Reversibility of Linear and Affine Transformations

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Dedicated

to

My Parents

Declaration

The work presented in this thesis has been carried out by me under the guidance of Prof. Krishnendu Gongopadhyay at the Indian Institute of Science Education and Research Mohali. This work has not been submitted in part or in full for a degree, a diploma, or a fellowship to any other university or institute. Whenever contributions of others are involved, every effort is made to indicate this clearly, with due acknowledgment of collaborative research and discussions. This thesis is a bonafide record of original work done by me and all sources listed within have been detailed in the bibliography.

Tejbir

In my capacity as the supervisor of the candidate's thesis work, I certify that the above statements by the candidate are true to the best of my knowledge.

Prof. Krishnendu Gongopadhyay

(Supervisor)

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Abstract

Let $\mathbb{D}$ be either $\mathbb{R}$, $\mathbb{C}$, or the real quaternion $\mathbb{H}$. Reversible elements in a group are those elements that are conjugate to their own inverses. Such elements appear naturally in different branches of mathematics. They are closely related to strongly reversible elements, which can be expressed as a product of two involutions. A strongly reversible element in a group is reversible, but the converse is not always true.

Classifying reversible and strongly reversible elements in a group has been a problem of broad interest. My thesis primarily focuses on investigating this problem in the context of the isometry group of Hermitian spaces over $\mathbb{C}$ and $\mathbb{H}$, as well as the general linear groups, the special linear groups, and the affine groups. More precisely, we have classified reversible and strongly reversible elements in the following groups:

1. $\mathrm{Sp}(n) \ltimes \mathbb{H}^n$, $\mathrm{U}(n) \ltimes \mathbb{C}^n$, and $\mathrm{SU}(n) \ltimes \mathbb{C}^n$,
2. $\mathrm{GL}(n, \mathbb{D})$,
3. $\mathrm{GL}(n, \mathbb{D}) \ltimes \mathbb{D}^n$,
4. $\mathrm{SL}(n, \mathbb{C})$ and $\mathrm{SL}(n, \mathbb{H})$.

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Chapter 1

Introduction

The study of reversibility focuses on the elements of a group that are conjugate to their own inverses. Reversible maps naturally appear in various areas, such as group theory, geometry, complex analysis, number theory, approximation theory, and functional equations; see [30] for more details. The origin of reversibility can be traced back to the classical works of Birkhoff, Arnol'd, and others, as seen in the literature such as [3, 1, 28].

The concept of symmetry is well-studied in group theory and dynamical systems. Understanding the symmetries and reversing symmetries (also known as time-reversal symmetries) of a dynamical system is beneficial in gaining information about its dynamics. Considering G as the group of automorphisms of some topological space, an automorphism $f \in G$ is called a *symmetry* of the automorphism $g \in G$, if it conjugates g to itself, i.e., f commutes with g. In group theoretic terms, f is an element in the *centralizer* of g. A *reversing symmetry* of $g \in G$ is an automorphism $f \in G$ which conjugates g to its inverse g^{-1}. The set of symmetries of an element $g \in G$ is always non-empty and forms a group. On the other hand, a priori, the existence of such a reversing symmetry is not apparent, and it depends on the particular choice of g. The set of all symmetries and reversing symmetries of $g \in G$ form a group called the *reversing symmetry* group of g. Further discussion on reversing symmetry groups can be found in [28, 5, 6, 30].

In the classical setup, an element g in a group G is called *reversible* if g has a reversing symmetry, i.e., if g is conjugate to g^{-1} in G. The notion of reversibility is significant in the theory of finite groups due to its close connection with representation theory. A classical theorem of Frobenius and Schur asserts that the number of real-valued complex irreducible characters of a finite group G is equal to the number of reversible conjugacy classes in G; see [30, Theorem 3.19]. Therefore, from an algebraic point of view, such elements are also referred to as *real* elements; see [26, 24, 38, 30]. In the literature, reversible elements are also known as *reciprocal* elements; see [33].

On the other hand, in classical geometries, it has been a problem of geometric interest to express geometric transformations as a product of *reflections*, which are self-reversing (or *involution*) in the sense that they are equal to their inverses. According to the well-known *three reflections theorem*: every isometry of the Euclidean space $\mathbb{R}^2$ can be expressed as a product of at most 3 reflections; see [37]. However, in higher dimensional geometries, there are more involutions (i.e., elements that are equal to their own inverse) that may not be reflections. Generalizing the decomposition problem, one can ask about the minimum number of involutions required to express an element g in a group as a product of involutions. An element g in a group is called *strongly reversible* or *strongly real* if it can be expressed as a product of two involutions. Equivalently, an element $g \in G$ is called *strongly reversible* or *strongly real* if it is conjugate to g^{-1} by an involution in G. Such elements are also known as *bireflectional* in the literature; see [41, 12, 26, 31]. Every strongly reversible element in G is reversible, but a reversible element may not be strongly reversible. For example, in the quaternion group $Q_8 := \{\pm 1, \pm \mathbf{i}, \pm \mathbf{j}, \pm \mathbf{k}\}$, every element is reversible but $\{\pm 1\}$ are the only strongly reversible elements in Q_8. This leads to the question of whether there is an equivalence between reversibility and strong reversibility in a group, which has been investigated in the literature for several groups; see [30]. However, a complete classification of reversible elements is not available in the literature other than a few families of groups.

The investigation of reversibility in groups has been a problem of broad interest. For an elaborate exposition of this theme, we refer to the monograph [30]. In the context of finite groups, Tiep and Zalesski [39] classified all finite simple groups in which all elements are real. The classification of finite simple strongly real groups, where every element of the group is strongly real, is given in [40, Theorem 3]. Several authors have worked on this theme in finite groups, including [39, 27, 38, 40, 22, 23]. Infinite groups where such classifications are known include the general linear groups over algebraically closed fields, the isometry groups of all the constant curvature geometries, the real rank one classical groups and compact Lie groups; see [41, 10, 12, 11, 26, 19, 36, 21, 30, 2].

Let $\mathbb{D}$ be either $\mathbb{R}$, $\mathbb{C}$, or the real quaternion $\mathbb{H}$. This thesis primarily focuses on the study of reversibility in affine groups and linear groups. In particular, we have investigated reversibility problem for the isometry group of Hermitian spaces over $\mathbb{C}$

and $\mathbb{H}$, the general linear groups over $\mathbb{D}$, the automorphism group of the affine space $\mathbb{D}^n$, and the special linear groups over $\mathbb{C}$ and $\mathbb{H}$.

Now we will briefly summarize the results of this thesis in the following sections.

1.1 Reversibility of Hermitian isometries

In [36], Short proved that all elements of the Euclidean isometry group $\mathbb{O}(n) \ltimes \mathbb{R}^n$ are strongly reversible. However, the situation is more intricate for the orientation-preserving Euclidean isometry group $\mathrm{SO}(n) \ltimes \mathbb{R}^n$, and Short also classified the strongly reversible elements in this group.

In this section, we will investigate the reversibility of elements in the isometry group of Hermitian spaces over $\mathbb{C}$ and $\mathbb{H}$. For the rest of this section, we assume that $\mathbb{F}$ is either $\mathbb{C}$ or $\mathbb{H}$.

Before stating the main results, we review the necessary background material. Let $\mathbb{V} := \mathbb{F}^n$ be equipped with the $\mathbb{F}$-Hermitian form

$$\Phi(z,w) := \bar{z}_1 w_1 + \cdots + \bar{z}_n w_n$$

where $z = (z_1, \ldots, z_n)$, $w = (w_1, \ldots, w_n) \in \mathbb{F}^n$. The group of linear transformations g that preserves this form, i.e., $\Phi(gz, gw) = \Phi(z,w)$ for all $z, w \in \mathbb{V}$, is the unitary group $\mathrm{U}(n,\mathbb{F})$. In matrix notation,

$$\mathrm{U}(n,\mathbb{F}) := \{g \in \mathrm{GL}(n,\mathbb{F}) \mid \bar{g}^\top g = g\bar{g}^\top = \mathrm{I}_n\},$$

where $\mathrm{GL}(n,\mathbb{F})$ is the group of invertible $n \times n$ matrices over $\mathbb{F}$ and I_n is the identity matrix of order n. Note that $\mathrm{U}(n,\mathbb{F})$ is a compact subgroup of $\mathrm{GL}(n,\mathbb{F})$. Following usual notation, we write $\mathrm{Sp}(n) := \mathrm{U}(n,\mathbb{H})$, $\mathrm{U}(n) := \mathrm{U}(n,\mathbb{C})$, and $\mathrm{SU}(n) := \{g \in \mathrm{U}(n,\mathbb{C}) : \det(g) = 1\}$. Note that all eigenvalues of $\mathrm{Sp}(n)$ and $\mathrm{U}(n)$ have unit modulus.

The Hermitian form Φ gives a natural metric $d(z,w) = \sqrt{\Phi(z-w, z-w)}$ on $\mathbb{V}$. We call $(\mathbb{V}, d)$, a *Hermitian space*. The group $\mathrm{U}(n,\mathbb{F}) \ltimes \mathbb{F}^n$ acts isometrically on $(\mathbb{V}, d)$ as affine transformations: $T : z \mapsto Az + v$, where $A \in \mathrm{U}(n,\mathbb{F})$ and $v \in \mathbb{F}^n$. This action identifies the isometry group $\mathrm{Isom}(\mathbb{V}, d)$ with $\mathrm{U}(n,\mathbb{F}) \ltimes \mathbb{F}^n$.

We classify the reversible and strongly reversible elements in the group $\mathrm{U}(n,\mathbb{F}) \ltimes \mathbb{F}^n$. The reversibility depends upon the underlying $\mathbb{F}$, and the results in the respective groups are very different. We begin with the group $\mathrm{Sp}(n) \ltimes \mathbb{H}^n$. It is proved that every element in $\mathrm{Sp}(n) \ltimes \mathbb{H}^n$ is reversible.

Theorem 1.1.1. *[14] Let g be an element of $\mathrm{Sp}(n) \ltimes \mathbb{H}^n$. Then g is reversible in $\mathrm{Sp}(n) \ltimes \mathbb{H}^n$.*

However, every element is not strongly reversible in this group, and the following theorem classifies strongly reversible elements in $\mathrm{Sp}(n) \ltimes \mathbb{H}^n$.

Theorem 1.1.2. *[14] Let $g = (A, w)$ be an element of $\mathrm{Sp}(n) \ltimes \mathbb{H}^n$. Then the following are equivalent.*

(1) g is strongly reversible in $\mathrm{Sp}(n) \ltimes \mathbb{H}^n$.

(2) A is strongly reversible in $\mathrm{Sp}(n)$.

(3) Every eigenvalue class of A is either ± 1 or of even multiplicity.

We recall the following definition.

Definition 1.1.3. Let $g \in \mathrm{U}(n)$. The characteristic polynomial $\chi_g(x)$ of g is called *self-dual* if whenever $\lambda \neq \pm 1$ is a root of $\chi_g(x)$, so is λ^{-1} with the same multiplicity.

In the group $\mathrm{U}(n) \ltimes \mathbb{C}^n$, every element is not reversible. The reversibility depends on the self-duality of the linear part, and we prove the following.

Theorem 1.1.4. *[14] Let $g = (A, w)$ be an element of $\mathrm{U}(n) \ltimes \mathbb{C}^n$. Then the following are equivalent.*

(1) g is reversible in $\mathrm{U}(n) \ltimes \mathbb{C}^n$.

(2) g is strongly reversible in $\mathrm{U}(n) \ltimes \mathbb{C}^n$.

(3) A is strongly reversible in $\mathrm{U}(n)$.

(4) The characteristic polynomial of A is self-dual.

We also have the following classification of reversible elements in $\mathrm{SU}(n) \ltimes \mathbb{C}^n$.

Theorem 1.1.5. *[14] Let $g = (A, w)$ be an element of $\mathrm{SU}(n) \ltimes \mathbb{C}^n$, where $A \neq \mathrm{I}_n$. Then the following are equivalent.*

(1) g is reversible in $\mathrm{SU}(n) \ltimes \mathbb{C}^n$.

(2) A is reversible in $\mathrm{SU}(n)$.

(3) The characteristic polynomial of A is self-dual.

Moreover, when $A = \mathrm{I}_n$ and $n \geq 2$, the element $g = (\mathrm{I}_n, w)$ is strongly reversible.

The following corollary follows from the above result.

Corollary 1.1.6. *[14] Let $g = (A, w)$ in $\mathrm{SU}(n) \ltimes \mathbb{C}^n$. Suppose A has an eigenvalue -1. Then the following are equivalent.*

(i) g is strongly reversible in $\mathrm{SU}(n) \ltimes \mathbb{C}^n$.

(ii) A is strongly reversible in $\mathrm{SU}(n)$.

(iii) The characteristic polynomial of A is self-dual.

When -1 is not an eigenvalue of A, the situation is more subtle in $\mathrm{SU}(n) \ltimes \mathbb{C}^n$ and g may not be strongly reversible even if A is so. We classify all such elements in $\mathrm{SU}(n) \ltimes \mathbb{C}^n$, which are not strongly reversible even if the linear part is strongly reversible.

Theorem 1.1.7. *[14] Let $g \in \mathrm{SU}(n) \ltimes \mathbb{C}^n$ be such that linear part of g is strongly reversible. Then g is not strongly reversible if and only if up to conjugacy g is of the form $g = (A, v)$ such that $A = \mathrm{diag}(e^{\mathbf{i}\theta_1}, e^{-\mathbf{i}\theta_1}, \dots, e^{\mathbf{i}\theta_{2r}}, e^{-\mathbf{i}\theta_{2r}}, 1)$ and $v = [0, 0, \dots, 0, v_1]$, where $v_1 \neq 0$, $r \geq 0$, and $\theta_k \in (0, \pi)$, $k = 1, \dots, r$.*

The classification of strongly reversible elements in $\mathrm{SU}(n) \ltimes \mathbb{C}^n$ follows from the above theorem.

Corollary 1.1.8. *[14] An element g in $\mathrm{SU}(n) \ltimes \mathbb{C}^n$ is strongly reversible if and only if the linear part of g is strongly reversible in $\mathrm{SU}(n)$ and g does not belong to the family given in Theorem 1.1.7.*

It should be noted that the classification of reversible and strongly reversible elements in $\mathrm{U}(n, \mathbb{F})$ is closely connected to the classification of reversibility in $\mathrm{U}(n, \mathbb{F}) \ltimes \mathbb{F}^n$. While the complete classifications of reversibility in $\mathrm{U}(n)$ and $\mathrm{SU}(n)$ are well-known and can be found in [21] and [30], the complete classification of strongly reversible elements in $\mathrm{Sp}(n)$ was only recently resolved. The problem of classifying strongly reversible elements in $\mathrm{Sp}(n)$ was posed as an open question in [30, p. 91]. In [2, Theorem 1.2], Bhunia and Gongopadhyay provided a solution to this problem using a geometric approach that relied on the concept of *projective points*. We will revisit this problem and present an alternative proof of the classification of strongly reversible elements in $\mathrm{Sp}(n)$ using only basic quaternionic linear algebra. Our proof may be of independent interest.

1.2 Reversibility in general linear groups

In this section, we will assume $\mathbb{D} = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$. The classification of reversible elements in the general linear groups over $\mathbb{R}$ and $\mathbb{C}$, and their equivalence with strongly reversible elements are well-known in the literature; see [41, 10, 30]. Extending these results to quaternions is not straightforward due to the non-commutativity of quaternions. Unlike the field case, reversible and strongly reversible elements are not equivalent in $\mathrm{GL}(n,\mathbb{H})$ in general. For example, let $g = (\mathbf{i}) \in \mathrm{GL}(1,\mathbb{H})$. Then g is reversible but not strongly reversible in $\mathrm{GL}(1,\mathbb{H})$; see Example 4.4.2

We revisit the reversibility problem in $\mathrm{GL}(n,\mathbb{D})$ by investigating the *reversing symmetry groups* in $\mathrm{GL}(n,\mathbb{D})$. This approach provides a uniform treatment over $\mathbb{D}$ and extends the understanding of reversibility to $\mathrm{GL}(n,\mathbb{H})$. We will classify the reversible elements in $\mathrm{GL}(n,\mathbb{D})$ and give a different proof of the equivalence of these notions in the groups $\mathrm{GL}(n,\mathbb{R})$ and $\mathrm{GL}(n,\mathbb{C})$; see Theorem 4.1.1 and Proposition 4.4.1. We have also proved the following sufficient criterion for the equivalence of reversible and strongly reversible elements in $\mathrm{GL}(n,\mathbb{H})$.

Theorem 1.2.1. *Let $A \in \mathrm{GL}(n,\mathbb{H})$ be an arbitrary reversible element. Suppose that in the Jordan decomposition of A, every Jordan block corresponding to non-real eigenvalue classes of unit modulus has even multiplicity. Then A is strongly reversible in* $\mathrm{GL}(n,\mathbb{H})$.

By using the notion of *Weyr canonical form*, cf. [34, 29, 35], we will show that the converse of Theorem 1.2.1 also holds. Thus, we classify the strongly reversible elements in $\mathrm{GL}(n,\mathbb{H})$; see Remark 7.1.1.

A crucial finding of our approach is the description of the reversing elements. For a group G, the *centralizer and reverser* of an element g of G are respectively defined as

$$\mathcal{Z}_G(g) := \{s \in G \mid sg = gs\}, \text{ and } \mathcal{R}_G(g) := \{r \in G \mid rgr^{-1} = g^{-1}\}.$$

The set $\mathcal{R}_G(g)$ of reversing elements of g is a right coset of the centralizer $\mathcal{Z}_G(g)$ of g. Thus, the reversing symmetry group or extended centralizer $\mathcal{E}_G(g) := \mathcal{Z}_G(g) \cup \mathcal{R}_G(g)$ is a subgroup of G in which $\mathcal{Z}_G(g)$ has index at most 2; see [30, Proposition 2.8]. Therefore, to find the reversing symmetry group $\mathcal{E}_G(g)$ of $g \in G$, it is enough to specify one reversing element of g, which is not in the centralizer. We refer to [5, 6, 20, 30] for an elaborate discussion on reversing symmetry groups.

In the literature, the centralizer for each element of the group $\mathrm{GL}(n,\mathbb{D})$ is well-known; see [17, Theorem 9.1.1, Theorem 12.5.1] and [32, Proposition 5.4.2]. Thus for finding the reversing symmetry group $\mathcal{E}_{\mathrm{GL}(n,\mathbb{D})}(A)$ of an arbitrary reversible element $A \in \mathrm{GL}(n,\mathbb{D})$, it is sufficient to find a reversing element for the Jordan form of A. Using some combinatorial identities, we will describe a reversing element for certain types of Jordan forms in $\mathrm{GL}(n,\mathbb{D})$, which are summarized in Table 1.1.

To state such an explicit description of a reversing element, we need some notations introduced in Chapter 2 and Chapter 4. For $\lambda \in \mathbb{D}$, the Jordan block $\mathrm{J}(\lambda,\, n) \in \mathrm{GL}(n,\mathbb{D})$ is defined in Definition 2.2.2. Similarly, for $\mu, \nu \in \mathbb{R}$, the real Jordan forms $\mathrm{J}_{\mathbb{R}}(\mu \pm \mathbf{i}\nu,\, 2n)$ and $\mathrm{J}_{\mathbb{R}}(\mu \mp \mathbf{i}\nu,\, 2n)$ in $\mathrm{GL}(2n,\mathbb{R})$ are defined in (2.2.1) and (2.2.2), respectively. We refer to Definition 4.2.1 and Definition 4.2.3 for the notations $\Omega(\lambda,n) \in \mathrm{GL}(n,\mathbb{D})$ and $\Omega_{\mathbb{R}}(K,2n) \in \mathrm{GL}(2n,\mathbb{R})$, respectively. Table 1.1 summarizes all the reversing involutions constructed in Section 4.3.

Table 1.1: Reversing element of Jordan forms in $\mathrm{GL}(n,\mathbb{D})$

Sr No.	Group	Jordan form (A)	Reversing element (g)
1	$\mathrm{GL}(n,\mathbb{D})$	$\mathrm{J}(\mu,\, n)$, $\mu \in \{\pm 1\}$.	$\Omega(\mu,n)$
2	$\mathrm{GL}(2n,\mathbb{D})$	$\begin{bmatrix} \mathrm{J}(\lambda,n) & \\ & \mathrm{J}(\lambda^{-1},n) \end{bmatrix}$, $\lambda \in \mathbb{D}\setminus\{\pm 1, 0\}$ for $\mathbb{D}=\mathbb{R},\mathbb{C}$; $\lambda \in \mathbb{C}, \mathrm{Im}(\lambda) \geq 0$, $\lvert\lambda\rvert \neq 1$, for $\mathbb{D}=\mathbb{H}$.	$\begin{bmatrix} & \Omega(\lambda,n) \\ \Omega(\lambda^{-1},n) & \end{bmatrix}$
3	$\mathrm{GL}(n,\mathbb{H})$	$\mathrm{J}(\mu,\, n)$, $\mu \in \mathbb{C}, \mathrm{Im}(\mu) > 0$, $\lvert\mu\rvert = 1$.	$\Omega(\mu,n)\,\mathbf{j}$
4	$\mathrm{GL}(2n,\mathbb{H})$	$\begin{bmatrix} \mathrm{J}(\mu,\, n) & \\ & \mathrm{J}(\mu,\, n) \end{bmatrix}$, $\mu \in \mathbb{C}, \mathrm{Im}(\mu) > 0$, $\lvert\mu\rvert = 1$.	$\begin{bmatrix} & \Omega(\mu,n)\,\mathbf{j} \\ \Big(\Omega(\mu,n)\,\mathbf{j}\Big)^{-1} & \end{bmatrix}$
5	$\mathrm{GL}(2n,\mathbb{R})$	$\mathrm{J}_{\mathbb{R}}(\mu \pm \mathbf{i}\nu,\, 2n)$, $\mu^2+\nu^2 = 1$.	$\Big(\Omega_{\mathbb{R}}(K,2n)\Big)\,\sigma$, $\sigma = \mathrm{diag}(1,\ldots,(-1)^{2n-1})$.
6	$\mathrm{GL}(4n,\mathbb{R})$	$\begin{bmatrix} \mathrm{J}_{\mathbb{R}}(\mu \pm \mathbf{i}\nu,\, 2n) & \\ & \mathrm{J}_{\mathbb{R}}(\frac{\mu}{\mu^2+\nu^2} \mp \mathbf{i}\frac{\nu}{\mu^2+\nu^2},\, 2n) \end{bmatrix}$, $\mu^2+\nu^2 \neq 1$.	$\begin{bmatrix} & \Omega_{\mathbb{R}}(K,2n) \\ \Omega_{\mathbb{R}}(K^{-1},2n) & \end{bmatrix}$

Note that all reversing elements in Table 1.1 are involutions except for the third one, i.e., $\Omega(\mu,n)\,\mathbf{j}$. Moreover, in view of Lemma 2.2.4 and Theorem 4.1.1, we can use Table 1.1 to construct a suitable reversing element in $\mathrm{GL}(n,\mathbb{D})$ for every reversible element $A \in \mathrm{GL}(n,\mathbb{D})$ that conjugates A to its inverse A^{-1}. Consequently, we have the reversing symmetry group $\mathcal{E}_{\mathrm{GL}(n,\mathbb{D})}(A)$ of an arbitrary reversible element $A \in \mathrm{GL}(n,\mathbb{D})$.

1.3 Reversibility in affine groups

Considering $\mathbb{D}^n$ as an affine space, where $\mathbb{D} = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$. The affine group of automorphisms of $\mathbb{D}^n$ is denoted by $\mathrm{Aff}(n,\mathbb{D})$ and given by $\mathrm{GL}(n,\mathbb{D}) \ltimes \mathbb{D}^n$. Understanding reversibility for the affine group $\mathrm{Aff}(n,\mathbb{D})$ is a natural problem of interest. We investigated this problem and proved the following result.

Theorem 1.3.1. *Let $g = (A,v) \in \mathrm{Aff}(n,\mathbb{D})$ be an arbitrary element, where $\mathbb{D} = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$. Then g is reversible (resp. strongly reversible) in $\mathrm{Aff}(n,\mathbb{D})$ if and only if A is reversible (resp. strongly reversible) in $\mathrm{GL}(n,\mathbb{D})$. Further, for $\mathbb{D} = \mathbb{R}$ or $\mathbb{C}$, the following statements are equivalent.*

(1) g is reversible in $\mathrm{Aff}(n,\mathbb{D})$.

(2) g is strongly reversible in $\mathrm{Aff}(n,\mathbb{D})$.

This theorem answers a problem raised in [30, p. 78–79]. Note that the classification of the reversible and the strongly reversible elements in $\mathrm{Aff}(n,\mathbb{D})$ is intimately related to the classification of reversibility in $\mathrm{GL}(n,\mathbb{D})$.

Recently, the concept of reversibility has been extended to semisimple Lie algebras using the adjoint representations of Lie groups in [18]. The infinitesimal notion of reversibility that has been introduced for the Lie algebras is called *adjoint reality*; see [18, Section 1.2]. To prove Theorem 1.3.1, we first investigate conjugacy in $\mathrm{Aff}(n,\mathbb{D})$ and then we show that reversibility in $\mathrm{Aff}(n,\mathbb{D})$ boils down to the case where the linear part of the affine transformation is unipotent. Then we apply the notion of adjoint reality to classify the strongly reversible elements in $\mathrm{Aff}(n,\mathbb{D})$ with unipotent linear parts.

The reversibility problem is closely related to the problem of finding the involution length of a group. The *involution length* of a group G is the least integer m so that any element of G can be expressed as a product of m involutions in G; see [30, p. 76].

Theorem 1.3.2. *Let $g = (A,v) \in \mathrm{Aff}(n,\mathbb{D})$ such that $\det(A) = \pm 1$. Then g can be written as a product of four involutions for $\mathbb{D} = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$.*

1.4 Reversibility in special linear groups

In this section, we will assume $\mathbb{F} = \mathbb{C}$ or $\mathbb{H}$. Note that if two matrices are conjugate by an element of $\mathrm{GL}(n,\mathbb{F})$, then by a suitable scaling of the conjugating element, we can assume that both the matrices are conjugate by an element of $\mathrm{SL}(n,\mathbb{F})$; see [30, p. 77] and Remark 2.2.5. Therefore, the classification of reversible elements in $\mathrm{SL}(n,\mathbb{F})$ follows from the classification of reversible elements in $\mathrm{GL}(n,\mathbb{F})$. We have classified the strongly reversible elements in $\mathrm{SL}(n,\mathbb{F})$.

A key tool used in studying strongly reversible elements of $\mathrm{SL}(n,\mathbb{F})$ is the notion of *Weyr canonical form*. Weyr canonical form is a block upper triangular matrix in which the diagonal blocks are scalar matrices (i.e., scalar multiples of identity matrices), the super-diagonal blocks contain identity matrices augmented by rows of zeros, and all the other blocks are zero. The centralizer (and hence reversing element) of the *Weyr canonical form* is more manageable than that of the Jordan canonical form. More precisely, for a reversible element of $\mathrm{SL}(n,\mathbb{F})$ written in the Weyr canonical form, every reversing element has a block upper triangular form; see [29] for more details. We have observed that Weyr canonical form is more efficient than the Jordan canonical form in the study of reversibility.

1.4.1 Strong reversibility in $\mathrm{SL}(n,\mathbb{C})$

The classification of *strongly reversible* elements in $\mathrm{SL}(n,\mathbb{C})$ is more subtle than the $\mathrm{GL}(n,\mathbb{C})$. There are reversible elements in $\mathrm{SL}(n,\mathbb{C})$ which are not strongly reversible. However, we do not know any literature for the complete classification of strongly reversible elements in $\mathrm{SL}(n,\mathbb{C})$. The best known results, however, for the classification problem of strongly reversible elements are [38, Theorem 3.1.1] and [18, Theorem 5.6].

To state our main results, we need some terminologies concerning partitioning a positive integer n. We will recall the notation introduced in [18, Section 3.3] for partitioning a positive integer n.

Definition 1.4.1. A *partition* of a positive integer n is an object of the form

$$\mathbf{d}(n) := [d_1^{t_{d_1}}, \dots, d_s^{t_{d_s}}],$$

where $t_{d_i}, d_i \in \mathbb{N}, 1 \leq i \leq s$, such that $\sum_{i=1}^{s} t_{d_i} d_i = n, t_{d_i} \geq 1$ and $d_1 > \cdots > d_s > 0$. Moreover, for a partition $\mathbf{d}(n) = [d_1^{t_{d_1}}, \dots, d_s^{t_{d_s}}]$ of n, define

(a) $\mathbb{N}_{\mathbf{d}(n)} := \{d_i \mid 1 \leq i \leq s\}$,

(b) $\mathbb{E}_{\mathbf{d}(n)} := \mathbb{N}_{\mathbf{d}(n)} \cap (2\mathbb{N})$, $\mathbb{O}_{\mathbf{d}(n)} := \mathbb{N}_{\mathbf{d}(n)} \setminus \mathbb{E}_{\mathbf{d}(n)}$, and

(c) $\mathbb{E}^2_{\mathbf{d}(n)} := \{\eta \in \mathbb{E}_{\mathbf{d}(n)} : \eta \equiv 2 \pmod 4\}$.

We prove the following theorem, which provides a complete classification of the strongly reversible elements in $\mathrm{SL}(n,\mathbb{C})$.

Theorem 1.4.2. *Let A be a reversible element of $\mathrm{SL}(n,\mathbb{C})$. Let p (resp. q) be the multiplicity of eigenvalue $+1$ (resp. -1) and let $\mathbf{d}(p)$ (resp. $\mathbf{d}(q)$) be the partition corresponding to the eigenvalue $+1$ (resp. -1) in the Jordan decomposition of A, where $p,q \in \mathbb{N}\cup\{0\}$. Then, A is strongly reversible if and only if at least one of the following conditions holds.*

(1) Either $\mathbb{O}_{\mathbf{d}(p)}$ or $\mathbb{O}_{\mathbf{d}(q)}$ is non-empty.

(2) $|\mathbb{E}^2_{\mathbf{d}(p)}| + |\mathbb{E}^2_{\mathbf{d}(q)}| + \dfrac{n-(p+q)}{2}$ is even.

1.4.2 Strong reversibility in $\mathrm{SL}(n,\mathbb{H})$

Note that if $g \in \mathrm{GL}(n,\mathbb{H})$ is an involution then $g \in \mathrm{SL}(n,\mathbb{H})$; see [32, Theorem 5.9.2]. Therefore, if an element of $\mathrm{SL}(n,\mathbb{H})$ is strongly reversible in $\mathrm{GL}(n,\mathbb{H})$, then it will be strongly reversible in $\mathrm{SL}(n,\mathbb{H})$. This establishes the equivalence between strongly reversible elements in $\mathrm{GL}(n,\mathbb{H})$ and $\mathrm{SL}(n,\mathbb{H})$. In Theorem 1.2.1, we gave a sufficient criterion for strong reversibility of the reversible elements in $\mathrm{GL}(n,\mathbb{H})$. In Chapter 7, we will show that these are also necessary conditions. We prove the following theorem.

Theorem 1.4.3. *Let $A \in \mathrm{SL}(n,\mathbb{H})$ be a reversible element. Then A is strongly reversible in $\mathrm{SL}(n,\mathbb{H})$ if and only if in the Jordan decomposition of A, every Jordan block corresponding to non-real eigenvalue classes of unit modulus has even multiplicity.*

Chapter 2

Preliminaries

In this thesis, $\mathbb{D}$ denotes either the field of real numbers $\mathbb{R}$, the field of complex numbers $\mathbb{C}$, or the division ring of Hamilton's quaternions $\mathbb{H}$. We will use the notation $\mathbb{F}$ to denote either $\mathbb{C}$ or $\mathbb{H}$.

In this section, we will fix some notation and recall necessary background that will be used throughout this thesis. Subsequently, we will introduce a few specialized notations as they become relevant

2.1 Linear algebra over the quaternions

Recall that $\mathbb{H} := \mathbb{R} + \mathbb{R}\mathbf{i} + \mathbb{R}\mathbf{j} + \mathbb{R}\mathbf{k}$ denotes the division algebra of Hamilton's quaternions. Every element $a \in \mathbb{H}$ can be expressed as $a = a_0 + a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$, where a_0, a_1, a_2, a_3 are real numbers, and $\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1$. The conjugate of a is given by $\bar{a} = a_0 - a_1\mathbf{i} - a_2\mathbf{j} - a_3\mathbf{k}$. We identify the real subspace $\mathbb{R} \oplus \mathbb{R}\mathbf{i}$ with the usual complex plane $\mathbb{C}$, and then one can write $\mathbb{H} = \mathbb{C} \oplus \mathbb{C}\mathbf{j}$. We consider $\mathbb{H}^n$ as a right $\mathbb{H}$-module. For an elaborate discussion on the linear algebra over the quaternions; see [32], [42].

Definition 2.1.1. Let $A \in \mathrm{M}(n, \mathbb{H})$, the algebra of $n \times n$ matrices over $\mathbb{H}$. A non-zero vector $v \in \mathbb{H}^n$ is said to be a (right) eigenvector of A corresponding to a (right) eigenvalue $\lambda \in \mathbb{H}$ if the equality $Av = v\lambda$ holds.

Eigenvalues of every $A \in \mathrm{M}(n, \mathbb{H})$ occur in similarity classes, i.e., if v is an eigenvector corresponding to eigenvalue λ, then $v\mu \in v\mathbb{H}$ is an eigenvector corresponding to eigenvalue $\mu^{-1}\lambda\mu$. Each similarity class of eigenvalues contains a unique complex number with non-negative imaginary part. Here, instead of similarity classes of eigenvalues, we will consider the *unique complex representatives* with non-negative imaginary parts as eigenvalues unless specified otherwise. In places where we need to distinguish between the similarity class and a representative, we shall write the similarity class of an eigenvalue representative λ by $[\lambda]$.

Definition 2.1.2. Let $A \in \mathrm{M}(n,\mathbb{H})$ and write $A = A_1 + A_2\mathbf{j}$, where $A_1, A_2 \in \mathrm{M}(n,\mathbb{C})$. Consider the embedding $\Phi : \mathrm{M}(n,\mathbb{H}) \longrightarrow \mathrm{M}(2n,\mathbb{C})$ defined by

$$\Phi(A) = \begin{pmatrix} A_1 & A_2 \\ -\overline{A_2} & \overline{A_1} \end{pmatrix}, \tag{2.1.1}$$

where $\overline{A_i}$ denotes the complex conjugate of A_i. Note that $\Phi(A)$ is also known as *complex adjoint* of A; see [42].

Definition 2.1.3 (cf. [32, p. 113])**.** Consider the embedding Φ defined in (2.1.1). The determinant of $A \in \mathrm{M}(n,\mathbb{H})$ is defined as $\det(A) := \det(\Phi(A))$. In view of the *Skolem-Noether theorem*, the above definition of quaternionic determinant is independent of the choice of the chosen embedding Φ. Note that for $A \in \mathrm{M}(n,\mathbb{H})$, $\det(\mathrm{A})$ is always a non-negative real number; see [32, Theorem 5.9.2].

Consider Lie groups $\mathrm{GL}(n,\mathbb{D}) := \{A \in \mathrm{M}(n,\mathbb{D}) \mid \det(A) \neq 0\}$ and $\mathrm{SL}(n,\mathbb{D}) := \{A \in \mathrm{GL}(n,\mathbb{D}) \mid \det(A) = 1\}$.

