Hyperbolicity and Cannon-Thurston maps for complexes of spaces

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To my parents

Declaration

The work presented in this thesis has been carried out by me under the guidance of Dr. Pranab Sardar at the Indian Institute of Science Education and Research Mohali. This work has not been submitted in part or in full for a degree, a diploma, or a fellowship to any other university or institute. Whenever contributions of others are involved, every effort is made to indicate this clearly, with due acknowledgement of collaborative research and discussions. This thesis is a bonafide record of original work done by me and all sources listed within have been detailed in the bibliography.

Rakesh Halder

In my capacity as the supervisor of the candidate's thesis work, I certify that the above statements by the candidate are true to the best of my knowledge.

Dr. Pranab Sardar
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Abstract

The concept of Cannon-Thurston maps in Geometric Group Theory was introduced by Mitra in [1] motivated by the fundamental work of Cannon and Thurston (see [2, 3]). Given Gromov hyperbolic groups $H < G$ (see [4]) one asks if the inclusion map $i : H \to G$ naturally induces the Cannon-Thurston (CT) map $\partial i : \partial H \to \partial G$ which is characterized by the property that for any sequence $\{h_n\}$ in H and $\xi \in \partial H$, $h_n \to \xi$ implies $h_n \to \partial i(\xi)$. It is well-known that such a map is continuous when it exists, but it may not, in general, exist (see [5]). In the first part of the thesis, among other things, we show the existence of CT maps for a pair of hyperbolic groups $H < G$ where (1) G is the fundamental group of a graph of hyperbolic groups $(\mathscr{G}, Y)$, say, satisfying qi embedded condition such that G is hyperbolic (see [6]), (2) H is the fundamental group of a subgraph of hyperbolic subgroups $(\mathscr{H}, Z)$, say, of $(\mathscr{G}, Y)$, (3) for any vertex v of Z, the inclusion of the vertex groups $H_v \to G_v$ of $(\mathscr{H}, Z)$ and $(\mathscr{G}, Y)$ admits the CT map and (4) for any edge e of Z, the edge group H_e of $(\mathscr{H}, Z)$ is same as the corresponding edge group G_e of $(\mathscr{G}, Y)$. (One is refered to [7, Corollary 1.14] for the definition of a subgraph of subgroups of a graph of groups.) This result is deduced by first proving an existence theorem for CT maps for certain morphisms of trees of hyperbolic metric spaces, which generalizes earlier results of M. Mitra ([8]), and (a special cases of) M. Kapovich and P. Sardar ([9, Theorem 8.11]). Moreover, in the course of this work, we also found a nonexistence theorem for CT maps which is similar to that of Baker-Riley ([5]) but is conceptually somewhat easier to understand.

In the second part of the thesis, we prove a combination theorem for trees of metric bundles extending the combination theorems for trees of hyperbolic metric spaces due to Bestvina-Feighn ([6]) and metric bundles due to Mj-Sardar ([10]). More precisely, we prove that if $\pi_B : B \to T$ is a tree of hyperbolic metric spaces whose edge spaces are points and $\pi_X : X \to B$ is a 1-Lipschitz surjective map then X is hyperbolic if the following holds:

1. The fibers of $\pi_B \circ \pi_X$ are hyperbolic metric spaces which are nonelementary (i.e., their barycentric maps are coarsely surjective as in [10, Section 2]) and are all uniformly properly embedded in X.

2. B is hyperbolic.

3. For all vertex u of T, let $B_u = \pi_B^{-1}(u)$ and $X_u = \pi_X^{-1}(B_u)$. Then the restriction of π_X to X_u gives a metric bundle $X_u \to B_u$ as defined by [10].

4. Suppose e is the edge in T joining two vertices u, v. Let e_B denote the (isometric) lift of e in B joining $b_u \in B_u$ and $b_v \in B_v$. Then π_X restricted to $\pi_X^{-1}(e_B)$ is a tree of metric spaces with the qi embedded condition over $e_B = [b_u, b_v]$ as defined in [8].

5. The parameters of (1), the bundles in (3) and the trees of metric spaces in (4) are uniform.

6. Bestvina-Feighn's hallway flaring condition holds for qi lifts in X of geodesics in B.

This theorem is then used to prove a combination theorem for certain complexes of hyperbolic groups.

Notations

$\mathbb{N}$: set of natural numbers.

$\mathbb{Z}$: set of integers.

$\mathbb{R}$: set of real numbers.

For a metric space X, the metric on X will be denoted by d_X or simply by d when X is understood.

For a subset $U \subseteq X$, $P_{X,U}$ (or P_{XU} or P_U): $X \to U$ is a nearest point projection map.

$Hd_X(A,B)$: Hausdorff distance between A and B for $A, B \subseteq X$

For $A \subseteq X$ and $r \geq 0$, $N_r(A) := \{x \in X : d_X(a,x) \leq r \text{ for some } a \in A\}$.

For $x, y \in X$, $[x,y]_X$ (or $[x,y]$): geodesic joining x and y (when X is understood).

Quasiconvex hull of $A \subseteq X$ is $\text{hull}(A) := \{[a,b] : a, b \in A\}$.

For trees of metric spaces:

$\pi : X \to T$, a tree of metric space

For a subtree $S \subseteq T$, $X_S := \pi^{-1}(S)$; in particular, for $u \in V(T)$, $X_u := \pi^{-1}(u)$.

For a quasiconvex subset $A \subseteq X_u$ ($u \in V(T)$), $\mathscr{F}l^X(A)$ is flow space determined by A.

For trees of metric bundles:

(X,B,T): tree of metric bundles along with maps $\pi_X : X \to B$, $\pi_B : B \to T$ and $\pi = \pi_B \circ \pi_X : X \to T$.

For a subtree $S \subseteq T$, $X_S := \pi^{-1}(S)$, $B_S := \pi_B^{-1}(S)$; in particular, for $u \in V(T)$, $X_u := \pi^{-1}(u)$, $B_u := \pi_B^{-1}(u)$. Fiber over $b \in B_u$ is $F_{b,u} := \pi_X^{-1}(b)$.

For an edge $[v,w]$ in T, we denote the edge joining $\mathfrak{v} \in B_v$ and $\mathfrak{w} \in B_w$ by $[\mathfrak{v},\mathfrak{w}]$. $F_{\mathfrak{v}\mathfrak{w}} := \pi_X^{-1}([\mathfrak{v},\mathfrak{w}])$ is δ'_0-hyperbolic, $F_{\mathfrak{v},v} \hookrightarrow F_{\mathfrak{v}\mathfrak{w}}$ and $F_{\mathfrak{w},w} \hookrightarrow F_{\mathfrak{v}\mathfrak{w}}$ are L'_0-qi embedding. $P_{\mathfrak{w}} := P_{F_{\mathfrak{v}\mathfrak{w}}F_{\mathfrak{w},w}} : F_{\mathfrak{v}\mathfrak{w}} \to F_{\mathfrak{w},w}$ is L'_1-coarse Lipschitz retraction. Any $2\delta_0$ quasiconvex subset of $F_{\mathfrak{v},v}$ or of $F_{\mathfrak{w},w}$ is λ'_0-quasiconvex in $F_{\mathfrak{v}\mathfrak{w}}$.

For $K \geq 1, C \geq 0$ and $\varepsilon \geq 0$, $\mathfrak{Y}$ is a (K,C,ε)-semicontinuous family with a central base $\mathfrak{B}$ over a central tree $\mathfrak{T}$. $T_{\mathfrak{Y}}$:=hull$(\pi(\mathfrak{Y}))$.

Flow space of X_u by $\mathscr{F}l_K(X_u)$ for $u \in T$.

Sometimes, we denote $\mathscr{U}_K := \mathscr{F}l_K(X_u)$ and $\mathscr{V}_K := \mathscr{F}l_K(X_v)$ for $u, v \in T$. Also $U_{KL} := N_L(\mathscr{U}_K) = N_L(\mathscr{F}l_K(X_u)) =: Fl_{KL}(X_u)$ for $L \geq 0$. Similarly, $V_{KL} = Fl_{KL}(X_v)$.

Ladder by $\mathscr{L}_K$ or simply by $\mathscr{L}$ and $\mathscr{L}_{a,v} := \mathscr{L} \cap F_{a,v}$ Similarly, $L_{KR} := N_R(\mathscr{L}_K)$ for $R \geq 0$.

$(\mathscr{G}, Y)$: graph of groups over an oriented connected graph Y

$\mathscr{G}(\mathscr{Y})$: complex of groups over a connected simplicial complex $\mathscr{Y}$

$\mathscr{G}(\mathscr{Y}, Y)$: complex of groups explained in setup $\mathscr{C}$ (see Introduction 1.2)

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Chapter 1

Introduction

This thesis has two parts. In the first part, we show existence of the Cannon-Thurston (CT) map for certain morphisms of trees of hyperbolic metric spaces. In the second part, we prove a combination theorem for complexes of hyperbolic spaces with some restrictions. The general setup that is required for both the parts is that of complexes of spaces, and the applications of these results are obtained in the context of complexes of hyperbolic groups. Now we elaborate on each of these two topics in the following two sections.

Remark 1.0.1. Results discussed in Section 1.1 are part of a preprint [11] and in Section 1.2 are submitted (see [12]).

1.1 Cannon-Thurson maps for morphisms of trees of hyperbolic spaces

A natural question in Geometric Group Theory is as follows.

Question 1. *Under what condition(s) can a map $f : Y \to X$ between hyperbolic metric spaces be extended continuously to their Gromov boundaries, $\partial f : \partial Y \to \partial X$?*

Such a continuous extension (if exists) is known as the *Cannon-Thurston* (CT) map as the first nontrivial examples of such maps was produced by Cannon and Thurston in ([2, 3]). The term 'Cannon-Thurston map' or 'CT map' was coined by Mahan Mitra (Mj) in [1] (and [8]) where he proved the existence of CT maps for any normal hyperbolic subgroup of a hyperbolic group (resp. vertex group of a graph of hyperbolic groups with qi embedded condition). Consequently, over the last two decades, many results on existence of CT maps have been proved. One is refered to [13] for a wondeful survey. However, the set of examples and nonexamples in this context are still very limited. A simple case of the questions addressed in this thesis is the following.

Question 2. *Suppose G_1, G_2 are two hyperbolic groups with a common quasiconvex subgroup H such that the free product with amalgamation $G = G_1 *_H G_2$ is hyperbolic. Suppose that $K_i < G_i$, $i = 1,2$ are hyperbolic subgroups where $H < K_i$, $i = 1,2$; let $K = K_1 *_H K_2$. Does the inclusion $K \to G$ admit the CT map?*

It follows from the work of Bestvina and Feighn ([14]) and Gersten ([15, Corollary 6.7]) that K is hyperbolic. However, it follows from the work of M. Kapovich and P. Sardar, ([9, Theorem 8.71]) that the answer to Question 2 is 'yes' if K_i is quasiconvex in G_i, $i = 1,2$. It easily follows from [8] that the existence of CT maps for the inclusions $K_i \to G_i$, $i = 1,2$ are necessary for the answer to Question 2 to be affirmative. One is referred to Definition 2.5.9 for graph of groups.

Definition 1.1.1 (**Subgraph of subgroups**, [7, Corollary 1.14])**.** Suppose $(\mathcal{G}', Y')$ is a graph of groups over Y'. For $v \in V(Y')$ and $e \in E(Y')$, let us denote the corresponding vertex group by G'_v and the edge group by G'_e. Let Y be a connected subgraph of Y'. A graph of groups $(\mathcal{G}, Y)$ is called a subgraph of subgroups if it is obtained as follows. For each $v \in V(Y)$, $G_v < G'_v$ and for each $e \in E(Y)$, we have $G_e < G'_e$ and the incidence homomorphisms for $(\mathcal{G}, Y)$ are simply the restrictions of those in $(\mathcal{G}', Y')$.

In the thesis, we prove the following.

Theorem 1.1.2. *Suppose $(\mathcal{G}', Y')$ is a graph of hyperbolic groups with the qi embedded condition such that the fundamental group $\pi_1(\mathcal{G}', Y')$ is hyperbolic. Let $(\mathcal{G}, Y)$ be a subgraph of subgroups over Y of $(\mathcal{G}', Y')$ as in Definition 1.1.1. We also assume the following.*

1. *For each $u \in V(Y)$, G_u is hyperbolic and the inclusion $G_u \hookrightarrow G'_u$ admits the CT map.*

2. *Let $e \in E(Y)$. Then:*

   (a) *$G_{i(e)} \cap i_e(G'_e) = i_e(G_e)$ and $G_{t(e)} \cap t_e(G'_e) = t_e(G_e)$.*

   (b) *The inclusions $i_e(G_e) \hookrightarrow G'_{i(e)}$ and $t_e(G_e) \hookrightarrow G'_{t(e)}$ are qi embedded.*

   (c) *There is $D \geq 0$ such that for all $g \in G_{i(e)}$,*

$$d_{G'_{i(e)}}(P_{G_{i(e)} i_e(G_e)}(g), P_{G'_{i(e)} i_e(G'_e)}(g)) \leq D$$

where $P_{G'_{i(e)} i_e(G'_e)} : G'_{i(e)} \to i_e(G'_e)$ is a nearest point projection map onto $i_e(G'_e)$ in the metric of $G'_{i(e)}$ and $P_{G_{i(e)} i_e(G_e)} : G_{i(e)} \to i_e(G_e)$ is that onto $i_e(G_e)$ in the metric of $G_{i(e)}$. With similar notations, for all $g \in G_{t(e)}$, we also have

$$d_{G'_{t(e)}}(P_{G_{t(e)} t_e(G_e)}(g), P_{G'_{t(e)} t_e(G'_e)}(g)) \leq D.$$

Then the fundamental group $\pi_1(\mathscr{G}, Y)$ of $(\mathscr{G}, Y)$ is hyperbolic and the natural homomorphism $\pi_1(\mathscr{G}, Y) \hookrightarrow \pi_1(\mathscr{G}', Y')$ is injective which admits the CT map.

Remark 1.1.3. (1) In Theorem 1.1.2, the injectivity of the inclusion $\pi_1(\mathscr{G}, Y) \hookrightarrow \pi_1(\mathscr{G}', Y')$ follows from [7, Corollary 1.14] and the hyperbolicity of $\pi_1(\mathscr{G}, Y)$ follows from [14] and [15, Corollary 6.7].

(2) In reference to Question 2 above, when the edge groups are same, i.e., $G'_e = G_e$ for all $e \in E(Y)$ then it is not hard to show that condition $(2)(c)$ follows from the mere fact that $G'_u \hookrightarrow G_u$ admits the CT map, whereas conditions $(2)(a)$ and $(2)(b)$ are trivially.

So as a consequence of Theorem 1.1.2, we have the following.

Theorem 1.1.4. *Suppose $(\mathscr{G}', Y')$ is a graph of hyperbolic groups with the qi embedded condition such that the fundamental group $\pi_1(\mathscr{G}', Y')$ is hyperbolic. Let $(\mathscr{G}, Y)$ be a subgraph of subgroups over Y of $(\mathscr{G}', Y')$ as in Definition 1.1.1 such that for each $u \in V(Y)$, G_u is hyperbolic and the inclusion $G_u \hookrightarrow G'_u$ admits the CT map. We also assume one of the followings.*

- *For each $e \in E(Y)$, $G_e = G'_e$.*
- *For each $e \in E(Y)$, we have $G_{i(e)} \cap i_e(G'_e) = i_e(G_e)$ and $G_{t(e)} \cap t_e(G'_e) = t_e(G_e)$; moreover, the inclusion $G_e \hookrightarrow G'_e$ is isomorphic onto finite index subgroup of the target group.*

Then the fundamental group $\pi_1(\mathscr{G}, Y)$ of $(\mathscr{G}, Y)$ is hyperbolic and the natural homomorphism $\pi_1(\mathscr{G}, Y) \hookrightarrow \pi_1(\mathscr{G}', Y')$ is injective which admits the CT map.

The above theorem (Theorem 1.1.6) for graphs of groups follows from a geometric result about trees of metric spaces on which we now elaborate. Suppose X is a tree of hyperbolic spaces over a tree T satisfying the qi embedded condition such that X is hyperbolic (see [6]). Mahan Mitra (Mj) showed the existence of CT map from any vertex space (resp. edge space) to X ([8]). Recently, in their book [9], M. Kapovich and P. Sardar proved the existence of CT map from a subtree of spaces to the ambient space in the setting of trees of hyperbolic spaces generalizing Mitra's work. We extend these results as follows. Let us first outline the setup.

We refer to Definition 2.3.1 for the definition of trees of metric spaces below and Section 2.1 for other terminologies.

1. Suppose $\pi : X \to T$ is a tree of hyperbolic metric spaces satisfying the qi embedded condition such that X is hyperbolic.

2. Let $Y \subseteq X$ be a hyperbolic subspace such that the inclusion $i : Y \hookrightarrow X$ is a proper embedding.

3. The restriction of π on Y, $\pi|_Y : Y \to S = \pi(Y)$ is a tree of hyperbolic metric spaces over S with the qi embedded condition.

4. For all $u \in V(S)$ and for all $e \in E(S)$, inclusions $Y_u \hookrightarrow X_u$ and $Y_e \hookrightarrow X_e$ admit the CT maps.

5. Both X and Y are proper metric spaces.

Remark 1.1.5. Under the above hypotheses hyperbolicity of Y is ensured. Indeed, since X is hyperbolic, $\pi : X \to T$ satisfies flaring condition which implies the same for Y. Basically the proof of [10, Proposition 5.8] works in this case too. Hence, by [6], Y is hyperbolic.

In addition to the above five hypotheses we shall need the following for Theorem 1.1.6.

Projection hypothesis: There is a constant $R_0 \geq 0$ such that for all $v \in V(S)$ and $e \in E(S)$ incident on v, and for all $x \in Y_v$ we have

$$d_{X_v}(P_{X_vX_{ev}}(x), P_{Y_vY_{ev}}(x)) \leq R_0$$

where $P_{X_vX_{ev}} : X_v \to X_{ev}$ is a nearest point projection map onto X_{ev} in the metric of X_v and $P_{Y_vY_{ev}} : Y_v \to Y_{ev}$ is that onto Y_{ev} in the metric of Y_v.

Theorem 1.1.6. *Suppose we have the hypotheses (1)-(5) above plus the following.*

1. *The inclusion $Y_e \hookrightarrow X_e$ is (uniform) qi embedding for all $e \in E(S)$.*

2. *The projection hypothesis holds.*

Then the inclusion $i : Y \hookrightarrow X$ admits the CT map.

A few words on the proof of Theorem 1.1.6: The proof for Theorem 1.1.6 runs by contradiction. For any two sequences $\{y_n\}$ and $\{y'_n\}$ of Y such that $\lim^Y_{n\to\infty} y_n = \lim^Y_{n\to\infty} y'_n$ and $\lim^X_{n\to\infty} y_n$, $\lim^Y_{n\to\infty} y'_n \in \partial X$, we show that $\lim^X_{n\to\infty} y_n = \lim^X_{n\to\infty} y'_n$. This completes the proof (see Lemma 2.2.43). We break the proof up into several cases depending on types of the sets hull$\{\pi(y_n) : n \in \mathbb{N}\}$ and hull$\{\pi(y'_n) : n \in \mathbb{N}\}$, and in each case, we compare the geodesics $[y_n, y'_n]_Y$ and $[y_n, y'_n]_X$. Comparing these geodesics is the main difficult task. For that we construct a quasiconvex subset in both X and Y containing y_n and y'_n using the flow spaces constructed in [9]. □

When the maps in the vertex space levels are uniform qi embeddings then we have the following stronger consequence.

Theorem 1.1.7. *Suppose we have the hypotheses (1)-(4) mentioned above. Moreover, suppose for all $u \in V(S)$ and for all $e \in E(S)$, the inclusions $Y_u \hookrightarrow X_u$ and $Y_e \hookrightarrow X_e$ are uniform qi embeddings and the projection hypothesis holds. Then*

1. *the inclusion $i : Y \hookrightarrow X_S$ is qi embedding where $X_S := \pi^{-1}(S)$, and*
2. *hence by* [9]*, the inclusion $i : Y \hookrightarrow X$ admits the CT map.*

A particular application of Theorem 1.1.2 is the following.

Example 1.1.8. Consider a hyperbolic group G' of the form $G' = N \rtimes Q$, where N is either the fundamental group of a closed orientable surface of genus at least 2 or a finitely generated free group of rank at least 3, and Q is a finitely generated free group of rank at least 2. Examples of this sort are well-known; e.g. see [16], [17]. It is easy to see that Q is a malnormal quasiconvex subgroup of G. Now let $F < Q$ be a malnormal free subgroup of rank at least 3 and let $\phi : F \to F$ be a hyperbolic automorphism. Suppose $H_1 = N \rtimes F < G'$. Then it follows from [6] that the HNN extensions $(< H_1, t : tat^{-1} = \phi(a), a \in F >=) H_2 = H_1 *_\phi < G = G' *_\phi (=< G', t : tat^{-1} = \phi(a), a \in F >)$, are both hyperbolic. (We note that H_1 is hyperbolic by the results of [10].)

However, it easily follows from Theorem 1.1.6 that the inclusion $H_2 \to G$ admits the CT map.

A nonexistence theorem for CT maps

Baker and Riley ([5]) were the first to produce an example of a free subgroup $\mathbb{F}$ of a hyperbolic group G for which the inclusion $\mathbb{F} \to G$ does not admit the CT map. This class of examples were obtained using small cancellation theory. Later, Matsuda and Oguni ([18]) using the examples of Baker-Riley showed that any non-elementary hyperbolic group can be embedded in another hyperbolic group for which there is no

CT map. In the current thesis, we prove a similar result (see Theorem 1.1.9) using geometry of trees of spaces. We feel that this is conceptually somewhat easier to understand than the ones obtained by Baker and Riley.

Theorem 1.1.9. *1. Suppose G' is a hyperbolic group, and Q and N are hyperbolic subgroups where Q is malnormal and quasiconvex in G, but N is not quasiconvex in G. Moreover, suppose that $Q \cap N = (1)$.*

2. Suppose $\phi : Q \to Q$ is an automorphism of Q such that the semidirect product $Q \rtimes_\phi \mathbb{Z}$ is hyperbolic. Let $G = G' *_Q$ be the HNN extension of G' along ϕ with stable letter t and let K be the subgroup of G generated by $N \cup \{t\}$.

3. Finally, suppose that there is a sequence $\{y_n\}$ in N such that $\lim^G_{n\to\infty} P_{G'Q}(y_n) = \lim^G_{n\to\infty} t^n$ where $P_{G'Q} : G' \to Q$ is a nearest point projection map from G' to Q.

Then $K = N * < t >$ is hyperbolic and the inclusion $K \to G$ does not admit the CT map.

As an application to the above theorem we have the following example.

Example 1.1.10. Consider the groups in Example 1.1.8 so that $F = Q$ and rank of Q is at least 3. Suppose t is the common stable letter for the HNN extensions under consideration. Let K be the subgroup of G generated by $N \cup \{t\}$. Clearly, $K = N * < t >$ whence it is hyperbolic. However, it is easy to verify the hypotheses of Theorem 1.1.9 for G and K (see Section 3.7). Thus the inclusion $K \to G$ does not admit the CT map.

One is referred to Definition 3.6.1 for *Cannon-Thurston* (CT) *lamination*. In this thesis, we also investigate the properties of the CT lamination in the situation where Theorem 1.1.6 holds. One of the main results proved in this connection is the following.

Theorem 1.1.11. *Suppose $i : Y \to X$ as in Theorem 1.1.6 such that $S = T$ and $\partial i_{YX} : \partial Y \to \partial X$ is the CT map. Let α be a geodesic line in Y. Suppose there is $w \in V(S)$ and $t_1, t_2 \in \mathbb{R}$ such that $T_1 = \pi(\alpha|_{(-\infty,t_1]})$ and $T_2 = \pi(\alpha|_{[t_2,\infty)})$ lie in two different components of $T \setminus \{w\}$. Then $\partial i_{YX}(\alpha(-\infty)) \neq \partial i_{YX}(\alpha(\infty))$.*

A few words on the proof of Theorem 1.1.11: The following dichotomy holds for the points of ∂X (where $\pi : X \to T$ is a tree of hyperbolic spaces as in Theorem 1.1.6): Either it is a conical limit point of some vertex space or it is not a conical limit point of any vertex space (see Remark 3.2.10). This is the main fact used in the proof of Theorem 1.1.11.

1.2 A Combination theorem for trees of metric bundles

Bestvina-Feighn ([6]) proved that the fundamental group of a finite graph of hyperbolic groups with the qi embedded condition and annuli flare condition is hyperbolic (see [14, Theorem 1.2]). Motivated by this work of Bestvina and Feighn, M. Kapovich asked whether one can extend this combination theorem for graphs of groups to complexes of groups (see [19, Problem 90]). (For more detailed exposition in complexes of groups, on is referred to [20], [21], [22], [23] or Section 2.5.) One may formulate the problem of M. Kapovich as follows.

Problem 1.2.1. *Suppose $\mathcal{G}(\mathcal{Y})$ is a developable complex of groups over a finite connected simplicial complex $\mathcal{Y}$ such that the following holds.*

1. *All the local groups are hyperbolic.*
2. *All the local maps are qi embeddings.*
3. *The universal cover of $\mathcal{G}(\mathcal{Y})$ is hyperbolic.*

Under what condition(s) the fundamental group $\pi_1(\mathcal{G}(\mathcal{Y}))$ is hyperbolic.

Here is brief history of the activities around this problem. Suppose $\mathcal{G}(\mathcal{Y})$ is a complex of groups with the condition as in Problem 1.2.1. If $\mathcal{G}(\mathcal{Y})$ is negatively curved and all the local groups are finite then $\pi_1(\mathcal{G}(\mathcal{Y}))$ is hyperbolic due to Gersten-Stallings ([20]). If the local maps are all isomorphisms onto finite index subgroups of the target groups, local groups are non-elementary hyperbolic and $\mathcal{G}(\mathcal{Y})$ satisfies Bestvina-Feighn's hallway flaring condition then it follows from the work of Mj-Sardar ([10]) that $\pi_1(\mathcal{G}(\mathcal{Y}))$ is hyperbolic. If the universal cover of $\mathcal{G}(\mathcal{Y})$ is $\mathrm{CAT}(0)$ and hyperbolic and the action of $\pi_1(\mathcal{G}(\mathcal{Y}))$ on the universal cover is acylindrical then $\pi_1(\mathcal{G}(\mathcal{Y}))$ is hyperbolic and local groups are quasiconvex in $\pi_1(\mathcal{G}(\mathcal{Y}))$ due to A. Martin ([24]). Apart from these extreme cases nothing is known. However, in this thesis ([12]), we attempt this question for yet another type of complexes of groups. Let us first outline the setup. We refer this as **setup** $\mathcal{C}$

1. Suppose Y is a finite connected graph and $p_Y : \mathcal{Y} \to Y$ is a graph of spaces where the edge spaces are points. We further assume that $\mathcal{Y}$ is a simplicial complex. Suppose $\mathcal{G}(\mathcal{Y})$ is a complex of groups over $\mathcal{Y}$ such that all the properties of Problem 1.2.1 hold with the following additional ones.

2. For all $v \in V(Y)$, $p_Y^{-1}(v) = \mathscr{Y}_v$, say, is a finite connected simplicial complex and the restriction of $\mathscr{G}(\mathscr{Y})$ on $\mathscr{Y}_v$ is a developable complex of groups, say, $\mathscr{G}_v(\mathscr{Y}_v)$ over $\mathscr{Y}_v$. Further, all the local maps in $\mathscr{G}_v(\mathscr{Y}_v)$ are isomorphisms onto finite index subgroups of the target groups.

3. Suppose u, v are two vertices in $\mathscr{Y}$ such that p_Y is injective when restricted to the edge e joining u, v. Then the local homomorphisms $G_e \to G_u$ and $G_e \to G_v$ are not necessarily isomorphisms onto finite index subgroups.

We denote $\mathscr{G}(\mathscr{Y})$ in this case as $\mathscr{G}(\mathscr{Y}, Y)$ to emphasize on the extra structure on $\mathscr{Y}$.

Then we have the following.

Theorem 1.2.2. ([12, Theorem 1.3]) *In addition, if $\mathscr{G}(\mathscr{Y}, Y)$ satisfies Bestvina-Feighn's hallway flare condition and in (2) of setup $\mathscr{C}$, all the local groups of $\mathscr{G}_v(\mathscr{Y}_v)$ are non-elementary (hyperbolic) then $\pi_1(\mathscr{G}(\mathscr{Y}, Y))$ is hyperbolic.*

The above Theorem 1.2.2 follows from a combination theorem for spaces. We now elaborate on this. In [6], Bestvina and Feighn proved that a tree of hyperbolic spaces is hyperbolic if it satisfy the qi embedded condition and hallway flaring condition. In [10], Mj and Sardar proved that if X is a metric bundle over B such that (1) fibers are uniformly hyperbolic and B is also hyperbolic, (2) the barycenter map for the fibers are uniformly coarsely surjective and (3) Bestvina-Feighn's hallway flaring condition holds then X is hyperbolic. The question that motivated us is if we can combine these two and still get hyperbolicity. Here is a baby version of the problem we are attempting to solve.

Question 1.2.3. *Suppose $\pi_i : X_i \to B_i$ are metric bundles for $i = 1, 2$. Suppose we join $b_1 \in B_1$ and $b_2 \in B_2$ by an edge, say, e. Let X_e be a new geodesic metric space. Let F_{b_1} and F_{b_2} be fibers over b_1 and b_2 of the bundles X_1 and X_2 respectively. Suppose there are qi embeddings $X_e \to F_{b_1}$ and $X_e \to F_{b_2}$ and we form a new space by gluing $X_e \times [0, 1]$ to $X_1 \sqcup X_2$ as follows: We attach $X_e \times \{0\}$ to F_{b_1} and $X_e \times \{1\}$ to F_{b_2} using the qi embeddings $X_e \to F_{b_1}$ and $X_e \to F_{b_2}$ respectively. When is the new space hyperbolic?*

In this thesis ([12]), we consider a general version of Question 1.2.3 and provide a combination theorem (Theorem 1.2.4). One is referred to Definition 2.4.2 for trees of metric bundles with the qi embedded condition.

Theorem 1.2.4. ([12, Theorem 1.1]) *Suppose (X, B, T) is a tree of metric bundles such that:*

1. *For $v \in V(T)$ and $a \in B_v$, the fibers, $F_{a,v}$ are uniformly hyperbolic geodesic metric spaces and the barycenter maps $\partial^3 F_{a,v} \to F_{a,v}$ are uniformly coarsely surjective.*

2. *Let $[v,w]$ be an edge in T and $\mathfrak{e} = [\mathfrak{v}, \mathfrak{w}]$ be the edge joining $\mathfrak{v} \in B_v$ and $\mathfrak{w} \in B_w$. Then π_X restricted to $\pi_X^{-1}(\mathfrak{e})$ is a tree of metric spaces with (uniform) qi embedded condition over $\mathfrak{e} = [\mathfrak{v}, \mathfrak{w}]$.*

3. *B is hyperbolic geodesic metric space.*

4. *Bestvina-Feighn's hallway flaring condition is satisfied.*

Then X is hyperbolic geodesic metric space.

Remark 1.2.5. The overall idea of the proof of Theorem 1.2.4 closely follows from that of [9] and makes crucial use of [10]. We are intellectually indebted to both of these works. However it is not a direct consequence of the combination theorems of [6] and [10] in any obvious way for the following reason. Let $v \in V(T)$. As the space X_v over B_v is a metric bundle satisfying all conditions of the main theorem of [10], X_v is (uniformly) hyperbolic. Now we can think of X as a tree of metric spaces over T where vertex spaces are these bundles and the edge spaces are inverse images under $\pi_B \circ \pi_X$ of the midpoints of the edges in T. But we can not apply the main theorem of [6] to this tree of spaces to conclude our theorem because in this case the edge spaces of the tree of spaces are not, in general, qi embedded in the corresponding vertex spaces.

Necessity of flaring

Gersten (see [15, Corollary 6.7]) showed that the annuli flaring is necessary for the fundamental group of a finite graph of hyperbolic groups to be hyperbolic provided the edge groups are qi embedded in the corresponding vertex groups. Mj and Sardar also showed that the (hallway) flaring condition is necessary for metric bundles to be hyperbolic provided fibers are uniformly hyperbolic (see [10, Proposition 5.8]). Let us briefly recall the idea of their proof. They first showed that small girth ladders bounded by two qi lifts satisfy flaring condition. Then a general ladder was subdivided into small girth ladders and summing them up showed that a general ladder satisfies flaring condition. In doing so they used a crucial lemma ([10, Lemma 5.9]) which is a specialization of the fact that geodesics diverge exponentially in hyperbolic metric spaces in the context of metric bundles; this lemma also holds true in trees of metric bundles. In trees of metric bundles, given two qi lifts over the same base, there is a special ladder (see Definition 2.4.11) bounded by these qi lifts (see Lemma 2.4.14). Therefore, the proof of the following remark is analogous to that of [10, Proposition 5.8], so we omit the full details.

Remark 1.2.6. Suppose (X,B,T) is a tree of metric bundles such that:

1. X is hyperbolic.
2. All the fibers are uniformly hyperbolic.
3. All the edge spaces in the corresponding fibers are uniformly qi embedded.

Then $\pi_X : X \to B$ satisfies Bestvina-Feighn's hallway flaring condition.

As a consequence of Remark 1.2.6, we have the following.

Corollary 1.2.7. ([12, Corollary 1.7]) *Suppose $\mathcal{G}(\mathcal{Y},Y)$ is a complex of groups as explained in the setup $\mathcal{C}$ such that the fundamental group $\pi_1(\mathcal{G}(\mathcal{Y},Y))$ of $\mathcal{G}(\mathcal{Y},Y)$ is hyperbolic. Then $\mathcal{G}(\mathcal{Y},Y)$ satisfies Bestvina-Feighn's hallway flare condition. (Note that we do not require the universal cover of $\mathcal{G}(\mathcal{Y})$ to be hyperbolic.)*

A few words on the proof of Theorem 1.2.4: (1) Motivated by that of [9], we construct semicontinuous families, ladders and flow spaces, and more general flow spaces in Section 5.1; whereas ladder was invented by Mitra in [8] for trees of metric spaces. Some properties of these subspaces, most importantly, Mitra's retraction of the whole space on these subspaces, are also discussed there. Main construction starts from here in Section 5.1.

(2) Most difficult job was to show the uniform hyperbolicity of ladders and flow spaces. In Section 5.2, we prove that the ladders are (uniformly) hyperbolic by dividing into two cases: small girth and general case. By invoking Bowditch criterion (see Proposition 2.2.6) for a metric space to be hyperbolic, we prove that small girth ladders are (uniformly) hyperbolic (Subsection 5.2.1). For general ladder, we first break it up into small girth ladder and then with the help of Proposition 2.2.7 we conclude its hyperbolicity (Subsection 5.2.2). Section 5.3 is devoted to prove the (uniform) hyperbolicity of flow spaces. To prove this, we follow the strategy elaborated in [9, Chapter 5].

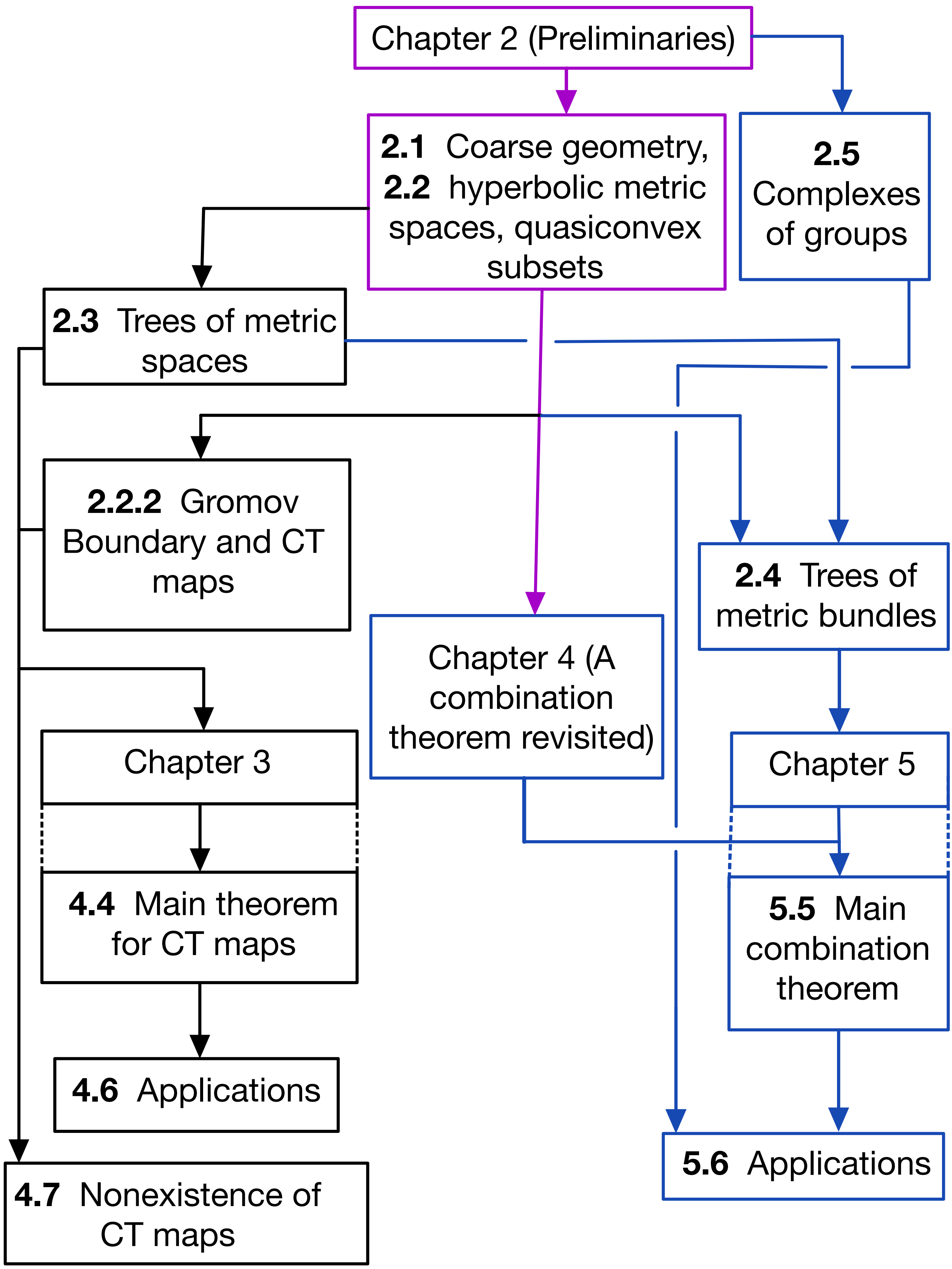
(3) In Section 5.4, we prove that union of uniform neighborhood of two intersecting flow spaces is uniformly hyperbolic; in the introduction of this section we elaborate what the properties are required from the earlier sections to prove this.

(4) Section 5.5 contains proof of Theorem 1.2.4 with the help of Theorem 4.0.1. Theorem 4.0.1 shows the hyperbolicity of total space for a tree of metric spaces within an axiomatic framework.

(5) The last Section 5.6, contains some applications to complexes of groups (Theorem 1.2.2 and Corollary 1.2.7).

Layout of the thesis: In Chapter 2, we recall basics definitions and results which are used in the subsequent chapters. We define trees of metric bundles in Section 2.4. In Chapter 3, we prove the main theorem for Cannon-Thurston maps (Section 3.4, Theorem 1.1.6). In the subsequent Sections 3.5, 3.6, we prove Theorem 1.1.7, Theorem 1.1.2, Theorem 1.1.4 and Theorem 1.1.11. We end Chapter 3, by proving Theorem 1.1.9 and Example 1.1.10 (Section 3.7). Chapter 4 is devoted to proving the hyperbolicity of trees of metric spaces within an axiomatic framework. In Chapter 5, we prove the main combination theorem (Section 5.5, Theorem 1.2.4). Theorem 1.2.2 and Corollary 1.1.5 are proven in Section 5.6.

1.3 Flowchart



Chapter 2

Preliminaries

2.1 Coarse geometric notions

Suppose X is a metric space. For $x,y \in X$, a **geodesic** joining them in X is an isometric embedding $\alpha : [0,d(x,y)] \to X$ with $\alpha(0) = x$ and $\alpha(d(x,y)) = y$. We refer to X as a **geodesic metric space** if there exists a geodesic in X joining every pair of points in X. We say that X is **proper metric space** if closed bounded balls in X are compact. In this thesis, it is assumed that graphs are connected, and their edges are isometric to a closed unit interval of $\mathbb{R}$. That makes the graph a geodesic metric space ([23, Section 1.9, I.1]). A **tree** is a connected graph without any embedded circle. For a tree T and $u,v \in V(T)$, by a **segment** or **interval** joining u,v in T, we mean an isometric embedding $\alpha : [n,m] \to T$ for $n,m \in \mathbb{Z}$ such that $\alpha(n) = u$, $\alpha(m) = v$. We denote $[u,v] := Im(\alpha)$, $(u,v] := Im(\alpha|_{[n+1,m]})$ and $(u,v) := Im(\alpha|_{[n+1,m-1]})$. **Degree** of a vertex $v \in V(T)$ in a tree T is defined to be the number of edges incident on v.

Let us recall some basic notions of large scale geometry (see [4], [25], [23], [26]). Let $(X,d_X),(Y,d_Y)$ be metric spaces and $\delta \geq 0, k \geq 1, \varepsilon \geq 0, r \geq 0, C \geq 0, L \geq 0, D \geq 0, R \geq 0$.

1. Let $A,B \subseteq X$. We say A is ***r*-dense** in X if $X = N_r(A) := \{x \in X : d_X(x,A) \leq r\}$. The **Hausdorff distance** between A and B is defined to be $\inf\{D : A \subseteq N_D(B), B \subseteq N_D(A)\}$ and is denoted by $Hd_X(A,B)$. The subset A is said to be *r-separated subset* if for all distinct $a,b \in A$, $d_X(a,b) \geq r$. We say A and B are ***R*-separated** if $d_X(a,b) > R, \forall\, a \in A$ and $\forall\, b \in B$.
2. A map $f : X \to Y$ is called **ε-coarsely surjective** if $N_\varepsilon(f(X)) = Y$; and, we say that f is **coarsely surjective** if it is ε-coarsely surjective for some ε.
3. A map $f : X \to Y$ is called ***C*-coarsely Lipschitz** if

$$d_Y(f(x), f(x')) \leq C d_X(x,x') + C$$

for all $x, x' \in X$. In particular, if $A \subseteq X$ and $f : X \to A$ is a C-coarsely Lipschitz map such that $f(a) = a$ for all $a \in A$ then we say f is a C-**coarsely Lipschitz retraction** of X on A.

4. A map $\phi : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to be (metrically) **proper** if the inverse image of bounded sets are bounded or equivalently $\lim_{r\to\infty} \phi(r) = \infty$. A function $f : X \to Y$ is called (metrically) ϕ-**proper embedding** (or simply **proper embedding** when ϕ is understood) for some proper map $\phi : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ if

$$d_Y(f(x), f(x')) \leq r \text{ implies } d_X(x, x') \leq \phi(r).$$

5. A map $f : X \to Y$ is said to be (k, ε)-**quasi-isometric embedding** (in short (k, ε)-**qi embedding**) if

$$\frac{1}{k} d_X(x, x') - \varepsilon \leq d_Y(f(x), f(x')) \leq k d_X(x, x') + \varepsilon \text{ for all } x, x' \in X.$$

   We say that f is a **quasi-isometric embedding** (in short **qi embedding**) if it is (k, ε)-qi embedding for some $k \geq 1$ and $\varepsilon \geq 0$. Lastly, by k-qi embedding, we mean (k, k)-qi embedding.

   We say that f is a (k, ε, r)-**quasi-isometry** if it is (k, ε)-qi embedding and r-coarsely surjective. In this case, X and Y are said to be quasi-isometric to each other. Lastly, by a k-quasi-isometry, we mean (k, k, k)-quasi-isometry. It is standard that if $f : X \to Y$ is a k-quasi-isometry then there is k' depending on k and a k'-quasi-isometry $g : Y \to X$ such that $d_X(g \circ f(x), x) \leq k'$ for all $x \in X$ and $d_Y(f \circ g(y), y) \leq k'$ for all $y \in Y$ (see [10, Lemma 1.1 (2)]). In this case, we say that f and g are coarse inverses to each other.

6. By a (k, ε)-**quasi-geodesic** (resp. k-quasi-geodesic) in X, we mean (k, ε)-qi embedding (resp. k-qi embedding) of an interval in $\mathbb{R}$. For a (k, ε)-quasi-geodesic (resp. a geodesic), say, α in X, most of the time we omit the domain (i.e. interval in $\mathbb{R}$) of α and work with its image in X.

7. We say $\alpha : I \subseteq \mathbb{R} \to X$ is a (k, ε, L)-**local quasi-geodesic** if the restriction of α on any subinterval $I' (\subseteq I)$ of length $\leq L$ is a (k, ε)-quasi-geodesic.

8. Let A be a closed subset of X. Let $x \in X$. A point $a \in A$ is called **nearest point projection** of x on A if

$$d_X(a, x) \leq d_X(a', x) \text{ for all } a' \in A.$$

   For a subset B, the set of nearest point projections of B on A is denoted by $P_{X,A}(B)$ or simply by $P_A(B)$ when X is understood.

9. Suppose U,V are closed subsets of X. We say that the pair (U,V) is *D*-**cobounded** in X if

$$max\{diam\{P_U(V)\},\ diam\{P_V(U)\}\} \leq D.$$

For the rest of the points, we assume that X is a **geodesic metric space** and a geodesic in X joining two points $a,b \in X$ is denoted by $[a,b] \subseteq X$.

