Study of Cauchy's basic equation and Convex functions

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A dissertation submitted for the partial fulfilment of

BS-MS dual degree in Science

IISER Mohali April 22, 2016

I would like to dedicate this thesis to my parents without whom nothing would have been possible.

Declaration

The work presented in this dissertation has been carried out by me under the guidance of Prof. H. L Vasudeva at the Indian Institute of Science Education and Research Mohali.

This work has not been submitted in part or in full for a degree, a diploma, or a fellowship to any other university or institute. Whenever contributions of others are involved, every effort is made to indicate this clearly, with due acknowledgement of collaborative research and discussions. I would also like to point out that all the results presented in this work are collected from different sources and what I have done is presenting it in a more structured and comprehensive way. This thesis is a bonafide record of work done by me and all sources listed within have been detailed in the bibliography.

Manu J
(candidate)

In my capacity as the supervisor of the candidate's project work, I certify that the aforesaid statements by the candidate are true to the best of my knowledge.

Prof. H. L. Vasudeva
(supervisor)

Dated: April 22, 2016

Certificate of Examination

This is to certify that the dissertation titled **Study of Cauchy's basic equation and Convex functions**, submitted by **Mr. Manu J** (Registration Number: MS11033) for the partial fulfilment of the BS-MS dual degree programme of the Indian Institute of Science Education and Research, Mohali, has been examined by the thesis committee duly appointed by the institute. The committee finds the work done by the candidate satisfactory and recommends that the report be accepted.

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Dated: April 22, 2016

Acknowledgements

I would like to thank Prof. H. L. Vasudeva for his invaluable guidance and his patience for correcting my writeup time to time. I am also grateful to Dr. Lingaraj Sahu for the help and insight he has provided me with during the time of my thesis work.

I would also like to thank my parents, teachers and friends for their immense support during my work. I specially thank my friend Atul Singh Arora for all the technical help he had provided during the writing of this work. Finally I am extending my acknowledgement to thank the authors of all the books and research articles whose ideas and results I have presented in this work.

Abstract

This work consists of two chapters. The initial part includes the study of the solutions of Cauchy's basic equation which are equations of the form $f(x+y) = f(x) + f(y)$. We start by looking at the solution for this equation when the given function has real domain and range. Various regularity and algebraic conditions leading to the linearity of the solution function are discussed in detail starting from continuity and generalizing it to the condition where only the measurability of the function is needed. Concept of almost additive functions are introduced and the existence of a unique additive function which coincides almost everywhere with almost additive function is proved. Stability of the solution of a Cauchy's equation is discussed in detail with the cases including $|f(x+y) - f(x) - f(y)|$ bounded and unbounded. Solution of the additive functions when the domain and range is extended to complex plane is also discussed. Finally the most general solution of Cauchy's basic equation is constructed using the existence of a Hamel basis for $\mathbb{R}$ over $\mathbb{Q}$ and the existence of a discontinuous solution for Cauchy's equation is shown. Then second chapter covers the study of convex functions. Various properties of convex functions are discussed. Concept of a weaker form of convexity namely mid convexity of function is introduced and sufficient conditions satisfied by the mid convex functions to be convex are discussed starting from continuity and generalizing it to the condition where the function only needs to be measurable. Finally, a more powerful form of convexity which is log convexity is introduced and the properties of such functions are discussed.

Basic knowledge of Measure theory, Functional Analysis and Fourier Analysis is assumed for understanding the topics presented in this work. Any of the non-standard results which are being used are carefully stated and proved.

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Chapter 1

Cauchy's basic equation.

Definition 1. Let $f : \mathbb{R} \to \mathbb{R}$ be real valued function satisfying:

$$f(x+y) = f(x) + f(y) \ \ , \forall x, y \in \mathbb{R}. \tag{1.1}$$

Equation (1.1) is called Cauchy's basic equation and function f satisfying it is called an additive function.

This equation was first treated by *A. M. Legendre* and *C. F. Gauss*. *A. L. Cauchy* found the general continuous solution of the same . We begin with a generalization of the Cauchy's theorem.

1.1 Solution of Cauchy's basic equation

1.1.1 Conditions implying continuity of the solution.

Theorem 2. *Let $f : \mathbb{R} \to \mathbb{R}$ be an additive function such that f is bounded in one side on a non-empty bounded open subset U of $\mathbb{R}$, then f has the general solution*

$$f(x) = cx \ \ \textit{where } c = f(1) \ . \tag{1.2}$$

Proof. We first assume that $f : \mathbb{R} \to \mathbb{R}$ is an additive function bounded above in a non-empty bounded open set U ; similar arguments will work for the proof in the case f is bounded below.

$$f(x) \leq M \quad ,\forall x \in U,\ M \in \mathbb{R}. \tag{1.3}$$

By induction, it follows from (1.1) that

$$f(x_1 + \cdots + x_n) = f(x_1) + \cdots + f(x_n).$$

Substituting $x_i = x$ in the above equation gives

$$f(nx) = nf(x)\ \forall n \in \mathbb{N}$$

Now

$$f(0) = f(0+0) = 2f(0)$$

$$\implies f(0) = 0.$$

Also

$$f(-x) = f(0-x) = f(0) - f(x) = -f(x) \quad ,\forall x \in \mathbb{R}\ . \tag{1.4}$$

Define the function $g : \mathbb{R} \to \mathbb{R}$ by

$$g(x) = f(x) - xf(1).$$

If we prove $g \equiv 0$, then the proof will be complete. Observe that g is an additive function and since f is bounded above in U , g is also bounded above in U. Indeed

$$\begin{aligned} g(x+y) &= f(x+y) - (x+y)f(1) \\ &= f(x) - xf(1) + f(y) - yf(1) = g(x) + g(y); \end{aligned}$$

$$g(x) = f(x) - xf(1) \leq M - xf(1) \leq M' \quad ,\forall x \in U,\ M' \in \mathbb{R} \tag{1.5}$$

using equation (1.3) and the fact that U is bounded.

Note that $g(x+1) = g(x) \quad ,\forall x \in \mathbb{R}$ which shows g has a period 1, so if U contains an interval (a,b) of length at least 1, then g will be bounded on $\mathbb{R}$. To the contrary, assume that U doesn't contain such an interval. Then since U is a non-empty open set in $\mathbb{R}$, it contains a non-empty interval (a,b) of length $\frac{1}{n}$ for some $n \in \mathbb{N}$. Since

$$g(1) = g(\frac{1}{n} + \cdots + \frac{1}{n}) = ng(\frac{1}{n}) = 0,$$

it follows that

$$g(\frac{1}{n}) = 0.$$

So

$$g(x + \frac{1}{n}) = g(x) \quad ,\forall x \in \mathbb{R}$$

which shows g has a period $\frac{1}{n}$ and since g is bounded in $(a,b) \subset U$, it follows that g is bounded all over $\mathbb{R}$. From equation (1.4) and equation (1.5) , we get

$$-B \leq g(x) \leq B \ ,\forall x \in \mathbb{R}.$$

If $B \leq 0$, then the above equation shows

$$B = 0$$

which implies $g(x) \equiv 0$. Now, assume that $B > 0$. Since $g(nx) = ng(x) \quad ,\forall n \in \mathbb{N}$,

$$-B \leq g(nx) = ng(x) \leq B \ ,\forall x \in \mathbb{R}$$

$$-\frac{B}{n} \leq g(x) \leq \frac{B}{n} \ ,\forall x \in \mathbb{R}.$$

Letting $n \to \infty$,we obtain $g(x) \equiv 0$, which completes the proof. □

The above theorem leads us to think that slight regularity assumption on the function f satisfying basic Cauchy equation (1.1) implies in fact a strong regularity which in turn yields $f(x) = cx$. Next corollary and theorem support the above assertion.

Corollary 3. *If $f : \mathbb{R} \to \mathbb{R}$ is an additive function that is continuous at a point then f has the general solution*

$$f(x) = cx \quad \textit{where } c = f(1)\,.$$

Proof. Let f be continuous at $x_0 \in \mathbb{R}$. Then for any $\varepsilon > 0$, $\exists \delta > 0$ such that

$$|f(t) - f(x_0)| < \varepsilon \quad \text{whenever } |t - x_0| < \delta.$$

This implies for $t \in (x_0 - \delta, x_0 + \delta)$

$$f(x_0) - \varepsilon < f(t) < f(x_0) + \varepsilon.$$

That is , f is bounded in the non-empty bounded interval $(x_0 - \delta, x_0 + \delta)$. The proof is complete on using the above theorem. □

Proposition 4. *If $f : \mathbb{R} \to \mathbb{R}$ is an additive function that is continuous at a point then f is continuous everywhere.*

Proof. We will first prove that if $f : \mathbb{R} \to \mathbb{R}$ is an additive function continuous at $x_0 \in \mathbb{R}$, then f is continuous at 0. Let $\{y_n\}$ be a sequence in $\mathbb{R}$ converging to 0. Now

$$f(x_0 + y_n) = f(x_0) + f(y_n),$$

letting $n \to \infty$

$$\lim_{n\to\infty} f(x_0 + y_n) = f(x_0) + \lim_{n\to\infty} f(y_n).$$

Since $x_0 + y_n$ converges to x_0, continuity of f at x_0 implies $\lim_{n\to\infty} f(x_0 + y_n) = f(x_0)$. This shows that

$$f(x_0) = f(x_0) + \lim_{n\to\infty} f(y_n)$$

which yields

$$\lim_{n\to\infty} f(y_n) = 0.$$

From the additive property of f we know, $f(0) = 0$; so

$$\lim_{n\to\infty} f(y_n) = f(0)$$

which proves that f is continuous at 0. Let $x \in \mathbb{R}$ be arbitrary and $\{x_n\}$ a sequence in $\mathbb{R}$ converging to x , then $x - x_n \to 0$ as $n \to \infty$. Since f is continuous at 0, it follows that $f(x - x_n) \to 0$ as $n \to \infty$. Also

$$f(x_n) = f(x_n - x + x) = f(x_n - x) + f(x).$$

So

$$\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} f(x_n - x) + f(x)$$

which implies

$$\lim_{n\to\infty} f(x_n) = f(x).$$

This proves the continuity of the function f at x. □

Now we will give some definitions and state some results without proofs which will be used in the subsequent sections.

Definition 5. (Convolution product): Let $f \in L^1(\mathbb{R})$, $g \in L^1(\mathbb{R})$. Then the convolution product $f * g$ is defined by

$$f * g(x) = \int_{-\infty}^{\infty} f(x-y)g(y)dy < \infty, x \in \mathbb{R}.$$

Proposition 6. *Let $f \in L^1(\mathbb{R})$, $g \in L^1(\mathbb{R})$. Then*

$$\int_{-\infty}^{\infty} |f(x-y)g(y)|dy < \infty$$

for almost all x and for these x, define

$$h(x) = (f * g)(x) = \int_{-\infty}^{\infty} f(x-y)g(y)dy.$$

Then $h \in L^1(\mathbb{R})$ *and*

$$\| h \|_1 \leq \| f \|_1 \| g \|_1 .$$

Definition 7. Let $f \in L^1(\mathbb{R})$, the Fourier transform $\hat{f}$ of f at any real t is defined by

$$\hat{f}(t) = \int_{-\infty}^{\infty} f(x) e^{-2\pi ixt} dx.$$

Since $f \in L^1(\mathbb{R})$ the above integral exists for any $t \in \mathbb{R}$. Indeed

$$\int_{-\infty}^{\infty} |f(x)e^{-2\pi ixt}| dx \leq \int_{-\infty}^{\infty} |f(x)| dx.$$

Proposition 8. *The Fourier transform* $\hat{f} : L^1(\mathbb{R}) \to C_0(\mathbb{R})$ *defined by the mapping* $f \longmapsto \hat{f}$ *is injective.*

Proposition 9. ***Lusin's Theorem:*** *Let f be a complex measurable function on* $\mathbb{R}$ *and* $A \subseteq \mathbb{R}$ *with* $\mu(A) < \infty$, $f(x) = 0$ *if* $x \notin A$. *Let* $\varepsilon > 0$ *be arbitrary. Then there exists a* $g \in C_c(\mathbb{R})$ *such that*

$$\mu(\{x : f(x) \neq g(x)\}) < \varepsilon$$

and

$$\sup_{x \in \mathbb{R}} |g(x)| \leq \sup_{x \in \mathbb{R}} |f(x)|.$$

The following lemma which was proved by *Hugo Steinhaus* is crucial for the proof of the subsequent theorem.