2.2 Jordan canonical forms in $\mathrm{M}(n,\mathbb{D})$, where $\mathbb{D} = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$

In this section, we will recall the Jordan canonical form in $\mathrm{M}(n,\mathbb{D})$; see [29, p. 39], [32, Theorem 15.1.1, Theorem 5.5.3] for more details.

Definition 2.2.1. Let $\psi \colon \mathbb{C} \longrightarrow \mathrm{M}(2,\mathbb{R})$ be an embedding given by

$$\psi(z) := \begin{pmatrix} \mathrm{Re}(z) & \mathrm{Im}(z) \\ -\mathrm{Im}(z) & \mathrm{Re}(z) \end{pmatrix}.$$

This induces the embedding $\Psi \colon \mathrm{M}(n,\mathbb{C}) \longrightarrow \mathrm{M}(2n,\mathbb{R})$ defined as

$$\Psi((z_{i,j})_{n\times n}) := \big(\psi(z_{i,j})\big)_{2n\times 2n}.$$

Definition 2.2.2 (cf. [32, p. 94])**.** A *Jordan block* $\mathrm{J}(\lambda, m)$ is an $m \times m$ matrix with $\lambda \in \mathbb{D}$ on the diagonal entries, 1 on all of the super-diagonal entries and 0 elsewhere.

Jordan block $\mathrm{J}(\lambda,m)$ is also known as a basic Jordan matrix with eigenvalue λ. We will refer to a block diagonal matrix where each block is a Jordan block as *Jordan form*.

We also consider the following block matrix as Jordan form over $\mathbb{R}$, cf. [17, p. 364], which corresponds to the case when eigenvalues of a matrix over $\mathbb{R}$ belong to $\mathbb{C}\setminus\mathbb{R}$. Recall the embedding Ψ as in Definition 2.2.1. Let $K := \Psi(\mu+\mathbf{i}\nu) = \left(\begin{smallmatrix}\mu & \nu\\ -\nu & \mu\end{smallmatrix}\right) \in \mathrm{M}(2,\mathbb{R})$, where μ, ν are real numbers with $\nu > 0$. Then define

$$\mathrm{J}_{\mathbb{R}}(\mu\pm\mathbf{i}\nu,\, n) := \Psi(\mathrm{J}(\mu+\mathbf{i}\nu,\, n)) = \begin{pmatrix} K & \mathrm{I}_2 & & & \\ & K & \mathrm{I}_2 & & \\ & & \ddots & \ddots & \\ & & & K & \mathrm{I}_2 \\ & & & & K \end{pmatrix} \in \mathrm{M}(2n,\mathbb{R}), \tag{2.2.1}$$

where I_2 denotes 2×2 identity matrix; see [32, Theorem 15.1.1], [17, Chapter 12]. Further, we also define

$$\mathrm{J}_{\mathbb{R}}(\mu\mp\mathbf{i}\nu,\, n) := \Psi(\mathrm{J}(\mu-\mathbf{i}\nu,\, n)) = \sigma\left(\Psi(\mathrm{J}(\mu+\mathbf{i}\nu,\, n))\right)\sigma^{-1}, \tag{2.2.2}$$

where $\sigma = \mathrm{diag}(1,-1,1,-1,\dots,(-1)^{2n-1}\,)_{2n\times 2n}$.

Remark 2.2.3. We will follow the notation $\mathrm{J}_{\mathbb{R}}(\mu\pm\mathbf{i}\nu,\, n)$ and $\mathrm{J}_{\mathbb{R}}(\mu\mp\mathbf{i}\nu,\, n)$ as defined in Equations (2.2.1) and (2.2.2) throughout this thesis. Note that $\{\mathrm{J}_{\mathbb{R}}(\mu\pm\mathbf{i}\nu,\, n)\}$ is a singleton by the above definition. □

The following notation will allow us to conveniently write block-diagonal square matrices with many blocks. For r-many square matrices $A_i \in \mathrm{M}_{m_i}(\mathbb{D})$, $1 \le i \le r$, the block diagonal square matrix of size $\sum m_i \times \sum m_i$, with A_i as the i-th block in the diagonal, is denoted by $A_1 \oplus \cdots \oplus A_r$.

The following lemma recalls the Jordan canonical form of matrices over $\mathbb{D}$.

Lemma 2.2.4 (Jordan form in $\mathrm{M}(n,\mathbb{D})$, cf. [32])**.** *For every $A \in \mathrm{M}(n,\mathbb{D})$, there is an invertible matrix $S \in \mathrm{GL}(n,\mathbb{D})$ such that SAS^{-1} has the following form:*

(1) For $\mathbb{D}=\mathbb{R}$,

$$SAS^{-1} = \mathrm{J}(\lambda_1,\, m_1)\oplus\cdots\oplus\mathrm{J}(\lambda_k,\, m_k)\oplus\mathrm{J}_{\mathbb{R}}(\mu_1\pm\mathbf{i}\nu_1,\, 2\ell_1)\cdots\oplus\mathrm{J}_{\mathbb{R}}(\mu_q\pm\mathbf{i}\nu_q,\, 2\ell_q), \tag{2.2.3}$$

where $\lambda_1,\dots,\lambda_k$; $\mu_1,\dots,\mu_q$; $\nu_1,\dots,\nu_q$ are real numbers (not necessarily distinct) and $\nu_1,\dots,\nu_q$ are positive.

(2) For $\mathbb{D} = \mathbb{C}$,

$$SAS^{-1} = \mathrm{J}(\lambda_1, m_1) \oplus \cdots \oplus \mathrm{J}(\lambda_k, m_k), \tag{2.2.4}$$

where $\lambda_1, \ldots, \lambda_k$ are complex numbers (not necessarily distinct).

(3) For $\mathbb{D} = \mathbb{H}$,

$$SAS^{-1} = \mathrm{J}(\lambda_1, m_1) \oplus \cdots \oplus \mathrm{J}(\lambda_k, m_k), \tag{2.2.5}$$

where $\lambda_1, \ldots, \lambda_k$ are complex numbers (not necessarily distinct) and have non-negative imaginary parts.

The forms (2.2.3), (2.2.4) and (2.2.5) are uniquely determined by A up to a permutation of Jordan blocks.

Remark 2.2.5. If two elements of $\mathrm{M}(n,\mathbb{F})$ are conjugate by an element of $\mathrm{GL}(n,\mathbb{F})$, then by a suitable scaling, we can assume that they are conjugate by an element of $\mathrm{SL}(n,\mathbb{F})$. To see this, let $h := g(a\mathrm{I}_n)$, where $a := \dfrac{1}{(\det(g))^{1/n}}$. Recall that $\det(g) \in \mathbb{R}$ and $\det(g) > 0$, for every $g \in \mathrm{GL}(n,\mathbb{H})$; see [32, Theorem 5.9.2.]. Therefore, $a\mathrm{I}_n$ lies in the center of $\mathrm{GL}(n,\mathbb{F})$, and hence $h \in \mathrm{SL}(n,\mathbb{F})$ such that $hAh^{-1} = B$. □

The Segre characteristic lists the sizes of the diagonal blocks in the Jordan form of a matrix and is defined as follows. We refer to Definition 6.1.1 for the notation of partition used in the following definition.

Definition 2.2.6 (cf. [29, p. 39])**.** Suppose $A \in \mathrm{M}(n,\mathbb{F})$ is similar to the Jordan form $\mathrm{J}(\lambda, n_1) \oplus \cdots \oplus \mathrm{J}(\lambda, n_r)$ such that $n_1 + n_2 + \cdots + n_r = n$ and $n_1 \geq n_2 \geq \cdots \geq n_r \geq 1$. Then the partition $(n_1, n_2, \cdots, n_r)$ of n is called *Jordan structure* (or *Segre characteristic*) of A.

2.3 Matrices commuting with Jordan blocks

In the following lemma, we recall the well-known Sylvester's theorem on solutions to the matrix equation $AX = XB$; see [29, Theorem 1.6.1], [32, Theorem 5.11.1] for more details.

Lemma 2.3.1 (cf. [29, 32])**.** *Let $A \in \mathrm{M}(m,\mathbb{F})$ and $B \in \mathrm{M}(n,\mathbb{F})$. Then the equation*

$$AX = XB$$

has only the trivial solution if and only if A and B have no common eigenvalues.

The following lemma is helpful in understanding the centralizer of a matrix in $\mathrm{M}(n,\mathbb{F})$.

Lemma 2.3.2 (cf. [29, Proposition 3.1.1]). *Let* $A = \begin{pmatrix} A_1 & \\ & A_2 \end{pmatrix} \in \mathrm{M}(n,\mathbb{F})$, *where* $A_1 \in \mathrm{M}(m,\mathbb{F})$ *(resp.* $A_2 \in \mathrm{M}(n-m,\mathbb{F})$*) has a single eigenvalue* λ_1 *(resp.* λ_2*) such that* $\lambda_1 \neq \lambda_2$. *If* $B \in \mathrm{M}(n,\mathbb{F})$ *such that* $BA = AB$, *then* B *has the following form*

$$B = \begin{pmatrix} B_1 & \\ & B_2 \end{pmatrix},$$

where $B_1 \in \mathrm{M}(m,\mathbb{F})$ *and* $B_2 \in \mathrm{M}(n-m,\mathbb{F})$ *such that*

$$B_1A_1 = A_1B_1 \text{ and } B_2A_2 = A_2B_2.$$

Proof. Let $B = \begin{pmatrix} B_{1,1} & B_{1,2} \\ B_{2,1} & B_{2,2} \end{pmatrix} \in \mathrm{M}(n,\mathbb{F})$ be an element having the same block structure as A such that $BA = AB$. Then we have

$$\begin{pmatrix} B_{1,1} & B_{1,2} \\ B_{2,1} & B_{2,2} \end{pmatrix} \begin{pmatrix} A_1 & \\ & A_2 \end{pmatrix} = \begin{pmatrix} A_1 & \\ & A_2 \end{pmatrix} \begin{pmatrix} B_{1,1} & B_{1,2} \\ B_{2,1} & B_{2,2} \end{pmatrix}.$$

This implies that

$$\begin{pmatrix} B_{1,1}A_1 & B_{1,2}A_2 \\ B_{2,1}A_1 & B_{2,2}A_2 \end{pmatrix} = \begin{pmatrix} A_1B_{1,1} & A_1B_{1,2} \\ A_2B_{2,1} & A_2B_{2,2} \end{pmatrix}.$$

Thus, we have $B_{1,2}A_2 = A_1B_{1,2}$ and $B_{2,1}A_1 = A_2B_{2,1}$. Since A_1 and A_2 have no common eigenvalue, using Lemma 2.3.1, we conclude that $B_{1,2}$ and $B_{2,1}$ are zero matrices. Consider $B_1 := B_{1,1}$ and $B_2 = B_{2,2}$. Then we have $B_1A_1 = A_1B_1$ and $B_2A_2 = A_2B_2$. This proves the lemma. □

To formulate results related to matrices commuting with Jordan blocks over $\mathbb{F}$, we need to introduce the notation for upper triangular *Toeplitz* matrices.

Definition 2.3.3. For $\mathbf{x} := (x_1, x_2, \dots, x_n) \in \mathbb{F}^n$, we define $\mathrm{Toep}_n(\mathbf{x}) \in \mathrm{M}(n,\mathbb{F})$ as

$$\mathrm{Toep}_n(\mathbf{x}) := [x_{i,j}]_{n\times n} = \begin{cases} 0 & \text{if } i > j \\ x_{j-i+1} & \text{if } i \leq j \end{cases}, \text{ where } 1 \leq i,j \leq n. \tag{2.3.1}$$

We can also write $\mathrm{Toep}_n(\mathbf{x})$ as

$$\mathrm{Toep}_n(\mathbf{x}) = \begin{pmatrix} x_1 & x_2 & \cdots & \cdots & \cdots & \cdots & x_n \\ & x_1 & x_2 & \cdots & \cdots & \cdots & x_{n-1} \\ & & x_1 & x_2 & \cdots & \cdots & x_{n-2} \\ & & & \ddots & \ddots & & \vdots \\ & & & & \ddots & \ddots & \vdots \\ & & & & & x_1 & x_2 \\ & & & & & & x_1 \end{pmatrix}. \qquad (2.3.2)$$

In the following lemma, we recall a basic result which gives matrices commuting with Jordan blocks in $\mathrm{M}(n,\mathbb{F})$; see [29, Proposition 3.1.2], [17, Theorem 9.1.1], and [32, Proposition 5.4.2] for more details.

Lemma 2.3.4 (cf. [29, 32]). *Let $B \in \mathrm{M}(n,\mathbb{F})$ such that $B\mathrm{J}(\lambda,n) = \mathrm{J}(\lambda,n)B$. Then B has the following form:*

(1) For $\mathbb{F} = \mathbb{C}$, we have $B = \mathrm{Toep}_n(\mathbf{x})$ for some $\mathbf{x} \in \mathbb{C}^n$,

(2) For $\mathbb{F} = \mathbb{H}$, we have $B = \mathrm{Toep}_n(\mathbf{x})$ for some $\mathbf{x} \in \begin{cases} \mathbb{H}^n & \text{if } \lambda \in \mathbb{R}, \\ \mathbb{C}^n & \text{if } \lambda \in \mathbb{C} \setminus \mathbb{R}. \end{cases}$

In particular, if $B \in \mathrm{M}(n,\mathbb{H})$ such that $B\mathrm{J}(e^{\mathbf{i}\theta},n) = \mathrm{J}(e^{\mathbf{i}\theta},n)B$, where $\theta \in (0,\pi)$, then $B = \mathrm{Toep}_n(\mathbf{x})$ for some $\mathbf{x} \in \mathbb{C}^n$.

2.4 Reversible elements in $\mathrm{SL}(n,\mathbb{C})$ and $\mathrm{SL}(n,\mathbb{H})$

Let $\mathbb{F}$ be either $\mathbb{C}$ or $\mathbb{H}$. In this section, we will classify the reversible elements in $\mathrm{SL}(n,\mathbb{F})$. The classification of reversible elements in $\mathrm{GL}(n,\mathbb{F})$ is given in Theorem 4.1.1. The following lemma establishes the equivalence between reversible elements in $\mathrm{GL}(n,\mathbb{F})$ and $\mathrm{SL}(n,\mathbb{F})$.

Lemma 2.4.1 (cf. [30, p. 77]). *An element A of $\mathrm{SL}(n,\mathbb{F})$ is reversible in $\mathrm{SL}(n,\mathbb{F})$ if and only if it is reversible in $\mathrm{GL}(n,\mathbb{F})$.*

Proof. To see this, suppose that $hAh^{-1} = A^{-1}$, where $h \in \mathrm{GL}(n,\mathbb{F})$. Either $h \in \mathrm{SL}(n,\mathbb{F})$ or $h \notin \mathrm{SL}(n,\mathbb{F})$. In the latter case, using Remark (2.2.5), we can construct $g \in \mathrm{SL}(n,\mathbb{F})$ such that $gAg^{-1} = A^{-1}$. Hence, the proof follows. $\square$

The following example shows that the above result does not hold for $\mathrm{SL}(n,\mathbb{R})$.

Example 2.4.2. Let $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in \mathrm{SL}(2,\mathbb{R})$ be a unipotent element. Let $g \in \mathrm{GL}(2,\mathbb{R})$ be such that $hgh^{-1} = g^{-1}$. Then $g = \begin{pmatrix} x & y \\ 0 & -x \end{pmatrix}$, where $x, y \in \mathbb{R}$. Thus, $\det(g) = -x^2$. Since there is no $x \in \mathbb{R}$ such that $x^2 = -1$, we have $g \notin \mathrm{SL}(2,\mathbb{R})$. Therefore, A is not reversible in $\mathrm{SL}(2,\mathbb{R})$. However, by considering $g = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, we find that A is reversible in $\mathrm{GL}(2,\mathbb{R})$. □

Combining Theorem 4.1.1 and Lemma 2.4.1, we get the classification of the reversible elements in $\mathrm{SL}(n,\mathbb{C})$ and $\mathrm{SL}(n,\mathbb{H})$.

Lemma 2.4.3. *Let $\mathbb{F} = \mathbb{C}$ or $\mathbb{H}$. An element $A \in \mathrm{SL}(n,\mathbb{F})$ with Jordan form as given in Lemma 2.2.4 is reversible if and only if the following hold:*

(1) For $\mathbb{F} = \mathbb{C}$, the blocks can be partitioned into pairs $\{\mathrm{J}(\lambda,s), \mathrm{J}(\lambda^{-1},s)\}$ or, singletons $\{\mathrm{J}(\mu,m)\}$, where $\lambda, \mu \in \mathbb{C} \setminus \{0\}$ and $\lambda \notin \{\pm 1\}, \mu \in \{\pm 1\}$.

(2) For $\mathbb{F} = \mathbb{H}$, the blocks can be partitioned into pairs $\{\mathrm{J}(\lambda,s), \mathrm{J}(\lambda^{-1},s)\}$ or, singletons $\{\mathrm{J}(\mu,m)\}$, where $\lambda, \mu \in \mathbb{C} \setminus \{0\}$ with non-negative imaginary parts such that $|\lambda| \neq 1, |\mu| = 1$.

The classification of strongly reversible elements in $\mathrm{SL}(n,\mathbb{C})$ and $\mathrm{SL}(n,\mathbb{H})$ is investigated in Chapter 6 and Chapter 7, respectively.

Chapter 3

Reversibility of Hermitian isometries

In this chapter, we will investigate the reversibility problem in the context of groups $\mathrm{Sp}(n) \ltimes \mathbb{H}^n$, $\mathrm{U}(n) \ltimes \mathbb{C}^n$, and $\mathrm{SU}(n) \ltimes \mathbb{C}^n$.

3.1 Reversibility in $\mathrm{Sp}(n)$

The classification of reversible and strongly reversible elements in $\mathrm{Sp}(n)$ has been obtained recently in [2], where it has been proved that every element in $\mathrm{Sp}(n)$ is reversible. The following theorem to classify the strongly reversible elements was also obtained in that paper.

Theorem 3.1.1. *[2, Theorem 1.2] An element g in $\mathrm{Sp}(n)$ is strongly reversible if and only if every eigenvalue class of g is either ± 1 or of even multiplicity.*

Here, we will give a different and simpler proof of the above theorem. To begin with, note the following well-known result.

Lemma 3.1.2. *[32, Theorem 5.3.6. (e)] If $A \in \mathrm{Sp}(n)$, then there exists $U \in \mathrm{Sp}(n)$ such that $UAU^{-1} = \mathrm{diag}(e^{\mathbf{i}\theta_1}, e^{\mathbf{i}\theta_2}, \dots, e^{\mathbf{i}\theta_n})$, where $\theta_s \in [0, \pi]$ for all $s \in \{1, 2, \dots, n\}$.*

The above lemma also holds for $\mathrm{U}(n)$ and $\mathrm{SU}(n)$; see [9, Corollary 32.9]. Now, we recall the following well-known primary result, which follows from the fundamental properties of quaternions.

Lemma 3.1.3. *Let $\theta \in [0, \pi]$ and $\alpha, \beta \in (-\pi, \pi)$. Then the following statements hold.*

(i) Let $a \in \mathbb{H}$ be such that $ae^{\mathbf{i}\theta} = e^{-\mathbf{i}\theta}a$. Then $a \in \begin{cases} \mathbb{C}\mathbf{j} & \text{if } \theta \neq 0, \pi; \\ \mathbb{H} & \text{if } \theta = 0, \pi. \end{cases}$

(ii) Let $a \in \mathbb{H}$ *be such that* $ae^{\mathbf{i}\theta} = e^{\mathbf{i}\theta}a$. *Then* $a \in \begin{cases} \mathbb{C} & \text{if } \theta \neq 0, \pi; \\ \mathbb{H} & \text{if } \theta = 0, \pi. \end{cases}$

(iii) Let $x \in \mathbb{H}$ *be such that* $xe^{\mathbf{i}\alpha} = e^{\mathbf{i}\beta}x$. *If* $\alpha \neq \pm\beta$ *then* $x = 0$.

Proof. Here, we will only prove the first part of the lemma. Using similar arguments, one can prove other parts of the lemma. Let $a = z + w\mathbf{j}$ be such that $a\,e^{\mathbf{i}\theta} = e^{-\mathbf{i}\theta}\,a$, where $z, w \in \mathbb{C}$. This implies

$$(z + w\mathbf{j})\,e^{\mathbf{i}\theta} = e^{-\mathbf{i}\theta}\,(z + w\mathbf{j}). \tag{3.1.1}$$

On comparing both sides of Equation (3.1.1) and using $\mathbf{j}\bar{x} = x\mathbf{j}$ for all $x \in \mathbb{C}$, we have

$$ze^{\mathbf{i}\theta} = e^{-\mathbf{i}\theta}z \text{ and } we^{-\mathbf{i}\theta} = e^{-\mathbf{i}\theta}w. \tag{3.1.2}$$

Due to commutativity of complex numbers, $we^{-\mathbf{i}\theta} = e^{-\mathbf{i}\theta}w$ holds for all $w \in \mathbb{C}$. The proof of (i) now follows from the equation $ze^{\mathbf{i}\theta} = e^{-\mathbf{i}\theta}z$. This completes the proof. □

Lemma 3.1.4. *Let* $A \in \mathrm{Sp}(n)$ *be such that the eigenvalue classes of A are either* ± 1 *or have even multiplicities. Then A is strongly reversible in* $\mathrm{Sp}(n)$.

Proof. Let $A \in \mathrm{Sp}(n)$. So by using Lemma 3.1.2, up to conjugacy in $\mathrm{Sp}(n)$, we can assume that:

$$A = e^{\mathbf{i}\theta_1}\mathrm{I}_{2k_1} \oplus e^{\mathbf{i}\theta_2}\mathrm{I}_{2k_2} \oplus \cdots \oplus e^{\mathbf{i}\theta_r}\mathrm{I}_{2k_r} \oplus -\mathrm{I}_s \oplus \mathrm{I}_t, \tag{3.1.3}$$

where $r, s, t \in \mathbb{N} \cup \{0\}$, and if $r \neq 0$ then $\theta_m \in (0, \pi)$ for all $m \in \{1, 2, \ldots, r\}$, and $2k_1 + 2k_2 + \cdots + 2k_r + s + t = n$.

Let $L = \begin{pmatrix} 0 & \mathbf{j} \\ -\mathbf{j} & 0 \end{pmatrix}$ in $\mathrm{Sp}(2)$. Then $L^2 = \mathrm{I}_2$ in $\mathrm{Sp}(2)$. Consider

$$B = \begin{pmatrix} L & & & & \\ & L & & & \\ & & \ddots & & \\ & & & L & \\ & & & & \mathrm{I}_{s+t} \end{pmatrix} \tag{3.1.4}$$

in $\mathrm{Sp}(n)$. Then $BAB^{-1} = A^{-1}$ and $B^2 = \mathrm{I}_n$. Hence A is strongly reversible in $\mathrm{Sp}(n)$. □

Lemma 3.1.5. *Suppose $A \in \mathrm{Sp}(n)$ has an eigenvalue class $[\lambda]$ of odd multiplicity k such that $\lambda \neq \pm 1$. Define $\mathcal{R}(A) := \{B \in \mathrm{GL}(n, \mathbb{H}) : BAB^{-1} = A^{-1}\}$. Then, up to conjugacy, every element B in $\mathcal{R}(A)$ has the form:*

$$\begin{pmatrix} B_1\mathbf{j} & \\ & B_2 \end{pmatrix},$$

where $B_1 \in \mathrm{GL}(k, \mathbb{C})$ and $B_2 \in \mathrm{GL}(n-k, \mathbb{H})$.

Proof. Let $\theta_1 \in (0, \pi)$ be such that $e^{\mathbf{i}\theta_1}$ is a chosen representative of the eigenvalue class $[\lambda]$ of odd multiplicity k. Let us first consider the case when $[\lambda]$ is the only eigenvalue class of A. Up to conjugacy in $\mathrm{Sp}(n)$, we can assume that :

$$A = [a_{s,t}]_{1 \leq s,t \leq n} = e^{\mathbf{i}\theta_1} \mathrm{I}_n.$$

Let $B = [b_{s,t}]_{1 \leq s,t \leq n} \in \mathcal{R}(A)$. Now, $BAB^{-1} = A^{-1}$ implies that $(b_{s,t})e^{\mathbf{i}\theta_1} = e^{-\mathbf{i}\theta_1}(b_{s,t})$ for all $1 \leq s,t \leq n$. So by using Lemma 3.1.3, we get $(b_{s,t}) = (w_{s,t})\mathbf{j}$ for some $w_{s,t} \in \mathbb{C}$. Therefore, we get $B = B_1\mathbf{j}$, where $B_1 = [w_{s,t}]_{1 \leq s,t \leq n} \in \mathrm{GL}(n, \mathbb{C})$.

Now, consider the case when A has at least two distinct eigenvalue classes. Up to conjugacy in $\mathrm{Sp}(n)$, we can assume that:

$$A = [a_{s,t}]_{1 \leq s,t \leq n} = e^{\mathbf{i}\theta_1} \mathrm{I}_k \oplus D, \text{where } D = \mathrm{diag}(e^{\mathbf{i}\theta_{k+1}}, e^{\mathbf{i}\theta_{k+2}}, \dots, e^{\mathbf{i}\theta_n}),$$

where $\theta_s \in [0, \pi]$, $\theta_1 \neq \theta_s$ for all $k+1 \leq s \leq n$. Note that $a_{s,t} = 0$ for all $s \neq t$.

Let $B = [b_{s,t}]_{1 \leq s,t \leq n} \in \mathcal{R}(A)$. From $BAB^{-1} = A^{-1}$, we get

$$BA = A^{-1}B \iff (b_{s,t})(a_{t,t}) = (a_{s,s}^{-1})(b_{s,t}) \text{ for all } 1 \leq s,t \leq n.$$

This implies

$$\begin{aligned} (b_{s,t})e^{\mathbf{i}\theta_1} &= e^{-\mathbf{i}\theta_1}(b_{s,t}) \text{ if } 1 \leq s,t \leq k; \\ (b_{s,t})e^{\mathbf{i}\theta_t} &= e^{-\mathbf{i}\theta_1}(b_{s,t}) \text{ if } 1 \leq s \leq k, k+1 \leq t \leq n; \\ (b_{s,t})e^{\mathbf{i}\theta_1} &= e^{-\mathbf{i}\theta_s}(b_{s,t}) \text{ if } k+1 \leq s \leq n, 1 \leq t \leq k. \end{aligned}$$

As we have chosen $\theta_1 \in (0, \pi)$, $\theta_s \in [0, \pi]$ such that $\theta_1 \neq \theta_s$ for all $k+1 \leq s \leq n, s \in \mathbb{N}$. Therefore, by using Lemma 3.1.3, we get the following:

$$\begin{cases} (b_{s,t}) = (w_{s,t})\mathbf{j} \text{ for some } w_{s,t} \in \mathbb{C} & \text{if } 1 \leq s,t \leq k; \\ b_{s,t} = 0 & \text{if } 1 \leq s \leq k, k+1 \leq t \leq n; \\ b_{s,t} = 0 & \text{if } k+1 \leq s \leq n, 1 \leq t \leq k. \end{cases}$$

This implies that the matrix B has the form:

$$B = \begin{pmatrix} B_1\mathbf{j} & \\ & B_2 \end{pmatrix},$$

where $B_1 = [w_{s,t}]_{1\leq s,t\leq k} \in \mathrm{GL}(k,\mathbb{C})$ and $B_2 \in \mathrm{GL}(n-k,\mathbb{H})$. This proves the lemma. $\square$

Lemma 3.1.6. *If $A \in \mathrm{Sp}(n)$ has an eigenvalue class $[\lambda]$, $\lambda \neq \pm 1$, of odd multiplicity, then A is not strongly reversible in* $\mathrm{Sp}(n)$.

Proof. Assume that A is strongly reversible in $\mathrm{Sp}(n)$. Then there exists B in $\mathrm{Sp}(n)$ such that $BAB^{-1} = A^{-1}$, $B^2 = \mathrm{I}_n$, and $BB^{\bigstar} = \mathrm{I}_n$, where $B^{\bigstar} := (\overline{B})^{\top}$, i.e., conjugate transpose of B. Since $B \in \mathcal{R}(A)$, so by using Lemma 3.1.5, we can write B in the following form:

$$B = \begin{pmatrix} B_1\mathbf{j} & \\ & B_2 \end{pmatrix},$$

where $B_1 \in \mathrm{GL}(k,\mathbb{C})$ and $B_2 \in \mathrm{GL}(n-k,\mathbb{H})$.

Now, we see that the relations $B^2 = \mathrm{I}_n$ and $BB^{\bigstar} = \mathrm{I}_n$ implies

$$(B_1\mathbf{j})(B_1\mathbf{j}) = \mathrm{I}_k \text{ and } (B_1\mathbf{j})(B_1\mathbf{j})^{\bigstar} = \mathrm{I}_k.$$

So by using $w\mathbf{j} = \mathbf{j}\bar{w}$ for all $w \in \mathbb{C}$, we get $(B_1)(\overline{B_1}) = -\mathrm{I}_k$ and $(B_1)(B_1^{\bigstar}) = \mathrm{I}_k$. In particular, $B_1 \in \mathrm{GL}(k,\mathbb{C})$ such that $B_1^{-1} = -\overline{B_1} = B_1^{\bigstar}$. This implies $B_1^{\top} = -B_1$. Therefore, we have $\det(B_1^{\top}) = \det(-B_1)$, i.e., $\det(B_1) = (-1)^k \det(B_1) = -\det(B_1)$, where k is odd. Consequently, $\det(B_1) = 0$, and hence B_1 is not invertible. This is a contradiction. So our initial assumption that A is strongly reversible can not be true. This proves the lemma. $\square$

Proof of Theorem 3.1.1: The proof follows from Lemma 3.1.4 and Lemma 3.1.6. $\square$

3.2 Reversibility in $\mathrm{SU}(n)$

In this section, we will revisit the classification of strongly reversible elements in the group $\mathrm{SU}(n)$. First, we shall note the following lemma that will be used in the proof of Theorem 1.1.7.

Lemma 3.2.1. *Suppose $A \in \mathrm{SU}(2n)$ has a self-dual characteristic polynomial such that no eigenvalue of A is equal to 1 or -1. Let $B \in \mathrm{GL}(2n,\mathbb{C})$ such that $BAB^{-1} = A^{-1}$ and $B^2 = \mathrm{I}_{2n}$. Then $\det(B) = (-1)^n$.*

Proof. We will prove this lemma by using strong induction. First, consider $n = 1$. Then up to conjugacy in $\mathrm{SU}(2)$, we can write A as: $A = \begin{pmatrix} e^{\mathbf{i}\theta} & \\ & e^{-\mathbf{i}\theta} \end{pmatrix}$, where $\theta \in (0,\pi)$. Let $B = [b_{p,q}]_{1 \leqslant p,q,\leqslant 2} \in \mathrm{GL}(2,\mathbb{C})$ be such that $BAB^{-1} = A^{-1}$ and $B^2 = \mathrm{I}_2$. Now, $BA = A^{-1}B$ implies

$$\begin{pmatrix} e^{\mathbf{i}\theta} b_{1,1} & e^{-\mathbf{i}\theta} b_{1,2} \\ e^{\mathbf{i}\theta} b_{2,1} & e^{-\mathbf{i}\theta} b_{2,2} \end{pmatrix} = \begin{pmatrix} e^{-\mathbf{i}\theta} b_{1,1} & e^{-\mathbf{i}\theta} b_{1,2} \\ e^{\mathbf{i}\theta} b_{2,1} & e^{\mathbf{i}\theta} b_{2,2} \end{pmatrix}.$$

Thus, $b_{1,1} = b_{2,2} = 0$ and $b_{1,2}$, $b_{2,1}$ are non-zero complex numbers. Further, $B^2 = \mathrm{I}_2$ implies that $b_{1,2}b_{2,1} = 1$. So $\det(B) = -b_{1,2}b_{2,1} = -1 = (-1)^1$. Thus lemma holds for $n = 1$. Next, assume that the lemma holds for all $n \leq k$. We will use the induction hypothesis to show that the lemma holds for $n = k+1$. Consider $n = k+1$. Let $e^{\mathbf{i}\theta_1}$ is an eigenvalue of A with multiplicity r, where $1 \leqslant r \leqslant n$ and $\theta_1 \in (0,\pi)$. Now there are two possible cases:

(1) Suppose that $n = r$. Then up to conjugacy in $\mathrm{SU}(2n)$, we can write A as: $A = \mathrm{diag}(\underbrace{e^{\mathbf{i}\theta_1},\dots,e^{\mathbf{i}\theta_1}}_{n\text{-times}},\underbrace{e^{-\mathbf{i}\theta_1},\dots,e^{-\mathbf{i}\theta_1}}_{n\text{-times}})$. Let $B \in \mathrm{GL}(2n,\mathbb{C})$ be such that $BAB^{-1} = A^{-1}$ and $B^2 = \mathrm{I}_{2n}$. Then $B = \begin{pmatrix} & P \\ Q & \end{pmatrix}$, where $P,Q \in \mathrm{GL}(n,\mathbb{C})$ such that $PQ = \mathrm{I}_n$. Note that $B = \begin{pmatrix} P & \\ & Q \end{pmatrix}\begin{pmatrix} & \mathrm{I}_n \\ \mathrm{I}_n & \end{pmatrix}$. This implies that $\det(B) = \det(P)\det(Q)(-1)^n$. Since $PQ = \mathrm{I}_n$, we have $\det(PQ) = \det(P)\det(Q) = \det(\mathrm{I}_n) = 1$. Therefore, $\det(B) = (-1)^n$.

(2) Suppose that $r < n$. Then up to conjugacy in $\mathrm{SU}(2n)$, we can write A as: $A = \begin{pmatrix} A_1 & \\ & A_2 \end{pmatrix}$ such that $A_1 = \mathrm{diag}(\underbrace{e^{\mathbf{i}\theta_1},\dots,e^{\mathbf{i}\theta_1}}_{r\text{-times}},\underbrace{e^{-\mathbf{i}\theta_1},\dots,e^{-\mathbf{i}\theta_1}}_{r\text{-times}}) \in \mathrm{SU}(2r)$ and $A_2 = \mathrm{diag}(e^{\mathbf{i}\theta_{r+1}}, e^{-\mathbf{i}\theta_{r+1}},\dots,e^{\mathbf{i}\theta_n},e^{-\mathbf{i}\theta_n}) \in \mathrm{SU}(2(n-r))$, where $\theta_1 \neq \theta_\ell$ and $\theta_\ell \in (0,\pi)$ for all $k \in \{r+1,r+2,\dots,n\}$. Let $B \in \mathrm{GL}(2n,\mathbb{C})$ be such that $BAB^{-1} = A^{-1}$ and $B^2 = \mathrm{I}_{2n}$. Then $B = \begin{pmatrix} B_1 & \\ & B_2 \end{pmatrix}$, where B_1 and B_2 satisfy the following conditions.

(*i*) $B_1 \in \mathrm{GL}(2r,\mathbb{C})$ such that $B_1A_1B_1^{-1} = A_1^{-1}$ and $B_1^2 = \mathrm{I}_{2r}$.

(*ii*) $B_2 \in \mathrm{GL}(2(n-r),\mathbb{C})$ such that $B_2A_2B_2^{-1} = A_2^{-1}$ and $B_2^2 = \mathrm{I}_{2(n-r)}$.

