Modeling Evolution of Spherical Over Densities in Cosmology

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Certificate of Examination

This is to certify that the dissertation titled **Modeling Evolution of Spherical Over densities in Cosmology** submitted by **Manvendra Pratap Rajvanshi** (Reg. No. MS11080) for the partial fulfillment of BS-MS dual degree programme of the Institute, has been examined by the thesis committee duly appointed by the Institute. The committee finds the work done by the candidate satisfactory and recommends that the report be accepted.

Dr. Manimala Mitra	Dr. Harvinder K. Jassal	Prof. Jasjeet S. Bagla (Supervisor)

Dated: April 22, 2016

Declaration

The work presented in this dissertation has been carried out by me under the guidance of Prof. Jasjeet Singh Bagla at the Indian Institute of Science Education and Research Mohali.

This work has not been submitted in part or in full for a degree, a diploma, or a fellowship to any other university or institute. Whenever contributions of others are involved, every effort is made to indicate this clearly, with due acknowledgement of collaborative research and discussions. This thesis is a bonafide record of original work done by me and all sources listed within have been detailed in the bibliography.

Manvendra Pratap Rajvanshi
(Candidate)

Dated: April 22, 2016

In my capacity as the supervisor of the candidates project work, I certify that the above statements by the candidate are true to the best of my knowledge.

Prof. Jasjeet Singh Bagla
(Supervisor)

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List of Figures

Abbreviations

ΛCDM	Cosmological Constant Cold Dark Matter
LTB	Lemaitre-Tolman-Bondi
FLRW	Friedmann Lemaitre Robertson Walker
GUT	Grand Unified Theories
GR	General Relativity

Abstract

The Universe that we see around is full of inhomogeneities. These inhomogeneities arise in a homogeneous universe via gravitational instability. Growth and collapse of perturbations in a homogeneous universe are studied to get insights into nature of dark energy, which can significantly affect structure formation. In this project we study growth and collapse of spherical density perturbations in different dark energy models. The two models of dark energy that we consider in this work are: Cosmological Constant and Quintessence.

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Chapter 1

Introduction

1.1 Motivation

Growth of large scale structure in universe is studied not only to get insights into formation of large scale structures like galaxy,galaxy clusters,etc but also to put constrains on cosmological theories. As a result, simplified models of growth & collapse of over-dense regions are made and studied in different cosmological settings. In this project we started by studying non-linear growth of spherical over densities in the Newtonian limit, studied growth of over densities using non-homogeneous metric called Lemaitre,Tolman & Bondi (TLB) metric with background having FRLW cosmologies and finally simulated isotropic perturbation growth with Quintessence models of dark energy. The growth & collapse of over densities can be affected by nature of Dark energy, so such studies are useful in exploring aspects of Dark energy.

1.2 A Brief Introduction to Cosmology

The quest for understanding dynamics of universe has been there in some form or other ever since the beginning of human intellect. The discovery of Kepler's law and Newton's theory of Gravitation allowed development of models based on concrete mathematical laws rather than those based on philosophical concepts. Newtonian Cosmology itself has been plagued with presumptions of steady state universe and several paradoxes. Any dynamics on such large scale has to be governed by a long range force like gravity, so cosmology is intimately dependent on the theory of gravity. When Einstein came up with his theory of General Relativity, he derived cosmological consequences of his theory. Preoccupied with the notion of a steady universe, he tweaked his original theory a bit by introducing a constant now called "Cosmological Constant" and thus was able to realize a steady universe cosmology based on GR. However, observational work carried by Slipher and Hubble gave strong evidence that universe is indeed expanding and hence ruled out any need for forcing the idea of a steady state universe in theoretical models. This took away the justification for a cosmological constant (Einstein later remarked cosmological constant as his "biggest blunder"). So one might have thought that things were then settled with respect to correspondence of theory of gravity with cosmology. But Nature never runs out of surprises !

1.2.1 Dark Matter

In 1930s Oort reported that the observed matter around the Sun ran short of explaining vertical oscillations of stars in Milky Way disk. It's called Oort discrepancy[7]. Oort's results were quickly followed by Zwicky's[8] interesting observation that observed velocity dispersions of galaxies in a galaxy cluster were too high for them to remain bound. While in Oort's discrepancy the required quantity of matter to explain the motions was roughly twice the observed mass, in case of Zwicky the required mass was 100 times larger than that observed. This discrepancy kept popping up again and again in various observations and analysis. Some other phenomena that require extra mass than observed mass are (this list is far from being exhaustive):

- Stability of galactic disks require unseen mass.[9]
- Flat rotation curves of spiral galaxies[10][11][16]. For objects moving in roughly circular orbit under Newton's law for gravity for a concentrated mass distribution the velocities of outermost stars are proportional to $1/\sqrt{R}$ and the rotation curve should fall as $1/\sqrt{R}$, but observed velocity rotation curves for spiral galaxies remain roughly flat with increasing radius.
- Observed temperature of gas in galaxy clusters is too high for observed mass to hold the gas.[12][20]

Above observations along with numerous others force us to consider either modifications of basic dynamical theories or the existence of a mysterious form of matter, termed "Dark Matter", which interacts primarily through gravity.

1.2.2 Dark Energy:Return of Cosmological Constant

In the late 1990s, observations[14][15] of Supernovae of type Ia at $z \sim 1$ provided evidence that the universe is not only expanding but the expansion rate is accelerating and it was understood that such an accelerated expansion can be nicely explained by keeping cosmological constant in Einstein's equations. Cosmological constant included in Einstein's equations along with hypothetical dark matter form the concordance model of cosmology called ΛCDM. Even before the accelerated expansion had it's observational verification, a number of theoretical results suggested that ΛCDM is needed for explaining many phenomena [19][17][18] . ΛCDM model was used by Ostriker and Steinhardt[13] to make for energy density(parameterΩ) reach the value required for spatially flat universe($\Omega = 1$) in coherence with inflation. It also resolved the contradictions between age of universe estimated from observed Hubble parameter and age predicted from independent methods. Physically cosmological constant corresponds to an energy density that is not diluted by expansion of universe. Current observational status is that roughly 68% of net content of universe is in form of dark energy, 27% in form of dark matter and approximately 5% is contributed by ordinary matter.

There are basically two approaches one can take in explaining accelerated expansion; hypothesis existence of a dark energy or modify Einstein's theory of gravity. Even in first approach there are several distinct lines of attack. Though the model that best fits the available data is ΛCDM but there is still enough motivation to look for alternatives like dynamic dark energy models e.g. quintessence,k-essence,Chaplygin gas,etc(see [5] and references within).

Chapter 2

Spherical Inhomogeneities in Flat FRLW universe

2.1 Introduction

The idea of isotropic and homogeneous universe is a prevalent notion in theoretical cosmology in the form of "Cosmological Principle". The model owes its popularity to its mathematical simplicity.
For an isotropic and homogeneous universe all points in space are equivalent in terms of metric at any particular moment of time.
The metric in (r, θ, ϕ, t) coordinates takes following form:

$$ds^2 = -\left(\frac{a^2}{1 - \kappa r^r}\right) dr^2 - a^2 r^2 (d\theta^2 + sin^2\theta d\phi^2) + dt^2 \quad (2.1)$$

where κ is constant representing curvature which is same everywhere in space.
One can get $a(t)$ by solving Einstein's Equations. Cosmologies governed by such homogeneous and isotropic metrics are collectively called Friedmann-Robertson-Lemaitre-Walker(FRLW) models.

2.2 Friedmann Equations for FRLW in ΛCDM cosmology

Using Einstein's equation:

$$G_{\mu\nu} - \Lambda g_{\mu\nu} = \frac{8\pi G T_{\mu\nu}}{c^4} \quad (2.2)$$

for the metric (2.1), we obtain following equations called Friedmann equations:

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho + \frac{p}{c^2}\right) + \frac{\Lambda c^2}{3} \tag{2.3}$$

$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{\kappa c^2}{a^2} = \frac{8\pi G}{3}\rho + \frac{\Lambda c^2}{3} \tag{2.4}$$

From observational evidence, κ is taken to be 0. With κ set to 0 in above equations we get dynamical equations for the flat FRLW universe. Structure formation and gravitational collapse in this particular model(flat FRLW) have been studied extensively. In this chapter we discuss dynamics of an spherical density perturbation (overs density) in a universe which has zero curvature i.e. its flat FRLW. The derivations shown here were originally carried out by Barrow and Saich in their article [2] in 1993.

2.3 Dynamical Equation

A spherical region with an over density can be considered as an isolated closed universe embedded in a flat universe. So if we assume that density in over dense region is σ then the Friedmann equation for dynamics of such a shell can be written as(with c=1):

$$\left(\frac{dR}{dt}\right) = \frac{8\pi G\sigma R^2}{3} + \frac{\Lambda R^2}{3} - 1 \tag{2.5}$$

While the background equation for evolution of scale factor is

$$\left(\frac{da}{dt}\right)^2 = \frac{8\pi G\rho a^2}{3} + \frac{\Lambda a^2}{3} \tag{2.6}$$

To solve this equation we have to specify initial conditions which we choose as following:

$$at \quad t = t_i \quad \text{we set:}$$
$$R_i = a_i$$
$$\dot{R}_i = \dot{a}_i$$

Second initial condition allows us to equate (2.5) and (2.6) at time $t = t_i$ and hence obtain following equation for shell dynamics

$$3\dot{R}^2 \;=\; \frac{\Lambda R^3 + 8\pi G\sigma_i R_i^3 - 8\pi G\rho_i\Delta_i R_i^2 R}{R} \tag{2.7}$$

where $\sigma_i \;=\; \rho_i(1+\Delta_i)$, Δ_i being initial density contrast.
At any time t the density contrast can be obtained from following relation:

$$(1+\Delta) \;=\; (1+\Delta_i)\left(\frac{a}{R}\right)^3 \tag{2.8}$$

The background equation has an anlytical solution:

$$a^3 = \frac{8\pi G\rho_i a_i^3}{\Lambda} sinh^2\left(\frac{t\sqrt{3\Lambda}}{2}\right) \tag{2.9}$$

2.4 Turn Around Radius and Critical Values for Turn Around

One can see from eqn.(2.7) that with the given initial conditions, the velocity (dR/dt) values are positive initially and the radius of the shell grows. The velocity may or may not become 0 at some point in time depending upon the competition between dark energy push away and gravitational attraction. Setting $\frac{dR}{dt} = 0$ in eqn.(2.7) we obtain expression for turn around radius by solving the equation obtained

$$R_T \;=\; \frac{3(1+\Delta_i)}{\Delta_i} R_i \left[\left(\frac{\Lambda_c}{\Lambda}\right)^{1/2} sin\left(\frac{1}{3} arcsin\left(\frac{\Lambda}{\Lambda_c}\right)^{1/2}\right)\right] \tag{2.10}$$

where

$$\frac{\Lambda}{\Lambda_c} = \frac{27\bar{\Omega}_\Lambda(1+\delta_i)^2 a_i^3}{4\bar{\Omega}_{nr}(\delta_i a_0)^3} \tag{2.11}$$

Above equation has real solutions only if $\Lambda \leq \Lambda_c$. So for a particular value of Λ , there is a lower cap on over density, anything which has over density less than a particular threshold can't collapse. This can be used as a test for validity of ΛCDM model. One can look for observational contradiction of phenomenon of critical over density. For details see [3].