10. Suppose $x,y,z \in X$. A **geodesic triangle** in X formed by these three points is the union of chosen geodesic segments $[x,y],[x,z]$ and $[y,z]$, and it is denoted by $\triangle(x,y,z)$. We call those geodesic segments as sides of the triangle. We say a geodesic triangle $\triangle(x,y,z)$ is δ-**slim** if any side is contained in the δ-neighborhood of the union of other two sides.
11. (C-**center** and C-**tripod**) Suppose $\triangle(x_1,x_2,x_3)$ is a geodesic triangle in X formed by $x_1,x_2,x_3 \in X$. A point $z \in X$ is called C-**center** of this triangle if $z \in N_C([x_i,x_j])$ for $i \neq j$ and $i,j \in \{1,2,3\}$. Sometimes we call $\cup_{i=1}^{3}[z,x_i]$ as the C-**tripod** in X with end points x_1,x_2,x_3.
12. Let $a,a' \in X$. A **discrete path** joining a and a' in X is a finite set of points with an order, say, $a = a_0 < a_1 < \cdots < a_n = a'$. A path joining a and a' based on a discrete path as above is $[a_0,a_1] \cup [a_1,a_2] \cup \cdots \cup [a_{n-1},a_n]$.

The following lemmata (Lemma 2.1.1, 2.1.2, 2.1.3, 2.1.4) are standard. So we omit the proofs.

Lemma 2.1.1. *Given a proper map $\phi : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ there is a proper function $g_{2.1.1} = g_{2.1.1}(\phi) : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ such that the following holds.*

Let X be a geodesic metric space and Y be a subspace of X. Suppose the inclusion $Y \hookrightarrow X$ is ϕ-proper embedding where Y is considered with the induced path metric from X. Then for all $y,y' \in Y$ and $r \in \mathbb{R}_{\geq 0}$, $d_Y(y,y') > r$ implies $d_X(y,y') > g_{2.1.1}(r)$.

Lemma 2.1.2. *Given $D \geq 0$ there is $C_{2.1.2}(D)$ such that the following holds.*

Suppose X is a geodesic metric space and Y is a subset of X (not necessarily connected) such that Y is 1-dense in X. Let $U \subseteq X$ and $\rho : Y \to U$ be a map such that $d_X(\rho(x),\rho(y)) \leq D$ for all $x,y \in Y$ with $d_X(x,y) \leq 1$. Then ρ can be extended to a map $\rho' : X \to U$ so that ρ' is $C_{2.1.2}$-coarsely Lipschitz.

Lemma 2.1.3. *Given a map $\phi : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ and constants $C > 0$, $R \geq 0$ there is a constant $L_{2.1.3} = L_{2.1.3}(\phi,C,R)$ such that we have the following.*

Suppose X is a geodesic metric space and $Y \subseteq X$ such that $N_R(Y)$ is path connected. Let $p : X \to Y$ be a C-coarsely Lipschitz retraction. Further, the inclusion

$i : N_R(Y) \hookrightarrow X$ is ϕ-proper embedding. Then $i : N_R(Y) \hookrightarrow X$ is $L_{2.1.3}$-qi embedding. We consider $N_R(Y)$ with its induced path metric from X.

Lemma 2.1.4. *Given $L \geq 1$, $D \geq 0$, there is a constant $L_{2.1.4} = L_{2.1.4}(L,D)$ such that we have following.*

Suppose X is a geodesic metric space and $Y \subseteq Z \subseteq X$ are geodesic subspaces such that the inclusion $Y \hookrightarrow X$ is L-qi embedding. Let $Z \subseteq N_D(Y)$. Then the inclusion $Z \hookrightarrow X$ is $L_{2.1.4}$-qi embedding.

2.2 Hyperbolic metric spaces

There are several equivalent definitions for hyperbolic geodesic metric spaces (see [27], [4]). We consider the following and refer this as Gromov hyperbolic space.

Definition 2.2.1. Suppose X is a geodesic metric space and $\delta \geq 0$. We say that X is δ-hyperbolic if all its geodesic triangles are δ-slim.

A geodesic metric space X is said to be hyperbolic if it is δ-hyperbolic for some $\delta \geq 0$.

For us hyperbolic metric spaces are geodesic (by definition above) and are of infinite diameter. In hyperbolic metric spaces, quasi-geodesics and geodesics with same end points are uniformly Hausdorff close. This is known as Morse lemma or stability of quasi-geodesic (see [23, Theorem 1.7, III.H]).

Lemma 2.2.2 (Stability of quasi-geodesic)**.** *Given $\delta \geq 0$, $k \geq 1$ and $\varepsilon \geq 0$ there is a constant $D_{2.2.2} = D_{2.2.2}(\delta, k, \varepsilon)$ such that the following holds.*

Suppose X is a δ-hyperbolic metric space. Then for any geodesic α and a (k, ε)-quasi-geodesic β in X with the same end points, $Hd_X(\alpha, \beta) \leq D_{2.2.2}$.

For a finitely generated group G with finite generating set S, the Cayley graph of G with respect to S is a graph whose vertex set is G and two vertices, say, $g,h \in G$ are joined by an edge if $g^{-1}h \in S \cup S^{-1}$.

Definition 2.2.3 (**Hyperbolic group).** A finitely generated group G is said to be hyperbolic if its Cayley graph with respect to some finite generating set is hyperbolic.

It is standard that given two finite generating sets, the Cayley graphs associated with them become quasi-isometric to each other. It is easy to prove from the stability of quasi-geodesic (Lemma 2.2.2) that the hyperbolicity is quasi-isometry invariant (see [23, Theorem 1.9, III.H]). Therefore, hyperbolic groups are well-defined.

Suppose X is a geodesic metric space and $\alpha : [s,t] \subseteq \mathbb{R} \to X$ is a continuous injective path. Let d_α be the induced path metric on $Im(\alpha)$ from X. Then we have the induced order on $Im(\alpha)$ from $[s,t]$. In other words, if $p,q \in [s,t]$ with $p \leq q$ then $\alpha(p) \leq \alpha(q)$ keeping in mind that $d_\alpha(\alpha(s),\alpha(p)) \leq d_\alpha(\alpha(s),\alpha(q))$. We have mentioned above that sometimes we forget the domain of quasi-geodesic (resp. geodesic) and work with their image. With this terminology, we have the following.

Lemma 2.2.4. *Given $\delta \geq 0$, $k \geq 1$ and $r \geq 0$, we have constants $L_{2.2.4} = L_{2.2.4}(\delta,k,r)$ and $k_{2.2.4} = k_{2.2.4}(\delta,k,r)$ such that the following hold.*

Suppose (X,d) is a δ-hyperbolic metric space. Let α and β be continuous injective k-quasi-geodesics in X joining points a_1,a_2 and b_1,b_2 respectively. Further, we suppose that $d(a_i,b_i) \leq r$, $i = 1,2$. Let $a_1 \leq a_2$ and $b_1 \leq b_2$ be the orders on α and β respectively. Then there is a monotonic (piece-wise linear) homeomorphism $\psi : (\alpha,d_\alpha) \to (\beta,d_\beta)$ such that $\psi(a_i) = b_i$ and $d(x,\psi(x)) \leq k_{2.2.4}$ for all $x \in \alpha$. Moreover, ψ is $L_{2.2.4}$-quasi-isometry.

Proof. Define a map $\phi : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ such that $\phi(t) = kt + k^2$. Then α,β are ϕ-properly embedded. Now by Lemma 2.2.2 and δ-slimness of geodesic triangles, we have $Hd(\alpha,\beta) \leq D_1$, where $D_1 = 2D_{2.2.2}(\delta,k,k) + 2\delta + r$. Thus by [9, Lemma 1.19], we have a map $g : \alpha \to \beta$ with $d(g(x),x) \leq D_1, \forall\, x \in \alpha \setminus \{a_1,a_2\}$ and $g(a_i) = b_i, i = 1,2$ such that g is L-quasi-isometry, where L depends on D_1 and ϕ. Again, by [9, Lemma 1.24], we have constants D_2,D_3 depending on L, and a monotonic (piece-wise linear) homeomorphism $\tilde{g} : \alpha \to \beta$ such that $\tilde{g}$ is D_2-quasi-isometry and $d(g(x),\tilde{g}(x)) \leq D_3$. So $d(x,\tilde{g}(x)) \leq d(x,g(x)) + d(g(x),\tilde{g}(x)) \leq D_1 + D_3$. Here $\tilde{g}$ serves as the required ψ.

Therefore, we can take $L_{2.2.4} = D_2$ and $k_{2.2.4} = D_1 + D_3$. □

We end this subsection by stating the following results (Lemma 2.2.5, Proposition 2.2.6 and Proposition 2.2.7). These results are very useful in Chapter 5. One can look at [28, Theorem 1.4, Chapter 3] for a proof of Lemma 2.2.5.

Lemma 2.2.5 (Local quasi-geodesic vs global quasi-geodesic)**.** *For all $\delta \geq 0$, $k \geq 1$ and $\varepsilon \geq 0$ there are constants $L_{2.2.5} = L_{2.2.5}(\delta,k,\varepsilon)$ and $\lambda_{2.2.5} = \lambda_{2.2.5}(\delta,k,\varepsilon)$ such that the following holds.*

Suppose X is a δ-hyperbolic metric space. Then any $(k,\varepsilon,L_{2.2.5})$-local quasi-geodesic in X is a $\lambda_{2.2.5}$-quasi-geodesic.

In [29, Proposition 3.1], Bowditch provided a criterion for hyperbolicity of a metric graph. Earlier, in [30], Hamenstadt also gave a similar criterion for a space to be hyperbolic. In Proposition 2.2.6, we consider Bowditch's version for space.

Proposition 2.2.6. ([9, Corollary 1.63]) *Given $D_0 \geq 1$, $D \geq 0$ and a proper map $\psi : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ there exist $\delta_{2.2.6} = \delta_{2.2.6}(\psi, D, D_0)$ and $K_{2.2.6} = K_{2.2.6}(\psi, D, D_0)$ such that the following holds.*

Suppose X is a geodesic metric space and $X_0 \subseteq X$ is a D_0-dense subset of X. Suppose for any pair (x,y) of distinct points in X_0 there is a continuous path $c(x,y)$ joining x and y. Further, for all $x,y,z \in X_0$ and $r \in \mathbb{R}_{\geq 0}$, we have

1. *$d(x,y) \leq r$ implies the length of the path $c(x,y)$ is bounded by $\psi(r)$, and*

2. *$c(x,y) \subseteq N_D(c(x,z) \cup c(y,z))$.*

Then X is $\delta_{2.2.6}$-hyperbolic metric space and the paths $c(x,y)$ are $K_{2.2.6}$-quasi-geodesic.

The proposition below is a very special case of the main theorem of [6] (see also [9, Theorem 2.59]). Here, the space is realized as a tree of metric spaces such that the tree is an interval. (One may look at [10, Corollary 1.52] for this result in metric graph.) However, it is true for an arbitrary tree also (see [31, Theorem 2]).

Proposition 2.2.7. [9, Theorem 2.59] *Given $\delta \geq 0$, $L \geq 1$, $D \geq 0$ there exists $\delta_{2.2.7} = \delta_{2.2.7}(\delta, L, D)$ such that the following holds.*

Suppose $X = \cup_{i=0}^{n-1} X_i$ is a geodesic metric space with X_i's are geodesic subspaces with the induced path metric from X such that:

1. *For $0 \leq i \leq n-1$, X_i is δ-hyperbolic metric space.*

2. *For $0 \leq i \leq n-2$, $Y_{i+1} = X_i \cap X_{i+1}$ is a path connected subspace, and the inclusions $Y_{i+1} \hookrightarrow X_i$ and $Y_{i+1} \hookrightarrow X_{i+1}$ are L-qi embeddings.*

3. *Y_{i+1} separates X_i and X_{i+1} in X in the sense that every path in X joining points in X_i and X_{i+1} passes through Y_{i+1}.*

4. *For $1 \leq i \leq n-2$, the pair (Y_i, Y_{i+1}) is D-cobounded in the metric X_i.*

5. *$d_{X_i}(Y_i, Y_{i+1}) \geq 1$ for $1 \leq i \leq n-1$.*

Then X is $\delta_{2.2.7}$-hyperbolic metric space.

Remark 2.2.8. In Proposition 2.2.7, if $n = 2$, we only need to check (1) and (2). In that case, X is $\delta_{2.2.8} = \delta_{2.2.8}(\delta, L)$-hyperbolic (see also Lemma 2.3.4).

2.2.1 Quasiconvex subsets

In this subsection, we will explore various basic results concerning quasiconvex subsets that will be useful in later discussions.

Definition 2.2.9. Suppose X is a geodesic metric space and $K \geq 0$. A subset U of X is said to be K-**quasiconvex** if $[a,b] \subseteq N_K(U)$ for all $a,b \in U$ and for all geodesics $[a,b]$ joining a,b in X. We say that a subset U of X is quasiconvex if it is K-quasiconvex for some $K \geq 0$.

In hyperbolic geodesic metric space, a common example of quasiconvex subset is the convex hull of any subset. This motivates us to define the following.

Definition 2.2.10. Suppose X is a geodesic metric space and $U \subseteq X$. The **quasiconvex hull** of U is defined as $\mathrm{hull}(U) := \{[a,b] : a,b \in U\}$.

Remark 2.2.11. Suppose X is a δ-hyperbolic metric space for some $\delta \geq 0$ and $A \subset X$ is any subset. Then $\mathrm{hull}(A)$ is 2δ-quasiconvex.

In hyperbolic metric space X, nearest point projection of a point on a quasiconvex subset, say, U is coarsely well-defined. (Here one requires U to be closed; which is the standard assumption for us for a quasiconvex subset.) We define a map $P_{X,U} : X \to U$ sending a point to its nearest point projection, called *a nearest point projection map* on U. Sometimes we denote this map by P_U if X is understood. Now we collect some facts related to quasiconvex subsets; some are well known and some are very easy to prove. The following is yet another way to obtain a quasiconvex subset.

Lemma 2.2.12. ([32, Lemma 4.2]) *Let $\delta \geq 0$ and $L \geq 0$. Suppose X is a δ-hyperbolic metric space and U is a subset of X such that there is a L-coarsely Lipschitz retraction $X \to U$. Then U is a K-quasiconvex subset of X where K depends on α and L.*

Lemma 2.2.13. ([9, Lemma 1.139, Lemma 1.127], [10, Lemma 1.35]) *Given $\delta \geq 0$, $\lambda \geq 0$ and $R \geq 0$, we have $R_{2.2.13} = R_{2.2.13}(\delta,\lambda) = 2\lambda + 5\delta$, $D_{2.2.13} = D_{2.2.13}(\delta,\lambda) = 2\lambda + 7\delta$ and $R'_{2.2.13} = 2\lambda + 3\delta + R$ such that the following hold.*

Suppose X is a δ-hyperbolic metric space. Let Y, $Z \subseteq X$ be two λ-quasiconvex subsets in X. Then we have the following.

1. *If Y, Z are $R_{2.2.13}$-separated then the pair (Y,Z) is $D_{2.2.13}$-cobounded.*

2. *If $d(Y,Z) \leq R$ then $P_Y(Z) \subseteq N_{R'_{2.2.13}}(Z) \cap Y$ and $Hd(P_Y(Z), P_Z(Y)) \leq R'_{2.2.13}$.*

One is referred to [9, Remark 1.142] for the upcoming remark. It is possible to minimize those constants, similar to what is mentioned in the remark.

Remark 2.2.14.
1. If Y and Z are geodesic segments in Lemma 2.2.13 (1), then one can take $D_{2.2.13} = 8\delta$ and $R_{2.2.13} = 5\delta$.
2. If Y and Z are geodesic segments in Lemma 2.2.13 (2), then one can take $R'_{2.2.13} = 4\delta + R$.

The following result follows from the stability of quasi-geodesic.

Lemma 2.2.15. *Given $\delta \geq 0$, $k \geq 0$ there is $D_{2.2.15} = D_{2.2.15}(\delta, k)$ such that the following holds.*

Suppose X is a δ-hyperbolic metric space and $U, V \subset X$ are k-quasiconvex in X. Let $U' = P_{X,U}(V)$. Let $x \in U, y \in V$ be any points. Then $[x,y]_X \cap N_{D_{2.2.15}}(U') \neq \emptyset$.

As a corollary we have the following.

Lemma 2.2.16. *Let $\delta \geq 0$, $k \geq 0$ and $D \geq 0$. Suppose X is a δ-hyperbolic metric space, and U and V are k-quasiconvex subsets of X. Further, suppose the pair (U,V) is D-cobounded. Then there are points $p \in U$, $q \in V$ and a constant D' depending on δ, k and D such that for any $x \in U$, $y \in V$, we have $p, q \in N_{D'}([x,y])$.*

The following Lemma 2.2.17 (1) follows from the very nature of quasiconvex subset and Lemma 2.2.16, whereas one can conclude (2) from Lemma 2.2.13 (1).

Lemma 2.2.17. *Let $\delta \geq 0$, $k \geq 1$ and $D \geq 0$. Suppose X is a δ-hyperbolic metric space, and U and V are k-quasiconvex subsets of X. We consider a subset A containing U and V as follows. (1) If the pair (U,V) is D-cobounded then $A = U \cup V \cup [x,y]$ for some $x \in U$ and $y \in V$. (2) If the pair (U,V) is not D-cobounded then $A = U \cup V$. Then there is a constant K depending on δ, k, D such that A is K-quasiconvex.*

Lemma 2.2.18. ([9, Corollary 1.140 (a)]) *Let $\delta \geq 0$, $k \geq 0$ and $D \geq 0$. Suppose X is a δ-hyperbolic metric space, and U and V are k-quasiconvex subsets of X. Further, we assume that diam $\{P_{X,U}(V)\} \leq D$. Then there is a constant $D' \geq D$ depending on δ, k and D such that diam $\{P_{X,V}(U)\} \leq D'$. In particular, the pair (U,V) is D'-cobounded.*

Lemma 2.2.19 (2) follows from (1) and the stability of quasi-geodesic in addition.

Lemma 2.2.19. *Let $\delta \geq 0$ and $k \geq 0$. Then there are constants $D_{2.2.19} = D_{2.2.19}(\delta, k)$ and $K_{2.2.19} = K_{2.2.19}(\delta, k)$ depending on δ, k such that we have the following. Suppose X is a δ-hyperbolic metric space and $U, V \subset X$ are k-quasiconvex subsets.*

(1) ([9, Lemma 1.113]) *For any* $x,y \in X$, $Hd_X(P_{X,U}([x,y]),[P_{X,U}(x),P_{X,U}(y)]) \leq D_{2.2.19}$.

(2) $P_{X,U}(V)$ *is* $K_{2.2.19}$*-quasiconvex in* X.

Lemma 2.2.20. *Given* $\delta \geq 0$, $k \geq 1$, $\lambda \geq 0$ *and* $D \geq 0$ *there is a constant* $C_{2.2.20} = C_{2.2.20}(\delta,k,\lambda,D)$ *such that the following holds.*

Let X *be a* δ*-hyperbolic metric space. Suppose* U *is a* λ*-quasiconvex in* X *and* $x,y \in X$ *such that* $d_X(P_U(x),P_U(y)) \leq D$. *Let* α *be a* k*-quasi-geodesic in* X *joining* x *and* y. *Then the pair* (α,U) *is* $C_{2.2.20}$*-cobounded.*

Proof. Since quasi-geodesics are quasiconvex subsets, so the lemma follows from Lemma 2.2.19 (1) and Lemma 2.2.18. □

One is referred to [9, Corollary 1.105] for a proof of Lemma 2.2.21 (2), and (3) easily follows from (1) and (2), so we omit the proof.

Lemma 2.2.21. *Given* $\delta \geq 0$, $K \geq 0$, $D \geq 0$ *and* $R \geq 0$ *there are constants* $C_{2.2.21} = C_{2.2.21}(\delta,K)$, $E_{2.2.21} = E_{2.2.21}(\delta,K,D)$ *and* $D_{2.2.21} = D_{2.2.21}(\delta,K,D,R)$ *such that the following hold.*

Suppose X *is a* δ*-hyperbolic metric space, and* U *and* V *are* K*-quasiconvex subsets of* X. *Then we have the following.*

1. ([4, Hyperbolic Groups, Lemma 7.3.*D*]) *Any nearest point projection map* $P_{X,U}: X \to U$ *is* $C_{2.2.21}$*-coarsely Lipschitz retraction.*

2. *Suppose* $x \in X$ *and* $Hd(U,V) \leq D$. *If* x_1 *and* x_2 *are nearest point projections of* x *on* U *and* V *respectively, then* $d(x_1,x_2) \leq E_{2.2.21}$.

3. *Suppose the pair* (U,V) *is* D*-cobounded. Then* $N_R(U)$ *and* $N_R(V)$ *are* $K_{2.2.21}$*-quasiconvex, and the pair* $(N_R(U),N_R(V))$ *is* $D_{2.2.21}$*-cobounded.*

Lemma 2.2.22. *For* $\delta \geq 0$, $K \geq 0$ *and* $L \geq 1$, *we have* $K_{2.2.22} = K_{2.2.22}(\delta,L,K)$ *and* $D_{2.2.22} = D_{2.2.22}(\delta,L,K)$ *such that the following hold.*

Suppose X *and* Y *are* δ*-hyperbolic metric spaces, and* $f: Y \to X$ *is a* L*-qi embedding. Let* U *be a* K*-quasiconvex subset of* Y *and* $y \in Y$. *Then we have the following.*

1. $f(U)$ *is* $K_{2.2.22}$*-quasiconvex in* X. *(For this, we do not need* Y *to be hyperbolic.)*

2. ([8, Lemma 3.5]) *If* y' *is a nearest point projection of* y *on* U *in* Y *and* x' *is that of* $f(y)$ *on* $f(U)$ *in* X. *Then* $d_Y(f(y'),x') \leq D_{2.2.22}$.

We saw that qi embedded subspaces are quasiconvex in hyperbolic metric spaces in Lemma 2.2.22 (1). Now by Lemma 2.2.21 (1) one can conclude Lemma 2.2.23 (1) for a converse; whereas (2) follows from (1) and Lemma 2.1.3 in addition.

Lemma 2.2.23. *Given a map $\phi:\mathbb{R}_{>0}\to\mathbb{R}_{>0}$ and constants $\delta\geq 0$, $k\geq 0$ and $R\geq k+1$ there are constants $L_{2.2.23}=L_{2.2.23}(\delta,k,R)$ and $L'_{2.2.23}=L'_{2.2.23}(\delta,k,R,\phi)$ such that we have the following.*

(1) Suppose X is a δ-hyperbolic metric space and A is a k-quasiconvex subset of X. Then $N^X_R(A)$ is path connected and with its induced path metric from X, the inclusion $N^X_R(A)\hookrightarrow X$ is $L_{2.2.23}$-qi embedding.

(2) Moreover, suppose $Y\subseteq X$ is a δ-hyperbolic subspace such that the inclusion $Y\hookrightarrow X$ is ϕ-proper embedding. Let $A\subseteq Y$ be k-quasiconvex in Y. Then $N^Y_R(A)$ is path connected and both the inclusions $N^Y_R(A)\hookrightarrow X$ and $N^Y_R(A)\hookrightarrow Y$ are $L'_{2.2.23}$-qi embedding.

Lemma 2.2.24. *Given $\delta\geq 0$, $L\geq 1$ and $K\geq 0$, we have constants*

$$K_{2.2.24}=K_{2.2.24}(\delta,L,K) \text{ and } D_{2.2.24}=D_{2.2.24}(\delta,L,K)$$

such that the following holds.

Suppose X is δ-hyperbolic metric space and $Y\subseteq X$ is a geodesic subspace such that the inclusion $i:(Y,d_Y)\hookrightarrow(X,d_X)$ is L-qi embedding where d_Y is the induced path metric on Y from X. Let $A\subseteq Y$ be K-quasiconvex in Y. Further, we assume that $y\in Y$, and y' is a nearest point projection of y on A in the metric Y and y'' is that of y on A in the metric X. Then A is $K_{2.2.24}$-quasiconvex in X and $d_X(y',y'')\leq D_{2.2.24}$.

Proof. By Lemma 2.2.22 (1), one can take $K_{2.2.24}=K_{2.2.22}(\delta,L,K)$.

For the second part, by [10, Lemma 1.31 (2)], we note that the arc-length parametrization of $[y,y']_Y\cup[y',y'']_Y$ is a $(3+2K)$-quasi-geodesic in Y and so is L_1-quasi-geodesic in X for some constant L_1 depending on $(3+2K)$ and L. Suppose $y_1\in[y,y'']_X$ such that $d_X(y',y_1)\leq D_{2.2.2}(\delta,L_1,L_1)$, and so $d_X(y_1,y'')\leq D_{2.2.2}(\delta,L_1,L_1)$. Therefore, $d_X(y',y'')\leq d_X(y',y_1)+d_X(y_1,y'')\leq 2D_{2.2.2}(\delta,L_1,L_1)=:D_{2.2.24}$. □

Here we recall from [9, Section 1.18], a small modification in nearest point projection on a path connected quasiconvex subset ([9, Definition 1.121]).

Definition 2.2.25 (Modified projection)**.** Suppose X is a geodesic metric space and U is a path connected quasiconvex subset of X. Then for any subset $A\subseteq X$, modified projection of A on U is defined as $\bar{P}_U(A):=\text{hull}(P_U(A))\subseteq U$, where the quasiconvex hull is taken in the induced path metric on U from X (see Definition 2.2.10 for notation).

Lemma 2.2.26. *Given $\delta \geq 0$, $L \geq 1$ and $\lambda \geq 0$ there are constants $\theta_{2.2.26} = \theta_{2.2.26}(\delta, L, \lambda)$ and $D_{2.2.26} = D_{2.2.26}(\delta, L, \lambda)$ such that the following hold.*

Suppose X is a δ-hyperbolic metric space and $Z \subseteq X$ such that Z with the induced path metric is L-qi embedded in X. Let Z be also δ-hyperbolic. Suppose $x_i \in Z$ ($i = 1,2,3$) and z is a δ-center of the triangle $\triangle(x_1,x_2,x_3)$ in Z giving a δ-tripod $Y = \cup_{i=1}^{3}[z,x_i]_Z$ in Z. Further, we assume that U is a λ-quasiconvex subset of X. Let Y be λ-quasiconvex in X. Let $P_Y : X \to Y$ and $P_U : X \to U$ be nearest point projection maps on Y and on U respectively. Then:

1. *$Hd(P_Y(U), \bar{P}_Y(U)) \leq \theta_{2.2.26}(\delta, L, \lambda)$.*

2. *Let $\bar{Y} = \bar{P}_Y(U)$ and $\bar{x}_i \in \bar{Y}$ be the closest to x_i in the intrinsic path metric on Y. Then:*

   (a) Let $\bar{Y} \nsubseteq [z,x_i]_Z$ for any $i \in \{1,2,3\}$. Then $d_X(P_U(x_i), P_U(\bar{x}_i)) \leq D_{2.2.26}$.

   (b) Let $\bar{Y} \subseteq [z,x_i]_Z$ for some $i \in \{1,2,3\}$. Note that $\bar{x}_{i+1} = \bar{x}_{i-1} = \bar{z}$ (say). Here $i \pm 1$ is calculated in modulo 3. Then

   $$d_X(P_U(\bar{x}_i), P_U(x_i)),\ d_X(P_U(z), P_U(\bar{z})) \text{ and } d_X(P_U(x_{i\pm 1}), P_U(\bar{z}))$$

   are bounded by $D_{2.2.26}$.

Proof. The proof of (1) follows from that of [9, Lemma 1.125]. We only proof (2) (b) since the proof for (2) (a) is a line by line argument of that of (2) (b). In (2) (b), we will specifically address $d_X(P_U(x_{i-1}), P_U(\bar{z}))$ as the other proofs are similar. We fix $i = 2$. Then $x_{i-1} = x_1$.

Let $P_U(x_1) = x_1'$ and $P_Y(x_1') = x_1''$. Note that $x_1'' \in [\bar{z}, x_2]_Z$. Since z is δ-center of the triangle $\triangle(x_1,x_2,x_3)$ in Z, so $d_Z(\bar{z}, [x_1,x_1'']_Z) \leq 2\delta$. Thus (by Lemma 2.2.2) $\exists\, z_1 \in [x_1,x_1'']_X$ such that $d_X(\bar{z}, z_1) \leq 2\delta + D_{2.2.2}(\delta, L, L)$. Again, $[x_1',x_1'']_X \cup [x_1'',x_1]_X$ is $(3+2\lambda)$-quasi-geodesic in X ([10, Lemma 1.31 (2)]), and so $\exists\, z_2 \in [x_1,x_1']_X$ such that $d_X(z_1,z_2) \leq D_{2.2.2}(\delta, 3+2\lambda, 3+2\lambda)$. Then by triangle inequality, $d_X(\bar{z}, z_2) \leq D$, where $D = D_{2.2.2}(\delta, 3+2\lambda, 3+2\lambda) + 2\delta + D_{2.2.2}(\delta, L, L)$. Notice that as $P_U(x) = x_1'$ and $z_2 \in [x_1,x_1']_X$, so x_1' is also a nearest point projection of z_2 on U in the metric of X. Then by Lemma 2.2.21 (1), $d_X(P_U(z_2), x_1') \leq C_{2.2.21}(\delta, \lambda)$. Hence, $d_X(P_U(\bar{z}, x_1') \leq d(P_U(\bar{z}), P_U(z_2)) + d_X(P_U(z_2), x_1') \leq C_{2.2.21}(\delta, \lambda)(D+2) =: D_{2.2.26}$. □

Remark 2.2.27. ([9, Remark 1.124]) In the above Lemma 2.2.26 (1), if both U and T are geodesic segments in X, one can bound $Hd(P_T(U), \bar{P}_T(U))$ by 4δ.

2.2.2 Gromov boundary and Cannon-Thurston maps

Suppose X is a Gromov hyperbolic metric space. Then the geodesic or visual boundary of X is defined as follows. Let $\mathcal{G}(X)$ be the set of all geodesic rays in X. One defines an equivalence relation on $\mathcal{G}(X)$ by setting $\alpha \sim \beta$ if $Hd(\alpha, \beta) < \infty$ for all $\alpha, \beta \in \mathcal{G}(X)$. The set of equivalence classes, denoted by ∂X, is called the geodesic boundary of X. The equivalence class of α is denoted by $\alpha(\infty)$. If $\alpha(0) = x$ then we say that α joins x to $\alpha(\infty)$. *In this Subsection 2.2.2 (consequently, in Chapter 3), we shall always assume that our spaces are proper hyperbolic metric spaces or trees.* We shall briefly recall all the properties of geodesic boundaries to be used in this thesis.

The barycenter map (For more details, one is referred to [10, Section 2]): Suppose X is a δ-hyperbolic geodesic metric space such that there are more than two elements in its Gromov boundary, ∂X. Then by [10, Lemma 2.4], for any $\eta, \eta', \eta'' \in \partial X$ such that $\eta' \neq \eta''$, there is a (uniform) quasi-geodesic ray starting at any point in X representing η and a (uniform) bi-infinite quasi-geodesic line whose one end represents η' and the other one represents η''. We denote such a line by (η', η''). Notice that we do not assume our space to be proper. Let $\partial^3 X = \{(\xi_1, \xi_2, \xi_3) \in \partial X \times \partial X \times \partial X : \xi_1 \neq \xi_2 \neq \xi_3 \neq \xi_1\}$. Now for $\xi = (\xi_1, \xi_2, \xi_3) \in \partial^3 X$, we consider an ideal quasi-geodesic triangle, say, $\triangle(\xi_1, \xi_2, \xi_3)$ formed by three (uniform) quasi-geodesic lines $\{(\xi_i, \xi_j) : i \neq j \text{ and } i, j \in \{1, 2, 3\}\}$. Then by [10, Lemma 2.7], there is a point, say, b_ξ in X uniformly close to each sides of $\triangle(\xi_1, \xi_2, \xi_3)$ and this b_ξ is coarsely well-defined. Thus in this way, we can (coarsely) define a map $\psi : \partial^3 X \to X$. Lastly, by [10, Lemma 2.9], the map $\psi : \partial^3 X \to X$ is coarsely unique and is called *the barycenter map*. Note that for such barycenter map, we always assume that ∂X has more than two elements.

Remark 2.2.28. We say a group G is non-elementary hyperbolic if the Gromov boundary of its Cayley graph with respect to some finite generating set contains more than two elements. It is a well known fact that for a non-elementary hyperbolic group the barycenter map is coarsely surjective.

Now we state a couple of results related to boundary. Since they are standard, we state them without proofs; one may find their proofs in [23, Chapter III.H].

Lemma 2.2.29. *If X is a proper hyperbolic metric space then $\partial X \neq \emptyset$.*

Lemma 2.2.30. ([23, Lemma 3.1, Lemma 3.2, Lemma 3.3, III.H]) *Let X be a proper δ-hyperbolic metric space or a tree for some $\delta \geq 0$. Then we have the following.*

1) If $x \in X$ and $\xi \in \partial X$ then there is a geodesic ray α in X with $\alpha(0) = x$ and $\alpha(\infty) = \xi$. If α' any other geodesic joining x to ξ then $Hd(\alpha, \alpha') \leq \delta$.

2) If $\xi_1 \neq \xi_2$ are two points of ∂X then there is a geodesic line γ in X joining ξ_1 to ξ_2. If γ' any other geodesic joining ξ_1 to ξ_2 then $Hd(\gamma, \gamma') \leq 2\delta$.

We note that one can define a Hausdorff topology on $\bar{X} = X \cup \partial X$ in a very natural way. However, since we do need it we skip the detailed discussion and we state the following features that will be used in Chapter 3.

The following lemmata (Lemma 2.2.31, Lemma 2.2.32) gives a geometric criteria for convergence and is well known among experts. One may look at [33, Lemma 2.45] for a proof of Lemma 2.2.32 (2).

Lemma 2.2.31. *Suppose $\{x_n\}$ is a sequence in $\bar{X}$ and $\xi \in \partial X$. Then $\{x_n\}$ converges to ξ iff the following holds: Suppose $x \in X$ is an arbitrary point and suppose α_n is a geodesic (ray or line according as $x_n \in X$ or $x_n \in \partial X$) in X joining x_n to ξ. Then $\lim_{n\to\infty} d(x, \alpha_n) = \infty$.*

Notation. Suppose $\{x_n\}$ is a sequence in X and $\xi \in \bar{X}$. Then we write $\lim^X_{n\to\infty} x_n = \xi$ to mean that $\{x_n\}$ converges to ξ in $\bar{X}$. If $\lim^X_{n\to\infty} x_n = \xi$ for some $\xi \in \partial X$ then we say that $\lim^X_{n\to\infty} x_n$ exists. Later in Chapter 3 we shall frequently encounter situations where there are two hyperbolic spaces $Y \subset X$ and sequences $\{y_n\}$ in Y. To differentiate between the limits of this sequences in $\bar{X}$ and $\bar{Y}$ we use superscript as above.

Lemma 2.2.32. *Suppose X is a proper hyperbolic metric space and $x \in X$. Then the following hold.*

1. *Any unbounded subsequence $\{x_n\}$ in X has a subsequence $\{x_{n_k}\}$ with $\lim_{k\to\infty} x_{n_k} \in \partial X$.*

2. *Suppose $\{x_n\}$ and $\{x'_n\}$ are two unbounded sequences in X such that $\lim^X_{n\to\infty} x_n \in \partial X$. If $d(x, [x_n, x'_n]) \to \infty$ as $n \to \infty$ then $\lim^X_{n\to\infty} x'_n \in \partial X$. Then in that case $\lim^X_{n\to\infty} x_n = \lim^X_{n\to\infty} x'_n$ if and only if $\lim_{n\to\infty} d(x, [x_n, x'_n]) = \infty$. Moreover, if $z_n \in [x_n, x'_n]$ then $\lim^X_{n\to\infty} z_n = \lim^X_{n\to\infty} x_n$.*

As a corollary of Lemma 2.2.32 (2), we have the following.

Lemma 2.2.33. *Suppose X is a hyperbolic metric space and $\{x_n\} \subseteq X$ such that $\lim^X_{n\to\infty} x_n$ exists. Let $x \in X$ and $x'_n \in [x, x_n]$ such that $d(x, x'_n) \to \infty$ as $n \to \infty$. Then $\lim^X_{n\to\infty} x_n = \lim^X_{n\to\infty} x'_n$.*

Definition 2.2.34. Suppose X is a hyperbolic metric space and $\{A_n\}$ is a sequence of (uniformly quasiconvex) subsets of X. Suppose $\xi \in \partial X$. We say that the sequence of subsets $\{A_n\}$ converges to ξ (in $\bar{X}$) if the following holds: Given $R > 0$ there is $N \in \mathbb{N}$ such that for all $n \geq N$ and $x_n \in A_n$ and any geodesic ray α joining x_n to ξ, we have $d(x, \alpha) > R$.

Suppose H is a subgroup of a finitely generated group G. It is a simple fact that, for any finite radius ball in the Cayley graph of G with respect to any finite generating set, there are only finitely many cosets of H that intersect with this ball. This observation motivates the following definition and has a significant impact in Chapter 3.

Definition 2.2.35. A family of subsets $\{A_\alpha\}_{\alpha\in\Lambda}$ in a metric space X is said to be locally finite if any finite radius ball in X intersects at most finitely many A_α's.

The proof of Lemma 2.2.36 follows from the definition; whereas Lemma 2.2.37 also follows from the very nature of quasiconvex subsets in addition. So we choose to omit their proofs.

Lemma 2.2.36. *Suppose $\{A_n : n \in \mathbb{N}\}$ is a locally finite collection of subsets in an infinite diameter metric space X. Then for any point $x \in X$, $d(x, A_n) \to \infty$ as $n \to \infty$.*

Proposition 2.2.37. *Suppose X is a proper hyperbolic metric space and $\{A_n\}$ is a sequence of uniformly quasiconvex subsets in X such that the collection $\{A_n : n \in \mathbb{N}\}$ is locally finite. Then there is a subsequence $\{A_{n_k}\}$ of $\{A_n\}$ that converges to a point of ∂X.*

Definition 2.2.38. (**Cannon-Thurston map**) Suppose $f : Y \to X$ is a (proper) embedding between hyperbolic metric spaces. We say that f admits the Cannon-Thurston (CT) map if there is a map $\partial f : \partial Y \to \partial X$ induced by f in the following sense:

For all $\xi \in \partial Y$ and for any sequence $\{y_n\}$ in Y with $\lim^Y_{n\to\infty} y_n = \xi$ one has $\lim^X_{n\to\infty} f(y_n) = \partial f(\xi)$.

In this case ∂f is called the CT map induced by f. We note that in the Definition 2.2.38, the existence of the CT map implies that it is also continuous (e.g. [33, Lemma 2.50]).

Lemma 2.2.39. ([9, Lemma 8.6]) *Suppose $f : Z \to Y$ and $g : Y \to X$ are maps between hyperbolic spaces both admitting the CT-maps. Then the composition $g \circ f : Z \to X$ admits the CT-map.*

In [8], Mitra gave the following criterion for the existence of CT-maps.

Lemma 2.2.40. (**Mitra's Criterion**, [8, Lemma 2.1]) *Suppose $f : Y \to X$ is a map between hyperbolic metric spaces. Fix $y_0 \in Y$ and let $x_0 = f(y_0)$. Then f admits the CT-map if there is a proper map $\phi : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ such that the following holds.*

Let $y, y' \in Y$ and $R \geq 0$. Suppose α is a geodesic in Y joining y, y' and β is that in X joining $f(y), f(y')$. Then $d_Y(y_0, \alpha) \geq R$ implies $d_X(x_0, \beta) \geq \phi(R)$.

In the situation of the above lemma we shall say that f *satisfies Mitra's criterion with respect to the base point* y_0 and we shall refer to the function ϕ to be a **CT parameter** for this base point. We note that Mitra's criterion implies that $f : Y \to X$ is a proper embedding. On the other hand it is easy to check that if Mitra's criterion holds for a map $f : Y \to X$ as above with respect to a base point $y_0 \in Y$, then the same will be true for any other base point in Y although in that case the CT parameter ϕ maybe different. However if there is a group G acting by isometries on both Y and X such that f is G-equivariant and the G-action on Y is transitive then the same function ϕ works for all base points in Y. Typically this is the case in group theoretic situations, i.e., when we have hyperbolic groups $H < G$ and f is an inclusion map between their Cayley graphs. This motivates the following.

Definition 2.2.41. Suppose $f : Y \to X$ is a (proper) embedding between hyperbolic metric spaces and that f satisfies Mitra's criterion with respect to a base point. We say that f satisfies a **uniform Mitra's criterion** if there is a function ϕ which works as a CT parameter for all base points in Y.

We note that although Mitra's criterion is not necessary for the existence of CT maps, it is a very reasonable sufficient condition for the existence of CT maps as the following lemma shows. Since this is quite standard we skip its proof.

Lemma 2.2.42. *Suppose X, Y are two proper hyperbolic metric spaces and $f : Y \to X$ is a proper embedding. If f admits the CT map, then $f : Y \to X$ satisfies Mitra's criterion.*

We note that all the spaces in consideration in this thesis, for which CT maps are to be discussed, are proper. The proofs will run by contradiction and for the same purpose the following lemma will be very useful.

Lemma 2.2.43. *Suppose X, Y are two proper hyperbolic metric spaces, and $f : Y \to X$ is a proper embedding which does not admit the CT map. Then there are two unbounded sequences $\{y_n\}$ and $\{y'_n\}$ in Y such that $\lim^Y_{n\to\infty} y_n = \lim^Y_{n\to\infty} y'_n$ but $\lim^X_{n\to\infty} f(y_n) \neq \lim^X_{n\to\infty} f(y'_n)$.*

Proof. Since $f: Y \to X$ does not admit the CT-map, it does not satisfies Mitra's criterion for any proper map $\phi : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ and any fixed point $y_0 \in Y$. Let $x_0 = f(y_0)$. Therefore, we get two sequences $\{y_n\}$ and $\{y'_n\}$ in Y such that $d_Y(y_0, [y_n, y'_n]_Y) \to \infty$ as $n \to \infty$ and $d_X(x_0, [f(y_n), f(y'_n)]_X) \leq D$ for all $n \in \mathbb{N}$ and for some $D \geq 0$. In particular, both the sequences $\{y_n\}$ and $\{y'_n\}$ are unbounded. Since f is proper embedding, both the sequences $\{f(y_n)\}$ and $\{f(y'_n)\}$ are unbounded also. Since both the spaces X and Y are proper, after passing to subsequences, if necessary, we assume that $\lim^Y_{n\to\infty} y_n,\ \lim^Y_{n\to\infty} y'_n \in \partial Y$ and $\lim^X_{n\to\infty} f(y_n),\ \lim^X_{n\to\infty} f(y'_n) \in \partial X$ (see Lemma 2.2.32 (1)). Now we through by Lemma 2.2.32 (2). □

The uniform Mitra's criterion plays a pivotal role in our main theorem (Theorem 1.1.6) proved in Chapter 3. We can now observe that having the uniform Mitra's criterion broadens the applicability of Lemma 2.2.24 to any hyperbolic subspace, instead qi embeddings.

Lemma 2.2.44. *Given $\delta \geq 0$, $K \geq 0$ and a proper function $\phi : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$, there is $D_{2.2.44} = D_{2.2.44}(\delta, K, \phi)$ such that the following holds.*

Suppose X is a proper δ-hyperbolic metric space. Let $Y \subseteq X$ be a proper δ-hyperbolic subspace with respect to induced path metric from X and $A \subseteq Y$ be K-quasiconvex in both X and Y. Further, we assume that the inclusion $i : Y \hookrightarrow X$ is ϕ-proper embedding and satisfies uniform Mitra's criterion with function ϕ. Suppose y_1 and y_2 are nearest point projections of $y \in Y$ on A in the metric X and Y respectively. Then $d_Y(y_1, y_2) \leq D_{2.2.44}$.

Proof. Note that the arc-length parametrization of $[y, y_1]_X \cup [y_1, y_2]_X$ is a $(3+2K)$-quasi-geodesic in X (see [10, Lemma 1.31 (2)]). Then there is $x \in [y, y_2]_X$ such that $d_X(y_1, x) \leq D$, where $D = D_{2.2.2}(\delta, 3+2K, 3+2K)$. Since ϕ is a proper function, we fix N_0 such that $\phi(N_0) > D$. Then it says that $d_Y(y_1, [y, y_2]_Y) \leq N_0$; otherwise, $d_Y(y_1, [y, y_2]_Y) > N_0$ implies $d_X(y_1, [y, y_2]_X) \geq \phi(N_0) > D$ (by Mitra's criterion with respect to base point y_1) and so $d_X(y_1, x) > D$ which is a contradiction.

Let $y_3 \in [y, y_2]_Y$ such that $d_Y(y_1, y_3) \leq N_0$. Since y_2 is a nearest point projection of y on A in the metric Y and $y_3 \in [y, y_2]_Y$, so $d_Y(y_3, y_2) \leq d_Y(y_3, y_1) \leq N_0$. Therefore, by triangle inequality, $d_Y(y_2, y_1) \leq 2N_0 =: D_{2.2.44}$. □

Lemma 2.2.45. *Suppose X is a proper hyperbolic space and Y is a proper hyperbolic subspace with respect to the induced path metric such that the inclusion $i : Y \hookrightarrow X$ is proper embedding and admits the CT-map $\partial i : \partial Y \to \partial X$. Let $A \subseteq Y$ be quasiconvex in both Y and X. Further, $d_X(P_{XA}(y), P_{YA}(y))$ is uniformly bounded for all $y \in Y$. Moreover, let $\alpha : [0, \infty) \to Y$ be a geodesic ray in Y such that $\partial i(\alpha(\infty)) \in \Lambda(A) \subseteq \partial X$. Then α is a quasi-geodesic ray in X.*

Proof. We fix a point $y_0 \in A$ and assume that $\alpha(0) = y_0$. Since the inclusion $Y \hookrightarrow X$ admits the CT-map, so $[y_0, \alpha(n)]_X$ converges to the geodesic ray, say, $\beta : [0, \infty) \to X$. Since $\partial i(\alpha(\infty)) = \beta(\infty) \in \Lambda(A)$, $\{P_{XA}([y_0, \alpha(n)]_X) : n \in \mathbb{N}\}$ is of infinite diameter. Hence by our assumption, $\{P_{YA}([y_0, \alpha(n)]_Y) : n \in \mathbb{N}\}$ is of infinite diameter. Since α is a geodesic ray in Y and A is quasiconvex in Y, α is in bounded neighborhood of A. Again since A is quasiconvex in X so α is a quasi-geodesic ray in X.

□

2.3 Trees of metric spaces

The notion of trees of metric spaces was introduced by Bestvina and Feighn in [6]. A coarsely equivalent definition was given by Mitra in [8]. We are going to adopt the latter definition.