Since $r < n = k+1$ and $(n-r) < n = k+1$, so by induction hypothesis, $\det(B_1) = (-1)^r$ and $\det(B_2) = (-1)^{n-r}$. This implies

$$\det(B) = \det(B_1)\det(B_2) = (-1)^r(-1)^{n-r} = (-1)^n.$$

Therefore, from both of the above cases, we can see that the lemma holds for $n = k+1$. Hence, by induction principle,we can conclude that the lemma holds for every $n \in \mathbb{N}$. $\square$

Now, we recall the following result from [21] that classifies strongly reversible elements in $\mathrm{SU}(n)$.

Lemma 3.2.2. *[21, Proposition 3.3] Suppose $A \in \mathrm{SU}(n)$ has a self-dual characteristic polynomial. Then A is not strongly reversible in $\mathrm{SU}(n)$ if and only if $n \equiv 2 \pmod 4$ and no eigenvalue of A is equal to 1 or -1.*

Note that if part of the above lemma follows directly from Lemma 3.2.1. For the converse part, see [30, p. 74].

3.3 Conjugacy in $\mathrm{U}(n,\mathbb{F}) \ltimes \mathbb{F}^n$, where $\mathbb{F} = \mathbb{C}$ or $\mathbb{H}$

Let G denote one of the groups $\mathrm{Sp}(n) \ltimes \mathbb{H}^n$, $\mathrm{U}(n) \ltimes \mathbb{C}^n$, or $\mathrm{SU}(n) \ltimes \mathbb{C}^n$. Let $L(G)$ denote the linear part of G. The identity element in the group $L(G)$ is denoted by I_n. The identity of G is denoted by I. The zero element of $\mathbb{F}^n$ is denoted by $\mathbf{0}$, where $\mathbb{F} = \mathbb{C}$ or $\mathbb{H}$.

Lemma 3.3.1. *Every element g in G, up to conjugacy, can be written as $g = (A, v)$ such that $A(v) = v$, $A = \mathrm{diag}(e^{\mathbf{i}\theta_1}, e^{\mathbf{i}\theta_2}, \dots, e^{\mathbf{i}\theta_r}, -\mathrm{I}_s, \mathrm{I}_t)$, and v has the form $v = [0, 0, \dots, 0, v_1, v_2, \dots, v_t]$, where $\theta_k \in (0, \pi)$, $k \in \{1, 2, \dots, r\}$, and $r, s, t \in \mathbb{N} \cup \{0\}$. Moreover, if 1 is not an eigenvalue of the linear part A of g, then, up to conjugacy, g has the form $g = (A, \mathbf{0})$.*

Proof. Let $g \in G$ be an arbitrary element. After conjugating g by a suitable element $(B, \mathbf{0})$ in G, we can assume $g = (A, w)$ such that $A = \mathrm{diag}(e^{\mathbf{i}\theta_1}, e^{\mathbf{i}\theta_2}, \dots, e^{\mathbf{i}\theta_r}, -\mathrm{I}_s, \mathrm{I}_t)$, where $\theta_k \in (0, \pi)$, $k \in \{1, 2, \dots, r\}$, and $r + s + t = n$. Now there are two possible cases:

(1) Suppose 1 is not an eigenvalue of A. Therefore, the linear transformation $A - \mathrm{I}_n$ is invertible. Consequently, we can choose $x_o = (A - \mathrm{I}_n)^{-1}(w) \in \mathbb{F}^n$.

Consider $h = (\mathrm{I}_n, x_o) \in G$, i.e., $h(x) = x + x_o$ for all $x \in \mathbb{F}^n$. Note that for all $x \in \mathbb{F}^n$, we have

$$hgh^{-1}(x) = hg(x - x_o) = h(Ax - Ax_o + w) = Ax + w - (A - \mathrm{I}_n)x_o.$$

Since $x_o = (A - \mathrm{I}_n)^{-1}(w)$, we have $hgh^{-1}(x) = A(x) + \mathbf{0}$ for all $x \in \mathbb{F}^n$. Therefore, by taking $v = \mathbf{0}$, we get $hgh^{-1} = (A, \mathbf{0})$ such that $A(\mathbf{0}) = \mathbf{0}$.

(2) Let 1 be an eigenvalue of A. In this case, $t > 0$ and $A - \mathrm{I}_n$ has rank $r + s = n - t < n$. Therefore, we can choose an element $u \in \mathbb{F}^n$ having the last $n - (r + s)$ coordinates zero such that $[(A - \mathrm{I}_n)(u)]_\ell = w_\ell$ for all $1 \leq \ell \leq r + s$, where $w = [w_\ell]_{1 \leq \ell \leq n}$. If $v = w - (A - \mathrm{I}_n)(u)$, then $v = [0, 0, \ldots, 0, w_{r+s+1}, w_{r+s+2}, \ldots, w_n]$ and $A(v) = v$. Consider $h = (\mathrm{I}_n, u) \in G$, i.e., $h(x) = x + u$ for all $x \in \mathbb{F}^n$. Note that for all $x \in \mathbb{F}^n$, we have

$$hgh^{-1}(x) = hg(x - u) = h(Ax - Au + w) = Ax + w - (A - \mathrm{I}_n)(u) = Ax + v.$$

This proves the lemma. □

In the following lemma, we get a sufficient condition for reversibility of $g \in G$.

Lemma 3.3.2. *Let $g = (A, v)$ in G be such that $A(v) = v$. If there exists an element B in $L(G)$ such that $BAB^{-1} = A^{-1}$ and $B(v) = -v$, then g is reversible in G.*

Proof. Consider $h = (B, \mathbf{0}) \in G$, where $\mathbf{0}$ is the zero element in $\mathbb{F}^n$. Then $hgh^{-1} = g^{-1}$. This proves the lemma. □

In the following lemma, we get a sufficient condition for strong reversibility of $g \in G$.

Lemma 3.3.3. *Let $g = (A, v)$ in G be such that $A(v) = v$. If there exists an element B in $L(G)$ such that $BAB^{-1} = A^{-1}$, $B(v) = -v$, and $B^2 = \mathrm{I}_n$, then g is strongly reversible in G.*

Proof. Consider $h = (B, \mathbf{0}) \in G$, where $\mathbf{0}$ is the zero element in $\mathbb{F}^n$. Then h is an involution in G such that $hgh^{-1} = g^{-1}$. This proves the lemma. □

In the following lemma, we get a necessary condition on the linear part of $g \in G$ for reversibility and strong reversibility of g.

Lemma 3.3.4. *The following hold.*

(i) If $g=(A,v)$ is reversible in G, then A is reversible in $L(G)$.

(ii) If $g=(A,v)$ is strongly reversible in G, then A is strongly reversible in $L(G)$.

Proof. Consider a group homomorphism $\phi : G \to L(G)$, which sends each element $g=(A,v)\in G$ to its linear part, i.e., $\phi(A,v)=A$. Let g be reversible in G. Then there exists h in G such that $hgh^{-1}=g^{-1}$. This implies $\phi(h)\phi(g)(\phi(h))^{-1}=(\phi(g))^{-1}$. Also, if h is an involution in G, then $\phi(h)$ will be an involution in $L(G)$. The lemma now follows from the above observations. $\square$

Remark 3.3.5. It is worth noting that the proof of Lemma 3.3.4 is purely group-theoretic and works under any homomorphism. Here, it is applied to a special situation of a semi-direct product. $\square$

3.4 Reversibility in $\mathrm{Sp}(n)\ltimes\mathbb{H}^n$

In this section, we will investigate reversibility in the group $\mathrm{Sp}(n)\ltimes\mathbb{H}^n$.

3.4.1 Proof of Theorem 1.1.1

Let $g\in\mathrm{Sp}(n)\ltimes\mathbb{H}^n$. By using Lemma 3.3.1, up to conjugacy, we can write $g=(A,v)$ such that $A(v)=v$, where $A=\mathrm{diag}(e^{\mathbf{i}\theta_1},e^{\mathbf{i}\theta_2},\dots,e^{\mathbf{i}\theta_r},-\mathrm{I}_s,\mathrm{I}_t)$, $\theta_k\in(0,\pi)$, $r,s,t\in\mathbb{N}\cup\{0\}$, $r+s+t=n$, and $v=[0,0,\dots,0,v_1,v_2,\dots,v_t]\in\mathbb{H}^n$.

If $t=0$, i.e., 1 is not an eigenvalue of A, then up to conjugacy, $g=(A,\mathbf{0})$, where $A\in\mathrm{Sp}(n)$. However, every element in $\mathrm{Sp}(n)$ is reversible; see [2, Proposition 3.1]. Therefore, A is reversible. Hence, g is reversible in $\mathrm{Sp}(n)\ltimes\mathbb{H}^n$.

If $t>0$, consider $B=\mathrm{diag}(\mathbf{j},\mathbf{j},\dots,\mathbf{j},-\mathrm{I}_{s+t})\in\mathrm{Sp}(n)$. Then $BAB^{-1}=A^{-1}$, $B(v)=-v$. Hence, the proof follows from Lemma 3.3.2. $\square$

Lemma 3.4.1. *Suppose that $g=(A,w)$ be an element of $\mathrm{Sp}(n)\ltimes\mathbb{H}^n$. Let every eigenvalue class of A is either ±1 or of even multiplicity. Then g is strongly reversible in $\mathrm{Sp}(n)\ltimes\mathbb{H}^n$.*

Proof. Up to conjugacy in $\mathrm{Sp}(n)\ltimes\mathbb{H}^n$, we can write g as: $g=(A,v)$ as in the Lemma 3.3.1. Further, we can assume that A has the form as given in Equation

(3.1.3). Now, consider

$$B = \begin{pmatrix} L & & & & & \\ & L & & & & \\ & & \ddots & & & \\ & & & L & & \\ & & & & \mathrm{I}_s & \\ & & & & & -\mathrm{I}_t \end{pmatrix} \tag{3.4.1}$$

in $\mathrm{Sp}(n)$, where $L = \begin{pmatrix} 0 & \mathbf{j} \\ -\mathbf{j} & 0 \end{pmatrix}$. Then $BAB^{-1} = A^{-1}$, $B(v) = -v$, and $B^2 = \mathrm{I}_n$. Hence, the proof follows from Lemma 3.3.3 . □

3.4.2 Proof of Theorem 1.1.2

The proof of the theorem follows from Theorem 3.1.1, Lemma 3.3.4, and Lemma 3.4.1. □

3.5 Reversibility in $\mathrm{U}(n) \ltimes \mathbb{C}^n$

In this section, we will prove the equivalence between reversible and strongly reversible elements of $\mathrm{U}(n) \ltimes \mathbb{C}^n$.

3.5.1 Proof of Theorem 1.1.4.

The implication $(2) \Rightarrow (1)$ is always true since every strongly reversible element in a group is always reversible. We shall prove the rest.

$(4) \Rightarrow (2)$: Suppose A has a self-dual characteristic polynomial. Therefore, by using Lemma 3.3.1, we can assume that $g = (A, v)$ such that $A(v) = v$ and $A = \mathrm{diag}(e^{\mathbf{i}\theta_1}, e^{-\mathbf{i}\theta_1}, \dots, e^{\mathbf{i}\theta_r}, e^{-\mathbf{i}\theta_r}, -\mathrm{I}_s, \mathrm{I}_t)$, $\theta_k \in (0, \pi)$, $k \in \{1, 2, \dots, r\}$, where $r, s, t \in \mathbb{N} \cup \{0\}$ and v has the form $v = [0, 0, \dots, 0, v_1, v_2, \dots, v_t]$. Let $K = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Consider

$$B = \begin{pmatrix} K & & & & & \\ & K & & & & \\ & & \ddots & & & \\ & & & K & & \\ & & & & \mathrm{I}_s & \\ & & & & & -\mathrm{I}_t \end{pmatrix} \tag{3.5.1}$$

in $\mathrm{U}(n)$. Then $BAB^{-1} = A^{-1}$, $B(v) = -v$, and $B^2 = \mathrm{I}_n$. Hence, the proof follows from Lemma 3.3.3.

(2) $\Rightarrow$ (3): Follows from Lemma 3.3.4.

(3) $\Leftrightarrow$ (4): This equivalence follows from [21, Proposition 3.1].

(1) $\Rightarrow$ (4): Follows from Lemma 3.3.4 and noting the fact that an element A in $\mathrm{U}(n)$ is reversible if and only if it has a self-dual characteristic polynomial; see [21, Corollary 3.2], [11, Theorem 8]. $\square$

3.6 Reversibility in $\mathrm{SU}(n) \ltimes \mathbb{C}^n$

In this section, we will classify the reversible and strongly reversible elements in $\mathrm{SU}(n) \ltimes \mathbb{C}^n$.

3.6.1 Proof of Theorem 1.1.5.

(a) (1) $\Rightarrow$ (2): Follows from the Lemma 3.3.4.

(b) (2) $\Rightarrow$ (3): This equivalence follows from [30, Theorem 4.22], [21, Proposition 3.3].

(c) (3) $\Rightarrow$ (1): Suppose A has a self-dual characteristic polynomial. Therefore, by using Lemma 3.3.1, we can assume that $g = (A, v)$ such that $A(v) = v$ and $A = \mathrm{diag}(e^{\mathbf{i}\theta_1}, e^{-\mathbf{i}\theta_1}, \dots, e^{\mathbf{i}\theta_r}, e^{-\mathbf{i}\theta_r}, -\mathrm{I}_s, \mathrm{I}_t)$, $\theta_k \in (0, \pi)$, $k \in \{1, 2, \dots, r\}$, and v has the form $v = [0, 0, \dots, 0, v_1, v_2, \dots, v_t]$, where $r, s, t \in \mathbb{N} \cup \{0\}$. Further, note that if $A \neq \mathrm{I}_n$, then $(r, s) \neq (0, 0)$.

Suppose that $A \neq \mathrm{I}_n$. Consider B as given in Equation (3.5.1). Note that $\det(K) = -1$ and $\det(B) = (-1)^{r+t}$. If $\det(B) = 1$, then choose this B. If not, then we choose $B \in \mathrm{SU}(n)$ in the following way. If $s \neq 0$, we choose $B \in \mathrm{SU}(n)$ as

$$B = \begin{pmatrix} K & & & & & & \\ & K & & & & & \\ & & \ddots & & & & \\ & & & K & & & \\ & & & & -1 & & \\ & & & & & \mathrm{I}_{s-1} & \\ & & & & & & -\mathrm{I}_t \end{pmatrix}. \tag{3.6.1}$$

Note that this B is also an involution in $\mathrm{SU}(n)$. When $r \neq 0$, the element B can also be chosen as

$$B = \begin{pmatrix} J & & & & & \\ & K & & & & \\ & & \ddots & & & \\ & & & K & & \\ & & & & \mathrm{I}_s & \\ & & & & & -\mathrm{I}_t \end{pmatrix}, \qquad (3.6.2)$$

where $J = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$. Then $\det(B) = 1$, although note that B is no more an involution. Therefore, we have constructed B in $\mathrm{SU}(n)$ such that $BAB^{-1} = A^{-1}$ and $B(v) = -v$. The theorem now follows from Lemma 3.3.2 and Corollary 3.6.4 proven below. $\square$

The following lemma follows by combining Corollary 1.1.6, Lemma 3.2.2 and Lemma 3.3.1.

Lemma 3.6.1. *Let $g = (A, w)$ in $\mathrm{SU}(n) \ltimes \mathbb{C}^n$. Assume that either one of the following holds:*

(a) A has an eigenvalue -1.

(b) $n \equiv 0 \pmod 4$ and A does not have any eigenvalue equal to 1 or -1.

Then the following are equivalent.

(i) g is strongly reversible in $\mathrm{SU}(n) \ltimes \mathbb{C}^n$.

(ii) A is strongly reversible in $\mathrm{SU}(n)$.

(iii) The characteristic polynomial of A is self-dual.

Note that if A has no eigenvalue -1, then g may not be strongly reversible even if A is so. The following example demonstrates this.

Example 3.6.2. Consider $g = (A, v) \in \mathrm{SU}(5) \ltimes \mathbb{C}^5$ such that $v = (0,0,0,0,1)$ and $A = \mathrm{diag}(e^{\mathbf{i}\theta}, e^{-\mathbf{i}\theta}, e^{\mathbf{i}\phi}, e^{-\mathbf{i}\phi}, 1)$, where $\theta, \phi \in (0, \pi)$ such that $\theta \neq \phi$. Note that A is strongly reversible in $\mathrm{SU}(5)$.

Assume that g is strongly reversible in $\mathrm{SU}(5) \ltimes \mathbb{C}^5$. Then there exists $h = (B, u)$ in $\mathrm{SU}(5) \ltimes \mathbb{C}^5$ such that $hgh^{-1} = g^{-1}$ and $h^2 = I$, where I is the identity in $\mathrm{SU}(5) \ltimes \mathbb{C}^5$. Using $hgh^{-1} = g^{-1}$ and $h^2 = I$, we have $B \in \mathrm{SU}(5)$ such that for all $x \in \mathbb{C}^5$,

$$BAB^{-1}(x) - BAB^{-1}(u) + B(v) + u = A^{-1}(x) - A^{-1}(v), \qquad (3.6.3)$$

$$B^2(x) + B(u) + u = \mathrm{I}_5(x). \tag{3.6.4}$$

This implies

$$BAB^{-1} = A^{-1}, (A^{-1} - \mathrm{I}_5)(v-u) = -(B+\mathrm{I}_5)(v), \tag{3.6.5}$$

$$B^2 = \mathrm{I}_5, (B+\mathrm{I}_5)(u) = 0. \tag{3.6.6}$$

From the equation $BA = A^{-1}B$, we have

$$\begin{pmatrix} e^{\mathbf{i}\theta}b_{1,1} & e^{-\mathbf{i}\theta}b_{1,2} & e^{\mathbf{i}\phi}b_{1,3} & e^{-\mathbf{i}\phi}b_{1,4} & b_{1,5} \\ e^{\mathbf{i}\theta}b_{2,1} & e^{-\mathbf{i}\theta}b_{2,2} & e^{\mathbf{i}\phi}b_{2,3} & e^{-\mathbf{i}\phi}b_{2,4} & b_{2,5} \\ e^{\mathbf{i}\theta}b_{3,1} & e^{-\mathbf{i}\theta}b_{3,2} & e^{\mathbf{i}\phi}b_{3,3} & e^{-\mathbf{i}\phi}b_{3,4} & b_{3,5} \\ e^{\mathbf{i}\theta}b_{4,1} & e^{-\mathbf{i}\theta}b_{4,2} & e^{\mathbf{i}\phi}b_{4,3} & e^{-\mathbf{i}\phi}b_{4,4} & b_{4,5} \\ e^{\mathbf{i}\theta}b_{5,1} & e^{-\mathbf{i}\theta}b_{5,2} & e^{\mathbf{i}\phi}b_{5,3} & e^{-\mathbf{i}\phi}b_{5,4} & b_{5,5} \end{pmatrix} = \begin{pmatrix} e^{-\mathbf{i}\theta}b_{1,1} & e^{-\mathbf{i}\theta}b_{1,2} & e^{-\mathbf{i}\theta}b_{1,3} & e^{-\mathbf{i}\theta}b_{1,4} & e^{-\mathbf{i}\theta}b_{1,5} \\ e^{\mathbf{i}\theta}b_{2,1} & e^{\mathbf{i}\theta}b_{2,2} & e^{\mathbf{i}\theta}b_{2,3} & e^{\mathbf{i}\theta}b_{2,4} & e^{\mathbf{i}\theta}b_{2,5} \\ e^{-\mathbf{i}\phi}b_{3,1} & e^{-\mathbf{i}\phi}b_{3,2} & e^{-\mathbf{i}\phi}b_{3,3} & e^{-\mathbf{i}\phi}b_{3,4} & e^{-\mathbf{i}\phi}b_{3,5} \\ e^{\mathbf{i}\phi}b_{4,1} & e^{\mathbf{i}\phi}b_{4,2} & e^{\mathbf{i}\phi}b_{4,3} & e^{\mathbf{i}\phi}b_{4,4} & e^{\mathbf{i}\phi}b_{4,5} \\ b_{5,1} & b_{5,2} & b_{5,3} & b_{5,4} & b_{5,5} \end{pmatrix}.$$

As $\theta, \phi \in (0, \pi)$ such that $\theta \neq \pm\phi$, so from the above matrix equation, we get that matrix B has the following block diagonal form:

$$B = \begin{pmatrix} 0 & a & & & \\ b & 0 & & & \\ & & 0 & c & \\ & & d & 0 & \\ & & & & \alpha \end{pmatrix}, \text{ where } a, b, c, d, \alpha \in \mathbb{C} \setminus \{0\}. \tag{3.6.7}$$

Using $(A^{-1} - \mathrm{I}_5)(v-u) = -(B+\mathrm{I}_5)(v)$ with Equation (3.6.7) and noting that $v = (0,0,0,0,1)$, we obtain $\alpha = -1$. Now, $B^2 = \mathrm{I}_5$ and Equation (3.6.7) implies $ab = cd = 1$. Therefore, we have

$$B = \begin{pmatrix} 0 & a & & & \\ b & 0 & & & \\ & & 0 & c & \\ & & d & 0 & \\ & & & & -1 \end{pmatrix},$$

where $a, b, c, d \in \mathbb{C} \setminus \{0\}$ such that $ab = cd = 1$. Consequently, $\det(B) = -abcd = -1$. This is a contradiction since $B \in \mathrm{SU}(5)$. Hence, g is not strongly reversible in $\mathrm{SU}(5) \ltimes \mathbb{C}^5$.

The above example demonstrates the need to classify the elements (A, v) in $\mathrm{SU}(n) \ltimes \mathbb{C}^n$ that are strongly reversible when A is strongly reversible. The following lemma proves this direction.

Lemma 3.6.3. *Let $g = (A, v) \in \mathrm{SU}(n) \ltimes \mathbb{C}^n$ be such that $A = \mathrm{diag}(e^{\mathbf{i}\theta_1}, e^{-\mathbf{i}\theta_1}, \dots, e^{\mathbf{i}\theta_r}, e^{-\mathbf{i}\theta_r}, \mathrm{I}_t)$ and $v = [0, 0, \dots, 0, v_1, v_2, \dots, v_t]$, where $\theta_k \in (0, \pi)$, $k \in \{1, 2, \dots, r\}$, $t = n - 2r$, and $r, t \in \mathbb{N}$. Assume that one of the following conditions holds:*

(1) Both r and t are either even or odd.

(2) There exists $m \in \{1, 2, \dots, t\}$ such that $v_m = 0$.

(3) For all $m \in \{1, 2, \dots, t\}$, $v_m \neq 0$, and one of the following conditions holds:

(i) r is odd and t is even.

(ii) r is even, t is odd, and $t \neq 1$.

Then g is strongly reversible in $\mathrm{SU}(n) \ltimes \mathbb{C}^n$.

Proof. To prove this lemma, it is sufficient to find an involution B in $\mathrm{SU}(n)$ such that $BAB^{-1} = A^{-1}$, $B^2 = \mathrm{I}_n$, and $B(v) = -v$. Then proof will follow from Lemma 3.3.3. Let $B_r = \begin{pmatrix} & K & & & & \\ & & K & & & \\ & & & \ddots & & \\ & & & & K & \\ & & & & & K \end{pmatrix} \in \mathrm{GL}(2r, \mathbb{C})$. Then $\det(B_r) = (-1)^r$. Note that we can construct the desired involution $B \in \mathrm{SU}(n)$ in the following way:

(1) Consider

$$B = B_r \oplus (-\mathrm{I}_t). \tag{3.6.8}$$

Then $\det(B) = (-1)^r(-1)^t = (-1)^{r+t}$. Therefore, if r and t are such that either both are even or both are odd, then $r + t$ will be even. Hence, $\det(B) = 1$.

(2) If $v = \mathbf{0}$, then g is conjugate to $(A, \mathbf{0})$ and hence strongly reversible. So, without loss of generality, we can assume that $v \neq \mathbf{0}$ and $v_t = 0$. Consider B as given in Equation (3.6.8). If $\det(B) = 1$, then choose this B. Otherwise, replace the nth entry of the nth row (i.e., the row corresponding to the zero entry v_t in v) of B with 1. That will make $\det(B) = 1$, and B will still satisfy the desired conditions making g strongly reversible.

(3) Let $P = \begin{pmatrix} 0 & -\frac{v_1}{v_2} \\ -\frac{v_2}{v_1} & 0 \end{pmatrix}$ and $Q = \begin{pmatrix} 0 & 0 & -\frac{v_1}{v_3} \\ 0 & -1 & 0 \\ -\frac{v_3}{v_1} & 0 & 0 \end{pmatrix}$. Then $\det(P) = -1$ and $\det(Q) = 1$. Therefore, by using P and Q, we can consider the following involution $B \in \mathrm{SU}(n)$:

In the sub-case (i), consider

$$B = B_r \oplus P \oplus (-\mathrm{I}_{t-2}),$$

where $t \in \mathbb{N}$ such that $t \geqslant 2$, and in the sub-case (ii), consider

$$B = B_r \oplus Q \oplus (-\mathrm{I}_{t-3}),$$

where $t \in \mathbb{N}$ such that $t \geqslant 3$.

Thus, in each of the above cases, we have constructed an *involution* B in $\mathrm{SU}(n)$ such that $BAB^{-1} = A^{-1}$ and $B(v) = -v$. Hence, the proof follows from Lemma 3.3.3. □

Corollary 3.6.4. *Let $g = (\mathrm{I}_n, v) \in \mathrm{SU}(n) \ltimes \mathbb{C}^n$ such that $v = [v_1, v_2, \dots, v_n]$. Assume $n \neq 1$. Then g is strongly reversible in $\mathrm{SU}(n) \ltimes \mathbb{C}^n$.*

3.6.2 Proof of Theorem 1.1.7

When $r = 0$, i.e., $n = 1$, then the result holds trivially. So we assume $n \geq 2$, $r \neq 0$.

Suppose that $g = (A, v)$ is strongly reversible in $\mathrm{SU}(n) \ltimes \mathbb{C}^n$. Then there exists $h = (B, u)$ in $\mathrm{SU}(n) \ltimes \mathbb{C}^n$ such that $hgh^{-1} = g^{-1}$ and $h^2 = I$, where I is the identity in $\mathrm{SU}(n) \ltimes \mathbb{C}^n$.

Using $hgh^{-1} = g^{-1}$ and $h^2 = I$, we have $B \in \mathrm{SU}(n)$ such that for all $x \in \mathbb{C}^n$,

$$BAB^{-1}(x) - BAB^{-1}(u) + B(v) + u = A^{-1}(x) - A^{-1}(v), \tag{3.6.9}$$

$$B^2(x) + B(u) + u = \mathrm{I}_n(x). \tag{3.6.10}$$

From this, we have

$$BAB^{-1} = A^{-1}, (A^{-1} - \mathrm{I}_n)(v - u) = -(B + \mathrm{I}_n)(v), \tag{3.6.11}$$

$$B^2 = \mathrm{I}_n, (B + \mathrm{I}_n)(u) = 0. \tag{3.6.12}$$

Further, on comparing nth row and nth column in the matrix equation $BAB^{-1} = A^{-1}$, we get that matrix B has the following form:

$$B = \begin{pmatrix} B_1 & \\ & \alpha \end{pmatrix}, \text{where } \alpha \in \mathbb{C} \setminus \{0\} \text{ and } B_1 \in \mathrm{GL}(n-1, \mathbb{C}) \tag{3.6.13}$$

such that $B_1 A_1 B_1^{-1} = A_1^{-1}$, where $A_1 = \mathrm{diag}(e^{\mathbf{i}\theta_1}, e^{-\mathbf{i}\theta_1}, \dots, e^{\mathbf{i}\theta_{2r}}, e^{-\mathbf{i}\theta_{2r}})$.

Note that $v = [0, 0, \dots, 0, v_1] \in \mathbb{C}^n$ is such that $v_1 \neq 0$. By using the above block diagonal form of B in Equation $(A^{-1} - \mathrm{I}_n)(v - u) = -(B + \mathrm{I}_n)(v)$ and on comparing the last rows, we get $\alpha = -1$. Now, $B^2 = \mathrm{I}_n$ and Equation (3.6.13) implies $B_1A_1B_1^{-1} = A_1^{-1}$ and $B_1^2 = \mathrm{I}_{n-1}$. Therefore, from Lemma 3.2.1, we have $\det(B_1) = (-1)^{n-1}$. As $n = 4r+1$ is an odd natural number, we have

$$\det(B) = \det(B_1)\det(\alpha) = (-1)^{n-1}(-1) = (-1)^n = -1.$$

Therefore, if g is of the form as given in the assertion, then $\det B = -1$, and hence, g can not be strongly reversible in $\mathrm{SU}(n) \ltimes \mathbb{C}^n$. If g is not of the given form, then strong reversibility of g follows from Lemma 3.2.2, Lemma 3.6.1, Lemma 3.6.3, and Corollary 3.6.4. This proves the theorem. $\square$

Chapter 4

Reversibility in general linear groups

In this chapter, we will revisit the reversibility problem in the group $\mathrm{GL}(n,\mathbb{D})$ and classify reversible and strongly reversible elements in $\mathrm{GL}(n,\mathbb{D})$, where $\mathbb{D}=\mathbb{R},\mathbb{C}$, or $\mathbb{H}$.

4.1 Reversibility in $\mathrm{GL}(n,\mathbb{D})$

The classification of reversible elements in $\mathrm{GL}(n,\mathbb{C})$ is given in [30, Theorem 4.2]. Using a similar line of argument, we have extended this classification to the case when $\mathbb{D}=\mathbb{R}$ or $\mathbb{H}$. We have also included the case $\mathbb{D}=\mathbb{C}$ here, as it will be used in Proposition 4.4.1.

Theorem 4.1.1. *An element $A \in \mathrm{GL}(n,\mathbb{D})$ with Jordan form as given in Lemma 2.2.4 is reversible if and only if the following hold:*

(1) For $\mathbb{D}=\mathbb{R}$, the Jordan blocks can be partitioned into pairs $\{\mathrm{J}(\lambda,s),\mathrm{J}(\lambda^{-1},s)\}$, $\{\mathrm{J}_{\mathbb{R}}(\mu\pm\mathbf{i}\nu,\,2t),\mathrm{J}_{\mathbb{R}}(\frac{\mu}{\mu^2+\nu^2}\mp\mathbf{i}\frac{\nu}{\mu^2+\nu^2},\,2t)\}$ or singletons $\{\mathrm{J}(\gamma,m)\}$, $\{\mathrm{J}_{\mathbb{R}}(\alpha\pm\mathbf{i}\beta,\,2\ell)\}$, where $\lambda,\mu,\nu\in\mathbb{R}$ such that $\lambda,\gamma\neq 0,\nu,\beta>0$ and $\lambda\neq\pm1$, $\mu^2+\nu^2\neq 1,\gamma=\pm1,\alpha^2+\beta^2=1$.

(2) For $\mathbb{D}=\mathbb{C}$, the Jordan blocks can be partitioned into pairs $\{\mathrm{J}(\lambda,s),\mathrm{J}(\lambda^{-1},s)\}$ or singletons $\{\mathrm{J}(\mu,m)\}$, where $\lambda,\mu\in\mathbb{C}\setminus\{0\}$ and $\lambda\neq\pm1,\mu=\pm1$.

(3) For $\mathbb{D}=\mathbb{H}$, the Jordan blocks can be partitioned into pairs $\{\mathrm{J}(\lambda,s),\mathrm{J}(\lambda^{-1},s)\}$ or singletons $\{\mathrm{J}(\mu,m)\}$, where $\lambda,\mu\in\mathbb{C}\setminus\{0\}$ with non-negative imaginary parts such that $|\lambda|\neq 1,|\mu|=1$.

Proof. Consider the case $\mathbb{D}=\mathbb{R}$. Using Lemma 2.2.4, A is conjugate to A^{-1} if and only if $\{\mathrm{J}(\lambda_1,m_1),\dots,\mathrm{J}(\lambda_k,m_k),\mathrm{J}_{\mathbb{R}}(\mu_1\pm\mathbf{i}\nu_1,\,2\ell_1),\dots,\mathrm{J}_{\mathbb{R}}(\mu_q\pm\mathbf{i}\nu_q,\,2\ell_q)\}$

$= \{\mathrm{J}(\lambda_1^{-1}, m_1), \ldots, \mathrm{J}(\lambda_k^{-1}, m_k), \mathrm{J}_{\mathbb{R}}(\frac{\mu_1}{\mu_1^2+\nu_1^2} \mp \mathbf{i}\frac{\nu_1}{\mu_1^2+\nu_1^2}, 2\ell_1), \ldots, \mathrm{J}_{\mathbb{R}}(\frac{\mu_q}{\mu_q^2+\nu_q^2} \mp \mathbf{i}\frac{\nu_q}{\mu_q^2+\nu_q^2}, 2\ell_q)\}$, and the result follows immediately for the case $\mathbb{D} = \mathbb{R}$.

Recall that for a unique complex representative $\lambda \in \mathbb{C}$ of an eigenvalue class of $A \in \mathrm{GL}(n, \mathbb{H})$, $[\lambda] = [\lambda^{-1}]$ if and only if $|\lambda| = 1$, i.e., $\lambda^{-1} = \overline{\lambda}$. Using the same line of argument as we use in the $\mathbb{D} = \mathbb{R}$ case, the result follows for the case $\mathbb{D} = \mathbb{H}$ or $\mathbb{C}$. For the proof of (2), one can also see [30, Theorem 4.2]. □

4.2 Reversing symmetry groups in $\mathrm{GL}(n, \mathbb{D})$

This section is devoted to working out certain details on the structures of the reversing symmetry group for certain types of Jordan forms in $\mathrm{GL}(n, \mathbb{D})$, which may be of independent interest.

Using such results, we will investigate strong reversibility in $\mathrm{GL}(n, \mathbb{D})$; see Section 4.4. We first introduce some additional notation that will be used in the rest of this chapter.