We solved the eqn.(2.7) numerically and reproduced few graphical results from Barrow & Saich [2] including following graph.

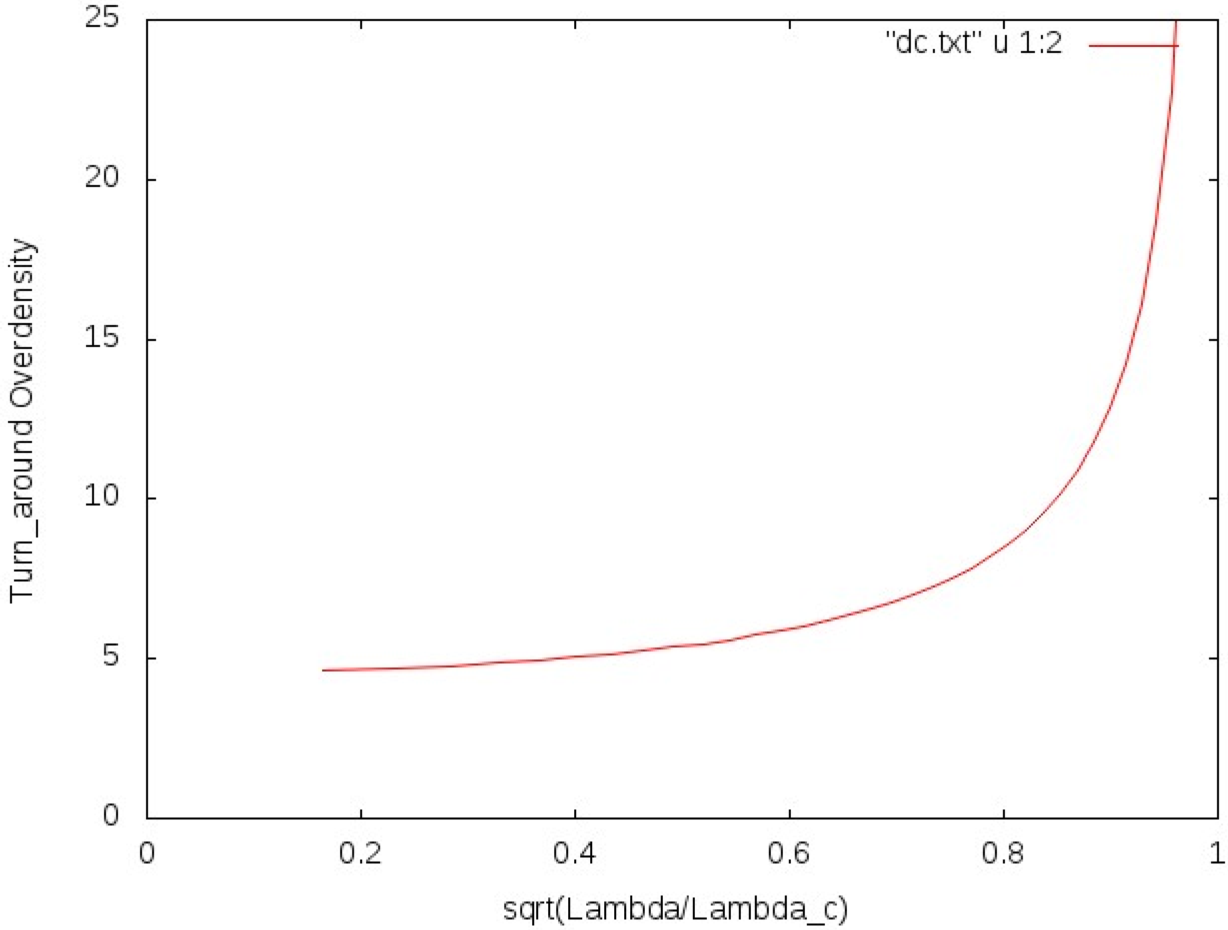

Figure 2.1: Λ/Λ_c vs δ_{turn} in ΛCDM
Here one can see that as $\Lambda \to \Lambda_c$, $\delta_{turn} \to$ inf.

In next graph we show evolution of scale factor for over density versus scale factor of background. One can see that over densities having less than a critical radius collapse while others don't.

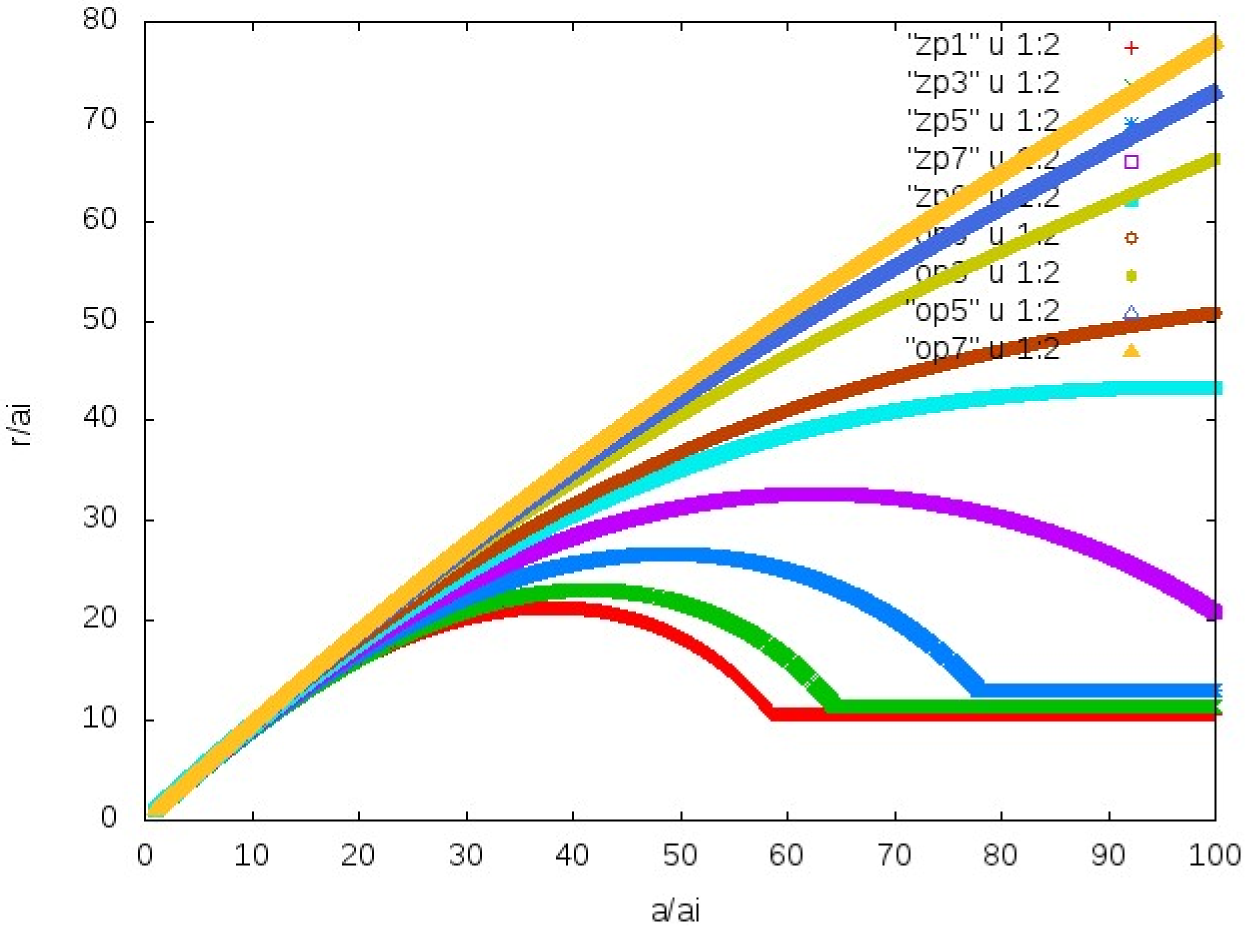

Figure 2.2: R vs a for different initial over densities
Here "zp1" means initial radius is 0.1 times critical radius while op1 stands for 1.1 times critical radius and so on. Initial radius values are between 0.1 and 1.7 times critical radius.

2.5 Virialization

Mathematical solutions are oscillatory in nature and the perturbation should shrink to singularity. But this does not happen in real physical system which we are trying to model. The system virializes at a particular radius where the average kinetic and potential energies satisfy an unique relationship stated by Virial theorem:

$$2T_{virial} = W_{virial} \tag{2.12}$$

where T is kinetic energy and W is potential energy. When we have Λ, then contribution from Λ is also considered and we have following relationship [21]
At virial radius

$$2T + W - 2W_{\Lambda} = 0 \tag{2.13}$$

We use this relationship to determine virial radius in ΛCDM model.

Chapter 3

Lemaitre Tolman Bondi Metric

Note: $c = 1$ in this chapter.

3.1 Introduction

Despite it's tremendous popularity and simplicity, the idea of homogeneous universe has been susceptible to investigation as well as doubt mainly due to following reasons:-
1) No prior physical/logical reason to assume homogeneity.
2) Observations show existence of inhomogeneities at scales of up to 100 Mpc
Work in this direction was pioneered by Lemaitre (1933), Tolman (1934), Bondi (1947), etc. In this project we use spherically symmetric LTB class of metrics to study isotropic perturbation growth in completely general relativistic regime with all non-linearity considered. Here we start from Tolmans formulation [1]

3.2 General form and Einstein's Equations

One starts with a general form of metric which assumes isotropy but does away the restriction of homogeneity. As a consequence the metric coefficients are only dependent on radial coordinate r and time t. The metric can be written in the following form (as prescribed by Tolman [1]):

$$ds^2 = -e^{\lambda}dr^2 - e^{\omega}(d\theta^2 + sin^2\theta d\phi) + dt^2 \tag{3.1}$$

where λ and ω are functions of t and r. We use this metric in Einstein equation:

$$G_{\mu\nu} = 8\pi G T_{\mu\nu} \tag{3.2}$$

to obtain following equations

$$8\pi G T_1^1 = e^{-\omega} - e^{-\lambda}\frac{\omega'^2}{4} + \ddot{\omega} + \frac{3}{4}\dot{\omega}^2 - \Lambda = 0 \tag{3.3}$$

$$8\pi G T_2^2 = 8\pi G T_3^3 = -e^{-\lambda}\left(\frac{\omega''}{2} + \frac{\omega'^2}{4} - \frac{\lambda'\omega'}{4}\right) + \frac{\ddot{\lambda}}{4} + \frac{\dot{\lambda}^2}{4} + \frac{\ddot{\omega}}{2} + \frac{\dot{\omega}^2}{4} + \frac{\dot{\lambda}\dot{\omega}}{4} - \Lambda = 0 \tag{3.4}$$

$$8\pi G T_0^0 = e^{-\omega} - e^{-\lambda}\left(\omega'' + \frac{3}{4}\omega'^2 - \frac{\lambda'\omega'}{2}\right) + \frac{\dot{\omega}^2}{4} + \frac{\dot{\lambda}\dot{\omega}}{2} - \Lambda = 8\pi G\rho \tag{3.5}$$

$$8\pi G e^{\lambda} T_0^1 = -8\pi G T_1^0 = \frac{\omega'\dot{\omega}}{2} - \frac{\dot{\lambda}\omega'}{2} + \dot{\omega}' = 0 \tag{3.6}$$

where $T_{\mu\nu}$ is taken is comoving in coordinates i.e.