Lemma 10. *Let* $E \subset \mathbb{R}$ *be a set of positive Lebesgue measure. Then* $F = E + E = \{x + y : x, y \in E\}$ *has non-empty interior.*

Proof. Without loss of generality, assume that E is of finite Lebesgue measure. Then the convolution product of the characteristic function χ_E of E with itself is

$$h(x) = \chi_E(x) * \chi_E(x) = \int_{-\infty}^{\infty} \chi_E(t) \chi_E(x - t) dt$$

$$= \int_E \chi_E(x-t)dt.$$

It follows from the proposition 6 that $h \in L^1(\mathbb{R})$. Moreover $supp(h) \subseteq F$. Indeed if $x \notin F$ and since x can be written as $(x-t)+t$ with $t \in E$, it follows that $x-t \notin E$ which in turn implies $x \notin supp(h)$.

We next show that h does not vanish almost everywhere. For this consider the Fourier transform of h ,

$$\begin{aligned}
\hat{h}(y) &= \int_{\mathbb{R}} e^{-2\pi iyx} h(x)dx \\
&= \int_{\mathbb{R}} e^{-2\pi iy(x-t+t)} \left[\int_{\mathbb{R}} \chi_E(t)\chi_E(x-t)dt \right] dx \\
&= \int_{\mathbb{R}} \int_{\mathbb{R}} e^{-2\pi iy(x-t)} \chi_E(x-t) e^{-2\pi iyt} \chi_E(t)dtdx \\
&= \int_{\mathbb{R}} \chi_E(t) e^{-2\pi iyt} dt \left[\int_{\mathbb{R}} \chi_E(x-t) e^{-2\pi iy(x-t)} dx \right] \\
&= \left(\int_{\mathbb{R}} \chi_E(t) e^{-2\pi iyt} dt \right) \left(\int_{\mathbb{R}} \chi_E(x) e^{-2\pi iyx} dx \right) \\
&= \left(\hat{\chi_E}(y) \right)^2 \quad \forall y \in \mathbb{R}.
\end{aligned}$$

Since E is a set of positive Lebesgue measure, χ_E does not vanish $a.e.$ So from proposition 8 it can be concluded that $\hat{\chi_E}$ is not the zero function. This in turn implies $\hat{h}$ is not the zero function.

Again using proposition 8, it follows that h does not vanish $a.e.$ Next we need to show that h is continuous. Indeed

$$h(x+\alpha) - h(x) = \int_E (\chi_E(x+\alpha-t) - \chi_E(x-t))\, dt$$

$$= \int_{x-E} (\chi_E(u+\alpha) - \chi_E(u))\, du.$$

χ_E is a complex measurable function such that $\mu(E) < \infty$ and $\chi_E(x) = 0$ for $x \notin E$. Therefore by Lusin's theorem , for any $\varepsilon > 0$ there exists a $g \in C_c(\mathbb{R})$ such that

$$\mu(\{x : \chi_E(x) \neq g(x)\}) < \varepsilon$$

and

$$\sup_{x\in\mathbb{R}}|g(x)| \leq \sup_{x\in\mathbb{R}}|\chi_E(x)|.$$

So

$$\begin{aligned}
|h(x+\alpha)-h(x)| &\leq |\int_{\mathbb{R}}(\chi_E(u+\alpha)-g(u+\alpha)-(\chi_E(u)-g(u)))+(g(u+\alpha)-g(u))du| \\
&\leq \int_{\mathbb{R}}(|\chi_E(u+\alpha)-g(u+\alpha)|+|\chi_E(u)-g(u)|+|g(u+\alpha)-g(u)|)\,du \\
&\leq \int_{\{x:\chi_E(x)\neq g(x)\}}(|\chi_E(u+\alpha)-g(u+\alpha)|+|\chi_E(u)-g(u)|du) \\
&\quad +\int_{\mathbb{R}}|g(u+\alpha)-g(u)|du \\
&\leq 2\varepsilon+\int_{\mathbb{R}}|g(u+\alpha)-g(u)|du.
\end{aligned}$$

Since g is uniformly continuous on its bounded support, we can find $\eta > 0$ such that $|\alpha| < \eta$ will imply $|g(u+\alpha)-g(u)| < \varepsilon$. So

$$\begin{aligned}
\int_{\mathbb{R}}|g(u+\alpha)-g(u)|du &= \int_{\{x:\chi_E(x)\neq g(x)\}}|g(u+\alpha)-g(u)|du \\
&\quad +\int_{\{x:\chi_E(x)=g(x)\}}|g(u+\alpha)-g(u)|du \\
&\leq \varepsilon^2+\varepsilon\mu(E)
\end{aligned}$$

which implies

$$|h(x+\alpha)-h(x)| < 2\varepsilon+\varepsilon^2+\varepsilon\mu(E).$$

Since ε was arbitrary, we have shown that h is continuous, and since $h(x)\neq 0$ for some $x\in E$, there exists some neighborhood around x in which h is non zero which is contained in $supp(h)\subset F$. This proves the existence of a non-empty open set inside F. □

Next we prove the following theorem.

Theorem 11. *Let $f:\mathbb{R}\to\mathbb{R}$ be an additive function that is bounded above by a Lebesgue measurable function $g:\mathbb{R}\to\mathbb{R}$ on a subset E of strictly positive Lebesgue measure. Then f has the general solution,*

$$f(x) = cx \quad \textit{where } c = f(1)$$

Proof. $g : \mathbb{R} \to \mathbb{R}$ is Lebesgue measurable and $\mu(E) > 0$. So for each integer n , the subset $E_n = \{x \in E : |g(x)| \leq n\}$ of E is measurable and

$$E_1 \subseteq E_2 \subseteq \ldots E_n \subseteq \ldots E.$$

Since $\{E_n\}$ is a sequence of monotonically increasing measurable sets converging to E ,

$$\lim_{n\to\infty} \mu(E_n) = \mu(\lim_{n\to\infty} E_n) = \mu(E) > 0 \,,$$

where the first equality follows from monotone convergence theorem. Therefore there exists some integer N such that

$$\mu(E_N) > 0.$$

This implies that g is bounded on a set of positive Lebesgue measure and so is f. So WLOG, we may assume that $\exists$ a constant $A \in \mathbb{R}$ such that

$$f(x) \leq M \quad \forall x \in E_N.$$

Finally let $z = x + y$ with $x, y \in E_N$, then

$$f(z) = f(x) + f(y) \leq 2M$$

which means there exists a non-empty open set inside $E_N + E_N$ (from Lemma 10) in which f is bounded above. So the proof is completed once *theorem* 1 is applied. □

Corollary 12. *A locally Lebesgue integrable function $f : \mathbb{R} \to \mathbb{R}$ satisfying Cauchy's equation has the general solution*

$$f(x) = cx \quad \textit{where } c = f(1).$$

The above corollary is a direct consequence of the theorem 11, we may also give an alternate independent proof for the same.

Proof. Given that f is locally integrable, so it makes sense to consider the following integral

$$\int_0^1 f(x+y)dy = \int_0^1 f(x)dy + \int_0^1 f(y)dy.$$

Using the substitution $x+y=t$, we will get

$$\int_x^{x+1} f(t)dt = f(x)+\alpha \text{ where } \alpha = \int_0^1 f(y)dy.$$

Differentiating both sides of the above equation and using *Fundamental theorem of calculus*, we obtain

$$f'(x) = f(x+1) - f(x)$$

$$= f(x-x+1) = f(1).$$

So

$$f(x) = cx+k \quad \text{where } c = f(1) \text{and } k \text{ is a constant .}$$

Since f is additive, $f(0)=0$ from which we get $k=0$. Hence we have proved

$$f(x) = cx \quad \text{where } c = f(1) \ .$$

□

The following generalization is due to *W.Sierpinski[refer Sur l'equation fonctionnelle f(x+y)=f(x)+f(y),Fundam. Mat., 116-120(1996)].*

Theorem 13. *Let $f:\mathbb{R}\to\mathbb{R}$ be an additive function. If f is Lebesgue measurable, then f is continuous.*

Proof. f is given to be both additive and Lebesgue measurable. Let $x_0\in\mathbb{R}$ be arbitrary and $\varepsilon>0$ and $(a.b)$ an arbitrary interval. By a version of ***Lusin's theorem****[refer G. Folland. Real Analysis: Modern Techniques and Their Applications, 2nd edition, 2nd Chapter]* , for every measurable function g(in this case f) and for every $\sigma>0$ there exists a continuous function

$F : \mathbb{R} \to \mathbb{R}$ such that

$$\mu\left(E = \{x \in (a,b) : f(x) \neq F(x)\}\right) < \sigma \tag{1.6}$$

where μ is the Lebesgue measure on $\mathbb{R}$.

Take σ to be $\frac{b-a}{3}$. Since F is continuous all over the real line, for every $\varepsilon > 0$ and any $x \in (a,b)$ there exists a $\delta(< \sigma)$ such that $|F(x+h) - F(x)| < \varepsilon$ whenever $|h| < \delta$. Since $f(x) = F(x)$ for $x \in (a,b) \backslash E$, $f(x+h) = F(x+h)$ is true for all $x \in E_1$, where $\mu(E_1) < \sigma + |h| < \sigma + \delta$ (using (1.6)). Therefore

$$\begin{aligned} \mu\left(\{x \in (a,b) : f(x) \neq F(x) \text{ and } f(x+h) \neq F(x+h)\}\right) &\leq \mu(E \cup E_1) < 2\sigma + \delta \\ &< 3\sigma < b - a. \end{aligned}$$

Hence there is a point $x \in (a,b)$ dependent on h for which $f(x) = F(x)$, $f(x+h) = F(x+h)$, $|F(x+h) - F(x)| < \varepsilon$ and therefore $|f(x+h) - f(x)| < \varepsilon$ is valid. Since $f(x+h) - f(x) = f(x_0+h) - f(x_0)$, for any $x_0 \in \mathbb{R}$

$$|f(x_0+h) - f(x_0)| < \varepsilon.$$

Which proves the continuity of f at x_0. Since $x_0 \in \mathbb{R}$ is arbitrary, it follows that f is continuous on $\mathbb{R}$. $\square$

1.1.2 Algebraic conditions implying linearity of additive functions

In what follows, we make additional algebraic assumption on additive function f and obtain the solution of the basis Cauchy's equation. In this direction, the following two results are interesting and are quite easy to prove.

Theorem 14. *Suppose $f : \mathbb{R} \to \mathbb{R}$ is an additive function satisfying the following algebraic condition*

$$f\left(\frac{1}{x}\right) = \frac{1}{x^2} f(x) \tag{1.7}$$

then f is continous and has the general solution

$$f(x) = cx \quad where\ c = f(1)\ .$$

Proof. Let $f : \mathbb{R} \to \mathbb{R}$ be an additive function satisfying the given algebraic condition 1.7. For $x = 0, \pm 1$, $f(x) = xf(1)$. Indeed,

$$\begin{aligned} f(0) = 0 &= 0 \times f(1) \text{ (additive property)} \\ f(1) &= 1 \times f(1) \\ f(-1) &= 1 \times f(-1) = -1 \times f(1) \text{(additive functions are odd functions)} \end{aligned}$$

Now let $x \in \mathbb{R} \backslash \{0, \pm 1\}$, then

$$\begin{aligned} f(x - \frac{1}{x}) &= f(\frac{x^2 - 1}{x}) \\ &= (\frac{x^2 - 1}{x})^2 f(\frac{x}{x^2 - 1}) \text{ (using the algebraic condition) .} \end{aligned}$$

Now consider the equation

$$f(\frac{2x}{x^2 - 1}) = f(\frac{x}{x^2 - 1} + \frac{x}{x^2 - 1}) = 2f(\frac{x}{x^2 - 1}).$$

So, we get

$$\begin{aligned}
f(x-\frac{1}{x}) &= (\frac{x^2-1}{x})^2 f(\frac{x}{x^2-1}) \\
&= \frac{(x^2-1)^2}{2x^2} f(\frac{2x}{x^2-1}) \\
&= \frac{(x^2-1)^2}{2x^2} f\left(\frac{1}{x-1}+\frac{1}{x+1}\right) \\
&= \frac{(x^2-1)^2}{2x^2}\left[\frac{1}{(x-1)^2} f(x-1)+\frac{1}{(x+1)^2} f(x+1)\right]
\end{aligned}$$