Definition 4.2.1. For non-zero $\lambda \in \mathbb{D}$, define $\Omega(\lambda, n) := [x_{i,j}]_{n \times n} \in \mathrm{GL}(n, \mathbb{D})$ such that:

(1) $x_{i,j} = 0$ for $1 \leq j < i \leq n$,

(2) $x_{i,n} = 0$ for all $1 \leq i \leq n-1$,

(3) $x_{n,n} = 1$,

(4) For all $1 \leq i \leq j \leq n-1$, define

$$\begin{aligned} x_{i,j} = & -\lambda^{-2} x_{i+1,j+1} + \lambda^{-3} x_{i+2,j+1} \\ & -\lambda^{-4} x_{i+3,j+1} + \cdots + (-1)^{(n-i)} \lambda^{-(n-i+1)} x_{n,j+1}. \end{aligned} \tag{4.2.1}$$

Equivalently, we can also write Equation (4.2.1) as:

$$x_{i,j} = -\lambda^{-2} x_{i+1,j+1} - \lambda^{-1} x_{i+1,j} \text{ for all } 1 \leq i \leq j \leq n-1. \tag{4.2.2}$$

Remark 4.2.2. For a non-zero $\lambda \in \mathbb{D}$, let $\Omega(\lambda, n) := [x_{i,j}]_{n\times n} \in \mathrm{GL}(n, \mathbb{D})$ be as defined in Definition 4.2.1. Then for $1 \leq i, j \leq n$, we have:

$$x_{i,j} = \begin{cases} 0 & \text{if } j < i \\ 0 & \text{if } j = n, i \neq n \\ (-1)^{n-i}\left(\lambda^{-2(n-i)}\right) & \text{if } j = i \\ (-1)^{n-i}\binom{n-i-1}{j-i}\left(\lambda^{-2n+i+j}\right) & \text{if } i < j, j \neq n \end{cases}, \tag{4.2.3}$$

where $\binom{n-i-1}{j-i}$ denotes the binomial coefficient. □

Definition 4.2.3. Let $K := \begin{pmatrix} \mu & \nu \\ -\nu & \mu \end{pmatrix} \in \mathrm{GL}(2, \mathbb{R})$, and μ, ν are real numbers with $\nu > 0$. Then define $\Omega_{\mathbb{R}}(K, 2n) := [X_{i,j}]_{1\leq i,j\leq n} \in \mathrm{GL}(2n, \mathbb{R})$, where $X_{i,j} \in \mathrm{M}(2, \mathbb{R})$ such that

(1) $X_{i,j} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ for $1 \leq j < i \leq n$,

(2) $X_{i,n} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ for all $1 \leq i \leq n-1$,

(3) $X_{n,n} = \mathrm{I}_2$,

(4) For $1 \leq i \leq j \leq n-1$, define

$$\begin{aligned} X_{i,j} = &- K^{-2}X_{i+1,j+1} + K^{-3}X_{i+2,j+1} \\ &- K^{-4}X_{i+3,j+1} + \cdots + (-1)^{(n-i)}K^{-(n-i+1)}X_{n,j+1}. \end{aligned} \tag{4.2.4}$$

Equivalently, we can also write Equation (4.2.4) as:

$$X_{i,j} = -K^{-2}X_{i+1,j+1} - K^{-1}X_{i+1,j} \text{ for all } 1 \leq i \leq j \leq n-1. \tag{4.2.5}$$

Remark 4.2.4. Observe that the matrix $\Omega(\lambda, n)$ as in Definition 4.2.1 is upper triangular matrix. On the other hand, $\Omega_{\mathbb{R}}(K, 2n) \in \mathrm{GL}(2n, \mathbb{R})$ as in Definition 4.2.3 is block upper triangular matrix. Further, both the above matrices are related by embedding Ψ given in Definition 2.2.1 as follow

$$\Omega_{\mathbb{R}}(\Psi(\mu + \mathbf{i}\nu), 2n) = \Omega_{\mathbb{R}}(K, 2n) = \Psi(-\Omega(\mu + \mathbf{i}\nu, n)),$$

where $K = \Psi(\mu + \mathbf{i}\nu)$. Thus, Definition 4.2.3 can be expressed in terms of Definition 4.2.1 and embedding Ψ. But for notational clarity, we include it separately. $\square$

In the next example, we get a relation between $\Omega(\lambda,4)$ and Jordan block $\mathrm{J}(\lambda,4)$.

Example 4.2.5. Consider $\Omega(\lambda,4)$ as defined in Definition 4.2.1, where $\lambda \in \mathbb{D} \setminus \{0\}$. Note that

$$\begin{aligned}
&\begin{pmatrix} -\lambda^{-6} & -2\lambda^{-5} & -\lambda^{-4} & 0 \\ & \lambda^{-4} & \lambda^{-3} & 0 \\ & & -\lambda^{-2} & 0 \\ & & & 1 \end{pmatrix} \begin{pmatrix} \lambda^{-1} & 1 & 0 & 0 \\ & \lambda^{-1} & 1 & 0 \\ & & \lambda^{-1} & 1 \\ & & & \lambda^{-1} \end{pmatrix} \\
&= \begin{pmatrix} -\lambda^{-7} & -3\lambda^{-6} & -3\lambda^{-5} & -\lambda^{-4} \\ & \lambda^{-5} & 2\lambda^{-4} & \lambda^{-3} \\ & & -\lambda^{-3} & -\lambda^{-2} \\ & & & \lambda^{-1} \end{pmatrix} \\
&= \begin{pmatrix} \lambda^{-1} & -\lambda^{-2} & \lambda^{-3} & -\lambda^{-4} \\ & \lambda^{-1} & -\lambda^{-2} & \lambda^{-3} \\ & & \lambda^{-1} & -\lambda^{-2} \\ & & & \lambda^{-1} \end{pmatrix} \begin{pmatrix} -\lambda^{-6} & -2\lambda^{-5} & -\lambda^{-4} & 0 \\ & \lambda^{-4} & \lambda^{-3} & 0 \\ & & -\lambda^{-2} & 0 \\ & & & 1 \end{pmatrix}.
\end{aligned}$$

This implies

$$\Omega(\lambda,4)\,\mathrm{J}(\lambda^{-1},4) = \mathrm{J}(\lambda,4)^{-1}\,\Omega(\lambda,4).$$

Further, note that if $\lambda = \pm 1$, then $\lambda^{-1} = \lambda$ and $\Omega(\lambda,4)$ is a involution. Therefore, for $\lambda = \pm 1$, $\mathrm{J}(\lambda,4)$ is a strongly reversible element in $\mathrm{GL}(4,\mathbb{D})$. $\square$

Now, we generalize the Example 4.2.5.

Lemma 4.2.6. *Let $\Omega(\lambda,n) \in \mathrm{GL}(n,\mathbb{D})$ and $\Omega_{\mathbb{R}}(K,2n) \in \mathrm{GL}(2n,\mathbb{R})$ be respectively defined as in Definition 4.2.1 and Definition 4.2.3. Then the following statements hold.*

(1) $\Omega(\lambda,n)\,\mathrm{J}(\lambda^{-1},n) = \Big(\mathrm{J}(\lambda,n)\Big)^{-1}\Omega(\lambda,n).$

(2) $\Omega_{\mathbb{R}}(K,2n)\,\mathrm{J}_{\mathbb{R}}\Big(\frac{\mu}{\mu^2+\nu^2} \mp \mathbf{i}\frac{\nu}{\mu^2+\nu^2},\, 2n\Big) = \Big(\mathrm{J}_{\mathbb{R}}(\mu \pm \mathbf{i}\nu,\, 2n)\Big)^{-1}\Omega_{\mathbb{R}}(K,2n).$

Proof. Write $\mathrm{J}(\lambda^{-1},n) = [a_{i,j}]_{1\le i,j\le n} \in \mathrm{GL}(n,\mathbb{D})$, where $a_{i,j} = \begin{cases} \lambda^{-1} & \text{if } j=i \\ 1 & \text{if } j=i+1 \\ 0 & \text{otherwise} \end{cases}.$

Also, $\Big(\mathrm{J}(\lambda,n)\Big)^{-1} = [b_{i,j}]_{n\times n} \in \mathrm{GL}(n,\mathbb{D})$, where $b_{i,j} = \begin{cases} \lambda^{-1} & \text{if } j=i \\ (-1)^k\lambda^{-(k+1)} & \text{if } j=i+k \\ 0 & \text{otherwise} \end{cases}.$

Recall the matrix $\Omega(\lambda,n) = [x_{i,j}]_{n\times n} \in \mathrm{GL}(n,\mathbb{D})$ as defined in the Definition 4.2.1. Note that $\Big(\Omega(\lambda,n)\,\mathrm{J}(\lambda^{-1},n)\Big)_{i,i} = \Big(\Big(\mathrm{J}(\lambda,n)\Big)^{-1}\Omega(\lambda,n)\Big)_{i,i} = \lambda^{-1}x_{i,i}$ for all $1 \le i \le n$. Since matrices under consideration are upper triangular, so it is enough to prove the following equality:

$$\Big(\Omega(\lambda,n)\,\mathrm{J}(\lambda^{-1},n)\Big)_{i,j} = \Big(\Big(\mathrm{J}(\lambda,n)\Big)^{-1}\Omega(\lambda,n)\Big)_{i,j} = \lambda^{-1}x_{i,j} + x_{i,j-1},$$

for all $1 \le i < j \le n$. To see this, note that for $1 \le i < j \le n$, we have

$$\begin{aligned}\Big(\Omega(\lambda,n)\,\mathrm{J}(\lambda^{-1},n)\Big)_{i,j} &= \sum_{r=1}^{n}(x_{i,r})\,(a_{r,j}) \\ &= \sum_{r=i}^{n}(x_{i,r})\,(a_{r,j}) = x_{i,j-1} + x_{i,j}\lambda^{-1} = \lambda^{-1}x_{i,j} + x_{i,j-1}.\end{aligned}$$

Here, we have used $x_{i,j}\lambda^{-1} = \lambda^{-1}x_{i,j}$ for all $1 \le i,j \le n$. Further, note that for $1 \le i < j \le n$, we have

$$\begin{aligned}\Big(\Big(\mathrm{J}(\lambda,n)\Big)^{-1}\Omega(\lambda,n)\Big)_{i,j} &= \sum_{r=1}^{n}(b_{i,r})\,(x_{r,j}) = \sum_{r=i}^{j}(b_{i,r})\,(x_{r,j}) \\ &= b_{i,i}x_{i,j} + b_{i,i+1}x_{i+1,j} + \cdots + b_{i,j}x_{j,j} \\ &= (\lambda^{-1}x_{i,j}) + (-\lambda^{-2}x_{i+1,j} + \cdots + (-1)^{(j-i)}\lambda^{-(j-i+1)}x_{j,j}).\end{aligned}$$

Using the equivalence between Equations (4.2.1) and (4.2.2), we get:

$$\Big(\Big(\mathrm{J}(\lambda,n)\Big)^{-1}\Omega(\lambda,n)\Big)_{i,j} = \lambda^{-1}x_{i,j} + x_{i,j-1} \text{ for all } 1 \le i < j \le n.$$

This proves the first part of this lemma. Next, we prove the second part of the lemma. Let $\mathrm{J}_{\mathbb{R}}\Big(\frac{\mu}{\mu^2+\nu^2} \mp \mathbf{i}\frac{\nu}{\mu^2+\nu^2}, 2n\Big) = [A_{i,j}]_{1\le i,j\le n}$ and $\Big(\mathrm{J}_{\mathbb{R}}(\mu \pm \mathbf{i}\nu, 2n)\Big)^{-1} = [B_{i,j}]_{1\le i,j\le n}$, where $A_{i,j}$

and $B_{i,j}$ are given by

$$A_{i,j} = \begin{cases} K^{-1} & \text{if } j = i \\ \mathrm{I}_2 & \text{if } j = i+1 \\ \mathrm{O}_2 & \text{otherwise} \end{cases}, \text{ and } B_{i,j} = \begin{cases} K^{-1} & \text{if } j = i \\ (-1)^k K^{-(k+1)} & \text{if } j = i+k \\ \mathrm{O}_2 & \text{otherwise} \end{cases}.$$

Recall the $\Omega_{\mathbb{R}}(K,n) = [X_{i,j}]_{1\leq i,j\leq n} \in \mathrm{GL}(2n,\mathbb{R})$, as defined in Definition 4.2.3. Now, using the same line of arguments as we used in the proof of the first part of this lemma, we have

$$\Big(\Omega_{\mathbb{R}}(K,2n)\,\mathrm{J}_{\mathbb{R}}\Big(\frac{\mu}{\mu^2+\nu^2} \mp \mathbf{i}\frac{\nu}{\mu^2+\nu^2}, 2n\Big)\Big)_{i,j} = \Big(\Big(\mathrm{J}_{\mathbb{R}}(\mu \pm \mathbf{i}\nu, 2n)\Big)^{-1}\Omega_{\mathbb{R}}(K,2n)\Big)_{i,j}$$

$$= \begin{cases} K^{-1}(X_{i,i}) & \text{if } j = i \\ X_{i,j-1} + K^{-1}(X_{i,j}) & \text{if } i < j \\ \mathrm{O}_2 & \text{if } i > j \end{cases}, \text{ for all } 1 \leq i,j \leq n.$$

Therefore, $\Omega_{\mathbb{R}}(K,2n)\,\mathrm{J}_{\mathbb{R}}\Big(\frac{\mu}{\mu^2+\nu^2} \mp \mathbf{i}\frac{\nu}{\mu^2+\nu^2}, 2n\Big) = \Big(\mathrm{J}_{\mathbb{R}}(\mu \pm \mathbf{i}\nu, 2n)\Big)^{-1}\Omega_{\mathbb{R}}(K,2n)$. This completes the proof. □

Next, we want to find relationship between $\Omega(\lambda,n)$ and $\Omega(\lambda^{-1},n)$, which will be used for constructing involutions in $\mathrm{GL}(n,\mathbb{D})$. For this, we will use some well-known combinatorial identities. We refer to [7, Section 1.2] for the basic notion related to the binomial coefficients. Recall the following well-known binomial identities regarding binomial coefficients:

Id.1 *Pascal's rule:* $\binom{n}{k} + \binom{n}{k-1} = \binom{n+1}{k}$ for all $1 \leq k \leq n$.

Id.2 *Newton's identity:* $\binom{n}{k}\binom{k}{r} = \binom{n}{r}\binom{n-r}{k-r}$ for all $0 \leq r \leq k \leq n$.

Id.3 For $n \geq 1$, we have $\sum_{k=0}^{n}(-1)^k\binom{n}{k} = 0$.

Lemma 4.2.7. *Let $\Omega(\lambda,n) \in \mathrm{GL}(n,\mathbb{D})$ and $\Omega_{\mathbb{R}}(K,2n) \in \mathrm{GL}(2n,\mathbb{R})$ be respectively defined as in Definition 4.2.1 and Definition 4.2.3. Then the following statements hold.*

(1) $\Big(\Omega(\lambda,n)\Big)^{-1} = \Omega(\lambda^{-1},n)$.

(2) $\Big(\Omega_{\mathbb{R}}(K,2n)\Big)^{-1} = \Omega_{\mathbb{R}}(K^{-1},2n)$.

Proof. Here, we will only prove the first part of the lemma. The second part of the lemma can be proved using the same line of arguments.

Let $\Omega(\lambda,n)=[x_{i,j}]_{n\times n}$ and $\Omega(\lambda^{-1},n)=[y_{i,j}]_{n\times n}$ in $\mathrm{GL}(n,\mathbb{D})$. Then observe that for all $1\le i,j\le n$, we have:

$$x_{i,j}=\begin{cases}0 & \text{if } j<i\\ 0 & \text{if } j=n, i\neq n\\ (-1)^{n-i}\left(\lambda^{-2(n-i)}\right) & \text{if } j=i\\ (-1)^{n-i}\binom{n-i-1}{j-i}\left(\lambda^{-2n+i+j}\right) & \text{if } i<j, j\neq n\end{cases}, \text{ and}$$

$$y_{i,j}=\begin{cases}0 & \text{if } j<i\\ 0 & \text{if } j=n, i\neq n\\ (-1)^{n-i}\left(\lambda^{2(n-i)}\right) & \text{if } j=i\\ (-1)^{n-i}\binom{n-i-1}{j-i}\left(\lambda^{-(-2n+i+j)}\right) & \text{if } i<j, j\neq n\end{cases}.$$

Here, condition (4.2.2) in Definition 4.2.1 can be checked using Pascal's rule (**Id**.1).

Let $g:=\Omega(\lambda,n)\,\Omega(\lambda^{-1},n)=[g_{i,j}]_{n\times n}$. Then g is an upper triangular matrix with diagonal entries equal to 1 such that $g_{i,j}=\sum_{k=1}^{n}(x_{i,k})\,(y_{k,j})=\sum_{k=i}^{j}(x_{i,k})\,(y_{k,j})$ for all $1\le i<j\le n$. This implies that for all $1\le i<j\le n$, we have

$$g_{i,j}=\sum_{k=i}^{j}(-1)^{n-i}\binom{n-i-1}{k-i}\left(\lambda^{-2n+i+k}\right)(-1)^{n-k}\binom{n-k-1}{j-k}\left(\lambda^{-(-2n+k+j)}\right).$$

Therefore, for all $1\le i<j\le n$, we have

$$g_{i,j}=\left(\lambda^{i-j}\right)\sum_{k=i}^{j}(-1)^{(-i-k)}\binom{n-i-1}{k-i}\binom{n-k-1}{j-k}. \tag{4.2.6}$$

By substituting $r=k-i$ in Equation (4.2.6), we get

$$g_{i,j}=\lambda^{i-j}\sum_{r=0}^{j-i}(-1)^{(-2i-r)}\binom{n-i-1}{r}\binom{(n-i-1)-r}{(j-i)-r}.$$

In view of the Newton's identity (**Id**.2) and identity (**Id**.3), we get

$$g_{i,j}=\lambda^{i-j}\binom{n-i-1}{j-i}\sum_{r=0}^{j-i}(-1)^{r}\binom{j-i}{r}=0 \text{ for all } 1\le i<j\le n.$$

Therefore, $g=\mathrm{I}_n$ in $\mathrm{GL}(n,\mathbb{D})$. This completes the proof. $\square$

Lemma 4.2.8. *Let* $\Omega(\mu,n) \in \mathrm{GL}(n,\mathbb{D})$ *be as defined in the Definition 4.2.1. If* $\mu = \pm 1$*, then* $\Omega(\mu,n)$ *is an involution in* $\mathrm{GL}(n,\mathbb{D})$.

Proof. In view of the Lemma 4.2.7, we have

$$\Big(\Omega(\mu,n)\Big)^{-1} = \Omega(\mu^{-1},n).$$

If $\mu = \pm 1$, then $\mu^{-1} = \mu$. Hence, the proof follows. □

Recall that $\mathrm{J}_{\mathbb{R}}(\mu \pm \mathbf{i}\nu,\, 2n)$ and $\mathrm{J}_{\mathbb{R}}(\mu \mp \mathbf{i}\nu,\, 2n)$ are defined in (2.2.1) and (2.2.2), respectively.

Lemma 4.2.9. *Let* $\Omega_{\mathbb{R}}(K,2n) \in \mathrm{GL}(2n,\mathbb{R})$ *be as defined in Definition 4.2.3. Let* $g := \Omega_{\mathbb{R}}(K,2n)\,\sigma$*, where* $\sigma = \mathrm{diag}(1,-1,1,-1,\dots,(-1)^{2n-1}\,)_{2n\times 2n}$*. If* $\det(K) = 1$*, that is,* $\mu^2+\nu^2 = 1$*, then the following statements hold.*

(1) $g\Big(\mathrm{J}_{\mathbb{R}}(\mu \pm \mathbf{i}\nu,\, 2n)\Big)g^{-1} = \Big(\mathrm{J}_{\mathbb{R}}(\mu \pm \mathbf{i}\nu,\, 2n)\Big)^{-1}.$

(2) g *is an involution in* $\mathrm{GL}(2n,\mathbb{R})$.

Proof. Note that using $\det(K) = 1$, i.e., $\mu^2+\nu^2 = 1$ and Lemma 4.2.6, we have

$$\Omega_{\mathbb{R}}(K,2n)\Big(\mathrm{J}_{\mathbb{R}}(\mu \mp \mathbf{i}\nu,\, 2n)\Big)(\Omega_{\mathbb{R}}(K,2n))^{-1} = \Big(\mathrm{J}_{\mathbb{R}}(\mu \pm \mathbf{i}\nu,\, 2n)\Big)^{-1}.$$

The proof of the first part of this lemma now follows from Equation (2.2.2).

Now, we prove the second part of the lemma. Write $\sigma = \mathrm{diag}(P,P,\dots,P)_{2n\times 2n}$, where $P := \left(\begin{smallmatrix}1 & 0\\ 0 & -1\end{smallmatrix}\right)$, and $g = \Omega_{\mathbb{R}}(K,2n)\,\sigma = [X_{i,j}]_{1\le i,j\le n}$ in $\mathrm{GL}(2n,\mathbb{R})$. Then observe that each entry $X_{i,j} \in \mathrm{M}(2,\mathbb{R})$ of g satisfies the following relation

$$X_{i,j} = \begin{cases} 0 & \text{if } j < i \\ 0 & \text{if } j = n, i \neq n \\ (-1)^{n-i}\Big(K^{-2(n-i)}\Big)P & \text{if } j = i \\ (-1)^{n-i}\binom{n-i-1}{j-i}\Big(K^{-2n+i+j}\Big)P & \text{if } i < j, j \neq n. \end{cases}$$

Since $\det(K) = 1$, we have $PKP^{-1} = K^{-1}$. This implies

$$K^m P = P K^{-m} \text{ for all } m \in \mathbb{N}.$$

Therefore, $\Omega_{\mathbb{R}}(K,2n)\,\sigma = \sigma^{-1}\,\Omega_{\mathbb{R}}(K,2n) = \sigma\,\Omega_{\mathbb{R}}(K,2n)$. This implies

$$g^2 = \Big(\Omega_{\mathbb{R}}(K,2n)\,\sigma\Big)\Big(\Omega_{\mathbb{R}}(K,2n)\,\sigma\Big) = \Omega_{\mathbb{R}}(K,2n)\,\Omega_{\mathbb{R}}(K^{-1},2n).$$

Now the proof of this lemma follows from Lemma 4.2.7. $\square$

In the following section, we will use Lemmas 4.2.6, 4.2.7, 4.2.8, and 4.2.9 to explicitly construct a reversing involution for certain reversible Jordan forms in $\mathrm{GL}(n,\mathbb{D})$.

4.3 Strong reversibility of Jordan forms in $\mathrm{GL}(n,\mathbb{D})$

In this section, we will prove that certain Jordan forms in $\mathrm{GL}(n,\mathbb{D})$ are strongly reversible. These results can also be proved using the notion of *adjoint reality*; see [15].

Lemma 4.3.1. *Let $A \in \mathrm{GL}(n,\mathbb{D})$ be the Jordan block $\mathrm{J}(\mu, n)$, where $\mu \in \{\pm 1\}$ and $\mathbb{D} = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$. Then A is strongly reversible in $\mathrm{GL}(n,\mathbb{D})$.*

Proof. Write $A = \mathrm{J}(\mu, n) \in \mathrm{GL}(n,\mathbb{D})$, where $\mu \in \{\pm 1\}$. Let $g := \Omega(\mu, n)$ be as defined in Definition 4.2.1. Then Lemma 4.2.6 and Lemma 4.2.8 implies that g is an involution in $\mathrm{GL}(n,\mathbb{D})$ such that $gAg^{-1} = A^{-1}$. This completes the proof. $\square$

Lemma 4.3.2. *Let $A \in \mathrm{GL}(2n,\mathbb{D})$ be the Jordan form $\mathrm{J}(\lambda, n) \oplus \mathrm{J}(\lambda^{-1}, n)$, where $\lambda \in \mathbb{D} \setminus \{\pm 1, 0\}$ for $\mathbb{D} = \mathbb{R}$ or $\mathbb{C}$, and for $\mathbb{D} = \mathbb{H}$, $\lambda \in \mathbb{C} \setminus \{0\}$ with non-negative imaginary part such that $|\lambda| \neq 1$. Then A is strongly reversible in $\mathrm{GL}(2n,\mathbb{D})$.*

Proof. Write $A = \begin{pmatrix} \mathrm{J}(\lambda,n) & \\ & \mathrm{J}(\lambda^{-1},n) \end{pmatrix} \in \mathrm{GL}(2n,\mathbb{D})$. Let $g = \begin{pmatrix} & \Omega(\lambda,n) \\ \Omega(\lambda^{-1},n) & \end{pmatrix}$, where $\Omega(\lambda, n) \in \mathrm{GL}(n,\mathbb{D})$ as defined in Definition 4.2.1. Then Lemma 4.2.7 implies that g is an involution in $\mathrm{GL}(2n,\mathbb{D})$. Moreover, $gAg^{-1} = A^{-1}$ if and only if $\Omega(\lambda, n)\,\mathrm{J}(\lambda^{-1}, n) = \Big(\mathrm{J}(\lambda, n)\Big)^{-1} \Omega(\lambda, n)$. The proof now follows from Lemma 4.2.6. $\square$

Lemma 4.3.3. *Let $A \in \mathrm{GL}(2n,\mathbb{H})$ be the Jordan block $\mathrm{J}(\mu, n) \oplus \mathrm{J}(\mu, n)$, where $\mu \in \mathbb{C} \setminus \{\pm 1\}$ with non-negative imaginary part such that $|\mu| = 1$. Then A is strongly reversible in $\mathrm{GL}(2n,\mathbb{H})$.*

Proof. Write $A = \begin{pmatrix} \mathrm{J}(e^{\mathbf{i}\theta},n) & \\ & \mathrm{J}(e^{\mathbf{i}\theta},n) \end{pmatrix}$, where $\theta \in (0, \pi)$. Recall that $\mathbf{j}Z = \overline{Z}\mathbf{j}$ for all $Z \in \mathrm{GL}(n,\mathbb{C})$, where $\overline{Z}$ is the matrix obtained by taking conjugate of each entry of complex matrix Z. This implies

$$\mathbf{j}\Big(\mathrm{J}(e^{\mathbf{i}\theta}, n)\Big)\mathbf{j}^{-1} = \overline{\mathrm{J}(e^{\mathbf{i}\theta}, n)} = \mathrm{J}(e^{-\mathbf{i}\theta}, n). \tag{4.3.1}$$

Let $g = \begin{pmatrix} & \Omega(e^{\mathbf{i}\theta},n)\,\mathbf{j} \\ \left(\Omega(e^{\mathbf{i}\theta},n)\,\mathbf{j}\right)^{-1} & \end{pmatrix}$, where $\Omega(e^{\mathbf{i}\theta},n) \in \mathrm{GL}(n,\mathbb{C})$ as defined in Definition 4.2.1. Note that g is an involution in $\mathrm{GL}(2n,\mathbb{H})$. Moreover, $gAg^{-1} = A^{-1}$ if and only if $\left(\Omega(e^{\mathbf{i}\theta},n)\,\mathbf{j}\right)\left(\mathrm{J}(e^{\mathbf{i}\theta},n)\right)\left(\Omega(e^{\mathbf{i}\theta},n)\,\mathbf{j}\right)^{-1} = \left(\mathrm{J}(e^{\mathbf{i}\theta},n)\right)^{-1}$. The proof now follows from Lemma 4.2.6 and Equation (4.3.1). □

Lemma 4.3.4. *Let* $A := \mathrm{J}_{\mathbb{R}}(\mu \pm \mathbf{i}\nu,\, 2n)$ *be the Jordan block in* $\mathrm{GL}(2n,\mathbb{R})$ *as defined in* (2.2.1). *If* $\mu^2+\nu^2 = 1$, *then* A *is strongly reversible in* $\mathrm{GL}(2n,\mathbb{R})$.

Proof. The proof follows from Lemma 4.2.9. □

Lemma 4.3.5. *Let* $A := \begin{pmatrix} \mathrm{J}_{\mathbb{R}}(\mu \pm \mathbf{i}\nu, 2n) & \\ & \mathrm{J}_{\mathbb{R}}\left(\frac{\mu}{\mu^2+\nu^2} \mp \mathbf{i}\frac{\nu}{\mu^2+\nu^2}, 2n\right) \end{pmatrix}$ *be the Jordan form in* $\mathrm{GL}(4n,\mathbb{R})$, *where* $\mu^2+\nu^2 \neq 1$. *Then* A *is strongly reversible in* $\mathrm{GL}(4n,\mathbb{R})$.

Proof. Let $g = \begin{pmatrix} & \Omega_{\mathbb{R}}(K,2n) \\ \Omega_{\mathbb{R}}(K^{-1},2n) & \end{pmatrix} \in \mathrm{GL}(4n,\mathbb{R})$, where $\Omega_{\mathbb{R}}(K,2n) \in \mathrm{GL}(2n,\mathbb{R})$ is as defined in Definition 4.2.3. Then Lemma 4.2.7 implies that g is an involution in $\mathrm{GL}(4n,\mathbb{R})$. Note that $gAg^{-1} = A^{-1}$ if and only if $\Omega_{\mathbb{R}}(K,2n)\,\mathrm{J}_{\mathbb{R}}(\frac{\mu}{\mu^2+\nu^2} \mp \mathbf{i}\frac{\nu}{\mu^2+\nu^2},\, 2n) = \left(\mathrm{J}_{\mathbb{R}}(\mu \pm \mathbf{i}\nu,\, 2n)\right)^{-1}\Omega_{\mathbb{R}}(K,2n)$. The proof now follows from the Lemma 4.2.6. □

Note the following lemma, which will be used to prove Table 1.1.

Lemma 4.3.6. *Let* $A \in \mathrm{GL}(n,\mathbb{H})$ *be the Jordan block* $\mathrm{J}(\mu,\, n)$, *where* $\mu \in \mathbb{C}\setminus\{\pm 1\}$ *with non-negative imaginary part such that* $|\mu| = 1$. *Then* A *is reversible in* $\mathrm{GL}(n,\mathbb{H})$.

Proof. Consider $\Omega(\mu,n) \in \mathrm{GL}(n,\mathbb{C})$, which is defined in Definition 4.2.1. Let $h := \Omega(\mu,n)\,\mathbf{j} \in \mathrm{GL}(n,\mathbb{H})$. Since $|\mu| = 1$, i.e., $\mu^{-1} = \overline{\mu}$, we have $\mathbf{j}\mu = \mu^{-1}\mathbf{j}$. This implies that $\mathbf{j}\,\mathrm{J}(\mu,n) = \mathrm{J}(\mu^{-1},n)\,\mathbf{j}$. Therefore, using Lemma 4.2.6, we have $hAh^{-1} = A^{-1}$. Thus, A is reversible in $\mathrm{GL}(n,\mathbb{H})$. □

Remark 4.3.7. In Table 1.1, we have summarized all the reversing symmetries constructed in this section. In view of Theorem 4.1.1, using the results of this section, for an arbitrary reversible element $A \in \mathrm{SL}(n,\mathbb{D})$, we can explicitly construct a reversing element g in $\mathrm{SL}(n,\mathbb{D})$ such that $gAg^{-1} = A^{-1}$. □

4.3.1 Proof of Table 1.1

The proof of Table 1.1 follows from Lemmas 4.3.1 − 4.3.6. □

4.4 Strong reversibility in $\mathrm{GL}(n,\mathbb{D})$

In this section, we will investigate strongly reversible elements in $\mathrm{GL}(n,\mathbb{D})$. Recall that every strongly reversible element in $\mathrm{GL}(n,\mathbb{D})$ is reversible. In the next result, we will prove that the converse holds when $\mathbb{D}=\mathbb{R}$ or $\mathbb{C}$.

Proposition 4.4.1 (cf. [41], [30, Theorem 4.7])**.** *Let $A \in \mathrm{GL}(n,\mathbb{D})$, where $\mathbb{D}=\mathbb{R}$ or $\mathbb{C}$. Then A is reversible in* $\mathrm{GL}(n,\mathbb{D})$ *if and only if A is strongly reversible in* $\mathrm{GL}(n,\mathbb{D})$.

Proof. In view of Theorem 4.1.1, the proof follows from Lemma 4.3.1, Lemma 4.3.2, Lemma 4.3.4, and Lemma 4.3.5. $\square$

The next example shows that Proposition 4.4.1 does not hold in the case $\mathbb{D}=\mathbb{H}$.

Example 4.4.2. Let $A := (\mathbf{i}) \in \mathrm{GL}(1,\mathbb{H})$. Then $gAg^{-1}=A^{-1}$, where $g=(\mathbf{j})$ in $\mathrm{GL}(1,\mathbb{H})$. Thus, A is reversible in $\mathrm{GL}(1,\mathbb{H})$. Assume that A is strongly reversible in $\mathrm{GL}(1,\mathbb{H})$. Let $g=(a)\in \mathrm{GL}(1,\mathbb{H})$ be an involution such that $gAg^{-1}=A^{-1}$. Then we get $a\mathbf{i}=-\mathbf{i}a$. This implies $a=w\mathbf{j}$ for some $w\in\mathbb{C}$, $w\neq 0$. Since g is an involution, we have $a^2=(w\mathbf{j})^2=1$, i.e., $|w|^2=-1$. This is a contradiction. Therefore, A is reversible in $\mathrm{GL}(1,\mathbb{H})$ but not strongly reversible. $\square$

In Theorem 1.2.1, we provide a sufficient criterion for the reversible elements in $\mathrm{GL}(n,\mathbb{H})$ to be strongly reversible. In Chapter 7, by using the notion of Weyr canonical form, cf. [29], we will prove that converse of Theorem 1.2.1 also holds.

4.4.1 Proof of Theorem 1.2.1

In view of Theorem 4.1.1, the proof follows from Lemma 4.3.1, Lemma 4.3.2 and Lemma 4.3.3. $\square$

Chapter 5

Reversibility of affine transformations

Let $\mathrm{Aff}(n,\mathbb{D})$ denote the affine group of all invertible affine transformations on the affine space $\mathbb{D}^n$, where $\mathbb{D}=\mathbb{R},\mathbb{C}$ or $\mathbb{H}$. Each element $g=(A,v)$ of $\mathrm{GL}(n,\mathbb{D})\ltimes\mathbb{D}^n$ acts on $\mathbb{D}^n$ as an affine transformation $g(x)=A(x)+v$, where $A\in\mathrm{GL}(n,\mathbb{D})$ is called the linear part of g and $v\in\mathbb{D}^n$ is called the translation part of g. This action identifies the affine group $\mathrm{Aff}(n,\mathbb{D})$ with $\mathrm{GL}(n,\mathbb{D})\ltimes\mathbb{D}^n$. We can embed $\mathbb{D}^n$ into $\mathbb{D}^{n+1}$ as the plane $\mathrm{P}:=\{(x,1)\in\mathbb{D}^{n+1}\mid x\in\mathbb{D}^n\}$. Consider the embedding $\Theta:\mathrm{Aff}(n,\mathbb{D})\longrightarrow\mathrm{GL}(n+1,\mathbb{D})$ defined as

$$\Theta((A,v))=\begin{pmatrix}A & v\\ \mathbf{0} & 1\end{pmatrix}, \tag{5.0.1}$$

where $\mathbf{0}\in\mathbb{D}^n$ is the zero vector. Note that action of $\Theta(\mathrm{Aff}(n,\mathbb{D}))$ on P is exactly same as action of $\mathrm{Aff}(n,\mathbb{D})$ on $\mathbb{D}^n$. In particular, for any $(x,1)\in\mathbf{P}$, we have

$$\begin{pmatrix}A & v\\ \mathbf{0} & 1\end{pmatrix}\begin{pmatrix}x\\ 1\end{pmatrix}=\begin{pmatrix}Ax+v\\ 1\end{pmatrix}.$$

In this chapter, we will classify reversible and strongly reversible elements in the affine group $\mathrm{Aff}(n,\mathbb{D})$. This chapter is based on [16]. We begin with an example.

Example 5.0.1. Let $g=(\mathrm{I}_n,v)\in\mathrm{Aff}(n,\mathbb{D})$. Consider $g_1=(-\mathrm{I}_n,\mathbf{0})$ and $g_2=(-\mathrm{I}_n,-v)$ in $\mathrm{Aff}(n,\mathbb{D})$. Then g_1 and g_2 are involutions in $\mathrm{Aff}(n,\mathbb{D})$ such that

$$g=g_1\,g_2,\ \text{i.e., } (\mathrm{I}_n,v)=(-\mathrm{I}_n,\mathbf{0})\,(-\mathrm{I}_n,-v).$$

Hence, g is strongly reversible in $\mathrm{Aff}(n,\mathbb{D})$. $\square$

In the following result, we obtain necessary and sufficient conditions for reversibility in $\mathrm{Aff}(n,\mathbb{D})$.