$$\mathbf{T} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \rho \end{pmatrix} \tag{3.7}$$

3.3 Equations to Solve

In order to proceed towards solving these equation, one needs to put them in a form in which they can be integrated either analytically or numerically. So we start by finding first integrals.
Multiplying eqn.(3.6) by $\frac{e^{\frac{\omega-\lambda}{2}}}{2}$ we get

$$\begin{aligned} \frac{e^{\frac{\omega-\lambda}{2}}}{2}\left(\frac{\omega'\dot{\omega}}{2} - \frac{\dot{\lambda}\omega'}{2} + \dot{\omega}'\right) &= 0 \\ \Rightarrow \quad \frac{\partial}{\partial t}\left(\frac{e^{\frac{\omega-\lambda}{2}}}{2}\omega'\right) &= 0 \\ \Rightarrow \quad \frac{e^{\frac{\omega}{2}}\omega'}{2e^{\frac{\lambda}{2}}} &= f(r) \qquad \text{(is a first integral)} \end{aligned}$$

$$\Rightarrow \quad e^{\lambda} = \frac{e^{\omega}\omega'^2}{4f^2(r)} \tag{3.8}$$

Putting eqn.(3.8) into (3.1) we get for metric

$$ds^2 = -\left(\frac{e^{\omega}\omega'^2}{4f^2(r)}\right)dr^2 - e^{\omega}(d\theta^2 + sin^2\theta d\phi) + dt^2 \tag{3.9}$$

Substituting eqn.(3.8) into (3.3), we get

$$f^2(r) \;=\; e^{\omega}\left[\ddot{\omega} + \frac{3}{4}\dot{\omega}^2 - \Lambda\right] + 1 \tag{3.10}$$

Multiplying eqn. (3.10) by $\dot{\omega}e^{\omega/2}$, we obtain

$$\begin{aligned}
&\dot{\omega}\ddot{\omega}e^{3\omega/2} + \frac{3}{4}\dot{\omega}^3e^{3\omega/2} + \dot{\omega}e^{3\omega/2}\Lambda + \dot{\omega}e^{\omega/2}(1 - f^2(r)) \;=\; 0\\
\Rightarrow\;& \frac{\partial}{\partial t}\left(e^{3\omega/2}\left(\frac{\dot{\omega}^2}{2} + \frac{2}{3}\Lambda\right) + 2e^{\omega/2}(1 - f^2)\right) \;=\; 0\\
\Rightarrow\;& e^{3\omega/2}\left(\frac{\dot{\omega}^2}{2} + \frac{2}{3}\Lambda\right) + 2e^{\omega/2}(1 - f^2) \;=\; F(r) \qquad \text{(another first integral)}
\end{aligned}$$

$$F(r) \;=\; e^{3\omega/2}\left[\frac{\dot{\omega}^2}{2} - \frac{2}{3}\Lambda\right] + 2e^{\omega/2}[1 - f^2] \tag{3.11}$$

Rearranging, we get

$$\frac{\partial\omega}{\partial t} \;=\; 2\left[e^{-3\omega/2}\frac{F}{2} - e^{-\omega}(1 - f^2) + \frac{\Lambda}{3}\right]^{\frac{1}{2}} \tag{3.12}$$

Integrating this leads to:

$$\int \frac{de^{\omega/2}}{\sqrt{f^2 - 1 + \frac{1}{2}Fe^{-\omega/2} + \frac{\Lambda}{3}e^{\omega}}} \;=\; t + \xi(r) \tag{3.13}$$

where $\xi(r)$ is another integral of motion obtained by integrating LHS of eqn.(3.13) at $t = 0$. Using eqn.3.8 with eqn.3.5 we have

$$8\pi G\rho \;=\; e^{-\omega}\left(1 - f^2 - \frac{4ff'}{\omega'}\right) + \frac{3}{4}\dot{\omega}^2 + \frac{\dot{\omega}\dot{\omega}'}{\omega'} - \Lambda \tag{3.14}$$

Using definition of f from eqn.(3.10) in (3.14) one can show that

$$8\pi G\rho \;=\; -3\ddot{\omega} - 2\frac{\ddot{\omega}'}{\omega'} - \frac{3}{2}\dot{\omega}^2 - 2\frac{\dot{\omega}\dot{\omega}'}{\omega'} + 2\Lambda \tag{3.15}$$

Bringing F from eqn. (3.11) into eqn.(3.14) we find

$$8\pi G\rho \;=\; \frac{e^{-3\omega/2}}{\omega'}\frac{\partial F}{\partial r} \tag{3.16}$$

3.4 Approach towards solving these equations

By stating $\omega(r)$, $\dot{\omega}(r)$ and $\ddot{\omega}(r)$ at initial time t_i, one can obtain $f^2(r)$, $F(r)$ and ξ at t_0 using eqns.(3.10), (3.11) and (3.13). Since these first integrals do not change with time eqn.(3.12) or eqn.(3.13) can be used to obtain $\omega(r)$ at any time and hence we can obtain metric, density(using (3.14) or (3.16)) and other relevant quantities at any time.

3.5 Equations in modern parlance

The equations derived can be rewritten in modern notation where in one can easily compare them with established FRLW cases and it is also helpful in setting initial conditions. Defining

$$A(r,t) \;=\; e^{\omega/2} \tag{3.17}$$

$$\kappa(r) = 1 - f^2(r) \tag{3.18}$$

where $A(r,t)$ is function of r and t while $\kappa(r)$ is function of r only.
Hence,

$$\dot{\omega} = \left(\frac{2\dot{A}}{A}\right) \quad ; \qquad \omega' = \left(\frac{2A'}{A}\right)$$

$$\ddot{\omega} = 2\left[\frac{\ddot{A}}{A} - \left(\frac{\dot{A}}{A}\right)^2\right] \quad ; \qquad \omega'' = \left[\frac{A''}{A} - \left(\frac{A''}{A}\right)^2\right]$$

and metric (3.1) becomes

$$ds^2 = -\frac{A'^2}{1-\kappa}dr^2 - A^2(d\theta^2 + sin^2\theta d\phi) + dt^2 \tag{3.19}$$

Now using above equations eqn.(3.11) can be written as

$$\left(\frac{\dot{A}}{A}\right)^2 = \frac{F}{2A^3} + \frac{\Lambda}{3} - \frac{\kappa}{A^2} \tag{3.20}$$

Note that above equation looks like the Friedmann equation for FRLW cosmology with curvature, except for the fact that curvature term as well as density dependent term are now functions of r besides being dependent on t. Generalized scale factor $A(r,t)$ depends on r and t.
Eqn.(3.14) in our new notation is

$$8\pi G\rho = \frac{\kappa + \dot{A}^2}{A^2} + \frac{2\dot{A}\dot{A}' + \kappa'}{AA'} - \Lambda \tag{3.21}$$

Combining eqn.(3.21) with differentiated eqn.(3.20) we get (3.16)

$$8\pi G\rho = \frac{1}{2A'A^2}\frac{\partial F}{\partial r} \tag{3.22}$$

We can define the generalized Hubble parameter $H(r,t)$ as

$$H(r,t) \equiv \left(\frac{\dot{A}}{A}\right)$$

If we set initial conditions at some time $t = t_i$ i.e. if we know $A_i = A(r,t_i)$, $\dot{A}_i = \dot{A}(r,t_i)$ and $F(r)$ then we can define:

$$\Omega_m = \frac{F}{2\dot{A}_i^{\,2} A_i} \qquad and \qquad \Omega_\Lambda = \frac{\Lambda A_i^2}{3\dot{A}_i^{\,2}} = \frac{\Lambda}{3H_i^2} \tag{3.23}$$

then

$$F(r) = H_i^2 \Omega_m A_i^3 \tag{3.24}$$

and from eqn.(3.20)

$$\kappa(r) = H_i^2(\Omega_m + \Omega_\Lambda - 1) * A_i^2 = -\Omega_c H_0^2 A_0^2 \tag{3.25}$$

where $\Omega_c = (1 - \Omega_\Lambda - \Omega_m)$.
Substituting above definitions in eqn.(3.20) we obtain

$$H^2 = H_i^2 \left[\Omega_m \left(\frac{A_i}{A} \right)^3 + \Omega_\Lambda + \Omega_c \left(\frac{A_i}{A} \right)^2 \right] \tag{3.26}$$

Now we have all equations that are needed to solve for a particular model. We can solve them in following manner:
We have the initial conditions $A_i(r)$, $\dot{A}_i(r)$ and density distribution $\rho_i(r)$, then
Step 1:
Obtain F using eqn.(3.22)

$$\frac{F}{2} = 8\pi G \int \rho_i A_i^2 d(A_i) \tag{3.27}$$

Step 2:
Get Ω_m and Ω_Λ from eqn.(3.23) and therefore Ω_c
Step 3:
Using eqn.(3.26) obtain

$$\frac{\partial A}{\partial t} = H_0 \sqrt{\Omega_m \frac{A_i^3}{A} + \Omega_\Lambda A^2 + \Omega_c A_i^2} \tag{3.28}$$

All that remains is to integrate this equation. Note that this partial differential equation can be solved for each r separately like an ordinary differential equation. One can either use numerical techniques or can search for analytical solutions of following integral:

$$\int_{A/A_i}^{1} \frac{dx}{H_i \sqrt{\frac{\Omega_m}{x} + \Omega_\Lambda x^2 + \Omega_c}} = \xi(r) + (t - t_i) \tag{3.29}$$

3.6 Applications

3.6.1 Friedmann Equations as a limiting form

In these cosmological models the universe is isotropic as well as homogeneous. Hence the density distribution is function of only time (t). Evolution is same everywhere in universe. Hence the metric coefficients for dr^2, dt^2 should be independent of all coordinates but time. At some initial time t_0 assuming following initial conditions:

$$\begin{aligned}
\rho(r,t_0) &= \rho_0 && \text{(constant everywhere)}\\
A(r,t_0) = A_0(r) &= a(t_0)r = a_0 r && \text{(a is a function of t only)}\\
\dot{A}(r,t_0) &= \dot{a}(t_0)r = \dot{a}_0 r
\end{aligned}$$

From density at initial time we get $F(r)$ using eqn.(3.27):

$$\frac{F}{2} = \frac{8\pi G\rho_0 A_0^3}{3} \tag{3.30}$$

Hence from eqn.(3.20) at initial time we obtain κ

$$\kappa = a_0^2 r^2 \left[\frac{8\pi G\rho_0}{3} + \frac{\Lambda}{3} - \left(\frac{\dot{a}_0}{a_0}\right)^2\right] \tag{3.31}$$

So κ can be written as a constant multiplied by r^2 i.e.

$$\kappa = \bar{\kappa} r^2$$

$$\bar{\kappa} = a_0 \left[\frac{8\pi G\rho_0}{3} + \frac{\Lambda}{3} - \left(\frac{\dot{a}_0}{a_0}\right)^2\right]$$

If $\frac{8\pi G\rho_0}{3} + \frac{\Lambda}{3} = \left(\frac{\dot{a}_0}{a_0}\right)^2$, then κ vanishes and we get a flat FRLW model.
We get following dynamical equation from eqn.(3.20) and above equations/initial conditions:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G\rho_0}{3} + \frac{\Lambda}{3} - \frac{\bar{\kappa}}{a^2} \tag{3.32}$$

And metric takes following form:

$$ds^2 = -\frac{a^2}{1-\bar{\kappa}r^2}dr^2 - a^2r^2(d\theta^2 + sin^2\theta d\phi) + dt^2$$

This is the familiar FRLW form.