(using the algebraic condition (1.7))

$$= \frac{(x^2-1)^2}{2x^2}\left[\frac{1}{(x-1)^2}(f(x)-f(1))+\frac{1}{(x+1)^2}(f(x)+f(1))\right]$$

(using additive property of f)

$$= \frac{(x+1)^2}{2x^2}(f(x)-f(1))+\frac{(x-1)^2}{2x^2}(f(x)+f(1)).$$

So,

$$f(x-\frac{1}{x}) = \frac{x^2+1}{x^2} f(x) - \frac{2}{x} f(1). \tag{1.8}$$

Also from the additive property and equation (1.7) it follows that

$$f(x-\frac{1}{x}) = f(x) - f(\frac{1}{x}) = f(x) - \frac{1}{x^2} f(x). \tag{1.9}$$

Consequently from equations (1.8) and (1.9) we get

$$\begin{aligned}
f(x)-\frac{1}{x^2}f(x) &= \frac{x^2+1}{x^2}f(x)-\frac{2}{x}f(1) \\
\frac{x^2-1}{x^2}f(x) &= \frac{x^2+1}{x^2}f(x)-\frac{2}{x}f(1) \\
\frac{x^2+1}{x^2}f(x)-\frac{2}{x^2}f(x) &= \frac{x^2+1}{x^2}f(x)-\frac{2}{x}f(1)
\end{aligned}$$

which implies

$$\begin{aligned}
\frac{2}{x^2}\quad f(x) &= \frac{2}{x}f(1) \\
f(x) &= xf(1).
\end{aligned}$$

This completes the proof. □

Theorem 15. *All ring homomorphisms from $\mathbb{R} \to \mathbb{R}$ are trivial, that is either identity or the zero map.*

Proof. Let $f : \mathbb{R} \to \mathbb{R}$ be a ring homomorphism; so $\forall x, y \in \mathbb{R}$,

$$f(x+y) = f(x) + f(y),$$

$$f(xy) = f(x)f(y). \tag{1.10}$$

That is, f is an additive function from $\mathbb{R} \to \mathbb{R}$ satisfying an extra algebraic condition, namely, equation(1.10). Putting $x = y$ in equation(1.10) gives

$$f(x^2) = f(x)^2.$$

That is if $x \geq 0$, then

$$f(x) = f(\sqrt{x})^2 \geq 0.$$

This implies that f is an additive function function which is bounded below in the interval $(0, \infty)$ and so from Theorem 2, it follows that f has the general solution of the form

$$f(x) = cx \quad \text{where } c = f(1) \ .$$

Now from equation equation(1.10), it follows that

$$f(1) = f(1 \times 1) = f(1)^2$$

which implies that

$$f(1) = 0 \text{ or } 1.$$

This completes the proof. □

Corollary 16. *Let $f : \mathbb{R} \to \mathbb{R}$ be an additive function satisfying the following algebraic condition*

$$f(x^2) = f(x)^2, \forall x \in \mathbb{R}$$

then f has the general solution of the form

$$f(x) = cx \quad \text{where } c = f(1).$$

Proof. Proof is immediate from the proof of Theorem 15 □

1.2 Complex valued additive functions

In this this section, we will consider additive functions with range as the field of complex numbers. Most of the theorems we have proved in the previous section can be generalized to the case of additive complex valued functions on $\mathbb{R}$. We give the proofs of a few.

Proposition 17. *Let $f:\mathbb{R}\to\mathbb{C}$ be a complex valued function satisfying the basic Cauchy's equation, that is*

$$f(x+y)=f(x)+f(y)\ ,\forall x,y\in\mathbb{R} \tag{1.11}$$

If f is continuous at a point, then f has the general solution of the form

$$f(x)=cx \ \ \textit{where } c=f(1).$$

Proof. Let $f(x)=f_1(x)+\imath f_2(x)$ where f_1 and f_2 are real valued functions on $\mathbb{R}$. Given that

$$f(x+y)=f(x)+f(y)\ ,\forall x,y\in\mathbb{R}.$$

if follows that

$$f_1(x+y)+\imath f_2(x+y) \quad = \quad f_1(x)+\imath f_2(x)+f_1(y)+\imath f_2(y).$$

On equating the real and imaginary part, we get

$$f_1(x+y)=f_1(x)+f_1(y),$$

$$f_2(x+y)=f_2(x)+f_2(y)$$

that is , both f_1 and f_2 are additive real valued functions on $\mathbb{R}$. Since f is continuous at a point, so are its real and imaginary parts. And so proposition 4 applies and we get

$$f_1(x)=c_1x \ \ \text{where } c_1=f_1(1),$$

$$f_2(x) = c_2x \quad \text{where } c_2 = f_2(1).$$

Consequently

$$f(x) = c_1x + \imath c_2, \quad \forall x \in \mathbb{R}$$

which can be written as

$$f(x) = cx \quad \forall x \in \mathbb{R}$$

where $c = c_1 + \imath c_2 x$. □

In the following theorem, we will generalize theorem 2 to the complex valued case.

Theorem 18. *Let $f : \mathbb{R} \to \mathbb{C}$ be a complex valued function satisfying the basic Cauchy's equation, that is*

$$f(x+y) = f(x) + f(y) \ , \forall x, y \in \mathbb{R}. \tag{1.12}$$

If f is bounded in a non-empty open interval of $\mathbb{R}$, then f has the general solution

$$f(x) = cx \quad \textit{where } c = f(1).$$

Proof. Let $f(x) = f_1(x) + \imath f_2(x)$ where f_1 and f_2 are real valued functions on $\mathbb{R}$. From the proof of the previous proposition, it can be concluded that both f_1 and f_2 are additive functions on $\mathbb{R}$. Once we prove that both f_1 and f_2 are bounded in some non-empty open interval of $\mathbb{R}$, theorem 2 can be applied. Let U be the non-empty open interval in which f is bounded. That is

$$\| f(x) \| < M, \text{ for some } M > 0 \text{ and } \forall x \in U.$$

That is

$$\sqrt{(f_1(x))^2 + (f_2(x))^2} \quad < \quad M,$$

which implies

$$|f_1(x)| < M \ , \forall x \in U$$

and

$$|f_2(x)| < M \,, \forall x \in U.$$

So both $f_1$and f_2 are additive real valued functions which are bounded in a non-empty open interval of $\mathbb{R}$ and it follows using theorem 2 that

$$f_1(x) = c_1 x \quad \text{where } c_1 = f_1(1)$$

$$f_2(x) = c_2 x \quad \text{where } c_2 = f_2(1).$$

Consequently, f can be written as

$$f(x) = c_1 x + \imath c_2 x, \quad \forall x \in \mathbb{R}$$

which, in turn ,can be expressed in the form

$$f(x) = cx \quad \forall x \in \mathbb{R}$$

where $c = c_1 + \imath c_2 x$. □

Now we will look at solution of the additive function with Complex domain, that is functions $f : \mathbb{C} \to \mathbb{C}$ satisfying

$$f(z+w) = f(z) + f(w), \ \forall z, w \in \mathbb{C}.$$

We will state a lemma which is crucial in the proof of the subsequent theorem.

Lemma 19. *If $f : \mathbb{R}^2 \to \mathbb{R}$ is an additive function, that is*

$$f(x+y) = f(x) + f(y), \ \ \forall x, y \in \mathbb{R}^2$$

then there exist additive functions $f_1 : \mathbb{R} \to \mathbb{R}$ and $f_2 : \mathbb{R} \to \mathbb{R}$ such that for any $x = (x_1, x_2) \in \mathbb{R}^2$

$$f(x) = f_1(x_1) + f_2(x_2).$$

Proof. Let $x = (x_1, x_2) \in \mathbb{R}^2$ and $y = (y_1, y_2) \in \mathbb{R}^2$. The equation

$$f(x+y) = f(x) + f(y)$$

implies

$$f((x_1+y_1),(x_2+y_2)) = f(x_1,x_2) + f(y_1,y_2). \tag{1.13}$$

Define $f_1, f_2 : \mathbb{R} \to \mathbb{R}$ as

$$f_1(x) = f(x,0),\ x \in \mathbb{R}$$

and

$$f_2(x) = f(0,x),\ x \in \mathbb{R}.$$

We claim that both f_1 and f_2 are additive functions on $\mathbb{R}$. Indeed

$$f_1(x_1+x_2) = f(x_1+x_2,0) = f(x_1,0) + f(x_2,0) = f_1(x_1) + f_1(x_2),$$

where the second equality follows from equation (1.13). Similarly,

$$f_2(x_1+x_2) = f(0,x_1+x_2) = f(0,x_1) + f(0,x_2) = f_2(x_1) + f_2(x_2).$$

Therefore

$$f(x) = f(x_1,x_2) = f(x_1,0) + f(0,x_2) = f_1(x_1) + f_2(x_2).$$

This completes the proof. □

Theorem 20. *If $f : \mathbb{C} \to \mathbb{C}$ is a continuous additive function then there exists complex constants c_1 and c_2 such that for any $z \in \mathbb{C}$*

$$f(z) = c_1 z + c_2 \bar{z}.$$

Proof. Let $f := f_1 + \imath f_2$, where $f_1 : \mathbb{C} \to \mathbb{R}$ and $f_2 : \mathbb{C} \to \mathbb{R}$. So

$$f(z+w) = f_1(z+w) + \imath f_2(z+w). \tag{1.14}$$

Since f is an additive function,

$$f(z+w) = f(z) + f(w) = f_1(z) + \imath f_2(z) + f_1(w) + \imath f_2(w) = (f_1(z) + f_1(w)) + \imath\,(f_2(z) + f_2(w));$$

equating right hand sides of the above equation and (1.14) , it follows

$$f_k(z+w) = f_k(z) + f_k(w), \text{ for } k = 1,2.$$

Consequently, both f_1 and $f_2$can be seen as additive functions from $\mathbb{R}^2 \to \mathbb{R}$. Therefore by lemma(19), there exists additive functions $f_{1k} : \mathbb{R} \to \mathbb{R}$ and $f_{2k} : \mathbb{R} \to \mathbb{R}$ for $k = 1,2$ such that

$$f_k(z) = f_{k1}(Re(z)) + f_{k2}(Im(z)) \quad \text{for } k = 1,2.$$

Consequently,

$$f(z) = f_{11}(Re(z)) + f_{12}(Im(z)) + \imath f_{21}(Re(z)) + \imath f_{22}(Im(z)).$$

Since f is a complex continuous function, $f_1$and f_2 being the real and imaginary component of f respectively are continuous. Since $f_k : \mathbb{R}^2 \to \mathbb{R}$ is continuous , the coordinate functions f_{k1} and f_{k2} are continuous for $k = 1,2$.

Therefore $f_{kj} : \mathbb{R} \to \mathbb{R}$ is continuous and additive for $k, j = 1,2$. Hence there exists real constants c_{kj} for $k, j = 1,2$ such that

$$f_{kj}(x) = c_{kj}x, \ \forall \text{x} \in \mathbb{R}.$$

Therefore

$$f(z) = c_{11}Re(z) + c_{12}Im(z) + \imath c_{21}Re(z) + \imath c_{22}Im(z)$$

$$= (c_{11} + \imath c_{21})Re(z) + (c_{12} + \imath c_{22})Im(z);$$

substitute $c_{11} + \imath c_{21} = a$ and $c_{12} + \imath c_{22} = b$ in the above equation to yield

$$f(z) = aRe(z) + bIm(z) = aRe(z) - \imath(\imath b)Im(z)$$

$$= \frac{a+\imath b}{2}Re(z) + \frac{a-\imath b}{2}Re(z) - \imath\frac{a+\imath b}{2}Im(z) + \imath\frac{a-\imath b}{2}Im(z)$$

$$= \frac{a-\imath b}{2}Re(z) + \imath\frac{a-\imath b}{2}Im(z) + \frac{a+\imath b}{2}Re(z) - \imath\frac{a+\imath b}{2}Im(z)$$

$$= \frac{a-\imath b}{2}(Re(z) + \imath Im(z)) + \frac{a+\imath b}{2}(Re(z) - \imath Im(z)).$$

This can be written as $f(z) = c_1 z + c_2 \bar{z}$, where $c_1 = \frac{a-\imath b}{2}$ and $c_2 = \frac{a+\imath b}{2}$. This completes the proof. □

Theorem 21. *Let $f : \mathbb{C} \to \mathbb{C}$ be an additive function. Then f is analytic if and only if there exists a complex constant c such that*

$$f(z) = cz;$$

that is , f is linear.