Lemma 5.0.2. *Let $g=(A,v)\in \mathrm{Aff}(n,\mathbb{D})$ be an arbitrary element. Then g is reversible in $\mathrm{Aff}(n,\mathbb{D})$ if and only if there exists an element $h=(B,w)\in \mathrm{Aff}(n,\mathbb{D})$ such that both the following conditions hold:*

(i) $BAB^{-1}=A^{-1}$,

(ii) $(A^{-1}-\mathrm{I}_n)(w)=(A^{-1}+B)(v)$.

Proof. Note that $g^{-1}(x)=A^{-1}(x)-A^{-1}(v)$, and $h^{-1}(x)=B^{-1}(x)-B^{-1}(w)$ for all $x\in\mathbb{D}^n$. This implies that for all $x\in\mathbb{D}^n$, we have

$$hgh^{-1}(x)=h(AB^{-1}(x)-AB^{-1}(w)+v)=BAB^{-1}(x)-BAB^{-1}(w)+B(v)+w.$$

Therefore, $hgh^{-1}=g^{-1}\Leftrightarrow BAB^{-1}=A^{-1}$, and $-A^{-1}(v)=-BAB^{-1}(w)+B(v)+w$. This proves the lemma. □

The following lemma gives necessary and sufficient conditions for strong reversibility in $\mathrm{Aff}(n,\mathbb{D})$.

Lemma 5.0.3. *Let $g=(A,v)\in \mathrm{Aff}(n,\mathbb{D})$ be an arbitrary element. Then g is strongly reversible in $\mathrm{Aff}(n,\mathbb{D})$ if and only if there exists an element $h=(B,w)\in \mathrm{Aff}(n,\mathbb{D})$ such that both the following conditions hold:*

(i) $BAB^{-1}=A^{-1}$, *and* $B^2=\mathrm{I}_n$,

(ii) $(B+\mathrm{I}_n)(w)=\mathbf{0}$, *and* $(B+A^{-1})(w-v)=\mathbf{0}$.

Proof. Note that $h=(B,w)\in \mathrm{Aff}(n,\mathbb{D})$ is an involution if and only if $h^2(x)=B^2(x)+B(w)+w=x$ for all $x\in\mathbb{D}^n$. This implies $B^2=\mathrm{I}_n$, and $(B+\mathrm{I}_n)(w)=\mathbf{0}$. Furthermore, in view of Lemma 5.0.2, $hgh^{-1}=g^{-1}$ if and only if conditions (i) and (ii) of Lemma 5.0.2 hold. Observe that the equation $(B+\mathrm{I}_n)(w)=\mathbf{0}$ and the equation $(A^{-1}-\mathrm{I}_n)(w)=(A^{-1}+B)(v)$ imply $(B+A^{-1})(w-v)=\mathbf{0}$. Hence, the proof follows. □

5.1 Conjugacy in the affine group $\mathrm{Aff}(n,\mathbb{D})$

In the affine group $\mathrm{Aff}(n,\mathbb{D})$, up to conjugacy, we can consider every element in a more simpler form. That is demonstrated by the next lemma. Recall that an element $U\in \mathrm{GL}(n,\mathbb{D})$ is called *unipotent* if U has only 1 as an eigenvalue. In our convention, we shall include identity as the only unipotent element, which is also semisimple.

Lemma 5.1.1. *Every element g in $\mathrm{Aff}(n,\mathbb{D})$, up to conjugacy, can be written as $g=(A,v)$ such that $A=T\oplus U$, where $T\in\mathrm{GL}(n-m,\mathbb{D})$, $U\in\mathrm{GL}(m,\mathbb{D})$ such that T does not have eigenvalue 1, U has only 1 as eigenvalue, and v is of the form $v=[0,0,\ldots,0,v_1,v_2,\ldots,v_m]\in\mathbb{D}^n$. Further, if 1 is not an eigenvalue of the linear part of g, then up to conjugacy, g is of the form $g=(A,\mathbf{0})$.*

Proof. Let $g\in\mathrm{Aff}(n,\mathbb{D})$ be an arbitrary element. In view of Lemma 2.2.4, after conjugating g by a suitable linear element $(B,\mathbf{0})$ in $\mathrm{Aff}(n,\mathbb{D})$, we can assume that $g=(A,w)$ such that

$$A=\begin{pmatrix} T & \\ & U\end{pmatrix},$$

where $T\in\mathrm{GL}(n-m,\mathbb{D})$, $U\in\mathrm{GL}(m,\mathbb{D})$ such that T does not have eigenvalue 1 and U is unipotent. There are two possible cases:

(1) Suppose 1 is not an eigenvalue of A. So the linear transformation $A-\mathrm{I}_n$ is invertible. Therefore, we can choose $x_o=(A-\mathrm{I}_n)^{-1}(w)\in\mathbb{D}^n$. Consider $h=(\mathrm{I}_n,x_o)\in\mathrm{Aff}(n,\mathbb{D})$. Then for all $x\in\mathbb{D}^n$, we have

$$hgh^{-1}(x)=hg(x-x_o)=h(Ax-Ax_o+w)=Ax+w-(A-\mathrm{I}_n)x_o.$$

This implies $hgh^{-1}(x)=A(x)+\mathbf{0}$ for all $x\in\mathbb{D}^n$, since $x_o=(A-\mathrm{I}_n)^{-1}(w)$.

(2) Suppose 1 is an eigenvalue of A. In this case, $m>0$ and $A-\mathrm{I}_n$ has rank $n-m<n$. So we can choose an element $u\in\mathbb{D}^n$ having the last m coordinates zero such that $[(A-\mathrm{I}_n)(u)]_i=w_i$ for all $1\le i\le n-m$, where $w=[w_i]_{1\le i\le n}$. Let $v=w-(A-\mathrm{I}_n)(u)$. Then $v=[0,0,\ldots,0,w_{r+s+1},w_{r+s+2},\ldots,w_n]\in\mathbb{D}^n$. Now consider $h=(\mathrm{I}_n,u)\in\mathrm{Aff}(n,\mathbb{D})$. Then for all $x\in\mathbb{D}^n$, we have

$$hgh^{-1}(x)=hg(x-u)=h(Ax-Au+w)=Ax+w-(A-\mathrm{I}_n)(u)=Ax+v.$$

This completes the proof. $\square$

Remark 5.1.2. The idea of the above proof is in a similar line of arguments as in Lemma 3.3.1 . However, here we have to deal with the subtle situation when the linear part of affine transformations contains a unipotent Jordan block. $\square$

5.2 Reversibility of affine transformations with a fixed point

Recall that if the linear part of an element in $\mathrm{Aff}(n,\mathbb{D})$ does not have eigenvalue 1, then it will have a fixed point in $\mathbb{D}^n$. In this case, reversibility in $\mathrm{Aff}(n,\mathbb{D})$ follows from reversibility in $\mathrm{GL}(n,\mathbb{D})$.

Proposition 5.2.1. *Let $g = (A,v) \in \mathrm{Aff}(n,\mathbb{D})$ be an arbitrary element such that 1 is not an eigenvalue of the linear part A of g. Then g is reversible (resp. strongly reversible) in $\mathrm{Aff}(n,\mathbb{D})$ if and only if A is reversible (resp. strongly reversible) in $\mathrm{GL}(n,\mathbb{D})$. Further, for $\mathbb{D} = \mathbb{R}$ or $\mathbb{C}$, the following are equivalent.*

(1) g is reversible in $\mathrm{Aff}(n,\mathbb{D})$.

(2) g is strongly reversible in $\mathrm{Aff}(n,\mathbb{D})$.

Proof. Using Lemma 5.1.1, up to conjugacy, we can assume that $g = (A,\mathbf{0})$. The proof now follows from Proposition 4.4.1. □

5.3 Reversibility of affine transformations with a unipotent linear part

In this section, we shall use the adjoint reality approach to show that every element of $\mathrm{Aff}(n,\mathbb{D})$ with a unipotent linear part is strongly reversible. In view of Lemma 5.1.1 and Proposition 5.2.1, classification of reversible and strongly reversible elements in $\mathrm{Aff}(n,\mathbb{D})$ reduces to the case when the linear part of affine group element is a unipotent element in $\mathrm{GL}(n,\mathbb{D})$. Moreover, due to Lemma 2.2.4, it is enough to consider the case when the linear part of an element $g \in \mathrm{Aff}(n,\mathbb{D})$ is equal to the unipotent Jordan block $\mathrm{J}(1,n)$.

Lemma 5.3.1. *Let $g = (A,v) \in \mathrm{Aff}(n,\mathbb{D})$ such that $A = \mathrm{J}(1,n)$. Then g is reversible in $\mathrm{Aff}(n,\mathbb{D})$.*

Proof. The inverse of $A = \mathrm{J}(1,n)$ is an upper triangular matrix and given as:

$$A^{-1} = \begin{pmatrix} 1 & -1 & \cdots & \cdots & (-1)^{n+1} \\ & 1 & -1 & \cdots & (-1)^n \\ & & \ddots & \ddots & \vdots \\ & & & \ddots & -1 \\ & & & & 1 \end{pmatrix}.$$

The rank of matrix $A^{-1} - \mathrm{I}_n$ is $n-1$ and last row of $A^{-1} - \mathrm{I}_n$ is zero row. Let $B := -(\Omega(1,n)) \in \mathrm{GL}(n,\mathbb{D})$, where $\Omega(1,n) \in \mathrm{GL}(n,\mathbb{D})$ is as defined in the Definition 4.2.1. Then Lemma 4.2.6 implies that B conjugates A to A^{-1}. Since B is such that only non-zero entry in the last row is $x_{n,n} = -1$, the last row of $A^{-1}+B$ is a *zero row.* This implies that for each element $v \in \mathbb{D}^n$, last coordinate of element $(A^{-1}+B)(v) \in \mathbb{D}^n$ is zero. Now consider the following equation

$$(A^{-1} - \mathrm{I}_n)(w) = (A^{-1}+B)(v). \tag{5.3.1}$$

Observe that the rank of the augmented matrix corresponding to the above equation is $n-1$ and it is independent of given $v \in \mathbb{D}^n$. Thus, Equation (5.3.1) is consistent and has a solution. Therefore, in view of the Lemma 5.0.2, we can choose a suitable $h = (B,w) \in \mathrm{Aff}(n,\mathbb{D})$ such that $hgh^{-1} = g^{-1}$. Hence, g is reversible in $\mathrm{Aff}(n,\mathbb{D})$. This proves the lemma. $\square$

Now, we will investigate the strong reversibility of affine transformations with a unipotent linear part. Note the following example

Example 5.3.2. Let $g = (A,v) \in \mathrm{Aff}(6,\mathbb{D})$ be such that $A = \mathrm{J}(1,6) \in \mathrm{GL}(n,\mathbb{D})$. We will show that g is strongly reversible in $\mathrm{Aff}(6,\mathbb{D})$. Let $B := -(\Omega(1,6))$ in $\mathrm{GL}(6,\mathbb{D})$, where $\Omega(1,6)$ is as defined in the Definition 4.2.1. Using Lemma 4.2.6 and Lemma 4.2.8, we have that B is an involution in $\mathrm{GL}(6,\mathbb{D})$ and it conjugates A to A^{-1}. Here, we have

$$A^{-1} = \begin{pmatrix} 1 & -1 & 1 & -1 & 1 & -1 \\ & 1 & -1 & 1 & -1 & 1 \\ & & 1 & -1 & 1 & -1 \\ & & & 1 & -1 & 1 \\ & & & & 1 & -1 \\ & & & & & -1 \end{pmatrix}, \text{ and } B = \begin{pmatrix} 1 & 4 & 6 & 4 & 1 & 0 \\ & -1 & -3 & -3 & -1 & 0 \\ & & 1 & 2 & 1 & 0 \\ & & & -1 & -1 & 0 \\ & & & & 1 & 0 \\ & & & & & -1 \end{pmatrix}.$$

This implies

$$B+\mathrm{I}_n = \begin{pmatrix} 2 & 4 & 6 & 4 & 1 & 0 \\ & 0 & -3 & -3 & -1 & 0 \\ & & 2 & 2 & 1 & 0 \\ & & & 0 & -1 & 0 \\ & & & & 2 & 0 \\ & & & & & 0 \end{pmatrix}, \text{ and } B+A^{-1} = \begin{pmatrix} 2 & 3 & 7 & 3 & 2 & -1 \\ & 0 & -4 & -2 & -2 & 1 \\ & & 2 & 1 & 2 & -1 \\ & & & 0 & -2 & 1 \\ & & & & 2 & -1 \\ & & & & & 0 \end{pmatrix}.$$

Note that corresponding diagonal entries of $B+\mathrm{I}_n$ and $B+A^{-1}$ are equal and have the same rank equal to 3. Now consider $h=(B,w)\in \mathrm{Aff}(6,\mathbb{D})$, where $w\in\mathbb{D}^n$ is defined as:

$$w=\begin{pmatrix} 4v_1+6v_2+10v_3+4v_2 \\ -2v_1-3v_2-7v_3-3v_4 \\ 2v_3+v_4 \\ -2v_3-v_4 \\ 0 \\ v_6-v_5 \end{pmatrix}.$$

Then h satisfies all the conditions of Lemma 5.0.3. Therefore, h is an involution such that $hgh^{-1}=g^{-1}$. Hence, g is strongly reversible in $\mathrm{Aff}(6,\mathbb{D})$. $\square$

If we follow the same approach, the computational complexity involved in Example 5.3.2 increases as n (the size of the Jordan block) increases. Therefore, when the linear part of g is $\mathrm{J}(1,n)$, generalizing the above construction to find a reversing involution for g seems to be intricate and not as straightforward as the reversible case shown in Lemma 5.3.1. To avoid these difficulties and provide a significantly simpler proof, we take a different path by considering the notion of *adjoint reality* in the Lie algebra setup; see Lemma 5.3.5.

First, let's introduce some notation that will be used in the next part of this section. As before, let $\mathbb{D}^n$ be the right $\mathbb{D}$-vector space. Consider $\mathbb{D}^n$ as an abelian Lie algebra. Then $\mathrm{Der}_{\mathbb{D}}\mathbb{D}^n\simeq\mathfrak{gl}(n,\mathbb{D})$. Thus we can make the semi-direct product on $\mathfrak{gl}(n,\mathbb{D})\oplus_\iota\mathbb{D}^n$ by setting $[(A,0),(0,v)]:=(0,Av)$; see [25, Chapter 1, §4, Example 2] for more details. As done for $\mathrm{Aff}(n,\mathbb{D})$ in (5.0.1), consider the embedding

$$\rho\colon\ \mathfrak{gl}(n,\mathbb{D})\oplus_\iota\mathbb{D}^n\longrightarrow\mathfrak{gl}(n+1,\mathbb{D})\ \text{ given by }\ \rho((X,w)):=\begin{pmatrix} X & w\\ \mathbf{0} & 0\end{pmatrix}. \quad (5.3.2)$$

Then the image has the usual Lie algebra structure, and $\mathfrak{aff}(n,\mathbb{D}):=\mathfrak{gl}(n,\mathbb{D})\oplus_\iota\mathbb{D}^n$ is the Lie algebra of the linear Lie group $\mathrm{Aff}(n,\mathbb{D})$. Note that the adjoint action of $G:=\mathrm{Aff}(n,\mathbb{D})$ on its Lie algebra $\mathfrak{g}:=\mathfrak{aff}(n,\mathbb{D})$ is given by:

$$\mathrm{Ad}\colon G\times\mathfrak{g}\longrightarrow\mathfrak{g}\quad;\quad \mathrm{Ad}(A,v)\cdot(X,w)=\big(AXA^{-1},-(AXA^{-1})v+Aw\big). \quad (5.3.3)$$

Now, we recall the notion of adjoint reality for a linear Lie group G, which was introduced in [18]. The adjoint action of a linear Lie group G on its Lie algebra $\mathfrak{g}$ is given by the conjugation, i.e., $\mathrm{Ad}(g)X:=gXg^{-1}$. An element $X\in\mathfrak{g}$ is called Ad_G-*real* if $-X=gXg^{-1}$ for some $g\in G$. An Ad_G-real element $X\in\mathfrak{g}$ is called *strongly* Ad_G*-real* if $-X = \tau X \tau^{-1}$ for some involution (i.e., element of order at most two) $\tau \in G$; see [18, Definition 1.1]. Observe that if $X \in \mathfrak{g}$ is Ad_G-real, then $\exp(X)$ is reversible in G, but the converse may not be true.

Recall that the Lie algebra $\mathfrak{aff}(n,\mathbb{D})$ of the affine group $\mathrm{Aff}(n,\mathbb{D})$ is given by $\mathfrak{gl}(n,\mathbb{D}) \oplus_\iota \mathbb{D}^n$. Now we will investigate the $\mathrm{Ad}_{\mathrm{Aff}(n,\mathbb{D})}$-real elements in the Lie algebra $\mathfrak{aff}(n,\mathbb{D})$. The next result gives necessary and sufficient conditions for the strongly $\mathrm{Ad}_{\mathrm{Aff}(n,\mathbb{D})}$-real elements in $\mathfrak{aff}(n,\mathbb{D})$. This can be thought of as a Lie algebra version of the Lemma 5.0.3.

Lemma 5.3.3. *Let* $(N,x) \in \mathfrak{gl}(n,\mathbb{D}) \oplus_\iota \mathbb{D}^n$ *be an arbitrary element. Then* (N,x) *is strongly* $\mathrm{Ad}_{\mathrm{Aff}(n,\mathbb{D})}$*-real if and only if there exists an element* $h = (B,w) \in \mathrm{Aff}(n,\mathbb{D})$ *such that both the following conditions hold:*

(i) $BNB^{-1} = -N$, *and* $B^2 = \mathrm{I}_n$.

(ii) $(B+\mathrm{I}_n)(w) = \mathbf{0}$, *and* $N(w) = -(B+\mathrm{I}_n)(x)$.

Proof. We omit the proof as it is identical to that of Lemma 5.0.3. □

The following result will be used in proving the Lemma 5.3.5.

Lemma 5.3.4. *Let* $(N,x) \in \mathfrak{gl}(n,\mathbb{D}) \oplus_\iota \mathbb{D}^n$ *such that* $N = \mathrm{J}(0,n)$, *where* $\mathbb{D} = \mathbb{R},\mathbb{C}$ *or* $\mathbb{H}$. *Then* (N,x) *is strongly* $\mathrm{Ad}_{\mathrm{Aff}(n,\mathbb{D})}$*-real.*

Proof. For the element (N,x), take $B := \mathrm{diag}((-1)^n,(-1)^{n-1},\dots,1,-1)_{n\times n}$. Then condition (i) of Lemma 5.3.3 holds. Further, by the choice of diagonal matrix B, the last row of N and $B+\mathrm{I}_n$ are equal to zero vector in $\mathbb{D}^n$. This implies that for every $x \in \mathbb{D}^n$, the last coordinate of $(B+\mathrm{I}_n)(x)$ is zero. Since rank of N is $n-1$, so equation $Nw = -(B+\mathrm{I}_n)(x)$ is consistent for given $x \in \mathbb{D}^n$ and has a solution. For proving this lemma, it is sufficient to choose a suitable $w \in \mathbb{D}^n$ so that the condition (ii) of Lemma 5.3.3 holds. This can be done in the following way:

(1) Let n be even. Then for $v = [v_k]_{n\times 1} \in \mathbb{D}^n$, take $w = [w_k]_{n\times 1} \in \mathbb{D}^n$ such that

$$w_{2k-1} = 0, \text{ and } w_{2k} = -2x_{2k-1}, \text{ where } k \in \{1,2,\cdots,\frac{n}{2}\}. \tag{5.3.4}$$

Here, we get unique w depending on v for our choice of B.

(2) Let n be odd. Then for $v = [v_k]_{n\times 1} \in \mathbb{D}^n$, take $w = [w_k]_{n\times 1} \in \mathbb{D}^n$ such that

$$w_1 \in \mathbb{D}, w_{2k} = 0, \text{ and } w_{2k+1} = -2x_{2k}, \text{ where } k \in \{1,2,\cdots,\frac{n-1}{2}\}. \tag{5.3.5}$$

Here, for our choice of B, we get no condition on w_1.

Then in view of Lemma 5.3.3, the element (N,x) is strongly $\mathrm{Ad}_{\mathrm{Aff}(n,\mathbb{D})}$-real. Hence the proof follows. $\square$

Now, we will generalize Example 5.3.2 to arbitrary n. The following lemma demonstrates that affine transformations with linear part conjugate to a unipotent Jordan block are strongly reversible.

Lemma 5.3.5. *Let $g = (A,v) \in \mathrm{Aff}(n,\mathbb{D})$ such that $A = \mathrm{J}(1,n)$. Then g is strongly reversible in* $\mathrm{Aff}(n,\mathbb{D})$.

Proof. Let $N := \mathrm{J}(0,n) \in \mathfrak{gl}(n,\mathbb{D})$. Then $(\sigma,y)\exp((N,x))(\sigma,y)^{-1} = (A,v)$ for some $(\sigma,y) \in \mathrm{Aff}(n,\mathbb{D})$. Recall that the Lie algebra $\mathfrak{aff}(n,\mathbb{D}) = \mathfrak{gl}(n,\mathbb{D}) \oplus_\iota \mathbb{D}^n$. Using Lemma 5.3.4, we have $(N,x) \in \mathfrak{aff}(n,\mathbb{D})$ is strongly $\mathrm{Ad}_{\mathrm{Aff}(n,\mathbb{D})}$-real. Let $(\alpha,z) \in \mathrm{Aff}(n,\mathbb{D})$ be an involution so that $(\alpha,z)(N,x)(\alpha,z) = -(N,x)$. By taking the exponential, we have $(\alpha,z)\exp((N,x))(\alpha,z)^{-1} = \exp(-(N,x))$. Let $h := (\sigma,y)(\alpha,z)(\sigma,y)^{-1}$. Then h is an involution in $\mathrm{Aff}(n,\mathbb{D})$ and $h(A,v)h^{-1} = (A,v)^{-1}$. Hence, g is strongly reversible in $\mathrm{Aff}(n,\mathbb{D})$. This proves the lemma. $\square$

The next result follows from Lemma 5.3.5, which will be crucially used in the proof of Theorem 1.3.1.

Proposition 5.3.6. *Let $g = (A,v) \in \mathrm{Aff}(n,\mathbb{D})$ such that A is a unipotent element. Then g is strongly reversible in* $\mathrm{Aff}(n,\mathbb{D})$.

Proof. In view of Lemma 2.2.4, up to conjugacy, we can assume that A has the following form:

$$A = \mathrm{I}_{m_0} \oplus \mathrm{J}(1,m_1) \oplus \cdots \oplus \mathrm{J}(1,m_k),$$

where $m_i \in \mathbb{N}$, for all $i \in \{0,1,2,\dots,k\}$.

Using Example 5.0.1 and Lemma 5.3.5, we can construct a suitable involution $h = (B,w) \in \mathrm{Aff}(n,\mathbb{D})$ such that $hgh^{-1} = g^{-1}$. Hence, g is strongly reversible in $\mathrm{Aff}(n,\mathbb{D})$. This completes the proof. $\square$

5.4 Proof of Theorem 1.3.1

Let $g \in \mathrm{Aff}(n,\mathbb{D})$ be an arbitrary element. Using Lemma 5.0.2 and Lemma 5.0.3, it follows that if g is reversible (resp. strongly reversible) in $\mathrm{Aff}(n,\mathbb{D})$, then A is reversible (resp. strongly reversible) in $\mathrm{GL}(n,\mathbb{D})$.

Conversely, using Lemma 5.1.1, up to conjugacy, we can assume that $g = (A, v) \in \mathrm{Aff}(n, \mathbb{D})$ such that

$$A = \begin{pmatrix} T & \\ & U \end{pmatrix}, \quad v = \begin{pmatrix} \mathbf{0}_{n-m} \\ \tilde{v} \end{pmatrix}, \tag{5.4.1}$$

where $\mathbf{0}_{n-m}$ denotes the zero vector in $\mathbb{D}^{n-m}$ and $T \in \mathrm{GL}(n-m, \mathbb{D})$, $U \in \mathrm{GL}(m, \mathbb{D})$ such that T does not have eigenvalue 1, U has only 1 as eigenvalue and $\tilde{v} = [v_1, v_2, \dots, v_m] \in \mathbb{D}^m$. Here, T and U do not have a common eigenvalue. This implies that if $B \in \mathrm{GL}(n, \mathbb{D})$ is such that $BAB^{-1} = A^{-1}$, then B has the following form

$$B = \begin{pmatrix} B_1 & \\ & B_2 \end{pmatrix}, \text{ where } B_1 \in \mathrm{GL}(n-m, \mathbb{D}), B_2 \in \mathrm{GL}(m, \mathbb{D}).$$

Therefore, if A is reversible (resp. strongly reversible) in $\mathrm{GL}(n, \mathbb{D})$, then $T \in \mathrm{GL}(n-m, \mathbb{D})$ and $U \in \mathrm{GL}(m, \mathbb{D})$ are reversible (resp. strongly reversible). Consider $h = (U, \tilde{v}) \in \mathrm{Aff}(m, \mathbb{D})$, where U is a unipotent element. Then Proposition 5.3.6 implies that h is strongly reversible in $\mathrm{Aff}(m, \mathbb{D})$. Proof of the converse part now follows from (5.4.1).

Further, for the case $\mathbb{D} = \mathbb{R}$ or $\mathbb{C}$, the Proposition 4.4.1 implies that g is reversible in $\mathrm{Aff}(n, \mathbb{D})$ if and only if g is strongly reversible in $\mathrm{Aff}(n, \mathbb{D})$. This completes the proof. □

5.5 Product of involutions in $\mathrm{Aff}(n, \mathbb{D})$

Recall that if $h = (B, v) \in \mathrm{Aff}(n, \mathbb{D})$ is an involution then B has to be an involution in $\mathrm{GL}(n, \mathbb{D})$; see Lemma 5.0.3. If an element of $\mathrm{GL}(n, \mathbb{D})$ is a product of involutions, then necessarily its determinant is either 1 or -1. Product of involutions in $\mathrm{GL}(n, \mathbb{D})$ has been studied in [13] and [30, Section 4.2.4] for the case $\mathbb{D} = \mathbb{R}$ or $\mathbb{C}$.

In the next result, we investigate the product of involutions in $\mathrm{GL}(n, \mathbb{D})$.

Lemma 5.5.1. *Every element of* $\mathrm{GL}(n, \mathbb{D})$ *with determinant* 1 *or* -1 *can be written as a product of four involutions.*

Proof. Note that using Lemma 2.2.4, up to conjugacy, we can assume that every element of $\mathrm{GL}(n, \mathbb{H})$ is in $\mathrm{GL}(n, \mathbb{C})$. The proof now follows from the fact that every element of $\mathrm{GL}(n, \mathbb{C})$ (resp. $\mathrm{GL}(n, \mathbb{R})$) with determinant 1 or -1 can be written as a product of four involutions; see [30, Theorem 4.9]. □

Next, we will prove the Theorem 1.3.2.

5.5.1 Proof of Theorem 1.3.2

Let $g = (A, v) \in \mathrm{Aff}(n, \mathbb{D})$ be such that determinant of A is either 1 or -1. Then using Lemma 5.1.1, up to conjugacy, we can assume that

$$A = \begin{pmatrix} T & \\ & U \end{pmatrix}, \quad v = \begin{pmatrix} \mathbf{0}_{n-m} \\ \tilde{v} \end{pmatrix}, \tag{5.5.1}$$

where $T \in \mathrm{GL}(n-m, \mathbb{D})$ and $U \in \mathrm{GL}(m, \mathbb{D})$ such that T does not have eigenvalue 1 and U has only 1 as eigenvalue. Here, $\mathbf{0}_{n-m}$ denotes the zero vector in $\mathbb{D}^{n-m}$ and $\tilde{v} = [v_1, v_2, \ldots, v_m] \in \mathbb{D}^m$.

Consider $h = (U, \tilde{v}) \in \mathrm{Aff}(m, \mathbb{D})$. Recall that if an element of a group G is strongly reversible, then it can be expressed as a product of two involutions in G; see [30, Proposition 2.12]. Using the Proposition 5.3.6, we have h is strongly reversible in $\mathrm{Aff}(m, \mathbb{D})$. Therefore, there exist involutions $h_1 = (P, u)$ and $h_2 = (Q, w)$ in $\mathrm{GL}(m, \mathbb{D}) \ltimes \mathbb{D}^m$ such that

$$h = h_1 h_2. \tag{5.5.2}$$

Further, note that the determinant of $T \in \mathrm{GL}(n-m, \mathbb{D})$ is either 1 or -1. In view of the Lemma 5.5.1, we have

$$T = B_1 B_2 B_3 B_4, \tag{5.5.3}$$

where B_i is an involution in $\mathrm{GL}(n-m, \mathbb{D})$ for all $i \in \{1, 2, 3, 4\}$. Here, B_i may be equal to I_{n-m} for some $i \in \{1, 2, 3, 4\}$. Now, consider the following elements in $\mathrm{Aff}(n, \mathbb{D})$:

(1) $f_1 := (B_1 \oplus \mathrm{I}_m, \mathbf{0}_n)$,

(2) $f_2 := (B_2 \oplus \mathrm{I}_m, \mathbf{0}_n)$,

(3) $f_3 := (B_3 \oplus P, \mathbf{0}_{n-m} \oplus u)$,

(4) $f_4 := (B_4 \oplus Q, \mathbf{0}_{n-m} \oplus w)$.

From the above construction, it is clear that f_1, f_2, f_3, and f_4 are involutions in $\mathrm{Aff}(n, \mathbb{D})$. Using Equations (5.5.1), (5.5.2), and (5.5.3), we have $g = f_1 f_2 f_3 f_4$. This completes the proof. $\square$

Chapter 6

Strong reversibility in $\mathrm{SL}(n,\mathbb{C})$

In this chapter, we will investigate strongly reversible elements in $\mathrm{SL}(n,\mathbb{C})$ and prove Theorem 1.4.2.

6.1 Background and notations

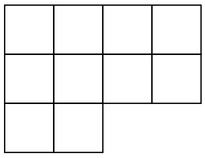
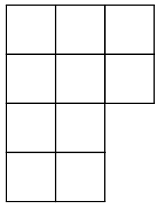
In this section, we fix some notations and introduce the notion of Weyr canonical form over $\mathbb{F}$, which will be crucially used in Chapter 6 and Chapter 7.

6.1.1 Notation for partition of n

In this section, we will recall a notation for partitioning a positive integer n. This notation will be used to introduce the Weyr canonical form in Section 6.1.3.

Definition 6.1.1 (cf. [29])**.** A *partition* of a positive integer n is a finite sequence $(n_1,n_2,\cdots,n_r)$ of positive integers such that $n_1+n_2+\cdots+n_r=n$ and $n_1 \geq n_2 \geq \cdots \geq n_r \geq 1$. Moreover, the *conjugate partition* (or *dual partition*) of the partition $(n_1,n_2,\cdots,n_r)$ of n is the partition $(m_1,m_2,\cdots,m_{n_1})$ such that $m_j = |\{i : n_i \geq j\}|$.

For any positive integer n, we can represent each of its partitions using a diagram called a Young diagram. The Young diagram of a specific partition $(n_1,n_2,\cdots,n_r)$ of n consists of n boxes arranged into r rows, where the length of the i-th row is n_i. We can obtain the Young diagram corresponding to the conjugate partition of a given partition of n by flipping the Young diagram of the given partition over its main diagonal from upper left to lower right. For example, the Young diagrams corresponding to the partition $(4,4,2)$ and its conjugate partition $(3,3,2,2)$ are given as follows:

 and .

Note that we have introduced two notations $(n_1, n_2, \cdots, n_r)$ and $\mathbf{d}(n)$ for the partition of a positive integer n; see Definition 1.4.1 and Definition 6.1.1. The following lemma provides the relationship between the partition $\mathbf{d}(n)$ and its conjugate partition $\overline{\mathbf{d}}(n)$. We omit the proof as it is straightforward.

Lemma 6.1.2. *Let* $\mathbf{d}(n) := [d_1^{t_{d_1}}, \ldots, d_s^{t_{d_s}}]$ *be a partition of a positive integer n. Then the conjugate partition* $\overline{\mathbf{d}}(n)$ *of* $\mathbf{d}(n)$ *has the following form:*

$$\overline{\mathbf{d}}(n) = [(t_{d_1} + t_{d_2} + \cdots + t_{d_s})^{d_s}, (t_{d_1} + t_{d_2} + \cdots + t_{d_{s-1}})^{d_{s-1} - d_s}, \cdots, (t_{d_1} + t_{d_2})^{d_2 - d_3}, (t_{d_1})^{d_1 - d_2}].$$

6.1.2 Block Matrices

We can partition the matrix $A \in \mathrm{M}(n, \mathbb{F})$ by choosing some horizontal partitioning of the rows and, independently, some vertical partitioning of the columns. If we use the same partitioning for both the rows and columns, we refer to the resulting partitioned matrix as a *block or blocked matrix*. For example:

$$A = \left(\begin{array}{ccc|cc} 2 & 0 & 0 & 1 & 0 \\ 0 & 2 & 0 & 0 & 1 \\ 0 & 0 & 2 & 0 & 0 \\ \hline 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{array}\right) = \left(\begin{array}{c|c} A_{1,1} & A_{1,2} \\ \hline A_{2,1} & A_{2,2} \end{array}\right) = (A_{i,j})_{1 \le i,j \le 2} \in \mathrm{M}(5, \mathbb{F}).$$

The symbol I_s represents the $s \times s$ identity matrix. When $r > s$, we will use the notation $\mathrm{I}_{r,s}$ to denote a matrix with r rows and s columns where the first s rows are identical to I_s, and the remaining $(r - s)$ rows are rows of zeros. For example,

$$\mathrm{I}_{3,2} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Note that the diagonal blocks $A_{i,i}$ in the block matrix $A = (A_{i,j})_{1 \le i,j \le m}$ are all square sub-matrices. Moreover, when specifying the block structure of matrix A, it is sufficient to specify only the sizes of the diagonal blocks $A_{i,i}$ since the (i, j)-th block $A_{i,j}$ must be a $n_i \times n_j$ matrix, where n_i and n_j are the sizes of the diagonal blocks $A_{i,i}$ and $A_{j,j}$, respectively. Therefore, if the diagonal blocks of A have decreasing size, we can uniquely specify the whole block structure of A by a partition $(n_1, n_2, \cdots, n_r)$ of n such that $n_1 + n_2 + \cdots + n_r = n$ and $n_1 \geq n_2 \geq \cdots \geq n_r \geq 1$. We refer to block matrix A as block upper triangular if $A_{i,j} = 0$ for all $i > j$. If all the off-diagonal blocks of a block matrix A are zero, then A is called a block diagonal matrix.

6.1.3 Weyr canonical form

Recall the Jordan canonical form over $\mathbb{F}$ introduced in Section 2.2. The Jordan canonical form is a well-known and highly studied matrix canonical form. In this section, we will recall Weyr canonical form introduced by the Czech mathematician Eduard Weyr in 1885. Weyr form is the preferred form over the Jordan form when it comes to problems concerning matrix centralizers. For an elaborate discussion on the theory of Weyr canonical form; see [29], [34], and [35].

In [29], Weyr canonical form of square matrices over algebraically closed fields are studied. Here, we will extend the notion of Weyr canonical form for quaternionic matrices.