3.6.2 Modeling Spherical Step Over density

Suppose we have a spherically over dense region of radius r_1 centered at origin ($r = 0$) followed by an isotropically under dense region between radius r_1 and r_2 at some initial time t_i. While the rest of universe is flat FRLW universe following dynamics

eqn.(3.32) with $\bar{\kappa} = 0$.
In this subsection we will be using different subscript/superscripts for initial time. Value of any variable X at initial time is represented by X_i while value of X at current time is denoted by X_0; so a variable subscripted/superscripted by 0 is its value at current time, **not** at initial time. Also anything with a bar represent background analogue of that quantity, e.g. $\bar{\rho}_i$ represents background density at initial time.
Now we explicitly specify initial conditions at time $t = t_i$
1) Density $\rho_i(r)$

$$\rho_i(r) = (1 + \delta_i(r))\bar{\rho}_i$$

where

$$\delta_i(r) = \begin{cases} \delta_1 & \text{if} \quad r \leq r_1, \\ -\delta_2 & \text{if} \quad r_1 < r \leq r_2, \\ 0 & \text{if} \quad r > r_2. \end{cases} \tag{3.33}$$

where δ_1 and δ_2 are positive constants and $\bar{\rho}_i$ is background density at initial time.
We further impose the condition that mass deficit in under dense region is compensated in the inner over dense region i.e.

$$\frac{4\pi}{3}\left(r_1^3\bar{\rho}_i(1+\delta_1) - r_1^3\bar{\rho}_i\right) = \frac{4\pi}{3}\left((r_2^3 - r_1^3)\bar{\rho}_i - (r_2^3 - r_1^3)\bar{\rho}_i(1-\delta_2)\right)$$

$$\Rightarrow \frac{\delta_1}{\delta_2} = \left(\frac{r_2}{r_1}\right)^3 - 1$$

For our case we take $\delta_1 = \delta_2 = \Delta_i$ and hence $r_2 = 2^{1/3}r_1$ and density becomes

$$\rho_i(r) = \begin{cases} \bar{\rho}_i(1+\Delta_i) & \text{if} \quad r \leq r_1, \\ \bar{\rho}_i(1-\Delta_i) & \text{if} \quad r_1 < r \leq r_2, \\ \bar{\rho}_i & \text{if} \quad r > r_2. \end{cases}$$

Also we assume that at initial time

$$A_i(r) = a_i r$$
$$\dot{A}_i = \dot{a}_i r \quad or \quad equivalently \quad H_i = \bar{H}_i$$

Using above initial settings, we obtain $F(r)$ using eqn.(3.27)

$$\frac{F}{2} = \frac{8\pi G a_i^3 \bar{\rho}_i r^3}{3} J_i \tag{3.34}$$

where

$$J_i = \begin{cases} (1+\Delta_i) & \text{if} \quad r \le r_1, \\ \left(1-\Delta_i+2\Delta_i(\frac{r_1}{r})^3\right) & \text{if} \quad r_1 < r \le r_2, \\ 1 & \text{if} \quad r > r_2. \end{cases} \tag{3.35}$$

And we obtain Ω_m at initial time using (3.23)

$$\Omega_m^i = \bar{\Omega}_m^i J_i \tag{3.36}$$

where (for background flat FRLW)

$$\bar{\Omega}_m^i = \frac{8\pi G\bar{\rho}_i}{3\bar{H}_i^2}$$

Using eqn.(3.26) we get

$$H^2 = H_i^2\left[\Omega_m^i\left(\frac{A_0}{A}\right)^3 + \Omega_\Lambda^i + \Omega_c^i\left(\frac{A_0}{A}\right)^2\right] \tag{3.37}$$

Since we have some knowledge about current background density and other background parameters, it is better to translate initial conditions into current background conditions. Substituting the values for Ω_m^i,Ω_Λ^i and A_i and using background density evolution relation $\bar{\rho}_0 a_0^3 = \bar{\rho}_i a_i^3$, we have

$$H^2 = \frac{8\pi G\bar{\rho}_0 r^3 J_i}{3A^3} + \frac{\Lambda}{3} - \frac{\kappa}{A^2} \tag{3.38}$$

$$\Rightarrow H^2 = \bar{H}_0^2\left[\frac{\bar{\Omega}_m^0 a_0^3 r^3 J_i}{A^3} + \bar{\Omega}_\Lambda^0 - \frac{\kappa}{A^2}\right] \tag{3.39}$$

where

$$\bar{\Omega}_m^0 = \frac{8\pi G\bar{\rho}_0}{3\bar{H}_0^2}$$

$$\bar{\Omega}_\Lambda^0 = \frac{\Lambda}{3\bar{H}_0^2}$$

Now we have to find κ from initial condition $H_i = \bar{H}_i$ i.e.

$$\frac{\bar{\Omega}_m^0 a_0^3 r^3 J_i}{A_i^3} + \bar{\Omega}_\Lambda^0 - \frac{\kappa}{A_i^2} = \bar{\Omega}_m^0\left(\frac{a_0}{a_i}\right)^3 + \bar{\Omega}_\Lambda^0 \tag{3.40}$$

$$\Rightarrow \quad -\kappa = \frac{\bar{\Omega}_m^0 a_0^3 r^2[1-J_i]}{a_i} \tag{3.41}$$

Substituting this value in (3.39) we get:

$$\left(\frac{\partial A}{\partial t}\right)^2 = \bar{H_0}^2\left[\frac{\bar{\Omega}^0_m a_0^3 r^3 J_i}{A} + \bar{\Omega}^0_\Lambda A^2 + \frac{\bar{\Omega}^0_m a_0^3 r^2(1-J_i)}{a_i}\right] \tag{3.42}$$

Dividing it by the eqn. for evolution of background scale factor

$$\left(\frac{\partial a}{\partial t}\right)^2 = \bar{H_0}^2\left[\frac{\bar{\Omega}^0_m a_0^3}{a} + \bar{\Omega}^0_\Lambda a^2\right]$$

we get

$$\left(\frac{\partial A}{\partial a}\right)^2 = \frac{\left[\frac{\bar{\Omega}^0_m a_0^3 r^3 J_i}{A} + \bar{\Omega}^0_\Lambda A^2 + \frac{\bar{\Omega}^0_m a_0^3 r^2(1-J_i)}{a_i}\right]}{\left[\frac{\bar{\Omega}^0_m a_0^3}{a} + \bar{\Omega}^0_\Lambda a^2\right]} \tag{3.43}$$

We integrate this equation numerically.

3.6.3 Modeling General Isotropic Over density

At any time we assume that the density can be written in following form:

$$\rho = \bar{\rho}(1+\delta)$$

where δ is a function of both time and r. Then using eq. (3.27) and background density evolution equation

$$\frac{F}{2} = \frac{8\pi G\bar{\rho}_0 a_0^3 A^3}{3a^3}\left[1 + \frac{3}{A^3}\int \delta A^2 dA\right]$$

or

$$\frac{F}{2} = \frac{8\pi G\bar{\rho}_0 a_0^3 A^3}{3a^3}J \tag{3.44}$$

where

$$J = \left[1 + \frac{3}{A^3}\int \delta A^2 dA\right] \tag{3.45}$$

It is evaluated by performing the integral at a particular instance of time (usually initial conditions) Now we specify initial conditions:

$$a = a_i \qquad A = A_i = a_i r \qquad \delta = \delta_i(r)$$

we also use $\bar{H}_i = H_i$

Using initial conditions we get for $t = t_i$ or $a = a_i$:

$$\frac{F}{2} = \frac{8\pi G\bar{\rho}_0 a_0^3 r^3}{3} J_i \tag{3.46}$$

where J_i is J evaluated at initial time using initial conditions.

$$J_i = \left[1 + \frac{3}{r^3}\int \delta_i(r) r^2 dr\right] \tag{3.47}$$

Using eqn. (3.20) at initial time:

$$\kappa = -\frac{8\pi G\bar{\rho}_0 a_0^3 r^2}{3a_i}\left[1 - J_i\right] \tag{3.48}$$

We get

$$\left(\frac{\partial A}{\partial t}\right)^2 = \frac{8\pi G\bar{\rho}_0 a_0^3 r^3 J_i}{3A} + \frac{\Lambda A^2}{3} + \frac{8\pi G\bar{\rho}_0 a_0^3 r^2}{3a_i}\left[1 - J_i\right] \tag{3.49}$$

$$\left(\frac{\partial A}{\partial t}\right)^2 = \bar{H}_0^2\left[\frac{\bar{\Omega}_m^0 a_0^3 r^3 J_i}{A} + \bar{\Omega}_\Lambda^0 A^2 + \frac{\bar{\Omega}_m^0 a_0^3 r^2(1 - J_i)}{a_i}\right] \tag{3.50}$$

$$\left(\frac{\partial A}{\partial t}\right)^2 = \bar{H}_0^2\left[\frac{\Omega_m}{A} + \Omega_\Lambda A^2 + \Omega_\kappa^0\right] \tag{3.51}$$

where we have defined

$$\Omega_m = \bar{\Omega}_m^0 a_0^3 r^3 J_i \qquad \Omega_\Lambda = \bar{\Omega}_\Lambda^0 \qquad \Omega_\kappa^0 = \frac{\bar{\Omega}_m^0 a_0^3 r^2(1 - J_i)}{a_i} \tag{3.52}$$

Hence for a particular value of r, we get

$$\left(\frac{dA}{da}\right)^2 = \frac{\left[\frac{\Omega_m}{A} + \Omega_\Lambda A^2 + \Omega_\kappa^0\right]}{\left[\frac{\bar{\Omega}_m^0 a_0^3}{a} + \bar{\Omega}_\Lambda^0 a^2\right]} \tag{3.53}$$

So for specified initial conditions, same equations work for different density contrast profiles, one has to just evaluate J_i using eqn. (3.47). Evolution of density contrast profile can be obtained using eqn.(3.30)

$$(1 + \delta) = \left(\frac{1}{A' A^2} r^2 a^3 (1 + \delta_i)\right) \tag{3.54}$$