Definition 2.3.1. Suppose T is a simplicial tree and X is a metric space. Then a 1-Lipschitz surjective map $\pi : X \to T$ is called a **tree of metric spaces** if there is a proper map $\phi : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ with the following properties:

1. For all $v \in V(T)$, $X_v := \pi^{-1}(v)$ is a geodesic metric space with the path metric d_v induced from X. Moreover, with respect to these metrics, the inclusion $X_v \hookrightarrow X$ is ϕ-proper embedding.

2. Suppose e is an edge in T joining $v, w \in V(T)$ and $m_e \in T$ is the midpoint of this edge. Then $X_e := \pi^{-1}(m_e)$ is a geodesic metric space with respect to the path metric d_e induced from X. Moreover, there is a map $\vartheta_e : X_e \times [0,1] \to \pi^{-1}(e) \subseteq X$ such that

   (a) $\pi \circ \vartheta_e$ is the projection map onto $[v, w]$.

   (b) ϑ_e restricted to $X_e \times (0,1)$ is an isometry onto $\pi^{-1}(int(e))$ where $int(e)$ denotes the interior of e.

   (c) ϑ_e restricted to $X_e \times \{0\} \simeq X_e$ and $X_e \times \{1\} \simeq X_e$ are ϕ-proper embeddings from X_e into X_v and X_w respectively with respect to their induced path metrics. Let us denote these restriction maps by $\vartheta_{e,v}$ and $\vartheta_{e,w}$ respectively.

Moreover, we say $\pi : X \to T$ is a **tree of hyperbolic metric spaces with the qi embedded condition** if additionally we have the following. There is $\delta_0 \geq 0$ and $L_0 \geq 1$ such that X_v's are δ_0-hyperbolic for all $v \in V(T)$ and in (2) (c), $\vartheta_{e,v}$ and $\vartheta_{e,w}$ are L_0-qi embedding.

Notations: Throughout the Chapter 3 and in this subsection, we will use the following notations. For any subtree T' of T and an edge $e=[v,w]$, we denote $X_{T'}$, X_{ev} and X_{ew} to mean $\pi^{-1}(T')$, $\vartheta_{e,v}(X_e)$ and $\vartheta_{e,w}(X_e)$; and we will use X_{vw} for $X_{[v,w]}$.

In the following lemma we see that if we restrict the tree of metric spaces on some subtree then the inclusion map is uniformly properly embedded.

Lemma 2.3.2. ([9, Proposition 2.17]) *Suppose $\pi : X \to T$ is a tree of metric spaces. Then there is a function $\eta_{2.3.2} = \eta_{2.3.2}(\phi) : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ depending on ϕ as in Definition 2.3.1 such that the following holds.*

Let T' be a subtree of T and $X_{T'} := \pi^{-1}(T')$. Then with respect to the path metric on $X_{T'}$ induced from X, the inclusion $X_{T'} \hookrightarrow X$ is $\eta_{2.3.2}$-proper embedding.

Since proof of the following result is standard, so we omit it.

Lemma 2.3.3. *Suppose $\pi : X \to T$ is a tree of spaces such that X is proper metric space. Let $u \in T$ and $\{e_\lambda : \lambda \in \Lambda\}$ be the collection of edges incident on u. Then the collection $\{X_{e_\lambda u} : \lambda \in \Lambda\}$ is locally finite in X_u.*

Convention: Unless otherwise specified, we always refer to the constants δ_0 and L_0 as in Definition 2.3.1 for a tree of hyperbolic metric spaces with the qi embedded condition.

Let us fix some secondary constants $\boldsymbol{\delta'_0, L'_0, \lambda'_0, L'_1}$ in the following Lemma 2.3.4 and these notations will be used through out the thesis.

Lemma 2.3.4. ([9, Corollary 2.62, Lemma 2.27]) *Suppose $\pi : Z \to [v,w]$ is a tree of metric spaces over an edge $e=[v,w]$ such that Z_v, Z_w are δ_0-hyperbolic and Z_e is L_0-qi embedded in both Z_v and Z_w. Then*

1. *Z is $\boldsymbol{\delta'_0}$-hyperbolic, and Z_v and Z_w are $\boldsymbol{L'_0}$-qi embedded in Z.*

2. *Suppose U is a $2\delta_0$-quasiconvex subset of the fiber Z_v or Z_w. Then U is $\boldsymbol{\lambda'_0}$-quasiconvex in Z (see Lemma 2.2.22 (1)), where $\lambda'_0 = K_{2.2.22}(\delta'_0, L'_0, 2\delta_0)$. In particular, Z_v, Z_w are λ'_0-quasiconvex in Z. Thus a nearest point projection map $P_{ZZ_w} : Z \to Z_w$ in the metric Z is $\boldsymbol{L'_1}$-coarsely Lipschitz retraction, where $L'_1 = C_{2.2.21}(\delta'_0, \lambda'_0)$.*

Lemma 2.3.5. *Given $k \geq 0$, $D \geq 0$ and $\varepsilon \geq 0$ there are constants $R_{2.3.5} = R_{2.3.5}(k,D,\varepsilon)$ and $R'_{2.3.5} = R'_{2.3.5}(k,D)$ such that the following holds.*

Suppose we have assumptions of Lemma 2.3.4. Let A_v be a k-quasiconvex subsets of Z_v in Z_v-metric and A_w be that of Z_w in Z_w-metric. Let $x \in Z_v$ and $y \in Z_w$ such *that $d_Z(x,y)=1$, and x' be a nearest point projection of x on A_v in Z_v-metric and y' be that of y on A_w in Z_w-metric. Then we have the following.*

(A) If $Hd_Z(P_{ZZ_w}(A_v),A_w)\leq \varepsilon$ and $d_Z(z,A_v)\leq D$ for all $z\in A_w$, then $d_Z(x',y')\leq R_{2.3.5}$.

(B) Suppose $A'_v=A_v\cap Z_{ev}$ is also k-quasiconvex in Z_v and $A'_w=A_w\cap Z_{ew}$ is that in Z_w. Let $Hd_{Z_v}(P_{Z_vZ_{ev}}(A_v),A'_v)\leq D$ and $Hd_{Z_w}(P_{Z_wZ_{ew}}(A_w),A'_w)\leq D$. Then $d_Z(x',y')\leq R'_{2.3.5}(k,D)$.

Proof. (A) Note that Z is δ'_0-hyperbolic, and Z_v and Z_w are L'_0-qi embedded in Z (see Lemma 2.3.4). Thus A_v and A_w are K-quasiconvex in Z for some $K=K_{2.2.22}(\delta'_0,L'_0,k)$. Suppose x_1 is a nearest point projection of x on A_v and y_1 is that of y on A_w in Z. Then by Lemma 2.2.24, $d_Z(x',x_1)\leq D_{2.2.24}(\delta'_0,L'_0,K)$ and $d_Z(y',y_1)\leq D_{2.2.24}(\delta'_0,L'_0,K)$. Hence it is enough to show a bound on $d_Z(x_1,y_1)$.

Let y_2 and y_3 be nearest point projections of y and y_1 on A_v respectively in Z. By given condition we have $d_Z(y_1,y_3)\leq D$ and by Lemma 2.2.21 (1) we have $d_Z(x_1,y_2)\leq 2C_{2.2.21}(\delta'_0,K)$. Consider the pair (A_v,Z_w) in Z. By Lemma [9, Lemma 1.127], if y'_2 is a nearest point projection of y_2 on Z_w, we have $d_Z(y_2,y'_2)\leq 2K+3\delta'_0+D$. Again by given condition we have $d_Z(y'_2,A_w)\leq\varepsilon$, and so $d_Z(y_2,A_w)\leq 2K+3\delta'_0+D+\varepsilon$. Now if y''_2 is a nearest point projection of y_2 on A_w, then $d_Z(y_2,y''_2)\leq 2K+3\delta'_0+D+\varepsilon=\varepsilon'$ (say).

Now we have $d_Z(y,y_3)\leq d_Z(y,y_1)+d_Z(y_1,y_3)\leq d_Z(y,y''_2)+d_Z(y_1,y_3)\leq d_Z(y,y_2)+d_Z(y_2,y''_2)+D\leq d_Z(y,y_2)+\varepsilon'+D$. Let $z\in[y,y_3]_Z$ such that $d_Z(y_2,z)\leq K+2\delta'_0$ (see [9, Lemma 1.102 (i)]). Then $d_Z(z,y_3)=d_Z(y,y_3)-d(y,z)\leq d_Z(y,y_2)+\varepsilon'+D-d(y,z)\leq d_Z(y_2,z)+\varepsilon'+D\leq K+2\delta'_0+\varepsilon'+D$. So $d_Z(y_2,y_3)\leq d_Z(y_2,z)+d_Z(z,y_3)\leq 2(K+2\delta'_0)+\varepsilon'+D=D_1$ (say).

Therefore, combining all inequalities, we have

$$\begin{aligned}d_Z(x',y')&\leq d_Z(x',x_1)+d_Z(x_1,y_2)+d_Z(y_2,y_3)+d_Z(y_3,y_1)\\&\qquad+d_Z(y_1,y')\\&\leq 2D_{2.2.24}(\delta'_0,L'_0,K)+2C_{2.2.21}(\delta'_0,K)+D_1+D=:R_{2.3.5}.\end{aligned}$$

(B) Abusing notation, we assume that Z_{ev}, Z_{ew} are also k-quasconvex in Z_v and A'_v, A'_w are that in Z. Let x_1 be a nearest point projection of x on A'_v in Z_v. Now we prove that $d_X(x',x_1)$ is uniformly bounded. Since $[x,x']_{Z_v}\cup[x',x_1]_{Z_v}$ is a $(3+2k)$-quasi-geodesic, by stability of quasi-geodesic in Z_v, there is $x_2\in[x,x_1]_{Z_v}$ such that $d_{Z_v}(x',x_2)\leq D_1$ for some uniform constant $D_1\geq 0$. Then there is $x_3\in Z_{ev}$ such that $d_{Z_v}(x_2,x_3)\leq k$, and so $d_{Z_v}(x',x_3)\leq D_1+k$, and so $d_{Z_v}(x',P_{Z_vZ_{ev}}(x'))\leq D_1+k$. Thus $d_{Z_v}(x_3,P_{Z_vZ_{ev}}(x'))\leq 2(D_1+k)$. Then by given condition, there is, $x'_3\in A'_v$

such that $d_{Z_v}(x_3', P_{Z_vZ_{ev}}(x')) \leq D$. This implies by triangle inequality, $d_{Z_v}(x_3, x_3') \leq 2(D_1+k)+D$; and again by triangle inequality, $d_{Z_v}(x_2, x_3') \leq 2D_1+3k+D$. Since x_1 is a nearest point projection of x on A_v' in the path metric Z_v and $x_2 \in [x, x_1]_{Z_v}$, $x_3' \in A_v'$, then $d_{Z_v}(x_2, x_1) \leq d_{Z_v}(x_2, x_3') \leq 2D_1+3k+D$. Hence $d_{Z_v}(x', x_1) \leq d_{Z_v}(x', x_2) + d_{Z_v}(x_2, x_1) \leq 3D_1+3k+D$.

Now let x_4 be a nearest point projection of x on A_v' in the path metric Z. Then by Lemma 2.2.44 $d_{Z_v}(x_1, x_4) \leq D_2$ for some uniform constant $D_2 \geq 0$.

Therefore, by triangle inequality, $d_Z(x', x_4) \leq 3D_1+3k+D_2 = D_3$ (say).

Now let y_4 be a nearest point projection of y on A_w' in the metric of Z. Then by the similar argument, we can conclude that $d_Z(y', y_4) \leq D_3$.

Again let y_5 be a nearest point projection of y on A_v' in the path metric Z. Since $d_Z(x,y)=1$, by Lemma 2.2.21 (1), $d_Z(x_4, y_5) \leq D_4$ for some uniform constant $D_4 \geq 0$. Since by given condition, $Hd_Z(A_v', A_w') \leq 2D$, by Lemma 2.2.21 (2), $d_Z(y_5, y_4) \leq D_5$ for some uniform constant D_5.

Therefore, by combining all the inequalities above, we get

$$\begin{aligned} d_Z(x', y') &\leq d_Z(x', x_4) + d_Z(x_4, y_5) + d_Z(y_5, y_4) + d_Z(y_4, y') \\ &\leq D_3 + D_4 + D_5 + D_3 = 2D_3 + D_4 + D_5 =: R'_{2.3.5}. \end{aligned}$$

Therefore, we are through. □

Lemma 2.3.6. *Given $k \geq 0$ there is $R_{2.3.6} = R_{2.3.6}(k)$ such that the following holds.*

Suppose we have assumptions of Lemma 2.3.4. Let A_v be a k-quasiconvex subset of Z_v in Z_v-metric. Let $x \in Z_v$ and $y \in Z_w$ such that $d_Z(x,y)=1$, and x' be a nearest point projection of x on A_v in Z_v-metric and y' be that of y on A_v in Z-metric. Then $d_Z(x', y') \leq R_{2.3.6}$.

Proof. From the first paragraph of the proof of Lemma 2.3.5, A_v is K-quasiconvex in Z where $K = K_{2.2.22}(\delta_0', L_0', k)$. If x_1 is a nearest point projection of x on A_v in Z-metric, then by Lemma 2.2.21 (1), $d_Z(x_1, y') \leq 2C_{2.2.21}(\delta_0', K)$. Again by Lemma 2.2.24, $d_Z(x', x_1) \leq D_{2.2.24}(\delta_0', L_0', K)$. Therefore, we can take $R_{2.3.6} := D_{2.2.24}(\delta_0', L_0', K) + C_{2.2.21}(\delta_0', K)$. □

Now we define Mitra's projection map on a subset of X in the following remark for later use.

Remark 2.3.7. **Mitra's projection map**: Suppose S is a subtree of T and $\mathscr{A} = \bigcup_{u \in V(S)} A_u$ where $A_u \subseteq X_u$ is any subset. Let $X_{vsp} = \bigcup_{u \in V(T)} X_u$. Now we define a map $\rho : X_{vsp} \to \mathscr{A}$ as follows. Suppose $x \in X_{vsp}$ and $\pi(x) = u$. If $u \in V(S)$ then we take $\rho(x)$ to be a nearest point projection of x on A_u in X_u. Now suppose

$u \notin V(S)$. Let v be the nearest point projection of u on S in T and $w \in [v,u]$ such that $d_T(v,w) = 1$. First we take a nearest point projection, say, x' of x on X_w in X and then $\rho(x)$ is defined to be a nearest point projection of x' on A_w in X_{vw}.

The following result gives us sufficient conditions for which the above map turns out to be a coarsely Lipschitz retraction.

Proposition 2.3.8. *Suppose $\mathscr{A}$ and ρ are as in Remark 2.3.7. We also assume the following for some constants $k, K, C, \varepsilon \geq 0$.*

(1) For all $v \in V(S)$, A_v's are k-quasiconvex in X_v.

Let $[v,w]$ be an edge in S such that $d_T(u,v) < d_T(u,w)$. Then:

(2) For $v \in V(S)$ and $w \notin V(S)$, the pair (A_v, X_w) is C-cobounded in X_{vw}.

(3) For $v,w \in V(S)$, $A_w \subseteq N_K(A_v)$ in X_{vw}.

(4) For $v,w \in V(S)$, $Hd_{X_{vw}}(P_{X_{vw}X_w}(A_v), A_w) \leq \varepsilon$.

There is a uniform constant $L_{2.3.8}$ depending on various constants above such that ρ can be extended to a $L_{2.3.8}$-coarsely Lipschitz retraction $X \to \mathscr{A}$.

Proof. Since X_{vsp} is 1-dense in X, by Lemma 2.1.2, it is enough to show $d_X(\rho(x), \rho(y))$ is uniformly bounded where $x, y \in X_{vsp}$ and $d_X(x,y) \leq 1$.

Let $\pi(x) = v$ and $\pi(y) = w$. We consider the following cases depending on the position of v, w.

Case 1: Suppose $v, w \in V(S)$. If $v = w$ then $d_X(\rho(x), \rho(y)) \leq C_{2.2.21}(\delta_0, k)$ (see Lemma 2.2.21 (1)). Now let $v \neq w$ and $d_T(u,v) < d_T(u,w)$. Then $d_T(v,w) \leq d_X(x,y) \leq 1$ implies $d_T(v,w) = 1$, $d_X(x,y) = 1$, $x \in X_v$, $y \in X_w$. Then by (3), (4) and Lemma 2.3.5 (1), $d_X(\rho(x), \rho(y)) \leq d_{X_{vw}}(\rho(x), \rho(y)) \leq R_{2.3.5}(k, K, \varepsilon)$.

Case 2: Without loss of generality, let $v \in V(S)$ and $w \notin V(S)$.Note that $d_X(x,y) = 1$ and $x \in X_v$, $y \in X_w$. Then by Lemma 2.3.6, $d_X(\rho(x), \rho(y)) \leq d_{X_{vw}}(\rho(x), \rho(y)) \leq R_{2.3.6}(k)$.

Case 3: Suppose $v, w \notin V(S)$. Since $d_T(v,w) \leq 1$, the nearest point projections of v and w on S are same; suppose that is v'. Let $w' \in [v', v]$ such that $d_T(v', w') = 1$. By (2), the pair $(A_{v'}, X_{w'})$ is C-cobounded in $X_{v'w'}$. Thus by the definition of ρ, $d_X(\rho(x), \rho(y)) \leq d_{X_{vw}}(\rho(x), \rho(y)) \leq C$.

Therefore, we are through. □

2.4 Trees of metric bundles and their properties

The notion of metric bundles (see Definition 2.4.1) was introduced by Mj and Sardar ([10]). Subsuming both metric bundles and trees of metric spaces, we define trees of metric bundles (Definition 2.4.2).

Definition 2.4.1. [10, Definition 1.2] Suppose (X,d) and (B,d_B) are geodesic metric spaces; let $c_0 \geq 1$ and $\phi : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ be a proper map. We say X is a (ϕ, c_0)-**metric bundle** over B if there is a 1-Lipschitz and surjective map $p : X \to B$ such that the following holds.

1. Let $z \in B$. Then $F_z := p^{-1}(z)$, called fiber, is a geodesic metric space with the induced path metric from X and the inclusion $F_z \hookrightarrow X$ is ϕ-proper embedding.
2. Let $z_1, z_2 \in B$ such that $d_B(z_1, z_2) \leq 1$ and α be a geodesic joining z_1 and z_2. Then for all $z \in \alpha$ and $x \in F_z$, there are paths in $p^{-1}(\alpha)$ of length at most c_0 joining x to points in F_{z_1} and F_{z_2}.

Definition 2.4.2 (Trees of metric bundles). Let (X,d) be a geodesic metric space. Suppose $\pi_B : (B, d_B) \to T$ is a tree of spaces over a tree T such that edge spaces are points. Let $\phi : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ be a proper map and $c_0 \geq 1$. A tree of metric bundles is a 1-Lipschitz surjective map $\pi_X : X \to B$ such that the following hold (see Figure 2.1).

1. For all $u \in V(T)$ let $B_u := \pi_B^{-1}(u)$ and $X_u := \pi_X^{-1}(B_u)$. Then X_u is geodesic metric space with the induced path metric and the restriction of π_X to X_u gives a (ϕ, c_0)-metric bundle $X_u \to B_u$ (see Definition 2.4.1).
2. Let $e = [v,w]$ be an edge in T, and $\mathfrak{e} = [\mathfrak{v}, \mathfrak{w}]$ be the lift of e joining $\mathfrak{v} \in B_v$ and $\mathfrak{w} \in B_w$. Then π_X restricted to $\pi_X^{-1}(\mathfrak{e})$ is a tree metric spaces over $\mathfrak{e}$ with parameter ϕ (see Definition 2.3.1)
3. For $u \in V(T)$ and $a \in B_u$, we denote the fiber corresponding to a by $F_{a,u} (:= \pi_X^{-1}(a))$. Then the inclusion $F_{a,u} \hookrightarrow X$ is ϕ-proper embedding.

Abusing terminology, we say (X,B,T) is a tree of metric bundles keeping the structural maps π_X, π_B and parameters ϕ, c_0 and other things implicit. We denote the composition of $\pi_X : X \to B$ and $\pi_B : B \to T$ by $\pi : X \to T$.

<u>**Disclaimer**</u>: The term 'trees of metric bundles' may be misleading for the map $\pi_X : X \to B$ since B is not a tree in general; but it is not misleading for $\pi = \pi_B \circ \pi_X : X \to T$. To maintain consistency with existing literature, we will adhere to our chosen nomenclature.

We will see some properties of a tree of metric bundles (X,B,T) that are used in the main proof. In our statements, we make the structural parameters ϕ, c_0 implicit.

For $u \in V(T)$, since fibers are ϕ-properly embedded in X, we can show (along the same line of arguments given in the proof of [9, Proposition 2.17]) that $X_u := \pi^{-1}(u)$ is uniformly properly embedded in X. Then considering (X,B,T) as a tree of metric

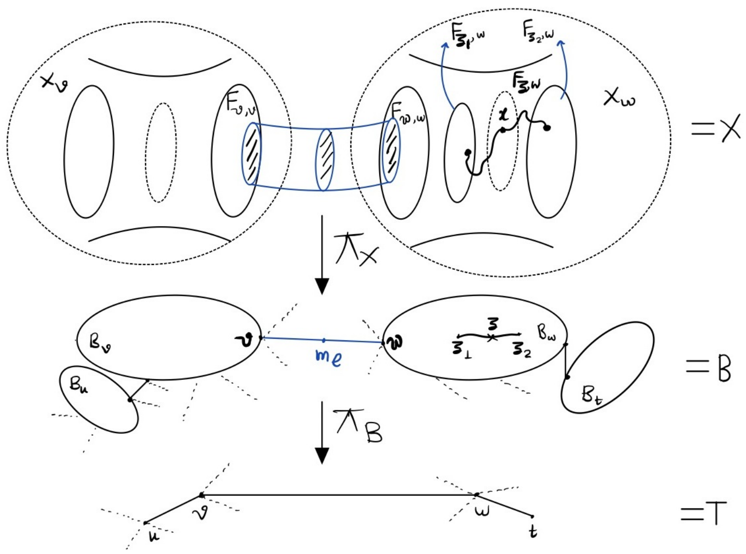


Figure 2.1: Trees of Metric Bundles

spaces $\pi := \pi_B \circ \pi_X : X \to T$, we get the following as corollary of [9, Proposition 2.17].

Proposition 2.4.3. *Suppose (X,B,T) is a tree of metric bundles. Let S be a subtree of T and $X_S := \pi^{-1}(S)$. We consider X_S with the path metric induced from X. Then there exists a proper function $\eta_{2.4.3} : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ depending only on the structural parameters of (X,B,T) such that the inclusion $i : X_S \hookrightarrow X$ is $\eta_{2.4.3}$-proper embedding.*

Definition 2.4.4 (Quasi-isometric (qi) section)**.** Let $K \geq 1$. Suppose (X,B,T) is a tree of metric bundles. Let B_1 be an isometrically embedded subspace in B and $X_1 \subseteq X$. We say X_1 is K-qi section in X over B_1 if there is a K-qi embedding $s : B_1 \to X$ such that $\pi_X \circ s = id$ on B_1 and $X_1 = Im(s)$. Further, we say that it is compatible if the following hold.

1. For all $w \in \pi_B(B_1)$, $X_1 \cap X_w$ is K-qi section over $B_1 \cap B_w$ in the path metric of X_w and $X_1 \cap X = Im(s|_{B_1 \cap B_w})$.

2. Suppose $[v,w] \subseteq \pi_B(B_1)$ is an edge, and $[\mathfrak{v},\mathfrak{w}]$ is the edge joining $\mathfrak{v} \in B_v$ and $\mathfrak{w} \in B_w$. Then $s(\mathfrak{v})$ and $s(\mathfrak{w})$ are K-apart in the path metric on $\pi_X^{-1}([\mathfrak{v},\mathfrak{w}])$ induced from X.

Definition 2.4.5. If B_1, in Definition 2.4.4, is a geodesic segment, say, $\alpha : [0,r] \subseteq \mathbb{R} \to B$, then we call the section a K-**qi lift** of the geodesic α. According to our definition, a K-qi lift of a geodesic $\alpha : [0,r] \subseteq \mathbb{R} \to B$ is $Im(\tilde{\alpha})$ where $\tilde{\alpha} : Im(\alpha) \subseteq B \to X$ is a K-qi embedding. We will simultaneously use $Im(\alpha)$ and $[0,r]$ as the domain of $\tilde{\alpha}$.

Existence of uniform qi section in metric bundle was one of the difficult jobs in [10]. We are going to use it frequently in our paper (see Lemma 2.4.12 (1)). For a short exact sequence, in a different way, it was proved earlier by Mosher [34]. In a hyperbolic geodesic metric space, geodesics (and hence quasi-geodesics) diverge exponentially. In a tree of metric bundles X, qi lifts are quasi-geodesics. So, they diverge exponentially provided X is hyperbolic. This property is captured in the following definition for special types of quasi-geodesics, namely, qi lifts (see also *necessity of flaring* in Introduction 1 to get more on this). This definition is a generalization of Bestvina-Feighn's hallway flaring condition ([6]) in a natural way as defined in [10, Definition 1.12] for metric bundles.

Definition 2.4.6. Suppose $k \geq 1$. A tree of metric bundles (X,B,T) is said to satisfy k-**flaring condition** (see Figure 2.2) if $\exists\ M_k > 0$, $n_k \in \mathbb{N}$ and $\lambda_k > 1$ depending on k such that the following holds.

For every pair (γ_0, γ_1) of k-qi lifts of a geodesic $\gamma : [-n_k, n_k] \to [a,b] \subseteq B$ joining a, b with $d^f(\gamma_0(0), \gamma_1(0)) > M_k$, we have,

$$\lambda_k d^f(\gamma_0(0), \gamma_1(0)) < max\{d^f(\gamma_0(-n_k), \gamma_1(-n_k)), d^f(\gamma_0(n_k), \gamma_1(n_k))\}$$

where d^f denotes the fiber distance in the corresponding fiber. Abusing terminology, we sometimes simply say that (X,B,T) satisfies k-flaring condition suppressing the constants M_k, n_k, λ_k. We say that (X,B,T) satisfies a flaring condition if it satisfies k-flaring condition for all $k \geq 1$.

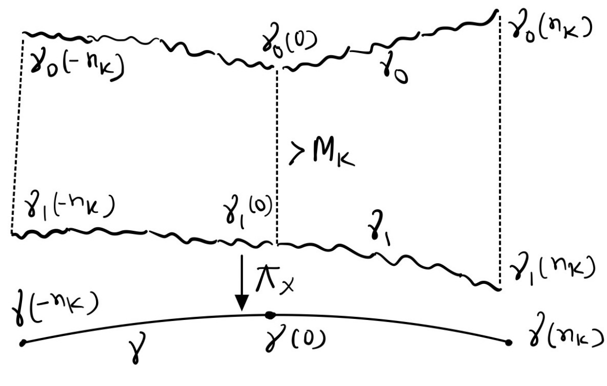


Figure 2.2: Flaring Condition

Now we will state the following Lemma 2.4.7 without a proof. These results correspond to [10, Lemma 2.17, Lemma 2.18] in metric bundles situation. One has a similar proofs in trees of metric bundles. The results (1) of Lemma 2.4.7 is defined as *uniform flaring condition* in the book [9], and (2) says that the neck are quasiconvex subset of the base in the sense of [10].

Lemma 2.4.7. *Let $k \geq 1$. Suppose (X,B,T) is a tree of metric bundles satisfying k-flaring condition with constants M_k, n_k, λ_k. Then for all $D \geq 0$ there is $\tau_{2.4.7} = \tau_{2.4.7}(k,D)$ and $R_{2.4.7} = R_{2.4.7}(k,D)$ satisfying the following.*

Let γ_0, γ_1 be two k-qi lifts of a geodesic $\gamma\colon [0,r] \to [a,b] \subseteq B$ joining $a,b \in B$ (resp. $a,b \in V(B)$). Let $0 = r_0 < r_1 < \cdots < r_n = r$ such that $r_{i+1} - r_i = 1$ for $0 \leq i \leq n-2$ and $r_n - r_{n-1} \leq 1$. Then:

1. *$d^f(\gamma_0(r_i), \gamma_1(r_i)) > M_k$ for all $1 \leq i \leq n-1$ and*

$$max\{d^f(\gamma_0(0), \gamma_1(0)), d^f(\gamma_0(r), \gamma_1(r))\} \leq D$$

   implies $r = d_B(a,b) \leq \tau_{2.4.7}$.

2. *$max\{d^f(\gamma_0(0), \gamma_1(0)), d^f(\gamma_0(r), \gamma_1(r))\} \leq D$ implies for all $0 \leq i \leq n$*

$$d^f(\gamma_0(r_i), \gamma_1(r_i)) \leq R_{2.4.7}.$$

Throughout the thesis, we directly will not use flaring condition in proving results instead Lemma 2.4.7. We make the following remark for later use, and it follows from the Definition 2.4.6.

Remark 2.4.8. Let $k \geq 1$. Suppose (X,B,T) is a tree of metric bundles satisfying k-flaring condition with constants M_k, n_k, λ_k. Let S be a subtree of T. Then (a) it satisfies k'-flaring condition for all $k' \leq k$ with the same constants, and (b) the restriction $\pi_X|_{X_S} : X_S \to B_S$ also satisfies k-flaring condition with the same constants.

Motivated by the main theorems of [6] and [10], we define the following.

Definition 2.4.9. We say that a tree of metric bundles (X,B,T) satisfies the hyperbolic axioms, in short, axiom **H** with parameters $\delta_0 \geq 0, N \geq 0$ and $L_0 \geq 1$ if the following hold.

1. Let $u \in V(T)$ and $a \in B_u$. Then $F_{a,u}$ is δ_0-hyperbolic and the barycenter map (see Subsection 2.2.2) $\partial^3 F_{a,u} \to F_{a,u}$ is N-coarsely surjective.

2. Let $e = [v,w]$ be an edge in T. Let $\mathfrak{e} = [\mathfrak{v}, \mathfrak{w}]$ be the edge joining $\mathfrak{v} \in B_v$ and $\mathfrak{w} \in B_w$, and $m_{\mathfrak{e}}$ be the mid point of $\mathfrak{e}$. Then the incident maps $\vartheta_{\mathfrak{e}\mathfrak{v}} : \pi_X^{-1}(m_{\mathfrak{e}}) \to F_{\mathfrak{v},v}$, $\vartheta_{\mathfrak{e}\mathfrak{w}} : \pi_X^{-1}(m_{\mathfrak{e}}) \to F_{\mathfrak{w},w}$ are L_0-qi embeddings.

3. Lastly, let B be δ_0-hyperbolic. This assumption is the same as B_v is δ_0-hyperbolic for all $v \in V(T)$.

2.4.1 Conventions and notations

1. Unless otherwise specified, our tree of metric bundles (X,B,T) always satisfies the axiom **H** with constants δ_0, N, L_0 as in Definition 2.4.9.

2. Suppose B' is connected subspace of B and S is subtree of T. Unless otherwise specified, by $b \in B'$ and $u \in S$, we always mean $b \in B' \cap B_{\pi_B(b)}$ and $u \in V(S)$ respectively.

3. If X_1 is a K-qi section over B_1, then it is a compatible $\eta_{2.4.3}(2K)$-qi section over B_1. Thus, now onward, by a K-qi section, we always mean a compatible K-qi section.

Notation 2.4.10. We use these notations throughout the Chapter 5. We denote the composition map $\boldsymbol{\pi_B \circ \pi_X}$ by $\boldsymbol{\pi}$. For a subtree $S \subseteq T$, $\boldsymbol{X_S := \pi^{-1}(S)}$ and $\boldsymbol{B_S := \pi_B^{-1}(S)}$. In particular, for $v \in V(T)$, $\boldsymbol{X_v := \pi^{-1}(v)}$ and $\boldsymbol{B_v := \pi_B^{-1}(v)}$. Let $v,w \in T$, $d_T(v,w) = 1$ and $[\mathfrak{v},\mathfrak{w}]$ is the edge joining $\mathfrak{v} \in B_v$ and $\mathfrak{w} \in B_w$. We denote $\boldsymbol{F_{\mathfrak{v}\mathfrak{w}} := \pi_X^{-1}([\mathfrak{v},\mathfrak{w}])}$. The induced path metric on $F_{\mathfrak{v}\mathfrak{w}}$ is denoted by $\boldsymbol{d_{\mathfrak{v}\mathfrak{w}}}$, and we use $\boldsymbol{N^{\mathfrak{v}\mathfrak{w}}}$ to mean neighborhood of subsets of $F_{\mathfrak{v}\mathfrak{w}}$ in the $d_{\mathfrak{v}\mathfrak{w}}$-metric. For a fiber $F_{a,u}$, where $u \in V(T)$ and $a \in B_u$, we simply use $\boldsymbol{d^f}$, $\boldsymbol{diam^f}$, $\boldsymbol{N^f}$ and $\boldsymbol{[x,y]^f}$ (or $\boldsymbol{[x,y]}_{F_{a,u}}$) to denote respectively the induced path metric on $F_{a,u}$, the diameter of a subset of $F_{a,u}$ in d^f-metric, the neighborhood of a subset of $F_{a,u}$ in d^f-metric and a geodesic inside $F_{a,u}$ joining $x,y \in F_{a,u}$. Since it will be clear from the context which fiber we are working with, we are not being more specific on d^f, N^f etc. Lastly, $\boldsymbol{P_{\mathfrak{w}} := P_{F_{\mathfrak{v}\mathfrak{w}},F_{\mathfrak{w},w}}}$.

We want to put metric bundle (Definition 2.4.1) structure on subspace of X. Suppose $X_1 \subseteq X$ and S is a subtree of T such that the restriction map $\pi_X|_{X_1} : X_1 \to B_S$ is surjective.

Definition 2.4.11 (K-metric bundle and special K-ladder)**.** With the above, we say that X_1 forms a ***K*-metric bundle** over B_S if there is a K-qi section through each point of X_1 over B_S such that the image lies inside X_1. Further, we say that X_1 forms a **special *K*-ladder** if X_1 forms a K-metric bundle along with two K-qi sections Σ_1, Σ_2 over B_S such that $X_1 = \bigcup_{v \in S,\ b \in B_v} [F_{b,v} \cap \Sigma_1, F_{b,v} \cap \Sigma_2]_{F_{b,v}}$. In this case, sometimes we denote X_1 by $\mathscr{L}_K(\Sigma_1,\Sigma_2)$ or simply by $\mathscr{L}(\Sigma_1,\Sigma_2)$ when K is understood.

Lemma 2.4.12 (2) is proved for metric graph bundles (see [10, Definition 1.5]) in [10, Lemma 3.1], although it holds for metric bundles. Since the same proof works, we omit it.

Lemma 2.4.12. ([10, Proposition 2.10, Proposition 2.12, Lemma 3.1, Lemma 3.3]) *Suppose (X,B,T) is a tree of metric bundles satisfying axiom* **H**. *(For* (3)*, axiom* **H** *is not required.) Then given $K \geq 1$ and $R \geq 2K$ there is $C_{2.4.12} = C_{2.4.12}(K) > K$ such that the following holds.*

(1) *There exists a constant $K_{2.4.12} \geq 1$ depending only on δ_0, N (constants of axiom* **H***) such that through each point $x \in X_v$ there is a $K_{2.4.12}$-qi section over B_v in the path metric of X_v, where $v \in V(T)$.*

(2) *Let $v \in V(T)$ and Σ_1, Σ_2 be two K-qi sections over B_v. Then $\mathscr{L}(\Sigma_1, \Sigma_2)$ is a special $C_{2.4.12}(K)$-ladder over B_v.*

(3) *Let Σ be a K-qi section over an isometrically embedded subspace $B_1 \subseteq B$. Let $s : [a,b] \to X$ be a qi lift of a geodesic segment $[a,b] \subseteq B_1$ such that $Im(s) \subseteq \Sigma$. Then $N_R(\Sigma)$ is path connected and if the induced path metric is d' then $d'(s(a), s(b)) \leq 2K d_B(a,b)$. Moreover, $N_{2K}(\Sigma)$ is $K(2K+1)$-qi embedded inside any geodesic subspace containing $N_R(\Sigma)$ (with respect to their induced path metric).*

Remark 2.4.13. In the view of Lemma 2.4.12 (2), for $i \in \mathbb{N}$, we denote the i^{th} iteration by $C^{(i)}_{2.4.12}(K) = C_{2.4.12}(C^{(i-1)}_{2.4.12}(K))$, where $C^{(0)}_{2.4.12}(K) = K$. In other words, if $\mathscr{L}(\Sigma_1, \Sigma_2)$ is bounded by two $C^{(i-1)}_{2.4.12}(K)$-qi sections Σ_1, Σ_2 over B_v, then $\mathscr{L}(\Sigma_1, \Sigma_2)$ is a special $C^{(i)}_{2.4.12}(K)$-ladder over B_v.

As an application of Lemma 2.4.12 (2) along with the fact that quadrilaterals are slim in hyperbolic spaces, we have the following. We omit the proof.

Lemma 2.4.14. *Given $K \geq 1$ there is $K_{2.4.14} = K_{2.4.14}(K)$ such that the following holds.*

Suppose (X,B,T) is a tree of metric bundles satisfying axiom **H**. *Let S be a subtree of T. Let $\mathscr{G}_{K,S} = \{\gamma : \gamma \text{ is a } K\text{-qi section over } B_S\} \neq \emptyset$. For $w \in S$, $b \in B_w$, let $H_{b,w} = hull\{\gamma(b) : \gamma \in \mathscr{G}_{K,S}\} \subseteq F_{b,w}$ and $H = \bigcup_{w \in S,\, b \in B_w} H_{b,w}$. (Here quasiconvex hull is considered in the corresponding fiber.) Then H is $K_{2.4.14}$-metric bundle over B_S. In particular, if $\mathscr{G}_{K,S} = \{\gamma_1, \gamma_2\}$ then $\bigcup_{w \in S,\, b \in B_w} [\gamma_1(b), \gamma_2(b)]^f$ forms a special $K_{2.4.14}$-ladder over B_S.*

Lemma 2.4.15. *Given $k \geq 1$ there exists $k_{2.4.15} = k_{2.4.15}(k)$ such that the following holds.*

Suppose (X,B,T) is a tree of metric bundles satisfying axiom **H**. *Let $v \in T$ and Σ_i be k-qi section over B_v for $i = 1,2,3$. Suppose $\{x^{(i)}_{b,v}\} = \Sigma_i \cap F_{b,v}, i = 1,2,3$ and $z_{b,v}$ is a δ_0-center of geodesic triangle $\triangle(x^{(1)}_{b,v}, x^{(2)}_{b,v}, x^{(3)}_{b,v}) \subseteq F_{b,v}$, $b \in B_v$. Then the map $s : B_v \to X$ defined by $b \mapsto z_{b,v}$ is a $k_{2.4.15}$-qi section over B_v.*

Proof. Let $b_1, b_2 \in B_v$ such that $d_B(b_1, b_2) \leq 1$. Since $\pi_X : X \to B$ is 1-Lipschitz, we only have to prove that $d_{X_v}(z_{b_1,v}, z_{b_2,v})$ is uniformly bounded. Define a map $\psi : F_{b_1,v} \to F_{b_2,v}$ by $\psi(x^{(i)}_{b_1,v}) = x^{(i)}_{b_2,v}$ for $i = 1,2,3$ and $\forall\, z \in F_{b_1,v} \setminus \{x^{(i)}_{b_1,v} : i = 1,2,3\}$, we take $\psi(z) \in F_{b_2,v}$ such that $d(z, \psi(z)) \leq c_0$ (as in Definition 2.4.2). Note that $d_{X_v}(x, \psi(x)) \leq 2k + c_0$ for all $x \in F_{b_1,v}$. Then by [10, Lemma 1.15], ψ extends to a $g(2k+c_0)$-quasi-isometry from $F_{b_1,v}$ to $F_{b_2,v}$ for some function $g : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$. Therefore, by [10, Lemma 1.29 (2)], $d^f(\psi(z_{b_1,v}), z_{b_2,v})$ is bounded by a constant D, depending only on δ_0 (hyperbolicity constant of $F_{b_i,v}$) and $g(2k+c_0)$. Hence, we can take $k_{2.4.15} := D + 2k$. □

Lemma 2.4.16. *Given $K \geq 1$ there exists $K_{2.4.16} = K_{2.4.16}(K)$ such that the following holds.*

Suppose (X, B, T) is a tree of metric bundles satisfying axiom **H**. *Let $v \in T$ and Σ_i be K-qi section over B_v for $i = 1,2,3,4$. Let $\mathscr{L}_1 = \mathscr{L}(\Sigma_1, \Sigma_2)$ and $\mathscr{L}_2 = \mathscr{L}(\Sigma_3, \Sigma_4)$ be special ladders over B_v formed by these sections. Let $a \in B_v$ and $\bar{P}_a : \mathscr{L}_1 \cap F_{a,v} \to \mathscr{L}_2 \cap F_{a,v}$ be modified projection in the metric $F_{a,v}$ (see Definition 2.2.25). Further, suppose $\bar{P}_a(\mathscr{L}_1 \cap F_{a,v}) = [p_a, q_a]_{F_{a,v}}$ such that p_a is closest to $\Sigma_3 \cap F_{a,v}$ and q_a is that to $\Sigma_4 \cap F_{a,v}$ in the metric $F_{a,v}$. Moreover, we define $s_1 : B_v \to \mathscr{L}_2$ and $s_2 : B_v \to \mathscr{L}_2$ by $a \mapsto p_a$ and $a \mapsto q_a$ respectively, where $a \in B_v$.*

Then s_1 and s_2 are $K_{2.4.16}$-qi sections over B_v lying inside $\mathscr{L}_2$.

Proof. Suppose $b, c \in B_v$ such that $d_B(b, c) \leq 1$. Let $\mathscr{L}_i \cap F_{b,v} = [x_i, y_i]$ and $\mathscr{L}_i \cap F_{c,v} = [s_i, t_i]$ for $i = 1,2$, where $\Sigma_1(b) = x_1$, $\Sigma_1(c) = s_1$, $\Sigma_2(b) = y_1$, $\Sigma_2(c) = t_1$ and $\Sigma_3(b) = x_2$, $\Sigma_3(c) = s_2$, $\Sigma_4(b) = y_2$, $\Sigma_4(c) = t_2$. Suppose $\bar{P}_b([x_1, y_1]) = [p_1, q_1]$ and $\bar{P}_c([s_1, t_1]) = [p_2, q_2]$ such that p_1, p_2 are closest to x_2, s_2 respectively in the metric $F_{b,v}$ and q_1, q_2 are closest to y_2, t_2 respectively in the metric $F_{c,v}$. Since $\pi_X : X \to B$ is 1-Lipschitz, we only need to show that $d_{X_v}(p_1, p_2)$ and $d_{X_v}(q_1, q_2)$ are uniformly bounded. We will show only the former one as a similar proof works for the later case.

Let $\bar{P}_b(x_1) = x'_1, \bar{P}_b(y_1) = y'_1, \bar{P}_c(s_1) = s'_1, \bar{P}_c(t_1) = t'_1$. Note that $x'_1 \in [p_1, y'_1] \subseteq [x_2, y_2]$ or $x'_1 \in [y'_1, q_1] \subseteq [x_2, y_2]$ and $s'_1 \in [p_2, t'_1] \subseteq [s_2, t_2]$ or $s'_1 \in [t'_1, q_2] \subseteq [s_2, t_2]$. Depending on the position on x'_1 and s'_1, we consider the following four cases.

Like in Lemma 2.4.15, we define a map $\psi : F_{b,v} \to F_{c,v}$ such that $\psi(x_i) = s_i$, $\psi(y_i) = t_i$, $i = 1,2$ and for all other points $x \in F_{b,v}$, we take $\psi(x) \in F_{c,v}$ such that $d_{X_v}(x, \psi(x)) \leq c_0$ (as in Definition 2.4.2). Then $d_{X_v}(x, \psi(x)) \leq 2K + c_0, \forall\, x \in F_{b,v}$, and so by [10, Lemma 1.15], ψ extends to a $g(2K+c_0)$-quasi-isometry from $F_{b,v}$ to $F_{c,v}$ for some function $g : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$. Thus by Lemma 2.2.22 (2), there is

$k = 2K + D_{2.2.22}(\delta_0, g(2K+c_0), \delta_0)$ such that

$$d_X(x'_1, s'_1) \leq k \text{ and } d_X(y'_1, t'_1) \leq k \qquad (2.4.\ 1)$$

Case 1: Suppose $x'_1 \in [p_1, y'_1]$ and $s'_1 \in [p_2, t'_1]$. Then by [9, Corollary 1.116], there is constant C_1 depending on δ_0 such that $d^f(p_1, x'_1) \leq C_1$ and $d^f(p_2, s'_1) \leq C_1$. Combining with inequation 2.4. 1, we have $d_{X_v}(p_1, p_2) \leq k + 2C_1$.

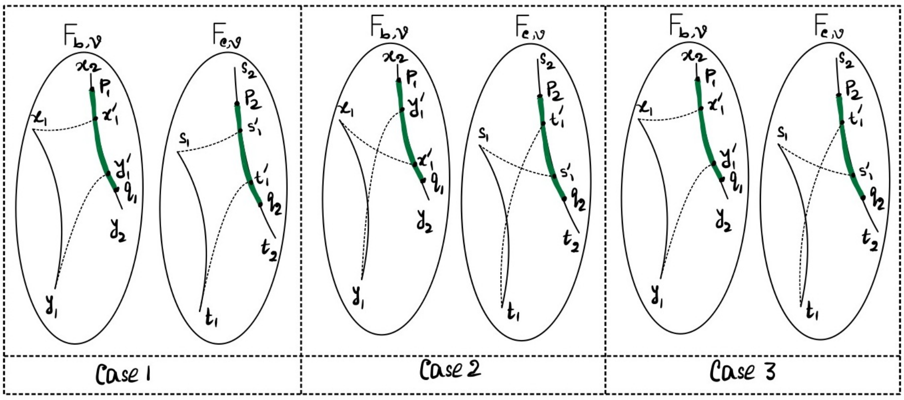


Figure 2.3

Case 2: Suppose $x'_1 \in [y'_1, q_1]$ and $s'_1 \in [t'_1, q_2]$. In this case, $y'_1 \in [p_1, x'_1]$ and $t'_1 \in [p_2, s'_1]$. Again by [9, Corollary 1.116], $d^f(p_1, y'_1) \leq C_1$ and $d^f(p_2, t'_1) \leq C_1$. Combining with inequation 2.4. 1, we get, $d_{X_v}(p_1, p_2) \leq k + 2C_1$.