Definition 6.1.3 (cf. [29, Definition 2.1.1])**.** A *basic Weyr matrix* with eigenvalue λ is a matrix $W \in \mathrm{M}(n, \mathbb{F})$ of the following form: There is a partition $(n_1, n_2, \ldots, n_r)$ of n such that, when W is viewed as an $r \times r$ blocked matrices (W_{ij}), where the (i, j)-th block W_{ij} is an $n_i \times n_j$ matrix, the following three features present:

(1) The main diagonal blocks $W_{i,i}$ are the $n_i \times n_i$ scalar matrices $\lambda \mathrm{I}_{n_i}$ for $i = 1, 2, \ldots, r$.

(2) The first super-diagonal blocks $W_{i,i+1}$ are the $n_i \times n_{i+1}$ matrices in reduced row-echelon (that is, an identity matrix followed by zero rows) for $i = 1, 2, \ldots, r-1$.

(3) All other blocks of W are zero (that is, $W_{ij} = 0$ when $j \neq i, i+1$).

In this case, we say that W has *Weyr structure* $(n_1, n_2, \ldots, n_r)$.

We can regard an $n \times n$ scalar matrix as a basic Weyr matrix with the trivial Weyr structure (n). At the other extreme, a basic Jordan matrix (i.e., Jordan block) is a basic Weyr matrix with Weyr structure $(1, 1, 1, \cdots, 1)$.

Definition 6.1.4 (cf. [29, Definition 2.1.5])**.** Let $W \in \mathrm{M}(n, \mathbb{F})$, and let $\lambda_1, \lambda_2, \ldots, \lambda_k$ be the distinct eigenvalues of W. We say that W is in Weyr form (or is a Weyr matrix) if W is a direct sum of basic Weyr matrices, one for each distinct eigenvalue. In other words, W has the following form:

$$W = W_1 \oplus W_2 \oplus \cdots \oplus W_k,$$

where W_i is a basic Weyr matrix with eigenvalue λ_i for $i = 1, 2, \cdots, k$.

In the case of the Jordan form, a Jordan matrix J with a single eigenvalue λ is a (direct) sum of Jordan blocks, each of which has λ as its eigenvalue. However, unlike Jordan matrices, a general Weyr matrix can not be expressed as the (direct) sum of basic Weyr blocks with the same eigenvalue. This is a significant difference between the Jordan and Weyr forms.

Theorem 6.1.5 (cf. [29, Theorem 2.2.4]). *Up to a permutation of basic Weyr blocks, each square matrix $A \in \mathrm{M}(n, \mathbb{F})$ is similar to a unique Weyr matrix W. The matrix W is called the Weyr canonical form of A.*

The Weyr structure $(n_1, n_2, \ldots, n_r)$ is called *homogeneous* if $n_1 = n_2 = \cdots = n_r$. For $A \in \mathrm{M}(n, \mathbb{C})$, we will use the notation A_W to denote the corresponding Weyr form. Let $A_W \in \mathrm{GL}(mk, \mathbb{C})$ be a basic Weyr matrix with a homogeneous Weyr structure $(k, k, \ldots, k)$ and having λ as only non-zero eigenvalue. Then A_W is easiest to picture because the blocks on the first super-diagonal are genuine $k \times k$ identity matrices, diagonal blocks are scalar matrices $\lambda \mathrm{I}_k$, and all the others blocks are zero. Then we use the notation $\mathfrak{J}_\lambda(\mathrm{I}_k, m)$ to denote A_W. In particular, if A_W is unipotent (i.e., $\lambda = 1$) then we denote A_W by $\mathfrak{J}(\mathrm{I}_k, m)$.

In the following result, we recall the centralizer of a basic Weyr matrix in $\mathrm{M}(n, \mathbb{C})$.

Proposition 6.1.6 (cf. [29, Proposition 2.3.3]). *Let $W \in \mathrm{M}(n, \mathbb{C})$ be an $n \times n$ basic Weyr matrix with the Weyr structure $(n_1, \ldots, n_r)$, where $r \geq 2$. Let K be an $n \times n$ matrix, blocked according to the partition $(n_1, \ldots, n_r)$, and let $K_{i,j}$ denote its (i, j)-th block (an $n_i \times n_j$ matrix) for $i, j \in \{1, \ldots, r\}$. Then W and K commute if and only if K is a block upper triangular matrix for which*

$$K_{i,j} = \begin{pmatrix} K_{i+1,j+1} & * \\ 0 & * \end{pmatrix}, \text{ for all } 1 \leq i \leq j \leq r-1.$$

Here, we have written $K_{i,j}$ as a blocked matrix where the zero block is $(n_i - n_{i+1}) \times n_{j+1}$. The asterisk entries $(*)$ indicate that there are no restrictions on the entries in that part of the matrix. The column of asterisks disappears if $n_j = n_{j+1}$, and the $\begin{pmatrix} 0 & * \end{pmatrix}$ row disappears if $n_i = n_{i+1}$.

6.1.4 Duality between the Jordan and Weyr Forms

In this section, we recall the duality between the Jordan and Weyr Forms. Each partition $(n_1, n_2, \cdots, n_r)$ of n determines a Young diagram. The Weyr structure

$(m_1, m_2, \cdots, m_{n_1})$ is the conjugate partition of the Jordan structure $(n_1, n_2, \cdots, n_r)$. Therefore, by transposing the Young diagram (writing its columns as rows) of partition $(n_1, n_2, \cdots, n_r)$, we get a Young tableau that corresponds to the conjugate partition $(m_1, m_2, \cdots, m_{n_1})$ of $(n_1, n_2, \cdots, n_r)$. More precisely, the number m_j is the number of n_i's that are greater than or equal to j; see Definition 6.1.1.

It is worth noting that if $\mathbf{d}(n)$ is a partition corresponding to the Jordan structure $(n_1, n_2, \cdots, n_r)$, then the corresponding Weyr structure $(m_1, m_2, \cdots, m_{n_1})$ can also be represented by the conjugate partition $\overline{\mathbf{d}}(n)$; see Lemma 6.1.2.

The following result establishes the duality between Jordan and Weyr structures of complex matrices.

Theorem 6.1.7 (cf. [29, Theorem 2.4.1]). *The Weyr and Jordan structures of a nilpotent $n \times n$ matrix A (more generally, a matrix with a single eigenvalue) are conjugate partitions of n. Moreover, the Weyr form and Jordan form of a square matrix are conjugate under a permutation transformation.*

Let us consider the following example, which illustrates the Proposition 6.1.6.

Example 6.1.8. Let $A = \left(\begin{array}{cccc|cccc|cc} 1&1&0&0&0&0&0&0&0&0\\ 0&1&1&0&0&0&0&0&0&0\\ 0&0&1&1&0&0&0&0&0&0\\ 0&0&0&1&0&0&0&0&0&0\\ \hline 0&0&0&0&1&1&0&0&0&0\\ 0&0&0&0&0&1&1&0&0&0\\ 0&0&0&0&0&0&1&1&0&0\\ 0&0&0&0&0&0&0&1&0&0\\ \hline 0&0&0&0&0&0&0&0&1&1\\ 0&0&0&0&0&0&0&0&0&1 \end{array}\right)$ be a Jordan matrix in $\mathrm{GL}(10, \mathbb{F})$ with Jordan structure $(4,4,2)$. Then the Weyr matrix A_W corresponding to the Jordan matrix A has the Weyr structure $(3,3,2,2)$ and can be given as follows:

$$A_W = \left(\begin{array}{ccc|ccc|cc|cc} 1&0&0&1&0&0&0&0&0&0\\ 0&1&0&0&1&0&0&0&0&0\\ 0&0&1&0&0&1&0&0&0&0\\ \hline 0&0&0&1&0&0&1&0&0&0\\ 0&0&0&0&1&0&0&1&0&0\\ 0&0&0&0&0&1&0&0&0&0\\ \hline 0&0&0&0&0&0&1&0&1&0\\ 0&0&0&0&0&0&0&1&0&1\\ \hline 0&0&0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&0&0&1 \end{array}\right).$$

Further, using Proposition 6.1.6, we have that a matrix $B \in \mathrm{GL}(10,\mathbb{F})$ commuting with the basic Weyr matrix A_W has the following form:

$$B = \left(\begin{array}{ccc|ccc|cc|cc} a & b & e & h & i & l & p & q & v & w \\ c & d & f & j & k & m & r & s & x & y \\ 0 & 0 & g & 0 & 0 & n & t & u & z & \alpha \\ \hline 0 & 0 & 0 & a & b & e & h & i & p & q \\ 0 & 0 & 0 & c & d & f & j & k & r & s \\ 0 & 0 & 0 & 0 & 0 & g & 0 & 0 & t & u \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & a & b & h & i \\ 0 & 0 & 0 & 0 & 0 & 0 & c & d & j & k \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a & b \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & c & d \end{array}\right). \tag{6.1.1}$$

Observe that partitions $(4,4,2)$ and $(3,3,2,2)$ representing the Weyr and Jordan structure of A, respectively, are conjugate (or dual) to each other; see Definition 6.1.1. □

6.1.5 Reverser set of certain Jordan forms over $\mathbb{C}$

In this section, we will compute the reverser set $\mathcal{R}_{\mathrm{GL}(n,\mathbb{C})}(A)$ for certain Jordan forms in $\mathrm{GL}(n,\mathbb{C})$. Recall the notation for upper triangular *Toeplitz* matrices introduced in Definition 2.3.3. We need the following definition to formulate our results.

Definition 6.1.9. For a non-zero $\lambda \in \mathbb{C}$ and $\mathbf{x} = (x_1, x_2, \dots, x_n) \in \mathbb{C}^n$ such that $x_1 \neq 0$, define upper triangular matrix $\Omega(\lambda,\mathbf{x},n) := [g_{i,j}]_{n\times n} \in \mathrm{GL}(n,\mathbb{C})$ such that

(1) $g_{i,j} = 0$ for $1 \leq i, j \leq n$ such that $i > j$,

(2) $g_{i,n} = x_{n-i+1}$ for all $1 \leq i \leq n$,

(3) For $1 \leq i \leq j \leq n-1$, we define

$$g_{i,j} = -\lambda^{-2} g_{i+1,j+1} - \lambda^{-1} g_{i+1,j}. \tag{6.1.2}$$

Note the following example. We refer to Definition 4.2.1 for the notation $\Omega(\lambda,n)$.

Example 6.1.10. Let $\lambda = 1$ and $\mathbf{x} = (x_1, x_2, \dots, x_5) \in \mathbb{C}^5$. Then we have

$$\mathrm{Toep}_5(\mathbf{x})\,\Omega(1,5) = \begin{pmatrix} x_1 & x_2 & x_3 & x_4 & x_5 \\ & x_1 & x_2 & x_3 & x_4 \\ & & x_1 & x_2 & x_3 \\ & & & x_1 & x_2 \\ & & & & x_1 \end{pmatrix} \begin{pmatrix} 1 & 3 & 3 & 1 & 0 \\ & -1 & -2 & -1 & 0 \\ & & 1 & 1 & 0 \\ & & & -1 & 0 \\ & & & & 1 \end{pmatrix}$$

$$= \begin{pmatrix} x_1 & 3x_1 - x_2 & 3x_1 - 2x_2 + x_3 & x_1 - x_2 + x_3 - x_4 & x_5 \\ & -x_1 & -2x_1 + x_2 & -x_1 + x_2 - x_3 & x_4 \\ & & x_1 & x_1 - x_2 & x_3 \\ & & & -x_1 & x_2 \\ & & & & x_1 \end{pmatrix} = \Omega(1, \mathbf{x}, 5).$$

This implies $\Omega(1,\mathbf{x},5) = \mathrm{Toep}_5(\mathbf{x})\,\Omega(1,5)$. □

In the following lemma, we will generalize Example 6.1.10.

Lemma 6.1.11. *Let $\lambda \in \mathbb{C}$ and $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{C}^n$ such that $\lambda \neq 0$ and $x_1 \neq 0$. Let $\Omega(\lambda, n)$, $\mathrm{Toep}_n(\mathbf{x})$, and $\Omega(\lambda, \mathbf{x}, n)$ be as defined in Definition 4.2.1, Definition 2.3.3, and Definition 6.1.9, respectively. Then the following statements hold.*

(i) $\Omega(\lambda,\mathbf{x},n) = \mathrm{Toep}_n(\mathbf{x})\,\Omega(\lambda,n)$.

(ii) $\Big(\Omega(\lambda,\mathbf{x},n)\Big)\,\mathrm{J}(\lambda^{-1},n) = \Big(\mathrm{J}(\lambda,n)\Big)^{-1}\Big(\Omega(\lambda,\mathbf{x},n)\Big)$.

Proof. Let $\mathrm{Toep}_n(\mathbf{x}) = [x_{i,j}]_{n\times n}$ and $\Omega(\lambda,n) = [y_{i,j}]_{n\times n}$ be as defined in Definition 2.3.3 and Definition 4.2.1, respectively. Then $y_{i,j}$ is given by Equation (4.2.3) for all $1 \leq i, j \leq n$.

Let $g := \mathrm{Toep}_n(\mathbf{x})\,\Omega(\lambda,n) = [g_{i,j}]_{n\times n}$. Then using Equations (4.2.3) and (2.3.1), we have that g is an upper triangular matrix such that for all $1 \leq i \leq j \leq n$,

$$\begin{aligned} g_{i,j} &= \sum_{k=1}^{n} (x_{i,k})\,(y_{k,j}) = \sum_{k=i}^{j} (x_{i,k})\,(y_{k,j}) \\ &= \sum_{k=i}^{j} (x_{k-i+1}) \Big((-1)^{n-k} \binom{n-k-1}{j-k} \lambda^{-2n+k+j} \Big). \end{aligned} \tag{6.1.3}$$

This implies for all $1 \leq i \leq n$, we have

$$g_{i,i} = (x_1)(-1)^{n-i}\Big(\lambda^{-2(n-i)}\Big), \text{ and } g_{i,n} = (x_{i,n})\,(y_{n,n}) = (x_{i,n})\,(1) = x_{n-i+1}. \tag{6.1.4}$$

Note that for all $1 \leq i < j \leq n-1$, we have

$$\begin{aligned} g_{i+1,j} &= \sum_{k=i+1}^{j} (x_{k-(i+1)+1})(-1)^{n-k} \binom{n-k-1}{j-k} \Big(\lambda^{-2n+k+j}\Big), \text{ and} \\ g_{i+1,j+1} &= \Big(\sum_{k=i+1}^{j} (x_{k-(i+1)+1})(-1)^{n-k} \binom{n-k-1}{(j+1)-k} \lambda^{-2n+k+j+1} \Big) \\ &\quad + \Big((x_{j-i+1})(-1)^{n-j-1} \lambda^{-2n+2j+2)} \Big). \end{aligned}$$

This implies, for all $1 \le i < j \le n-1$, we have

$$\lambda^{-2} g_{i+1,j+1} + \lambda^{-1} g_{i+1,j} = (x_{j-i+1})(-1)^{n-j-1}\left(\lambda^{-2n+2j}\right)$$

$$+\sum_{k=i+1}^{j}\left(x_{k-(i+1)+1}(-1)^{n-k}\lambda^{-2n+k+j-1}\right)\left(\binom{n-k-1}{(j-k)+1}+\binom{n-k-1}{j-k}\right).$$

Using Pascal's identity (**Id**.1) for all $1 \le i < j \le n-1$, we have

$$\lambda^{-2} g_{i+1,j+1} + \lambda^{-1} g_{i+1,j} = (-1)^{n-j-1}(x_{j-i+1})\left(\lambda^{-2n+2j}\right)$$

$$+\sum_{k=i+1}^{j}(-1)^{n-k}(x_{k-(i+1)+1})\left(\lambda^{-2n+k+j-1}\right)\binom{n-k}{j-k+1}.$$

By substituting $r = k-1$, for all $1 \le i < j \le n-1$, we have

$$\lambda^{-2} g_{i+1,j+1} + \lambda^{-1} g_{i+1,j} = (-1)^{n-j-1}(x_{j-i+1})\left(\lambda^{-2n+2j}\right)$$

$$+\sum_{r=i}^{j-1}(-1)^{n-r-1}(x_{r-i+1})\left(\lambda^{-2n+r+j}\right)\binom{n-r-1}{j-r}.$$

This implies

$$\lambda^{-2} g_{i+1,j+1} + \lambda^{-1} g_{i+1,j} = \sum_{r=i}^{j}(-1)^{n-r-1}(x_{r-i+1})\left(\lambda^{-2n+r+j}\right)\binom{n-r-1}{j-r}, \tag{6.1.5}$$

where $1 \le i < j \le n-1$. Therefore, from Equations (6.1.3) and (6.1.5), we have

$$g_{i,j} = -(\lambda^{-2} g_{i+1,j+1} + \lambda^{-1} g_{i+1,j}) \text{ for all } 1 \le i < j \le n-1. \tag{6.1.6}$$

Using Equations (6.1.4) and (6.1.6), we have

$$g = \mathrm{Toep}_n(\mathbf{x})\,\Omega(\lambda,n) = \Omega(\lambda,\mathbf{x},n).$$

This proves the first part of the lemma.

Now, in view of Lemma 2.3.4, it follows that the Toeplitz matrix $\mathrm{Toep}_n(\mathbf{x})$ commutes with the Jordan block $\mathrm{J}(\lambda,n)$ and its inverse. Therefore, using Lemma 4.2.6, we have

$$\left(\mathrm{Toep}_n(\mathbf{x})\,\Omega(\lambda,n)\right)\mathrm{J}(\lambda^{-1},n) = \mathrm{Toep}_n(\mathbf{x})\left(\Omega(\lambda,n)\,\mathrm{J}(\lambda^{-1},n)\right)$$

$$= \mathrm{Toep}_n(\mathbf{x})\left(\left(\mathrm{J}(\lambda,n)\right)^{-1}\Omega(\lambda,n)\right) = \left(\mathrm{J}(\lambda,n)\right)^{-1}\left(\mathrm{Toep}_n(\mathbf{x})\,\Omega(\lambda,n)\right).$$

Hence, $\Omega(\lambda,\mathbf{x},n)\,\mathrm{J}(\lambda^{-1},n) = \Big(\mathrm{J}(\lambda,n)\Big)^{-1}\,\Omega(\lambda,\mathbf{x},n)$. This completes the proof. $\square$

The following result gives us the reverser set for certain reversible Jordan forms in $\mathrm{GL}(n,\mathbb{C})$.

Theorem 6.1.12. *Let $\mu,\lambda\in\mathbb{C}$ such that $\mu\in\{\pm1\}$ and $\lambda\notin\{\pm1\}$. Then the following statements hold.*

(1) $\mathcal{R}_{\mathrm{GL}(n,\mathbb{C})}(\mathrm{J}(\mu,n)) = \Big\{\Omega(\mu,\mathbf{x},n)\in\mathrm{GL}(n,\mathbb{C}) \mid \mathbf{x}\in\mathbb{C}^n \textit{ with } x_1\neq 0\Big\}.$

(2) $\mathcal{R}_{\mathrm{GL}(2n,\mathbb{C})}(\mathrm{J}(\lambda,n)\oplus\mathrm{J}(\lambda^{-1},n)) =$

$$\Big\{\begin{pmatrix} & \Omega(\lambda,\mathbf{x},n)\\ \Omega(\lambda^{-1},\mathbf{y},n) & \end{pmatrix}\in\mathrm{GL}(2n,\mathbb{C}) \mid \mathbf{x},\mathbf{y}\in\mathbb{C}^n \textit{ with } x_1,y_1\neq 0\Big\}.$$

Proof. Let $g\in\mathcal{R}_{\mathrm{GL}(m,\mathbb{C})}(A)$ be an arbitrary element, where $A\in\mathrm{GL}(m,\mathbb{C})$ is reversible. Recall that for $A\in\mathrm{GL}(m,\mathbb{C})$, the set $\mathcal{R}_{\mathrm{GL}(m,\mathbb{C})}(A)$ of reversers of A is a right coset of the centralizer $\mathcal{Z}_{\mathrm{GL}(m,\mathbb{C})}(A)$ of A. Therefore, if $h\in\mathcal{R}_{\mathrm{GL}(m,\mathbb{C})}(A)$ is such that $hgh^{-1}=g^{-1}$, then

$$\mathcal{R}_{\mathrm{GL}(m,\mathbb{C})}(A) = \mathcal{Z}_{\mathrm{GL}(m,\mathbb{C})}(A)\,h.$$

Proof of (1). Let $g\in\mathcal{R}_{\mathrm{GL}(n,\mathbb{C})}(\mathrm{J}(\mu,n))$, and $h:=\Omega(\mu,n)$ be as defined in Definition 4.2.1. Then Lemma 4.2.6 implies $h\in\mathcal{R}_{\mathrm{GL}(n,\mathbb{C})}(\mathrm{J}(\mu,n))$. Using Lemma 2.3.4, there exists a Toeplitz matrix $\mathrm{Toep}_n(\mathbf{x})$, such that

$$g=\mathrm{Toep}_n(\mathbf{x})\,h,\ \text{ where } \mathbf{x}=(x_1,x_2,\dots,x_n)\in\mathbb{C}^n \text{ such that } x_1\neq 0.$$

The proof of the first part of the theorem follows from Lemma 6.1.11.

Proof of (2). Let $g\in\mathcal{R}_{\mathrm{GL}(2n,\mathbb{C})}(\mathrm{J}(\lambda,n)\oplus\mathrm{J}(\lambda^{-1},n))$. Let $h:=\begin{pmatrix} & \Omega(\lambda,n)\\ \Omega(\lambda^{-1},n) & \end{pmatrix}$, where $\Omega(\lambda,n)$ and $\Omega(\lambda^{-1},n)$ are in $\mathrm{GL}(n,\mathbb{C})$ as defined in Definition 4.2.1. Using Lemma 4.2.6 and Lemma 4.2.7, we have $h\in\mathcal{R}_{\mathrm{GL}(2n,\mathbb{C})}(\mathrm{J}(\lambda,n)\oplus\mathrm{J}(\lambda^{-1},n))$. Since $\lambda\notin\{\pm1\}$, using Lemma 2.3.2 and Lemma 2.3.4, we have that there exists Toeplitz matrices $\mathrm{Toep}_n(\mathbf{x})$ and $\mathrm{Toep}_n(\mathbf{y})$ in $\mathrm{GL}(n,\mathbb{C})$ such that

$$g=\begin{pmatrix}\mathrm{Toep}_n(\mathbf{x}) & \\ & \mathrm{Toep}_n(\mathbf{y})\end{pmatrix}\begin{pmatrix} & \Omega(\lambda,n)\\ \Omega(\lambda^{-1},n) & \end{pmatrix},$$

where $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ and $\mathbf{y} = (y_1, y_2, \ldots, y_n)$ are in $\mathbb{C}^n$ such that x_1 and y_1 both are non-zero. Now, the Lemma 6.1.11 implies that g has the following form

$$g = \begin{pmatrix} & \Omega(\lambda, \mathbf{x}, n) \\ \Omega(\lambda^{-1}, \mathbf{y}, n) & \end{pmatrix}.$$

This completes the proof. $\square$

For more clarity, using Definition 6.1.9, we can rewrite the Theorem 6.1.12 (1) as in the following result.

Corollary 6.1.13. *Let $g = [g_{i,j}]_{1 \leq i,j \leq n} \in \mathcal{R}_{\mathrm{GL}(n,\mathbb{C})}(\mathrm{J}(\mu, n))$, where $\mu \in \{\pm 1\}$. The entries of matrix g satisfy the following conditions*

(1) $g_{i,j} = 0$ *for* $1 \leq i, j \leq n$ *such that* $i > j$,

(2) $g_{n,n}$ *is a non-zero complex number* $\{\because \det(g) \neq 0 \Rightarrow g_{n,n} \neq 0\}$,

(3) $g_{i,n} \in \mathbb{C}$ *is an arbitrary element for all* $1 \leq i, j \leq n-1$,

(4) For $1 \leq i \leq j \leq n-1$, *we have*

$$g_{i,j} = -\mu^{-2} g_{i+1,j+1} - \mu^{-1} g_{i+1,j}.$$

6.1.6 Strong reversibility of certain Jordan forms in $\mathrm{SL}(n,\mathbb{C})$

We will now examine the determinant of involutions in the reverser sets obtained in the Theorem 6.1.12.

Lemma 6.1.14. *Let $\mu, \lambda \in \mathbb{C}$ such that $\mu \in \{-1, +1\}$ and $\lambda \notin \{-1, +1\}$. Then the following statements hold.*

(1) Let $g \in \mathcal{R}_{\mathrm{GL}(n,\mathbb{C})}(\mathrm{J}(\mu, n))$ *be an involution. Then*

$$\det(g) = \begin{cases} +1 & \text{if } n = 4k, \\ -1 & \text{if } n = 4k+2, \\ \pm 1 & \text{if } n = 4k+1, 4k+3, \end{cases}$$

where $k \in \mathbb{N} \cup \{0\}$.

(2) Let $g \in \mathcal{R}_{\mathrm{GL}(2n,\mathbb{C})}(\mathrm{J}(\lambda,n) \oplus \mathrm{J}(\lambda^{-1},n))$ *be an involution. Then*

$$\det(g) = (-1)^n = \begin{cases} +1 & \text{if } n \text{ is even,} \\ -1 & \text{if } n \text{ is odd.} \end{cases}$$

Proof. Proof of (1). Using Theorem 6.1.12 (1), we can write $g = [g_{i,j}]_{1 \leq i,j \leq n} = \Omega(\mu, \mathbf{x}, n)$, where $\mathbf{x} = (x_1, x_2, \dots, x_n) \in \mathbb{C}^n$ such that $x_1 \neq 0$. Given that $\mu = \pm 1$, using Equation (6.1.4), we can conclude that g is an upper triangular matrix with diagonal entries $g_{i,i} = (x_1)(-1)^{(n-i)}$ for all $1 \leq i \leq n$. This implies $\det(g) = (x_1)^n \prod_{i=1}^{n}(-1)^{(n-i)} = (x_1)^n(-1)^{(\Sigma_{k=1}^{n}(n-i))} = (x_1)^n \ (-1)^{\frac{n(n-1)}{2}}$. Therefore, $\det(g)$ depend only on x_1 and n. Given that g is an involution having an upper triangular form, it follows that $(g_{n,n})^2 = (x_1)^2 = 1$. This implies

$$x_1 \in \{-1, +1\}.$$

The proof now follows from the equation $\det(g) = (x_1)^n \ (-1)^{\frac{n(n-1)}{2}}$.

Proof of (2). Using Theorem 6.1.12 (2), we can write

$$g = \begin{pmatrix} & \Omega(\lambda, \mathbf{x}, n) \\ \Omega(\lambda^{-1}, \mathbf{y}, n) & \end{pmatrix}, \tag{6.1.7}$$

where $\mathbf{x} = (x_1, x_2, \dots, x_n)$ and $\mathbf{y} = (y_1, y_2, \dots, y_n)$ are in $\mathbb{C}^n$ such that x_1 and y_1 both are non-zero. Let $\Omega(\lambda, \mathbf{x}, n) = [a_{i,j}]_{1 \leq i,j \leq n}$ and $\Omega(\lambda^{-1}, \mathbf{y}, n) = [b_{i,j}]_{1 \leq i,j \leq n}$.

Note that $\Omega(\lambda, \mathbf{x}, n)$ and $\Omega(\lambda^{-1}, \mathbf{y}, n)$ both are upper triangular matrix with diagonal entries $a_{i,i} = (x_1)(-\lambda^{-2})^{(n-i)}$ and $b_{i,i} = (y_1)(-\lambda^{2})^{(n-i)}$ respectively, where $1 \leq i \leq n$. This implies

$$\det(\Omega(\lambda, \mathbf{x}, n)) = (x_1)^n \Big[\prod_{i=1}^{n} (-\lambda^{-2})^{(n-i)} \Big] = (x_1)^n \Big[(-\lambda^{-2})^{(\Sigma_{k=1}^{n}(n-i))} \Big].$$

Thus, $\det(\Omega(\lambda, \mathbf{x}, n)) = (x_1)^n \Big[(-\lambda^{-2})^{\frac{n(n-1)}{2}} \Big]$. Similarly, we can show

$$\det(\Omega(\lambda^{-1}, \mathbf{y}, n)) = (y_1)^n \Big[(-\lambda^{2})^{\frac{n(n-1)}{2}} \Big].$$

Since $\det(g) = \det \begin{pmatrix} & \Omega(\lambda, \mathbf{x}, n) \\ \Omega(\lambda^{-1}, \mathbf{y}, n) & \end{pmatrix}$, we have

$$\det(g) = \det \begin{pmatrix} \Omega(\lambda, \mathbf{x}, n) & \\ & \Omega(\lambda^{-1}, \mathbf{y}, n) \end{pmatrix} \begin{pmatrix} & \mathrm{I}_n \\ \mathrm{I}_n & \end{pmatrix}.$$

Thus, $\det(g) = \det(\Omega(\lambda, \mathbf{x}, n)) \det(\Omega(\lambda^{-1}, \mathbf{y}, n))(-1)^n$. This implies

$$\det(g) = (-1)^n(-x_1)^n(-y_1)^n = (-1)^n(x_1y_1)^n.$$

But g is an involution in $\mathrm{GL}(2n, \mathbb{C})$ and has the form as given in Equation (6.1.7), where both $\Omega(\lambda, \mathbf{x}, n)$ and $\Omega(\lambda^{-1}, \mathbf{y}, n)$ are upper triangular matrices in $\mathrm{GL}(n, \mathbb{C})$. Therefore, we have $(g^2)_{2n,2n} = x_1y_1 = 1$. This implies

$$\det(g) = (-1)^n(x_1y_1)^n = (-1)^n.$$

This completes the proof. □

The following result helps us understand the strong reversibility in $\mathrm{SL}(n, \mathbb{C})$.

Proposition 6.1.15. *Let $\mu, \lambda \in \mathbb{C}$ such that $\mu \in \{-1, +1\}$ and $\lambda \notin \{-1, +1\}$. Let $A \in \mathrm{SL}(n, \mathbb{C})$ and $B \in \mathrm{SL}(2n, \mathbb{C})$ denote the Jordan forms $\mathrm{J}(\mu, n)$ and $\mathrm{J}(\lambda, n) \oplus \mathrm{J}(\lambda^{-1}, n)$ respectively. Then both of the following statements hold.*

(1) A is reversible in $\mathrm{SL}(n, \mathbb{C})$ for all n. Moreover, A is strongly reversible in $\mathrm{SL}(n, \mathbb{C})$ if and only if $n \neq 4k + 2$, where $k \in \mathbb{N} \cup \{0\}$.

(2) B is reversible in $\mathrm{SL}(2n, \mathbb{C})$ for all n. Moreover, B is strongly reversible in $\mathrm{SL}(2n, \mathbb{C})$ if and only if n is even.

Proof. Proof of (1). Let $g = \Omega(\mu, \mathbf{x}, n)$, where $\mathbf{x} = (x_1, 0, \dots, 0) \in \mathbb{C}^n$ such that

$$x_1 = \begin{cases} +1 & \text{if } n = 4k, 4k+1 \\ -1 & \text{if } n = 4k+3 \\ \sqrt{-1} & \text{if } n = 4k+2 \end{cases}, \text{ where } k \in \mathbb{N} \cup \{0\}.$$

Then $gAg^{-1} = A^{-1}$ and $g \in \mathrm{SL}(n, \mathbb{C})$. Thus, A is reversible in $\mathrm{SL}(n, \mathbb{C})$ for all $n \in \mathbb{N}$. Further, observe that if $n \neq 4k + 2$, then g is also an involution in $\mathrm{SL}(n, \mathbb{C})$, where $k \in \mathbb{N} \cup \{0\}$. This implies that A is strongly reversible in that case.

Next, consider the case when $n = 4k + 2$, where $k \in \mathbb{N} \cup \{0\}$. Suppose that A is strongly reversible. Then there exists $h \in \mathrm{SL}(n, \mathbb{C})$ such that $hAh^{-1} = A^{-1}$

and $h^2 = \mathrm{I}_n$. In view of the Lemma 6.1.14 (1), we have $\det(h) = -1$, which is a contradiction. Hence, the proof follows.

Proof of (2). Let $g = \begin{pmatrix} & \Omega(\lambda,\mathbf{x},n) \\ \Omega(\lambda^{-1},\mathbf{y},n) & \end{pmatrix}$, where $\mathbf{x} = (x_1, 0, \ldots, 0)$ & $\mathbf{y} = (y_1, 0, \ldots, 0)$ are in $\mathbb{C}^n$ such that

$$x_1 = 1, \text{ and } y_1 = \begin{cases} 1 & \text{if } n \text{ is even} \\ -1 & \text{if } n \text{ is odd} \end{cases}.$$

Then $gAg^{-1} = A^{-1}$ and $g \in \mathrm{SL}(2n,\mathbb{C})$. Thus A is reversible in $\mathrm{SL}(2n,\mathbb{C})$ for all $n \in \mathbb{N}$. Further, observe that if n is even, then g is also an involution in $\mathrm{SL}(2n,\mathbb{C})$. This implies that A is strongly reversible in $\mathrm{SL}(2n,\mathbb{C})$ if n is even.

Next, consider the case when n is odd. Suppose that A is strongly reversible. Then there exists $h \in \mathrm{SL}(2n,\mathbb{C})$ such that $hAh^{-1} = A^{-1}$ and $h^2 = \mathrm{I}_{2n}$. In view of Lemma 6.1.14 (2), we have $\det(h) = -1$, which is a contradiction. This completes the proof. $\square$

6.2 Strong reversibility of semisimple elements

First, we classify the semisimple strongly reversible elements in $\mathrm{SL}(n,\mathbb{C})$.

Lemma 6.2.1. *A reversible semisimple element in* $\mathrm{SL}(n,\mathbb{C})$ *is strongly reversible if and only if either* $\{\pm 1\}$ *is an eigenvalue or* $n \not\equiv 2 \pmod 4$.

Proof. Let $A \in \mathrm{SL}(n,\mathbb{C})$ be a reversible semisimple element. If either 1 or -1 is an eigenvalue of A, then we can find a suitable involution in $\mathrm{SL}(n,\mathbb{C})$ which conjugate A to A^{-1}, and hence A is strongly reversible in $\mathrm{SL}(n,\mathbb{C})$. Suppose 1 and -1 are not eigenvalues of A. Further, if $n \not\equiv 2 \pmod 4$, then $n = 4m$ for some $m \in \mathbb{N}$, and we can assume $A = \mathrm{diag}(\lambda_1, \ldots, \lambda_{2m}, \lambda_1^{-1}, \ldots, \lambda_{2m}^{-1})$. Consider involution $g = \begin{pmatrix} & \mathrm{I}_{2m} \\ \mathrm{I}_{2m} & \end{pmatrix}$ in $\mathrm{SL}(n,\mathbb{C})$. Then $gAg^{-1} = A^{-1}$. Hence, A is strongly reversible.

Suppose that A is strongly reversible such that 1 and -1 are not eigenvalue of A. Let $g \in \mathrm{SL}(n,\mathbb{C})$ is an involution such that $gAg^{-1} = A^{-1}$. Moreover, up to conjugacy, we can assume that

$$A = \Big((\lambda_1)\mathrm{I}_{m_1} \oplus (\lambda_1)^{-1}\mathrm{I}_{m_1}\Big) \oplus \Big((\lambda_2)\mathrm{I}_{m_2} \oplus (\lambda_2)^{-1}\mathrm{I}_{m_2}\Big) \oplus \cdots \oplus \Big((\lambda_k)\mathrm{I}_{m_k} \oplus (\lambda_k)^{-1}\mathrm{I}_{m_k}\Big),$$

where $\sum_{i=1}^{k} 2(m_i) = n, \lambda_s \neq \lambda_t$ or $(\lambda_t)^{-1}$ for $s \neq t$, and $\lambda_i \notin \{\pm 1\}$ for all $i,s,t \in \{1,2,\ldots k\}$.