3.7 Numerical Results

These results are for over density which has following profile at $z \approx 1000$

$$\delta_i(r) = I \frac{e^{\frac{-r^2}{2\sigma^2}}}{\sqrt{2\pi}\sigma^3} \left[\frac{r^2}{\sigma^2} - 1 \right] \tag{3.55}$$

where I is amplitude.
Here are some of the results from simulations

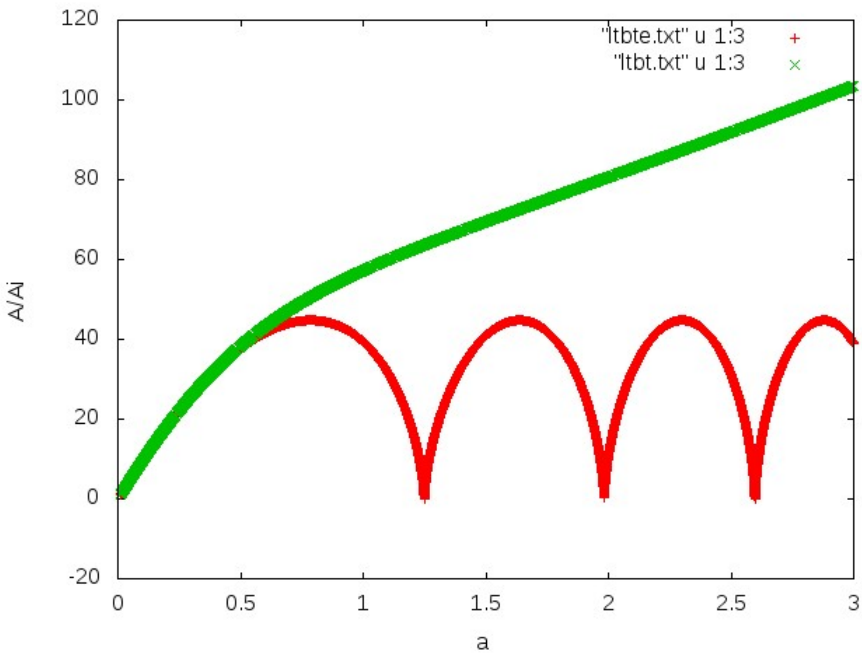

Figure 3.1: R/Ri vs a for two different over densities

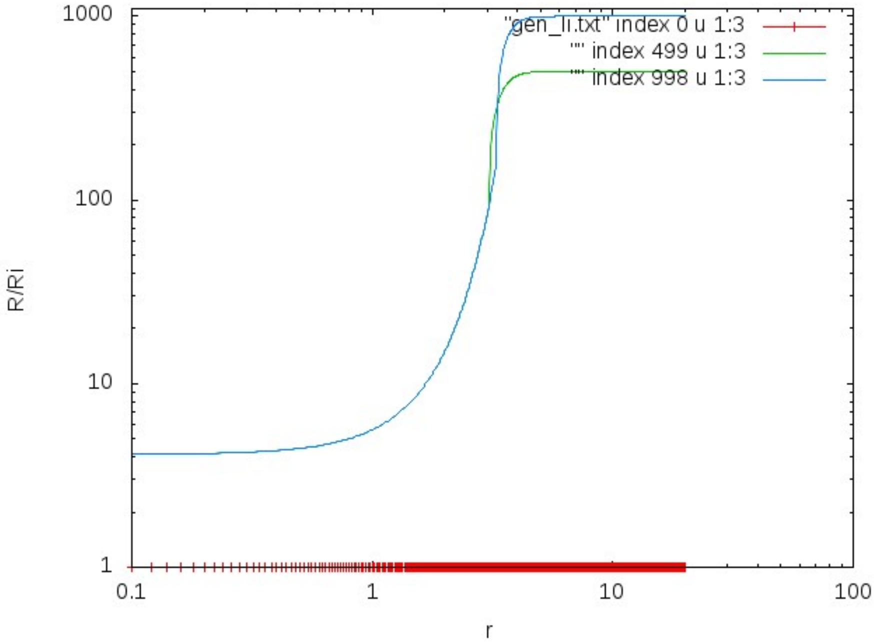

Figure 3.2: Spatial profile R/R_i of over dense region at redshifts near 1000,500 and 0
Note : scale is log-log
Here line marked by 998 gives profile at redshift near 0, that marked by 499 gives profile at redshift near 500 while profile at $z \approx 1000$ is given by line marked by 0.

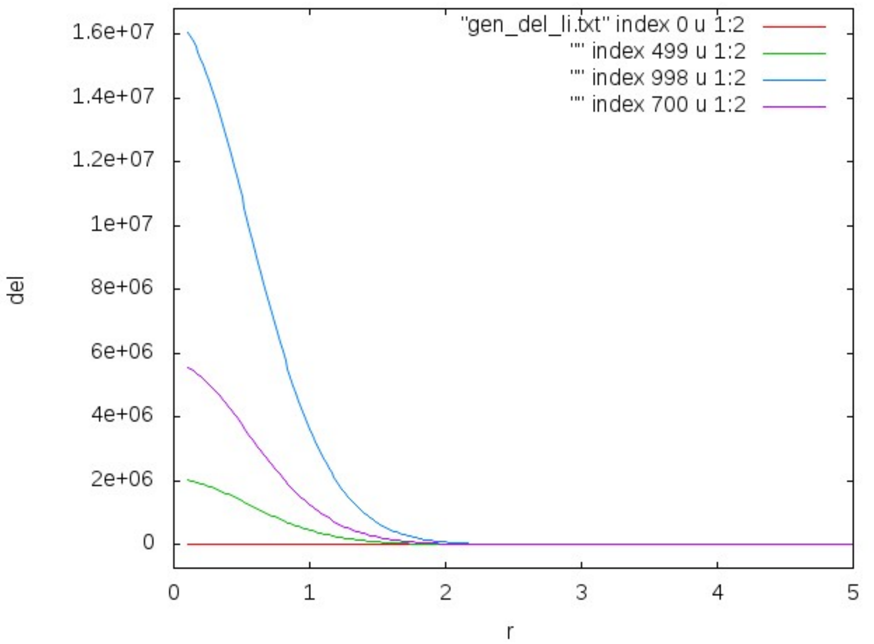

Figure 3.3: Spatial profile of δ at $z \approx 0, 300, 500, 1000$
Here line indexed 700 is at $z \approx (1000 - 700) = 300$ and so on.

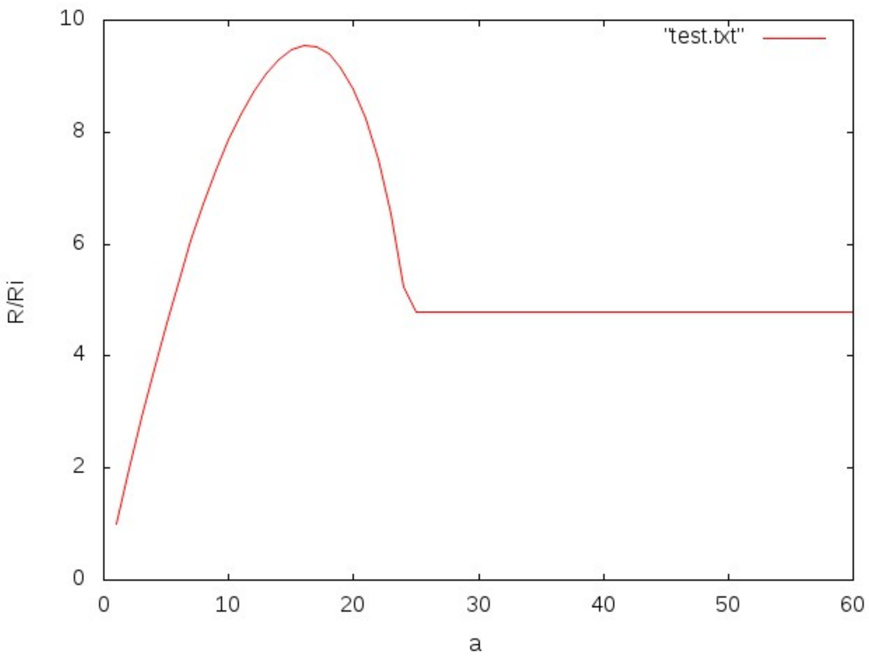

Figure 3.4: Evolution of Radius(R) for point within the over density

Chapter 4

Quintessence

Note: In this chapter dot represent $\frac{\partial}{\partial t}$ and dash/prime represents $\frac{\partial}{\partial r}$

4.1 Motivation for studying alternatives to ΛCDM

Despite its tremendous success w.r.t to observations ΛCDM model has a few challenges. The two biggest problems are:

4.1.1 Fine-Tuning Problem

A Λ energy density indeed has a theoretical basis in Particle Physics in form of vacuum energy of empty space. But the predicted value in GUT models is very different from what is observed.

$$\rho_\Lambda \approx 10^{-47}\, GeV^4$$
$$\rho_{vacuum} \approx 10^{74}\, GeV^4$$

So there is huge discrepancy of factor 10^{121} between the theoretical prediction and observed values. This is called fine-tuning problem and it existed even before discovery of accelerated expansion, but in a slightly different form. At the time it was considered that cosmological constant is zero and one had to explain vanishing of cosmological constant from the context of particle physics theories. Even if some other theory is considered in place of Cosmological constant, it has to explain the vanishing/very small cosmological constant. See [4] for introductory discussion.

4.1.2 Coincidence Problem

Current energy densities for matter and dark energy are of same order. Both energy densities evolve differently and one needs special initial conditions to get the observed energy densities today. Coincidence problem consists of explaining this unique initial ratio between two energy densities. Models like Quintessence try to address this problem via a dynamical approach wherein existence of attractors can allow a larger space of initial conditions converging to a common trajectory.

These two challenges have led people to try alternatives to cosmological constant resulting in a number of models for dark energy: Quintessence,k-essence, Chaplygin gas model,Modified gravity theories,etc(see [5] and references within). Here we study spherical collapse in Quintessence model.

4.2 Quintessence

Quintessence is a scalar field minimally coupled to metric which interacts with other components via gravity only. The action for Quintessence model is:

$$I = \int dx^4 \sqrt{-g} \left\{ \frac{c^3}{16\pi G} R + L_\psi \right\} \tag{4.1}$$

where L_ψ is Lagrangian density for field:

$$L_\psi = \left[\frac{1}{2} g^{\mu\nu} \partial_\mu \psi \partial_\nu \psi \; - \; V(\psi) \right]$$

In FRLW background, one can show that pressure and energy density are:

$$\rho_\psi = \frac{\dot{\psi}^2}{2} + V(\psi) \tag{4.2}$$

$$P_\psi = \frac{\dot{\psi}^2}{2} - V(\psi) \tag{4.3}$$

and equation of state parameter(w_ψ) is given by

$$w_\psi = \frac{P_\psi}{\rho_\psi} = \frac{\dot{\psi}^2 - 2V}{\dot{\psi}^2 + 2V} \tag{4.4}$$