Proof. Proof of one direction of the theorem is obvious. We will prove the only if part.

Assume that $f : \mathbb{C} \to \mathbb{C}$ is analytic and additive. Since f is analytic, it is differentiable. Now differentiating

$$f(z_1 + z_2) = f(z_1) + f(z_2) \tag{1.15}$$

with respect to z_2, we get

$$f'(z_1 + z_2) = f'(z_2)$$

for all $z_1, z_2 \in \mathbb{C}$. Therefore letting $z_1 = z$ and $z_2 = 0$, we get

$$f'(z) = c,$$

where $c = f'(0)$ is a complex constant. The above equation implies that

$$f(z) = cz + b, \tag{1.16}$$

where b is a complex constant. Putting $z_1 = z_2 = 0$ in equation (1.15) gives

$$f(0+0) = 2f(0)$$

which implies

$$f(0) = 0. \tag{1.17}$$

From equations (1.16) and (1.17) it follows that $b = 0$. This completes the proof. □

1.3 Stability of Cauchy's equation

In this section, we are looking at functions which may not be exact additive functions but are very close to being additive. We will state and prove some conditions in which these functions will have the same family of solutions as additive functions.

The problem of stability of an equation arises when we actually want to compute the value of functions in real life using numerical techniques. Since there are no actual real number representation on a computer, the numerical solution will be in almost all the cases an approximation of actual one. So stability analysis is essential to make sure that the numerical solution is not far off from the exact one.

The proof of the following theorem was first presented by Hyers, Isac, and Rassias's [refer D.H. Hyers, G. Isac, and Th.M. Rassias, Stability of Functional Equations in Several Variables, Birkh¨auser, Boston (1998)]

Theorem 22. *Let X and Y be Banach spaces and $f : X \to Y$ be such that*

$$\| f(x+y) - f(x) - f(y) \| \leq \delta \tag{1.18}$$

for some $\delta > 0$ and $\forall x, y \in X$. Define

$$A(x) = \lim_{n\to\infty} 2^{-n} f(2^n x).$$

Then the above limit exists for each $x \in X$ and $A : X \to Y$ is the unique additive function such that

$$\| f(x) - A(x) \| \leq \delta \text{ for any } x \in X. \tag{1.19}$$

Moreover, if $f(tx) : \mathbb{R} \to Y$ is continuous in t for each fixed $x \in X$, then A is linear. Also continuity of f at a point in X implies that A is linear($A(x) = cx$, for some $c \in X$).

Proof. Let $x \in X$, then from 1.18 , we have

$$\begin{aligned} \| f(x+x) - f(x) - f(x) \| &\leq \delta, \\ \| f(2x) - 2f(x) \| &\leq \delta, \end{aligned}$$

and

$$\| \frac{1}{2}f(2x) - f(x) \| \leq \frac{\delta}{2}. \tag{1.20}$$

On using induction, it follows that

$$\| 2^{-n}f(2^n x) - f(x) \| \leq (1 - 2^{-n})\delta. \tag{1.21}$$

Indeed, replace x in 1.20 by $2x$ to get

$$\begin{aligned} \| \frac{1}{2}f(2^2x) - f(2x) \| &\leq \frac{\delta}{2}, \\ \| \frac{1}{2}f(2^2x) - 2f(x) - f(2x) + 2f(x) \| &\leq \frac{\delta}{2} \end{aligned}$$

By applying triangle inequality in LHS of the above inequality, we will get

$$\begin{aligned} \| \frac{1}{2}f(2^2x) - 2f(x) \| - \| f(2x) - 2f(x) \| &\leq \frac{\delta}{2}, \\ \| \frac{1}{2}f(2^2x) - 2f(x) \| &\leq \frac{\delta}{2} + \| f(2x) - 2f(x) \|, \\ &\leq \frac{\delta}{2} + \delta \end{aligned}$$

which implies

$$\| \frac{1}{4}f(2^2x) - f(x) \| \leq \delta(\frac{1}{2} + \frac{1}{2^2}).$$

So we conclude by induction that

$$\begin{aligned} \| 2^{-n}f(2^n x) - f(x) \| &\leq \delta(\frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^n}) \\ &= (1 - 2^{-n})\delta. \end{aligned}$$

Now we will show that the sequence $\{2^{-n}f(2^n x)\}$ is a Cauchy sequence in Y for each $x \in X$. Let $m, n \in \mathbb{Z}_+$ with $m > n$. Then

$$\| 2^{-m}f(2^m x) - 2^{-n}f(2^n x) \| = 2^{-n} \| 2^{-(m-n)}f(2^{m-n} \times 2^n x) - f(2^n x) \| .$$

Right hand side of the above equation by is $2^{-n}\delta(1 - 2^{-(m-n)})$ using(1.21). So ,

$$\| 2^{-m}f(2^m x) - 2^{-n}f(2^n x) \| \leq 2^{-n}\delta(1 - 2^{-(m-n)})$$

$$\| 2^{-m} f(2^m x) - 2^{-n} f(2^n x) \| \leq \delta(\frac{1}{2^n} - \frac{1}{2^m}) \tag{1.22}$$

which proves that $\{2^{-n} f(2^n x)\}$ is a Cauchy sequence in Y for each $x \in X$.

Since Y is a Banach space, $\lim_{n\to\infty} 2^{-n} f(2^n x)$ exists and equals $A(x)$, say. We next show that A is an additive function. Replace x and y by $2^n x$ and $2^n y$ respectively in 1.18 to get ,

$$\| 2^{-n} f(2^n(x+y)) - 2^{-n} f(2^n x) - 2^{-n} f(2^n y) \| \leq 2^{-n}\delta.$$

On letting $n \to \infty$ to get

$$\| A(x+y) - A(x) - A(y) \| = 0$$

which implies

$$A(x+y) = A(x) + A(y).$$

To prove the assertion 1.19 , let $n \to \infty$ in 1.21 to get

$$\begin{aligned} \| \lim_{n\to\infty} 2^{-n} f(2^n x) - f(x) \| &\leq \lim_{n\to\infty} (1 - 2^{-n})\delta, \\ \| A(x) - f(x) &\leq \delta, \ \forall x \in X. \end{aligned}$$

Next we will prove that A is unique. Suppose $A' : X \to Y$ is another additive function satisfying equation(1.19). Then for $x \in X$,

$$\| A(x) - A'(x) \| = n^{-1} \| A(nx) - f(nx) - A'(nx) + f(nx) \|$$

$$\leq 2\delta n^{-1}$$

(equation(1.19)). Letting $n \to \infty$, we get $A' = A$.

Suppose f is continous at $y \in X$. Let $\{x_n\}$ be a sequence in X such that $x_n \to 0$ as $n \to \infty$. Then for any integer $m \in \mathbb{Z}_+$,

$$\| A(x_n + y) - A(y) \| = \| A(x_n) \|$$

$$= \frac{1}{m}[\| A(mx_n + y) - f(mx_n + y) + f(mx_n + y) - f(y) + f(y) - A(y) \|]$$

$$\leq \frac{2\delta + \varepsilon}{m}$$

(using triangle inequality, equation 1.19 and continuity of f at y for large n and any integer m); therefore A is also continuous at y. For a fixed $x \in X$, if $f(tx)$ is continuous in t, then it follows that $A(tx)$ is continuous at t and hence A is linear. This completes the proof of the theorem. □

T. M. Rassias tried to weaken the boundedness hypothesis in the previous theorem and succeeded in proving what is now known to be the *Hyers-Ulam-Rassias stability for the additive Cauchy equation.* This terminology is due to the fact that theorem proved by *Rassias* influenced many Mathematicians to study stability problems of functional equations. We state and prove this theorem in the following section.

Theorem 23. *Let $E_1$and E_2be Banach spaces and let $f : E_1 \to E_2$ be a function satisfying the functional inequality*

$$\| f(x+y) - f(x) - f(y) \| \leq \theta(\| x \|^p + \| y \|^p) \tag{1.23}$$

for some $\theta > 0$, $p \in [0,1)$, and for all $x, y \in E_1$.Then there exists a unique additive function $A : E_1 \to E_2$ such that

$$\| f(x) - A(x) \| \leq \frac{2\theta}{2 - 2^p} \| x \|^p \tag{1.24}$$

for any $x \in E_1$. Moreover, if $f(tx)$ is continuous in t for each fixed $x \in E_1$,then A linear.

Proof. First we prove the following inequality using induction

$$\| 2^{-n} f(2^n x) - f(x) \| \leq \theta \| x \|^p \sum_{m=0}^{n-1} 2^{m(p-1)}. \tag{1.25}$$

Putting $x = y$ in (1.23) and dividing by 2 gives

$$\| 2^{-1}f(2x) - f(x) \| \leq \theta \| x \|^p . \tag{1.26}$$

This proves (1.25) for n=1. Assume that (1.25) is true for some $n \geq 1$. Replace x by $2x$ and divide throughout by 2 in equation (1.25) to get,

$$\| 2^{-n-1}f(2^n 2x) - \frac{f(2x)}{2} \| \leq \theta 2^{-1} \| 2x \|^p \sum_{m=0}^{n-1} 2^{m(p-1)},$$

that is ,

$$\| 2^{-n-1}f(2^{n+1}x) - \frac{f(2x)}{2} \| \leq \theta 2^{-1} 2^p \| x \|^p \sum_{m=0}^{n-1} 2^{m(p-1)},$$

$$\| 2^{-n-1}f(2^{n+1}x) - \frac{f(2x)}{2} \| = \theta \| x \|^p \sum_{m=1}^{n} 2^{m(p-1)}. \tag{1.27}$$

Consider the equation

$$\| 2^{-n-1}f(2^{n+1}x) - f(x) \| = \| 2^{-n-1}f(2^{n+1}x) - \frac{f(2x)}{2} + \frac{f(2x)}{2} - f(x) \| .$$

On using triangle inequality on the right hand side of the above equality, we have

$$\begin{aligned} \| 2^{-n-1}f(2^{n+1}x) - f(x) \| &= \| 2^{-n-1}f(2^{n+1}x) - \frac{f(2x)}{2} + \frac{f(2x)}{2} - f(x) \| \\ &\leq \| 2^{-n-1}f(2^{n+1}x) - \frac{f(2x)}{2} \| + \| \frac{f(2x)}{2} - f(x) \| \\ &\leq \theta \| x \|^p \sum_{m=0}^{n} 2^{m(p-1)} \end{aligned}$$

which follows from equations (1.26) and (1.27). This proves (1.25). Since sum of the geometric series $\sum_{m=0}^{n} 2^{m(p-1)}$ is $\frac{2}{2-2^p}$ for $p \in [0,1)$,

$$\| 2^{-n}f(2^n x) - f(x) \| \leq \frac{2\theta}{2-2^p} \| x \|^p . \tag{1.28}$$

Consider the sequence $\{2^{-n}f(2^n x)\}$. For $m > n > 0$, we have

$$\| 2^{-m}f(2^m x) - 2^{-n}f(2^n x) \| = 2^{-n} \| 2^{-(m-n)}f(2^{m-n}2^n x) - f(2^n x) \|$$

$$\leq 2^{n(p-1)} \frac{2\theta}{2-2^p} \| x \|^p .$$

Therefore, $\{2^{-n}f(2^n x)\}$is a Chauchy sequence for each $x \in E_1$. Since E_2 is complete, the series converges in $E_2$to the limit defined as the following

$$A(x) = \lim_{n\to\infty} 2^{-n}f(2^n x).$$

From (1.23) it follows that

$$\| f(2^n(x+y) - f(2^n x) - f(2^n y) \| \leq 2^{np}\theta(\| x \|^p + \| y \|^p).$$

Dividing by 2^n and letting $n \to \infty$ in the last expression($p < 1$) yields

$$A(x+y) = A(x+y)$$

which proves that A is an additive function. Letting $n \to \infty$ in equation (1.28) proves (1.24).