Case 3: Suppose $x'_1 \in [p_1, y'_1]$ and $s'_1 \in [t'_1, q_2]$. In this case, $t'_1 \in [p_2, s'_1]$. Applying [9, Corollary 1.116], we get, $d_{X_v}(p_1, x'_1) \leq C_1$ and $d_{X_v}(p_2, t'_1) \leq C_1$. We define a map (like above) $\psi : F_{b,v} \to F_{c,v}$ such that $\psi(x'_1) = s'_1$, $\psi(y_2) = t_2$ and for all other points $x \in F_{b,v}$, we take $\psi(x) \in F_{c,v}$ such that $d_{X_v}(x, \psi(x)) \leq c_0$ (as in Definition 2.4.2). Then $\forall\ x \in F_{b,v}$, $d_{X_v}(\psi(x), x) \leq k + c_0$ (see inequation 2.4. 1). Thus by [10, Lemma 1.15], ψ extends to a $g(k+c_0)$-quasi-isometry from $F_{b,v}$ to $F_{c,v}$ for some function $g : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$. Applying Morse Lemma 2.2.2, $\exists\ \zeta \in [s'_1, t_2] \subseteq [s_2, t_2]$ such that $d_{X_v}(y'_1, \zeta) \leq c_0 + D_{2.2.2}(\delta_0, g(k+c_0), g(k+c_0)) = k'$ (say). Then $d_{X_v}(\zeta, t'_1) \leq d_{X_v}(\zeta, y'_1) + d_{X_v}(y'_1, t'_1) \leq k + k'$. Since fibers are ϕ-properly embedded in X, so $d^f(\zeta, t'_1) \leq \phi(k+k')$. In particular, $d^f(s'_1, t'_1) \leq \phi(k+k')$. Thus $d_{X_v}(p_1, p_2) \leq d_{X_v}(p_1, x'_1) + d_{X_v}(x'_1, s'_1) + d_{X_v}(s'_1, t'_1) + d_{X_v}(t'_1, p_2) \leq 2C_1 + k + \phi(k+k')$.

Case 4: Lastly, we assume that $x'_1 \in [y'_1, q_1]$ and $s'_1 \in [p_2, t'_1]$. But this is same as Case 3.

Therefore, we can take $K_{2.4.16} := 2C_1 + k + \phi(k+k')$ (maximum of all constants we get in the above four cases). □

2.5 Complexes of groups

Bass and Serre in [35] introduced graphs of groups to study infinite groups by their action on simplicial trees. Given a finite graph of groups there is a simplicial tree, called *Bass-Serre tree*, on which the fundamental group of the given graph of groups acts without inversion such that the quotient is the given graph. This theory also says the converse that if a group acts by cocompact on a simplicial tree without inversion then it corresponds to a graph of groups whose fundamental group is isomorphic to the given group and there is an equivariant isomorphism from the Bass-Serre tree to the given simplicial tree. With this one wants to know whether this theory can be generalized to the higher dimension. In other words, suppose a group G acts on a simply connected simplicial complex $\tilde{X}$ without inversion such that the quotient $G \setminus \tilde{X} = X$ is a finite simplicial complex. Can we get back $\tilde{X}$ and the action from the information on X?

This motivates us to study complexes of groups. Gersten and Stallings studied triangle of groups, i.e., when X is 2-dimensional ([20]). Later, Haefliger [21] studied higher dimensional case in a more general setting, called *small category without loops* (abbreviated *scwol*) and Corson [22] studied the 2-dimensional case independently. Now we will briefly recall some definitions and results for complexes of groups. For a more comprehensive understanding of the concepts presented here and the overall theory of complexes of groups over scwol, we refer the reader to [23].

Definition 2.5.1 (Small category without loops (scwol)). A *small category without loop* is a set $\mathscr{X}$ which is the disjoint union of a set $V(\mathscr{X})$ called the vertex set of $\mathscr{X}$ and a set $E(\mathscr{X})$ called the edge set of $\mathscr{X}$ along with two maps

$$i : E(\mathscr{X}) \to V(\mathscr{X}) \text{ and } t : E(\mathscr{X}) \to V(\mathscr{X}).$$

For $a \in E(\mathscr{X})$, $i(a)$ and $t(a)$ are called initial vertex and terminal vertex of a respectively.

Let $E^{(2)}(\mathscr{X})$ denote the set of pairs $(a,b) \in E(\mathscr{X}) \times E(\mathscr{X})$ such that $i(a) = t(b)$. A third map

$$E^{(2)}(\mathscr{X}) \to E(\mathscr{X})$$

is given that associates to each pair (a,b) an edge ab called their *composition* (and we say that a, b are composable). These maps are required to satisfy the following conditions:

1. For all $(a,b) \in E^{(2)}(\mathscr{X})$, we have $i(ab) = i(b)$ and $t(ab) = t(a)$.

2. For all $a,b,c \in E(\mathscr{X})$, if $i(a) = t(b)$ and $i(b) = t(c)$, then $(ab)c = a(bc)$. Thus we can denote it simply abc and this is called associativity.

3. No loops condition: for each $a \in E(\mathscr{X})$, we have $i(a) \neq t(a)$.

Example 2.5.2. Suppose Q is a poset. Then we can associate a scwol as follows. The set of vertices is Q and the edges are pairs $(\tau,\sigma) \in Q \times Q$ such that $\tau < \sigma$. Define $i((\tau,\sigma)) := \sigma$, $t((\tau,\sigma)) := \tau$ and the composition of $(\tau,\sigma)(\sigma,\rho) = (\tau,\rho)$.

Suppose $k \in \mathbb{N}$ and $E^{(k)}(\mathscr{X})$ is the set of sequences $(a_1,a_2,\cdots,a_k)$ such that $(a_i,a_{i+1}) \in E^{(2)}(\mathscr{X})$ for all $i = 1,2,\cdots,k-1$. By convention $E^{(0)}(\mathscr{X}) = V(\mathscr{X})$. The dimension of $\mathscr{X}$ is defined to be the supremum of $k \geq 0$ for which $E^{(k)}(\mathscr{X})$ is non-empty. To each scwol $\mathscr{X}$, one can associate a polyhedral complex, denoted by $|\mathscr{X}|$, called geometric realization of $\mathscr{X}$. Roughly speaking it is disjoint union of standard k-simplices for each $(a_1,a_2,\cdots,a_k) \in E^{(k)}(\mathscr{X})$ with a natural relation. One is referred to [23, Chapter III.C, 1.3] for details. Note that in general, the intersection of two simplices in $|\mathscr{X}|$ is not a common face, rather union faces. So $|\mathscr{X}|$ might not be a simplicial complex in general. A scwol $\mathscr{X}$ is connected if $|\mathscr{X}|$ is connected with respect to quotient topology.

Example 2.5.3. Suppose K is a M_k-polyhedral complex (see [23, Definition 7.37, I.7]). Now we construct a scwol $\mathscr{X}$ from K as follows. The set of vertices of $\mathscr{X}$ is the simplices of K (equivalently, the set of barycentres of the cells of K). The edges of $\mathscr{X}$ are the 1-simplices of the barycentric subdivision K' of K: each 1-simplex of K' corresponds to a pair of cells $T \subseteq S$; we define $i(a)$ to be the barycentre of S and $t(a)$ to be that of T.

Definition 2.5.4 (Morphisms of scwols). Suppose $\mathscr{X}$ and $\mathscr{Y}$ are two scwols. A *non-degenerate morphism* $f : \mathscr{X} \to \mathscr{Y}$ is a map that sends $V(\mathscr{X})$ to $V(\mathscr{Y})$ and $E(\mathscr{X})$ to $E(\mathscr{Y})$ such that the following hold.

1. For all $a \in E(\mathscr{X})$, $i(f(a)) = f(i(a))$ and $t(f(a)) = f(t(a))$.

2. For each $(a,b) \in E^{(2)}(\mathscr{X})$, $f(ab) = f(a)f(b)$.

3. For each vertex $\sigma \in V(\mathscr{X})$, the restriction of f to the set of edges with initial vertex σ is a bijection onto the set of edges of $\mathscr{Y}$ with initial vertex $f(\sigma)$.

An automorphism of a scwol $\mathscr{X}$ is a morphism $f : \mathscr{X} \to \mathscr{X}$ that has an inverse, i.e., there is $f^{-1} : \mathscr{X} \to \mathscr{X}$ such that $ff^{-1} = f^{-1}f$ is the identity on $\mathscr{X}$.

Definition 2.5.5 (Group actions on scwols). An action of a group G on a scwol $\mathscr{X}$ is a homomorphism from G to the automorphism of $\mathscr{X}$ such that the following conditions hold.

1. For all $a \in E(\mathscr{X})$ and $g \in G$, $g.i(a) \neq t(a)$.
2. For all $a \in E(\mathscr{X})$ and $g \in G$, if $g.i(a) = i(a)$ then $g.a = a$.

Example 2.5.6. Suppose K is a M_k-simplicial complex. A simplicial action of a group G on K is said to be *without inversion* if an element $g \in G$ sends a simplex of K to itself then g fixes that simplex pointwise. Let $\mathscr{X}$ be the corresponding scwol of K as in Example 2.5.3. Then G-action on K induces a natural action on $\mathscr{X}$. The action of G on K is without inversion if and only if G acts on $\mathscr{X}$ as in Definition 2.5.5.

Now we define complexes of groups.

Definition 2.5.7 (Complex of groups). Suppose $\mathscr{Y}$ is a scwol. A complex of groups $\mathscr{G}(\mathscr{Y}) = (G_\sigma, \psi_a, g_{a,b})$ over $\mathscr{Y}$ is given by the following data:

1. for each $\sigma \in V(\mathscr{Y})$, a group G_σ called the *local group* at σ,
2. for each $a \in E(\mathscr{Y})$, an injective homomorphism $\psi_a : G_{i(a)} \to G_{t(a)}$,
3. for each pair of composable edges $(a,b) \in E^{(2)}(\mathscr{Y})$, a twisting element $g_{a,b} \in G_{t(a)}$,

   with the following compatibility conditions:

   (a) $\mathrm{Ad}(g_{a,b})\psi_{ab} = \psi_a\psi_b$, where $\mathrm{Ad}(g_{a,b})$ is the conjugation by the element $g_{a,b}$ in $G_{t(a)}$ and

   (b) for each triple $(a,b,c) \in E^{(3)}(\mathscr{Y})$ of composable edges we have the cocycle condition,

$$\psi_a(g_{b,c})g_{a,bc} = g_{a,b}g_{ab,c}.$$

Suppose Y is a (Euclidean) simplicial complex and $\mathscr{Y}$ is the associated scwol (see Example 2.5.3). Then by a complex of groups over Y, we mean a complex of groups over $\mathscr{Y}$.

Remark 2.5.8. 1. A *simple* complex of groups over $\mathscr{Y}$ is a complex of groups over $\mathscr{Y}$ such that all the twisting elements $g_{a,b}$ are trivial.

2. The condition $(3),(a)$ is empty if $\mathscr{Y}$ is 1-dimensional and the condition $(3),(b)$ is empty if $\mathscr{Y}$ is 2-dimensional.

3. Let $\mathscr{Y}$ be 1-dimensional (equivalently, $E^{(2)}(\mathscr{Y})$ is empty). In this case, the notion of complex of groups over $\mathscr{Y}$ restrict to the notion of graph of groups introduced by Bass-Serre ([35]).

For our reference, we define the graph of groups below.

Definition 2.5.9 (**Graph of groups** ([35])). Let Y be an oriented, connected graph with vertex set $V(Y)$ and edge set $E(Y)$ ([35]). So we have maps $i : E(Y) \to V(Y)$ sending an edge to its initial vertex and $t : E(Y) \to V(Y)$ sending an edge to its terminal vertex. A *graph of groups* $(\mathscr{G}, Y)$ over Y consists of following data:

1. For each vertex $u \in V(Y)$ there is a group G_u and for each edge $e \in E(Y)$ there is a group G_e.

2. Let $e \in E(Y)$. Then there are monomorphisms $i_e : G_e \to G_{i(e)}$ and $t_e : G_e \to G_{t(e)}$.

The groups G_u in (1) are referred to as the *vertex groups*, and groups G_e are called the *edge groups* of the graph of groups and the homomorphisms in (2) are called the *incidence homomorphisms*.

Definition 2.5.10 (**Morphisms of complexes of groups**). Let $\mathscr{G}(\mathscr{Y}) = (G_\sigma, \psi_a, g_{a,b})$ and $\mathscr{G}(\mathscr{Y}') = (G_{\sigma'}, \psi_{a'}, g_{a',b'})$ be two complexes of groups over scwols $\mathscr{Y}$ and $\mathscr{Y}'$ respectively. Suppose $f : \mathscr{Y} \to \mathscr{Y}'$ is a (possibly degenerate) morphism of scwols. A *morphism* $\phi = (\phi_\sigma, \phi(a))$ from $\mathscr{G}(\mathscr{Y})$ to $\mathscr{G}(\mathscr{Y}')$ over f consists of following data:

1. There is a homomorphism $\phi_\sigma : G_\sigma \to G_{f(\sigma)}$ of groups for each $\sigma \in V(\mathscr{Y})$.

2. There is an element $\phi(a) \in G_{t(f(a))}$ for each $a \in E(\mathscr{Y})$ such that

   (a) $$\mathrm{Ad}(\phi(a))\psi_{f(a)}\phi_{i(a)} = \phi_{t(a)}\psi_a,$$

   and for all $(a,b) \in E^{(2)}(\mathscr{Y})$,

   (b) $$\phi_{t(a)}(g_{a,b})\phi(ab) = \phi(a)\psi_{f(a)}(\phi(b))g_{f(a),f(b)}$$

If f is an isomorphism of scwols and ϕ_σ is an isomorphism for every $\sigma \in V(\mathscr{Y})$, then ϕ is called an isomorphism.

Let us restate the above Definition 2.5.10 for an important case when $\mathscr{Y}'$ is a single vertex.

Definition 2.5.11. A morphism $\phi = (\phi_\sigma, \phi(a))$ from a complex of groups $\mathscr{G}(\mathscr{Y}) = (G_\sigma, \psi_a, g_{a,b})$ to a group G consists of a homomorphism $\phi_\sigma : G_\sigma \to G$ for each $\sigma \in V(\mathscr{Y})$ and an element $\phi(a) \in G$ for each $a \in E(\mathscr{Y})$ such that the following hold.

$$\phi_{t(a)}\psi_a = \mathrm{Ad}(\phi(a))\phi_{i(a)} \text{ and } \phi_{t(a)}(g_{a,b})\phi(ab) = \phi(a)\phi(b)$$

We say ϕ is *injective on local groups* if ϕ_σ's are injective for all $\sigma \in V(\mathscr{Y})$.

2.5.1 The complex of groups associated to an action

Suppose a group G acts on a scwol $\mathscr{X}$ (see Definition 2.5.5) such that quotient $G \setminus \mathscr{X} = \mathscr{Y}$ is a scwol. Let $p : \mathscr{X} \to \mathscr{Y}$ be the natural projection.

For each $\sigma \in V(\mathscr{Y})$, we choose $\tilde{\sigma} \in V(\mathscr{X})$ such that $p(\tilde{\sigma}) = \sigma$. For each $a \in E(\mathscr{Y})$ with $i(a) = \sigma$, by condition (2) of Definition 2.5.5, there is an unique $\tilde{a} \in E(\mathscr{X})$ such that $i(\tilde{a}) = \tilde{\sigma}$ and $p(\tilde{a}) = a$. Note that if $\tau = t(a)$, in general, $\tilde{\tau} \neq t(\tilde{a})$. Then there is $h_a \in G$ such that $h_a.t(\tilde{a}) = \tilde{\tau}$. For each $\sigma \in V(\mathscr{Y})$, let G_σ be the isotropy subgroup of $\tilde{\sigma}$. Then for all $a \in E(\mathscr{Y})$, we define a map $\psi_a : G_{i(a)} \to G_{t(a)}$ by

$$\psi_a(g) = h_a g h_a^{-1}.$$

Again condition (2) of Definition 2.5.5 tells that ψ_a is a well-defined homomorphism. For composable edges $(a,b) \in E^{(2)}(\mathscr{Y})$, we define

$$g_{a,b} = h_a h_b h_{ab}^{-1}.$$

It is not hard to check that $g_{a,b} \in G_{t(a)}$.

The complex of groups over $\mathscr{Y}$ associated to the action of G on $\mathscr{X}$ (along with the above choices) is

$$\mathscr{G}(\mathscr{Y}) = (G_\sigma, \psi_a, g_{a,b}).$$

One can easily check that conditions (3) (a) and (3) (b) of Definition 2.5.7 are satisfied.

We also note that there is a natural homomorphism associated to this action

$$\phi : \mathscr{G}(\mathscr{Y}) \to G,$$

$\phi = (\phi_\sigma, \phi(a))$, where $\phi_\sigma : G_\sigma \to G$ is the natural inclusion and $\phi(a) = h_a$. This is a morphism which is injective on local groups.

Again another choices of vertices $\tilde{\sigma}$ of $\sigma \in V(\mathscr{Y})$ will give a complex of groups which is isomorphic to the previous one (see [23], [21] for details).

Definition 2.5.12 (**Developability).** A complex of groups $\mathscr{G}(\mathscr{Y})$ is said to be developable if it is isomorphic to a complex of groups associated to an action (in the sense above) of a group G on a scwol $\mathscr{X}$ with $G \backslash \mathscr{X} = \mathscr{Y}$.

Theorem 2.5.13. ([23, Theorem 2.13, III.*C* (**The Basic Construction**)]) *Suppose $\mathscr{G}(\mathscr{Y}) = (G_\sigma, \psi_a, g_{a,b})$ is a complex of groups over a scwol $\mathscr{Y}$.*

1. *Suppose G is a group. Canonically associated to each morphism $\phi : \mathscr{G}(\mathscr{Y}) \to G$ there is an action of G on a scwol $D(\mathscr{Y}, \phi)$ with quotient $\mathscr{Y}$. ($D(\mathscr{Y}, \phi)$ is called the development of $\mathscr{Y}$ with respect to ϕ.) If ϕ is injective on local groups then $\mathscr{G}(\mathscr{Y})$ is the complex of groups associated to the action of G on $D(\mathscr{Y}, \phi)$ (with respect to canonical choices) and $\mathscr{G}(\mathscr{Y}) \to G$ is the associated morphism.*

2. *If $\mathscr{G}(\mathscr{Y})$ is a complex of groups associated to an action of G on a scwol $\mathscr{X}$ (with respect to some choices) and if $\phi : \mathscr{G}(\mathscr{Y}) \to G$ is the associated morphism, then there is a G-equivariant isomorphism $D(\mathscr{Y}, \phi) \to \mathscr{X}$ which projects to the identity on $\mathscr{Y}$.*

We get an immediate corollary of the basic construction as follows. This is an algebraic condition for a complex of groups to be developable. One notes that not all complex of groups is developable (see [23, Chapter II.12, Examples 12.17, (5) and (6)]).

Corollary 2.5.14. ([23, Corollary 2.15]) *A complex of groups $\mathscr{G}(\mathscr{Y})$ is developable if and only if there exists a morphism ϕ from $\mathscr{G}(\mathscr{Y})$ to some group G that is injective on local groups.*

2.5.2 The fundamental group of a complex of groups

Suppose $\mathscr{G}(\mathscr{Y}) = (G_\sigma, \psi_a, g_{a,b})$ is a complex of groups over a connected scwol $\mathscr{Y}$. Let $|\mathscr{Y}|$ be the geometric realization of $\mathscr{Y}$ and T be a maximal tree in the 1-skeleton $|\mathscr{Y}|^{(1)}$ of $|\mathscr{Y}|$. Then 'the' fundamental group of $\mathscr{G}(\mathscr{Y})$ is a group, denoted by $\pi_1(\mathscr{G}(\mathscr{Y}), T)$ and is generated by

$$\bigsqcup_{\sigma \in V(\mathscr{Y})} G_\sigma \bigsqcup E^{\pm}(\mathscr{Y})$$

subject to the relations

$$\begin{cases} \text{the relations in the groups } G_\sigma \\ (a^+)^- = a^- \text{ and } (a^-)^- = a^+ \\ (ab)^+ = b^+ a^+ g_{a,b}, \ \forall\ (a,b) \in E^{(2)}(\mathscr{Y}) \\ \psi_a(g) = a^- g a^+, \ \forall\ g \in G_{i(a)} \\ a^+ = 1, \ \forall\ a \in T \end{cases}$$

Theorem 2.5.15. ([23, Proposition 3.9, III.$\mathcal{C}$]) *A complex of groups $\mathscr{G}(\mathscr{Y})$ over a connected scwol $\mathscr{Y}$ is developable if and only if each of the natural homomorphisms $G_\sigma \to \pi_1(\mathscr{G}(\mathscr{Y}),T)$ is injective.*

Now we will talk about the universal covering of a developable complex of groups over a simplicial complex Y (see Definition 2.5.7).

Definition 2.5.16 (**The Universal covering of a developable complex of groups**). Suppose $\mathscr{G}(\mathscr{Y})$ is a developable complex of groups over a connected simplicial complex Y, where $\mathscr{Y}$ is the corresponding scwol of Y. Suppose T is a maximal tree in $|\mathscr{Y}|^{(1)}$ (equivalently, in the first barycentric subdivision of Y) and $G = \pi_1(\mathscr{G}(\mathscr{Y}),T)$ is the fundamental group of $\mathscr{G}(\mathscr{Y})$. Let $\tau \subseteq \sigma \subseteq Y$ correspond to an edge a. Then we assume that $i_T : \mathscr{G}(\mathscr{Y}) \to G$ is the natural morphism mapping each element of the local group G_σ to the corresponding generator of G and each edge a to the generator $a^- = i_T(a)$ (see Proposition 2.5.15). Let

$$B := \bigsqcup_{\sigma \subseteq Y} (G \times \sigma)/\sim .$$

where for $g \in G,\ g' \in G_\sigma,\ x \in \sigma$ and $\sigma \subseteq Y$, we have $(g,x) \sim (gi_T(g'),x)$; also if $\tau \subseteq \sigma$ correspond to the edge a and $i_{\tau,\sigma} : \tau \to \sigma$ is the natural inclusion then for $g \in G,\ y \in \tau$, we have $(g, i_{\tau,\sigma}(y)) \sim (gi_T(a), y)$.

There is a natural left multiplication action of G on the first factor of B. Here B is the universal cover of $\mathscr{G}(\mathscr{Y})$.

We are interested mostly in combination theorem for complexes of groups. More precisely, we prove in Chapter 5 (see Section 5.6) that under certain restrictions the fundamental group of a complex of groups is hyperbolic. So our main object of study the fundamental group $\pi_1(\mathscr{G}(\mathscr{Y}),T)$ and the natural morphism $i_T : \mathscr{G}(\mathscr{Y}) \to \pi_1(\mathscr{G}(\mathscr{Y}),T)$ rather than the arbitrary morphism which is injective on local groups (see Theorem 2.5.13). We will end this subsection by stating the following two theorems and we will recap some of the significant points regarding complexes of groups via algebraic topology in the application in Section 5.6.

Theorem 2.5.17. ([23, Theorem 3.13, III.C]) *The development $B = D(\mathscr{Y}, i_T)$ above is connected and simply connected simplicial complex.*

Theorem 2.5.18. ([23, Corollary 3.15, III.C]) *Suppose G is a group acting on a simply connected scwol $\mathscr{X}$ with quotient $\mathscr{Y} = G \setminus \mathscr{X}$. Also, we assume that $\mathscr{G}(\mathscr{Y})$ is the complex of groups associated to this action (with respect to some choices). Let T be a maximal tree in the 1-skeleton of the geometric realization of $\mathscr{Y}$. Then G is isomorphic to $\pi_1(\mathscr{G}(\mathscr{Y}), T)$ and $\mathscr{X}$ is equivariantly isomorphic to $D(\mathscr{Y}, i_T)$.*

Chapter 3

Cannon-Thurston maps for morphisms of trees of hyperbolic spaces

This paper makes substantial use of the results and proof techniques of the book [9]. In this books three general constructions are used repeatedly, namely 'flow spaces', 'ladders' and 'boundary flow'. Therefore, we shall briefly recall them here for our reference.

3.1 Flow spaces and their properties

Definition 3.1.1. *(***Flow spaces***) Suppose $u \in V(T)$ and $A \subset X_u$ is a k-quasiconvex subset for some $k \geq 1$. Then by Lemma 2.3.4 and Lemma 2.2.22, A is k'-quasiconvex in $X_{uv} := \pi^{-1}([u,v])$ where $k' = K_{2.2.22}(\delta_0', L_0', k)$. Suppose $R \geq R'_{2.2.13}(\delta_0', k')(\geq R_{2.2.13}(\delta_0', \lambda_0'))$ is fixed. Then the flow space determined by A, with constants k, R, is denoted by $\mathscr{F}l^X(A)$ and is defined inductively as follows: $\mathscr{F}l^X(A)$ consists of a collection $\{A_v : v \in V(S)\}$ where*

- *S is a subtree of T containing u,*
- *$A_u = A$ and*
- *each A_v, $v \in V(S)$, $v \neq u$, is a $2\delta_0$-quasiconvex subset of X_v.*

The induction is on distance from u in T, and S and the sets A_v's are simultaneously constructed in the process.

Base of induction: *For each $v \in V(T)$ which is connected by an edge e to u we check if $A'_v := N_R(A) \cap X_v \neq \emptyset$ (neighborhood is considered in X_{uv}) then we include the segment $[u,v]$ in S and we let $A_v = hull(A'_v)$ where hull is considered in X_v;*

otherwise, we do not include $[u,v]$ in S. Thus by the first step of induction we get a subtree of T contained in $N_1(u)$.

Induction step: *Suppose $v \in V(S)$ with $d(u,v) = n$. Then for each $w \in V(T)$ which is connected to v by an edge e', say, such that $d_T(u,w) = n+1$ we check if $A'_w := N_R(A) \cap X_w \neq \emptyset$ (neighborhood is considered in X_{vw}), then we include the segment $[v,w]$ in S and define $A_w = hull(A'_w)$ where hull is considered in X_w; otherwise, we do not include $[v,w]$ in S.*

Let us see three fundamental properties of $\mathscr{F}l^X(A)$ as follows. Suppose $[v,w]$ is an edge in T such that $d_T(u,v) < d_T(u,w)$.

Property 1: Suppose $v \in S$ and $w \notin S$. Then by construction $N_R(A_v) \cap X_w = \emptyset$ (in X_{vw}-metric). In particular, A_v and X_w are $R_{2.2.13}(\delta'_0, \lambda'_0)$-separated in X_{vw}. Again A_v and X_w are λ'_0-quasiconvex in X_{vw} (see Lemma 2.3.4 (2)). Then by Lemma 2.2.13, the pair (A_v, X_w) is $C := D_{2.2.13}(\delta'_0, \lambda'_0)$-cobounded in X_{vw}.

Property 2: Suppose $v, w \in S$. Then $A_w \subseteq N_K(A_v)$ for some uniform constant K depending on k, R, where the neighborhood is considered in X_{vw}.

Proof Let $x \in A_w$. Then $\exists\, x_1, x_2 \in A'_w$ and $x \in [x_1, x_2]_{X_w}$. Let $y_1, y_2 \in A_v$ such that $d_{X_{vw}}(x_i, y_i) \leq R,\ i = 1,2$. Note that if $v = u$, then by Lemma 2.3.4 and Lemma 2.2.22, $A_u = A$ is $K_{2.2.22}(\delta'_0, L'_0, k)$-quasiconvex in X_{vw}, and if $v \neq u$ then A_v is $K_{2.2.22}(\delta'_0, L'_0, 2\delta_0)$-quasiconvex in X_{vw}. Let

$$K' = max\{K_{2.2.22}(\delta'_0, L'_0, k), K_{2.2.22}(\delta'_0, L'_0, 2\delta_0)\}.$$

Note that A_w is also K'-quasiconvex in X_{vw}. Then by slimness of quadrilateral in X_{vw} with vertices x_1, x_2, y_1 and y_2, there is $x' \in A_v$ such that $d_{X_{vw}}(x, x') \leq D_{2.2.2}(\delta'_0, L'_0, L'_0) + K' + R + 2\delta'_0 =: K$ (say).

Property 3: Suppose $v, w \in S$. Then $Hd_{X_{vw}}(P_{X_{vw}X_w}(A_v), A_w) \leq \varepsilon$ for some uniform constant ε depending on k, R.

Proof: Property (2) says that $A_w \subseteq N_{2K'}(P_{X_{vw}X_w}(A_v))$ (in X_{vw}-metric). Again by construction $P_{X_{vw}X_w}(A_v) \subseteq A_w$. So $Hd_{X_{vw}}(P_{X_{vw}X_w}(A_v), A_w) \leq 2K' =: \varepsilon$.

As a consequence of Proposition 2.3.8 we have the following

Proposition 3.1.2. *Consider the map ρ as in Remark 2.3.7 for the subset $\mathscr{F}l^X(A)$. Then there is a constant $L_{3.1.2}(k)$ depending on k such that ρ can be extended to a $L_{3.1.2}(k)$-coarsely Lipschitz retraction $X \to \mathscr{F}l^X(A)$.*

3.1.1 Ladders

Ladder is a special type of flow space whose fibers are geodesic segments in the respective fibers. Construction of a ladder given any geodesic segment is similar to that of flow space. For a geodesic segment α, let us denote the end points of α by α^- and α^+.

Definition 3.1.3 (Ladder). Suppose $u \in V(T)$ and α is a geodesic segment in X_u. Since α is δ_0-quasiconvex, we fix $R = R_{2.2.13}(\delta_0', \delta_0)$. Now the ladder determined by α is denoted by $\mathfrak{L}^X(\alpha)$ and is defined inductively as follows: $\mathfrak{L}^X(\alpha)$ consists of a collection $\{\alpha_v : v \in V(S)\}$ where

- S is a subtree of T containing u,
- $\alpha_u = \alpha$,
- each α_v, $v \in V(S)$, $v \neq u$ is a geodesic segment of X_v.

The induction is on distance from u in T, and S and α_v's are simultaneously constructed in the process.

Base of induction: For each $v \in V(T)$ which is connected by an edge to u we check if $N_R(\alpha) \cap X_v \neq \emptyset$ (neighborhood is considered in X_{uv}) then we let $\alpha_v := [P_{X_{uv}X_v}(\alpha^-), P_{X_{uv}X_v}(\alpha^+)]_{X_v}$ and we include the segment $[u,v]$ in S; otherwise, we do not include $[u,v]$ in S. Thus by the first step of induction we get a subtree of T contained in $N_1(u)$.

Induction step: Suppose $v \in V(S)$ with $d_T(u,v) = n$. Then for each $w \in V(T)$ which is connected to v by an edge such that $d_T(u,w) = n+1$, we check if $N_R(\alpha_v) \cap X_w \neq \emptyset$ (neighborhood is considered in X_{vw}), then we include the edge $[v,w]$ in S and define $\alpha_w := [P_{X_{vw}X_w}(\alpha_v^-), P_{X_{vw}X_w}(\alpha_v^+)]_{X_w}$; otherwise, we do not include $[v,w]$ in S.

We have the following three fundamental properties for ladder as we had for flow space. The proof is similar to that of flow space. Suppose $[v,w]$ is an edge in T such that $d_T(u,v) < d_T(u,w)$.

Property 1: Suppose $v \in V(S)$ and $w \notin V(S)$. Then the pair (α_v, X_w) is C-cobounded in X_{vw} where $C = D_{2.2.13}(\delta_0', \delta_0)$.

Property 2: Let $v, w \in V(S)$. Then $\alpha_w \subseteq N_K(\alpha_v)$ for some uniform constant K.

Property 3: Suppose $v, w \in V(S)$. Then $Hd_{X_{vw}}(P_{X_{vw}X_w}(\alpha_v), \alpha_w) \leq \varepsilon$ for some uniform constant ε.

Hence as a consequence of Proposition 2.3.8 we have the following.

Proposition 3.1.4. *Consider the map ρ as in Remark 2.3.7 for the subset $\mathfrak{L}^X(\alpha)$. Then there is a uniform constant $L_{3.1.4}$ such that ρ can be extended to a $L_{3.1.4}$-coarsely Lipschitz retraction $X \to \mathfrak{L}^X(\alpha)$.*

In a similar way one can define ladder determined by a geodesic ray or line. Since we will not directly use ladder determined by a geodesic line in this thesis, so we will define the same only for geodesic ray as follows.

Definition 3.1.5 (Semi-infinite ladder). Suppose α is a geodesic ray in X_u for some $u \in V(T)$. Construction is same as ladder with following changes.

Suppose $v \in V(T)$ such that $d_T(u,v) = 1$. If $N_R(\alpha) \cap X_v \neq \emptyset$ and is of finite diameter in X_{uv}, then take a point, say, $\alpha_v^+ \in N_R(\alpha) \cap X_v$ which is farthest from $P_{X_{uv}X_v}(\alpha(0))$. Now we set $\alpha_v := [P_{X_{uv}X_v}(\alpha(0)), \alpha_v^+]_{X_v}$. If $N_R(\alpha) \cap X_v \neq \emptyset$ and is of infinite diameter in X_{uv}, then it is not hard to see that $\alpha(\infty) \in \Lambda(X_{eu})$. Then we set α_v to be a geodesic ray in X_v joining $P_{X_{uv}X_v}(\alpha(0))$ and $\partial\vartheta_{ev}(\partial\vartheta_{eu}^{-1}(\alpha(\infty)))$. Let us denote the semi-infinite ladder determined by α by $\mathfrak{L}^X(\alpha, \alpha(\infty))$. We put $\alpha(\infty)$ to emphasize that α is a geodesic ray.

Remark 3.1.6. Conditions $(1)-(4)$ of Proposition 2.3.8 are satisfied by the subset $\mathfrak{L}^X(\alpha, \alpha(\infty))$ for some uniform constants.

3.2 Boundary of X

In general it is difficult to describe the geodesic rays in X. However, one of the main result of this subsection is the following theorem that gives a rough understanding of the points of ∂X.

Theorem 3.2.1. *Suppose $\xi \in \partial X$. Then there is a sequence $\{x_n\}$ in X such that $\lim^X_{n\to\infty} x_n = \xi$ along with one of the following additional properties:*
(1) $\{\pi(x_n)\}$ is a constant sequence or
(2) there is a geodesic ray α in T such that $\pi(x_n) \in \alpha$ for all $n \in \mathbb{N}$ and $\lim^T_{n\to\infty} \pi(x_n) = \alpha(\infty)$.

We postpone the proof of the theorem to collect a couple of facts needed for the proof.

Lemma 3.2.2. *Suppose $\{x_n\}$ is an unbounded sequence in X such that $\lim^X_{n\to\infty} x_n \in \partial X$. Suppose S is the convex hull in T of the set $\{\pi(x_n) : n \in \mathbb{N}\}$ and that there is a vertex of infinite degree in S. Suppose u is any such vertex and $\{x_{n_k}\}$ is any subsequence of $\{x_n\}$ such that $[u, \pi(x_{n_k})]_T \cap [u, \pi(x_{n_l})]_T = \{u\}$ for $k \neq l$. Let e_k be the edge on $[u, \pi(x_{n_k})]_T$ incident on u. Then for all $k \in \mathbb{N}$, there is $x'_k \in X_{e_k u}$ such that $\lim^X_{n\to\infty} x_n = \lim^X_{n\to\infty} x'_n$.*

Moreover, suppose that the subsequence $\{x_{n_k}\}$ *is chosen (*see Remark 3.2.3*) in such a way that the sets* $X_{e_k u}$ *converges to a point of* ∂X_u*, and suppose* x''_k *is an arbitrary point of* $X_{e_k u}$ *for all* $k \in \mathbb{N}$*. Then* $\lim^X_{n\to\infty} x_n = \lim^X_{n\to\infty} x''_n$*.*

Remark 3.2.3. We note that in the case (2) of Lemma 3.2.2, $\{X_{e_k u} : k \in \mathbb{N}\}$ is an infinite, locally finite collection (see Lemma 2.3.3) of uniformly quasiconvex subsets of X_u. Hence, by Lemma 2.2.37 we can always extract a subsequence of $\{X_{e_k u}\}$ satisfying the cobdition of the second part of the lemma.

Proof of Lemma 3.2.2. Fix $x \in X_u$. Then for all $k \in \mathbb{N}$, $[x, x_{n_k}]_X \cap X_{e_k u} \neq \emptyset$. Let x'_k be any point of $[x, x_{n_k}]_X \cap X_{e_k u}$. Since $\{X_{e_k u}\}$ is a locally finite collection of subsets in X_u, by Lemma 2.2.36, $d_{X_u}(x, X_{e_k u}) \to \infty$ as $k \to \infty$. It follows that $d_{X_u}(x, x'_k) \to \infty$ as $k \to \infty$. Then by Lemma 2.2.33, we have $\lim^X_{n\to\infty} x'_n = \lim^X_{n\to\infty} x_n$.

Moreover if $X_{e_k u} \to \xi \in \partial X_u$ as $k \to \infty$ then for any choices of $x''_k \in X_{e_k u}$, $k \in \mathbb{N}$, we have $\lim^{X_u}_{n\to\infty} x'_n = \lim^{X_u}_{n\to\infty} x''_n$. Since the inclusion $X_u \to X$ admits the CT map we have $\lim^X_{n\to\infty} x''_n = \lim^X_{n\to\infty} x'_n = \lim^X_{n\to\infty} x_n$. □

Proof of Theorem 3.2.1: Let $\{x'_n\}$ be any sequence in X such that $\lim^X_{n\to\infty} x'_n = \xi$. Let S be the convex hull in T of the set $\{\pi(x'_n) : n \in \mathbb{N}\}$. There are two cases to consider.

Case 1: Suppose S is a locally finite tree, i.e. all its vertices are of finite degree. Note that if S is bounded then there is a subsequence $\{x'_{n_k}\}$ of $\{x'_n\}$ such that $x'_{n_k} \in X_u$ for some $u \in T$. Now suppose S is unbounded. Then, by Lemma 2.2.29, there is a geodesic ray $\alpha : [0, \infty) \to S$. Let $u = \alpha(0)$ and let $\{x'_{n_k}\}$ be subsequence of $\{x'_n\}$ such that $\lim_{k\to\infty} \pi(x'_{n_k}) = \alpha(\infty)$. Fix $x \in X_u$. Let v_k be the nearest point projection of $\pi(x'_{n_k})$ on α. We note that $[x, x'_{n_k}]_X \cap X_{v_k} \neq \emptyset$ for all $k \in \mathbb{N}$. Let $x_k \in [x, x'_{n_k}]_X \cap X_{v_k}$ for all $k \in \mathbb{N}$. Then by Lemma 2.2.33 $\lim^X_{n\to\infty} x'_n = \lim_{k\to\infty} x'_{n_k} = \lim_{k\to\infty} x_k$.

Case 2: Suppose S has a vertex of infinite degree. Then we are done by Lemma 3.2.2.

In the rest of the subsection we prove a few other related results which come to use in the later part of the paper.

Lemma 3.2.4. *Suppose* $\{x_n\}$ *is an unbounded sequence in X such that* $\lim^X_{n\to\infty} x_n$ *exists. Let* $u_n = \pi(x_n)$ *for all* $n \in \mathbb{N}$ *and suppose* $d_T(u, u_n) \to \infty$ *as* $n \to \infty$*. Let* e_n *be the edge on* $[u, u_n]$ *incident on* u_n *for all* $n \in \mathbb{N}$ *and let* x'_n *be a nearest point projection of* x_n *on* $X_{e_n u_n}$ *in* X_{u_n}*. Then* $\lim^X_{n\to\infty} x'_n = \lim^X_{n\to\infty} x_n$*.*

Proof. We will show that there are uniformly quasiconvex subsets, say, Z_n of X containing both x_n and x'_n such that for a fixed $x \in X$, $d_X(x, Z_n) \to \infty$ as $n \to \infty$. Then by Lemma 2.2.23 (1), there is uniform D such that $N_D(Z_n)$ is uniformly qi embedded in X. So geodesics in $N_D(Z_n)$ are uniform quasigeodesic in X. Note that $d_X(x, N_D(Z_n)) \to \infty$ as $n \to \infty$. Hence by stability of quasigeodesic (Lemma 2.2.2), $d_X(x, [x_n, x'_n]_X) \to \infty$ as $n \to \infty$. Therefore, by Lemma 2.2.32, $\lim^X_{n\to\infty} x_n = \lim^X_{n\to\infty} x'_n$.

Finding Z_n: Let $v_n \in [u, u_n]$ such that $d_T(v_n, u_n) = 1$. Since $X_{e_n u_n}$ is uniformly quasiconvex in X_{u_n}, by Lemma 2.2.19 (1), $\text{diam}\{P_{X_{u_n} X_{e_n u_n}}([x_n, x'_n]_{X_{u_n}})\}$ is uniformly bounded. Then it follows that the pair $(X_{v_n}, [x_n, x'_n]_{X_{u_n}})$ is uniformly cobounded in $X_{v_n u_n}$. Let T_n be the connected component of $T \setminus \{v_n\}$ containing u_n. Let $Z_n = \mathfrak{L}^X(\alpha) \cap X_{T_n}$. Consider the map ρ as in Remark 2.3.7 for the subset Z_n. Note that Z_n satisfies Property (1), (2), (3) (as in Definition 3.1.3) for uniform constants. Then it follows from Proposition 2.3.8 (and in addition Lemma 2.1.2) that ρ can be extended to a uniformly coarsely Lipschitz retraction $X \to Z_n$. Since X is hyperbolic, Z_n is uniformly quasiconvex in X (see Lemma 2.2.12). □

Given $\xi \in \partial X$, by Theorem 3.2.1 there is a sequence $\{x_n\}$ such that either $\{\pi(x_n)\}$ is constant or $\lim_{n\to\infty} \pi(x_n) \in \partial T$ and $\lim^X_{n\to\infty} x_n = \xi$. However, these two possibilities are not mutually exclusive, i.e. we may have two different sequences $\{x_n\}$ and $\{x'_n\}$ such that $\lim^X_{n\to\infty} x_n = \lim^X_{n\to\infty} x'_n = \xi$ where $\{\pi(x_n)\}$ is constant but $\{\pi(x'_n)\}$ converges to a point of ∂T. The following lemma records the implication of such an instance.

Lemma 3.2.5. *Suppose $\{x_n\}$, $\{x'_n\}$ are two unbounded sequences in X such that $\lim^X_{n\to\infty} x_n = \lim^X_{n\to\infty} x'_n \in \partial X$, and $\lim^T_{n\to\infty} \pi(x_n) = \xi \in \partial T$. Suppose that the nearest point projection of each $\pi(x'_n)$ on the geodesic ray $[u, \xi)$ is u for some $u \in V(T)$. Then $\mathscr{F}l^X(X_u)$ and $\mathscr{F}l^X(X_v)$ are not cobounded for any vertex $v \in (u, \xi)$.*

Proof. Suppose $v \in [u, \xi)$ is such that $\mathscr{F}l^X(X_u))$ and $\mathscr{F}l^X(X_v)$ are cobounded. Since X is hyperbolic metric space, by Lemma 3.1.2 and Lemma 2.2.12, $\mathscr{F}l^X(X_u)$ and $\mathscr{F}l^X(X_v)$ are uniformly quasiconvex in X. Let v_n be the nearest point projection of $\pi(x_n)$ on $[u, \xi)$. Since $\lim^T_{n\to\infty} \pi(x_n) = \xi$, there is $N \in \mathbb{N}$ such that for $v_n \in [v, \xi)$ for all $n \geq N$, $n \in \mathbb{N}$. Therefore, any geodesic segment $[x'_n, x_n]_X$ has a subsegment, say, α_n joining a point in X_u to a point in X_v. Since $\mathscr{F}l^X(X_u)$, $\mathscr{F}l^X(X_v)$ are cobounded, by Lemma 2.2.16 there is a point of $\mathscr{F}l^X(X_u)$ uniformly close to α_n, for all $n \geq N$. In particular, for any $x \in X$, $d_X(x, [x'_n, x_n]_X)$ is bounded. This is a contradiction, by Lemma 2.2.32 (2), as $\lim^X_{n\to\infty} x_n = \lim^X_{n\to\infty} x'_n \in \partial X$. Hence we are done. □

3.2.1 Boundary flow

Definition 3.2.6. [36, Definition 4.3] (1) Suppose $u, v \in V(T)$ are connected by an edge e. If $\xi_u \in \partial X_u$ is in the iamge of $\partial \vartheta_{eu} : \partial X_e \to \partial X_u$, then $\partial \vartheta_{ev}((\partial \vartheta_{eu})^{-1}(\xi_u)) = \xi_v$, say, is called the boundary flow of ξ_u to X_v (or more precisely ∂X_v).

(2) Suppose $u, v \in V(T)$ are any two vertices and $u_0 = u, u_1, \cdots, u_n = v$ are the consecutive vertices on $[u, v]$. Suppose $\xi_0 \in \partial X_u$ and $\xi_n \in \partial X_v$. We say that ξ_n is the boundary flow of ξ_0 if there are $\xi_i \in \partial X_{u_i}$, $1 \leq i \leq n-1$ such that ξ_i is the boundary flow of ξ_{i-1} for all $1 \leq i \leq n$.

In this case we say that ξ_0 can be flowed to X_v. Clearly boundary flow of a point of ∂X_u to ∂X_v is unique if it exists.

Lemma 3.2.7. [36, Lemma 4.4] *Suppose $u, v \in V(T)$ are any two vertices. Suppose $\xi_u \in \partial X_u$ and $\xi_v \in \partial X_v$. Suppose $\alpha_u \subset X_u$ and $\alpha_v \subset X_v$ are geodesic rays in these vertex spaces such that $\alpha_u(\infty) = \xi_u$ and $\alpha_v(\infty) = \xi_v$. If ξ_v is the boundary flow of ξ_u then $Hd_X(\alpha_u, \alpha_v) < \infty$.*

Proposition 3.2.8. [37, Proposition 2.3] *Suppose $\xi_u \in \partial X_u$ and $\xi_v \in \partial X_v$ are mapped to the same point of ∂X under the CT maps $\partial i_{X_u X} : \partial X_u \to \partial X$ and $\partial i_{X_v X} : \partial X_v \to \partial X$. Then there is a vertex $w \in [u, v]$ such that both ξ_u, ξ_v can be boundary flowed to X_w.*

Definition 3.2.9 (Conical limit). Suppose Z is a subset of a hyperbolic geodesic metric space W. A point $p \in \partial W$ is said to be a conical limit point of Z if some (any) (quasi) geodesic ray, say, $\alpha : [0, \infty) \to Z$ such that $\alpha(\infty) = p$, there is $R \geq 0$ and a sequence $\{z_n\} \subseteq N_R(\alpha) \cap Z$ such that $\lim^W_{n\to\infty} z_n = p \in \partial W$.