Using usual procedure we can get Einstein's equation for FRLW-Quintessence background:

$$H^2 = \frac{8\pi G}{3}\left[\rho_m + \rho_\psi\right] = \frac{8\pi G}{3}\left[\rho_m + \frac{\dot{\psi}^2}{2} + V(\psi)\right] \tag{4.5}$$

$$\dot{H} = -4\pi G\left(\rho_m + P_m + \dot{\psi}^2\right) \tag{4.6}$$

While Klein-Gordon equation for field dynamics is:

$$\ddot{\psi} + 3H\dot{\psi} + V_{,\psi} = 0 \tag{4.7}$$

4.2.1 Quintessence Models

Tracking Behavior

Some Quintessence models are endowed with behavior that may the resolve coincidence problem. In this class of models called trackers, trajectories from a very large space of initial conditions are attracted to a common path, and hence allow flexibility in initial conditions. The density parameter for field closely "tracks" background fluid(radiation or matter). Mathematical solutions with potential satisfying following conditions(see [5] or [6]) give rise to tracker behavior:

$$\frac{VV_{,\psi\psi}}{V_{,\psi}^2} = \Gamma > 1 \tag{4.8}$$

$$\Omega_\psi = 3(1+w)/\lambda^2 \tag{4.9}$$

where $\lambda = -M_p V_\psi / V$ and $M_p = 1/\sqrt{8\pi G}$

Quintessence models can be roughly classified into two classes based on evolution of w_ψ:

Freezing Models

In this class of models, w_ψ is slowly stabilizing towards -1 i.e. $\dot{w_\psi} < 0$. Example potentials for these models are:

$$V(\psi) = M^{4+n}\psi^{-n} \;\; for\ n > 0 \tag{4.10}$$

$$V(\psi) = M^{4+n}\psi^{-n} exp\left(\alpha\psi^2 G\right) \tag{4.11}$$

Thawing Models

In these models, the field is initially almost frozen by friction term of H in eqn.(4.7). It's only later that w_ψ starts increasing from 1 and hence $\dot{w_\psi} > 0$. Example potentials are:

$$V(\psi) = M^{4-n}\psi^n \quad for\ n > 0 \tag{4.12}$$

$$V(\psi) = M^4 cos^2(\psi/f) \tag{4.13}$$

4.3 Modeling background in Quintessence DE models

We solve background equations (4.5),(4.6) and (4.7) for two potentials: $exp(-\psi)$ and ψ^2 potential. The results:

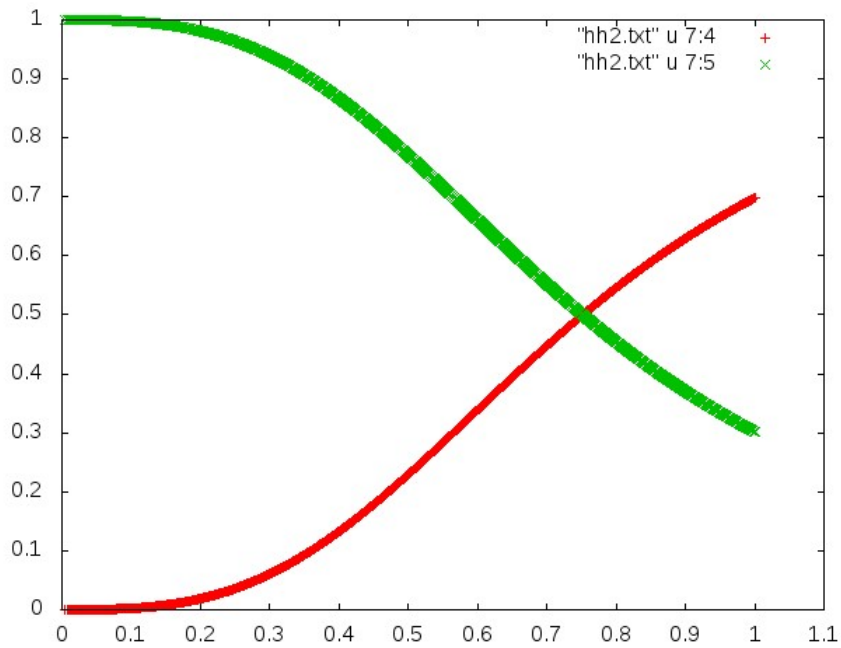

Figure 4.1: The evolution of Ω_m(in green) and Ω_ψ(in red) for ψ^2 potential as function of a

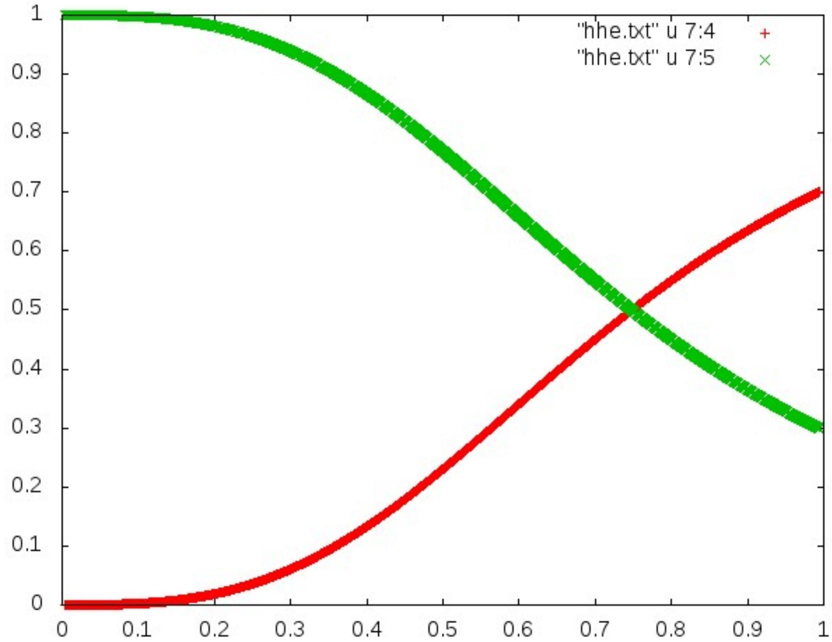

Figure 4.2: The evolution of Ω_m(in green) and Ω_ψ(in red) for $exp(-\psi)$ potential as function of a

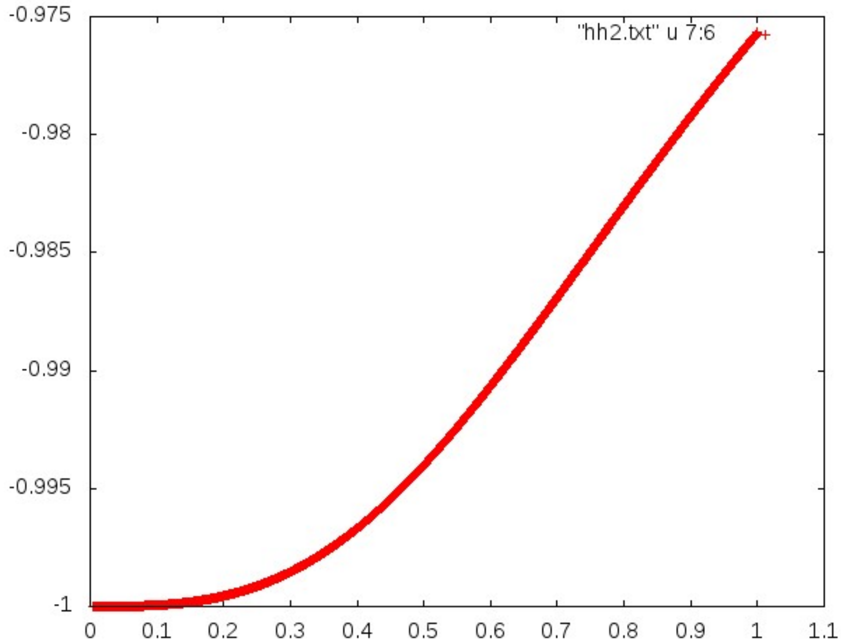

Figure 4.3: The evolution of w v/s a for ψ^2 potential

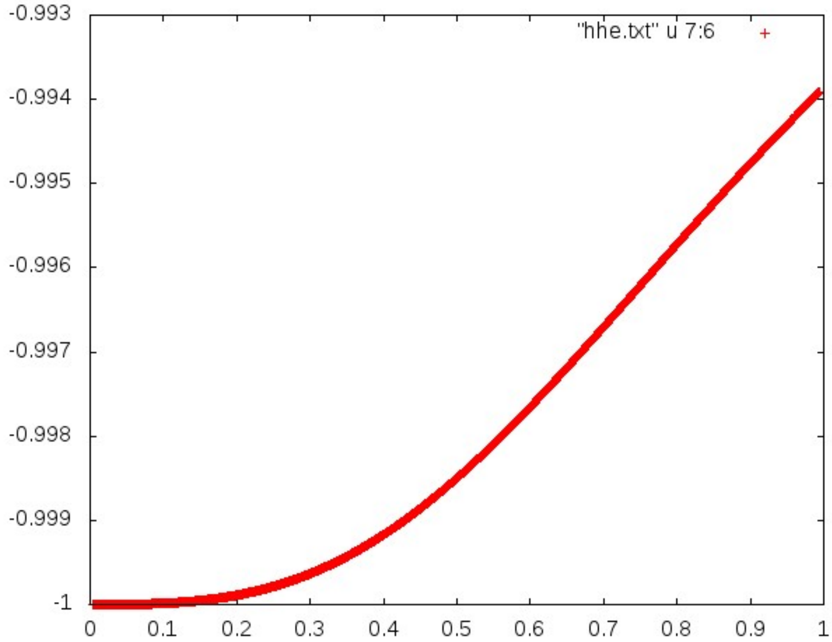

Figure 4.4: The evolution of w v/s a for $exp(-\psi)$ potential

4.4 Spherical Perturbations in Quintessence

4.4.1 Introduction

Our aim in this sections is to model non-linear evolution of matter perturbation with Quintessence models of dark energy. When we have a scalar field in addition to matter there are off diagonal terms in stress-energy tensor and we have to consider the spherically symmetric metric in it's general form and cannot resort to simplified LTB metric (3.9). Hence we don't have the advantage we had because of first integrals and have to solve equations numerically.

4.4.2 General Spherically Symmetric Metric

Since we are going to deal with only isotropic perturbations, we start with a completely general isotropic metric which consists of two undetermined functions $B(t, r)$ and $R(t, r)$ which are functions of both space(r) and time(t). From hereon $B(t, r)$ and $R(t, r)$ are written as B and R with assumption that it is understood that B and R are functions of r and t. The metric in (r, θ, ϕ, t) coordinates has following form:

$$ds^2 = -e^{(2B)}dr^2 - R^2(d\theta^2 + sin^2\theta d\phi^2) + dt^2$$

4.4.3 Action

The action I can be written as sum of action for scalar field and Einstein-Hilbert action (The actual action also has a component for matter density summed up, but here we take care of that term by using the well known Stress-Energy tensor for variation of that term).

$$I = I_{Ein-Hilb} + I_{\psi}$$

Dynamical equations for different variables can be obtained by varying the action w.r.t to that variable, equating it to 0 and thus obtaining Euler-Lagrange equation for that variable.

$$\delta I = \delta I_{Ein-Hilb} + \delta I_{\psi} = 0$$

Equations for Dynamics of Scalar Field ψ

Since Einstein-Hilbert part of action is independent of field coordinates, variation of total action w.r.t field ψ is

$$\delta I = \delta I_{\psi} = 0$$

where

$$I_{\psi} = \int (drd\theta d\phi dt)\sqrt{-g}L_{\psi}$$

and

$$L_{\psi} = [\frac{1}{2}g^{\mu\nu}\partial_{\mu}\psi\partial_{\nu}\psi \; - \; V(\psi)] \tag{4.14}$$

Using the standard procedure for obtaining Euler-Lagrange equations for this Lagrange density, we obtain:

$$\ddot{\psi} = c^2\left[-\frac{\partial V}{\partial\psi} + e^{-2B}\left\{\psi'' - \left(B' - \frac{2R'}{R}\right)\psi'\right\}\right] - \left(\dot{B} + \frac{2\dot{R}}{R}\right)\dot{\psi} \tag{4.15}$$

We will see in upcoming sections that same equation can be obtained by setting four divergence of stress-energy tensor for field to 0.