We will now show the uniqueness of A. Let $A' : E_1 \to E_2$ be another additive function such that for constants $\varepsilon \geq 0$ and $q \in [0,1)$

$$\| A'(x) - f(x) \| \leq \varepsilon \| x \|^q .$$

$$\begin{aligned}
\| A(x) - A'(x) \| &= \frac{1}{n} \| A(nx) - A'(nx) \| \text{ (using additive property of } A \text{ and } A') \\
&\leq \frac{1}{n} \left(\| A(nx) - f(x) \| + \| f(x) - A'(nx) \| \right) \text{ (using triangular inequality)} \\
&\leq \frac{1}{n} \left(\frac{2\theta}{2-2^p} \| nx \|^p + \varepsilon \| nx \|^q \right) \\
&= n^{p-1} \frac{2\theta}{2-2^p} \| x \|^p + n^{q-1}\varepsilon \| x \|^q
\end{aligned}$$

Since left hand side of the above inequality is independent of n , letting $n \to \infty$, we get $A(x) = A'(x)$ for all $x \in E_1$.

Assume that $f(tx)$ is continuous in t for any fixed $x \in E_1$.Since A is additive in E_1, $A(qx) = qA(x)$ for any $q \in \mathbb{Q}$. Fix an $x_0$in E_1 and $\rho \in E_2^*$(dual space of E_2). Define a function $\phi : \mathbb{R} \to \mathbb{R}$ by

$$\phi(t) = \rho(A(tx_0)).$$

ϕ is an additive real valued function, indeed

$$\phi(t_1+t_2) = \rho(A((t_1+t_2)x_0)) = \rho(A(t_1x_0)+A(t_2x_0)) \text{ (using additive property of A)}$$

$$= \rho(A(t_1x_0)) + \rho(A(t_2x_0)) \text{ (using linearity of } \rho)$$

$$= \phi(t_1) + \phi(t_2).$$

Moreover, ϕ is a Borel measurable function by the following reasoning:

Let $\phi(t) = \lim_{n\to\infty} 2^{-n}\rho(f(2^n tx_0))$ and $\phi_n(t) = 2^{-n}\rho(f(2^n tx_0))$. Then $\phi_n(t)$ are continuous functions for each n. $\phi(t)$ is the point wise limit of continuous functions. Point wise limit of Borel measurable functions is bore measurable. Therefore ϕ is a Borel measurable function. Therefore ϕ is linear[theorem 13] and hence continuous. Let $a \in \mathbb{R}$. Then $a = \lim_{n\to\infty} q_n$ for some sequence of real numbers $\{q_n\}$.

$$\phi(at) = \phi(t \lim_{n\to\infty} q_n) = \lim_{n\to\infty} \phi(tq_n) \text{ (using continuity of } \phi)$$

$$= \lim_{n\to\infty} q_n\phi(t) \text{ (using additive property of } \phi)$$

$$= a\phi(t)$$

Therefore, $\phi(at) = a\phi(t)$, for any $a \in \mathbb{R}$. This implies $A(ax) = aA(x)$ for any $a \in \mathbb{R}$. This proves A is linear. □

Remark 24. The above theorem is a generalization of theorem(22). Indeed, putting $p=0$ and replacing δ by $\frac{\delta}{2}$ will yield theorem(22).

Z.Gajda showed that the theorem(23) is not valid when $p=1$ by constructing the following counter example.

For fixed $\theta > 0$ and $\mu := \frac{1}{6}\theta$, define $f:\mathbb{R}\to\mathbb{R}$ by

$$f(x) = \sum_{n=0}^{\infty} 2^{-n}\phi(2^n x), x\in\mathbb{R},$$

where the function $\phi:\mathbb{R}\to\mathbb{R}$ is given by

$$\phi(x) = \begin{cases} \mu & \text{if } 1 \leq x < \infty \\ \mu x & \text{if } -1 < x < 1 \\ -\mu & \text{if } -\infty < x \leq -1. \end{cases}$$

Proof. ϕ is continuous and bounded uniformly($|\phi(x)| \leq \mu$). Therefore the function $f(x)$ is well defined. Since f is defined in terms of uniformly convergent series of continuous functions, f itself is continuous. Moreover f is uniformly bounded over whole of real line. Indeed

$$|f(x)| \leq \sum_{n=0}^{\infty} \frac{\mu}{2^n} = 2\mu, x\in\mathbb{R}.$$

Now we will show that the function f indeed satisfies (1.23) with $p=1$. That is

$$|f(x+y)-f(x)-f(y)| \leq \theta(|x|+|y|). \tag{1.29}$$

If $x=y=0$, then f trivially satisfies (1.29). Next assume that

$$0 < |x|+|y| < 1.$$

Then there exists $N \in \mathbb{N}$ such that

$$2^{-N} \leq |x| + |y| < 2^{-N+1}.$$

Hence from the second inequality above , it follows that $|2^{N-1}x| < 1, |2^{N-1}y| < 1$ and $|2^{N-1}(x+y)| < 1$. Therefore for each $n \in \{0, 1 \ldots, N-1\}$, the numbers $2^n x$, $2^n y$ and $2^n(x+y)$ lie in the interval $(-1,1)$. Since ϕ is linear on $(-1,1)$, it follows that

$$\phi(2^n(x+y)) - \phi(2^n x) - \phi(2^n y) = 0,$$

for $n = 0, 1, 2, \ldots, N-1$.Therefore

$$\begin{aligned}
\frac{|f(x+y) - f(x) - f(y)|}{|x| + |y|} &\leq \sum_{n=0}^{\infty} \frac{|\phi(2^n(x+y)) - \phi(2^n x) - \phi(2^n y)|}{2^n(|x| + |y|)} \\
&= \sum_{n=N}^{\infty} \frac{|\phi(2^n(x+y)) - \phi(2^n x) - \phi(2^n y)|}{2^n(|x| + |y|)} \\
&\leq \sum_{k=0}^{\infty} \frac{3\mu}{2^k 2^N(|x| + |y|)} \\
&\leq \sum_{k=0}^{\infty} \frac{3\mu}{2^k} = 6\mu = \theta.
\end{aligned}$$

which implies f satisfies (1.29). Finally, assume that $|x| + |y| \geq 1$, then from boundedness of f it follows that

$$\frac{|f(x+y) - f(x) - f(y)|}{|x| + |y|} \leq 6\mu = \theta.$$

Now contrary to claim, suppose that there exists a $\delta \in [0, \infty)$ and an additive function $T : \mathbb{R} \to \mathbb{R}$ such that

$$|f(x)| - T(x) \leq \delta |x|.$$

Letting $x \to \infty$

$$\lim_{x \to 0} |f(x) - T(x)| \leq \lim_{x \to 0} \delta |x|.$$

$$|f(\lim_{x \to 0} x) - \lim_{x \to 0} T(x)| \leq \lim_{x \to 0} \delta |x|$$

This implies T is bounded in a neighborhood of 0. So by 2, there exists a real constant c such that

$$T(x) = cx,\ \forall x \in \mathbb{R}.$$

Therefor

$$|f(x) - cx| \le \delta |x|,\ \forall x \in \mathbb{R}$$

which implie

$$|\frac{f(x)}{x}| \le \delta + |c|,\ \forall x \in \mathbb{R}. \tag{1.30}$$

On the other hand, choose $N \in \mathbb{Z}$ large enough so that $N\mu > \delta + |x|$. Then picking an x from the interval $(0, \frac{1}{2^{N-1}})$, we have $2^n x \in (0,1)$ for each $n \in \{0, 1, \ldots, N-1\}$.Consequently for such an x, we have

$$\begin{aligned} |\frac{f(x)}{x}| = \frac{f(x)}{x} &= \sum_{n=0}^{\infty} \frac{\phi(2^n x)}{2^n x} \ge \sum_{n=0}^{N-1} \frac{\phi(2^n x)}{2^n x} \\ &= \sum_{n=0}^{N-1} \mu \frac{2^n x}{2^n x} \quad (\text{since } 2^n x \in (0,1)) \\ &= N\mu > \delta + |x| \end{aligned}$$

which contradicts (1.30). Therefore such a δ as assumed cannot exist. □

1.4 Almost additive functions

Definition 25. Let $f : \mathbb{R} \to \mathbb{R}$ be a real valued function such that

$$f(x+y) = f(x) + f(y)$$

for almost every pair $(x,y) \in \mathbb{R}^2$. That is the set $E = \{(x,y) \in \mathbb{R}^2 : f(x+y) \neq f(x) + f(y)\}$ has planar Lebesgue measure zero. Such a function f is called an almost additive function.

Note that in the previous definition the domain of f can also be $\mathbb{R} \backslash X$ where X is a set of linear Lebesgue measure zero.

P.Erdos asked whether there exists an additive function which is almost everywhere equal to an almost additive function. The following theorem not only answers the above question positively but also proves its uniqueness of such an additive function.

1.4.1 Existence of additive function almost everywhere equal to almost additive function

Theorem 26. *[W. B. Jukrat] Let $f:\mathbb{R}\to\mathbb{R}$ be an almost additive function. Then there exists an additive function $F:\mathbb{R}\to\mathbb{R}$ satisfying*

$$F(x+y)=F(x)+F(y) \tag{1.31}$$

for all $(x,y)\in\mathbb{R}^2$ such that the set $K=\{x\in\mathbb{R}:f(x)\neq F(x)\}$ has linear Lebesgue measure zero. Also such an F is uniquely determined.

Proof. Let $E=\{(x,y)\in\mathbb{R}^2:f(x+y)\neq f(x)+f(y)\}$. Then $\mu_{\mathbb{R}^2}(E)=0$. Define $E_x=\{y\in\mathbb{R}:(x,y)\in E\}$ and $H=\{x\in\mathbb{R}:(x,y)\in E\}$. Then by Fubini's theorem,

$$\mu_{\mathbb{R}^2}(E)=\mu_{\mathbb{R}}(E_x)\times\mu_{\mathbb{R}}(H)=0$$

which implies $\mu_{\mathbb{R}}(E_x)=0$ for almost all $x\in\mathbb{R}$. Therefore the set $N=\{x:\mu_{\mathbb{R}}(E_x)\neq 0\}$ has linear Lebesgue measure zero. Also if $x\notin N$ and $y\notin E_x$, then equation (1.31) is satisfied for (x,y). Denote N^C by M. We claim that equation (1.31) holds if $x,y,x+y\in M$. To prove this claim, select z such that $z\notin E_{x+y}$, $z\notin E_y$ and $y+z\notin E_x$. This is possible because E_{x+y},E_y and E_x-y are null sets and hence the z can be selected from the complement of the union of these null sets. Therefore such a selection of z implies

$$\begin{aligned}
f(x+y+z) &= f(x+y)+f(z)\\
f(y+z) &= f(y)+f(z)\\
f(x+y+z) &= f(x)+f(y+z).
\end{aligned}$$

On equating the right hand sides of the first and third equations above and substituting the value of $f(y+z)$ from the second equation, we get

$$f(x+y) = f(x) + f(y),\ x, y \in M. \tag{1.32}$$

We next claim that if $x_1, y_1, x_2, y_2 \in M$ and $x_1 + y_1 = x_2 + y_2$ then

$$f(x_1) + f(y_1) = f(x_2) + f(y_2). \tag{1.33}$$

This claim can be proved by selecting $z \in M$ such that $y_1 - z$, $y_2 - z$, $x_1 + y_1 - z$, $x_2 + y_2 - z \in M$. This is possible because the sets N, $y_1 - N$, $y_2 - N$ and $x_1 + y_1 - N$ are null sets and hence z can be selected from the complement of union of these null sets. Denote $y_1 - z$ by y_1' and $y_2 - z$ by y_2' respectively. Then $y_1', y_2', x_1 + y_1' = x_2 + y_2' \in M$ and therefore by property (1.32) we get

$$\begin{aligned} f(y_1) &= f(y_1') + f(z) \\ f(x_1 + y_1') &= f(x_1) + f(y_1') \\ f(y_2) &= f(y_2') + f(z) \\ f(x_2 + y_2') &= f(x_2) + f(y_2'). \end{aligned}$$

From first and second equations above, we get $f(x_1 + y_1') + f(z) = f(x_1) + f(y_1)$ and from third and fourth equations above, we get $f(x_2 + y_2') + f(z) = f(x_2) + f(y_2)$. Therefore on using the fact that $x_1 + y_1' = x_2 + y_2'$ we get

$$f(x_1) + f(y_1) = f(x_2) + f(y_2).$$

Our final claim is if $x_1, x_2 x_2 \in M$, there exist $y_1, y_2 \in M$ such that

$$x_1 + x_2 + x_3 = y_1 + y_2$$

and

$$f(x_1) + f(x_2) + f(x_3) = f(y_1) + f(y_2). \tag{1.34}$$

To prove this claim select $z \in M$ such that $z' = x_3 - z$, $x_1 + z$, $x_2 + x_3 - z = x_2 + z' \in M$. This is possible by selecting z avoiding the union of null sets $N, x_3 - N, N - x_1$ and $x_2 + x_3 - N$. Define $y_1 = x_1 + z$ and $y_2 = x_2 + z'$. On application of property (1.32) we get

$$\begin{aligned} f(x_3) &= f(z) + f(z') \\ f(y_1) &= f(x_1) + f(z) \\ f(y_2) &= f(x_2) + f(z') \end{aligned}$$

from which we can conclude property (1.34).