Remark 3.2.10. Suppose $\pi : X \to T$ is the tree of hyperbolic spaces under consideration. For any $\xi \in \partial X$, either it is a conical limit point of some vertex space or it is not a conical limit point of any vertex space.

Lemma 3.2.11. *Suppose $\alpha : [0, \infty) \to X$ is a geodesic ray such that $\alpha(\infty)$ is not a conical limit point of any vertex space of X. Then $\pi(\alpha[0, \infty))$ is a infinite locally finite subtree in T and hence contains a geodesic ray in T. Moreover, this ray is unique.*

Proof. As we will see in the proof that this is a result for trees of hyperbolic metric spaces under consideration. So we proof only for $\pi : X \to T$. On contrary, suppose $\exists\, u \in \pi(\alpha[0, \infty))$ such that u is a vertex of infinite degree in $\pi(\alpha[0, \infty))$. Then there is a subsequence $\{r_n\} \subseteq \mathbb{N}$ such that $\alpha(r_n) \in X_u$ for all $n \in \mathbb{N}$. Since X is hyperbolic, $\alpha(\infty)$ is a conical limit point of X_u $-$ which contradicts to our assumption. Hence $\pi(\alpha[0, \infty))$ is a locally finite subtree in T.

Again if $\pi(\alpha[0,\infty))$ is of finite diameter then we have a vertex $u \in V(T)$ with the same conclusion above.

By Lemma 2.2.29, we have $\partial\pi(\alpha[0,\infty)) \neq \emptyset$. Now we show that the geodesic ray in $\pi(\alpha([0,\infty))$ is unique. If it is not and since α is a geodesic ray, then we have a vertex $u \in V(T)$ with the same conclusion as in first paragraph of the proof.

Therefore, we are done. $\square$

Lemma 3.2.12. *Suppose β is a geodesic ray in X_u and α is that in X such that $\alpha(\infty)$ is not a conical limit point of any vertex space. Further, we assume that $\lim^X_{n\to\infty}\alpha(n) = \lim^X_{n\to\infty}\beta(n)$. Then $\beta(\infty)$ has boundary flow in X_v for all vertex $v \in [u,\xi)$ where $\xi \in \partial\pi(\alpha)$.*

Proof. Note that by Lemma 3.2.11, $\pi(\alpha)$ contains an unique ray, say, $[u,\xi)$ for some $\xi \in \partial T$. For the sake of contradiction, let $v,w \in [u,\xi)$ be a vertices such that $d_T(v,w) = 1$ and $\beta(\infty)$ flows in X_v but does not flow in X_w. We will find a k-quasiconvex subset, say, Z in X containing β such that $\pi(Z)\cap[u,\xi) = [u,w]$ for some $w \in V(T)$. Then we will be done as follows. Since $\alpha(\infty)$ is not a conical limit point of any vertex space, we take $r \in \mathbb{R}$ such that $\pi(\alpha|_{[r,\infty)})\cap[u,\xi) = [w,\xi)$ and $d_T(w,\pi(\alpha(r))) > R_{2.2.13}(\delta,k) = R$ where X is δ-hyperbolic. In particular, $d_X(Z,\alpha|_{[r,\infty)}) > R$, and so by Lemma 2.2.13, the pair $(Z,\alpha|_{[r,\infty)})$ is $D_{2.2.13}(\delta,k)$-cobounded in X. Then by Lemma 2.2.16, for all large $n \in \mathbb{N}$, every geodesic joining $\beta(n)$ and $\alpha(n)$ passes through a fixed point in X. This is a contradiction to $\lim^X_{n\to\infty}\alpha(n) = \lim^X_{n\to\infty}\beta(n)$ (see Lemma 2.2.32 (2)).

***Finding** Z*: Note that $\mathfrak{L}^X(\beta,\beta(\infty))\cap X_v$ is a geodesic ray in X_v but $\mathfrak{L}^X(\beta,\beta(\infty))\cap X_w$ is a finite geodesic segment in X_w. Let $w' \in [w,\xi)$ such that $d_T(w,w') = 1$. Suppose T_1 is the connected component of $T\setminus\{w\}$ containing w'. Set $Z = \mathfrak{L}^X(\beta)\cap X_{T_1}$. Consider the map ρ as in Remark 2.3.7 for the subset Z. Since $\mathfrak{L}^X(\beta,\beta(\infty))\cap X_w$ is a finite geodesic segments, conditions $(1)-(4)$ of Proposition 2.3.8 are satisfied by the subset Z (see also Remark 3.1.6) for some uniform constants. Hence by Proposition 2.3.8 that ρ can be extended to uniformly coarsely Lipschitz retraction $X \to Z$. Since X is hyperbolic, Z is a uniformly quasiconvex in X (see Lemma 2.2.12). $\square$

3.3 Morphisms of trees of spaces

Definition 3.3.1. (I) Suppose $\pi_1 : X_1 \to T_1$ and $\pi_2 : X_2 \to T_2$ are two trees of metric spaces. A *morphism* of trees of spaces from X_1 to X_2, for us, consists of the following data:

1. An isometric embedding $\iota : T_1 \to T_2$.

2. A coarsely Lipschitz map $f : X_1 \to X_2$ such that diagram below commutes.

$$\begin{array}{ccc} X_1 & \xrightarrow{f} & X_2 \\ {\scriptstyle \pi_1}\downarrow & \curvearrowright & \downarrow{\scriptstyle \pi_2} \\ T_1 & \xrightarrow{\iota} & T_2 \end{array}$$

(II) A morphism $(f,\iota) : (X_1,T_1) \to (X_2,T_2)$ between trees of metric spaces will be called an isomorphism if the following hold:

1. ι is an isometry,

2. f is a quasiisometry,

3. there is a morphism $(g,\iota^{-1}) : (X_2,T_2) \to (X_1,T_1)$ where g is a quasiisometry,

4. f,g are coarse invereses to each other.

A trivial way to construct examples of morphisms is to take restrictions as defined below.

Example 3.3.2. Suppose $\pi : X \to T$ is a tree of space and $S \subset T$ is a subtree. Let $X_S := \pi^{-1}(S)$. Let $\pi|_{X_S} : X_S \to S$ be the restriction of $\pi : X \to T$ to X_S. We note that (1) in this case X_S is given the induced length metric from X whence the inclusion $X_S \to X$ is Lipschitz; and (2) the inclusions $S \to T$, $X_S \to X$ give a morphism of trees of spaces.

Although the results sought after this section can be formulated and proved for more general morphisms of trees of spaces, we will deal with only a very special type of morphisms as described below. This will include all the examples coming from graphs of groups.

3.3.1 Induced trees of spaces

Suppose $\pi : X \to T$ is a tree of metric spaces, S is a subtree of T and $Y \subset X_S$ such that the restriction of π to Y gives a tree of metric spaces; or equivalently suppose that there is a tree of metric space $\pi_1 : Y \to S$ and a morphism $(f,\iota) : (Y,S) \to (X,T)$ where f and ι are inclusion maps. Then we will say that Y has an *induced tree of metric space* structure from X or simply that Y is an induced tree metric space (obtained from X).

For the rest of this section and the next section we fix the following notation and convention.

Convention 3.3.3.

1. $\pi : X \to T$ is a tree of hyperbolic metric spaces with parameters ϕ, δ_0, L_0 as defined in Definition 2.3.1.
2. $S \subset T$ is a subtree and $\pi_Y : Y \to S$ is a tree of hyperbolic metric spaces with parameters ϕ, δ_0, L_0.
3. $Y \subset X$, π_Y is the restriction of π on Y, and the inclusion $Y \to X$ is ϕ-proper embedding.
4. The inclusions $Y_v \to X_v$, $v \in V(S)$ admit the CT-map.
5. The inclusions $Y_e \to X_e$, $e \in E(S)$, are L-qi embeddings for a constant $L \geq 1$.
6. Both X and Y are proper hyperbolic geodesic metric spaces.

We shall refer to $\pi_Y : Y \to S$ where $Y \subset X$ as above as an ***induced (sub)tree of spaces*** **satisfying property $\mathscr{HC}$.**

Remark 3.3.4. (1) Under the above hypotheses in Convention 3.3.3 hyperbolicity of Y is ensured. Indeed, since X is hyperbolic, $\pi : X \to T$ satisfies Bestvina-Feighn's flaring condition (see [6]) which implies the same for Y as $Y \to X$ is proper embedding. Basically the proof of [10, Proposition 5.8] works in this case too. Hence, by [6], Y is hyperbolic.

(2) Since $Y \to X$ is proper embedding, then by Lemma 2.3.2, for all $v \in V(S)$, the inclusions $Y_v \to X_v$ are uniformly properly embedded.

Lemma 3.3.5. *Suppose $\{y_n\}, \{y'_n\}$ are two unbounded sequences of points in Y_u such that $\lim^{Y_u}_{n\to\infty} y_n$, $\lim^{Y_u}_{n\to\infty} y'_n \in \partial Y_u$. If $\lim^{Y}_{n\to\infty} y_n = \lim^{Y}_{n\to\infty} y'_n \in \partial Y$ then $\lim^{X}_{n\to\infty} y_n = \lim^{X}_{n\to\infty} y'_n \in \partial X$.*

Proof. Since the CT maps for the inclusions $Y_u \to X_u$ and $X_u \to X$ exist, by the functoriality of CT-maps (see Lemma 2.2.39), we see that $\lim^{X}_{n\to\infty} y_n$ and $\lim^{X}_{n\to\infty} y'_n$ exist; and they are equal if $\lim^{Y_u}_{n\to\infty} y_n = \lim^{Y_u}_{n\to\infty} y'_n$. Suppose $\lim^{Y_u}_{n\to\infty} y_n \neq \lim^{Y_u}_{n\to\infty} y'_n$. Let α be a geodesic line in Y_u such that $\alpha(-\infty) = \lim^{Y_u}_{n\to\infty} y_n$ and $\alpha(\infty) = \lim^{Y_u}_{n\to\infty} y'_n$. Then by [9, Proposition 8.54 (1)], there is a geodesic ray $[u, \xi)$ in T such that both $\alpha(-\infty)$ and $\alpha(\infty)$ flow in Y_v for all vertex $v \in [u, \xi)$. Since edge spaces are uniformly qi embedded in corresponding edge spaces of X, so both $\alpha(-\infty)$ and $\alpha(\infty)$ have boundary flow in X_v for all vertex $v \in [u, \xi)$ and they are not equal in ∂X_v. Therefore, α is a uniform quasi-geodesic in both X_u and Y_u. Fix $y \in Y_u$. Then by the description of uniform quasi-geodesic given in [9, Proposition 8.49] joining $\alpha(n)$ and $\alpha(-n)$, we can conclude the following. If γ_n's are uniform quasi-geodesic joining $\alpha(-n)$ and $\alpha(n)$ in Y and γ'_n's are that in X, then $Hd_X(\gamma_n, \gamma'_n)$ is uniformly bounded for all $n \in \mathbb{N}$. Since $\lim^Y_{n\to\infty} \alpha(n) = \lim^Y_{n\to\infty} \alpha(-n)$, then by stability of quasi-geodesic, $d_Y(y, \gamma_n) \to \infty$ as $n \to \infty$ (see Lemma 2.2.32 (2)). Since Y is properly embedded in X, $d_X(y, \gamma'_n) \to \infty$ as $n \to \infty$. Thus, by the same lemma, $\lim^X_{n\to\infty} \alpha(n) = \lim^X_{n\to\infty} \alpha'(n)$. Since $Y_u \to X_u$ and $X_u \to X$ admit the CT-maps (see [8] for later one) and by the functoriality property of CT-map (see Lemma 2.2.39), we are through. □

A generalization of Lemma 3.3.5 is the following.

Lemma 3.3.6. *Suppose $\{y_n\}$ is an unbounded sequence in Y such that both $\lim^X_{n\to\infty} y_n$, and $\lim^Y_{n\to\infty} y_n$ exist. Let T_1 be the convex hull of $\{\pi(y_n) : n \in \mathbb{N}\}$ in T. Suppose, moreover, one of the following holds:*

(1) $\{\pi(y_n)\}$ is bounded.

(2) There is a vertex of infinite degree in T_1.

Then there is a point $u \in V(T_1)$ and a sequence $\{y'_n\}$ in Y_u such that $\lim^Y_{n\to\infty} y'_n = \lim^Y_{n\to\infty} y_n$ and $\lim^X_{n\to\infty} y_n = \lim^X_{n\to\infty} y'_n$.

Proof. We note that in case $\{\pi(y_n)\}$ is bounded, either there is constant subsequence of $\{\pi(x_n\}$ or T_1 has a vertex of infinite degree. Hence we may divide the proof into the following two cases:

Case 1. Suppose there is a constant subsequence $\{\pi(y_{n_k})\}$ of $\{\pi(y_n)\}$. Let $u = \pi(y_{n_k})$ for all $k \in \mathbb{N}$. Then $y_{n_k} \in Y_u$ for all $k \in \mathbb{N}$. Let $y'_k = y_{n_k}$, $k \in \mathbb{N}$. We note that $\lim^X_{n\to\infty} y'_n = \lim^X_{n\to\infty} y_n$ and $\lim^Y_{n\to\infty} y'_n = \lim^Y_{n\to\infty} y_n$.

Case 2. Suppose T_1 has a vertex u of infinite degree. Suppose $\{y_{n_k}\}$ is a subsequence of $\{y_n\}$ such that $[u, y_{n_k}]_T \cap [u, y_{n_l}]_T = \{u\}$ for all $k \neq l$. For all $k \in \mathbb{N}$, let e_k be the edge on $[u, y_{n_k}]_T$ which is incident on u. By passing to a further subsequence we may assume that the sequence of quasiconvex sets $Y_{e_k u}$ converges to a point of ∂Y_u and the sequence of quasiconvex sets $\{X_{e_k u}\}$ converges to a point of ∂X_u as $k \to \infty$ (see Remark 3.2.3). Therefore, if we take $y'_k \in Y_{e_k u} \subset X_{e_k u}$ for all $k \in \mathbb{N}$ then, by the last part of Lemma 3.2.2, we get $\lim^X_{n\to\infty} y'_n = \lim^X_{n\to\infty} y_n$ and $\lim^Y_{n\to\infty} y'_n = \lim^Y_{n\to\infty} y_n$. □

Proposition 3.3.7. *Suppose $\{y_n\}, \{z_n\}$ are two unbounded sequences in Y converging to the same point of ∂Y which satisfy the property (1) or (2) of Lemma 3.3.6. If $\lim^X_{n\to\infty} y_n$, $\lim^X_{n\to\infty} z_n$ exist then they are equal.*

Proof. Consider the sequence $\{y_n\}$. By Lemma 3.3.6 we can find a vertex $u \in V(T)$ and a sequence $\{y'_n\}$ in Y_u such that $\lim^Y_{n\to\infty} y_n = \lim^Y_{n\to\infty} y'_n$ and $\lim^X_{n\to\infty} y_n = \lim^X_{n\to\infty} y'_n$. Similarly, we can find a vertex $v \in V(T)$ and a sequence $\{z'_n\}$ in Y_v

such that $\lim^{Y}_{n\to\infty} z_n = \lim^{Y}_{n\to\infty} z'_n$ and $\lim^{X}_{n\to\infty} z_n = \lim^{X}_{n\to\infty} z'_n$. Passing to further subsequences, if necessary, we may assume that $\lim^{Y_u}_{n\to\infty} y'_n$ and $\lim^{Y_v}_{n\to\infty} z'_n$ exist.

Now, it is enough to show that $\lim^{X}_{n\to\infty} y'_n = \lim^{X}_{n\to\infty} z'_n$. Let α_u and α_v be two geodesic rays in Y_u and Y_v respectively such that $\lim^{Y_u}_{n\to\infty} y'_n = \alpha_u(\infty)$ and $\lim^{Y_v}_{n\to\infty} z'_n = \alpha_v(\infty)$. Since the inclusion map $Y_u \to Y$ admits the CT map we have $\lim^{Y}_{n\to\infty} y'_n = \lim^{Y}_{n\to\infty} \alpha_u(n)$ and $\lim^{Y}_{n\to\infty} z'_n = \lim^{Y}_{n\to\infty} \alpha_v(n)$. Similarly, since the inclusion maps $Y_u \to X_u$ and $X_u \to X$ admit the CT maps we have $\lim^{X}_{n\to\infty} y'_n = \lim^{X}_{n\to\infty} \alpha_u(n)$ and $\lim^{X}_{n\to\infty} z'_n = \lim^{X}_{n\to\infty} \alpha_v(n)$. Hence, we are reduced to showing that $\lim^{X}_{n\to\infty} \alpha_u(n) = \lim^{X}_{n\to\infty} \alpha_v(n)$. Note that $\lim^{Y}_{n\to\infty} \alpha_u(n) = \lim^{Y}_{n\to\infty} \alpha_v(n)$ as $\lim^{Y}_{n\to\infty} y'_n = \lim^{Y}_{n\to\infty} z'_n$. This means that $\partial i_{Y_uY}(\alpha_u(\infty)) = \partial i_{Y_vY}(\alpha_v(\infty))$. So by Proposition 3.2.8 there is a vertex $w \in [u,v]$ such that both $\alpha_u(\infty)$ and $\alpha_v(\infty)$ boundary flow to ∂Y_w. Let β and β' be geodesic rays in Y_w such that the boundary flows of $\alpha_u(\infty)$ and $\alpha_v(\infty)$ in ∂Y_w are respectively $\beta(\infty)$ and $\beta'(\infty)$. Then by Lemma 3.2.7 we have $Hd_Y(\alpha_u,\beta) < \infty$, $Hd_Y(\alpha_v,\beta') < \infty$. This implies $Hd_X(\alpha_u,\beta) < \infty$, and $Hd_X(\alpha_v,\beta') < \infty$ respectively since the inclusion $Y \to X$ is Lipschitz. However, $Hd_Y(\alpha_u,\beta) < \infty$ implies $\lim^{Y}_{n\to\infty} \alpha_u(n) = \lim^{Y}_{n\to\infty} \beta(n)$. Similarly we have $\lim^{Y}_{n\to\infty} \alpha_v(n) = \lim^{Y}_{n\to\infty} \beta'(n)$, $\lim^{X}_{n\to\infty} \alpha_u(n) = \lim^{X}_{n\to\infty} \beta(n)$ and $\lim^{X}_{n\to\infty} \alpha_v(n) = \lim^{X}_{n\to\infty} \beta'(n)$. Thus it is enough to show that $\lim^{X}_{n\to\infty} \beta(n) = \lim^{X}_{n\to\infty} \beta'(n)$. Note that $\lim^{Y}_{n\to\infty} \beta(n) = \lim^{Y}_{n\to\infty} \beta'(n)$ as we had $\lim^{Y}_{n\to\infty} \alpha_u(n) = \lim^{Y}_{n\to\infty} \alpha_v(n)$. Therefore, we can apply Lemma 3.3.5 to the sequences $\{\beta(n)\}$ and $\{\beta'(n)\}$ to finish the proof. □

3.3.2 Induced trees of spaces with projection hypothesis

Projection hypothesis: There is a constant $R_0 \geq 0$ such that for all $v \in V(S)$ and $e \in E(S)$ where e is incident on v, and for all $x \in Y_v$ we have

$$d_{X_v}(P_{X_vX_{ev}}(x), P_{Y_vY_{ev}}(x)) \leq R_0.$$

Remark 3.3.8. (1) For results proved in Section 3.4 we shall explicitly mention where the projection hypothesis is needed. mention it all the time.

(2) Although the projection hypothesis may seem unnatural it holds in the following situations:

- If for all $v \in V(S)$, Y_v is uniformly qi embedded (or equivalently uniformly quasiconvex) in X_v then the projection hypothesis holds See Lemma 2.2.24.
- Suppose for all $e \in E(S)$, the qi embeddings $Y_e \to X_e$ are all uniform quasi-isometries and there is a proper map $\phi : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ such that for all $v \in V(S)$,

the inclusions $Y_v \to X_v$ satisfy uniform Mitra's criterion with the function ϕ (see Definition 2.2.41). Then the projection hypothesis holds. Indeed, by Lemma 2.2.44, it is enough to show that for all $v \in V(S)$ the inclusions $Y_v \to X_v$ are uniformly properly embedded. This fact is actually Remark 3.3.4 (2).

3.4 Proof of Theorem 1.1.6

Note that X_S is hyperbolic by [6] where $X_S := \pi^{-1}(S)$. By [9, Theorem 8.11], the CT map exists for the inclusion $i : X_S \hookrightarrow X$. Therefore, it is enough to show that the CT map exists for the inclusion $i : Y \to X$ where both X and Y have same base T. We are going to assume this for the rest of the proof.

Convention 3.4.1. For the rest of the section we shall assume that the trees of spaces X and Y satisfy the following:
(1) Properties mentioned in Convention 3.3.3.
(2) The projection hypothesis (see Subsection 3.3.2).
(3) $S = T$.

Lemma 3.4.2. *Let $u \in V(T)$ and $e' \in E(T)$ be an edge incident on u. Suppose $A \subseteq Y_{e'u}$ is a k-quasiconvex in Y_u for some $k \geq 0$. Let $\mathscr{F}l^Y(A)$ be the flow space of A inside Y with constants $k, R \geq R'_{2.2.13}(\delta'_0, k')$ as in Definition 3.1.1. Let $e = [v,w]$ be an edge in T such that $d_T(u,v) < d_T(u,w)$. Then we have the following.*

(1) Suppose $v \in \pi(\mathscr{F}l^Y(A))$ and $w \notin \pi(\mathscr{F}l^Y(A))$. Then the pair (A_v, X_w) is C_1-cobounded for some uniform constant C_1.

Suppose $v, w \in \pi(\mathscr{F}l^Y(A))$.

(2) Then $A_w \subseteq N_{K_1}(A_v)$ (in X_{vw}-metric) for some uniform constant K_1, and

(3) $Hd_{X_{vw}}(P_{X_{vw}X_w}(A_v), A_w) \leq \varepsilon_1$ for some uniform constant ε_1.

Proof. First of all, we note that $P_{Y_{vw}Y_w}(A_v) = \vartheta_{ew}(\vartheta_{ev}^{-1}(P_{Y_vY_{ev}}(A_v)))$ and $P_{X_{vw}X_w}(A_v) = \vartheta_{ew}(\vartheta_{ev}^{-1}(P_{X_vX_{ev}}(A_v)))$. By Projection hypothesis $Hd_{X_v}(P_{X_vX_{ev}}(A_v), P_{Y_vY_{ev}}(A_v))$ is uniformly bounded.

(1) Property (1) of $\mathscr{F}l^Y(A)$ (as in Definition 3.1.1) says that the pair (A_v, Y_w) is C-cobounded in Y_{vw} where $C = D_{2.2.13}(\delta'_0, k')$, i.e., $\text{diam}\{P_{Y_{vw}Y_w}(A_w)\} \leq C$. Since ϑ_{ev} and ϑ_{ew} are uniformly qi embeddings, then by the first paragraph of the proof, we have a uniform bound on $P_{Y_vY_{ev}}(A_v)$. It then follows that $\text{diam}\{P_{X_vX_{ev}}(A_v)\}$ is uniformly bounded, and so $P_{X_{vw}X_w}(A_v)$ is uniformly bounded. Therefore, by Lemma 2.2.18 we are done.

(2) Since $Y_{vw} \subseteq X_{vw}$, we can take $K = K_1$ as in Property (2) of Definition 3.1.1.

(3) Note that $Hd_{Y_{vw}}(P_{Y_{vw}Y_w}(A_v), A_w) \leq \varepsilon$ (Property (3) of $\mathscr{F}l^Y(A)$). Then by first paragraph of the proof and the fact that ϑ_{ev} and ϑ_{ew} are uniformly qi embeddings, we are through. □

Lemma 3.4.3. *Suppose $\mathscr{F}l^Y(A)$ is the flow space as in Lemma 3.4.2. Consider the map ρ as in Remark 2.3.7 for the subset $\mathscr{F}l^Y(A)$. Then:*

(1) ρ can be extended to a uniformly coarsely Lipschitz retraction $X \to \mathscr{F}l^Y(A)$.

(2) $\mathscr{F}l^Y(A)$ is uniformly quasiconvex in Y as well as in X.

Proof. Since X and Y are hyperbolic metric spaces, (2) follows from (1) and Proposition 3.1.2 in addition to Lemma 2.2.12.

We first observe that $Y_w \cap \mathscr{F}l^Y(A)$ is a uniformly quasiconvex subset of X_w for all $w \in \pi(\mathscr{F}l^Y(A))$. This is clear for $u = w$ since Y_{eu} is uniformly qi embedded in X_u. Let $v \in [u,w]$ such that $d_T(v,w) = 1$. Note that A_w is $2\delta_0$-quasiconvex in Y_w and $Hd_{Y_{vw}}(P_{Y_{vw}Y_w}(A_v), A_w) \leq \varepsilon$. So $P_{Y_{vw}Y_w}(A_v)$ is uniformly quasiconvex in Y_w. Since the edge spaces of Y are uniformly qi embedded in the corresponding vertex spaces of X and $P_{Y_{vw}Y_w}(A_v) \subseteq Y_{ew}$, so $P_{Y_{vw}Y_w}(A_v)$ is uniformly quasiconvex in X_w. Hence, $Hd_{X_{vw}}(P_{Y_{vw}Y_w}(A_v), A_w) \leq Hd_{Y_{vw}}(P_{Y_{vw}Y_w}(A_v), A_w) \leq \varepsilon$ implies that A_w is uniformly quasiconvex in X_w. Let the uniform quasiconvexity constant be k_1.

Therefore, by Lemma 3.4.2, conditions $(1)-(4)$ of Proposition 2.3.8 are satisfied by $\rho : \cup_{v \in v(T)} X_v \to \mathscr{F}l^Y(A)$ for some uniform constants where ρ is as in Remark 2.3.7; whence (1) follows. □

The following theorem gives a comparison between X-geodesics and Y-geodesics joining the same pair of end points.

Theorem 3.4.4. *There are constants D, D' such that the following hold:*
Suppose $u, v \in T$ and $w \in [u,v]$. Let e be the edge on $[w,v]$ (or $[u,w]$) incident on w.

(1) If $y \in Y_u$ and $y' \in Y_v$ belong to some edge spaces in the respective vertex spaces then $N_D(Y_{ew}) \cap [y,y']_X \neq \emptyset$

(2) If $z \in Y_v$ belong to some edge space of Y_v and $z' \in Y_{ew}$ then $d(u, \pi([z,z']_X)) \geq d(u, \pi(\mathscr{F}l^Y(Y_{ew}))) - D'$.

Proof. We first construct a subspace which is the union of flow spaces in Y in the directions away from $[u,v]$. This then will be a uniformly quasiconvex set in X containing the various points given in the theorem. The very nature of the quasiconvex set will help us to prove the theorem. We shall assume that e is on $[w,v]$ since the proof for the other case is absolutely analogous.

For any edge e incident on u, we assume that Y_{eu} is k-quasiconvex in both X_u and Y_u. We also fix $R' = R_{2.2.13}(\delta_0, k) = 2k + 5\delta_0$.

Suppose $u = u_1, u_2, \cdots, u_n = v$ is the sequence of vertices on the geodesic $[u_1, u_n] \subset T$ with $d(u_1, u_i) = i-1$, $1 \le i \le n$. Suppose e_i is the edge joining u_i and u_{i+1}. Then for all $1 \le i \le n$ we first define a uniformly quasiconvex subset $A_i \subset X_{u_i}$ as follows:

Construction of A_i's:

Let us fix two edges e_0 incident on u_1 and e_n incident on u_n such that $y \in Y_{e_0 u_1}$ and $y' \in Y_{e_n u_n}$.

Type 1. For $1 \le i \le n$ if $Y_{e_i u_i}$ and $Y_{e_{i-1} u_i}$ are R'-separated in X_{u_i} then we let $z_{i-1} \in P_{X_{u_i} Y_{e_i u_i}}(Y_{e_{i-1} u_i})$ and $z_i \in P_{X_{u_i} Y_{e_{i-1} u_i}}(Y_{e_i u_i})$. (Note that projection hypothesis says that the pair $(Y_{e_{i-1} u_i}, Y_{e_i u_i})$ is uniformly cobounded in X_{u_i} if and only if it is so in Y_{u_i}.) In this case we define $A_i = Y_{e_i u_u} \cup Y_{e_{i-1} u_i} \cup [y_i, z_i]_{X_{u_i}}$.

Type 2. On the other hand if $d_{X_{u_i}}(Y_{e_i, u_i}, Y_{e_{i-1} u_i}) \le R'$ then we define $A_i = Y_{e_i, u_i} \cup Y_{e_{i-1} u_i}$.

Properties of A_i's:

Property 1. *First of all, clearly A_i is K_1-quasiconvex in X_{u_i} for some uniform constant K_1 depending on k, δ_0.* This follows from the fact that the edge spaces of Y are k-quasiconvex in the corresponding vertex spaces of X and Lemma 2.2.17.

Property 2. *$Hd_{X_{u_i}}(P_{X_{u_i} X_{e_i u_i}}(A_i), Y_{e_i u_i})$ and $Hd_{X_{u_i}}(P_{X_{u_i} X_{e_{i-1} u_i}}(A_i), Y_{e_{i-1} u_i})$ are uniformly small.* This is clear if A_i is of type 2 using the projection hypothesis. When A_i is of type 1 one has to use Lemma 2.2.19 (1) in addition.

Property 3. Suppose e' is an edge incident on u_i which is not on $[u, v]$. Let $P_{X_{u_i} X_{e' u_i}}(A_i) = B^X_{i,e'}$ and $P_{Y_{u_i} Y_{e' u_i}}(Y_{e_i u_i} \cup Y_{e_{i-1} u_i}) = B^Y_{i,e'}$. *Then by Lemma 2.2.19 (2), $B^X_{i,e'}$ is K_2-quasiconvex in X_{u_i}* for some K_2 depending on k, δ_0. By projection hypothesis and (1) of Lemma 2.2.19, $Hd_{X_{u_i}}(B^X_{i,e'}, B^Y_{i,e'}) \le R_1$ for some R_1 depending on R_0, k, δ_0 where R_0 is coming from projection hypothesis. *Hence $B^Y_{i,e'}$ is K_3-quasiconvex in both Y_{u_i} and X_{u_i} for some K_3 depending on K_2, R_1, δ_0.*

Let K be maximum of all quasiconvexity constants we have above and $R = R_{2.2.13}(\delta'_0, K) = 2K + 5\delta'_0 > 2k + 5\delta_0 = R'$. Given an edge e incident on u and a K-quasiconvex subset $A \subseteq Y_{eu}$, we fix $\mathscr{F}l^Y(A)$ is flow space determined by A in Y as in Definition 3.1.1 with constant k, R.

Construction of the flow spaces

Now we construct (some modified) flow spaces of the various A_i's in the direction away from $[u, v]$. Let T_i be the maximal subtree of T such that $T_i \cap [u, v] = u_i$. The modified flow space $\mathscr{A}_i$ of A_i is defined as follows.

Case 1. $i = 1$ and $e_0 = e_1$ or $i = n$ and $e_n = e_{n-1}$: In this case $A_i = Y_{e_{i-1} u_i}$ and we let $\mathscr{A}_i = \mathscr{F}l^Y(A_i) \cap Y_{T_i}$ where $Y_{T_i} = \pi_Y^{-1}(T_i)$.

Case 2: In all other situation we proceed as follows. Suppose e' is an edge connecting u_i to u_i', say, such that e' is not on $[u,v]$. Let $T_{i,e'}$ be the maximal subtree of T containing u_i' and not containing u_i. Now if the pair (A_i, X_{e,u_i}) is R-separated in X_{u_i}, we define the 'flow of A_i' in the direction of u_i' to be $\mathscr{A}_{i,e'} = \emptyset$. Otherwise, we let $A_{i,e'} = \vartheta_{e'u_i'}(\vartheta_{e'u_i}^{-1}(B_{i,e'}^Y))$ and $\mathscr{A}_{i,e'} = \mathscr{F}l^Y(A_{i,e'}) \cap Y_{T_{i,e'}}$. (Note that $Hd_{Y_{u_iu_i'}}(B_{i,e'}^Y, A_{i,e'}) \leq 1$, $B_{i,e'}^Y$ is K_3-quasiconvex in Y_{u_i}. Then by Lemma 2.3.4, one can conclude that $A_{i,e'} \subseteq Y_{e'u_i'}$ is uniformly quasiconvex in $Y_{u_i'}$. Without introducing another constant, we assume that it is K-quasiconvex as above.) We let $\mathscr{A}_i = \cup \mathscr{A}_{i,e'}$'s where the union is taken over all the edges e' incident on u_i, other than e_i, e_{i-1}.

We claim that $\mathfrak{Q}_{[u,v]} = (\cup_i A_i) \cup (\cup_i \mathscr{A}_i)$ is quasiconvex in X. Since X is hyperbolic, by Lemma 2.2.12, it is enough to provide a coarsely Lipschitz retraction $X \to \mathfrak{Q}_{[u,v]}$.

Let $X_{vsp} = \cup_{s \in V(T)} X_s$. We consider the map $\rho : X_{vsp} \to \mathfrak{Q}_{[u,v]}$ as in Remark 2.3.7. Since X_{vsp} is 1-dense in X, by Lemma 2.1.2, we need to show a uniform bound on $d_X(\rho(x), \rho(y))$ where $x, y \in X_{vsp}$ such that $d_X(x,y) \leq 1$. Let $\pi(x) = v'$ and $\pi(y) = w'$. We consider the following cases depending on position of v', w'.

Case 1: Suppose both $v', w' \notin [u,v]$. Since $d_T(v', w') \leq 1$, let u_i be the nearest point projection of v' and w' on $[u,v]$. If $\mathscr{A}_{i,e'} \neq \emptyset$ then by Lemma 3.4.3 (1), $d_X(\rho(x), \rho(y))$ is uniformly bounded. If $\mathscr{A}_{i,e'} = \emptyset$ then by construction of $\mathscr{A}_{i,e'}$, the pair $(A_i, X_{e'u_i})$ is R-separated in X_{u_i}. Then by Lemma 2.2.13, the pair $(A_i, X_{e'u_i})$ is D-cobounded in X_{u_i} for $D = 2K + 7\delta_0$ whence the pair $(A_i, X_{e'u_i'})$ is so in $X_{u_iu_i'}$. Then by definition of ρ, $d_X(\rho(x), \rho(y)) \leq D$.

Case 2: Suppose both $v', w' \in [u,v]$. If $v' = w'$, then by Lemma 2.2.21 (1), $d_X(\rho(x), \rho(y)) \leq C_{2.2.21}(\delta_0, K)$. If $v' \neq w'$. then $d_T(v', w') \leq d_X(x,y) \leq 1$ implies $d_T(v', w') = 1$ and $x \in X_{v'}, y \in X_{w'}$. It follows from property (2) and Lemma 2.3.5 (2) that $d_X(\rho(x), \rho(y))$ is uniformly bounded.

Case 3: Lastly, without loss of generality, we assume that $v' \in [u,v]$ and $w' \notin [u,v]$. If $v' = u$ or $v' = v$, then by Lemma 3.4.3 (1), $d_X(\rho(x), \rho(y))$ is uniformly bounded. Now let $v' = u_i$ and $w' = u_i'$ for some i and $e' = [u_i, u_i']$ to make the notation consistent above. If $\mathscr{A}_{i,e'} = \emptyset$ then we are through as explained in Case 1. Now suppose $\mathscr{A}_{i,e'} \neq \emptyset$. By construction $d_{X_{u_i}}(A_i, X_{e'u_i}) \leq R$. Then by Lemma 2.2.13 (2), $B_{i,e'}^X \subseteq N_{R_2}(A_i)$ in X_{u_i} for some $R_2 = 2K + 3\delta_0 + R$ whence by Property (3), $B_{i,e'}^Y \subseteq N_{R_1+R_2}(A_i)$ in X_{u_i}. Thus it follows that $A_{i,e'} \subseteq N_{R_1+R_2+1}(A_i)$ in $X_{u_iu_i'}$. Also from Property (3), it follows that $Hd_{X_{u_iu_i'}}(P_{X_{u_iu_i'}X_{u_i'}}(A_i), A_{i,e'})$ is uniformly bonded. Hence by Lemma 2.3.5 (1), we are through.

By abusing notation, we assume that $\rho : X \to \mathfrak{Q}_{[u,v]}$ is L-coarsely Lipschitz retraction for some uniform constant L. Therefore, $\mathfrak{Q}_{[u,v]}$ is K'-quasiconvex in X where K' depends on L and the hyperbolicity constant of X.

Now we are ready to prove the two statements of the theorem.

For (1) we note that $[y,y']_X \cap X_{ew} \neq \emptyset$. Let $y_1 \in [y,y']_X \cap X_{ew}$. By the quasiconvexity of $\mathfrak{Q}_{[u,v]}$ there is a point $y_2 \in \mathfrak{Q}_{[u,v]}$ such that $d_X(y_1,y_2) \leq K'$. Let $w = u_i$. Since $\rho : X \to \mathfrak{Q}_{[u,v]}$ is L-coarsely Lipschitz retraction then $d_X(\rho(y_1),\rho(y_2)) \leq LK' + K$ where $\rho(y_2) = y_2$. It follows that $d_X(y_1,\rho(y_1)) \leq K' + LK' + L = L'$ (say). This means $d_{X_w}(y_1,\rho(y_1)) \leq \phi(L')$. We note that in this case $\rho(y_1) \in A_i$ and $Y_{ew} = Y_{e_iu_i}$. Now, if $\rho(y_1) \in Y_{ew}$ then we are done. However, otherwise, by Lemma 2.2.15, $[y_1,\rho(y_1)]_{X_{u_i}}$ goes through a uniformly small neighborhood of Y_{ew} in X_{u_i} whence (1) follows.

For (2) we appeal to the set $\mathfrak{Q}_{[w,v]}$ instead of the whole collection. Note that $[z,z']_X \subseteq N_{K'}(\mathfrak{Q}_{[w,v]})$. Therefore, $\pi([z,z']_X) \subseteq N_{K'}(\pi(\mathfrak{Q}_{[w,v]}))$ in T. Hence, $d(u,\pi([z,z']_X)) \geq d(u,\pi(\mathfrak{Q}_{[w,v]})) - K'$. However, it is clear from the construction of $\mathfrak{Q}_{[w,v]}$ that $d(u,\pi(\mathfrak{Q}_{[w,v]})) \geq d(u,\pi(\mathscr{F}l^Y(A_{ew})))$ whence (2) follows with $D' = K'$. □

Let $e \in E(T)$ be an edge incident on $u \in V(T)$ and Y_{eu} is k-quasiconvex in Y_u. For the rest of the proof we assume $\mathscr{F}l^Y(Y_{eu})$ is the flow space as in Definition 3.1.1 with constant k and $R = R_{2.2.13}(\delta'_0,k) = 2k + 5\delta'_0$.

Proposition 3.4.5. *Suppose $u \in T$ is a vertex and $y \in Y_u$. Suppose $\{y_n\}$ is an unbounded sequence in Y such that $\lim^Y_{n\to\infty} y_n$ and $\lim^X_{n\to\infty} y_n$ both exist. Let $u_n = \pi(y_n)$ and suppose that $\lim^T_{n\to\infty} u_n = \xi$. Let c_n be the nearest point projection of u_n on $[u,\xi)$ and let e_n be the edge on $[u,c_n]$ incident on c_n. Suppose $z_n \in Y_{e_nc_n} \cap [y,y_n]_Y$. Then $\lim^Y_{n\to\infty} y_n = \lim^Y_{n\to\infty} z_n$ and $\lim^X_{n\to\infty} y_n = \lim^X_{n\to\infty} z_n$.*

Proof. We note that $d_Y(y,z_n) \geq d_T(u,c_n) = d_T(u,[u_n,\xi))$. Since $u_n \to \xi$ we have $d_T(u,[u_n,\xi)) \to \infty$, whence $d_Y(y,z_n) \to \infty$ as $n \to \infty$. Then by Lemma 2.2.33 $\lim^Y_{n\to\infty} z_n = \lim^Y_{n\to\infty} y_n$ as $z_n \in [y,y_n]_Y$. Hence, it remains to prove the second limit.

For all $n \in \mathbb{N}$, let e'_n be the edge on $[u,u_n]$ incident on u_n, and let $e_{n,1},e_{n,2},\cdots,e_{n,t(n)} = e'_n$ be all the successive edges on the geodesic from c_n to u_n when $c_n \neq u_n$, i.e. $u_n \notin [u,\xi)$. First we do the following reduction.

Reduction step: Let y'_n be a nearest point projection of y_n on $Y_{e'_nu_n}$ in Y_{u_n}. Then by Lemma 3.2.4 $\lim^Y_{n\to\infty} y_n = \lim^Y_{n\to\infty} y'_n$, and also by the same lemma and the projection hypothesis we have $\lim^X_{n\to\infty} y_n = \lim^X_{n\to\infty} y'_n$. Therefore, we are reduced to proving $\lim^X_{n\to\infty} y'_n = \lim^X_{n\to\infty} z_n$ and so far we have $\lim^Y_{n\to\infty} y'_n = \lim^Y_{n\to\infty} z_n$.

The following recurring argument in the proof.

Claim: Suppose for all $n \in \mathbb{N}$ there is a set $Z_n \subset Y$ which is uniformly quasiconvex in both X and Y such that $y'_n, z_n \in Z_n$. Then $\lim^X_{n\to\infty} y'_n = \lim^X_{n\to\infty} z_n$.

Proof of claim: Suppose Z_n is k_1-quasiconvex in Y for all $n \in \mathbb{N}$. By Lemma 2.2.23 (1), $N_D^Y(Z_n)$ is uniformly qi embedded in Y for $D = k_1 + 1$. Since Y is properly embedded in X and $N_D^Y(Z_n)$ is uniformly qi embedded in Y, it follows that $N_D^Y(Z_n)$ is uniformly properly embedded in X. Thus by Lemma 2.2.23 (2), $N_D^Y(Z_n)$ is uniformly qi embedded in X. Thus a geodesic, say α_n, joining y'_n, z_n in $N_D^Y(Z_n)$ is a uniform quasigeodesic in both X and Y. Now, $d_Y(y, [y'_n, z_n]_Y) \to \infty$ as $n \to \infty$ since $\lim^Y_{n\to\infty} y'_n = \lim^Y_{n\to\infty} z_n$ (see Lemma 2.2.32 (2)). Therefore, by the stability of quasigeodesics (Lemma 2.2.2) $d_Y(y, \alpha_n) \to \infty$ as $n \to \infty$. Since Y is properly embedded in X, it follows that $d_X(y, \alpha_n) \to \infty$ as $n \to \infty$ whence $d_X(y, [y'_n, z_n]_X) \to \infty$ as $n \to \infty$ again by the stability of quasigeodesics. Thus $\lim^X_{n\to\infty} y_n = \lim^X_{n\to\infty} y'_n = \lim^X_{n\to\infty} z_n$.

The proof is divided into several cases. First we discuss three special cases and then finally we prove the general case using them. Let $e \in E(T)$ incident to w and Y_{ew} is k-quasiconvex in Y_w. Then we consider below the flow space of Y_{ew} as in Definition 3.1.1 with constant k and $R = R_{2.2.13}(\delta_0, k)$.

Case 1. Suppose $u_n \in [u, \xi)$ for all $n \in \mathbb{N}$. In this case $e_n = e'_n$ and the reduction step yields $y'_n \in Y_{e_n c_n}$. Now note that $y'_n, z_n \in Y_{e_n c_n}$ and $\mathscr{F}l^Y(Y_{e_n c_n})$ is uniformly quasiconvex in both Y and X by Lemma 3.4.3 (2). Hence, by the above claim we have $\lim^X_{n\to\infty} y'_n = \lim^X_{n\to\infty} z_n$.

Case 2. Suppose $u_n \notin [u, \xi)$ but $\mathscr{F}l^Y(Y_{e_{n,t(n)} u_n}) \cap Y_{e_n c_n} \neq \emptyset$ for all $n \in \mathbb{N}$. In this case $\mathscr{F}l^Y(Y_{e_{n,t(n)} u_n}) \cup \mathscr{F}l^Y(Y_{e_n c_n}) = Z_n$, say, is uniformly quasiconvex in Y as well in X by Lemma 3.4.3 (2) and the fact that union of two intersecting quasiconvex subsets is quasiconvex in a hyperbolic metric space. Therefore, we are done by the above claim.

Case 3. Suppose $u_n \notin [u, \xi)$ and $\mathscr{F}l^Y(Y_{e_{n,t(n)} u_n}) \cap Y_{e_n c_n} = \emptyset$ for all $n \in \mathbb{N}$. Suppose $c_n = v_{n,1}, v_{n,2}, \cdots$ are the consecutive vertices on geodesic joining c_n to u_n so that each edge $e_{n,j}$ joins $v_{n,j}$ and $v_{n,j+1}$. Suppose $e_{n,i}$ is the closest edge from c_n for which $Y_{e_n c_n} \cap \mathscr{F}l^Y(Y_{e_{n,i} v_{n,i+1}}) = \emptyset$. Let $y''_n \in Y_{e_{n,i} v_{n,i+1}}$ be any point. Now, by the second part of Theorem 3.4.4 (2) there is a uniform constant D' depending only on the parameter of the tree of spaces under consideration such that $d(u, \pi([y'_n, y''_n]_X) \geq d(u, \mathscr{F}l^Y(Y_{e_{n,i} v_{n,i+1}})) - D'$. However, since $Y_{e_n c_n} \cap \mathscr{F}l^Y(Y_{e_{n,i} v_{n,i+1}}) = \emptyset$, so we have $d(u, \mathscr{F}l^Y(Y_{e_{n,i} v_{n,i+1}})) \geq d(u, c_n)$. Thus $\lim_{n\to\infty} d_X(y, [y'_n, y''_n]_X) = \infty$. Hence, by Lemma 2.2.32 (2) $\lim^X_{n\to\infty} y'_n = \lim^X_{n\to\infty} y''_n$. Applying Theorem 3.4.4 (2) to the case $Y = X$ we similarly get $\lim^Y_{n\to\infty} y'_n = \lim^Y_{n\to\infty} y''_n$. Let $Im_Y(y''_n)$ denote the image of y''_n in $Y_{e_{n,i} v_{n,i}}$. Since $d(y''_n, Im(y''n)) = 1$, it follows that $\lim^X_{n\to\infty} y'_n = \lim^X_{n\to\infty} y''_n = \lim^X_{n\to\infty} Im(y''_n)$ and $\lim^Y_{n\to\infty} y'_n = \lim^Y_{n\to\infty} y''_n = \lim^Y_{n\to\infty} Im(y''_n)$.