Stress Energy tensor for scalar field ψ

Equations for unknown metric coefficients can be obtained by varying the action w.r.t to metric coefficients.

$$\delta I = \delta I_{Ein-Hilb} + \delta I_{\psi} = 0$$

$$\delta I_\psi = \int (drd\theta d\phi dt)\delta\left[\sqrt{-g}L_\psi\right]$$

$$= \int (drd\theta d\phi dt)\left[\delta(\sqrt{-g})L_\psi + \sqrt{-g}(\delta L_\psi)\right]$$

It can be shown that

$$\delta(\sqrt{-g}) = -\frac{1}{2}\sqrt{-g}g_{\mu\nu}\delta g^{\mu\nu}$$

then

$$\delta I_\psi = \int (drd\theta d\phi dt)\sqrt{-g}\left[\frac{\partial L_\psi}{\partial g^{\mu\nu}} - \frac{1}{2}L_\psi g_{\mu\nu}\right]\delta g^{\mu\nu}$$

In order to get Einstein's equation in the familiar form, we have to define stress-energy tensor as follows:

$$T_{\mu\nu} = -2c\left[\frac{\partial L_\psi}{\partial g^{\mu\nu}} - \frac{1}{2}L_\psi g_{\mu\nu}\right]$$

Owing to spherical symmetry we get the following non-vanishing components for T^ν_μ:

$$T^\nu_\mu = c\left[\partial_\mu\psi\partial_\nu\psi \;-\; L_\psi g_{\mu\nu}\right]$$

$$T^0_0 = c\left[\frac{\dot\psi^2}{2c^2} + \frac{e^{-2B}\psi'^2}{2} + V\right] \tag{4.16}$$

$$T^1_1 = -c\left[\frac{\dot\psi^2}{2c^2} + \frac{e^{-2B}\psi'^2}{2} - V\right] \tag{4.17}$$

$$T^2_2 = T^3_3 = -c\left[\frac{\dot\psi^2}{2c^2} - \frac{e^{-2B}\psi'^2}{2} - V\right] \tag{4.18}$$

$$T^1_0 = -ce^{-2B}\dot\psi\psi' \tag{4.19}$$

$$T^0_1 = \frac{\dot\psi\psi'}{c} \tag{4.20}$$

Vanishing of four divergence of stress energy tensor gives:

$$T^\mu_0{}_{,\mu} = c\dot\psi\left[e^{-2B}\left(B'\psi' - \psi'' - 2\frac{R'}{R}\psi'\right) + \frac{\dot B\dot\psi}{c^2} + \frac{2\dot\psi\dot R}{Rc^2} + \frac{\ddot\psi}{c^2} + V_{,\psi}\right] = 0$$

$$= c\frac{\psi'}{\dot\psi}T^\mu_1{}_{,\mu} = 0$$

This gives us the Euler-Lagrange equation(4.15) for scalar field dynamics.

Variation of Einstein-Hilbert part and Einstein's Equations

Variation of $I_{Ein-Hilb}$ gives:

$$\delta I_{Ein-Hilb} = \frac{c^3}{16\pi G}\int (drd\theta d\phi dt)\sqrt{-g}\left[R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R_E\right]\delta g^{\mu\nu}$$

where Ricci scalar is represented as R_E to distinguish it from metric coefficient R. Combining this variation with the stress- energy tensor for ψ in previous sub-subsection, we get Einstein's equations

$$G^\mu_\nu = R^\mu_\nu - \frac{1}{2}\delta^\mu_\nu R_E = \frac{8\pi G}{c^4}T^\mu_\nu$$

$\binom{1}{1}$ component

$$\left[\frac{1}{R^2} - e^{-2B}\frac{R'^2}{R^2} + \frac{\dot{R}^2}{c^2R^2} + \frac{2\ddot{R}}{c^2R}\right] = -\frac{8\pi G}{c^3}\left[\frac{\dot{\psi}^2}{2c^2} + \frac{e^{-2B}\psi'^2}{2} - V\right] \tag{4.21}$$

$\binom{2}{2}$ and $\binom{3}{3}$ component

$$e^{-2B}\left[\frac{R'B'}{R} - \frac{R''}{R}\right] + \frac{1}{c^2}\left[\frac{\dot{R}\dot{B}}{R} + \frac{\ddot{R}}{R} + \dot{B}^2 + \ddot{B}\right] = -\frac{8\pi G}{c^3}\left[\frac{\dot{\psi}^2}{2c^2} - \frac{e^{-2B}\psi'^2}{2} - V\right] \tag{4.22}$$

$\binom{0}{0}$ or $\binom{4}{4}$ component

$$-e^{-2B}\left[\left(\frac{R'}{R}\right)^2 - \frac{2R'B'}{R} + \frac{2R''}{R}\right] + \frac{1}{R^2} + \frac{\dot{R}^2}{c^2R^2} + \frac{2\dot{R}\dot{B}}{c^2R} = \frac{8\pi G\rho}{c^2} + \frac{8\pi G}{c^3}\left[\frac{\dot{\psi}^2}{2c^2} + \frac{e^{-2B}\psi'^2}{2} + V\right] \tag{4.23}$$

$\binom{1}{0}$ and $\binom{0}{1}$ components yield same equation

$$R'\dot{B} - \dot{R}' = \frac{4\pi G}{c^3}\dot{\psi}\psi' R \tag{4.24}$$

Combining equations for $\binom{0}{0}$,$\binom{1}{1}$ and $\binom{2}{2}$ components, we obtain:

$$\ddot{B} = \frac{8\pi G}{c}\left[e^{-2B}\psi'^2 + V + \frac{\rho c}{2}\right] + 2e^{-2B}c^2\left[\frac{R''}{R} - \frac{R'B'}{R}\right] - \frac{2\dot{B}\dot{R}}{R} - \dot{B}^2 \tag{4.25}$$

or equivalently we can obtain

$$\ddot{B} = -c^2 e^{-2B}\frac{R'^2}{R^2} + \frac{c^2}{R^2} + \frac{\dot{R}^2}{R^2} - \dot{B}^2 - 4\pi G\rho - \frac{8\pi G}{c}\left[\frac{\dot{\psi}^2}{2c^2} - e^{-2B}\frac{\psi'^2}{2}\right] \tag{4.26}$$

and from $\binom{1}{1}$, we already have eqn.(4.21). Rewriting it again

$$\frac{\ddot{R}}{R} = -\frac{4\pi G}{c}\left[\frac{\dot{\psi}^2}{2c^2} + \frac{e^{-2B}\psi'^2}{2} - V\right] - \frac{1}{2}\frac{\dot{R}^2}{R^2} + \frac{c^2}{2}\left[e^{-2B}\frac{R'^2}{R^2} - \frac{1}{R^2}\right] \tag{4.27}$$

and we have the equation of motions for the scalar field (4.15)

$$\ddot{\psi} = c^2\left[-\frac{\partial V}{\partial \psi} + e^{-2B}\left\{\psi'' - \left(B' - \frac{2R'}{R}\right)\psi'\right\}\right] - \left(\dot{B} + \frac{2\dot{R}}{R}\right)\dot{\psi}$$

Equations to be solved numerically

For dynamics of scalar field we have (4.15)

$$\ddot{\psi} = c^2\left[-\frac{\partial V}{\partial \psi} + e^{-2B}\left\{\psi'' - \left(B' - \frac{2R'}{R}\right)\psi'\right\}\right] - \left(\dot{B} + \frac{2\dot{R}}{R}\right)\dot{\psi}$$

for evolution of R we use eqn.(4.27)

$$\frac{\ddot{R}}{R} = -\frac{4\pi G}{c}\left[\frac{\dot{\psi}^2}{2c^2} + \frac{e^{-2B}\psi'^2}{2} - V\right] - \frac{1}{2}\frac{\dot{R}^2}{R^2} + \frac{c^2}{2}\left[e^{-2B}\frac{R'^2}{R^2} - \frac{1}{R^2}\right]$$

for B we can either use either eqn.(4.25) or eqn.(4.26), but eqn.(4.26) needs less spatial derivative evaluations, hence we use eqn.(4.26) for simulation

$$\ddot{B} = -c^2 e^{-2B}\frac{R'^2}{R^2} + \frac{c^2}{R^2} + \frac{\dot{R}^2}{R^2} - \dot{B}^2 - 4\pi G\rho - \frac{8\pi G}{c}\left[\frac{\dot{\psi}^2}{2c^2} - e^{-2B}\frac{\psi'^2}{2}\right]$$

While for evolution of matter density, we get following equation from four divergence of matter stress-energy tensor

$$\dot{\rho_m} = -\left(\dot{B} + \frac{2\dot{R}}{R}\right)\rho_m \tag{4.28}$$

For convenience, we do a scaling by multiplying all of above equations by $\frac{1}{H_i^2}$ and scaling r and t as follows:

$$
\begin{aligned}
r &\rightarrow rH_i \\
t &\rightarrow tH_i \\
&and\ \ hence \\
R &\rightarrow RH_i
\end{aligned}
$$

Setting up Initial Conditions

Like we did in case of cosmological constant we assume that at initial time t_i, the H parameter for perturbation is same everywhere and is equal to that of background. Using this with initial condition $R_i = a_i r$, we can obtain initial conditions for all variables except ψ and $\dot{\psi}$:

$$B_i = ln(a_i) - \frac{1}{2} ln \left[1 - \frac{3}{r}\frac{\bar{\Omega}_{im} a_i^2}{c^2} \int drr^2 \delta(r)\right] \tag{4.29}$$

$$\dot{B}_i = 1 \tag{4.30}$$

$$\dot{R}_i = R_i = a_i r \tag{4.31}$$

$$R'' = 0 \tag{4.32}$$

For field we start with w=-1;

$$\dot{\psi}_i = 0 \tag{4.33}$$

$$\psi_i = 1 \tag{4.34}$$

4.5 Numerical Results

These results are for over density which has following profile at $z \approx 1000$

$$\delta_i(r) = M \frac{e^{\frac{-r^2}{2\sigma^2}}}{\sqrt{2\pi}\sigma^3} \left[\frac{r^2}{\sigma^2} - 1\right] \tag{4.35}$$

where M is amplitude. The potential we try is

$$V(\psi) = V_0 \psi^2 \tag{4.36}$$

Here for simplicity, we have taken the virialization condition to be as follows:

$$R_{virial} = \frac{R_{max}}{1.8} \tag{4.37}$$

And we also assume that scalar field ψ also virializes and hence we stop the evolution of both metric terms and field terms when we reach virial radius(R_virial). Above stated conditions are considered just to test the code and get a robust program. Once we have a reliable program, it can be used to consider various virialization conditions and also different potentials. For these settings we got following primitive results:

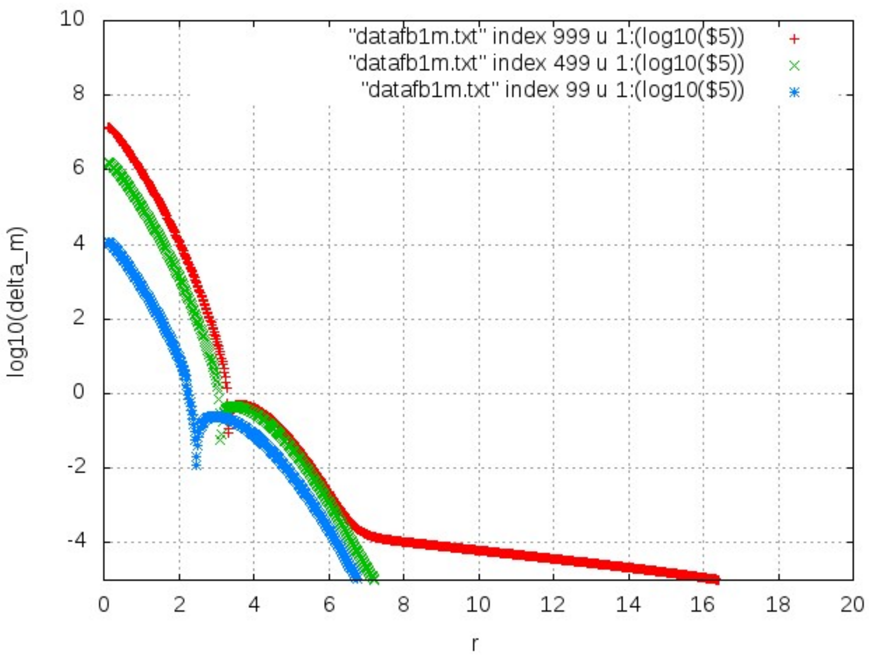

Figure 4.5: Spatial density contrast(matter) for $z \approx 900, 500, 0$ in Quintessence model Here data set indexed as 499 is at $z \approx 1000 - 499 \approx 500$ and so on

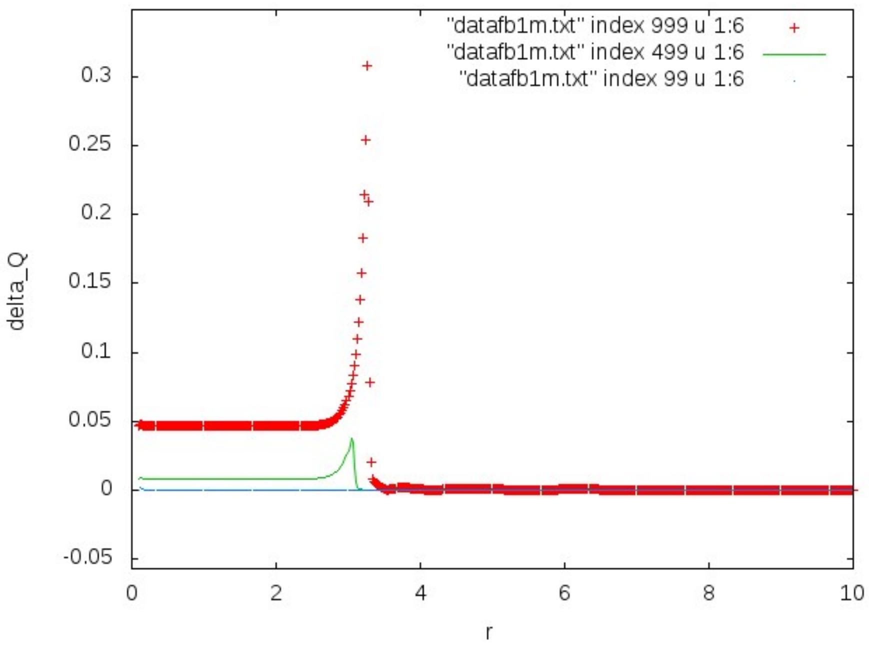

Figure 4.6: Spatial density contrast(field) for $z \approx 900, 500, 0$ in Quintessence model Here data set indexed as 499 is at $z \approx 1000 - 499 \approx 500$ and so on

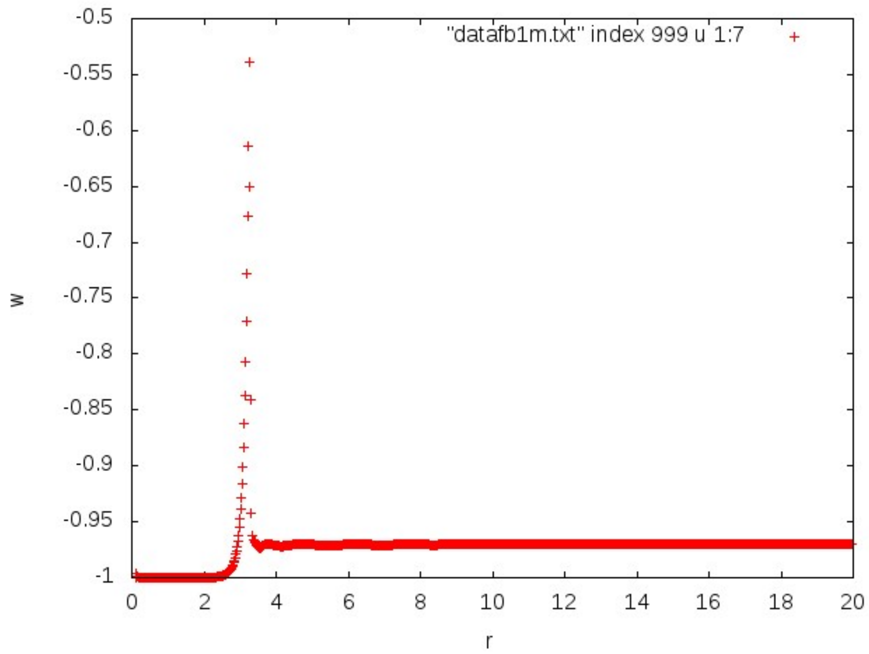

Figure 4.7: Equation of state parameter (w) for field at $z \approx 0$ after evolution from $z \approx 1000$

There is a spatial discontinuity in w and δ_ψ, that appears in these primitive simulations. It is a matter of further investigation if this discontinuity is a numerical artifact or result of imposed virialization condition.

We also solved ΛCDM model using Einstein's equation by replacing field energy density and pressure of scalar field with that of cosmological constant and compared it with results from previous chapters(where we used first integrals). A comparison

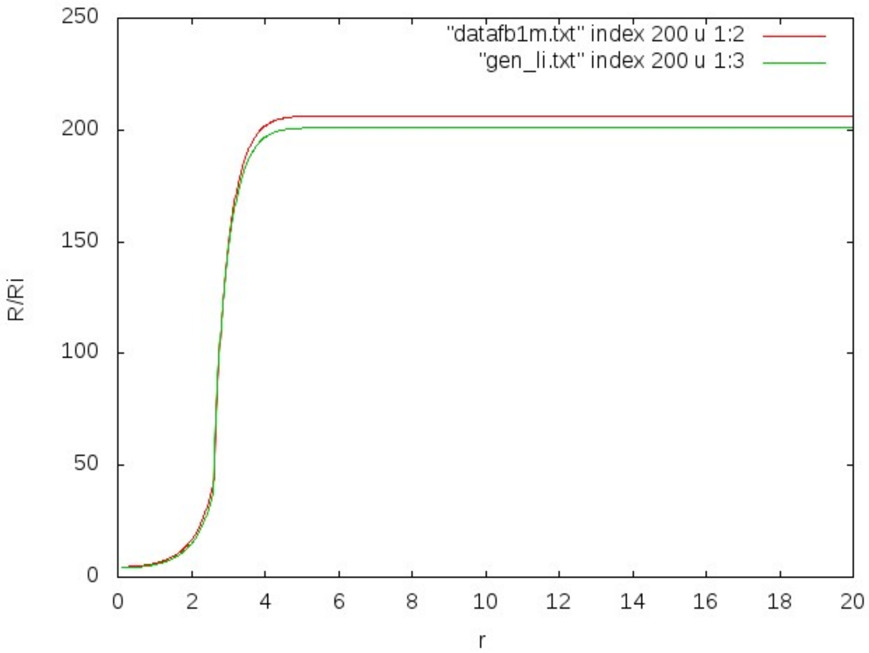

Figure 4.8: R/R_i for ΛCDM after evolving through $z \approx 200$

between ΛCDM solution from previous chapter(green) and solution using equations from this chapter(red). Small discrepancy is there because the initial conditions are not exactly same and also both use different algorithms .

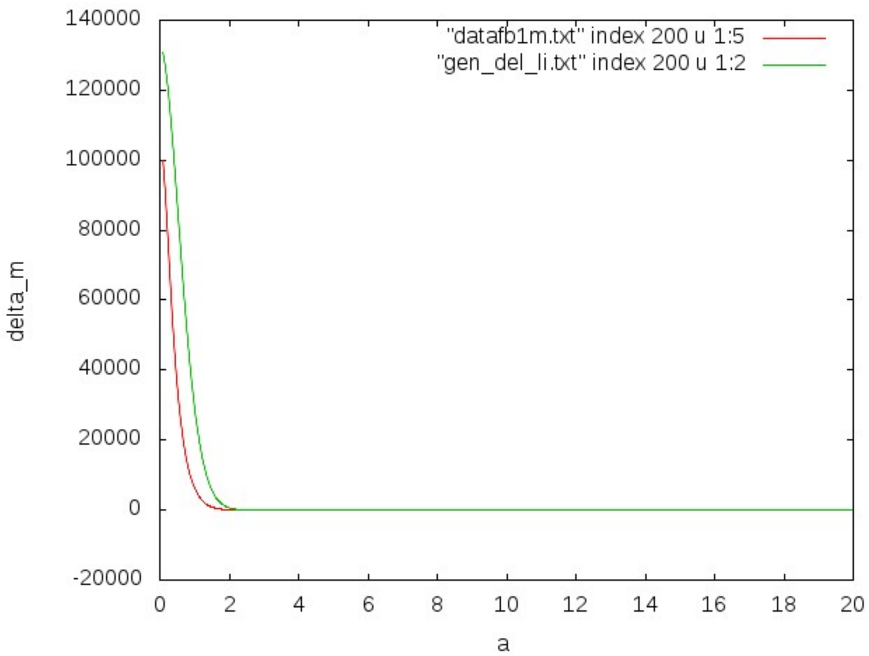

Figure 4.9: $\delta_m(r)$ for direct equations (red) and for equations using first integral(green)

Chapter 5

Summary

During this project, we studied and modeled evolution of spherically over dense regions:

- in Newtonian limit
- in flat FRLW cosmology using LTB metric with and without cosmological constant

In models with Λ there is a lower limit on initial density contrast for collapse to happen.
The aim of the project was to extend the analysis done for Λ model to quintessence models. This constituted the second half of project. Until the submission of this thesis we have had limited success in this aspect and for Quintessence models, We are able to

- model background.
- model spherical perturbation for limited set of initial conditions and virial conditions. More work needs to be done to validate the code and interpret the results.

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