To define the required additive function F, we use the fact that every real number z is of the form $x + y$ with $x, y \in M$. This sum can be formed by simply selecting $x \in M$ such that $y = z - x \in M$. Such an x exists because N is a null set and such a y exists because $N - x$ is a null set for all x. Define

$$F(z) := f(x) + f(y).$$

F is a well defined function because of the property (1.33). If $z \in M$ then by property (1.32), $f(z) = f(x) + f(y)$ and hence $f(z) = F(z)$ for all z except a set of linear Lebesgue measure zero. To prove that F is additive , let z_1 and $z_2$be arbitrary real numbers. Let $x_1, y_1, x_2, y_2 \in M$ be such that $z_1 = x_1 + y_1$ and $z_2 = x_2 + y_2$. By applying property (1.34) twice, we will get $p_1, p_2 \in M$ such that

$$x_1 + y_1 + x_2 + y_2 = p_1 + p_2$$

and

$$f(x_1) + f(y_1) + f(x_2) + f(y_2) = f(p_1) + f(p_2).$$

But the left hand side of the second equation above equals $F(z_1) + F(z_2)$ by definition and right hand side equals $F(p_1 + p_2) = F(z_1 + z_2)$. Since $z_1$and z_2 were arbitrary, this proves F is additive.

To show the uniqueness of the additive function F, let F_1 and F_2be additive functions which coincide with each other almost everywhere except on a set of measure zero. Let M be the set on which F_1agrees with F_2. $F_1 - F_2 = 0$ for all $z \in M$. But every real number z can be written as the sum $x + y$ with $x, y \in M$ and hence $F_1 - F_2$ vanishes everywhere on the real line. This proves the uniqueness of the additive function F. □

1.5 Most general solution for Cauchy equations

In *1905, G.Hamel* used the existence of *Hamel Basis* to find the most general solution of an additive function on $\mathbb{R}$. In this section we will look at the construction of additive functions using a Hamel basis of $\mathbb{R}$ over $\mathbb{Q}$.

1.5.1 Construction of additive functions using Hamel basis

Theorem 27. *Let B be a Hamel basis of $\mathbb{R}$ over $\mathbb{Q}$ and $f : B \to \mathbb{R}$ be any arbitrary function. Then there exists an additive function $F : \mathbb{R} \to \mathbb{R}$ such that $F|_B = f$.*

Proof. Let B be a Hamel basis of $\mathbb{R}$ over $\mathbb{Q}$ and $f : B \to \mathbb{R}$ be any arbitrary function. Every $x \in \mathbb{R}$ can be written uniquely as finite linear combination of elements of B. That is

$$x = \sum_{i=1}^{n} q_i b_i,$$

where $q_i \in \mathbb{Q}$ and $b_i \in B$. Define the new function $F : \mathbb{R} \to \mathbb{R}$ by

$$F(x) = \sum_{i=1}^{n} q_i f(b_i). \tag{1.35}$$

Clearly, $F|_B = f$. Moreover, F is additive also. Indeed if $x_1 = \sum_{i=1}^{n} q_{1i} b_i$ and $x_2 = \sum_{i=1}^{n} q_{2i} b_i$ are two real numbers with $q_{1i}, q_{2i}, \in \mathbb{Q}$ and $b_i \in \mathbb{B}$. Then

$$\begin{aligned} F(x_1 + x_2) &= F(\sum_{i=1}^{n} q_{1i} b_i + \sum_{i=1}^{n} q_{2i} b_i) \\ &= F(\sum_{i=1}^{n} (q_{1i} + q_{2i}) b_i) \\ &= \sum_{i=1}^{n} (q_{1i} + q_{2i}) f(b_i) \\ &= \sum_{i=1}^{n} q_{1i} f(b_i) + \sum_{i=1}^{n} q_{2i} f(b_i) \\ &= F(x_1) + F(x_2) \end{aligned}$$

which follows from the definition of F. This proves that F is additive. □

Corollary 28. *Basic Cauchy Equation has discontinuous solutions.*

Proof. Let B be a Hamel basis of $\mathbb{R}$ over $\mathbb{Q}$ and $f : B \to \mathbb{R}$ be a function such that there exists distinct $b_1$and b_2 in B such that

$$\frac{f(b_1)}{b_1} \neq \frac{f(b_2)}{b_2}. \tag{1.36}$$

Such a b_1 and b_2 exist whenever there does not exists a constant $c \in \mathbb{R}$ such that $f(b_i) = cb_i$ for all b_i in B. Let F be as defined in the previous theorem. Then F is additive and $F|_B = f$. From Corollary 3 of Theorem 2, continuity of F at a single point will imply that

$$f(x) = cx \quad \forall x \in \mathbb{R} \text{ ,where } c = f(1)$$

which is not possible due to condition(1.36). Thus F is discontinuous everywhere which proves our Corollary. □

Chapter 2

Convex functions and inequalities.

2.1 Convex functions on $\mathbb{R}$.

Definition 29. Let I be an open interval in $\mathbb{R}$. A function $f : \mathbb{R} \to \mathbb{R}$ is called *convex* if

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y) \tag{2.1}$$

for all $x, y \in I$ and $\lambda \in [0,1]$.

If the inequality in (2.1) is strict, then f is called *strictly convex.*

In all our further discussions in this section, f and λ are defined according to 29 unless otherwise stated.

2.1.1 Geometric interpretation of convex functions.

If A, B and C are three distinct points on graph of a convex function f with B lying between A and C, then B lies on or below chord AC. This can be stated in terms of slopes as

$$\text{slope}(AB) \leq \text{slope}(AC) \leq \text{slope}(BC) \tag{2.2}$$

with strict inequalities in case of strictly convex functions.

2.1.2 Properties of Convex functions.

In this subsection, we will discuss the implications or necessary conditions for a convex function.

Theorem 30. *A finite convex function f defined on a closed interval $[a,b]$ is bounded above by $M = max\{f(a), f(b)\}$ and bounded below by $m = 2f(\frac{a+b}{2}) - M$.*

Proof. For any $z = \lambda x + (1-\lambda)y$ in $[a,b]$, we have

$$f(z) \le \lambda f(a) + (1-\lambda)y \le \lambda M + (1-\lambda)M = M$$

where the first inequality follows from the convexity of f. Also we can write any arbitrary point in $[a,b]$ in the form $\frac{a+b}{2} + t$ for some $t \in \mathbb{R}$ and from the convexity of f, it follows that

$$f(\frac{a+b}{2}) \le \frac{1}{2} f(\frac{a+b}{2} + t) + \frac{1}{2} f(\frac{a+b}{2} - t)$$

which implies

$$f(\frac{a+b}{2} + t) \ge 2f(\frac{a+b}{2}) - f(\frac{a+b}{2} - t).$$

Now using $-f(\frac{a+b}{2} - t) \ge -M$ we get

$$f(\frac{a+b}{2} + t) \ge 2f(\frac{a+b}{2}) - M = m.$$

□

Convex functions can be discontinuous on end points of the interval in which they are defined. To show this consider the example of $f : [0,1] \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 0 & 0 \le x < 1 \\ 1 & x = 1 \end{cases}.$$

f is clearly discontinuous at $x = 1$ and still convex.

Our next theorem shows how they behave in the interior of the domain interval.

Theorem 31. *Let $f : I \to \mathbb{R}$ be a convex function , then f satisfies Lipschitz condition on any closed interval $[a,b]$in interior $I^0$of I. That is there exists a real constant k such that*

$$|f(x) - f(y)| \leq k|x-y| \tag{2.3}$$

for all $x,y \in [a,b]$. Consequently f is absolutely continuous on $[a,b]$.

Proof. Let $\varepsilon > 0$ be such that $a-\varepsilon$, $b+\varepsilon \in I$, this also implies $[a,b] \in I^o$. Let x and y be distinct points in $[a,b]$. Define

$$z = y + \frac{\varepsilon(y-x)}{|y-x|},$$

and

$$\lambda = \frac{|y-x|}{\varepsilon + |y-x|}.$$

Then clearly $z \in [a,b]$ and $y = \lambda z + (1-\lambda)x$. Indeed

$$\lambda z + (1-\lambda)x = \frac{|y-x|}{\varepsilon+|y-x|}y + \frac{\varepsilon}{\varepsilon+|y-x|}(y-x) + \frac{\varepsilon}{\varepsilon+|y-x|}x$$

$$= \frac{\varepsilon+|y-x|}{\varepsilon+|y-x|}y = y.$$

Therefore from convexity of f we have

$$f(y) \leq \lambda f(z) + (1-\lambda)f(x) = \lambda(f(z) - f(x)) + f(x)$$

which implies

$$f(y) - f(x) \leq \lambda(M-m) < \frac{|y-x|}{\varepsilon}(M-m)$$

where M and m are the upper and lower bounds of f in $[a,b]$ respectively. Since this is true for any $x,y \in [a,b]$, putting $k = \frac{M-m}{\varepsilon}$ in the above inequality gives

$$|f(y) - f(x)| \leq k|y-x|$$

as desired.

Now for any $\varepsilon_1 > 0$, we can define $\delta = \varepsilon_1/k$ so that for any collection $\{(a_i,b_i)\}_1^n$ of disjoint open subintervals of $[a,b]$ with $\sum_{i=1}^n (b_i - a_i) < \delta$, $\sum_{i=1}^n |f(b) - f(a_i)| < \varepsilon_1$. Thus we have proved the uniform continuity of f on $[a,b]$. □

Definition 32. Left and right derivatives of a function are defined respectively as

$$f'_-(x) = \lim_{y \uparrow x} \frac{f(y) - f(x)}{y - x}$$

and

$$f'_+(x) = \lim_{y \downarrow x} \frac{f(y) - f(x)}{y - x}.$$

Theorem 33. *Let $f : I \to \mathbb{R}$ be a convex function, then both $f'_-(x)$ and $f'_+(x)$ exist and are increasing(strictly increasing if f is a strict convex function).*

Proof. Let $w,x,y,z \in I^o$ with $w < x < y < z$ and A,B,C and D be $(w, f(w)), (x, f(x)), (y, f(y))$ and $(z, f(z))$ on the graph of f. From inequality (2.2), we get

$$\text{slope}(AB) \leq \text{slope}(AC) \leq \text{slope}(BC) \leq \text{slope}(BD) \leq \text{slope}(CD). \tag{2.4}$$

Therefore

$$\frac{f(x) - f(y)}{x - y} \leq \frac{f(z) - f(y)}{z - y}. \tag{2.5}$$

Also it can be seen that $slope(BC)$ increases as $x \uparrow y$ and $slope(CD)$ decreases as $z \downarrow y$. So the LHS of inequality(2.5) increases as $x \uparrow y$ and RHS of inequality(2.5) increases as $x \downarrow y$. Thus $f'_-(y)$ and $f'_+(y)$ exist and satisfy

$$f'_-(y) \leq f'_+(y). \tag{2.6}$$

Also from inequality(2.4),

$$f'_-(w) \leq \frac{f(x) - f(w)}{x - w} \leq \frac{f(y) - f(x)}{y - x} \leq f'_-(y) \tag{2.7}$$

which combined with inequality(2.6) gives

$$f'_-(w) \leq f'_+(w) \leq f'_-(y) \leq f'_+(y)$$

for all $w < y$ in I^o. This proves that f'_+ and f'_- are increasing functions.

Replacing all the inequalities in the above discussion with strict inequalities, it follows that f'_+ and f'_- are strictly increasing functions when f is strictly convex. □

We will now look at the existence of the derivative of a convex function.

Theorem 34. *Let $f : I \to \mathbb{R}$ be a convex(strict convex) function defined on an open interval I, then the set*

$$E = \left\{x \in I : f'(x)\, fail\, to\, exists\right\}$$

is countable and $f'(x)$ is continuous on $I \backslash E$.