Next, we may apply the reduction step to $Im(y_n'')$ to find $y_n''' \in Y_{e_{n,i-1}v_{n,i}}$ for each $n \in \mathbb{N}$ so that $\lim^X_{n\to\infty} Im(y_n'') = \lim^X_{n\to\infty} y_n'''$ and $\lim^Y_{n\to\infty} Im(y_n'') = \lim^Y_{n\to\infty} y_n''' = \lim^Y_{n\to\infty} z_n$. Finally, since $\mathscr{F}l^Y(Y_{e_{n,i-1}v_{n,i}}) \cap Y_{e_nc_n} \neq \emptyset$ (by our choice of i) for all $n \in \mathbb{N}$ we have $\lim^X_{n\to\infty} z_n = \lim^X_{n\to\infty} y_n'''$ by Case 2.

Case 4. The general case: Let $S_1 = \{n \in \mathbb{N} : u_n \in [u,\xi)\}$, $S_2 = \{n \in \mathbb{N} \setminus S_1 : \mathscr{F}l^Y(Y_{e_{n,t(n)}u_n}) \cap Y_{e_nc_n} \neq \emptyset\}$, and let $S_3 = \mathbb{N} \setminus (S_1 \cup S_2)$. Now, if any S_i, $1 \le i \le 3$, is infinite then we have have a subsequence $\{n_{ik}\}_{k\in\mathbb{N}}$ of the sequence of natural numbers such that $S_i = \{n_{ik} : k \in \mathbb{N}\}$. Then Case i applies to the subsequence $\{y'_{n_{ik}}\}$ of $\{y_n'\}$ to give $\lim^X_{k\to\infty} y'_{n_{ik}} = \lim^X_{k\to\infty} z_{n_{ik}}$. Thus it follows that $\lim^X_{n\to\infty} y_n' = \lim^X_{n\to\infty} z_n$.

Continuation of the proof of Theorem 1.1.6: Suppose $\{y_n\}, \{y_n'\}$ are two arbitrary unbounded sequences in Y.

Compatible sequences: We will say that $\{y_n\}, \{y_n'\}$ are *compatible* if $\lim^Y_{n\to\infty} y_n = \lim^Y_{n\to\infty} y_n' \in \partial Y$, and $\lim^X_{n\to\infty} y_n, \lim^X_{n\to\infty} y_n'$ both exist.

Therefore, to prove the theorem one has to show that for any compatible sequences $\{y_n\}, \{y_n'\}$ one has $\lim^X_{n\to\infty} y_n = \lim^X_{n\to\infty} y_n' \in \partial X$ (see Lemma 2.2.43). The idea of the proof is that given two compatible sequences $\{y_n\}, \{y_n'\}$ we find a new pair of compatible sequences, say $\{w_n\}$, $\{w_n'\}$, with additional properties so that checking if $\lim^X_{n\to\infty} w_n = \lim^X_{n\to\infty} w_n'$ is easier whereas by construction we have $\lim^X_{n\to\infty} w_n = \lim^X_{n\to\infty} y_n$ and $\lim^X_{n\to\infty} w_n' = \lim^X_{n\to\infty} y_n'$. Sometimes we may need to do this a number of times.

Now, to start the proof suppose $\{y_n\}$ and $\{y_n'\}$ are two compatible sequences. Let $b_n = \pi(y_n)$ and $b_n' = \pi(y_n')$ for all $n \in \mathbb{N}$. Fix $u \in T$. Let $S_1 = Hull(\{b_n : n \in \mathbb{N}\})$ and $S_2 = Hull(\{b_n' : n \in \mathbb{N}\})$. Now, we have the following three possibilities:

Case 1: Both $\{b_n\}$ and $\{b_n'\}$ satisfy the property (1) or (2) of Lemma 3.3.6. In this case the proof follows from Proposition 3.3.7.

Case 2: Exactly one of the sequences $\{b_n\}$ or $\{b_n'\}$ satisfies the property (1) or (2) of Lemma 3.3.6. Without loss of generality, suppose $\{b_n'\}$ satisfies the property (1) or (2) of Lemma 3.3.6 but $\{b_n\}$ does not.

Now, first of all, using Lemma 3.3.6 we can find a vertex u and sequence of points $\{z_n'\}$ in Y_u such that $\lim^Y_{n\to\infty} z_n' = \lim^Y_{n\to\infty} y_n'$ and $\lim^X_{n\to\infty} z_n' = \lim^X_{n\to\infty} y_n'$. Therefore, we shall replace the sequence $\{y_n'\}$ by $\{z_n'\}$ for the purpose of the proof.

Secondly, we note that the subtree S is locally finite and unbounded. Hence, $\partial S_1 \neq \emptyset$ by Lemma 2.2.29. Up to passing to a subsequence, if necessary, we may assume that b_n converges to $\xi \in \partial S_1$ where c_n is the nearest point projection of b_n on $[u,\xi)$. Let e_n be the edge on $[u,c_n]$ incident on c_n. Then by Proposition 3.4.5, there is a point $p_n \in Y_{e_n}$ for all $n \in \mathbb{N}$ such that $\lim^Y_{n\to\infty} p_n = \lim^Y_{n\to\infty} y_n$ and

$\lim^X_{n\to\infty} p_n = \lim^X_{n\to\infty} y_n$. Thus we may replace $\{y_n\}$ by $\{p_n\}$ for the sake of the proof. Note that $\{z'_n\}$, $\{p_n\}$ is a pair of compatible sequences and it is enough to show that $\lim^X_{n\to\infty} z'_n = \lim^X_{n\to\infty} p_n$.

A part of the remaining arguments is summarized as a lemma below.

Lemma 3.4.6. *Suppose $\{y_n\}$, $\{y'_n\}$ are unbounded sequences of points in Y such that the following hold:*

1. $\lim^Y_{n\to\infty} y_n = \lim^Y_{n\to\infty} y'_n$.
2. *The point $v_n = \pi(y_n)$ is on a geodesic ray $[v,\xi)$ in T such that* $\lim^T_{n\to\infty} v_n = \xi$.
3. *Nearest point projection of the set $\{\pi(y'_n)\}$ on $[v,\xi)$ is bounded.*

Then there is a sequence of points $\{q_n\}$ in Y such that

1. $\lim^Y_{n\to\infty} y_n = \lim^Y_{n\to\infty} q_n$,
2. *$\{\pi(q_n)\}$ is bounded and*
3. *$d_X(y_1,[q_n,y_n]_X) \to \infty$ as $n\to\infty$.*

In particular if $\lim^X_{n\to\infty} y_n$ *exists then* $\lim^X_{n\to\infty} y_n = \lim^X_{n\to\infty} q_n$.

Proof. Fix $u \in [v,\xi)$ such that $[\pi(y'_n), v_n]_T$ passes through u for all $n \in \mathbb{N}$. Let e be the edge on $[u,\xi)$ incident on u. Let q_n be a point on $Y_{eu} \cap [y_n, y'_n]_Y$. Since $\lim^Y_{n\to\infty} y_n = \lim^Y_{n\to\infty} y'_n$, we have $\lim^Y_{n\to\infty} y_n = \lim^Y_{n\to\infty} q_n$ by Lemma 2.2.32 (2), and again by the same lemma we have $d_Y(y_1,[q_n,y_n]_Y) \to \infty$ as $n\to\infty$. Now, applying Lemma 3.2.5 to the sequences $\{y_n\},\{y'_n\}$ in Y and Lemma 2.2.13, we see that $\mathscr{F}l^Y(Y_{eu}) \cup \mathscr{F}l^Y(Y_{e_n c_n})$ is uniformly quasiconvex in Y and X. Then by Lemma 2.2.23 $N^Y_{D_0}(\mathscr{F}l^Y(Y_{eu}) \cup \mathscr{F}l^Y(Y_{e_n c_n})) = Z_n$, say, is uniformly qi embedded in both Y and X for some uniform D_0. Thus $[q_n,y_n]_{Z_n}$ is a uniform quasigeodesic in both Y and X. Since $d_Y(y_1,[q_n,y_n]_Y) \to \infty$ as $n\to\infty$, it follows, by stability of quasigeodesics (Lemma 2.2.2), that $d_X(y_1,[q_n,y_n]_X) \to \infty$ as $n\to\infty$. Thus $\lim^X_{n\to\infty} y_n = \lim^X_{n\to\infty} q_n$. □

Applying Lemma 3.4.6 to the sequences $\{z'_n\},\{p_n\}$, we find a new sequence $\{q_n\}$ and the proof boils down to deal with the compatible sequences $\{z'_n\}$ and $\{q_n\}$. However, $\{z'_n\}$ and $\{q_n\}$ satisfy the conditions in Case 1. Hence we are done in this case too.

Case 3: Suppose neither $\{b'_n\}$ nor $\{b_n\}$ satisfies the property (1) or (2) of Lemma 3.3.6. In this case both S_1 and S_2 are unbounded and locally finite whence ∂S_1 and ∂S_2 are both nonempty (see Lemma 2.2.29). After passing to subsequences, if necessary, we can assume that $\pi(y_n) = b_n \to \xi \in \partial S_1$, and $\pi(y'_n) = b'_n \to \xi' \in \partial S_2$

and that the hypotheses of Proposition 3.4.5 are satisfied by both $\{y_n\}$ and $\{y'_n\}$. Then applying Proposition 3.4.5 and passing to further subsequences if necessary, we may also assume that (1) b_n's are on the geodesic $[u,\xi)$, (2) b'_n's are on the geodesic $[u,\xi')$, (3) the sequences $\lim^T_{n\to\infty} b_n = \xi$ and $\lim^T_{n\to\infty} b'_n = \xi'$ and the following: Let e_n be the edge on $[u,b_n]$ which is incident on b_n and let e'_n be the edge on $[u,b'_n]$ which is incident on b'_n. Then (4) we may assume that $y_n \in Y_{e_n b_n}$ and $y'_n \in Y_{e'_n b'_n}$. This last statement will be used for dealing with Subcase 3B below.

We note that we have natual inclusions $\partial S_1 \to \partial T$ and $\partial S_2 \to \partial T$. In particular, $\xi, \xi' \in \partial T$. The rest of the proof is divided into two subcases.

Subcase 3A: Suppose $\xi \neq \xi'$. In this case the nearest point projection of $[u,\xi)$ on $[u,\xi')$ is a finite diameter set. Thus by applying Lemma 3.4.6 we can find a new sequence $\{p_n\}$ in Y such that $\{\pi(p_n)\}$ is bounded, and $\{p_n\}$ and $\{y'_n\}$ for which we need to show that $\lim^X_{n\to\infty} y'_n = \lim^X_{n\to\infty} p_n$. However, this now follows from Case 2.

Subcase 3B: Suppose $\xi = \xi'$. After passing through a subsequence, we assume that $d_T(u,b_n) \le d_T(u,b'_n) \le d_T(u,b_{n+1})$.

Let $t_n = d_T(u, \pi(\mathscr{F}l^Y(Y_{e_n b_n})))$. Let $y \in Y_u$. Suppose $\lim^X_{n\to\infty} y_n \neq \lim^X_{n\to\infty} y'_n$. Then there is a $R \in \mathbb{N}$ such that $d_X(y,[y_n,y'_n]_X) \le R$ for all $n \in \mathbb{N}$. This implies $d_T(u, \pi([y_n,y'_n]_X) \le R$ since π is 1-Lipschitz. Now it follows from Theorem 3.4.4 (2) that $t_n \le R + D'$ where D' as in Theorem 3.4.4.

It then follows that for all large $n \in \mathbb{N}$, $\mathscr{F}l^Y(Y_{e_n b_n}) \cap \mathscr{F}l^Y(Y_{e'_n b'_n}) \neq \emptyset$. Hence by Lemma 2.2.23, there is a uniform constant D_0 such that $N^Y_{D_0}(\mathscr{F}l^Y(Y_{e_n b_n}) \cup \mathscr{F}l^Y(Y_{e'_n b'_n})) = W_n$, say, is uniformly qi embedded in both X and Y. Thus $[y_n,y'_n]_{Z_n}$ is a uniform quasi-geodesic in both Y and X. Since $d_Y(y,[y_n,y'_n]_Y) \to \infty$ as $n \to \infty$, it follows, by stability of quasi-geodesics (Lemma 2.2.2), that $d_X(y,[y_n,y'_n]_X) \to \infty$ as $n \to \infty$ − which is a contradiction by Lemma 2.2.32 (2). Therefore, $\lim^X_{n\to\infty} y_n = \lim^X_{n\to\infty} y'_n$. ☐

3.5 Proof of Theorem 1.1.7

Let us restate the theorem for readers' references.

Theorem 3.5.1 (Theorem 1.1.7). *Additionally, suppose in Convention 3.3.3, we have the following. (Here we do not require the spaces X and Y to be proper.) For all $v \in V(S)$, the inclusions $Y_v \to X_v$ are uniformly qi embedded. If moreover the projection hypothesis holds then we have the following:*

1. *The inclusion $Y \to X_S := \pi^{-1}(S)$ is (uniformly) qi embedded.*

2. The inclusion $Y \to X$ admits the CT map.

Proof. Note that X_S is hyperbolic by [6]. Then (2) follows from (1), Theorem [9, Theorem 8.11] and the functoriality property of CT-maps (see Lemma 2.2.39). So we proof only (1). For that we prove the existence of a coarse Lipschitz retraction $X_S \to Y$ where $Z = X_S$ is with its induced path metric from X. If that is done then the inclusion $Y \hookrightarrow X$ is ϕ-proper embedding implies that the inclusion $Y \hookrightarrow Z$ is also ϕ-proper embedding. Therefore, $Y \hookrightarrow Z$ is uniformly qi embedded (see Lemma 2.1.3).

By given condition, for all $e \in E(S)$ incident on $u \in V(S)$, Y_u's and Y_{eu}'s are uniformly qi embedded in X_u. Hence they are uniformly quasiconvex in X_u (see Lemma 2.2.22 (1)).

Let $Z_{vsp} = \cup_{u \in V(S)} X_u$ and $\rho : Z_{vsp} \to Y$ be the map as defined in Remark 2.3.7. since Z_{vsp} is 1-dense in Z, by Lemma 2.1.2, we need to show $d_Z(\rho(x), \rho(y))$ is uniformly bounded where $x, y \in Z_{vsp}$ and $d_Z(x,y) \leq 1$.

Suppose $\pi(x) = u, \pi(y) = v$. If $u = v$ then by Lemma 2.2.21 (1), $d_{X_u}(\rho(x), \rho(y))$ is uniformly bounded and so is $d_Z(\rho(x), \rho(y))$. Now suppose $u \neq v$. Note that $d_T(u,v) \leq d_Z(x,y) \leq 1$ implies $d_T(u,v) = 1$ and so $x \in X_{eu}, y \in X_{ev}$, where $e = [u,v] \in E(S)$.

Note that Y_u and Y_v are uniformly quasiconvex in X_u and X_v respectively. Since $P_{Y_u Y_{eu}}(Y_u) = Y_{eu}$, by projection hypothesis, $Hd_{X_u}(P_{X_u X_{eu}}(Y_u), Y_{eu}) \leq R_0$ where R_0 is coming from projection hypothesis. Similarly, we have $Hd_{X_v}(P_{X_v X_{ev}}(Y_v), Y_{ev}) \leq R_0$. Since Y_{eu} is uniformly quasiconvex in X_u, so is $P_{X_u X_{eu}}(Y_u)$. Similarly $P_{X_u X_{eu}}(Y_u)$ is also quasiconvex in X_v. Again $Hd_{X_{uv}}(Y_{eu}, Y_{ev}) = 1$ implies $Hd_{X_{uv}}(P_{X_u X_{eu}}(Y_u), P_{X_v X_{ev}}(Y_v)) \leq 2R_0 + 1$. Then, by Lemma 2.3.5 (2), $d_{X_{uv}}(\rho(x), \rho(y))$ is uniformly bounded and so is $d_Z(\rho(x), \rho(y))$.

Therefore, we are through.

□

3.6 Applications and related results

In this section, we will see two main applications of Theorem 1.1.6 (see Theorem 1.1.2 and Theorem 1.1.11).

Proof of Theorem 1.1.2:

It is standard that for a graph of groups $(\mathcal{G}', Y')$, there is tree of metric spaces $\pi : X \to T$ where T is the Bass-Serre tree of $(\mathcal{G}', Y')$ such that $\pi_1(\mathcal{G}', Y')$ acts on X properly and cocompactly; and so the orbit map $\pi_1(\mathcal{G}', Y') \to X$ is quasi-isometry for any finite generating set for $\pi_1(\mathscr{G}',Y')$ (see Section 5.6 for detailed explanation). By hypotheses of Theorem 1.1.2, it turns out that $\pi : X \to T$ is a tree of hyperbolic metric spaces with the qi embedded condition.

Suppose $\pi_Y : Y \to S$ is the tree of hyperbolic metric spaces with the qi embedded condition corresponding to a subgraph of subgroups $(\mathscr{G},Y)$ as in Theorem 1.1.2. We consider the orbit map $\pi_1(\mathscr{G},Y) \to Y$ which is a quasi-isometry.

By condition 2 (a) of Theorem 1.1.2, it follows from [7, Proposition 2.7, Corollary 1.14, see also 2.15] that the natural inclusion $\pi_1(\mathscr{G},Y) \to \pi_1(\mathscr{G}',Y')$ is an injective homomorphism and $S \to T$ is an embedding of trees. We also can think of $\pi_Y : Y \to S$ as induced subtree of spaces in $\pi : X \to T$ over $S \subseteq T$ (via the embedding above). (See Subsection 3.3.1 for induced subtree of spaces.)

Moreover, the inclusion $Y \to X$ is $\pi_1(\mathscr{G},Y)$-equivariant. Therefore, proving the CT-map for $\pi_1(\mathscr{G},Y) \to \pi_1(\mathscr{G}',Y')$, it is enough to show the same for the inclusion $Y \to X$.

By condition 2 (c) in Theorem 1.1.2, the induced subtree of spaces under consideration satisfies the projection hypothesis whence satisfies all the hypothesis of Theorem 1.1.6.

Therefore, we are through. $\square$

Proof of Theorem 1.1.4:

We will to apply Theorem 1.1.6. For that we only need to prove the projection hypothesis (see Remark 1.1.3 (2)). Since $G_u \to G'_u$ admits the CT map, so it satisfies uniform Mitra's criterion whence the theorem follows from Remark 3.3.8 (2). $\square$

3.6.1 Lamination

Definition 3.6.1. If a map between hyperbolic spaces $f : Y \to X$ admits the CT map then the Cannon-Thurston (CT) lamination Λ_{CT} ([38]) for f is defined to be

$$\Lambda_{CT} = \{\xi_1, \xi_2) \in \partial Y \times \partial Y : \xi_1 \neq \xi_2, \partial f(\xi_1) = \partial f(\xi_2)\}$$

where $\partial f : \partial Y \to \partial X$ is the CT map.

In this thesis, we also investigate the properties of the CT lamination in the situation where Theorem 1.1.6 holds. We will now prove a couple of result related to this; which will be used to prove Theorem 1.1.11.

Notation and convention: Suppose the inclusion $i : Z \hookrightarrow W$ of hyperbolic metric spaces admits the CT-map. We denote the CT-map by $\partial i_{Z,W} : \partial Z \to \partial W$. In this subsection, we assume that base for both Y and X are same, i.e., $S = T$ in Theorem 1.1.6.

In the following lemma we prove converse of Lemma 3.3.5. Proof goes along the same line as Lemma 3.3.5.

Lemma 3.6.2. *Let e, e' be edges in T incident on u. Suppose γ and γ' two quasi-geodesic rays in Y_u such that $\gamma \subseteq Y_{e'u}, \gamma' \subseteq Y_{eu}$. Then $\lim^X_{n\to\infty} \gamma(n), \lim^X_{n\to\infty} \gamma'(n)$ exist and if $\lim^X_{n\to\infty} \gamma(n) = \lim^X_{n\to\infty} \gamma'(n)$ then $\lim^Y_{n\to\infty} \gamma(n) = \lim^Y_{n\to\infty} \gamma'(n)$.*

Proof. Since the inclusions $Y_u \to X_u$ and $X_u \to X$ admit the CT-maps (see [8] for later one), and so by functoriality property of CT-maps (*see* Lmma 2.2.39), so $\lim^X_{n\to\infty} \gamma(n), \lim^X_{n\to\infty} \gamma'(n)$ exist. For the second part, if $\lim^{Y_u}_{n\to\infty} \gamma(n) = \lim^{Y_u}_{n\to\infty} \gamma'(n)$, then we are done. Now Suppose $\lim^{Y_u}_{n\to\infty} \gamma(n) \neq \lim^{Y_u}_{n\to\infty} \gamma'(n)$. Note that γ and γ' are also qausi-geodesic in X_u since edge spaces of Y are uniformly qi embedded in the corresponding vertex spaces of X. So $\lim^{X_u}_{n\to\infty} \gamma(n) \neq \lim^{X_u}_{n\to\infty} \gamma'(n)$, otherwise, $\lim^{Y_u}_{n\to\infty} \gamma(n) = \lim^{Y_u}_{n\to\infty} \gamma'(n)$.

Suppose α is a geodesic line X_u such that $\alpha(-\infty) = \lim^{X_u}_{n\to\infty} \gamma(n)$ and $\alpha(\infty) = \lim^{X_u}_{n\to\infty} \gamma'(n)$. Hence by given condition, we have $\lim^X_{n\to\infty} \alpha(-n) = \lim^X_{n\to\infty} \alpha(n)$. Then by [9, Proposition 8.54 (1)], there is ray geodesic ray $[u, \xi)$ in T such that both $\alpha(-\infty)$ and $\alpha(\infty)$ flow in X_v for all vertex $v \in [u, \xi)$. Let β is geodesic line in Y_u such that $Hd_{X_u}(\beta, \alpha)$ is uniformly bounded. Fix $y \in Y_u$. Then by the description of uniform quasi-geodesic given in [9, Proposition 8.49] joining $\beta(n)$ and $\beta(-n)$, we can conclude the following. If β_n's are uniform quasi-geodesic joining $\beta(-n)$ and $\beta(n)$ in Y and β'_n's are that in X, then $Hd_X(\beta_n, \beta'_n)$ is uniformly bounded. Since $Hd_x(\alpha, \beta) < \infty$ and $\lim^X_{n\to\infty} \beta(n) = \lim^X_{n\to\infty} \beta(-n)$ then $d_X(y, \beta'_n) \to \infty$ as $n \to \infty$ (see Lemma 2.2.32 (2)). Hence, $d_Y(y, \beta_n) \to \infty$ as $n \to \infty$ since Y is properly embedded in X. This shows that $\lim^Y_{n\to\infty} \beta(n) = \lim^Y_{n\to\infty} \beta'(n)$. Note that $\lim^{Y_u}_{n\to\infty} \beta(n) = \lim^{Y_u}_{n\to\infty} y_n$ and $\lim^{Y_u}_{n\to\infty} \beta(-n) = \lim^{Y_u}_{n\to\infty} y'_n$. Since $Y_u \to Y$ admits the CT-map, we are through. □

A generalization of Lemma 3.6.2 is the following.

Lemma 3.6.3. *Suppose α and α' are geodesic rays in Y_u and Y_v respectively. Let $u \neq v$. Then $\lim^Y_{n\to\infty} \alpha(n) \neq \lim^Y_{n\to\infty} \alpha'(n)$ implies $\lim^X_{n\to\infty} \alpha(n) \neq \lim^X_{n\to\infty} \alpha'(n)$.*

Proof. On contrary, suppose $\lim^X_{n\to\infty} \alpha(n) = \lim^X_{n\to\infty} \alpha'(n)$. Let β and β' be geodesic rays in X_u and X_v respectively such that $\partial i_{Y_u,X_u}(\alpha(\infty)) = \beta(\infty)$ and $\partial i_{Y_v,X_v}(\alpha'(\infty)) = \beta'(\infty)$. Then by Proposition 3.2.8, there is a vertex $w \in [u, v]$ such that both $\beta(\infty)$ and $\beta'(\infty)$ have boundary flow in X_w. Now we consider two cases depending on the position of w.

Case 1: Suppose $w \in [u,v] \setminus \{u,v\}$. Let $e \subseteq [u,w]$ and $e' \subseteq [w,v]$ be edges adjacent to w. Let β_1 and β_1' be geodesic rays in X_{ew} and $X_{e'w}$ representing the boundary flow of $\beta(\infty)$ and $\beta'(\infty)$ respectively. Since we have projection hypothesis, then by repeated application of Lemma 2.2.45, we conclude that $\alpha(\infty)$ flows in Y_{ew}, $\alpha'(\infty)$ flows in $Y_{e'w}$ and $Hd_X(\alpha,\beta) < \infty$, $Hd_X(\alpha',\beta') < \infty$. Now by Lemma 3.2.7, $Hd_X(\beta,\beta_1) < \infty$ and $Hd_X(\beta',\beta_1') < \infty$. So $Hd_X(\alpha,\beta_1) < \infty$ and $Hd_X(\alpha',\beta_1') < \infty$. By replacing β_1 and β_1' by some quasi-geodesic rays, say, γ and γ' respectively such that $\gamma \subseteq Y_{ew}$ and $\gamma' \subseteq Y_{e'w}$, and $Hd_X(\beta_1,\gamma) < \infty$ and $Hd_X(\beta_1',\gamma') < \infty$. Then $Hd_X(\alpha,\gamma) < \infty$ and $Hd_X(\alpha',\gamma') < \infty$, and since Y is properly embedded in X, so $Hd_Y(\alpha,\gamma) < \infty$ and $Hd_Y(\alpha',\gamma') < \infty$.

Now $\lim^X_{n\to\infty} \alpha(n) = \lim^X_{n\to\infty} \alpha'(n)$ implies $\lim^X_{n\to\infty} \gamma(n) = \lim^X_{n\to\infty} \gamma'(n)$. Then by Lemma 3.6.2, $\lim^Y_{n\to\infty} \gamma(n) = \lim^Y_{n\to\infty} \gamma'(n)$. Hence $\lim^Y_{n\to\infty} \alpha(n) = \lim^Y_{n\to\infty} \alpha'(n)$ - which is a contradiction.

Case 2: Without loss of generality, we assume that $w = u$. Since $\beta'(\infty)$ flows in X_u, with the same notation as in Case 1, we have the following facts. (1) α is a geodesic ray in Y_u, (2) γ' is a quasi-geodesic ray in Y_u such that $\gamma' \subseteq Y_{e'u}$, and so is a quasi-geodesic ray in X_u, (3) $\lim^X_{n\to\infty} \gamma'(n) = \lim^X_{n\to\infty} \alpha'(n)$ and $\lim^Y_{n\to\infty} \gamma'(n) = \lim^Y_{n\to\infty} \alpha'(n)$.

Now if $\lim^{X_u}_{n\to\infty} \gamma'(n) = \lim^{X_u}_{n\to\infty} \alpha(n)$, then (since we have projection hypothesis) by Lemma 2.2.45, α is a quasi-geodesic ray in X_u. Hence $Hd_{X_u}(\alpha,\gamma') < \infty$, and so $Hd_{Y_u}(\alpha,\gamma') < \infty$. This contradicts to $\lim^Y_{n\to\infty} \alpha(n) \neq \lim^Y_{n\to\infty} \alpha'(n)$.

So we assume that $\lim^{X_u}_{n\to\infty} \gamma'(n) \neq \lim^{X_u}_{n\to\infty} \alpha(n)$. Let γ_0 be a geodesic line in X_u such that $\lim^{X_u}_{n\to\infty} \gamma_0(-n) = \lim^{X_u}_{n\to\infty} \gamma(n)$ and $\lim^{X_u}_{n\to\infty} \gamma_0(n) = \lim^{X_u}_{n\to\infty} \alpha(n)$. Now, since $\lim^X_{n\to\infty} \gamma_0(-n) = \lim^X_{n\to\infty} \gamma_0(n)$, by [9, Proposition 8.54 (1)], there is a geodesic ray $[u,\xi)$ in T such that both $\gamma_0(-\infty)$ and $\gamma_0(\infty)$ have flow in X_v for all vertex $v \in [u,\xi)$. Then by Lemma 2.2.45, α is a quasi-geodesic ray in X_u. Then we can replace γ_0 by a quasi-geodesic line γ_1 such that $\gamma_1 \subseteq Y_{e'u}$ and $\lim^X_{n\to\infty} \gamma_1(-n) = \lim^X_{n\to\infty} \gamma_1(n)$ and $\lim^Y_{n\to\infty} \gamma_1(-n) = \lim^Y_{n\to\infty} \alpha'(n)$ and $\lim^Y_{n\to\infty} \gamma_1(n) = \lim^Y_{n\to\infty} \alpha(n)$. Therefore, by Lemma 3.6.2, $\lim^Y_{n\to\infty} \gamma_1(-n) = \lim^Y_{n\to\infty} \gamma_1(n)$ - which leads to a contradiction that $\lim^Y_{n\to\infty} \alpha(n) \neq \lim^Y_{n\to\infty} \alpha'(n)$. □

One can easily verify the following lemma from definition of conical limit point (see Definition 3.2.9) and the existence of CT-maps.

Lemma 3.6.4. *Suppose α is a geodesic ray in Y such that $\alpha(\infty)$ is a conical limit point of a vertex space Y_u for some vertex $u \in \pi(\alpha)$. Then there is a subsequence $\{r_i\} \subseteq \mathbb{N}$ such that $\alpha(r_i) \in Y_u$, $\lim^{Y_u}_{n\to\infty} \alpha(r_i) \in \partial Y_u$ and $\lim^X_{n\to\infty} \alpha(r_i) = \partial i_{Y,X}(\alpha(\infty))$.*

In particular, there is a geodesic ray α' in Y_u such that $\lim^Y_{n\to\infty}\alpha(n) = \lim^Y_{n\to\infty}\alpha'(n)$ *and* $\lim^X_{n\to\infty}\alpha(n) = \lim^X_{n\to\infty}\alpha'(n)$.

Lemma 3.6.5. *Suppose α is a geodesic ray in Y_u and α' is that in Y. Further, we assume that $\alpha'(\infty)$ is a conical limit point of the vertex space Y_v for some vertex $v \in \pi(\alpha)$. Let $u \neq v$. Then* $\lim^Y_{n\to\infty}\alpha(n) \neq \lim^Y_{n\to\infty}\alpha'(n)$ *implies* $\lim^X_{n\to\infty}\alpha(n) \neq \lim^X_{n\to\infty}\alpha'(n)$.

Proof. It follows from Lemma 3.6.4 and Lemma 3.6.3. □

Lemma 3.6.6. *Suppose α and α' are geodesic rays in Y. Further, we assume that $\alpha(\infty)$ and $\alpha'(\infty)$ are conical limit points of vertex spaces Y_u and Y_v respectively for some $u \in \pi(\alpha)$ and $v \in \pi(\alpha')$. Let $u \neq v$. Then* $\lim^Y_{n\to\infty}\alpha(n) \neq \lim^Y_{n\to\infty}\alpha'(n)$ *implies* $\lim^X_{n\to\infty}\alpha(n) \neq \lim^X_{n\to\infty}\alpha'(n)$.

Proof. By Lemma 3.6.4 and Lemma 3.6.3, we are done. □

Lemma 3.6.7. *Suppose α is a geodesic ray in Y_u and α' is that in Y such that $\alpha'(\infty)$ is not a conical limit point of any vertex space. Then* $\lim^Y_{n\to\infty}\alpha(n) = \lim^Y_{n\to\infty}\alpha'(n)$ *if and only if* $\lim^X_{n\to\infty}\alpha(n) = \lim^X_{n\to\infty}\alpha'(n)$.

Proof. Note that 'only if' part follows from CT-map $\partial i_{Y,X} : \partial Y \to \partial X$. Now we proof 'if' part, i.e., we have $\lim^X_{n\to\infty}\alpha(n) = \lim^X_{n\to\infty}\alpha'(n)$. We will find uniformly quasiconvex subset, say, Z_n in both X and Y containing $\alpha(n)$ and $\alpha'(n)$ for all large $n \in \mathbb{N}$. Then by Lemma 2.2.23, there is a uniform constant $D \geq 0$ such that $N^Y_D(Z_n)$ is, with the induced path metric from Y, uniformly qi embedded in both X and Y. Then by stability of quasi-geodesic and Lemma 2.2.32 (2), $\lim_{n\to\infty} d_X(x,\gamma_n) = \infty$ where γ_n is a geodesic in $N^Y_D(Z_n)$. Therefore, by same lemmas, $\lim_{n\to\infty} d_Y(y,[\alpha(n),\alpha'(n)]_Y) = \infty$ and $\lim^Y_{n\to\infty}\alpha(n) = \lim^Y_{n\to\infty}\alpha'(n)$.

Finding Z_n: By Lemma 3.2.11, there is a geodesic ray $[u,\xi) \subseteq S \cap \pi(\alpha')$. Let $\{e_1, e_2, \cdots\}$ be successive edges directed away from u on the ray $[u,\xi)$, and $u_1 = u$ and e_i joins u_i and u_{i+1} for all $i \in \mathbb{N}$. Suppose $\{r_i\} \subseteq \mathbb{N}$ is a subsequence such that $\alpha'(r_i) \in Y_{e_i}$. Suppose β is a geodesic ray in X_u such that $\partial i_{Y_u,X_u}(\alpha(\infty)) = \beta(\infty)$. So $\lim^X_{n\to\infty}\beta(n) = \lim^X_{n\to\infty}\alpha'(n) = \lim^X_{n\to\infty}\alpha'(r_n)$. Suppose $Y_{e_iu_i}$'s are k-quasiconvex in Y_{u_i}. Let $\mathscr{F}l^Y(Y_{e_iu_i})$ be the flow space for a fixed $R = R_{2.2.13}(\delta'_0,k)$ as in Definition 3.1.1. Now by Lemma 3.2.12, $\beta(\infty)$ has a boundary flow in X_v for all vertex $v \in [u,\xi)$. Hence by Lemma 2.2.45, we can conclude that $\alpha(\infty)$ has a boundary flow in Y_v for all vertex $v \in [u,\xi)$. In particular, we have $\mathscr{F}l(Y_{e_1u}) \cap Y_{e_iu_i} \neq \emptyset$. By Lemma 3.4.3, $Z_n = \mathscr{F}l^Y(Y_{e_1u}) \cup \mathscr{F}l^Y(Y_{e_nu_n})$ is uniformly quasiconvex in both X and Y containing both $\alpha(n)$ and $\alpha'(r_n)$. □

Lemma 3.6.8. *Suppose α and α' are geodesic rays in Y such that $\alpha(\infty)$ is a conical limit of a vertex space Y_u for some $u \in \pi(\alpha)$ and $\alpha'(\infty)$ is not a conical limit point of any vertex space of Y. Then $\lim^Y_{n\to\infty} \alpha(n) = \lim^Y_{n\to\infty} \alpha'(n)$ if and only if $\lim^X_{n\to\infty} \alpha(n) = \lim^X_{n\to\infty} \alpha'(n)$.*

Proof. It follows from Lemma 3.6.4 and Lemma 3.6.7. □

Lemma 3.6.9. *Suppose α and α' are geodesic rays in Y such that both $\alpha(\infty)$ and $\alpha'(\infty)$ are not conical limit points of any vertex space of Y. Then $\lim^Y_{n\to\infty} \alpha(n) \neq \lim^Y_{n\to\infty} \alpha'(n)$ implies $\lim^X_{n\to\infty} \alpha(n) \neq \lim^X_{n\to\infty} \alpha'(n)$.*

Proof. Let $[u,\xi)$ and $[u,\xi')$ be geodesic rays in $\pi(\alpha)$ and $\pi(\alpha')$ (see Lemma 3.2.11). On contrary, suppose $\lim^X_{n\to\infty} \alpha(n) = \lim^X_{n\to\infty} \alpha'(n)$. Let $\{e_1, e_2, \cdots\}$ be successive edges on $[u,\xi)$ directed away from u and $\{e'_1, e'_2, \cdots\}$ be that on $[u,\xi')$. For $\in \mathbb{N}$, let $u_i, u'_i \in V(T)$ such that e_i joins u_i and u_{i+1}, and e'_i joins u'_i and u'_{i+1}. Suppose $\{r_i\}$ and $\{t_i\}$ are subsequences of $\mathbb{N}$ such that $\alpha(r_i) \in Y_{e_iu_i}$ and $\alpha'(t_i) \in Y_{e'_iu'_i}$. Now we consider two cases depending on $\xi \neq \xi'$ and $\xi = \xi'$.

Case 1: Suppose $\xi \neq \xi'$. Let j be the smallest for which $e_j \neq e'_j$. Then by Theorem 3.4.4 (1), for all large i, a uniform quasi-geodesic in X joining $\alpha(r_i)$ and $\alpha'(t_i)$ passes through $Y_{e_ju_j}$. We take a point $x_i \in Y_{e_ju_j}$ on that quasi-geodesic for all large i. If necessary, after passing through a subsequence, we assume that $\lim^Y_{n\to\infty} x_n$ exists. Since $\lim^X_{n\to\infty} \alpha(n) = \lim^X_{n\to\infty} \alpha'(n)$, we have $\lim^X_{n\to\infty} \alpha(n) = \lim^X_{n\to\infty} x_n$ and $\lim^X_{n\to\infty} \alpha'(n) = \lim^X_{n\to\infty} x_n$. Then by Lemma 3.6.7, $\lim^Y_{n\to\infty} \alpha(n) = \lim^Y_{n\to\infty} x_n$ and $\lim^Y_{n\to\infty} \alpha'(n) = \lim^Y_{n\to\infty} x_n$. This implies that $\lim^Y_{n\to\infty} \alpha(n) = \lim^Y_{n\to\infty} \alpha'(n)$ – which is a contradiction.

Case 2: Suppose $\xi = \xi'$. Note that $e_i = e'_i$ for all $i \in \mathbb{N}$. Then by Lemma 3.4.3 (2), we have a quasiconvex subset, namely, $\mathscr{F}l^Y(Y_{e_iu_i})$ in both X and Y containing $\alpha(r_i)$ and $\alpha'(t_i)$. Then by Lemma 2.2.23 (2), there is a uniform constant $D \geq 0$ such that $N^Y_D(\mathscr{F}l^Y(Y_{e_iu_i}))$ is uniformly qi embedded in both X and Y. Now by stability of quasi-geodesic, Lemma 2.2.32 (2) and $\lim^X_{n\to\infty} \alpha(r_n) = \lim^X_{n\to\infty} \alpha'(t_n)$, we have $\lim^Y_{n\to\infty} \alpha(r_n) = \lim^Y_{n\to\infty} \alpha'(t_n)$. Thus $\lim^Y_{n\to\infty} \alpha(n) \neq \lim^Y_{n\to\infty} \alpha'(n)$ – which is a contradiction. □

Proof of Theorem 1.1.11:

By Remark 3.2.10, each of $\alpha(-\infty)$ and $\alpha(\infty)$ have two possibilities. So we have the following cases.

Case 1: Both $\alpha(-\infty)$ and $\alpha(\infty)$ are conical limit points of some vertex spaces. Then by Lemma 3.6.6 we are done.

Case 2: Both $\alpha(-\infty)$ and $\alpha(\infty)$ are not conical of any vertex spaces. This is Lemma 3.6.9.

Case 3: Without loss of generality, we assume that $\alpha(-\infty)$ is a conical limit point of some vertex space and $\alpha(\infty)$ is not conical limit point of any vertex spaces. Then we through by Lemma 3.6.8. □

3.7 Nonexistence of Cannon-Thurston maps

In this section we prove Theorem 1.1.9 and verify the conditions of this theorem for Example 1.1.10. We will not rewrite the statement again.

Proof of Theorem 1.1.9: Let $A = \{t^n : n \in \mathbb{Z}\}$ and let $h_n = P_{G'Q}(y_n)$. It is a standart fact A is a quasigeodesic line in G. Let $t^{a(n)}$ be a nearest point projection of h_n on A where $a(n) \in \mathbb{Z}$. Since $\lim^G_{n\to\infty} h_n = \lim^G_{n\to\infty} t^n$, we have $\lim_{n\to\infty} a(n) = \infty$. Now, this means $\lim^G_{n\to\infty} t^{-a(n)} = \lim^G_{n\to\infty} t^{-n}$. On the other hand, a nearest point projection of $t^{-a(n)}h_n = x_n$, say, on $t^{-a(n)}A = A$ is $1 \in G$. Since t^nG', $n \in \mathbb{Z}$ form a geodesic line in the Bass-Serre tree for the HNN extension $G'*_Q$ (see Figure 3.1) and $t^{-a(n)}h_n \in t^{-a(n)}G'$, so $\{d_G(1,x_n) : n \in \mathbb{N}\}$ is unbounded. Suppose $\{n_k\}$ is a subsequence of the sequence of $\mathbb{N}$ such that $\lim^G_{n\to\infty} x_{n_k} = \xi \in \partial G$. Then clearly nearest point projection of the geodesic ray $[1,\xi)$ on A is a bounded set. In particular, $\xi \neq \lim^G_{n\to\infty} t^{-n}$. Thus

$$\lim^G_{k\to\infty} t^{-a(n_k)} h_{n_k} \neq \lim^G_{n\to\infty} t^{-n}. \tag{3.7. 1}$$

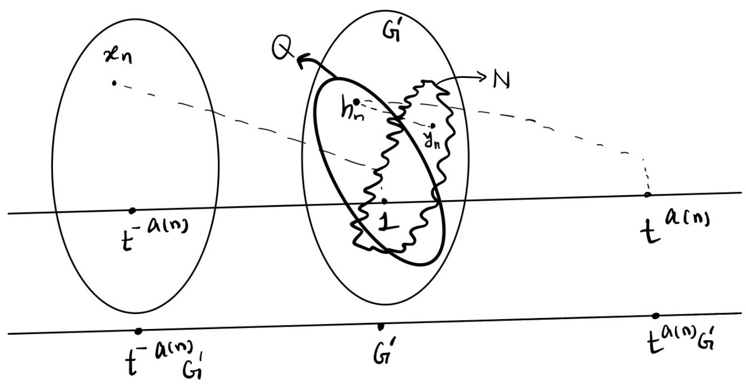


Figure 3.1

Claim: $\lim^G_{k\to\infty} t^{-a(n_k)} h_{n_k} = \lim^G_{k\to\infty} t^{-a(n_k)} y_{n_k}$.

Proof of claim: We consider the geodesic line t^nG', $n \in \mathbb{Z}$ in the Bass-Serre tree for the HNN extension under consideration. The vertex space over t^nG' is the coset t^nG'

of G and the edge space of t^nG' which is gluing to $t^{n-1}G'$ and $t^{n+1}G'$ is t^nQ. Since h_n is the nearest point projection of y_n in G', t^mh_n is the nearest point projection of t^my_n in the vertex space t^mG' for any $n \in \mathbb{N}$, $m \in \mathbb{Z}$. This verifies the conditions of Lemma 3.2.4 for $X = G$, $u = G'$, $u_k = t^{-a(n_k)}G'$, $e_k = [t^{n_k-1}G', t^{-a(n_k)+1}G']$ and $X_{e_ku_k}$ is the image of $t^{-a(n_k)}Q$ in $t^{-a(n_k)}G'$, and of course, $x_k = t^{-a(n_k)}h_{n_k}$ and $x'_k = t^{-a(n_k)}y_{n_k}$ for all $k \in \mathbb{N}$. Hence we are done.

It follows from Inequality 3.7. 1 and the claim that

$$\lim^G_{k\to\infty} t^{-a(n_k)}y_{n_k} \neq \lim^G_{n\to\infty} t^{-n}. \tag{3.7. 2}$$

However, we note that $\lim^K_{n\to\infty} t^{-a(n_k)}y_{n_k} = \lim^K_{n\to\infty} t^{-a(n_k)}$. This can be seen as follows. Since $Q \cap N = (1)$, we have $K = N* < t > < G$ and K obtains an induced tree of spaces structure from G. Again $t^{-a(n_k)}y_{n_k}, t^{-a(n_k)} \in t^{-a(n_k)}N \subseteq t^{-a(n_k)}G'$ which is a vertex space for K for all $k \in \mathbb{N}$. Since the edge spaces are points in this case we have an acylindrical tree of spaces. Thus as $t^{-n}G'$ are successive vertices on a geodesic ray in the corresponding Bass-Serre tree we are through.

Hence we get two sequences, namely, $\{t^{-a(n_k)}y_{n_k}\}$, $\{t^{-k}\} \subseteq K$ such that they limit to the same point in ∂K but not in ∂G. Thus the inclusion $K \to G$ does not satisfy Mitra's criterion. Therefore, we are done with the theorem by Lemma 2.2.42. □

Now we will show that Example 1.1.10 (see Introduction 1.1) satisfy the condition of Theorem 1.1.9.

Proof of Example 1.1.10: Note that Q is malnormal and quasiconvex subgroup of G', and N is non-quasiconvex hyperbolic subgroups of G' such that $Q \cap N = \{1\}$. Since N is infinite index normal subgroup, we have $\Lambda_{G'}(N) = \partial G'$ where $\partial G'$ is the Gromov boundary of G' and $\Lambda_{G'}(N)$ denotes the accumulation points of N in G'. Now Q is quasiconvex in G' and $\Lambda_{G'}(N) = \partial G'$ imply that $P_{G'Q}(N)$ and Q are Hausdorff close in G. It then follows that

$$\Lambda_G(P_{G'Q}(N)) \quad = \quad \Lambda_G(Q) \tag{3.7. 3}$$

Let ϕ be a hyperbolic automorphism of Q and $G = G' *_Q = G' \rtimes_\phi < t >$ be the HNN extension of G' over Q along ϕ where t is the stable letter; let $H = Q*_Q = Q \rtimes_\phi < t >$ be the restriction of that to Q. Note that Q is a normal subgroup of infinite index in H. Hence, $\Lambda_H(Q) = \partial H$. In particular, $\lim^H_{n\to\infty} t^{\pm n} \in \Lambda_H(Q)$. Since the inclusions $Q \to G' \to G$ (by [8]) and $H \to G$ (by Theorem 1.1.2) admit the CT maps, we see that

$$\lim^G_{n\to\infty} t^{\pm n} \quad \in \quad \Lambda_G(Q) \tag{3.7. 4}$$

It is clear from Inequalities 3.7. 3 and 3.7. 4 that the hypotheses of Theorem 1.1.9 are satisfied. Let K be the subgroup of G generated by $N \cup \{t\} = N * < t >$ (free product). Therefore, the inclusion $K \to G$ does not admit the CT map. $\square$

Chapter 4

A combination theorem for trees of metric spaces revisited

Suppose $\pi : X \to T$ is a tree of metric spaces (see Definition 2.3.1). In this chapter we prove the hyperbolicity of X within an axiomatic framework. As a consequence, we get a proof of Theorem 1.2.4 in Section 5.5. Now we will explain the hypotheses. Unless otherwise specified, by $u \in S$ (or $v \in S$ or $w \in S$) where S is a subtree of T, we always mean u (or v or w) to be a vertex of T. We use the notation $X_S := \pi^{-1}(S)$.