By comparing each entry of the matrix equation $gA = A^{-1}g$ and using the conditions satisfied by each λ_i, we obtain that g has the following block diagonal form:

$$g = \begin{pmatrix} & g_1 \\ \tilde{g}_1 & \end{pmatrix} \oplus \begin{pmatrix} & g_2 \\ \tilde{g}_2 & \end{pmatrix} \oplus \cdots \oplus \begin{pmatrix} & g_k \\ \tilde{g}_k & \end{pmatrix}, \text{ where } g_k \in \mathrm{GL}(m_k, \mathbb{C}) \text{ for all } k.$$

Since g is an involution, we have $\tilde{g}_k = (g_k)^{-1}$. This implies

$$\det(g) = \prod_{i=1}^{k} \det \begin{pmatrix} & g_i \\ (g_i)^{-1} & \end{pmatrix} = \prod_{i=1}^{k} \det \begin{pmatrix} g_i & \\ & (g_i)^{-1} \end{pmatrix} \det \begin{pmatrix} & \mathrm{I}_{m_i} \\ \mathrm{I}_{m_i} & \end{pmatrix}.$$

Therefore, $\det(g) = \prod_{i=1}^{k} (-1)^{m_i} = (-1)^{\sum_{i=1}^{k}(m_i)} = (-1)^{\frac{n}{2}}$. As $\det g = 1$, it follows that $\frac{n}{2} \in 2\mathbb{N}$. This completes the proof. $\square$

6.3 Strong reversibility of unipotent elements

In this section, we will consider unipotent elements in $\mathrm{SL}(n, \mathbb{C})$. Before proceeding, we will look at an example demonstrating the complexities of the strong reversibility in $\mathrm{SL}(n, \mathbb{C})$.

Example 6.3.1. Let $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \bigoplus \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \bigoplus \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ be a unipotent element in $\mathrm{SL}(6, \mathbb{C})$. Then A is not strongly reversible in $\mathrm{SL}(6, \mathbb{C})$. To see this, suppose that A is strongly reversible. Then there exists $g \in \mathrm{SL}(6, \mathbb{C})$ such that $gAg^{-1} = A^{-1}$ and $g^2 = \mathrm{I}_6$. Using equation $gA = A^{-1}g$, we have

$$g = \begin{pmatrix} x_{11} & x_{12} & x_{13} & x_{14} & x_{15} & x_{16} \\ 0 & -x_{11} & 0 & -x_{13} & 0 & -x_{15} \\ x_{31} & x_{32} & x_{33} & x_{34} & x_{35} & x_{36} \\ 0 & -x_{31} & 0 & -x_{33} & 0 & -x_{35} \\ x_{51} & x_{52} & x_{53} & x_{54} & x_{55} & x_{56} \\ 0 & -x_{51} & 0 & -x_{53} & 0 & -x_{55} \end{pmatrix}. \tag{6.3.1}$$

Note that after suitably permuting rows and columns of matrix g, we get the following:

$$\tilde{g} = SgS^{-1} = \begin{pmatrix} x_{11} & x_{13} & x_{15} & x_{12} & x_{14} & x_{16} \\ x_{31} & x_{33} & x_{35} & x_{32} & x_{4} & x_{36} \\ x_{51} & x_{53} & x_{55} & x_{52} & x_{54} & x_{56} \\ 0 & 0 & 0 & -x_{11} & -x_{13} & -x_{15} \\ 0 & 0 & 0 & -x_{31} & -x_{33} & -x_{35} \\ 0 & 0 & 0 & -x_{51} & -x_{53} & -x_{55} \end{pmatrix}, \text{ where } S \in \mathrm{GL}(6, \mathbb{C}).$$

Thus, we can write $\tilde{g} = \left(\begin{array}{c|c} P & Q \\ \hline & -P \end{array}\right)$, where $P \in \mathrm{GL}(3,\mathbb{C})$ and $Q \in \mathrm{M}(3,\mathbb{C})$. Since g is an involution in $\mathrm{SL}(6,\mathbb{C})$, $\tilde{g} = SgS^{-1}$ implies

$$\tilde{g} \in \mathrm{SL}(6,\mathbb{C}) \text{ and } (\tilde{g})^2 = \mathrm{I}_6.$$

Using $(\tilde{g})^2 = \mathrm{I}_6$, i.e., $P^2 = \mathrm{I}_3$, we have

$$\det(\tilde{g}) = \det(P)\det(-P) = \det(P^2)(-1)^3 = (-1)^3.$$

This is a contradiction. Hence, A is not strongly reversible in $\mathrm{SL}(6,\mathbb{C})$. $\square$

In Example 6.3.1, we transform the reversing element g into a block upper triangular form by appropriately permuting its rows and columns. This step is crucial in proving Example 6.3.1. If we consider a unipotent matrix with Jordan blocks of unequal sizes or increase the number of diagonal Jordan blocks, finding an involution in $\mathcal{R}_{\mathrm{GL}(n,\mathbb{C})}(A) \cap \mathrm{SL}(n,\mathbb{C})$ for a reversible element $A \in \mathrm{SL}(n,\mathbb{C})$ in Jordan form becomes challenging.

Therefore, using the Jordan canonical form of a matrix to study strongly reversible elements in $\mathrm{SL}(n,\mathbb{C})$ is not an efficient approach. Instead, we can use the Weyr canonical form, which provides a canonical form of matrices with a more manageable centralizer (and reversing element) than the Jordan canonical form. In particular, if a reversible element of $\mathrm{SL}(n,\mathbb{C})$ is expressed in the Weyr canonical form, every reversing element is of a block upper triangular form. In the next section, we will apply the notion of Weyr canonical form to generalize Example 6.3.1; see Lemma 6.3.2 for details.

6.3.1 Reverser set for homogeneous unipotent Weyr forms

In this section, we consider unipotent matrices that have homogeneous Jordan (and hence Weyr) structures. Let A_W be a unipotent basic Weyr matrix in $\mathrm{SL}(mk,\mathbb{C})$ with Weyr structure $(\underbrace{k,k,\ldots,k}_{m\text{-times}})$; see Section 6.1.3 for the definition. Then we can write A_W as follows:

$$A_W := \mathfrak{J}(\mathrm{I}_k, m) = \begin{pmatrix} \mathrm{I}_k & \mathrm{I}_k & & & & \\ & \mathrm{I}_k & \mathrm{I}_k & & & \\ & & \ddots & \ddots & & \\ & & & \ddots & \ddots & \\ & & & & \mathrm{I}_k & \mathrm{I}_k \\ & & & & & \mathrm{I}_k \end{pmatrix}. \tag{6.3.2}$$

Now, consider $\Omega(\mathrm{I}_k, m) := [X_{i,j}]_{m\times m} \in \mathrm{GL}(mk, \mathbb{C})$ such that

(1) $X_{i,j} = \mathrm{O}_k$ for all $1 \le i, j \le m$ such that $i > j$, where O_k denotes the $k \times k$ zero matrix.

(2) $X_{i,m} = \mathrm{O}_k$ for all $1 \le i \le m-1$,

(3) $X_{m,m} = \mathrm{I}_k$,

(4) $X_{i,j} = -X_{i+1,j+1} - X_{i+1,j}$ for all $1 \le i \le j \le m-1$.

Then, by using a similar argument as in Lemma 4.2.6, we have

$$\Omega(\mathrm{I}_k, m) A_W = (A_W)^{-1}\, \Omega(\mathrm{I}_k, m). \tag{6.3.3}$$

Further, using Proposition 6.1.6, we have that if $B \in \mathrm{M}(mk, \mathbb{C})$ commutes with A_W, then B has the following form:

$$B = \mathrm{Toep}_m(\mathbf{K}) := \begin{pmatrix} K_{1,1} & K_{1,2} & \cdots & \cdots & \cdots & K_{1,m} \\ & K_{1,1} & K_{1,2} & \cdots & \cdots & K_{1,m-1} \\ & & \ddots & \ddots & & \vdots \\ & & & \ddots & \ddots & \vdots \\ & & & & K_{1,1} & K_{1,2} \\ & & & & & K_{1,1} \end{pmatrix}, \tag{6.3.4}$$

where $\mathbf{K} := (K_{1,1}, K_{1,2}, \cdots, K_{1,m})$ is a m-tuple of matrices and $K_{1,j} \in \mathrm{M}(k, \mathbb{C})$ for all $j = 1, 2, \cdots, m$.

Let g be in $\mathrm{GL}(mk, \mathbb{C})$ such that $gA_Wg^{-1} = (A_W)^{-1}$. Since set of reversing elements of A_W is a right coset of the centralizer of A_W, using Equations (6.3.3) and (6.3.4), we have

$$g = \mathrm{Toep}_m(\mathbf{K})\, \Omega(\mathrm{I}_k, m), \tag{6.3.5}$$

where $\det(K_{1,1}) \neq 0$.

The following lemma generalizes Example 6.3.1.

Lemma 6.3.2. *Let $A = \bigoplus_{i=1}^{k} \mathrm{J}(1, 2m)$ be a unipotent Jordan form in $\mathrm{SL}(2mk, \mathbb{C})$. Then the following statements hold.*

(i) If $gAg^{-1} = A^{-1}$ and $g^2 = \mathrm{I}_{2mk}$, then $\det(g) = (-1)^{mk}$.

(ii) If m and k both are odd, then A can not be strongly reversible in $\mathrm{SL}(2mk, \mathbb{C})$.

Proof. Let $A_W := \mathfrak{J}(\mathrm{I}_k, 2m)$ be the Weyr form corresponding to Jordan form A. Then using Theorem 6.1.7, we have

$$A_W = \tau A \tau^{-1} \text{ for some } \tau \in \mathrm{GL}(2mk, \mathbb{C}).$$

Let $g \in \mathrm{GL}(2mk,\mathbb{C})$ such that $gAg^{-1} = A^{-1}$ and $g^2 = \mathrm{I}_{2mk}$. This implies

$$(\tau g \tau^{-1}) A_W (\tau g \tau^{-1})^{-1} = (A_W)^{-1}.$$

Let $h = \tau g \tau^{-1} = [Y_{i,j}] \in \mathrm{GL}(2mk,\mathbb{C})$, where $Y_{i,j} \in \mathrm{M}(k,\mathbb{C})$ for all $1 \leq i, j \leq 2m$. Using Equation (6.3.5), we have

$$h = \mathrm{Toep}_{2m}(\mathbf{K})\Omega(\mathrm{I}_k, 2m), \text{ and } h^2 = \mathrm{I}_{2mk},$$

where $\mathbf{K} := (K_{1,1}, K_{1,2}, \cdots, K_{1,2m})$ such that $K_{1,j} \in \mathrm{M}(k,\mathbb{C})$ and $\det(K_{1,1}) \neq 0$. Therefore, h is a block upper-triangular matrix with diagonal blocks $Y_{i,i} = (-1)^{(2m-i)} K_{1,1}$ for all $i = 1, 2, \cdots, 2m$ such that $(K_{1,1})^2 = \mathrm{I}_k$. This implies

$$\det(h) = \prod_{i=1}^{2m} \det\Big((-1)^{(2m-i)} K_{1,1}\Big) = ((-1)^k)^m \Big(\det\Big((K_{1,1})^2\Big)\Big)^m = (-1)^{km}.$$

Since $h = \tau g \tau^{-1}$, we have $\det(g) = (-1)^{mk}$. Moreover, if m and k are odd then $\det(g) = -1$, and hence A is not strongly reversible in $\mathrm{SL}(2mk,\mathbb{C})$. This completes the proof. $\square$

6.3.2 Reverser set for non-homogeneous unipotent Weyr forms

In this section, we will consider non-homogeneous unipotent Weyr matrices. Note the following example.

Example 6.3.3. Let A be a unipotent element in $\mathrm{SL}(10,\mathbb{C})$ with Jordan structure $(4,4,2)$ as given in Example 6.1.8. Then A will not be strongly reversible in $\mathrm{SL}(10,\mathbb{C})$. To see this, recall that A has the Weyr structure $(3,3,2,2)$. So the Weyr form A_W of A and its inverse A_W^{-1} can be given as

$$A_W = \left(\begin{array}{c|c|c|c} \mathrm{I}_3 & \mathrm{I}_3 & \mathrm{O}_{3,2} & \mathrm{O}_{3,2} \\ \hline \mathrm{O}_{3,3} & \mathrm{I}_3 & \mathrm{I}_{3,2} & \mathrm{O}_{3,2} \\ \hline \mathrm{O}_{2,3} & \mathrm{O}_{2,3} & \mathrm{I}_2 & \mathrm{I}_2 \\ \hline \mathrm{O}_{2,3} & \mathrm{O}_{2,3} & \mathrm{O}_{2,2} & \mathrm{I}_2 \end{array}\right), \text{ and } A_W^{-1} = \left(\begin{array}{c|c|c|c} \mathrm{I}_3 & -\mathrm{I}_3 & \mathrm{I}_{3,2} & -\mathrm{I}_{3,2} \\ \hline \mathrm{O}_{3,3} & \mathrm{I}_3 & -\mathrm{I}_{3,2} & \mathrm{I}_{3,2} \\ \hline \mathrm{O}_{2,3} & \mathrm{O}_{2,3} & \mathrm{I}_2 & -\mathrm{I}_2 \\ \hline \mathrm{O}_{2,3} & \mathrm{O}_{2,3} & \mathrm{O}_{2,2} & \mathrm{I}_2 \end{array}\right).$$

Using equation (6.1.1), we have that any element $\tau_1 \in \mathrm{GL}(10,\mathbb{C})$ satisfying the equation $\tau_1 A_W = A_W \tau_1$ is an upper triangular block matrix and has the following form

$$\tau_1 = \left(\begin{array}{c|c|c|c} \begin{pmatrix} P & * \\ & Q \end{pmatrix} & * & * & * \\ \hline & \begin{pmatrix} P & * \\ & Q \end{pmatrix} & * & * \\ \hline & & P & * \\ \hline & & & P \end{array}\right),$$

where $P = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}(2,\mathbb{C})$ and $Q = \begin{pmatrix} g \end{pmatrix} \in \mathrm{GL}(1,\mathbb{C})$.

Consider $\tau_2 = \left(\begin{array}{c|c|c|c} -\mathrm{I}_3 & -2\mathrm{I}_3 & -\mathrm{I}_{3,2} & \mathrm{O}_{3,2} \\ \hline \mathrm{O}_{3,3} & \mathrm{I}_3 & \mathrm{I}_{3,2} & \mathrm{O}_{3,2} \\ \hline \mathrm{O}_{2,3} & \mathrm{O}_{2,3} & -\mathrm{I}_2 & \mathrm{O}_{2,2} \\ \hline \mathrm{O}_{2,3} & \mathrm{O}_{2,3} & \mathrm{O}_{2,2} & \mathrm{I}_2 \end{array}\right)$ in $\mathrm{GL}(10,\mathbb{C})$. Then we have

$$\tau_2 A_W = A_W^{-1}\tau_2 = \left(\begin{array}{c|c|c|c} -\mathrm{I}_3 & -3\mathrm{I}_3 & -3\mathrm{I}_{3,2} & -\mathrm{I}_{3,2} \\ \hline \mathrm{O}_{3,3} & \mathrm{I}_3 & 2\mathrm{I}_{3,2} & \mathrm{I}_{3,2} \\ \hline \mathrm{O}_{2,3} & \mathrm{O}_{2,3} & -\mathrm{I}_2 & -\mathrm{I}_2 \\ \hline \mathrm{O}_{2,3} & \mathrm{O}_{2,3} & \mathrm{O}_{2,2} & \mathrm{I}_2 \end{array}\right).$$

Therefore, $\tau_2 A_W \tau_2^{-1} = A_W^{-1}$. Since set of reversing elements of A_W is a right coset of the centralizer of A_W, every reversing element τ of A_W has the following form:

$$\tau = \tau_1\tau_2 = \left(\begin{array}{c|c|c|c} \begin{pmatrix} -P & * \\ & -Q \end{pmatrix} & * & * & * \\ \hline & \begin{pmatrix} P & * \\ & Q \end{pmatrix} & * & * \\ \hline & & -P & * \\ \hline & & & P \end{array}\right).$$

This implies that

$$\det(\tau) = ((-1)^2)^2 \det(P^4)(-1)^1 \det(Q^2) = (-1)\det(P^4)\det(Q^2).$$

Moreover, if τ is an involution, then P and Q will also be involutions. Thus, $\det(P^4) = \det(Q^2) = 1$. Therefore, if $\tau \in \mathrm{GL}(10,\mathbb{C})$ such that $\tau A_W \tau^{-1} = (A_W)^{-1}$ and $\tau^2 = \mathrm{I}_{10}$, then $\det(\tau) = -1$. Hence, A is not strongly reversible in $\mathrm{SL}(10,\mathbb{C})$. $\square$

Now, we will generalize Example 6.3.3. The following lemma follows from the proof of [18, Theorem 5.6]. However, we will provide another proof using the notion of Weyr canonical form. We refer to Definition 1.4.1 and Section 6.1.1 for the notations of partition used in the following lemma.

Lemma 6.3.4. *Let $A \in \mathrm{SL}(n,\mathbb{C})$ be a unipotent element such that Jordan form of A is represented by the partition $\mathbf{d}(n) = [d_1^{t_{d_1}}, \dots, d_s^{t_{d_s}}]$, where d_k is even for all $1 \le k \le s$. Suppose that g is an involution in $\mathrm{GL}(n,\mathbb{C})$ such that $gAg^{-1} = A^{-1}$. Then*

$$\det(g) = (-1)^{\sum_{d_k \in \mathbb{E}^2_{\mathbf{d}(n)}} t_{d_k}}.$$

Proof. Let A_W denote the Weyr form of A. Then $A_W = \tau A \tau^{-1}$ for some $\tau \in \mathrm{GL}(n,\mathbb{C})$. Consider $h = \tau g \tau^{-1}$ in $\mathrm{GL}(n,\mathbb{C})$. Since g is an involution in $\mathrm{GL}(n,\mathbb{C})$ such that $gAg^{-1} = A^{-1}$, we have that h is an involution in $\mathrm{GL}(n,\mathbb{C})$ such that

$$hA_W h^{-1} = (A_W)^{-1}, \text{ and } \det(h) = \det(g). \tag{6.3.6}$$

Using Lemma 6.1.2, we have that partition $\overline{\mathbf{d}}(n)$ representing Weyr form A_W is given by

$$\overline{\mathbf{d}}(n) = [(t_{d_1} + t_{d_2} + \cdots + t_{d_s})^{d_s}, (t_{d_1} + t_{d_2} + \cdots + t_{d_{s-1}})^{d_{s-1}-d_s}, \cdots, (t_{d_1} + t_{d_2})^{d_2-d_3}, (t_{d_1})^{d_1-d_2}].$$

Therefore, A_W is a block matrix with $(d_1)^2$ many blocks and for all $1 \leq i \leq d_1$, the size n_i of i-th diagonal block is given by

$$n_i = \begin{cases} t_{d_1} + t_{d_2} + \cdots + t_{d_s} & \text{if } 1 \leq i \leq d_s \\ t_{d_1} + t_{d_2} + \cdots + t_{d_{s-r-1}} & \text{if } d_{s-r} + 1 \leq i \leq d_{s-r-1}, \text{where } 0 \leq r \leq s-2 \end{cases}.$$

This implies that (i,j)-th block of A_W has the size $n_i \times n_j$; see Section 6.1.2. Further, (i,j)-th block of Weyr form A_W and its inverse $(A_W)^{-1}$ can be given as follow:

$$(A_W)_{i,j} = \begin{cases} \mathrm{I}_{n_i} & \text{if } j = i \\ \mathrm{I}_{n_i \times n_{i+1}} & \text{if } j = i+1 \\ \mathrm{O}_{n_i \times n_j} & \text{otherwise} \end{cases}, \text{ and } ((A_W)^{-1})_{i,j} = \begin{cases} \mathrm{I}_{n_i} & \text{if } j = i \\ (-1)^{(j-i)}\, \mathrm{I}_{n_i \times n_j} & \text{if } j > i \\ \mathrm{O}_{n_i \times n_j} & \text{otherwise} \end{cases},$$

where $1 \leq i,j \leq d_1$. Consider $\Omega_W := [X_{i,j}]_{d_1 \times d_1} \in \mathrm{GL}(n,\mathbb{C})$ such that

$$X_{i,j} = \begin{cases} \mathrm{O}_{n_i \times n_j} & \text{if } i > j \\ \mathrm{O}_{n_i \times n_j} & \text{if } j = d_1, i \neq d_1 \\ (-1)^{d_1-i}\mathrm{I}_{n_i} & \text{if } j = i \\ (-1)^{d_1-i}\binom{d_1-i-1}{j-i}\mathrm{I}_{n_i \times n_j} & \text{if } j > i, j \neq d_1 \end{cases}, \tag{6.3.7}$$

where $\binom{d_1-i-1}{j-i}$ denotes the binomial coefficient. Then, by using a similar argument as in Lemma 4.2.6, we have

$$\Omega_W A_W (\Omega_W)^{-1} = (A_W)^{-1}.$$

Let $f = [P_{i,j}]_{d_1 \times d_1}$ be an $n \times n$ matrix commuting with Weyr form A_W, where block structure of both f and A is according to the partition $\overline{\mathbf{d}}(n)$. Then using Proposition 6.1.6, we can conclude that f is an upper triangular block matrix, and the i-th diagonal block $P_{i,i}$ of f has the following form:

(1) if $1 \leq i \leq d_s$, then $P_{i,i} = \begin{pmatrix} P_1 & * & * & * & \cdots & * \\ & P_2 & * & \cdots & \cdots & * \\ & & \ddots & \ddots & & \vdots \\ & & & \ddots & \ddots & \vdots \\ & & & & P_{s-1} & * \\ & & & & & P_s \end{pmatrix}$,

(2) if $d_{s-r}+1 \leq i \leq d_{s-r-1}$, then $P_{i,i} = \begin{pmatrix} P_1 & * & * & * & \cdots & * \\ & P_2 & * & \cdots & \cdots & * \\ & & \ddots & \ddots & & \vdots \\ & & & \ddots & \ddots & \vdots \\ & & & & P_{s-r-2} & * \\ & & & & & P_{s-r-1} \end{pmatrix}$,

(3) if $d_2+1 \leq i \leq d_1$, then $P_{i,i} = P_1$,

where $0 \leq r \leq s-3$ and $P_k \in \mathrm{GL}(t_{d_k}, \mathbb{C})$ for all $1 \leq k \leq s$. It is worth noting that there are d_k many times matrices P_k are repeated in the diagonal blocks of matrix f, where $1 \leq k \leq s$.

Now, suppose that $h = [Y_{i,j}]_{d_1 \times d_1} \in \mathrm{GL}(n,\mathbb{C})$ such that $hA_W h^{-1} = (A_W)^{-1}$. Since the set of reversing elements of A_W is a right coset of the centralizer of A_W, we have

$$h = f\,\Omega_W.$$

This implies that h is an upper triangular block matrix such that diagonal blocks of h are given by

$$Y_{i,i} = (-1)^{d_1-i} P_{i,i} \text{ for all } 1 \leq i \leq d_1.$$

Recall that we have also assumed that d_k is even for all $1 \leq k \leq s$. Therefore,

$$\det(h) = \prod_{i=1}^{d_1} \det((-1)^{d_1-i} P_{i,i})$$

$$= ((-1)^{t_{d_1}})^{\frac{d_1}{2}} \det((P_1)^2)^{\frac{d_1}{2}} ((-1)^{t_{d_2}})^{\frac{d_2}{2}} \det((P_2)^2)^{\frac{d_2}{2}} \cdots ((-1)^{t_{d_s}})^{\frac{d_s}{2}} \det((P_s)^2)^{\frac{d_s}{2}}.$$

Further, since h is an involution, we have that P_k is an involution for all $1 \leq k \leq s$. This implies that

$$\det(h) = ((-1)^{t_{d_1}})^{\frac{d_1}{2}} ((-1)^{t_{d_2}})^{\frac{d_2}{2}} \cdots ((-1)^{t_{d_s}})^{\frac{d_s}{2}}.$$

Observe that if $d_k = 0 \pmod 4$ for some $1 \leq k \leq s$, then $((-1)^{t_{d_k}})^{\frac{d_k}{2}} = 1$. Thus we have

$$\det(h) = \prod_{d_k = 2 \pmod 4} ((-1)^{t_{d_k}})^{\frac{d_k}{2}} = \prod_{d_k = 2 \pmod 4} (-1)^{t_{d_k}} = (-1)^{\sum_{d_k \in \mathbb{E}^2_{\mathbf{d}(n)}} t_{d_k}},$$

where $\mathbb{E}^2_{\mathbf{d}(n)} := \{d_i \in \mathbb{E}_{\mathbf{d}(n)} \mid d_i \equiv 2 \pmod 4\}$ and $\mathbb{E}_{\mathbf{d}(n)} := \{d_i \mid d_i \text{ is even}\}$. The proof of lemma now follows from Equation (6.3.6). $\square$

In the following lemma, we have classified all the strongly reversible unipotent elements in $\mathrm{SL}(n,\mathbb{C})$. This lemma is also proved in [18] by using an infinitesimal version of the notion of the classical reversibility or reality, known as *adjoint reality*. We refer to Definition 1.4.1 for the notations used in the following result.

Lemma 6.3.5 ([18, Theorem 5.6]). *Let $A \in \mathrm{SL}(n,\mathbb{C})$ be a unipotent element, and $\mathbf{d}(n)$ be the corresponding partition in the Jordan decomposition of A. Then A is strongly reversible if and only if at least one of the following conditions holds:*

(1) $\mathbb{O}_{\mathbf{d}(n)}$ *is non-empty,*

(2) $|\mathbb{E}^2_{\mathbf{d}(n)}|$ *is even.*

Proof. Let A is strongly reversible and $\mathbf{d}(n) = [d_1^{t_{d_1}}, \dots, d_s^{t_{d_s}}]$ be the partition representing the Jordan form of A. Suppose that $\mathbb{O}_{\mathbf{d}(n)}$ is empty, i.e., d_k is even for all $1 \le k \le s$. If g is an involution in $\mathrm{SL}(n,\mathbb{C})$ such that $gAg^{-1} = A^{-1}$, then, using Lemma 6.3.4, we have

$$\det(g) = (-1)^{\sum_{d_k \in \mathbb{E}^2_{\mathbf{d}(n)}} t_{d_k}} = 1.$$

This implies that $\sum_{d_k \in \mathbb{E}^2_{\mathbf{d}(n)}} t_{d_k}$ is even. Therefore, $|\mathbb{E}^2_{\mathbf{d}(n)}|$ is even.

Conversely, if either of the conditions (1) or (2) of Lemma 6.3.5 holds, using Lemma 2.2.4 and Proposition 6.1.15, we can construct a suitable involution g in $\mathrm{SL}(n,\mathbb{C})$ such that $gAg^{-1} = A^{-1}$. Therefore, A is strongly reversible in $\mathrm{SL}(n,\mathbb{C})$. This proves the lemma. $\square$

To illustrate Lemma 6.3.5, we can consider the following examples.

Example 6.3.6. Note that $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in \mathrm{SL}(2,\mathbb{C})$ is not strongly reversible, but $B = A \oplus A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \bigoplus \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ is strongly reversible in $\mathrm{SL}(4,\mathbb{C})$. To see this, consider $g = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \bigoplus \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, which is an involution in $\mathrm{SL}(4,\mathbb{C})$ such that $gBg^{-1} = B^{-1}$.

Example 6.3.7. Let $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \bigoplus \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \bigoplus \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in \mathrm{SL}(6,\mathbb{C})$. Then A is not strongly reversible in $\mathrm{SL}(6,\mathbb{C})$; see Example 6.3.1. But $B = A \oplus (1) \in \mathrm{SL}(7,\mathbb{C})$ is strongly reversible in $\mathrm{SL}(7,\mathbb{C})$. To see this, consider $g = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \bigoplus \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \bigoplus \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \bigoplus (-1)$, which is an involution in $\mathrm{SL}(7,\mathbb{C})$ such that $gBg^{-1} = B^{-1}$.

Remark 6.3.8. Let $A \in \mathrm{SL}(n,\mathbb{C})$ be a unipotent element, where $n \not\equiv 2 \pmod 4$. Then Lemma 6.3.5 implies that A is strongly reversible in $\mathrm{SL}(n,\mathbb{C})$.

In the following lemma, we classify strongly reversible elements in $\mathrm{SL}(n,\mathbb{C})$ having only -1 as an eigenvalue.

Lemma 6.3.9. *Let A be an element of* $\mathrm{SL}(n,\mathbb{C})$ *with* -1 *as its only eigenvalue, and* $\mathbf{d}(n)$ *be the corresponding partition in the Jordan decomposition of A. Then A is strongly reversible if and only if at least one of the following conditions holds:*

(1) $\mathbb{O}_{\mathbf{d}(n)}$ *is non-empty,*

(2) $|\mathbb{E}^2_{\mathbf{d}(n)}|$ *is even.*

Proof. The proof follows using a similar line of argument as done in Lemma 6.3.5. □

6.4 Strong reversibility of elements having non-unit modulus eigenvalues

In this section, we will investigate the strong reversibility of reversible elements of $\mathrm{SL}(n,\mathbb{C})$ having eigenvalues λ and λ^{-1}, where $\lambda \neq \pm 1$.

Lemma 6.4.1. *Let $\lambda \neq \pm 1$, and $A = A_1 \oplus A_2$ be an element of* $\mathrm{SL}(2n,\mathbb{C})$ *such that* $A_1 \in \mathrm{GL}(n,\mathbb{C})$ *and* $A_2 \in \mathrm{GL}(n,\mathbb{C})$ *have eigenvalues λ and λ^{-1}, respectively. Let g be an involution in* $\mathrm{GL}(n,\mathbb{C})$ *such that* $gAg^{-1} = A^{-1}$*. Then* $\det(g) = (-1)^n$

Proof. Let A_W denote the Weyr form of A. Then $A_W = \tau A \tau^{-1}$ for some $\tau \in \mathrm{GL}(2n,\mathbb{C})$. Consider $h = \tau g \tau^{-1}$ in $\mathrm{GL}(2n,\mathbb{C})$. Since g is an involution in $\mathrm{GL}(2n,\mathbb{C})$ such that $gAg^{-1} = A^{-1}$, we have that h is an involution in $\mathrm{GL}(2n,\mathbb{C})$ such that

$$hA_W h^{-1} = (A_W)^{-1}, \text{ and } \det(h) = \det(g). \tag{6.4.1}$$

Up to conjugacy, we can assume that A_1 and A_2 are in Jordan form. Since $A = A_1 \oplus A_2$ is reversible, in view of Lemma 2.4.3, we have that A_1 and A_2 have the same Jordan structure. Let $(A_1)_W$ and $(A_2)_W$ denote the corresponding Weyr forms having the same Weyr structure, say $(n_1, n_2, \cdots, n_r)$; see Definition 6.1.3. Then the Weyr form A_W of A can be given as

$$A_W = (A_1)_W \oplus (A_2)_W,$$

where $((A_1)_W)_{i,j} = \begin{cases} \lambda \mathrm{I}_{n_i} & \text{if } j = i \\ \mathrm{I}_{n_i \times n_{i+1}} & \text{if } j = i+1 \\ \mathrm{O}_{n_i \times n_j} & \text{otherwise} \end{cases}$, and $((A_2)_W)_{i,j} = \begin{cases} \lambda^{-1} \mathrm{I}_{n_i} & \text{if } j = i \\ \mathrm{I}_{n_i \times n_{i+1}} & \text{if } j = i+1 \\ \mathrm{O}_{n_i \times n_j} & \text{otherwise} \end{cases}$. Then (i,j)-th block of upper triangular block matrix $((A_1)_W)^{-1}$ can be written as follows

$$((A_1)_W)^{-1}_{i,j} = \begin{cases} \lambda^{-1} \mathrm{I}_{n_i} & \text{if } j = i \\ (-1)^k \lambda^{-(k+1)} \mathrm{I}_{n_i \times n_{i+k}} & \text{if } j = i+k, 1 \le k \le r-i \\ \mathrm{O}_{n_i \times n_j} & \text{otherwise} \end{cases}, \text{where } 1 \le i, j \le r.$$

Similarly, we can write

$$((A_2)_W)^{-1}_{i,j} = \begin{cases} \lambda \mathrm{I}_{n_i} & \text{if } j = i \\ (-1)^k \lambda^{(k+1)} \mathrm{I}_{n_i \times n_{i+k}} & \text{if } j = i+k, 1 \le k \le r-i \\ \mathrm{O}_{n_i \times n_j} & \text{otherwise} \end{cases}, \text{where } 1 \le i, j \le r.$$

Consider $\Omega_W = \begin{pmatrix} & \Omega_1 \\ \Omega_2 & \end{pmatrix}$ in $\mathrm{GL}(2n,\mathbb{C})$ such that $\Omega_1 = [X_{i,j}]_{r\times r} \in \mathrm{GL}(n,\mathbb{C})$ and $\Omega_2 = [Y_{i,j}]_{r\times r} \in \mathrm{GL}(n,\mathbb{C})$ are defined as follows

$$X_{i,j} = \begin{cases} \mathrm{O}_{n_i \times n_j} & \text{if } i > j \\ \mathrm{O}_{n_i \times n_j} & \text{if } j = r, i \neq r \\ (-1)^{r-i} \left(\lambda^{-2(r-i)}\right) \mathrm{I}_{n_i} & \text{if } j = i \\ (-1)^{r-i} \binom{r-i-1}{j-i} \left(\lambda^{-2r+i+j}\right) \mathrm{I}_{n_i \times n_j} & \text{if } j > i, j \neq r \end{cases}, \text{ and}$$

$$Y_{i,j} = \begin{cases} \mathrm{O}_{n_i \times n_j} & \text{if } j < i \\ \mathrm{O}_{n_i \times n_j} & \text{if } j = r, i \neq r \\ (-1)^{r-i} \left(\lambda^{2(r-i)}\right) \mathrm{I}_{n_i} & \text{if } j = i \\ (-1)^{r-i} \binom{r-i-1}{j-i} \left(\lambda^{2r-i-j}\right) \mathrm{I}_{n_i \times n_j} & \text{if } i < j, j \neq r \end{cases},$$

where $\binom{r-i-1}{j-i}$ denotes the binomial coefficient. Then by using a similar argument as in Lemma 4.2.6, we have

$$\Omega_W A_W (\Omega_W)^{-1} = (A_W)^{-1}.$$

Suppose that $f \in \mathrm{M}(2n,\mathbb{C})$ such that $fA = Af$. Since $A = A_1 \oplus A_2$ such that A_1 and A_2 have no common eigenvalues, using Lemma 2.3.2, we have

$$f = \begin{pmatrix} B_1 & \\ & B_2 \end{pmatrix},$$

where $B_1, B_2 \in \mathrm{M}(n,\mathbb{C})$ such that $B_1A_1 = A_1B_1$ and $B_2A_2 = A_2B_2$. Moreover, since A_1 and A_2 are basic Weyr matrices with the same Weyr structure, using Proposition 6.1.6, B_1 and B_2 are block upper triangular matrices with the same block structure $(n_1, n_2, \cdots, n_r)$.