Proof. Let w,x,y,z be as in the proof of the previous theorem. Then from inequality(2.4),

$$f_+(x) \leq \frac{f(y)-f(x)}{y-x}.$$

Also from the previous theorem, f_+ is a monotone function, so the right hand limit and left hand limit exists. Therefore

$$\lim_{x \downarrow w} f'_+(x) \leq \lim_{x \downarrow w} \frac{f(y)-f(x)}{y-x} = \frac{f(y)-f(w)}{y-w}, \tag{2.8}$$

where the equality follows from the continuity of f in I^o. Also

$$\lim_{y \downarrow w} \frac{f(y)-f(w)}{y-x} = f'_+(w)$$

and combining this inequality with inequality(2.8) gives

$$\lim_{x \downarrow w} f'_+(x) \leq f'_+(w).$$

But $w < x$ implies

$$f'_+(w) \leq f'_+(x)$$

and therefore implies

$$\lim_{x \downarrow w} f'_+(x) = f'_+(w). \tag{2.9}$$

Similarly we can show that

$$\lim_{x\uparrow w} f'_+(x) = f'_-(w). \tag{2.10}$$

From equations (2.9) and (2.10), it is clear that $f'_+(w) = f'_-(w)$ if and only if f'_+ is continuous at w. Therefore existence of f'at a point is same as the continuity of f'_+ at that point. Since f'_+ is a monotone functions, all its discontinuities are jumps and hence the number of discontinuities are countable. Therefore on $I\backslash E$, f'_+ is continuous and hence f' which agrees with f'_+ on $I\backslash E$ is continuous. □

Since we have established the countability of discontinuities of a convex functions, now we will look into what happens when the derivative exists.

Theorem 35. *Let $f : (a,b) \to \mathbb{R}$ be a differentiable function. Then f is convex(strictly convex) if and only if f' is an increasing(strictly increasing) function.*

Proof. Let $f : (a,b) \to \mathbb{R}$ be differentiable and convex(strictly convex), then from theorem 33, it is obvious that f' is an increasing(strictly increasing) function. Now for the converse suppose that $f : (a,b) \to \mathbb{R}$ is differentiable with increasing derivative. From the fundamental theorem of calculus,

$$f(y) - f(x) = \int_x^y f'(t)dt \tag{2.11}$$

for any $x, y \in (a,b)$. Now let a and b be arbitrary positive reals such that $a+b = 1$. For proving convexity of f , it is enough to show that $f(ax+by) \leq af(x)+bf(y)$ for any $x,y \in (a,b)$. Assume that $x < y$, then from (2.11), it follows that

$$\begin{aligned}
af(x)+bf(y)-(a+b)f(ax+by) &= b(f(y)-f(ax+by))-a(f(ax+by)-f(x)) \\
&= b\int_{ax+by}^{y} f'(t)dt - a\int_{x}^{ax+by} f'(t)dt \\
&\geq bf'(ax+by)\int_{ax+by}^{y} dt - af'(ax+by)\int_{x}^{ax+by} dt \\
&= f'(ax+by)\left[ax+by-(ax+by)(a+b)\right] \\
&= 0.
\end{aligned}$$

which implies

$$f(ax+by) \leq af(x)+bf(y).$$

This completes the proof for the convex functions. Replacing inequalities in the above discussions with strict inequalities proves the case for strictly convex functions. □

Theorem 36. *Let $f:(a,b)\to\mathbb{R}$ is a twice differentiable function on (a,b). Then f is convex if and only if $f''(x)\geq 0$, also if $f''(x)>0$on (a,b), then f is strictly convex on the interval.*

Proof. Given that $f''(x)$ exists for all x in (a,b), therefore $f'(x)$ exists for all x in (a,b). But $f''(x)\geq 0$ if and only if $f'(x)$ is an increasing function which happens if and only if f is a convex function from the previous theorem. Also if $f''(x)>0$, then $f'(x)$ is strictly increasing and again from the previous theorem, it follows that f is a strictly convex function. This completes the proof. □

It is interesting to note that there exists function f which is strictly convex , but $f''(x)=0$. To see this, take the example of $f(x)=x^n$ where n is any positive even integer greater than 2 defined on any open interval (a,b) containing 0. It is clear that $f(x)$ is strictly increasing, but

$$f''(x)=n(n-1)x^{n-2}$$

$$f''(0)=0.$$

Now we will define a weaker condition than convexity.

2.2 Mid convex functions

Definition 37. A function $f:(a,b)\to\mathbb{R}$ is called **mid convex** or **weakly convex** if

$$f(\frac{x+y}{2}) \leq \frac{1}{2}(f(x)+f(y)) \tag{2.12}$$

for all $x,y\in(a,b)$.

2.2.1 Properties of Mid convex functions

Theorem 38. *Finite sum of mid convex functions which are defined on the same interval is mid convex. Also let $f_1, f_2, \cdots$ be an infinite sequence of mid convex functions defined on the same interval such that $\{f_n(x)\}$ has a point wise limit f on the interval, then f is mid convex.*

Proof. Let $f_1, f_2, \cdots, f_n$ be a finite collection of mid convex functions defined on an interval (a,b). Then for any $x, y \in (a,b)$,

$$(f_1+f_2)(x+y) = f_1(x+y)+f_2(x+y) \leq \frac{1}{2}(f_1(x)+f_1(y))+\frac{1}{2}(f_2(x)+f_2(y))$$

which follows from the mid convexity of f_1 and f_2. This indeed imply

$$(f_1+f_2)(x+y) \leq \frac{1}{2}((f_1+f_2)(x)+(f_1+f_2)(y)).$$

This proves that f_1+f_2 is mid convex. Now using induction it can be proven that $f_1+f_2+\cdots+f_n$ is a mid convex function. Now let $f_1, f_2, \cdots$ be an infinite sequence of mid convex functions defined on the interval (a,b) such that $\{f_n(x)\}$ has a point wise limit f on (a,b). Then for any $x, y \in (a,b)$, mid convexity of f_n gives

$$f_n(\frac{x+y}{2}) \leq \frac{1}{2}(f_n(x)+f_n(y))$$

and on taking limits on both sides

$$\lim_{n\to\infty} f_n(\frac{x+y}{2}) \leq \frac{1}{2}\left(\lim_{n\to\infty} f_n(x)+\lim_{n\to\infty} f_n(y)\right),$$

which indeed gives

$$(\frac{x+y}{2}) \leq \frac{1}{2}(f(x)+f(y)).$$

This completes the proof. □

It should be noted that convexity implies mid convexity(take $\lambda = \frac{1}{2}$), but the converse is not true.

Consider $\mathbb{R}$ as a vector space over $\mathbb{Q}$ with the Hamel basis $\{h_i\}$. Every real number x can be written as a

$$x = \sum c_i(x)h_i,$$

where the summation is taken over all elements of the Hamel basis and $c_i(x)$ corresponds to the rational coefficient of h_i in th summation. c can be seen as a function of x taking rational values. Also c is additive, indeed if

$$x = \sum c_i(x)h_i;$$

$$y = \sum c_i(y)h_i,$$

then

$$x+y = \sum c_i(x) + c_i(y)h_i$$

which implies

$$c(x+y) = c(x) + c(y).$$

Also $\frac{1}{2}$ is a rational which implies

$$c(\frac{x+y}{2}) = \frac{1}{2}(c(x)+c(y))$$

proving that c is mid convex. On the other hand , c cannot be convex. Indeed c is a non-constant function taking only rational values and therefore discontinuous everywhere, hence by theorem 31 c cannot be convex.

We shall prove this after looking at the following set of conditions which has to be satisfied by weakly convex functions to be convex.

Lemma 39. *Let $f : (a,b) \to \mathbb{R}$ be a mid convex function and $x_1, x_2, \ldots, x_n \in (a,b)$. Also let $r_1, r_2, \ldots, r_n \in \mathbb{Q}$ such that $r_i x_i \in (a,b)$ for $i = 1,2,\ldots,n$ and $\sum_{i=1}^n r_i = 1$. Then*

$$f(r_1x_1 + \cdots + r_nx_n) \leq r_1 f(x_1) + \cdots + r_n f(x_n). \tag{2.13}$$

Proof. Let f and $x_1,x_2,\dots,x_n \in (a,b)$ be as in the hypothesis of the theorem. Then we will first show

$$f(\frac{x_1+x_2+\cdots+x_n}{n}) \leq \frac{1}{n} f(x_1+x_2+\cdots+x_n). \tag{2.14}$$

Let inequality (2.14) be true for some $n \in \mathbb{N}$, then it is also true for $2n$. Indeed

$$\begin{aligned} f(\frac{x_1+\cdots+x_n+x_{n+1}+\cdots x_{2n}}{2n}) &= f(\frac{\frac{x_1+\cdots+x_n}{n}+\frac{x_{n+1}+\cdots x_{2n}}{n}}{2}) \\ &\leq \frac{1}{2} f(\frac{x_1+\cdots+x_n}{n})+f(\frac{x_{n+1}+\cdots x_{2n}}{n}) \\ &\leq \frac{1}{2} f(\frac{x_1+\cdots+x_n+x_{n+1}+\cdots+x_{2n}}{n}) \end{aligned}$$

where the first inequality follows from the mid convexity of f and the second inequality follows from the assumption that inequality (2.14) is true for n. Next we will show that if inequality (2.14) holds for some $n+1 \in \mathbb{N}$, then it holds for n as well. Let $x_1,x_2,\dots,x_n \in (a,b)$ and define $x_{n+1} = \frac{x_1+x_2+\ldots+x_n}{n}$. It is clear that $x_{n+1} \in (a,b)$ since $\min\{x_1,\dots,x_n\} \leq x_{n+1} \leq \max\{x_1,\dots,x_n\}$. Also

$$\begin{aligned} f(x_{n+1}) &= f(\frac{nx_{n+1}+x_{n+1}}{n+1}) \\ &= f(\frac{x_1+x_2+\cdots+x_n+x_{n+1}}{n+1}) \\ &\leq \frac{1}{n+1}[f(x_1)+f(x_2)+\cdots+f(x_n)+f(x_{n+1})] \\ f(x_{n+1})-\frac{1}{n+1}f(x_{n+1}) &\leq \frac{1}{n+1}[f(x_1)+f(x_2)+\cdots+f(x_n)] \\ \frac{n}{n+1}f(x_{n+1}) &\leq \frac{1}{n+1}[f(x_1)+f(x_2)+\cdots+f(x_n)] \\ f(\frac{x_1+x_2+\cdots+x_n}{n}) &\leq \frac{1}{n}f(x_1)+f(x_2)+\cdots+f(x_n) \end{aligned}$$

where the first inequality follows from our assumption that inequality (2.14) holds for $n+1$. Hence (2.14) is true for n when it is true for $n+1$. Since $2n$ can be as large as required, we have proved the result (2.14) for all $n \in \mathbb{N}$.