For each vertex $u \in T$ there is a subspace, say, $\mathscr{M}(X_u)$ containing X_u and satisfying the following properties $(\mathscr{P}0) - (\mathscr{P}4)$.

$(\mathscr{P}0)$ Suppose $u, v \in T$ and e is the edge on $[u, v]$ incident on v. Let T' be the maximal subtree of T containing v but not containing e. Then $\mathscr{M}(X_u) \cap X_{T'} \subseteq \mathscr{M}(X_v) \cap X_{T'}$.

$(\mathscr{P}1)$ Let $L' \geq 0$. For each $u \in T$, there is a L'-coarsely Lipschitz retraction $\rho_u : X \to \mathscr{M}(X_u)$. We also have an extra property of ρ_u as follows. Let $T_u = \pi(\mathscr{M}(X_u))$ and e be an edge in T intersecting T_u at a vertex. Suppose v is the vertex adjacent to e not in T_u and S is the maximal subtree of T containing v but not containing e. Then $\mathrm{diam}\{\rho_u(X_S)\} \leq C$ for some uniform constant $C \geq 0$.

$(\mathscr{P}2)$ There is a threshold constant $L_0 \geq 0$ such that for $L \geq L_0$, $N_L(\mathscr{M}(X_u))$ is path connected with the induced path metric from X and the inclusion $N_L(\mathscr{M}(X_u)) \hookrightarrow X$ is $\eta(L)$-proper embedding.

For $u, v \in T$, we say $[u, v] \subseteq T$ is a *special interval* if either $\mathscr{M}(X_u) \cap X_v \neq \emptyset$ or $X_u \cap \mathscr{M}(X_v) \neq \emptyset$. If $[u, v]$ is a special interval then for $L \geq L_0$,

$(\mathscr{P}3)$ the inclusion $N_L(\mathscr{M}(X_u)) \cup N_L(\mathscr{M}(X_v)) \hookrightarrow X$ is $\eta'(L)$-proper embedding, and

$(\mathscr{P}4)$ $N_L(\mathscr{M}(X_u)) \cup N_L(\mathscr{M}(X_v))$ is $\delta(L)$-hyperbolic metric space.

Theorem 4.0.1. *Suppose $\pi : X \to T$ is a tree of metric spaces satisfying properties $(\mathscr{P}0)-(\mathscr{P}4)$. Then X is hyperbolic metric space.*

Some remarks on Theorem 4.0.1: (1) Lemma 4.0.5 below, says that $\mathscr{M}_L(X_u)$ is hyperbolic (by Theorem 4.0.1 or one can use property $\mathscr{P}4$).

(2) Suppose $\pi : X \to T$ is a trees of hyperbolic metric spaces with the qi embedded condition and Bestvina-Feighn's flaring condition ([6]). Now we think of $\pi : X \to T$ as trees of metric bundles. We set $\mathscr{M}(X_u) = \mathscr{F}l_K(X_u)$ (flow space of X_u with certain fixed parameters) (see Subsection 5.1.1). In Section 5.5, it is shown, in this case, that $\mathscr{M}(X_u)$ satisfies all the conditions $\mathscr{P}0-\mathscr{P}4$. This shows that Theorem 4.0.1 covers the combination theorem for trees of metric spaces considered in [6] and particularly, acylindrical trees of metric spaces ([39]).

Definition 4.0.2. For a subtree S of T, we define $\mathscr{M}(X_S) := \cup_{w\in S}\mathscr{M}(X_w)$. Also for finitely many vertices $u_1, u_2, \cdots, u_n$, we define $\mathscr{M}(X_{\{u_1,u_2,\cdots,u_n\}}) := \mathscr{M}(X_{u_1}) \cup \mathscr{M}(X_{u_2}) \cup \cdots \cup \mathscr{M}(X_{u_n})$.

Notation: For a given subtree $S \subseteq T$, we denote $\mathscr{M}_L(X_S)$ to mean $N_L(\mathscr{M}(X_S))$.

The proof of Theorem 4.0.1 is divided into two parts as follows.

(1) Hyperbolictiy of $\mathscr{M}_L(X_I)$ where I is a special interval in T.

(2) Hyperbolicity of $\mathscr{M}_L(X_I)$ where I is any interval in T.

Finally, using (2), we conclude the proof. Before going into the proof of (1), let us first prove some lemmata which are required in (1) and (2).

Lemma 4.0.3. *Let S be a subtree of T. There is a uniform constant $L_{4.0.3}$ for which we have a $L_{4.0.3}$-coarsely Lipschitz retraction $\rho_S : X \to \mathscr{M}(X_S)$.*

Proof. Let us first define $\rho_S : X \to \mathscr{M}(X_S)$. Let $x \in X$ and u be the nearest point projection of $\pi(x)$ onto S. Then $\rho_S(x)$ is defined to be $\rho_u(x)$. Note that if $x \in \mathscr{M}(X_S)$, then by $(\mathscr{P}0)$, $\rho_S(x) = x$.

Let $X_{vsp} = \cup_{u\in T} X_u$ and $x, y \in X_{vsp}$ such that $d_X(x,y) \le 1$. Then by Lemma 2.1.2, we need to show a uniform bound on $d_X(\rho_S(x), \rho_S(y))$. Let u, v be the nearest point projections of $\pi(x), \pi(y)$ on S respectively. If $u = v$, then by definition of ρ_S, $\rho_S(x) = \rho_u(x)$ and $\rho_S(y) = \rho_u(y)$. So by $(\mathscr{P}1)$, $d_X(\rho_S(x), \rho_S(y)) = d_X(\rho_u(x), \rho_u(y)) \le 2L'$. Now let $u \neq v$. Since $d_T(\pi(x), \pi(y)) \le 1$, we have $x, y \in X_S$. So by definition of ρ_S, $d_X(\rho_S(x), \rho_S(y)) \le 1$. Therefore, we can take $L_{4.0.3} := C_{2.1.2}(\max\{2L', 1\})$. □

Lemma 4.0.4. *Let S be a subtree of T. Then for all $L \ge L_0$, there is a proper function $\eta_{4.0.4} = \eta_{4.0.4}(L) : \mathbb{R}_{\ge 0} \to \mathbb{R}_{\ge 0}$ such that the inclusion $\mathscr{M}_L(X_S) \to X$ is $\eta_{4.0.4}$-proper embedding.*

Proof. We denote the metric on $\mathscr{M}_L(X_S)$ by d'. Suppose $x_1, y_1 \in \mathscr{M}_L(X_S)$ such that $d_X(x_1, y_1) \leq r$ for some $r \in \mathbb{R}_{\geq 0}$. Then there are $x, y \in \mathscr{M}(X_S)$ such that $d'(x_1, x) \leq L$, $d'(y_1, y) \leq L$, and so $d_X(x, y) \leq r + 2L$. Let $\pi(x) = u$ and $\pi(y) = v$, and so $d_T(u, v) \leq r$. Let u' be the nearest point projection of u on S and v' be that of v on S. We consider the following two cases depending on whether $u' = v'$ or $u' \neq v'$.

Case 1: Suppose $u' \neq v'$. Let $x' \in X_{u'} \cap [x, y]$ and $y' \in X_{v'} \cap [x, y]$. Then $d_X(x, x') \leq r + 2L$, $d_X(y, y') \leq r + 2L$, and also $d_X(x', y') \leq r + 2L$. Now $x, x' \in \mathscr{M}_L(X_{u'})$ implies $d'(x, x') \leq \eta(r + 2L)$ (by ($\mathscr{P}$2)). Similarly, $d'(y, y') \leq \eta(r + 2L)$. Again $x', y' \in X_S$ implies $d'(x', y') \leq d_{X_S}(x', y') \leq \eta_{2.3.2}(r + 2L)$ (see Lemma 2.3.2). Hence by triangle inequality, $d'(x, y) \leq 2\eta(r + 2L) + \eta_{2.3.2}(r + 2L)$.

Case 2: Suppose $u' = v'$. Then by ($\mathscr{P}$0), $x, y \in \mathscr{M}(X_{u'})$. Hence $d'(x, y) \leq \eta(r + 2L)$.

Therefore, by triangle inequality, in both the cases, $d'(x_1, y_1) \leq 2L + 2\eta(r + 2L) + \eta_{2.3.2}(r + 2L) =: \eta_{4.0.4}(L)(r)$. □

Hence by Lemma 2.1.3, we have the following.

Lemma 4.0.5. *Let S be a subtree of T. Then for all $L \geq L_0$, there is $L_{4.0.5} = L_{4.0.5}(L)$ such that the inclusion $\mathscr{M}_L(X_S) \to X$ is $L_{4.0.5}$-qi embedding.*

Proposition 4.0.6 (Horizontal Subdivision)**.** *Let $J = [u, v] \subseteq T$ be an interval and $n_0 \in \mathbb{N}$. Then we can subdivide J into subintervals $J = J_0 \cup J_1 \cup \ldots \cup J_{n-1}$ such that $J_i = [w_i, w_{i+1}]$, $w_0 = u$, $w_n = v$ and each J_i is further subdivided into subintervals, $J_i = [w_i, w_{i,1}] \cup [w_{i,1}, w_{i,2}] \cup [w_{i,2}, w_{i,3}] \cup [w_{i,3}, w_{i+1}]$ such that the following hold.*

1. $\pi(\mathscr{M}(X_{w_i})) \cap J_i = [w_i, w_{i,1}]$, $\forall\, 0 \leq i \leq n-1$.

2. *For all i except possibly $i = n-1$, $d_T(w_{i,1}, w_{i,2}) \leq 2n_0$. Also, $[w_i, w_{i,1}], [w_{i,2}, w_{i,3}]$ and $[w_{i,3}, w_{i+1}]$ are special intervals. Moreover, $d_T(w_{i,3}, w_{i+1}) = 1$.*

3. $d_T(\pi(\mathscr{M}(X_{w_i})), \pi(\mathscr{M}(X_{w_{i+1}}))) > 2n_0$, $\forall\, 1 \leq i \leq n-2$.

Proof. The proof is by induction. Suppose we have constructed J_{i-1} and we want to construct J_i.

Case 1: Suppose $\mathscr{M}(X_{w_i}) \cap X_v \neq \emptyset$. Then we stop the process and set $n - 1 = i$, $J_{n-1} = [w_{n-1}, v]$ and $w_{n-1,s} = v = w_n$ for $s = 1, 2, 3$.

Case 2: Suppose $\mathscr{M}(X_{w_i}) \cap X_v = \emptyset$. Consider the vertex $w_{i,1} \in (w_i, v]$ in T, which is the farthest from w_i such that $\pi(\mathscr{M}(X_{w_i})) \cap [w_i, v] = [w_i, w_{i,1}]$. Now we consider the following two subcases.

Subcase (2A): Suppose $d_T(w_{i,1},\pi(\mathcal{M}(X_v))) \leq 2n_0$. Then we consider $w_{i,2} \in [w_{i,1},v]$ such that $X_{w_{i,2}} \cap \mathcal{M}(X_v) \neq \emptyset$ and $d_T(w_{i,1},w_{i,2}) \leq 2n_0$. Then we stop the process and set $n-1 = i$ and $w_{n-1,3} = v = w_n$.

Subcase (2B): Suppose $d_T(w_{i,1},\pi(\mathcal{M}(X_v))) > 2n_0$. We take $w_{i+1} \in [w_{i,1},v]$ is the farthest from v such that $d_T(w_{i,1},\pi(\mathcal{M}(X_{w_{i+1}}))) > 2n_0$. Let $w_{i,3} \in [w_{i,1},w_{i+1}]$ such that $d_T(w_{i,3},w_{i+1}) = 1$. Then by our choices, $d_T(w_{i,1},\pi(\mathcal{M}(X_{w_{i,3}}))) \leq 2n_0$. Now we fix $w_{i,2} \in [w_{i,1},w_{i,3}]$ such that $d_T(w_{i,1},w_{i,2}) \leq 2n_0$ and $X_{w_{i,2}} \cap \mathcal{M}(X_{w_{i,3}}) \neq \emptyset$. We also note that $d_T(\pi(\mathcal{M}(X_{w_i})),\pi(\mathcal{M}(X_{w_{i+1}}))) > 2n_0$, otherwise,

$$d_T(w_{i,1},\pi(\mathcal{M}(X_{w_{i+1}}))) \leq 2n_0.$$

Therefore, we get $J_i = [w_i,w_{i,1}] \cup [w_{i,1},w_{i,2}] \cup [w_{i,2},w_{i,3}] \cup [w_{i,3},w_{i+1}]$ with the required properties.

The induction stops at $(n-1)^{\text{th}}$ step if $w_n = v$. Therefore, we are through. □

Lemma 4.0.7. *Suppose $S_i, i = 1,2$ are two subtrees in T such that $S_1 \cap S_2 = \{u\}$. Then $\forall\, L \geq 0$, $\mathcal{M}_L(X_{S_1}) \cap \mathcal{M}_L(X_{S_2}) = \mathcal{M}_L(X_u)$.*

Proof. It is clear that $\mathcal{M}_L(X_u) \subseteq \mathcal{M}_L(X_{S_1}) \cap \mathcal{M}_L(X_{S_2})$. For the reverse inclusion, let $x \in \mathcal{M}_L(X_{S_1}) \cap \mathcal{M}_L(X_{S_2})$. Then there exists $x_i \in \mathcal{M}(X_{S_i})$ such that $d_X(x,x_i) \leq L$ for $i = 1,2$. Let $\pi(x_i) = t_i$ for $i = 1,2$. Now by $(\mathcal{P}0)$, if $d_T(t_2,S_1) \leq d_T(t_2,S_2)$ then $x_2 \in \mathcal{M}(X_u)$ or if $d_T(t_1,S_2) \leq d_T(t_1,S_1)$ then $x_1 \in \mathcal{M}(X_u)$. In either case, $x \in \mathcal{M}_L(X_u)$. Now suppose $d_T(t_2,S_1) > d_T(t_2,S_2)$ and $d_T(t_1,S_2) > d_T(t_1,S_1)$. Then since T is a tree, at least one of the geodesics $[x,x_1]_X$ or $[x,x_2]_X$ has to pass through X_u. Hence $d_X(x,X_u) \leq L$ and so $x \in \mathcal{M}_L(X_u)$. Therefore, we are done. □

The proof of the upcoming lemma follows from a similar line of reasoning as Lemma 4.0.7. So we omit the proof.

Lemma 4.0.8. *Suppose u,v and w lie on an interval in T such that $d_T(u,v) \leq d_T(u,w)$. Then $\forall\, L \geq 0$, $\mathcal{M}_L(X_{\{u,v\}}) \cap \mathcal{M}_L(X_{\{v,w\}}) = \mathcal{M}_L(X_v)$.*

Lemma 4.0.9. *For all $L \geq L_0$ there are constants $\delta_{4.0.9} = \delta_{4.0.9}(L)$ and $K_{4.0.9} = K_{4.0.9}(L)$ satisfying the following. Suppose u,v and w lie on an interval in T such that $d_T(u,v) \leq d_T(u,w)$ and $\mathcal{M}(X_u) \cap X_v \neq \emptyset$, $\mathcal{M}(X_v) \cap X_w \neq \emptyset$. Then $\mathcal{M}_L(X_{\{u,v,w\}})$ is $\delta_{4.0.9}$-hyperbolic metric space with the induced path metric. Further, the union of any two intersecting spaces among $\{\mathcal{M}_L(X_u), \mathcal{M}_L(X_v), \mathcal{M}_L(X_w)\}$ is $K_{4.0.9}$-quasiconvex in $\mathcal{M}_L(X_{\{u,v,w\}})$.*

Proof. For the first part, we will apply Proposition 2.2.7 for $n = 2$ (see Remark 2.2.8). Since $\mathcal{M}(X_u) \cap X_v \neq \emptyset$ and $\mathcal{M}(X_v) \cap X_w \neq \emptyset$, by $(\mathcal{P}4)$, $\mathcal{M}_L(X_{\{u,v\}})$ and $\mathcal{M}_L(X_{\{v,w\}})$

are $\delta(L)$-hyperbolic. Now by Lemma 4.0.8, $\mathscr{M}_L(X_{\{u,v\}}) \cap \mathscr{M}_L(X_{\{v,w\}}) = \mathscr{M}_L(X_v)$. Again, by Lemma 4.0.5, $\mathscr{M}_L(X_v)$ is $L_{4.0.5}(L)$-qi embedded in X and so is in both $\mathscr{M}_L(X_{\{u,v\}})$ and $\mathscr{M}_L(X_{\{v,w\}})$ with respect to their corresponding path metric. Thus by Remark 2.2.8, $\mathscr{M}_L(X_{\{u,v\}}) \cup \mathscr{M}_L(X_{\{v,w\}}) = \mathscr{M}_L(X_{\{u,v,w\}})$ is $\delta_{4.0.9}$-hyperbolic, where $\delta_{4.0.9} := \delta_{2.2.8}(\delta(L), L_{4.0.5}(L))$.

For the second part, we prove that $\mathscr{M}_L(X_{\{u,w\}})$ is a quasiconvex in $\mathscr{M}_L(X_{\{u,v,w\}})$ provided $\mathscr{M}_L(X_u) \cap \mathscr{M}_L(X_w) \neq \emptyset$, and the proof is similar for other intersections. By Lemma 4.0.5, $\mathscr{M}_L(X_u)$ and $\mathscr{M}_L(X_w)$ are $L_{4.0.5}(L)$-qi embedded in X and so are in $\mathscr{M}_L(X_{\{u,v,w\}})$. Then by Lemma 2.2.22 (1), $\mathscr{M}_L(X_u)$ and $\mathscr{M}_L(X_w)$ are K_1–quasiconvex in $\mathscr{M}_L(X_{\{u,v,w\}})$, where $K_1 = K_{2.2.22}(\delta_{4.0.9}(L), L_{4.0.5}(L), 0)$. Therefore, $\mathscr{M}_L(X_u) \cup \mathscr{M}_L(X_w)$ is $K_{4.0.9}$-quasiconvex subset of $\mathscr{M}_L(X_{\{u,v,w\}})$, where $K_{4.0.9} := K_1 + \delta_{4.0.9}$. □

Hyperbolicity of $\mathscr{M}_L(X_I)$ where I is a special interval:

Proposition 4.0.10. *Let I be a special interval in T. Then for all $L \geq L_0$ there is $\delta_{4.0.10} = \delta_{4.0.10}(L)$ such that $\mathscr{M}_L(X_I)$ is a $\delta_{4.0.10}$-hyperbolic metric space with the induced path metric from X.*

Proof. We will apply Proposition 2.2.6. Let $I = [u', v']$. Without loss of generality, we assume that $\mathscr{M}(X_{u'}) \cap X_{v'} \neq \emptyset$.

Choices: For a given $x \in \mathscr{M}(X_I)$, we fix once and for all $u_x \in I$ corresponding to x such that $x \in \mathscr{M}(X_{u_x})$. For a pair (x, y) of distinct points $\mathscr{M}(X_I)$, without loss of generality, we assume that $d_T(u', u_x) \leq d_T(u', u_y)$. Since $\mathscr{M}(X_{u'}) \cap X_{v'} \neq \emptyset$, so by ($\mathscr{P}0$), $\mathscr{M}(X_{u_x}) \cap X_{u_y} \neq \emptyset$. We take $c(x, y)$ a fixed geodesic path in $\mathscr{M}_L(X_{\{u_x, u_y\}})$. These paths serve as family of paths for Proposition 2.2.6.

Note that $\mathscr{M}(X_I)$ is L-dense subset in $\mathscr{M}_L(X_I)$. Let $x, y, z \in \mathscr{M}(X_I)$. Without loss of generality, we assume that $x \in \mathscr{M}(X_u)$, $y \in \mathscr{M}(X_v)$, $z \in \mathscr{M}(X_w)$ for $u, v, w \in [u', v']$ and $d_T(u', u) \leq d_T(u', v) \leq d_T(u', w)$. So by ($\mathscr{P}0$), $\mathscr{M}(X_u) \cap X_v \neq \emptyset$, $\mathscr{M}(X_v) \cap X_w \neq \emptyset$ and $\mathscr{M}(X_u) \cap X_w \neq \emptyset$.

Condition (1): Let $s, t \in \{u, v, w\}$ and $s \neq t$. By ($\mathscr{P}3$), $\mathscr{M}_L(X_{\{s,t\}})$ is $\eta'(L)$-properly embedded in X and so is in $\mathscr{M}_L(X_I)$. Hence the family of paths are $\eta'(L)$-properly embedded in $\mathscr{M}_L(X_I)$.

Condition (2): We want to prove that the triangle formed by paths $c(x, y)$, $c(y, z)$ and $c(x, z)$ is uniformly slim. Then by Lemma 4.0.9, $\mathscr{M}_L(X_{\{u,v,w\}}) \subseteq \mathscr{M}_L(X_I)$ is $\delta_{4.0.9}(L)$-hyperbolic with the induced path metric and for all distinct $s, t \in \{u, v, w\}$, $\mathscr{M}_L(X_{\{s,t\}})$ is K_1-quasiconvex in $\mathscr{M}_L(X_{\{u,v,w\}})$, where $K_1 = K_{4.0.9}(L)$. Now we will show that $\mathscr{M}_L(X_{\{s,t\}})$ is uniformly qi embedded in $\mathscr{M}_L(X_{\{u,v,w\}})$ where $s \neq t$ and $s, t \in \{u, v, w\}$. Let $N'_{K_1+1}(\mathscr{M}_L(X_{\{s,t\}})) \subseteq \mathscr{M}_L(X_{\{u,v,w\}})$ denote $(K_1 + 1)$-neighborhood of

$\mathscr{M}_L(X_{\{s,t\}})$ in metric of $\mathscr{M}_L(X_{\{u,v,w\}})$. Hence by Lemma 2.2.23 (1), $N'_{(K_1+1)}(\mathscr{M}_L(X_{\{s,t\}}))$ is L_1-qi embedded in $\mathscr{M}_L(X_{\{u,v,w\}})$ for some L_1 depending on $\delta_{4.0.9}(L)$ and K_1. Hence by Lemma 2.1.4, the inclusion $\mathscr{M}_L(X_{\{s,t\}}) \hookrightarrow \mathscr{M}_L(X_{\{u,v,w\}})$ is L_2-qi embedding, where $L_2 = L_{2.1.4}(\delta_{4.0.9}(L), K_1+1)$.

So by the stability of quasi-geodesic (see Lemma 2.2.2), there is constant $D = 2D_{2.2.2}(\delta_{4.0.9}, L_2, L_2) + \delta_{4.0.9}(L)$ such that the triangle formed by paths $c(x,y)$, $c(y,z)$, $c(x,z)$ is D-slim in the metric of $\mathscr{M}_L(X_{\{u,v,w\}})$ and so is in the metric of $\mathscr{M}_L(X_I)$.

Therefore, by Proposition 2.2.6, $\mathscr{M}_L(X_I)$ is $\delta_{4.0.10}$-hyperbolic, where $\delta_{4.0.10} = \delta_{2.2.6}(\eta'(L), D, L)$. □

As an iterated application of Proposition 4.0.10 along with Proposition 2.2.7 for $n = 2$ (see Remark 2.2.8), we obtain the following. The proof is omitted.

Lemma 4.0.11. *Given $L \geq L_0$ and $l \in \mathbb{N}$ there is a constant $\delta_{4.0.11} = \delta_{4.0.11}(L,l)$ satisfying the following. Let $I = [u,v]$ for $u,v \in T$ such that $d_T(u,v) \leq l$. Then $\mathscr{M}_L(X_I)$ is a $\delta_{4.0.11}$-hyperbolic metric space with the induced path metric from X.*

Hyperbolicity of $\mathscr{M}_L(X_I)$ where I is any interval: Before going into the proof, we first prove the following two lemmata which will be used in the proof.

Let $l \in \mathbb{N}$. Suppose $J = \cup_{i=1}^{4} J_i \subseteq T$ is an interval in T such that length of $J_2 \leq l$. Further J_1, J_3, J_4 are special intervals, and $J_i \cap J_{i+1}$ is a single vertex for $1 \leq i \leq 3$.

Lemma 4.0.12. *Suppose J is as described above. For all $l \in \mathbb{N}$ and $L \geq L_0$ there exists $\delta_{4.0.12} = \delta_{4.0.12}(L,l)$ such that $\mathscr{M}_L(X_J)$ is $\delta_{4.0.12}$-hyperbolic metric space with the induced path metric from X.*

Proof. We will apply Proposition 2.2.7 for $n = 2$ three times, successively on pairs $(\mathscr{M}_L(X_{J_1}), \mathscr{M}_L(X_{J_2}))$, $(\mathscr{M}_L(X_{J_1 \cup J_2}), \mathscr{M}_L(X_{J_3}))$ and $(\mathscr{M}_L(X_{J_1 \cup J_2 \cup J_3}), \mathscr{M}_L(X_{J_4}))$. Since J_1, J_3, J_4 are special interval, by Proposition 4.0.10, $\mathscr{M}_L(X_{J_i})$ is $\delta_{4.0.10}(L)$-hyperbolic for $i = 1,3,4$; and by Lemma 4.0.11, $\mathscr{M}_L(X_{J_2})$ is $\delta_{4.0.11}(L,l)$-hyperbolic. Suppose $\delta_1 = max\{\delta_{4.0.10}(L), \delta_{4.0.11}(L,l)\}$. Let $\{u\} = J_1 \cap J_2$. Then by Lemma 4.0.7, $\mathscr{M}_L(X_u) = \mathscr{M}_L(X_{J_1}) \cap \mathscr{M}_L(X_{J_2})$. Again by Lemma 4.0.5, $\mathscr{M}_L(X_u)$ is $L_{4.0.5}(L)$-qi embedded in X and so is in both $\mathscr{M}_L(X_{J_1})$ and $\mathscr{M}_L(X_{J_2})$ in their respective path metric. Therefore, by Remark 2.2.8, $\mathscr{M}_L(X_{J_1}) \cup \mathscr{M}_L(X_{J_2})$ is $\delta_{2.2.8}(\delta_1, L_{4.0.5}(L))$-hyperbolic.

Applying the similar argument as above on the remaining pairs we have mentioned, we conclude that $\mathscr{M}_L(X_{J_1 \cup J_2 \cup J_3}) \cup \mathscr{M}_L(X_{J_4}) = \mathscr{M}_L(X_J)$ is uniformly hyperbolic metric space with the induced path metric from X. Let the uniform constant be $\delta_{4.0.12} = \delta_{4.0.12}(L,l)$. □

Lemma 4.0.13. *Given $\delta \geq 0$, $L \geq L_0$ and a proper function $g : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$, there is a constant $D_{4.0.13} = D_{4.0.13}(\delta, L, g)$ such that the following holds.*

Let $Y \subseteq X$ be a δ-hyperbolic subspace of X such that $\mathscr{M}_L(X_u) \cup \mathscr{M}_L(X_v) \subseteq Y$ and $\pi(\mathscr{M}(X_u)) \cap \pi(\mathscr{M}(X_v)) = \emptyset$. Suppose Y is g-properly embedded in X. Then the pair $(\mathscr{M}_L(X_u), \mathscr{M}_L(X_v))$ is $D_{4.0.13}$-cobounded in Y.

Proof. We first note that by Lemma 4.0.5, $\mathscr{M}_L(X_u)$ is $L_{4.0.5}(L)$-qi embedded in X and so is in Y. Then by Lemma 2.2.22 (1), $\mathscr{M}_L(X_u)$ is K_1-quasiconvex in Y, where $K_1 = K_{2.2.22}(\delta_1, L_{4.0.5}(L), 0)$. Hence $\mathscr{M}(X_u)$ is K_2-quasiconvex in Y, where $K_2 = K_1 + L$. Similarly, $\mathscr{M}(X_v)$ is K_2-quasiconvex in Y.

Let $p : Y \to \mathscr{M}(X_u)$ be a nearest point projection map on $\mathscr{M}(X_u)$ in the metric of Y. Let $x, y \in \mathscr{M}(X_v)$ such that $p(x) = x_1,\ p(y) = y_1$. By Lemma 2.2.21 (3) and the symmetry of the proof, it is enough to show that $d_Y(x_1, y_1)$ is uniformly bounded.

Now by [10, Lemma 1.31 (2)], the arc-length parametrizations of $[x, x_1]_Y \cup [x_1, y_1]_Y$ and $[y, y_1]_Y \cup [y_1, x_1]_Y$ are $(3 + 2K_2)$-quasi-geodesic in Y. If $d_Y(x_1, y_1) \leq L_{2.2.5}(\delta, 3 + 2K_2, 3 + 2K_2)$, then we are done. Suppose $d_Y(x_1, y_1) > L_{2.2.5}(\delta, 3 + 2K_2, 3 + 2K_2)$. So by Lemma 2.2.5, $[x, x_1]_Y \cup [x_1, y_1]_Y \cup [y_1, y]_Y$ is λ_1-quasi-geodesic in Y, where $\lambda_1 = \lambda_{2.2.5}(\delta, 3 + 2K_2, 3 + 2K_2)$. Therefore, by stability of quasi-geodesic (see Lemma 2.2.2) and K_2-quasiconvexity of $\mathscr{M}(X_v)$ in Y, there exist $x_2, y_2 \in \mathscr{M}(X_v)$ such that $d_Y(x_1, x_2) \leq D_{2.2.2}(\delta, \lambda_1, \lambda_1) + K_2 = L_1$ (say) and $d_Y(y_1, y_2) \leq L_1$. Since $\pi(\mathscr{M}(X_u)) \cap \pi(\mathscr{M}(X_v)) = \emptyset$, by $(\mathscr{P}1)$, $\text{diam}\{\rho_u(\mathscr{M}(X_v))\} \leq C$ in X. Therefore, $x_2, y_2 \in \mathscr{M}(X_v)$ implies $d_X(\rho_u(x_2), \rho_u(y_2)) \leq C$. Again ρ_u is L'-coarsely Lipschitz retraction of X on $\mathscr{M}(X_u)$ (see $(\mathscr{P}1)$). Since $x_1, y_1 \in \mathscr{M}(X_u)$, so $\rho_u(x_1) = x_1,\ \rho_u(y_1) = y_1$. So $d_X(x_1, \rho_u(x_2)) = d_X(\rho_u(x_1), \rho_u(x_2)) \leq L'L_1 + L' = L_2$ (say). Similarly, $d_X(y_1, \rho_u(y_2)) \leq L_2$. Therefore, $d_X(x_1, y_1) \leq d_X(x_1, \rho(x_2)) + d_X(\rho_u(x_2), \rho_u(y_2)) + d_X(\rho_u(y_2), y_1) \leq 2L_2 + C \Rightarrow d_Y(x_1, y_1) \leq g(2L_2 + C)$ since Y is g-properly embedded in X.

Hence, $\text{diam}\{p(\mathscr{M}(X_v))\} \leq L_3$ in Y, where $L_3 = \max\{g(2L_2 + C),\ L_{2.2.5}(\delta, 3 + 2K_2, 3 + 2K_2)\}$. Therefore, (by the symmetry of the proof) the pair $(\mathscr{M}(X_u), \mathscr{M}(X_v))$ is L_3-cobounded in Y. Then by Lemma 2.2.21 (3), the pair $(\mathscr{M}_L(X_u), \mathscr{M}_L(X_v))$ is $D_{4.0.13}$-cobounded in Y, where $D_{4.0.13} = D_{2.2.21}(\delta_1, K_2, L_3, L)$. ☐

Now we are ready to proof the main result.

Proposition 4.0.14. *Let I be an interval in T. Then for all $L \geq L_0$ there is $\delta_{4.0.14} = \delta_{4.0.14}(L)$ such that $\mathscr{M}_L(X_I)$ is $\delta_{4.0.14}$-hyperbolic metric space with the induced path metric from X.*

Proof. Let $I = J_0 \cup J_1 \cup \cdots \cup J_{n-1}$ be a subdivision of the interval I coming from horizontal subdivision, Proposition 4.0.6, with $n_0 = [L]+2$, where $[L]$ is the greatest integer not greater than L. We refer to Proposition 4.0.6 for the description of $J_i = [w_i, w_{i+1}]$. Then by Lemma 4.0.12, $\mathscr{M}_L(X_{J_i})$ is $\delta_{4.0.12}(L, 2n_0)$-hyperbolic metric space for all $i \in \{0, 1, \cdots, n-1\}$.

Now we will verify all the conditions of Proposition 2.2.7. Let $X_i = \mathscr{M}_L(X_{J_i})$ for $0 \le i \le n-1$ and $Y_{i+1} = X_i \cap X_{i+1} = \mathscr{M}_L(X_{w_{i+1}})$ for $0 \le i \le n-2$ (see Lemma 4.0.7).

(1) For $0 \le i \le n-1$, X_i is δ_1-hyperbolic metric space, where $\delta_1 = \delta_{4.0.12}(L, 2n_0)$.

(2) By Lemma 4.0.5, $Y_{i+1} = \mathscr{M}_L(X_{w_{i+1}})$ is $L_{4.0.5}(L)$-qi embedded in X so is in both X_i and X_{i+1} for $0 \le i \le n-2$.

(3) Note that by $(\mathscr{P}0)$, if $x \in X_i \setminus Y_{i+1}$ and $y \in X_{i+1} \setminus Y_{i+1}$ then $w_{i+1} \in [\pi(x), \pi(y)] \setminus \{\pi(x), \pi(y)\}$. Hence every path in $\mathscr{M}_L(X_I)$ joining points X_i and X_{i+1} passes through Y_{i+1}.

(4) Suppose $i \in \{1, 2, \cdots, n-2\}$. Note that $\pi(\mathscr{M}(X_{w_i})) \cap \pi(\mathscr{M}(X_{w_{i+1}})) = \emptyset$ (by Proposition 4.0.6 (3)). Again X_i is $\eta_{4.0.4}(L)$-properly embedded in X and so is in $\mathscr{M}_L(X_i)$. Also, $\mathscr{M}_L(X_{w_i}) \cup \mathscr{M}_L(X_{w_{i+1}}) \subseteq X_i$. Then by Lemma 4.0.13, there is D depending on δ_1, L and $\eta_{4.0.4}(L)$ such that the pair (Y_i, Y_{i+1}) is D-cobounded in X_i.

(5) Let $1 \le i \le n-2$ and $d_{X_i}(Y_i, Y_{i+1}) \le 1$. Then $d_X(\mathscr{M}(X_{w_i}), \mathscr{M}(X_{w_{i+1}})) \le 2L+1$, and so $d_T(\pi(\mathscr{M}(X_{w_i})), \pi(\mathscr{M}(X_{w_{i+1}}))) \le 2L+1 \le 2n_0$ (by our choice of n_0). This contradicts to (3) of Proposition 4.0.6.

Therefore, by Proposition 2.2.7, $\mathscr{M}_L(X_I)$ is $\delta_{4.0.14}$-hyperbolic, where $\delta_{4.0.14} = \delta_{2.2.7}(\delta_1, L_{4.0.5}(L), D)$. □

As a consequence of Proposition 4.0.14 along with Proposition 2.2.7, we obtain following. We omit the proof.

Lemma 4.0.15. *Given $L \ge L_0$ there is $\delta_{4.0.15} = \delta_{4.0.15}(L)$ satisfying the following. Let $u, v, w \in T$ and T_{uvw} be the tripod in T with vertices u, v, w. Then $\mathscr{M}_L(X_{T_{uvw}})$ is $\delta_{4.0.15}$-hyperbolic metric space with the induced path metric from X.*

Proof of Theorem 4.0.1: We fix $L = L_0$. We show that X satisfies all the conditions of Proposition 2.2.6. Let $X_{vsp} = \cup_{u \in T} X_u$. Note that X_{vsp} is a 1-dense subspace of X. So given any two points $x, y \in X_{vsp}$, we define path joining them as follows:

Let $x \in X_u$ and $y \in X_v$ for some $u, v \in T$. Note that $X_{[u,v]} = \pi^{-1}([u, v])$. We fix once and for all, a geodesic path $c(x, y)$ in $\mathscr{M}_L(X_{[u,v]})$ joining x and y. These paths serve as family of paths for Proposition 2.2.6.

Let $x, y, z \in V(X)$ such that $\pi(x) = u$, $\pi(y) = v$ and $\pi(z) = w$.

Condition (1): For all distinct $s,t \in \{u,v,w\}$, by Proposition 4.0.4, $\mathscr{M}_L(X_{[s,t]})$ is $\eta_{4.0.4}(L)$-properly embedded in X and so are the paths.

Condition (2): Let T_{uvw} be the tripod in T with vertices $\{u,v,w\}$. By Lemma 4.0.15, $\mathscr{M}_L(X_{T_{uvw}})$ is $\delta_{4.0.15}(L)$-hyperbolic metric space. Again by Lemma 4.0.5, $\mathscr{M}_L(X_{[u,v]})$, $\mathscr{M}_L(X_{[v,w]})$ and $\mathscr{M}_L(X_{[u,w]})$ are $L_{4.0.5}(L)$-qi embedded subspaces of X and so are of $\mathscr{M}_L(X_{T_{uvw}})$. Then the hyperbolicity of $\mathscr{M}_L(X_{T_{uvw}})$ and the stability of quasi-geodesic in $\mathscr{M}_L(X_{T_{uvw}})$ (see Lemma 2.2.2) imply that the triangle formed by paths $c(x,y)$, $c(y,z)$ and $c(x,z)$ is D-slim in $\mathscr{M}_L(X_{T_{uvw}})$ and so is in X, where

$$D = 2D_{2.2.2}(\delta_{4.0.15}(L), L_{4.0.5}(L), L_{4.0.5}(L)) + \delta_{4.0.15}(L).$$

Therefore, by Proposition 2.2.6, X is $\delta_{2.2.6}(\eta_{4.0.4}(L), D, 1)$-hyperbolic. This completes the proof. $\square$

Chapter 5

A Combination Theorem for Trees of Metric Bundles

In this chapter we will prove Theorem 1.2.4. Our standard assumptions for this chapter are that the tree of metric bundles (X,B,T) must satisfy axiom **H** and a flaring condition. However, by Remark 2.4.8 (a) one can observe that k-flaring condition for a large k is enough (see introduction of Section 5.2, Section 5.3 and Section 5.4).

5.1 Semicontinuous families: flow space and ladder

Motivated by the construction of semicontinuous families of spaces in [9, Chapter 3], we build subspaces analogous to that in trees of metric bundles. We follow the same terminology used in [9]. Also, following [9], we will see two special kinds of subspaces: flow spaces and ladders. These are the building blocks, which will be shown to be hyperbolic, towards proving Theorem 1.2.4.

Suppose (X,B,T) is a tree of metric bundles as in Definition 2.4.2. Suppose $\mathfrak{Y} = \bigcup_{v\in T_{\mathfrak{Y}}, b\in B_v} Q_{b,v}$ where $Q_{b,v} \subseteq F_{b,v}$ and $T_{\mathfrak{Y}} := \text{hull}(\pi(\mathfrak{Y}))$. With this we define the following.

Definition 5.1.1 (Semicontinuous subspace)**.** Let $K \geq 1, C \geq 0$ and $\varepsilon \geq 0$. We say that $\mathfrak{Y} \subseteq X$ is a (K,C,ε)-semicontinuous family in X with a central base $\mathfrak{B} = \pi_B^{-1}(\mathfrak{T})$ for some central subtree $\mathfrak{T}$ in $T_{\mathfrak{Y}}$ if the following hold.

1. Let $v \in T_{\mathfrak{Y}}$ and $b \in B_v$. Then $Q_{b,v}$ is a $2\delta_0$-quasiconvex subset of $F_{b,v}$ and $\bigcup_{b\in B_v} Q_{b,v} \subseteq X_v$ forms a K-metric bundle (see Definition 2.4.11) over B_v. Moreover, $\bigcup_{v\in T_{\mathfrak{Y}},\, b\in B_v} Q_{b,v}$ forms a K-metric bundle over $\mathfrak{B}$ in X.

2. Let $v,w \in T_{\mathfrak{Y}}$ such that $w \notin \mathfrak{T}, d_T(v,w) = 1, d_T(\mathfrak{T},w) > d_T(\mathfrak{T},v)$. Let $[\mathfrak{v},\mathfrak{w}]$ be the edge joining $\mathfrak{v} \in B_v$ and $\mathfrak{w} \in B_w$. Then $Hd_{\mathfrak{v}\mathfrak{w}}(P_{\mathfrak{w}}(Q_{\mathfrak{v},v}), Q_{\mathfrak{w},w}) \leq \varepsilon$ (see Notation 2.4.10 for $P_{\mathfrak{w}}$) and $d_{\mathfrak{v}\mathfrak{w}}(x, Q_{\mathfrak{v},v}) \leq K, \forall\, x \in Q_{\mathfrak{w},w}$. Moreover, if both $v,w \in \mathfrak{T}$ then $Hd_{\mathfrak{v}\mathfrak{w}}(Q_{\mathfrak{v},v}, Q_{\mathfrak{w},w}) \leq K$.

3. Suppose $w \notin T_{\mathfrak{Y}}, v \in T_{\mathfrak{Y}}$ such that $d_T(v,w) = 1$. Let $[\mathfrak{v},\mathfrak{w}]$ be the edge joining $\mathfrak{v} \in B_v$ and $\mathfrak{w} \in B_w$. Then the pair $(Q_{\mathfrak{v},v}, F_{\mathfrak{w}.w})$ is C-cobounded in the metric $F_{\mathfrak{v}\mathfrak{w}}$.

4. Additionally, let $B' \subseteq \pi_B^{-1}(T_{\mathfrak{Y}})$ be $(1,6\delta_0)$-qi embedded subspace in B. Suppose $v \in T_{\mathfrak{Y}}$ and $B'_v := B_v \cap B'$. Then $\forall\, v \in T_{\mathfrak{Y}}$ and $\forall\, b \in B_v \setminus B'_v$, $\mathrm{diam}^f(Q_{b,v}) \leq C$. Let $\mathfrak{Y}' := \pi_X^{-1}(B') \cap \mathfrak{Y}$.

Remark 5.1.2. (a) The condition (4) is used in Section 5.3 (more precisely, in Lemma 5.3.14), otherwise, all the time $B' = \pi_B^{-1}(T_{\mathfrak{Y}})$ and so $\mathfrak{Y}' = \mathfrak{Y}$.

(b) If T is a single vertex, say, $\{u\}$ then $\mathfrak{Y}$ is K-metric bundle over B_u.

(c) If $\pi_B : B \to T$ is a graph isomorphism, then $\mathfrak{Y}$ is the same as the semicontinuous family defined in the book [9, Chapter 3].

(d) (**Maximality**) 'Moreover part' in conditions (1) and (2) are equivalent provided first parts of (1) and (2) hold. Let $z \in \mathfrak{Y}$ and t_z be the nearest point projection of $\pi(z)$ on $\mathfrak{T}$. Suppose $B_z = \pi_B^{-1}([t_z, \pi(z)]) \cup \mathfrak{B}$. Then it follows from the conditions (1) and (2) that there is a compatible K-qi section (see Definition 2.4.4), say, Σ_z over B_z lying inside $\mathfrak{Y}$. Sometimes (more precisely, in Subsection 5.2.2), we work with maximal qi sections through points in $\mathfrak{Y}$ as follows. Let S be a subtree of $T_{\mathfrak{Y}}$ containing $\mathfrak{T} \cup [t_z, \pi(z)]$. Note that $B_z \subseteq B_S$. Let $\mathscr{G} = \{\eta :$ η is a compatible K-qi section over B_S through z lying inside $\mathfrak{Y}\}$. We put an order '$\leq$' (inclusion) on $\mathscr{G}$ as follows. For $\eta, \eta' \in \mathscr{G}$, we say $\eta \leq \eta'$ if and only if $\eta \subseteq \eta'$. This order '$\leq$' makes $\mathscr{G}$ a poset. It is easy to see every chain has an upper bound in $\mathscr{G}$. Therefore, by Zorn's lemma, we get a maximal compatible K-qi section through z over a base, say, B_S containing B_z and contained in $\pi_B^{-1}(T_{\mathfrak{Y}})$. By abusing notation, we still denote the base for this maximal section by B_z.

(e) One also can introduce other (uniform) constants for quasiconvexity of sets $Q_{b,v}$ in $F_{b,v}$ and qi embedding of B' in $\pi_B^{-1}(T_{\mathfrak{Y}})$ instead of $2\delta_0$ and $(1,6\delta_0)$ respectively. However, for simplicity, we will exclusively work with these constants.

(f) Later on, in our statements, we suppress the dependence on the constants C, ε and the other structural constants of the tree of metric bundles when dealing with semicontinuous family.

Now we will see nice properties (see Theorem 5.1.3, Proposition 5.1.4 and Corollary 5.1.5) enjoyed by semicontinuous families. In the proof, we use the same notations as in the Definition 5.1.1. First, we prove that there is uniformly coarsely Lipschitz retraction of X on semicontinuous families. This is motivated by Mitra's retraction in [1] (see also [10, Theorem 3.2] and [9, Theorem 3.3]). In this thesis, we refer this retraction as *Mitra's retraction*.

Theorem 5.1.3. *Given $K \geq 1, \varepsilon \geq 0$ and $C \geq 0$ there is a constant $L_{5.1.3} = L_{5.1.3}(K)$ such that the following holds.*

If $\mathfrak{Y}$ is a (K,C,ε)-semicontinuous family (as in Definition 5.1.1) in X, then there is $L_{5.1.3}$-coarsely Lipschitz retraction $\rho_{5.1.3} = \rho_{\mathfrak{Y}'} : X \to \mathfrak{Y}'$ of X on $\mathfrak{Y}'$.