Now, suppose that $h \in \mathrm{GL}(2n,\mathbb{C})$ such that $hA_Wh^{-1} = (A_W)^{-1}$. Since the set of reversing elements of A_W is a right coset of the centralizer of A_W, we have $h = f\,\Omega_W$. This implies

$$h = \begin{pmatrix} B_1 & \\ & B_2 \end{pmatrix} \begin{pmatrix} & \Omega_1 \\ \Omega_2 & \end{pmatrix} = \begin{pmatrix} & B_1\Omega_1 \\ B_2\Omega_2 & \end{pmatrix}.$$

Then $\det(h) = \det\begin{pmatrix} B_1\Omega_1 & \\ & B_2\Omega_2 \end{pmatrix} \det\begin{pmatrix} & \mathrm{I}_n \\ \mathrm{I}_n & \end{pmatrix} = (-1)^n \det\begin{pmatrix} B_1\Omega_1 & \\ & B_2\Omega_2 \end{pmatrix}$. Since h is an involution, we have $\det\begin{pmatrix} B_1\Omega_1 & \\ & B_2\Omega_2 \end{pmatrix} = \det(B_1\Omega_1)\det(B_2\Omega_2) = 1$. Therefore, $\det(h) = (-1)^n$. The proof of lemma now follows from Equation (6.4.1). □

The following lemma generalizes Proposition 6.1.15 (2).

Lemma 6.4.2. *Let $\lambda \neq \pm 1$, and $A = A_1 \oplus A_2$ be a reversible element of $\mathrm{SL}(2n,\mathbb{C})$ such that all eigenvalues of $A_1 \in \mathrm{GL}(n,\mathbb{C})$ and $A_2 \in \mathrm{GL}(n,\mathbb{C})$ are λ and λ^{-1}, respectively. Then A is strongly reversible if and only if n is even.*

Proof. If A is strongly reversible in $\mathrm{SL}(2n,\mathbb{C})$, then there exists an involution g in $\mathrm{SL}(n,\mathbb{C})$ such that $gAg^{-1} = A^{-1}$. Using Lemma 6.4.1, we have that $\det g = (-1)^n = 1$. Therefore, n is even.

Conversely, let n be even. Since A is reversible, using Lemma 2.4.3, we can partitioned the blocks in the Jordan form of A into pairs $\{\mathrm{J}(\lambda,r),\mathrm{J}(\lambda^{-1},r)\}$. Further, since n is even, there are even number of pairs $\{\mathrm{J}(\lambda,r),\mathrm{J}(\lambda^{-1},r)\}$ with r is odd. Therefore, using Proposition 6.1.15, we can construct a suitable involution g in $\mathrm{SL}(n,\mathbb{C})$ such that $gAg^{-1} = A^{-1}$. Therefore, A is strongly reversible in $\mathrm{SL}(n,\mathbb{C})$. This proves the lemma. □

6.5 Proof of Theorem 1.4.2

In view of Lemma 2.4.3, the Jordan blocks in the Jordan decomposition of A can be partitioned into pairs $\{\mathrm{J}(\lambda,r),\mathrm{J}(\lambda^{-1},r)\}$ or singletons $\{\mathrm{J}(\mu,s)\}$, where $\lambda \notin \{\pm 1\}, \mu \in \{\pm 1\}$. Let $\lambda_1,\lambda_2,\cdots,\lambda_t, (\lambda_1)^{-1},(\lambda_2)^{-1},\cdots,(\lambda_t)^{-1}$ are distinct eigenvalues of A such that $\lambda_k \notin \{\pm 1\}$ for all $1 \leq k \leq t$. Further, suppose that both λ_k and $(\lambda_k)^{-1}$ have multiplicity m_k. Thus, up to conjugacy, we can assume that

$$A = P \oplus Q \oplus \begin{pmatrix} R_1 & \\ & R'_1 \end{pmatrix} \oplus \begin{pmatrix} R_2 & \\ & R'_2 \end{pmatrix} \oplus \cdots \oplus \begin{pmatrix} R_t & \\ & R'_t \end{pmatrix}, \tag{6.5.1}$$

where $P \in \mathrm{GL}(p,\mathbb{C})$, $Q \in \mathrm{GL}(q,\mathbb{C})$, $R_k \in \mathrm{GL}(m_k,\mathbb{C})$ and $R'_k \in \mathrm{GL}(m_k,\mathbb{C})$ are Jordan matrices corresponding to eigenvalues $+1$, -1, λ_k and $(\lambda_k)^{-1}$, respectively. Then the Weyr form A_W corresponding to the Jordan form A can be given by

$$\begin{aligned} A_W = & P_W \oplus Q_W \oplus \begin{pmatrix} (R_1)_W & \\ & (R'_1)_W \end{pmatrix} \\ & \oplus \begin{pmatrix} (R_2)_W & \\ & (R'_2)_W \end{pmatrix} \oplus \cdots \oplus \begin{pmatrix} (R_t)_W & \\ & (R'_t)_W \end{pmatrix}, \end{aligned} \tag{6.5.2}$$

where $P_W, Q_W, (R_k)_W$ and $(R'_k)_W$ are basic Weyr matrices corresponding to Jordan matrices P, Q, R_k and R'_k, respectively. Since diagonal block matrices in Weyr form A_W do not have common eigenvalues, using Lemma 2.3.2, we have that any $f \in \mathrm{M}(n,\mathbb{C})$ commuting with A_W has the following form:

$$f = B \oplus C \oplus \begin{pmatrix} D_1 & \\ & D'_1 \end{pmatrix} \oplus \begin{pmatrix} D_2 & \\ & D'_2 \end{pmatrix} \oplus \cdots \oplus \begin{pmatrix} D_t & \\ & D'_t \end{pmatrix}, \tag{6.5.3}$$

where $B \in \mathrm{M}(p,\mathbb{C}), C \in \mathrm{M}(q,\mathbb{C}), D_k \in \mathrm{M}(m_k,\mathbb{C})$ and $D'_k \in \mathrm{M}(m_k,\mathbb{C})$ such that $BP_W = P_W B$, $CQ_W = Q_W C$, $D_k(R_k)_W = (R_k)_W D_k$ and $D'_k(R'_k)_W = (R'_k)_W D'_k$, respectively. Moreover, since the set of reversing elements is a right coset of the centralizer, we can find the general form of a reversing element of A_W by using Equation (6.5.3) and finding a suitable reversing element of A_W, as done in Lemma 6.3.4 and Lemma 6.4.1.

Let A is strongly reversible in $\mathrm{SL}(n,\mathbb{C})$. Then there exists an involution $g \in \mathrm{SL}(n,\mathbb{C})$ such that $gAg^{-1} = A^{-1}$. Since $A_W = SAS^{-1}$ for some $S \in \mathrm{SL}(n,\mathbb{C})$, there exists an involution $h = SgS^{-1} \in \mathrm{SL}(n,\mathbb{C})$ such that $hA_W h^{-1} = (A_W)^{-1}$. Suppose that both $\mathbb{O}_{\mathbf{d}(p)}$ and $\mathbb{O}_{\mathbf{d}(q)}$ are empty. Then using a similar argument as in Lemma 6.3.4 and Lemma 6.4.1, we have that

$$\det(h) = (-1)^{|\mathbb{E}^2_{\mathbf{d}(p)}|}(-1)^{|\mathbb{E}^2_{\mathbf{d}(q)}|}\prod_{k=1}^{t}(-1)^{m_k} = (-1)^{|\mathbb{E}^2_{\mathbf{d}(p)}|+|\mathbb{E}^2_{\mathbf{d}(q)}|+\sum_{k=1}^{t} m_k}.$$

Since $p+q+2\sum_{k=1}^{t} m_k = n$ and $\det(h) = 1$, we have

$$\det(h) = (-1)^{|\mathbb{E}^2_{\mathbf{d}(p)}|+|\mathbb{E}^2_{\mathbf{d}(q)}|+\frac{n-(p+q)}{2}} = 1.$$

This implies that $|\mathbb{E}^2_{\mathbf{d}(p)}| + |\mathbb{E}^2_{\mathbf{d}(q)}| + \dfrac{n-(p+q)}{2}$ is even. Therefore, if A is strongly reversible in $\mathrm{SL}(n,\mathbb{C})$ such that both $\mathbb{O}_{\mathbf{d}(p)}$ and $\mathbb{O}_{\mathbf{d}(q)}$ are empty, then $|\mathbb{E}^2_{\mathbf{d}(p)}| + |\mathbb{E}^2_{\mathbf{d}(q)}| + \dfrac{n-(p+q)}{2}$ is even.

Conversely, up to conjugacy, we can assume that A is in Jordan form given by Equation (6.5.1). If either of the conditions (1) and (2) holds, then using a similar argument as in Proposition 6.1.15, we can construct a suitable involution g in $\mathrm{SL}(n,\mathbb{C})$ such that $gAg^{-1} = A^{-1}$. Therefore, A is strongly reversible in $\mathrm{SL}(n,\mathbb{C})$. This completes the proof. $\square$

Chapter 7

Strong reversibility in $\mathrm{SL}(n,\mathbb{H})$

In this chapter, we will investigate strong reversibility in $\mathrm{SL}(n,\mathbb{H})$ and prove Theorem 1.4.3, which classifies strongly reversible elements in $\mathrm{SL}(n,\mathbb{H})$. The following lemma establishes that strong reversibility in $\mathrm{GL}(n,\mathbb{H})$ and $\mathrm{SL}(n,\mathbb{H})$ is equivalent.

Lemma 7.0.1. *An element A of $\mathrm{SL}(n,\mathbb{H})$ is strongly reversible in $\mathrm{SL}(n,\mathbb{H})$ if and only if it is strongly reversible in $\mathrm{GL}(n,\mathbb{H})$.*

Proof. Suppose that $gAg^{-1} = A^{-1}$, where $g \in \mathrm{GL}(n,\mathbb{H})$ is an involution. Note that $\Phi(g) \in \mathrm{GL}(2n,\mathbb{C})$ is an involution, so $\det(\Phi(\mathrm{g})) \in \{\pm 1\}$. Moreover, $\det(\mathrm{g}) := \det(\Phi(\mathrm{g}))$ is always a non-negative real number; see Definition 2.1.3. Therefore, we can conclude that $\det(\mathrm{g}) = 1$, i.e., $g \in \mathrm{SL}(n,\mathbb{H})$. This proves the lemma. $\square$

Now, we will investigate strong reversibility of matrices in $\mathrm{SL}(n,\mathbb{H})$ having a single eigenvalue $e^{\mathbf{i}\theta}$, where $\theta \in (0,\pi)$.

Lemma 7.0.2. *Let $A := \mathrm{J}(e^{\mathbf{i}\theta}, n) \in \mathrm{SL}(n,\mathbb{H})$ be the Jordan block, where $\theta \in (0,\pi)$. Then A is reversible but not strongly reversible in $\mathrm{SL}(n,\mathbb{H})$.*

Proof. Consider $\Omega(e^{\mathbf{i}\theta}, n) \in \mathrm{GL}(n,\mathbb{C})$, which is defined in Definition 4.2.1. Let $h := \Omega(e^{\mathbf{i}\theta}, n)\,\mathbf{j} \in \mathrm{GL}(n,\mathbb{H})$. Since $\mathbf{j}\,\mathrm{J}(e^{\mathbf{i}\theta}, n) = \mathrm{J}(e^{-\mathbf{i}\theta}, n)\,\mathbf{j}$, using Lemma 4.2.6, we have $hAh^{-1} = A^{-1}$. Thus A is reversible in $\mathrm{SL}(n,\mathbb{H})$. Let $f \in \mathrm{GL}(n,\mathbb{H})$ such that $fA = Af$. Then using Lemma 2.3.4, we have

$$f = \mathrm{Toep}_n(\mathbf{x}),$$

where $\mathrm{Toep}_n(\mathbf{x}) \in \mathrm{GL}(n,\mathbb{C})$ is as defined in Definition 2.3.3 and $\mathbf{x} := (x_1, x_2, \dots, x_n) \in \mathbb{C}^n$ with $x_1 \neq 0$.

Suppose that A is strongly reversible in $\mathrm{SL}(n,\mathbb{H})$. Then there exists an involution $g = [g_{i,j}]_{1\le i,j\le n} \in \mathrm{SL}(n,\mathbb{H})$ such that $gAg^{-1} = A^{-1}$. Since the set of reversing elements of A is a right coset of the centralizer of A, we have

$$g = \mathrm{Toep}_n(\mathbf{x})\Omega(e^{\mathbf{i}\theta}, n)\,\mathbf{j}.$$

This implies that $g_{1,1} = z\mathbf{j}$, where $z = (-1)^{n-1}e^{-2\mathbf{i}(n-1)\theta}x_1 \in \mathbb{C}$. Since g is an involution with an upper triangular form, we have $(g_{1,1})^2 = 1$. Therefore, $|z|^2 = -1$. This is a contradiction. Hence, A is not strongly reversible in $\mathrm{SL}(n,\mathbb{H})$. $\square$

Although, the Jordan block $\mathrm{J}(e^{\mathbf{i}\theta},n) \in \mathrm{SL}(n,\mathbb{H})$ is not strongly reversible in $\mathrm{SL}(n,\mathbb{H})$. But Jordan forms in $\mathrm{SL}(n,\mathbb{H})$ having such Jordan blocks may be strongly reversible.

Lemma 7.0.3. *Let* $A := \begin{pmatrix} \mathrm{J}(e^{\mathbf{i}\theta},n) & \\ & \mathrm{J}(e^{\mathbf{i}\theta},n) \end{pmatrix} \in \mathrm{SL}(2n,\mathbb{H})$ *be the Jordan form, where* $\theta \in (0,\pi)$. *Then* A *is strongly reversible in* $\mathrm{SL}(2n,\mathbb{H})$.

Proof. Consider $\Omega(e^{\mathbf{i}\theta},n) \in \mathrm{GL}(n,\mathbb{C})$, which is defined in Definition 4.2.1. Let $g = \begin{pmatrix} & \Omega(e^{\mathbf{i}\theta},n)\,\mathbf{j} \\ \left(\Omega(e^{\mathbf{i}\theta},n)\,\mathbf{j}\right)^{-1} & \end{pmatrix}$. Then g is an involution in $\mathrm{SL}(2n,\mathbb{H})$ such that $gAg^{-1} = A^{-1}$. This proves the lemma. $\square$

Lemma 7.0.4. *Let* $A \in \mathrm{SL}(n,\mathbb{H})$ *be an element having only eigenvalue* $e^{\mathbf{i}\theta}$, *where* $\theta \in (0,\pi)$, *and* $\mathbf{d}(n) = [d_1^{t_{d_1}}, \ldots, d_s^{t_{d_s}}]$ *be the corresponding partition in the Jordan decomposition of* A. *Then* A *is strongly reversible in* $\mathrm{SL}(n,\mathbb{H})$ *if and only if* t_{d_ℓ} *is even for all* $1 \leq \ell \leq s$.

Proof. ($\Rightarrow$) In view of Lemma 2.2.4, up to conjugacy, we can assume A in Jordan form with complex entries given by Equation (2.2.5). Let A_W denote the Weyr form of A. Then $A_W = \delta A \delta^{-1}$ for some $\delta \in \mathrm{GL}(n,\mathbb{H})$. Using Lemma 6.1.2, we have that partition $\overline{\mathbf{d}}(n)$ representing Weyr form A_W is given by

$$\overline{\mathbf{d}}(n) = [(t_{d_1}+t_{d_2}+\cdots+t_{d_s})^{d_s}, (t_{d_1}+t_{d_2}+\cdots+t_{d_{s-1}})^{d_{s-1}-d_s}, \cdots, (t_{d_1}+t_{d_2})^{d_2-d_3}, (t_{d_1})^{d_1-d_2}].$$

Therefore, A_W is a block matrix with $(d_1)^2$ many blocks and for all $1 \leq i \leq d_1$, the size n_i of i-th diagonal block is given by

$$n_i = \begin{cases} t_{d_1}+t_{d_2}+\cdots+t_{d_s} & \text{if } 1 \leq i \leq d_s \\ t_{d_1}+t_{d_2}+\cdots+t_{d_{s-r-1}} & \text{if } d_{s-r}+1 \leq i \leq d_{s-r-1}, \text{where } 0 \leq r \leq s-2 \end{cases}.$$

This implies that (i,j)-th block of A_W has the size $n_i \times n_j$; see Section 6.1.2. Further, (i,j)-th block of Weyr form A_W and its inverse $(A_W)^{-1}$ can be given as follow:

$$(A_W)_{i,j} = \begin{cases} e^{\mathbf{i}\theta}\mathrm{I}_{n_i} & \text{if } j = i \\ \mathrm{I}_{n_i \times n_{i+1}} & \text{if } j = i+1 \\ \mathrm{O}_{n_i \times n_j} & \text{otherwise} \end{cases}, \text{ and}$$

$$((A_W)^{-1})_{i,j} = \begin{cases} e^{-\mathbf{i}\theta}\mathrm{I}_{n_i} & \text{if } j = i \\ (-1)^k e^{-(k+1)\mathbf{i}\theta}\mathrm{I}_{n_i \times n_{i+k}} & \text{if } j = i+k, 1 \leq k \leq d_1 - i \\ \mathrm{O}_{n_i \times n_j} & \text{otherwise} \end{cases},$$

where $1 \le i, j \le d_1$. Consider $\Omega_W := [X_{i,j}]_{d_1 \times d_1} \in \mathrm{GL}(n, \mathbb{C})$ such that

$$X_{i,j} = \begin{cases} \mathrm{O}_{n_i \times n_j} & \text{if } i > j \\ \mathrm{O}_{n_i \times n_j} & \text{if } j = d_1, i \neq d_1 \\ (-1)^{d_1 - i} \left(e^{-2(d_1 - i)\mathbf{i}\theta} \right) \mathrm{I}_{n_i} & \text{if } j = i \\ (-1)^{d_1 - i} \binom{d_1 - i - 1}{j - i} \left(e^{-(2d_1 + i + j)\mathbf{i}\theta} \right) \mathrm{I}_{n_i \times n_j} & \text{if } j > i, j \neq d_1 \end{cases}, \tag{7.0.1}$$

where $\binom{d_1 - i - 1}{j - i}$ denotes the binomial coefficient. Let $\tau = \Omega_W \mathbf{j} \in \mathrm{GL}(n, \mathbb{H})$. Then by using a similar argument as in Lemma 4.2.6 and Lemma 7.0.2, we have

$$\tau A_W \tau^{-1} = (A_W)^{-1}. \tag{7.0.2}$$

Let $f = [P_{i,j}]_{d_1 \times d_1} \in \mathrm{M}(n, \mathbb{H})$ be an $n \times n$ matrix commuting with Weyr form A_W, where both f and A_W are blocked according to the partition $\overline{\mathbf{d}}(n)$. Then, using a similar argument as in Lemma 2.3.4, we can conclude that f is a complex matrix. Thus, $f(e^{\mathbf{i}\theta}\mathrm{I}_n) = (e^{\mathbf{i}\theta}\mathrm{I}_n)f$. Then $fA_W = A_W f$ implies that $fN = Nf$, where $N = A_W - e^{\mathbf{i}\theta}\mathrm{I}_n$ is a nilpotent Weyr matrix with Weyr structure $\overline{\mathbf{d}}(n)$. Therefore, using Proposition 6.1.6, we have that $f \in \mathrm{M}(n, \mathbb{C})$ is an upper triangular block matrix such that i-th diagonal block $P_{i,i}$ of f is given as follows:

(1) if $1 \le i \le d_s$, then $P_{i,i} = \begin{pmatrix} P_1 & * & * & * & \cdots & * \\ & P_2 & * & \cdots & \cdots & * \\ & & \ddots & \ddots & & \vdots \\ & & & \ddots & \ddots & \vdots \\ & & & & P_{s-1} & * \\ & & & & & P_s \end{pmatrix}$,

(2) if $d_{s-r} + 1 \le i \le d_{s-r-1}$, then $P_{i,i} = \begin{pmatrix} P_1 & * & * & * & \cdots & * \\ & P_2 & * & \cdots & \cdots & * \\ & & \ddots & \ddots & & \vdots \\ & & & \ddots & \ddots & \vdots \\ & & & & P_{s-r-2} & * \\ & & & & & P_{s-r-1} \end{pmatrix}$,

(3) if $d_2 + 1 \le i \le d_1$, then $P_{i,i} = P_1$,

where $0 \le r \le s - 3$ and $P_k \in \mathrm{M}(t_{d_k}, \mathbb{C})$ for all $1 \le k \le s$.

Since A is strongly reversible in $\mathrm{SL}(n, \mathbb{H})$, there exists an involution g in $\mathrm{SL}(n, \mathbb{H})$ such that $gAg^{-1} = A^{-1}$. Consider $h = [Y_{i,j}]_{d_1 \times d_1} := \delta g \delta^{-1}$ in $\mathrm{SL}(n, \mathbb{H})$. Then h is an involution in $\mathrm{SL}(n, \mathbb{H})$ such that

$$hA_W h^{-1} = (A_W)^{-1}. \tag{7.0.3}$$

Note that the set of reversing elements of A_W is a right coset of the centralizer of A_W. Therefore, using Equations (7.0.3) and (7.0.2), we have

$$h = f\tau.$$

This implies that h is an upper triangular block matrix such that diagonal blocks of h are given by

$$Y_{1,1} = \begin{pmatrix} Q_1\mathbf{j} & * & * & * & \cdots & * \\ & Q_2\mathbf{j} & * & \cdots & \cdots & * \\ & & \ddots & \ddots & & \vdots \\ & & & \ddots & \ddots & \vdots \\ & & & & Q_{s-1}\mathbf{j} & * \\ & & & & & Q_s\mathbf{j} \end{pmatrix},$$

where $Q_k = (-1)^{d_1-1}\left(e^{-2(d_1-1)\mathbf{i}\theta}\right)P_k \in \mathrm{GL}(t_{d_k},\mathbb{C})$ for all $1 \le k \le s$. Since h is an involution, we have $Y_{1,1}$, and hence $Q_k\mathbf{j}$ is an involution for all $1 \le k \le s$. Then we have

$$Q_k\overline{Q_k} = -\mathrm{I}_{t_{d_k}}.$$

This implies $|\det(Q_k)|^2 = (-1)^{t_{d_k}}$. Therefore, t_{d_k} is even for all $1 \le k \le s$.

($\Leftarrow$) The proof of converse part of the lemma follows using Lemma 7.0.3. This proves the lemma. $\square$

7.1 Proof of Theorem 1.4.3

($\Leftarrow$) The proof follows from Theorem 1.2.1, Remark 2.2.5, and Lemma 7.0.1.

($\Rightarrow$) Let A be a strongly reversible element in $\mathrm{SL}(n,\mathbb{H})$. Assume that A has a non-real unit modulus eigenvalue $\mu = e^{\mathbf{i}\theta_o}$ with multiplicity m, where $\theta_o \in (0,\pi)$. Using the Jordan decomposition Lemma 2.2.4, we can write A as

$$A = \begin{pmatrix} A_1 & \\ & A_2 \end{pmatrix},$$

where $A_1 \in \mathrm{GL}(m,\mathbb{H})$ and $A_2 \in \mathrm{GL}(n-m,\mathbb{H})$. Moreover, A_1 has a single eigenvalue $e^{\mathbf{i}\theta_o}$, and for each eigenvalue λ of A_2, we have $[e^{\mathbf{i}\theta_o}] \neq [\lambda]$. Let $B \in \mathrm{GL}(n,\mathbb{H})$ such that $BA = AB$. Then Lemma 2.3.2 implies that B has the following form

$$B = \begin{pmatrix} B_1 & \\ & B_2 \end{pmatrix},$$

where $B_1 \in \mathrm{M}(m,\mathbb{H})$ and $B_2 \in \mathrm{M}(n-m,\mathbb{H})$ such that $B_1A_1 = A_1B_1$ and $B_2A_2 = A_2B_2$. Since A is reversible, using Lemma 2.4.3, Lemma 7.0.2, and Table 1.1, we can find a reverser h of A which has the form

$$h = \begin{pmatrix} h_1 & \\ & h_2 \end{pmatrix},$$

where $h_1 \in \mathrm{GL}(m,\mathbb{H})$ and $h_2 \in \mathrm{GL}(n-m,\mathbb{H})$ such that $h_iA_i(h_i)^{-1} = (A_i)^{-1}$ for $i \in \{1,2\}$.

Let $g \in \mathrm{SL}(n,\mathbb{H})$ be an involution such that $gAg^{-1} = A^{-1}$. Since the set of reversing elements of A is a right coset of the centralizer of A, we have that g has the following form

$$g = \begin{pmatrix} g_1 & \\ & g_2 \end{pmatrix},$$

where $g_1 \in \mathrm{GL}(m,\mathbb{H})$ and $g_2 \in \mathrm{GL}(n-m,\mathbb{H})$ are involutions such that $g_i A_i (g_i)^{-1} = (A_i)^{-1}$ for $i \in \{1,2\}$. Thus we have an involution $g_1 \in \mathrm{GL}(m,\mathbb{H})$ such that $g_1 A_1 (g_1)^{-1} = (A_1)^{-1}$. The proof now follows from Lemma 7.0.4. □

Remark 7.1.1. In Theorem 1.2.1, we gave a sufficient criterion for strong reversibility of the reversible elements in $\mathrm{GL}(n,\mathbb{H})$. In view of Lemma 7.0.1 and Theorem 1.4.3, it follows that converse of Theorem 1.2.1 also holds. This gives a complete classification of strongly reversible elements in $\mathrm{GL}(n,\mathbb{H})$ and $\mathrm{SL}(n,\mathbb{H})$. □

Chapter 8

Future directions

8.1 Reversibility in $\mathrm{U}(n,1;\mathbb{D}) \ltimes \mathbb{D}^{n+1}$

Let $\mathbb{D} := \mathbb{R}, \mathbb{C}$, or $\mathbb{H}$. Let $\mathbb{V} := \mathbb{D}^{n,1}$ be the (right) vector space $\mathbb{D}^{n+1}$, together with the unitary structure defined by a non-degenerate and indefinite $\mathbb{D}$-Hermitian form Φ of signature $(n,1)$:

$$\Phi(z,w) = \bar{z}_1 w_1 + \cdots + \bar{z}_n w_n - \bar{z}_{n+1} w_{n+1},$$

where $z = (z_1, \ldots, z_{n+1})$, $w = (w_1, \ldots, w_{n+1}) \in \mathbb{D}^{n+1}$. The group of linear transformations g that preserves this form, i.e., $\Phi(gz, gw) = \Phi(z,w)$ for all $z, w \in \mathbb{V}$, is the unitary group $\mathrm{U}(n,1;\mathbb{D})$; see [8]. The Isometric group $\mathrm{U}(n,1;\mathbb{D}) \ltimes \mathbb{D}^{n+1}$ of the Hermitian space $(\mathbb{V}, d)$ acts on $\mathbb{V}$ as affine transformations: $T : z \mapsto Az + v$, where $A \in \mathrm{U}(n,1;\mathbb{D})$, $v \in \mathbb{D}^{n+1}$ and metric d on $\mathbb{V}$ is usual Bergman metric. In this project, we are interested in classifying the reversible and strongly reversible elements in the group $\mathrm{U}(n,1;\mathbb{D}) \ltimes \mathbb{D}^{n+1}$.

8.2 Linking in $\mathrm{U}(n,\mathbb{F}) \ltimes \mathbb{F}^n$

Two strongly reversible elements g and h in a group G is called *linked* if $g = i_1 i_2$ and $h = i_2 i_3$ for involutions i_1, i_2, i_3 in G. One of the main problems in geometry is to classify *linked pairs* in a group of isometries; see [4]. In [4], authors study the linking problem in the group of orientation preserving isometries of space forms; the $(n-1)$- sphere, Euclidean n-space, and hyperbolic n-space. We hope to use our results in [14] to classify linked pairs in $\mathrm{U}(n,\mathbb{F}) \ltimes \mathbb{F}^n$, where $\mathbb{F} = \mathbb{C}$ or $\mathbb{H}$.

8.3 Reversibility in discrete groups

The reversibility in the group $\mathrm{SL}(2,\mathbb{Z})$ is discussed in [30, Chapter 7]. We would like to pursue the analogous questions regarding the classification of reversibility in the discrete matrix group $\mathrm{SL}(n,\mathbb{Z})$ for $n > 2$.

In the monograph [30], several alternative directions for exploring the reversibility problem were proposed, offering numerous possibilities for investigation that I would like to pursue.

Bibliography

[1] V. I. Arnol'd, A. Avez, *Ergodic problems of classical mechanics.* Translated from the French by A. Avez W. A. Benjamin, Inc., New York-Amsterdam 1968.

[2] S. Bhunia, K. Gongopadhyay, Reversible quaternionic hyperbolic isometries. *Linear Algebra Appl*. 591 (2020), 268–283.

[3] G.D. Birkhoff, The restricted problem of three bodies. *Rend. Circ. Mat. Palermo,* 39 (1915), 265–334.

[4] A. Basmajian, B. Maskit, Space form isometries as commutators and products of involutions. *Trans. Amer. Math. Soc.* 364 (2012), no. 9, 5015–5033.

[5] M. Baake, J. A. G. Roberts, Reversing symmetry group of $\mathrm{Gl}(2,\mathbb{Z})$ and $\mathrm{PGl}(2,\mathbb{Z})$ matrices with connections to cat maps and trace maps, *J. Phys. A*, 30 (1997), no. 5, 1549–1573.

[6] M. Baake, J. A. G. Roberts, The structure of reversing symmetry groups, *Bull. Austral. Math. Soc.*, 73(2006), 445–459.

[7] D. M. Burton, *Elementary number theory,* McGraw-Hill, 7th edition (2011).

[8] S. S. Chen, L. Greenberg, *Hyperbolic spaces*. Contributions to analysis (a collection of papers dedicated to Lipman Bers), pp. 49–87. Academic Press, New York, 1974.

[9] C. W. Curtis, *Linear algebra. An introductory approach*. Fourth edn. Undergraduate Texts in Mathematics. New York: Springer-Verlag. 1984.

[10] D. Ž. Djoković, Product of two involutions. *Arch. Math. (Basel)* 18 (1967), 582–584.

[11] E.W. Ellers, Cyclic decomposition of unitary spaces. *J. Geom*. 21 (2) (1983) 101–107.

[12] E. W. Ellers, W. Nolte, Bireflectionality of orthogonal and symplectic groups. *Arch. Math. (Basel)* 39 (1982), no. 2, 113–118.

[13] W. H. Gustafson, P. R. Halmos, H. Radjavi, Products of involutions. *Linear Algebra Appl.*, 13 (1976), 157–162.

[14] K. Gongopadhyay, T. Lohan, Reversibility of Hermitian isometries. *Linear Algebra Appl.* 639 (2022), 159–176.

[15] K. Gongopadhyay, T. Lohan, C. Maity, Reversibility and real adjoint orbits of linear maps. *Essays in Geometry*, dedicated to Norbert A'Campo (ed. A. Papadopoulos), European Mathematical Society Press, Berlin, 2023, DOI: 10.4171/IRMA/34/15.

[16] K. Gongopadhyay, T. Lohan, C. Maity, *Reversibility of affine transformations*, 2022, preprint, arXiv:2211.11606.

[17] I. Gohberg, P. Lancaster, L. Rodman, *Invariant Subspaces of Matrices with Applications*. John Wiley, New York, 1986; republication SIAM, Philadelphia, 2006.

[18] K. Gongopadhyay, C. Maity, Reality of unipotent elements in classical Lie groups. *Bull. Sci. math.* 185 (2023), Paper No. 103261.

[19] K. Gongopadhyay, Conjugacy classes in Möbius groups. *Geom. Dedicata* 151 (2011), 245–258.

[20] G. R. Goodson, Inverse conjugacies and reversing symmetry groups. *Amer. Math. Monthly* 106 (1999), no. 1, 19–26.

[21] K. Gongopadhyay, J. Parker, Reversible complex hyperbolic isometries. *Linear Algebra Appl*. 438 (6), (2013), 2728–2739.

[22] N. Gill, A. Singh, Real and strongly real classes in $\mathrm{PGL}_n(q)$ and quasi-simple covers of $\mathrm{PSL}_n(q)$. *J. Group Theory* 14 (2011), no. 3, 461–489.

[23] N. Gill, A. Singh, Real and strongly real classes in $\mathrm{SL}_n(q)$. *J. Group Theory* 14 (2011), no. 3, 437–459.

[24] G. James, M. Liebeck, *Representations and characters of groups.* Second edition. Cambridge University Press, New York, 2001.

[25] A. W. Knapp, *Lie groups beyond an introduction* Second edition. Progress in Mathematics, Vol. 140, Birkhäuser. Boston, Inc., Boston, MA, 2002.

[26] F. Knüppel, K. Nielsen, On products of two involutions in the orthogonal group of a vector space, *Linear Algebra Appl*. 94 (1987) 209–216.

[27] S. G. Kolesnikov, J. N. Nuzhin, On strong reality of finite simple groups. *Acta Appl. Math.* 85 (2005), no. 1-3, 195–203.

[28] J. S. W. Lamb, J. A. G. Roberts, Time-reversal symmetry in dynamical systems: a survey. *Phys. D* 112 (1998), no. 1–2, 1–39.

[29] K. C. O'Meara, J. Clark, C. I. Vinsonhaler, *Advanced topics in linear algebra: Weaving matrix problems through the Weyr form*, Oxford University Press, Oxford, 2011, MR2849857.

[30] A. G. O'Farrell, I. Short, *Reversibility in Dynamics and Group Theory*. London Mathematical Society Lecture Note Series, vol.416, Cambridge University Press, Cambridge, 2015.

[31] C. d. S. Pazzis, Products of involutions in the stable general linear group. *J. Algebra* 530 (2019), 235–289.

[32] L. Rodman. *Topics in quaternion linear algebra*. Princeton Series in Applied Mathematics. Princeton University Press, Princeton, NJ, 2014.

[33] P. Sarnak, Reciprocal geodesics. *Analytic number theory,* 217–237, Clay Math. Proc., 7, Amer. Math. Soc., Providence, RI, 2007.

[34] H. Shapiro, The Weyr Characteristic. *American Mathematical Monthly*, 106, 1999, 919–929, MR1732500.

[35] H. Shapiro, *Linear algebra and matrices: Topics for a second course.* Pure and Applied Undergraduate Texts, 24. American Mathematical Society, Providence, RI, 2015, MR3408220.

[36] I. Short, Reversible maps in isometry groups of spherical, Euclidean and hyperbolic space. *Math. Proc. R. Ir. Acad*. 108 (2008), no. 1, 33–46.

[37] J. Stillwell, *Geometry of surfaces*. Universitext. Springer-Verlag, New York, 1992.

[38] A. Singh, M. Thakur, Reality properties of conjugacy classes in algebraic groups. *Israel J. Math.* 165 (2008), 1–27.

[39] P. H. Tiep, A. E. Zalesski, Real conjugacy classes in algebraic groups and finite groups of Lie type. *J. Group Theory* 8 (2005), no. 3, 291–315.

[40] E. P. Vdovin, A. A. Gal't, Strong reality of finite simple groups. *Sib. Math. J.* 51 (2010), no. 4, 610–615.

[41] M. J. Wonenburger, Transformations which are products of two involutions, *J. Math. Mech.* 16 (1966) 327–338.

[42] F. Zhang, Quaternions and matrices of quaternions. *Linear Algebra Appl.*, 251 (1997), 21–57.