerated frame(taking Unruh effect into account). Although, their calculation wasn't reliable, since to calculate decay rate in an inertial frame they did not take final state as energy-momentum eigenstate. Hence, their final state didn't have an asymptotic limit as time $\to \infty$, which is required for them to use S-matrix method of quantum field theory.[V.S05] Instead one possible way to do the calculation is to calculate the decay rate in case of a certain neutrino mass eigenstate and the sum the decay rates for all possible final mass eigenstates. Now we'll show that in this decay rates would be same in both frames.

Amplitue of a process including neutrino mass eigenstate '*i*' as a final product in inertial frame using equations (4.1) and (3.23)

$$\mathcal{A}^i_{in} = |U_{ie}|^2\mathcal{A}_{in}(\nu_e \to \nu_i) \tag{4.34}$$

where $\nu_e \to \nu_i$ means replacement of ν_e by ν_i in equation (3.23). So then the decay rate of only this process will be

$$\Gamma^i_{in} = |U_{ie}|^4 \Gamma_{in}(\nu_e \to \nu_i) \tag{4.35}$$

Γ_{in} is defined in equation(3.37). Then decay rate due to transition to all of the mass eigenstates will be

$$\Gamma^{total}_{in} = \sum_i |U_{ie}|^4 \Gamma^i_{in}(\nu_e \to \nu_i) \tag{4.36}$$

We' get the same factor of $|U_{ie}|^4$ for each of the three reactions (3.40) in accelerated frame. And total decay rates calculated in both frames would match.

$$\Gamma^{total}_{in} = \Gamma^{total}_{acc} \tag{4.37}$$

Although, the question still persists, about this equality in case of final state having neutrino of definite flavor. In case of accelerated initial particle the $\delta^{(4)}$ functions in equation (4.9) need not be there. So we are left with the question about how to write final state. If we want to calculate the decay rate to a particular final momentum eigenstate, then we can take state in (4.9) as final state and then integrate over all possible Q values to get the total decay rate. However, Q in this case isn't the change in the four momentum of hadron, as it's in the case of inertial hadron (for example in section 4.0).

Appendices

A

Some Important Definitions

- **Cauchy Surface** - Cauchy surface is a surface on spacetime, which is intersected by all the causal curves on the spacetime only and necessarily once.[@Wik16a]

- **Globally Hyperbolic Spacetime** - A smooth Lorentzian spacetime, which has a Cauchy surface.[@Wik16c]

- **Killing Vector Field** - A coordinate transformation $x^{\mu} \to x'^{\mu}$ under which new metric $g'_{\mu\nu}(x) = g_{\mu\nu}(x)$ is called an *isometry*. An infinitesimal isometry can be written as $x^{\mu} \to x^{\mu} + \varepsilon X^{\mu}(x)$, with ε being infinitesimal parameter. Then X^{μ} is called a Killing vector field of the metric $g_{\mu\nu}(x)$. X^{μ} obeys following set of equations[Wei72]

$$X_{\mu,\nu} + X_{\nu,\mu} = 0 \tag{A.1}$$

- **Friedmann-Robertson-Walker(FRW) Spacetime** - FRW spacetime metric is an exact solution to Einstein's field equations.

$$g_{\mu\nu} dx^{\mu} dx^{\nu} = dt^2 - a(t)^2 (dx^2 + dy^2 + dz^2) \tag{A.2}$$

  $a(t)$ is known as scale factor. (x, y, z) are called comoving coordinates and observers with constant (x, y, z) coordinates are called comoving observers.

- **Conformal transformation of metric** - A conformal transformation of a metric $g_{\mu\nu}(x)$ produces another metric $g'_{\mu\nu}(x)$ such that

$$g'_{\mu\nu}(x) = \rho^2(x) g_{\mu\nu}(x) \tag{A.3}$$

B

Dirac Equation in a General Background Spacetime

Due to Einstein's equivalence principle, in a general curved spacetime one can locally erect flat Minkowski coordinates with local metric being η_{ab}. If the metric of space is $g_{\mu\nu}$, then Vierbein or tetrad fields $X^a_\nu(x)$ is defined by

$$g_{\mu\nu}(x) = X^a_\mu(x) X^b_\nu(x) \eta_{ab} \tag{B.1}$$

Vierbeins are arbitrary upto a local Lorentz transformation, which would leave local metric unchanged. Under local Lorentz transformation it transforms as contravariant vector

$$X^a_\mu(x) \rightarrow \Lambda^a_b X^b_\mu(x) \tag{B.2}$$

Where Λ^a_b is Lorentz transformation matrix. While under general coordinate transformation ($x^\mu \rightarrow x^{'\mu}$) it transforms as covariant vector

$$X^a_\mu \rightarrow \frac{\partial x^\nu}{\partial x^{'\nu}} X^a_\nu \tag{B.3}$$

One can use vierbeins to make a local tensor of a general tensor. eg- $V_a = X^\nu_a V_\nu$ Under general coordinate transformations V_a transforms as a set of four scalars. There is no spinor representation of diffeomorphism group(group of general coordinate transformations), like there is for Lorentz group $SO(3,1)$. Under local Lorentz transformation a spinor $\psi(y) \rightarrow (1 - \frac{i}{2}\theta_{ab}\Sigma^{ab})\psi(y)$, with y being local inertial coordinates. θ_{ab} are parameters of transformation and Σ^{ab} is generator of Lorentz transformation of spinors

$$\Sigma^{ab} = \frac{i[\gamma^a, \gamma^b]}{4} \tag{B.4}$$

To, formulate a theory of spinors in curved spacetime one needs to link together locally defined spinor fields at different spacetime points. This can be done by defining spinor covariant derivative. [Wei72]

$$\begin{aligned} \nabla_\mu &= \partial_\mu + \Delta_\mu \\ \text{where} \quad \Delta_\mu &= \frac{1}{2}\Sigma^{ab} X_a^\nu [X_{b\mu,\nu}] \end{aligned} \tag{B.5}$$

Spinor covariant derivative $\nabla_\mu \psi$ is a tensor under general coordinate transformations. Now we can write a scalar Lagrangian for ψ

$$\mathcal{L}(x) = \sqrt{-g}[\frac{1}{2}i\{\bar{\psi}(x)\gamma^a X_a^\mu \nabla_\nu \psi(x) - X_a^\mu \nabla_\nu \bar{\psi}(x)\gamma^a \psi(x)\} - m\bar{\psi}(x)\psi(x)] \tag{B.6}$$

And Euler-Lagrange equation of motion for ψ is

$$(iX_a^\nu \gamma^a \nabla_\nu)\psi = 0 \tag{B.7}$$

Often, one defines $X_a^\nu \gamma^a$ as *curved space Dirac gamma matrices* $\bar{\gamma}^\nu$. And

$$\{\bar{\gamma}^\mu, \bar{\gamma}^\nu\} = 2g^{\mu\nu} \tag{B.8}$$

For more details about vierbeins,reader can refer to [Dav82; Wei72].

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