Now let $r_1,r_2,\dots,r_n \in \mathbb{Q}$ such that $r_i x_i \in (a,b)$ for $i=1,2,\dots,n$ and $\sum_{i=1}^n r_i = 1$. Let l be the least common denominator of $r_1,r_2,\dots,r_n$. Then

$$r_1x_1+\cdots+r_nx_n = \frac{1}{l}(lr_1x_1+\cdots+lr_nx_n) = \frac{1}{l}(s_1x_1+\cdots+s_nx_n)$$

where $s_i = lr_i \in \mathbb{Z}$ and $\sum_i s_i = l$ for $i = 1, \ldots, n$. Therefore

$$\begin{aligned} f(r_1x_1 + \cdots + r_nx_n) &= f(\frac{s_1x_1 + \cdots + s_nx_n}{l}) \\ &= f(\frac{\sum_{i=1}^{s_1} x_1 + \cdots + \sum_{i=1}^{s_n} x_n}{l}) \\ &\leq \frac{1}{l}(s_1 f(x_1) + \cdots + s_n f(x_{n)}) \\ &= r_1 f(x_1) + \cdots + r_n f(x_n) \end{aligned}$$

where the first inequality follows from the property (2.14) applied to each s_i. This completes the proof. □

2.2.2 Sufficient conditions for mid convex functions that implies Convexity

Theorem 40. *Let $f : (a,b) \to \mathbb{R}$ be a continuous mid convex function, then f is convex.*

Proof. Let f be as in the hypothesis of the theorem. Let $\lambda \in (0,1)$, by denseness of $\mathbb{Q}$, there exists a rational sequence $\{r_n\}$ converging to λ. Also for any $x, y \in (a,b)$

$$\begin{aligned} f(\lambda x + (1-\lambda)y) &= f(\lim_{n\to\infty} r_n x + (1 - \lim_{n\to\infty} r_n)y) \\ &= \lim_{n\to\infty} f(r_n x + (1-r_n)y) \\ &\leq \lim_{n\to\infty} r_n f(x) + (1-r_n) f(y) \\ &= \lambda f(x) + (1-\lambda) f(y). \end{aligned}$$

where the first equality follows from the continuity of f and first inequality follows from lemma 39. This completes the proof. □

Theorem 41. *Let $f : (a,b) \to \mathbb{R}$ be a mid convex function that is bounded above in (a,b) then f is convex.*

Proof. Let f be as in the hypothesis of the theorem and be bounded by $M \in \mathbb{R}$ in (a,b). Suppose to the contrary assume that f is not convex. Then there exists $\lambda \in (0,1)$ and $x, y \in (a,b)$ such that

$$f(\lambda x + [1-\lambda]y) - (\lambda f(x) + [1-\lambda] f(y)) = h > 0.$$

Choose $n \in \mathbb{N}$ big enough so that $M - \min\{f(x), f(y)\} < nh$. Let $a \in \mathbb{R}$ such that

$$\lambda - a \in \mathbb{Q} \tag{2.15}$$
$$\lambda + na \in \mathbb{Q}. \tag{2.16}$$

Such an a exists because of the denseness of rational numbers. Also

$$\frac{1}{n+1}((\lambda+na)x+(1-[\lambda+na])y)+\frac{n}{n+1}((\lambda-a)x+(1-[\lambda-a])y)$$

$$=\lambda x+[1-\lambda]y,$$

on applying lemma 39 gives

$$\frac{1}{n+1}f((\lambda+na)x+(1-[\lambda+na])y)+\frac{n}{n+1}f((\lambda-a)x+(1-[\lambda-a])y)$$

$$\geq f(\lambda x+[1-\lambda]y);$$

which implies

$$f((\lambda+na)x+(1-[\lambda+na])y)$$

$$\geq (n+1)(\lambda f(x)+[1-\lambda]f(y)+h)-nf((\lambda-a)x+(1-[\lambda-a])y). \tag{2.17}$$

Also from (2.15) and lemma 39 gives

$$f((\lambda-a)x+(1-[\lambda-a])y) \leq (\lambda-a)f(x)+(1-[\lambda-a])f(y). \tag{2.18}$$

On combining equations (2.17) and (2.18), we will get

$$f((\lambda+na)x+(1-[\lambda+na])y) \geq f(x)\,[n\lambda+\lambda-n\lambda+na]+f(y)\,[n-n\lambda+1-\lambda-n+n\lambda-na]+(n+1)h$$

$$
\begin{aligned}
&= [\lambda + na] f(x) + f(y)[1-(\lambda+na)] + (n+1)h \\
&\geq \min\{f(x), f(y)\} + n(n+1)h \\
&> M,
\end{aligned}
$$

which contradicts our assumption that f is bounded above by M. Therefore f should be convex and this completes the proof. □

Now we shall give a weaker sufficiency condition for a mid convex function to be convex.

Theorem 42. *Let $f : (a,b) \to \mathbb{R}$ be a mid convex function that is Lebesgue measurable, then f is continuous and hence convex.*

Proof. To the contrary, we start with the assumption that f is not continuous. Let $x_0 \in (a,b)$ be a point of discontinuity. Choose c such that $(x_0 - 2c, x_0 + 2c) \subseteq (a,b)$ and define

$$B_n := \{x \in (a,b) : f(x) > n\}.$$

for $n \in \mathbb{N}$. Note that $\{B_n\}$ is a collection of measurable sets since f is measurable. For a fixed n, choose $u \in B_n \cap (x_0 - c, x_0 + c)$. Such a u exists for each n since otherwise f will be bounded in a neighborhood of $x_0$and hence by theorem 41 f will be continuous at x_0. Select $\lambda \in [0,1]$. Then

$$n < f(u) = f[\frac{u+\lambda c}{2} + \frac{u-\lambda c}{2}] \leq \frac{1}{2}[f(u+\lambda c) + f(u-\lambda c)] \tag{2.19}$$

where the inequality follows from the mid convexity of f. From equation (2.19) it follows that either $f(u+\lambda c) > n$ or $f(u-\lambda c) > n$ which implies either $u+\lambda c \in B_n$ or $u - \lambda c \in B_n$. This is equivalent to saying if $M_n = \{x : x = y - u, y \in B_n\}$, then either $\lambda c \in M_n$ or $-\lambda c \in M_n$. Since λ was arbitrary chosen, it is true for all $\lambda \in [0,1]$. Now we shall show that

$$c \leq \mu(M_n) = \mu(B_n). \tag{2.20}$$

Equality in the above equation follows from the translation invariant property of Lebesgue measure. Also for a fixed n, λc or $-\lambda c$ lies in M_n for all $\lambda \in [0,1]$. Therefore if $A_1 = M_n \cap [-c,0]$ and $A_2 = M_n \cap [0,c]$, then $-A_1 \cup A_2 = [0,c]$. Therefore

$$c = \mu[0,c] \leq \mu(-A_1) + \mu(A_2) = \mu(A_1) + \mu(A_2) = \mu(A_1 \cup A_2) \leq \mu(M_n),$$

where the first inequality follows from the sub additivity of measure, second equality follows from the translation invariant property of measure and third inequality follows from the additive property of measure. This proves equation (2.20) which implies

$$c \leq \lim_{n\to\infty} \mu(B_n) = \lim_{n\to\infty} \mu(\cap_{n=1}^{\infty} B_n),$$

where the equality follows from the fact that $\{B_n\}$ is a sequence of decreasing measurable set. This implies $\cap_{n=1}^{\infty} B_n$ is nonempty and hence there exists a point $v \in (a,b)$ such that $f(v) > n$ for every $n \in \mathbb{N}$ which contradicts the definition of f as a function to the real line. This shows our assumption that f is not a continuous function is wrong which completes the proof. □

Now we will look at a stronger condition than convexity.

2.3 Log convex functions

Definition 43. A positive function $f : (a,b) \to \mathbb{R}$ is *called a log convex function* if $\log \circ f$ is a convex function.

It is easy to note that a log convex function is always convex, indeed if f is a log convex function then the function itself is the composition of exponential function with $\log \circ f$. However a convex function may not be log convex. Consider the example of $f(x) = x$ on an interval $(0,1)$. $f(x)$ is convex but $\log \circ f(x) = \log x$ is not convex on $(0,1)$.

Now we shall look at some properties of log convex functions.

2.3.1 Properties of Log convex functions.

Theorem 44. *Finite product of log convex functions is log convex. Let $f_1, f_2, \cdots$ be an infinite sequence of log convex functions defined on the same interval such that $\{f_n(x)\}$ has a point wise limit f which is positive on the same interval. Then f is a log convex function.*

Proof. Let $f_1, f_2, \cdots, f_n$ be a finite collection of log convex functions defined on an interval (a,b). Then

$$\log(f_1 f_2 \dots f_n) = \log(f_1) + \log(f_2) + \cdots + \log(f_n).$$

Since finite sum of convex functions is convex , $\log(f_1 f_2 \dots f_n)$ is convex from the above expression and hence $f_1 f_2 \dots f_n$ is log convex by definition. Let $f_1, f_2, \cdots$ be an infinite sequence of log convex functions defined on the same interval (a,b) such that $\{f_n(x)\}$ has a point wise limit f which is positive on the same interval. Then for $x, y \in (a,b)$ and $\lambda \in [0,1]$,

$$\log \circ f_n(\lambda x + (1-\lambda)y) \leq \lambda \log \circ f_n(x) + (1-\lambda) \log \circ f_n(y).$$

On taking limit of n on both sides ,

$$\lim_{n\to\infty} \log \circ f_n(\lambda x + (1-\lambda)y) \leq \lambda \lim_{n\to\infty} \log \circ f_n(x) + (1-\lambda) \lim_{n\to\infty} \log \circ f_n(y)$$

which implies

$$\log \circ \lim_{n\to\infty} f_n(\lambda x + (1-\lambda)y) \leq \lambda \log \circ \lim_{n\to\infty} f_n(x) + (1-\lambda) \log \circ \lim_{n\to\infty} f_n(y)$$

where the limit is taken inside due to the continuity of log function. Therefore

$$\log \circ f(\lambda x + (1-\lambda)y) \leq \lambda \log \circ f(x) + (1-\lambda) \log \circ f(y)$$

which shows that f is a log convex function and hence completing the proof. □

Theorem 45. *Let $f : (a,b) \to \mathbb{R}$ is a strictly positive function satisfying the following inequal-*

ity

$$\frac{f(x)f''(x)-(f'(x))^2}{(f(x))^2} \geq 0,$$

then f is log convex.

Proof. Define $g(x) = \log \circ f(x)$. Then

$$g'(x) = \frac{f'(x)}{f(x)}$$

and

$$g''(x) = \frac{f(x)f''(x)-(f'(x))^2}{(f(x))^2}.$$

Therefore $\frac{f(x)f''(x)-(f'(x))^2}{(f(x))^2} \geq 0$ implies $g''(x) \geq 0$ and theorem 36 shows that g is a convex function and hence f is a log convex function. □

Next we present a result which is not so obvious from the definition of log convex functions.

Theorem 46. *Let f and g are log convex functions on the same interval, then $f+g$ is also log convex on the same interval.*

Proof. Let f and g be log convex functions defined on $(a.b)$. We shall show that $\log(f+g)$ is mid convex and hence from the continuity of $\log(f+g)$, it will follow that $\log(f+g)$ is convex.

Let $x_1, x_2 \in (a,b)$, then the log convexity of f and g gives the mid convexity of $\log \circ f$ and $\log \circ g$.

$$\log(f(\frac{x_1+x_2}{2})) \leq \frac{1}{2}(\log(f(x_1)) + \log(f(x_2))),$$

on taking the exponential of both LHS and RHS,

$$f(\frac{x_1+x_2}{2}) \leq (f(x_1)f(x_2))^{\frac{1}{2}}$$

which implies

$$f(\frac{x_1+x_2}{2})^2 \leq f(x_1)f(x_2). \tag{2.21}$$

Similarly we get

$$g(\frac{x_1+x_2}{2})^2 \leq g(x_1)f(x_2). \tag{2.22}$$

To show the mid convexity of $f+g$, we have to show given conditions (2.21) and (2.22),

$$(f(\frac{x_1+x_2}{2})+g(\frac{x_1+x_2}{2}))^2 \leq (f(x_1)+g(x_1))(f(x_2)+g(x_2)).$$

This is proven once we show that if a_1,a_2,b_1,b_2,c_1,c_2 are positive real numbers with $a_ic_i - b_i^2 \geq 0$ for $i=1,2$, then

$$(a_1+a_2)(c_1+c_2)-(b_1+b_2)^2 \geq 0.$$

Consider the real quadratic form $a_ix^2+2b_ixy+c_iy^2$ for $i=1,2$ where $a_i>0$. Then

$$\begin{aligned} a_i(a_ix^2+2b_ixy+c_iy^2) &= a_i^2x^2+2a_ib_ixy+a_ic_iy^2 \\ &= (a_ix+b_iy)^2+(a_ic_i-b_i^2)y^2. \end{aligned}$$

From the above expression its clear that if $a_ic_i-b_i^2 \geq 0$, the quadratic form $a_ix^2+2b_ixy+c_iy^2$ can never take negative values for any x,y. That is

$$a_ix^2+2b_ixy+c_iy^2 \geq 0$$

for $i=1,2$ which implies

$$(a_1+a_2)x^2+2(b_1+b_2)xy+(c_1+c_2)y^2 \geq 0.$$

Therefore the determinant of the matrix corresponding to the quadratic form $(a_1+a_2)x^2+2(b_1+b_2)xy+(c_1+c_2)y^2$ is non negative. In other words

$$(a_1+a_2)(c_1+c_2)-(b_1+b_2)^2 \geq 0.$$

This proves the mid convexity of $f+g$. Since both f and g are convex and hence continuous functions inside the defined interval, $f+g$ is also continuous inside the interval and hence by theorem 40 , $f+g$ is convex. This completes the proof. $\square$

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