Proof. Let $X_{vsp} = \cup_{u \in T} X_u$ and $x, y \in X_{vsp}$ such that $d_X(x,y) \leq 1$. Then by Lemma 2.1.2, it is enough to define a map $\rho : X_{vsp} \to \mathfrak{Y}'$ for which $d_X(\rho(x), \rho(y))$ is uniformly bounded.

Let us define $\rho : X_{vsp} \to \mathfrak{Y}'$ as follows. Suppose $x \in X_{vsp}$ and $b = \pi_X(x), u = \pi(x)$. If $b \in B'$, then $\rho(x)$ is defined to be a nearest point projection on $Q_{b,u}$ in metric $F_{b,u}$. Now suppose $b \notin B'$. Let a be a nearest point projection of b on B' and $\pi_B(a) = v$. Since B' is $(1, 6\delta_0)$-qi embedded in $\pi_B^{-1}(T_{\mathfrak{Y}})$, so is in B. Then B' is K'-quasiconvex in B, where $K' = K_{2.2.22}(\delta_0, max\{1, 6\delta_0\}, 0)$. Note that a is coarsely well defined. We also assume that $a_- \in [a,b]_B$ such that $a \neq a_-$ and $d_B(a, a_-) \leq 1$, and let $\pi_B(a_-) = w$. Let x' be a nearest point projection of x on $F_{a_-,w}$ in the metric X. Then we define $\rho(x)$ as nearest point projection of x' on $Q_{a,v}$ in the path metric $F_{aa_-} := \pi_X^{-1}([a, a_-]_B)$.

Now we prove $d_X(\rho(x), \rho(y))$ is uniformly bounded where $x, y \in X_{vsp}$ and $d_X(x,y) \leq 1$. Let $\pi_X(x) = a$, $\pi_X(y) = b$ and $\pi(x) = v$, $\pi(y) = w$. We consider the following cases, depending on the position of a, b, v and w.

Case 1: Suppose $a, b \in B'$. We consider two subcases, depending on whether $v = w$ or $v \neq w$.

Subcase (1A): Suppose $v = w$. We proof it by dividing into two parts, when $a = b$ and $a \neq b$.

Subsubcase (1AA): Suppose $a = b$. Since $Q_{a,v}$ is $2\delta_0$-quasiconvex in $F_{a,v}$, by Lemma 2.2.21 (1), $d^f(\rho(x), \rho(y)) \leq 2C_{2.2.21}(\delta_0, 2\delta_0) = L_1$ (say).

Subsubcase (1AB): Suppose $a \neq b$. Note that $d_B(a,b) \leq d_X(x,y) \leq 1$. Now through each point in $\mathfrak{Y} \cap X_v$ there is K-qi section over B_v lying inside $\mathfrak{Y} \cap X_v$. Define a map $\psi : F_{a,v} \to F_{b,v}$ as follows. For $z \in Q_{a,v}$ take $\psi(z) \in Q_{b,v}$ such that $d_{X_v}(\psi(z), z) \leq 2K$. For $z \notin Q_{a,v}$ take $\psi(z) \in F_{b,v}$ such that $d(\psi(z), z) \leq c_0$ (as in Definition 2.4.2). In either case, $d_{X_v}(\psi(z), z) \leq 2K, \forall\, z \in F_{a,v}$. Then by [10, Lemma 1.15], ψ is $g(2K + c_0)$-quasi-isometry for some function $g : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$. Again $\forall\, \xi \in Q_{b,v}, \exists\, \eta \in Q_{a,v}$ such that $d_{X_v}(\xi, \eta) \leq 2K$ and η is further $2K$-close to a point in $\psi(Q_{a,v})$, i.e. $Q_{b,v} \subseteq N_{4K}(\psi(Q_{a,v}))$ in the metric of X_v. Since fibers are ϕ-properly embedded, $Q_{b,v} \subseteq N^f_{\phi(4K)}(Q_{a,v})$ (see 2.4.10 for notation). Then $\psi(Q_{a,v}) \subseteq Q_{b,v}$ implies $Hd^f(\psi(Q_{a,v}), Q_{b,v}) \leq \phi(4K)$ in the metric of $F_{b,v}$.

Let y_1 be a nearest point projection of y on $\psi(Q_{a,v})$ in the metric of $F_{b,v}$. Now by Lemma 2.2.22 (1), there is a constant $K_1 = K_{2.2.22}(\delta_0, g(2K+c_0), 2\delta_0) \geq 2\delta_0$ such that $\psi(Q_{a,v})$ is K_1-quasiconvex in $F_{a,v}$, and so by Lemma 2.2.21 (2), $d^f(y_1, \rho(y)) \leq E_{2.2.21}(\delta_0, K_1, \phi(4K))$. Again Lemma 2.2.22 (2) says that there is $D = D_{2.2.22}(\delta_0, g(2K+c_0), 2\delta_0)$ for which $d_{X_v}(\psi(\rho(x)), y_1) \leq D$. By the definition of ψ, we also have $d_{X_v}(\rho(x), \psi(\rho(x))) \leq 2K$. Hence combining these four inequalities, we have $L_2 = 2K + D_{2.2.22}(\delta_0, g(2K+c_0), 2\delta_0) + E_{2.2.21}(\delta_0, K_1, \phi(4K))$ such that $d_{X_v}(\rho(x), \rho(y)) \leq L_2$.

Subcase (1B): Suppose $v \neq w$. Then it follows that $d_X(x,y) = 1$, $d_B(a,b) = 1$, $x \in F_{\mathfrak{v},v}$, $y \in F_{\mathfrak{w},w}$ and $a \in B_v$, $b \in B_w$. To make things notationally consistent, we assume that $a = \mathfrak{v}, b = \mathfrak{w}$. Note that $[\mathfrak{v}, \mathfrak{w}] \subseteq B' \subseteq \pi_B^{-1}(T_{\mathfrak{Y}})$. Now irrespective of whether $[\mathfrak{v}, \mathfrak{w}]$ is an edge in $\mathfrak{B}$ or not, we have $Hd_{\mathfrak{v}\mathfrak{w}}(P_{\mathfrak{w}}(Q_{\mathfrak{v},v}), Q_{\mathfrak{w},w}) \leq max\{2K, \varepsilon\}$. Then by Lemma 2.3.5 (1), $d_X(\rho(x), \rho(y)) \leq d_{\mathfrak{v}\mathfrak{w}}(\rho(x), \rho(y)) \leq R_{2.3.5}(2\delta_0, K, max\{2K, \varepsilon\}) = L_3$ (say).

Case 2: Suppose one of a, b belongs to B'. Without loss of generality, we assume that $a \in B'$ and $b \notin B'$. Here we also consider the following subcases, depending on whether $v = w$ or $v \neq w$.

Subcase (2A): Suppose $v = w$. Let $\pi_X(\rho(y)) = a'$. Then $d_B(a, a') \leq 2$, and so $Hd_{X_v}(Q_{a,v}, Q_{a',v}) \leq 2K + K = 3K$. Again, since $\text{diam}^f(Q_{b,v}) \leq C$ then $\text{diam}(Q_{a,v}) \leq 4K + C$. Thus $\text{diam}^f(Q_{a,v}) \leq \phi(4K+C)$, and so $d_X(\rho(x), \rho(y)) \leq 3K + \phi(4K+C) = L_4$ (say).

Subcase (2B): Suppose $v \neq w$. For the consistency of notation, we assume that $a = \mathfrak{v}, b = \mathfrak{w}$. Without loss of generality, we let $d_T(\pi_B(\mathfrak{B} \cap B'), v) < d_T(\pi_B(\mathfrak{B} \cap B'), w)$. Note that in this case, $\rho(y)$ is nearest point projection of y on $Q_{\mathfrak{v},v}$ in the metric of $F_{\mathfrak{v}\mathfrak{w}}$. Then by Lemma 2.3.6, $d_X(\rho(x), \rho(y)) \leq d_{\mathfrak{v}\mathfrak{w}}(\rho(x), \rho(y)) \leq R_{2.3.6}(2\delta_0) = L_5$ (say).

Case 3: Suppose $a, b \notin B'$. Let $a' = \pi_X(\rho(x)), b' = \pi_X(\rho(y))$ and $a'_- \in [a', a]_B$, $b'_- \in [b', b]_B$ such that $a'_- \neq a'$, $b'_- \neq b'$ and $d_B(a', a'_-) \leq 1, d_B(b', b'_-) \leq 1$. Since $d_X(x,y) \leq 1$, then $\pi_B(a') = \pi_B(b')$ and $\pi_B(a'_-) = \pi_B(b'_-)$. Let us rename $\pi_B(a')$ as v and $\pi_B(a'_-)$ as w not to make notation-heavy. We consider the following subcases depending on whether $v = w$ or $v \neq w$.

Subcase (3A): Suppose $v = w$. Since B' is K'-quasiconvex in B (where $K' = K_{2.2.22}(\delta_0, max\{1, 6\delta_0\}, 0)$) and $d_B(a,b) \leq d_X(x,y) \leq 1$, so by Lemma 2.2.21 (1), $d_B(a', b') \leq 2C_{2.2.21}(\delta_0, K')$. Note that $\mathfrak{Y} \cap X_v$ forms a K-metric bundle and so $Hd_{X_v}(Q_{a',v}, Q_{b',v}) \leq 2KC_{2.2.21}(\delta_0, K') + K$. Since $a'_-, b'_- \in B_v$ then $\text{diam}^f(Q_{a',v}) \leq$

$\phi(4K+C)$ and $\mathrm{diam}^f(Q_{b',v}) \leq \phi(4K+C)$ (see *Subcase* (2A) for instance). Hence $d_X(\rho(x),\rho(y)) \leq 2KC_{2.2.21}(\delta_0,K') + K + \phi(4K+C) = L_6$ (say).

Subcase (3B): Suppose $v \neq w$. For the consistency of notation, we assume that $a' = b' = \mathfrak{v}$ and $a'_- = b'_- = \mathfrak{w}$. In the same way, we consider the following two subcases.

Subsubcase (3BA): Let $[\mathfrak{v},\mathfrak{w}]$ be an edge in $\pi_B^{-1}(T_{\mathfrak{Y}})$. Without loss of generality, we assume that $d_B(\pi_B(\mathfrak{B}\cap B'),v) < d_B(\pi_B(\mathfrak{B}\cap B'),w)$. By the assumption $\mathrm{diam}^f(Q_{\mathfrak{w},w}) \leq C$ and so $diam(P_{\mathfrak{w}}(Q_{\mathfrak{v},v})) \leq 2\varepsilon + C$ in the metric of $F_{\mathfrak{v}\mathfrak{w}}$. Then by Lemma 2.2.18, there is a constant C_1 depending on λ'_0, δ'_0 and $2\varepsilon + C$ such that the pair $(Q_{\mathfrak{v},v}, F_{\mathfrak{w},w})$ is C_1-cobounded in the metric of $F_{\mathfrak{v}\mathfrak{w}}$. Therefore, $d_X(\rho(x),\rho(y)) \leq C_1 = L_7$ (say).

Subsubcase (3BB): Suppose $[\mathfrak{v},\mathfrak{w}]$ is not an edge in $\pi_B^{-1}(T_{\mathfrak{Y}})$. Then by definition of semicontinuous family $\mathfrak{Y}$, the pair $(Q_{\mathfrak{v},v}, F_{\mathfrak{w},w})$ is C-cobounded in the path metric of $F_{\mathfrak{v}\mathfrak{w}}$. So $d_X(\rho(x),\rho(y)) \leq C$.

Suppose $L = max\{L_i, C : 1 \leq i \leq 7\} = max\{L_i : 1 \leq i \leq 7\}$. Therefore, by Lemma 2.1.2, one can take $L_{5.1.3} = C_{2.1.2}(L)$. □

Next we show that a uniform neighborhood of semicontinuous families are path connected in X and uniformly properly embedded in X with the induced path metric from X. As a consequence, we will see that it is also (uniformly) qi embedded in X (see Corollary 5.1.5).

Proposition 5.1.4. *Suppose $K \geq 1, C \geq 0$ and $\varepsilon \geq 0$. Then for all $L \geq max\{2\delta_0 + 1, 2K\}$ there exists $\eta_{5.1.4} = \eta_{5.1.4}(K,L) : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ such that the following holds.*

If $\mathfrak{Y}$ is a (K,C,ε)-semicontinuous family (as in Definition 5.1.1) in X, then $N_L(\mathfrak{Y}')$ is path connected and with the path metric on $N_L(\mathfrak{Y})$ induced from X, the inclusion $i : N_L(\mathfrak{Y}') \hookrightarrow X$ is $\eta_{5.1.4}$-proper embedding.

Proof. It is clear that $N_L(\mathfrak{Y}')$ is path connected. We denote the path metric on $N_L(\mathfrak{Y}')$ induced from X by d'.

For second part, we first show that for $r \in \mathbb{R}_{\geq 0}, x,y \in \mathfrak{Y}'$ and $d_X(x,y) \leq r$ we have bound on $d'(x,y)$ in terms of r; in the end, we show the same for points in $N_L(\mathfrak{Y})$. Fix $u \in \pi_B(\mathfrak{B}\cap B')$. We take t, the center of the tripod in T with vertices $\pi(x)$, $\pi(y)$, u if $[\pi(x),\pi(y)]_T \cap \pi_B(\mathfrak{B}\cap B') = \emptyset$; otherwise, $t \in [\pi(x),\pi(y)]_T \cap \pi_B(\mathfrak{B}\cap B')$ arbitrary. Let $a = \pi_X(x), b = \pi_X(y)$. Then $d_B(a,b) \leq d_X(x,y) \leq r$. Since the inclusion $B' \hookrightarrow \pi_B^{-1}(T_{\mathfrak{Y}})$ is $(1,6\delta_0)$-qi embedding, $d_{B'}(a,b) \leq r + 6\delta_0$. Let $c \in B_t \cap [a,b]_{B'}$ be arbitrary. Then $d_{B'}(a,c) \leq r + 6\delta_0$ and $d_{B'}(c,b) \leq r + 6\delta_0$. By taking K-qi lifts of geodesics $[a,c]_{B'}$ and $[c,b]_{B'}$ in $\mathfrak{Y}$ (more precisely, in $\mathfrak{Y}'$), we get, $x_1, y_1 \in Q_{c,t}$ such that $d'(x,x_1) \leq 2K(r+6\delta_0)$ and $d'(y,y_1) \leq 2K(r+6\delta_0)$ (see Lemma 2.4.12 (3)).

Now $d_X(x_1,y_1) \leq d_X(x_1,x) + d_X(x,y) + d_X(y,y_1) \leq d'(x_1,x) + d_X(x,y) + d'(y,y_1) \leq 4K(r+6\delta_0) + r \Rightarrow d^f(x_1,y_1) \leq \phi(4K(r+6\delta_0)+r)$.

Note that $N^f_{2\delta_0+1}(Q_{c,t}) \subseteq N_L(\mathfrak{Y}') \cap F_{c,t}$ as $L \geq 2\delta_0+1$, and $x_1,y_1 \in Q_{c,t}$. Then by Lemma 2.2.23 (1), there is $D(r)$ depending on r such that $d'(x_1,y_1) \leq D(r)$. Hence $d'(x,y) \leq d'(x,x_1) + d'(x_1,y_1) + d'(y_1,y) \leq 4K(r+6\delta_0) + D(r)$.

Now suppose $x,y \in N_L(\mathfrak{Y}')$ such that $d_X(x,y) \leq r$. Then $\exists\, x_1,y_1 \in \mathfrak{Y}'$ such that $d'(x,x_1) \leq L$ and $d'(y,y_1) \leq L$. So $d_X(x_1,y_1) \leq 2L+r$. Thus by above, $d'(x_1,y_1) \leq 4K(2L+r+6\delta_0) + D(2L+nr)$. Hence combining these inequalities, we get, $d'(x,y) \leq 4K(2L+r+6\delta_0) + D(2L+r) + 2L$.

Therefore, we can take $\eta_{5.1.4} : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ sending $r \mapsto 4K(2L+r+6\delta_0) + D(2L+r) + 2L$. □

As a consequence we have the following corollary (see Lemma 2.1.3).

Corollary 5.1.5. *Suppose $K \geq 1, C \geq 0$ and $\varepsilon \geq 0$. Then for all $L \geq max\{2\delta_0 + 1, 2K\}$ there exists $L_{5.1.5} = L_{5.1.5}(K,L) := L_{2.1.3}(\eta_{5.1.4}(K,L), L_{5.1.3}(K), L)$ such that the following holds.*

If $\mathfrak{Y}$ is a (K,C,ε)-semicontinuous family (as in Definition 5.1.1) in X, then the inclusion $i : N_L(\mathfrak{Y}') \hookrightarrow X$ is $L_{5.1.5}$-qi embedding in X.

Remark 5.1.6. Conclusion of Theorem 5.1.3, Proposition 5.1.4 and Corollary 5.1.5 hold for $\mathfrak{Y}$ as well.

5.1.1 Flow space

Suppose (X,B,T) is a tree of metric bundles as in Definition 2.4.2. Let $u \in T, a \in B_u$. Given a subset $A_{a,u}$ of $F_{a,u}$, we define (rather, construct) the flow space of $A_{a,u}$, which is a semicontinuous family in X with central base (possibly bigger than) B_u. The construction of this flow space is by induction as follows. Let $k \geq K_{2.4.12}$ be fixed.

Step 1: $\mathscr{G}_{a,u} = \{\gamma : \gamma$ is a k-qi section over B_u through a point in $A_{a,u}\}$. Let $b \in B_u$ and $Q_{b,u} = \text{hull}\{\gamma(b) : \gamma \in \mathscr{G}_{a,u}\} \subseteq F_{b,u}$; where quasiconvex hull is considered in the corresponding fiber. Note that $Q_{b,u}$ is $2\delta_0$-quasiconvex in $F_{b,u}$ and by Lemma 2.4.12 (2), $\bigcup_{b \in B_u} Q_{b,u}$ forms a $C_{2.4.12}(k)$-metric bundle over B_u.

Step 2: We extend this to other X_v by induction on $d_T(u,v)$. Suppose we have extended it till X_v, where $d_T(u,v) = n$. Let $w \in T$ such that $d_T(u,w) = n+1$ and $d_T(v,w) = 1$. Let $[\mathfrak{v},\mathfrak{w}]$ be the edge joining $\mathfrak{v} \in B_v$ and $\mathfrak{w} \in B_w$. We denote $Q_{a,t}$ as the intersection of the flow space we are constructing with $F_{a,t}$ for $t \in T, a \in B_t$. We first flow $Q_{\mathfrak{v},v}$ in $F_{\mathfrak{w},w}$ and then by Step 1 above in the entire X_w, provided $Q_{\mathfrak{w},w} \neq \emptyset$.

Let us fix $R \geq R_{2.2.13}(\delta_0', \lambda_0')$ and let $R' = R'_{2.2.13}(\delta_0', \lambda_0', R)$. Note that $Q_{\mathfrak{v},v}$ is $2\delta_0$-quasiconvex in $F_{\mathfrak{v},v}$ and so is λ_0'-quasiconvex in $F_{\mathfrak{vw}}$ (see Lemma 2.3.4 (2)). Suppose $N_R^{\mathfrak{vw}}(Q_{\mathfrak{v},v}) \cap F_{\mathfrak{w},w} \neq \emptyset$. Then by Lemma 2.2.13 (2), $P_{\mathfrak{w}}(Q_{\mathfrak{v},v}) \subseteq N_{R'}^{\mathfrak{vw}}(Q_{\mathfrak{v},v}) \cap F_{\mathfrak{w},w} =: Q'_{\mathfrak{w},w}$ (say). Let $Q_{\mathfrak{w},w} := \text{hull}(Q'_{\mathfrak{w},w}) \subseteq F_{\mathfrak{w},w}$, where quasiconvex hull is considered in $F_{\mathfrak{w},w}$. Note that $Q_{\mathfrak{w},w}$ is $2\delta_0$-quasiconvex in $F_{\mathfrak{w},w}$. Now we apply Step 1 to $Q_{\mathfrak{w},w}$ by considering all k-qi section over B_w through points in $Q_{\mathfrak{w},w}$.

If $N_R^{\mathfrak{vw}}(Q_{\mathfrak{v},v}) \cap F_{\mathfrak{w},w} = \emptyset$, then we will not 'flow' $Q_{\mathfrak{v},v}$ in that direction. In other words, let S be the component of $T \setminus \{v\}$ containing w. Then $\forall\, t \in S$ and $\forall\, b \in B_t$, we have, $Q_{b,t} = \emptyset$.

Now we prove the following properties which verify that the subspace we are constructing is a semicontinuous family. Let $v, w \in T$ such that $d_T(u,v) < d_T(u,w)$.

Property 1: Suppose $Q_{\mathfrak{v},v} \neq \emptyset$ and $N_R^{\mathfrak{vw}}(Q_{\mathfrak{v},v}) \cap F_{\mathfrak{w},w} = \emptyset$. Then the pair $(Q_{\mathfrak{v},v}, F_{\mathfrak{w},w})$ is $C := D_{2.2.13}(\delta_0', \lambda_0')$-cobounded in $F_{\mathfrak{vw}}$. Indeed, $Q_{\mathfrak{v},v}$ and $F_{\mathfrak{w},w}$ are λ_0'-quasiconvex in $F_{\mathfrak{vw}}$ (see Lemma 2.3.4 (2)). So by Lemma 2.2.13 we are done.

Property 2: Suppose $Q_{\mathfrak{v},v}$ and $Q_{\mathfrak{w},w}$ are nonempty. Then $Q_{\mathfrak{w},w} \subseteq N_{K'}^{\mathfrak{vw}}(Q_{\mathfrak{v},v})$ for some uniform constant K'.

Proof. Let $x \in Q_{\mathfrak{w},w}$. Then $\exists\, x_1, x_2 \in Q'_{\mathfrak{w},w}$ and $x' \in [x_1,x_2]_{F_{\mathfrak{w},w}}$ such that $d^f(x,x') \leq \delta_0$. Let $y_1, y_2 \in Q_{\mathfrak{v},v}$ such that $d_{\mathfrak{vw}}(x_i,y_i) \leq R'$, $i = 1,2$. Note that $Q_{\mathfrak{v},v}$ is L_0'-qi embedded in $F_{\mathfrak{vw}}$ (Lemma 2.3.4). Then by slimness of quadrilateral in $F_{\mathfrak{vw}}$ with vertices x_1, x_2, y_1 and y_2, there is $x'' \in Q_{\mathfrak{v},v}$ such that $d_{\mathfrak{vw}}(x',x'') \leq 2D_{2.2.2}(\delta_0', L_0', L_0') + R' + 2\delta_0' + 2\delta_0$. Thus $d(x,x'') \leq 2D_{2.2.2}(\delta_0', L_0', L_0') + R' + 2\delta_0' + 2\delta_0 + \delta_0 =: K'$.

Property 3: Suppose both $Q_{\mathfrak{v},v}$ and $Q_{\mathfrak{w},w}$ are nonempty. Then $Hd_{\mathfrak{vw}}(P_{\mathfrak{w}}(Q_{\mathfrak{v},v}), Q_{\mathfrak{w},w}) \leq \varepsilon$ for some uniform constant ε.

Proof. Property (2) tells that $Q_{\mathfrak{w},w} \subseteq N_{2K'}^{\mathfrak{vw}}(P_{\mathfrak{w}}(Q_{\mathfrak{v},v}))$. Again by construction $P_{\mathfrak{w}}(Q_{\mathfrak{v},v}) \subseteq Q_{\mathfrak{w},w}$. So $Hd_{\mathfrak{vw}}(P_{\mathfrak{w}}(Q_{\mathfrak{v},v}), Q_{\mathfrak{w},w}) \leq 2K' =: \varepsilon$.

We denote $\mathscr{F}l_K(A_{a,u}) := \bigcup\limits_{v \in T, b \in B_v} Q_{b,v}$; where $K = max\{K', C_{2.4.12}(k)\}$.

Definition 5.1.7 (Flow space). We say $\mathscr{F}l_K(A_{a,u}) = \bigcup_{v \in T, b \in B_v} Q_{b,v}$ is the flow space of $A_{a,u}$ with parameters $k \geq K_{2.4.12}$ and $R \geq R_{2.2.13}(\delta_0', \lambda_0')$. It is clear from the construction that $\mathscr{F}l_K(A_{a,u})$ is a (K, C, ε)-semicontinuous family, where $K = K_{5.1.7}(k,R) = max\{K', C_{2.4.12}(k)\}, C = D_{2.2.13}(\delta_0', \lambda_0')$ and $\varepsilon = \varepsilon_{5.1.7}(R)$ are as in above properties.

In particular, for any $u \in T$ and $a \in B_u$, suppose $\mathscr{F}l_K(F_{a,u})$ is the flow space of $F_{a,u}$ with parameters $k \geq K_{2.4.12}$, $R \geq R_{2.2.13}(\delta_0', \lambda_0')$, where $K = K_{5.1.7}(k,R)$. Then by Lemma 2.4.12 (1), $X_u \subseteq \mathscr{F}l_K(F_{a,u})$. In this case, we denote the flow space by $\mathscr{F}l_K(X_u)$ and we say $\mathscr{F}l_K(X_u)$ is the flow space of X_u with parameters k, R.

Although $\mathscr{F}l_K(A_{a,u})$ depends on the constants C,ε and the other structural constants of (X,B,T), we make them implicit in our notation.

We have defined the flow space of a subset of a fiber and of X_u for $u \in T$ in Definition 5.1.7. Below we make it a bit general and will use this in Section 5.4.

Definition 5.1.8 (Flow space of metric bundles). Fix $k \geq K_{2.4.12}$. Let S be a subtree of T and $R \geq max\{R_{2.2.13}(\delta_0', \lambda_0'), k\}$. Suppose H is a k-metric bundle over B_S (see Definition 2.4.11). Let $H_{b,u} := H \cap F_{b,u}$ for $u \in S, b \in B_u$. We also assume that $H_{b,u}$ is $2\delta_0$-quasiconvex in $F_{b,u}$. Suppose $\mathscr{S}(S,1) = \{w \in T : d_T(S,w) = 1\}$. Let $w \in \mathscr{S}(S,1)$ and $v \in S$ such that $d_T(v,w) = 1$. Let T_{wv} be the connected component of $T \setminus \{v\}$ containing w along with the edge $[v,w]$. Suppose $[\mathfrak{v},\mathfrak{w}]$ is the edge joining $\mathfrak{v} \in B_v$ and $\mathfrak{w} \in B_w$. Let $\mathscr{F}l_K^{T_{wv}}(H_{\mathfrak{v},v})$ is the flow space of $H_{\mathfrak{v},v}$ only inside $X_{T_{wv}}$ (with the parameters k,R) such that $\mathscr{F}l_K^{T_{wv}}(H_{\mathfrak{v},v}) \cap X_v = H \cap X_v$, where $K = K_{5.1.7}(k,R)$. We define $\mathscr{F}l_K(H) := \bigcup_{w \in \mathscr{S}(S,1)} \mathscr{F}l_K^{T_{wv}}(H_{\mathfrak{v},v})$ as flow space of H.

It is clear that $\mathscr{F}l_K(H)$ is a (K,C,ε)-semicontinuous family with a central base B_S, where $K = K_{5.1.7}(k,R)$, $C = D_{2.2.13}(\delta_0', \lambda_0')$, $\varepsilon = \varepsilon_{5.1.7}(R)$.

Let us record some constants from the above discussion in the following lemma.

Lemma 5.1.9. *Given $k \geq K_{2.4.12}$ and $R \geq max\{R_{2.2.13}(\delta_0', \lambda_0'), k\}$ there are constants $K_{5.1.9} = K_{5.1.9}(k,R) = K_{5.1.7}(k,R), \varepsilon_{5.1.9} = \varepsilon_{5.1.9}(R) = \varepsilon_{5.1.7}(R)$ and $C_{5.1.9} = D_{2.2.13}(\delta_0', \lambda_0')$ such that the following hold.*

1. *Let $u \in T$. Then $\mathscr{F}l_{K_{5.1.9}}(X_u)$ is a $(K_{5.1.9}, C_{5.1.9}, \varepsilon_{5.1.9})$-semicontinuous family with a central base B_u.*

2. *Let S be a subtree of T and H be a k-metric bundle over B_S. Then $\mathscr{F}l_{K_{5.1.9}}(H)$ is a $(K_{5.1.9}, C_{5.1.9}, \varepsilon_{5.1.9})$-semicontinuous family with a central base B_S.*

Note that in Lemma 5.1.9, (2) needs the condition $R \geq max\{R_{2.2.13}(\delta_0', \lambda_0'), k\}$, whereas (1) holds for $R \geq R_{2.2.13}(\delta_0', \lambda_0')$.

Consider the flow spaces with parameters k,R as taken in Lemma 5.1.9. Let $\boldsymbol{K} = K_{5.1.9}(k,R)$, $\boldsymbol{C} = C_{5.1.9}$ and $\boldsymbol{\varepsilon} = \varepsilon_{5.1.9}(R)$. Flow spaces being semicontinuous families, we have and restate the following three results for flow spaces (see Proposition 5.1.3, Proposition 5.1.4, Corollary 5.1.5 and also Remark 5.1.6), as they will be utilized extensively in Section 5.3 and Section 5.4.

Proposition 5.1.10. *There exists $L_{5.1.10} = L_{5.1.10}(K)$-coarsely Lipschitz retraction $\rho_{5.1.10} : X \to \mathscr{F}l_K(Z)$ where $Z \in \{X_u, H\}$.*

Proposition 5.1.11. *Given $L \geq max\{2K, 2\delta_0 + 1\}$ there is $\eta_{5.1.11} = \eta_{5.1.11}(K,L) : \mathbb{N} \to \mathbb{N}$ such that the inclusion $i : N_L(\mathscr{F}l_K(Z)) \hookrightarrow X$ is $\eta_{5.1.11}$-proper embedding in X, where $Z \in \{X_u, H\}$.*

Corollary 5.1.12. *Given $L \geq max\{2K, 2\delta_0 + 1\}$, there is $L_{5.1.12} = L_{5.1.12}(K,L)$ such that the inclusion $i : N_L(\mathscr{F}l_K(Z)) \hookrightarrow X$ is $L_{5.1.12}$-qi embedding, where $Z \in \{X_u, H\}$.*

Flow space being semicontinuous family, the fundamental and crucial property is the existence of qi sections through each point over the respective domain. If one carefully analyses the construction, one will realize the following. Given a qi section, one can construct, by taking larger neighborhood at the junction (more precisely, in $F_{\mathfrak{vw}}$), a flow space containing the qi section. The following lemma captures this property. Since it is straightforward, we omit the proof.

Lemma 5.1.13. *Given $K \geq K_{5.1.9}(K_{2.4.12}, R_{2.2.13}(\delta'_0, \lambda'_0))$ there are constants $K_{5.1.13} = K_{5.1.13}(K) = K_{5.1.9}(K,K), C_{5.1.13} = C_{5.1.9}(\delta'_0, \lambda'_0)$ and $\varepsilon_{5.1.13} = \varepsilon_{5.1.13}(K) = \varepsilon_{5.1.9}(K)$ such that the following holds.*

Suppose $S \subseteq T$ is a subtree and $u \in S$. Let γ be a K-qi section over B_S. Suppose $\mathscr{F}l_{K_{5.1.13}}(X_u)$ is the flow space of X_u with parameters K and K. Then $\gamma \subseteq \mathscr{F}l_{K_{5.1.13}}(X_u)$ and $\mathscr{F}l_{K_{5.1.13}}(X_u)$ is a $(K_{5.1.13}, C_{5.1.13}, \varepsilon_{5.1.13})$-semicontinuous family.

Notation 5.1.14. We define a function $\kappa^{(i)} \longmapsto \kappa^{(i+1)}$ to measure the iteration in the above Lemma 5.1.13. In other words, suppose $\kappa \geq K_{5.1.9}(K_{2.4.12}, R_{2.2.13}(\delta'_0, \lambda'_0))$. Define $\kappa^{(0)} = \kappa, \kappa^{(i+1)} = K_{5.1.13}(\kappa^{(i)}, \kappa^{(i)})$. Then for the flow space $\mathscr{F}l_{\kappa^{(i)}}(X_u)$ of X_u with parameters $k = \kappa^{(i-1)}$ and $R = \kappa^{(i-1)}$, we have, $\mathscr{F}l_{\kappa^{(i)}}(X_u) \subseteq \mathscr{F}l_{\kappa^{(i+1)}}(X_u)$.

5.1.2 Ladder

Suppose (X, B, T) is a tree of metric bundles. Let $K \geq 1, C \geq 0, \varepsilon \geq 0$. A ladder $\mathscr{L} \subseteq X$ is a (K, C, ε)-semicontinuous family with a central base, say, $\mathfrak{B}$ such that fibers are geodesic segments. However, in addition, we also have the following extra properties.

(L1): For all $v \in T_{\mathscr{L}}$ (where $T_{\mathscr{L}} = \text{hull}(\pi(\mathscr{L}))$), $\mathscr{L} \cap X_v$ is a special K-ladder (see Definition 2.4.11) over B_v. Moreover, $\mathscr{L} \cap X_{\mathfrak{B}}$ forms a special K-ladder over $\mathfrak{B}$.

We also have the following orientation on fiber geodesic.

Notation 5.1.15. $\mathfrak{T} := \pi_B(\mathfrak{B})$, $\mathscr{L}_{a,v} := \mathscr{L} \cap F_{a,v}\ \forall\ a \in B_v, \forall\ v \in T_{\mathscr{L}}$.

We fix $u \in \mathfrak{T}$ once and for all. As $\mathscr{L} \cap X_u$ is bounded by two K-qi sections over $\mathfrak{B}$, we set one of them as top and the other one as bot to give an orientation on $\mathscr{L} \cap X_u$; where the abbreviation top and bot is coming from 'top' and 'bottom' respectively.

So we have an orientation for each fiber geodesics of $\mathscr{L}\cap X_u$ as '*bot* to *top*'. We put orientation on $\mathscr{L}$ by induction on $d_T(u,v)$ as follows, where $v\in T_{\mathscr{L}}$. Let $v,w\in T_{\mathscr{L}}$ such that $d_T(u,v)<d_T(u,w)$ and $d_T(v,w)=1$. We also assume that $w\notin\mathfrak{T}$; and it is mentioned in the end for the case $w\in\mathfrak{T}$. The orientation to fiber geodesics of $\mathscr{L}\cap X_w$ depend on that of $\mathscr{L}\cap X_v$. Let $[\mathfrak{v},\mathfrak{w}]$ be the edge joining $\mathfrak{v}\in B_v$ and $\mathfrak{w}\in B_w$. Let $\mathscr{L}_{\mathfrak{v},v}:=[x_{\mathfrak{v},v},y_{\mathfrak{v},v}]^f$ and $\mathscr{L}_{\mathfrak{w},w}:=[x_{\mathfrak{w},w},y_{\mathfrak{w},w}]^f$ such that $top(\mathscr{L}_{\mathfrak{v},v})=x_{\mathfrak{v},v}$ and $bot(\mathscr{L}_{\mathfrak{v},v})=y_{\mathfrak{v},v}$. Let $\bar{x}_{\mathfrak{v},v},\bar{y}_{\mathfrak{v},v}\in\mathscr{L}_{\mathfrak{v},v}$ such that $d_{\mathfrak{v}\mathfrak{w}}(\bar{x}_{\mathfrak{v},v},x_{\mathfrak{w},w})\le K$ and $d_{\mathfrak{v}\mathfrak{w}}(\bar{y}_{\mathfrak{v},v},y_{\mathfrak{w},w})\le K$. Let $h_{wv}:\mathscr{L}_{\mathfrak{w},w}\to\mathscr{L}_{\mathfrak{v},v}$ be a monotonic map (see Lemma 2.2.4) sending $x_{\mathfrak{w},w}$ to $\bar{x}_{\mathfrak{v},v}$ and $y_{\mathfrak{w},w}$ to $\bar{y}_{\mathfrak{v},v}$ such that $d_{\mathfrak{v}\mathfrak{w}}(h_{wv}(x),x)\le k_{2.2.4}(\delta_0',L_0',K)$ for all $x\in\mathscr{L}_{\mathfrak{w},w}$. We fix this h_{wv} once and for all for such v,w. The orientation in $\mathscr{L}\cap X_w$ depends on the order of how $\bar{x}_{\mathfrak{v},v}$ and $\bar{y}_{\mathfrak{v},v}$ appear in $\mathscr{L}_{\mathfrak{v},v}$. Let the K-qi sections γ_1 and γ_2 bound $\mathscr{L}\cap X_w$ such that $\gamma_1(\mathfrak{w})=x_{\mathfrak{w},w}$ and $\gamma_2(\mathfrak{w})=y_{\mathfrak{w},w}$. If $y_{\mathfrak{v},v}\le\bar{y}_{\mathfrak{v},v}<\bar{x}_{\mathfrak{v},v}\le x_{\mathfrak{v},v}$, then we set γ_1 to be *top* and γ_2 to be *bot* for $\mathscr{L}\cap X_w$. In other words, $top(\mathscr{L}_{a,w})=\gamma_1(a)$ and $bot(\mathscr{L}_{a,w})=\gamma_2(a)$, $a\in B_w$. If $y_{\mathfrak{v},v}\le\bar{x}_{\mathfrak{v},v}<\bar{y}_{\mathfrak{v},v}\le x_{\mathfrak{v},v}$, then we set γ_2 to be *top* and γ_1 to be *bot* for $\mathscr{L}\cap X_w$. In other words, $top(\mathscr{L}_{a,w})=\gamma_2(a)$ and $bot(\mathscr{L}_{a,w})=\gamma_1(a)$, $a\in B_w$. However, by renaming, we always denote $\bar{x}_{\mathfrak{v},v}\in Im(h_{wv})$ for the closest point (in the induced metric on $\mathscr{L}_{\mathfrak{v},v}$) to $x_{\mathfrak{v},v}$ and $\bar{y}_{\mathfrak{v},v}\in Im(h_{wv})$ for the closest point to $y_{\mathfrak{v},v}$. Otherwise, i.e., if $\bar{x}_{\mathfrak{v},v}=\bar{y}_{\mathfrak{v},v}$, then we set any one of γ_1,γ_2 as *top* and the other one as *bot*. If $w\in\mathfrak{T}$ then $v\in\mathfrak{T}$. Then the monotonic map $h_{wv}:\mathscr{L}_{\mathfrak{w},w}\to\mathscr{L}_{\mathfrak{v},v}$ sends $x_{\mathfrak{w},w}$ to $x_{\mathfrak{v},v}$ and $y_{\mathfrak{w},w}$ to $y_{\mathfrak{v},v}$. We let $top(\mathscr{L}):=\cup_{a\in B_v,v\in T_{\mathscr{L}}}top(\mathscr{L}_{a,v})$ and $bot(\mathscr{L}):=\cup_{a\in B_v,v\in T_{\mathscr{L}}}top(\mathscr{L}_{a,v})$.

(L2) **Quasi-isometric (qi) section in $\mathscr{L}$**: Let $x\in\mathscr{L}$ such that $t=\pi(x)$. Suppose s is the nearest point projection of t on $\mathfrak{T}$. Then there is a K-qi section through x lying inside $\mathscr{L}$ over $B_x:=\mathfrak{B}\cup B_{[s,t]}$. By a qi section in $\mathscr{L}$, we always mean that it obeys the order at the junction between two metric bundles given by the family $\{h_{wv}\}$. In other words, suppose $u\in\mathfrak{T}$ is fixed (as mentioned above) and Σ is a qi section in $\mathscr{L}$ over, say, B_1. Let $[\mathfrak{v},\mathfrak{w}]$ be an edge in B_1 joining $\mathfrak{v}\in B_v$ and $\mathfrak{w}\in B_w$ corresponding to the edge $[v,w]$ in T such that $d_T(u,v)<d_T(u,w)$. Then $\Sigma(\mathfrak{v})=h_{wv}(\Sigma(\mathfrak{w}))$. Thus $d_{\mathfrak{v}\mathfrak{w}}(\Sigma(\mathfrak{v}),\Sigma(\mathfrak{w}))\le k_{2.2.4}(\delta_0',L_0',K)$. As $max\{K,k_{2.2.4}(\delta_0',L_0',K)\}=k_{2.2.4}(\delta_0',L_0',K)$, Σ would be a $k_{2.2.4}(\delta_0',L_0',K)$-qi section. By abusing notation, we still say Σ is K-qi section.

Definition 5.1.16 (Ladder). A ladder $\mathscr{L}\subseteq X$ with parameters $K\ge1,C\ge0,\varepsilon\ge0$ is a (K,C,ε)-semicontinuous family with a central base, say, $\mathfrak{B}$ along with a family of monotonic maps $\{h_{wv}\}$ (as described above) and $(L1)$, $(L2)$.

We refer $\mathscr{L}$ as (K,C,ε)-ladder with a central base $\mathfrak{B}$. Sometimes we denote $\mathscr{L}$ by $\mathscr{L}_K$ to emphasise K.

One can think of $x_{\mathfrak{w},w}$ and $y_{\mathfrak{w},w}$ as uniformly close to nearest point projections (in $d_{\mathfrak{v}\mathfrak{w}}$-metric) on $F_{\mathfrak{w},w}$ of the points $x_{\mathfrak{v},v}$ and $y_{\mathfrak{v},v}$ respectively. (This uniform bound is measured in terms of ε.) Again it follows from the definition of ladder that the pairs $([x_{\mathfrak{v},v},\bar{x}_{\mathfrak{v},v}]^f,F_{\mathfrak{w},w})$ and $([y_{\mathfrak{v},v},\bar{y}_{\mathfrak{v},v}]^f,F_{\mathfrak{w},w})$ are uniformly cobounded in $F_{\mathfrak{v}\mathfrak{w}}$. This is proved in the lemma below.

Lemma 5.1.17. *Given $K\geq 1, C\geq 0, \varepsilon\geq 0$ there exist $C_{5.1.17}=C_{5.1.17}(K,C,\varepsilon)$ and $\varepsilon_{5.1.17}=\varepsilon_{5.1.17}(K,C,\varepsilon)$ such that the following holds.*

Suppose $\mathscr{L}$ is a (K,C,ε)-ladder with a central base $\mathfrak{B}$. Let $[v,w]$ be an edge in T. Suppose $[\mathfrak{v},\mathfrak{w}]$ is the edge joining $\mathfrak{v}\in B_v$ and $\mathfrak{w}\in B_w$ such that $w\notin\mathfrak{T}$ and $d_T(\mathfrak{T},v)<d_T(\mathfrak{T},w)$. Let $z\in[\bar{x}_{\mathfrak{v},v},\bar{y}_{\mathfrak{v},v}]^f\subseteq\mathscr{L}_{\mathfrak{v},v}$ and $h_{wv}(z')=z$ for $z'\in\mathscr{L}_{\mathfrak{w},w}$ (with the notation used in the Definition 5.1.16). Then:

(1) *The pairs $([x_{\mathfrak{v},v},\bar{x}_{\mathfrak{v},v}]^f,F_{\mathfrak{w},w})$ and $([y_{\mathfrak{v},v},\bar{y}_{\mathfrak{v},v}]^f,F_{\mathfrak{w},w})$ are $C_{5.1.17}$-cobounded in $F_{\mathfrak{v}\mathfrak{w}}$.*

(2) $Hd_{\mathfrak{v}\mathfrak{w}}(P_{\mathfrak{w}}([x_{\mathfrak{v},v},z]^f),[x_{\mathfrak{w},w},z']^f)\leq\varepsilon_{5.1.17}$, $Hd_{\mathfrak{v}\mathfrak{w}}(P_{\mathfrak{w}}([y_{\mathfrak{v},v},z]^f),[y_{\mathfrak{w},w},z']^f)\leq\varepsilon_{5.1.17}$.

Proof. (1) We prove only for the pair $([x_{\mathfrak{v},v},\bar{x}_{\mathfrak{v},v}]^f,F_{\mathfrak{w},w})$ as the other one has a similar proof. For ease of notation, let $x_1=x_{\mathfrak{v},v}, x_2=x_{\mathfrak{w},w}, x_3=\bar{x}_{\mathfrak{v},v}$. Let $x_1'=P_{\mathfrak{w}}(x_1)$ and $x_1''\in\mathscr{L}_{\mathfrak{w},w}$ such that $d_{\mathfrak{v}\mathfrak{w}}(x_1',x_1'')\leq\varepsilon$. Suppose $x_4=h_{wv}(x_1'')$ and $x_4'=P_{\mathfrak{w}}(x_4)$. Note that $x_3\in[x_4,x_1]^f$. Now $d_{\mathfrak{v}\mathfrak{w}}(x_1'',x_4)\leq K$ implies $d_{\mathfrak{v}\mathfrak{w}}(x_1'',x_4')\leq 2K$. So $d_{\mathfrak{v}\mathfrak{w}}(x_1',x_4')\leq\varepsilon+2K$. Again, $[x_1,x_4]^f$ is a L_0'-quasi-geodesic in $F_{\mathfrak{v}\mathfrak{w}}$ (see Lemma 2.3.4). Then by [9, Corollary 1.116], there is a constant C_1 depending on $\delta_0',\lambda_0',L_0'$ such that $Hd_{\mathfrak{v}\mathfrak{w}}(P_{\mathfrak{w}}([x_1,x_4]^f),[x_1',x_4']_{F_{\mathfrak{v}\mathfrak{w}}})\leq C_1$. Let $x_3'=P_{\mathfrak{w}}(x_3)$. Then $d_{\mathfrak{v}\mathfrak{w}}(x_3',x_1')\leq d_{\mathfrak{v}\mathfrak{w}}(x_3',[x_1',x_4']_{F_{\mathfrak{v}\mathfrak{w}}})+d_{\mathfrak{v}\mathfrak{w}}(x_1',x_4')\leq C_1+\varepsilon+2K$. Again applying [9, Corollary 1.116] to $[x_1,x_3]^f$, the diameter of $P_{\mathfrak{w}}([x_1,x_3]^f)$ in the metric $F_{\mathfrak{v}\mathfrak{w}}$ is bounded by $\leq 2C_1+(C_1+\varepsilon+2K)=3C_1+\varepsilon+2K$. Therefore, by Lemma 2.2.18, we can take a constant depending on λ_0',δ_0' and $3C_1+\varepsilon+2K$ as our required constant $C_{5.1.17}$. So, we are done.

(2) We only prove that $Hd_{\mathfrak{v}\mathfrak{w}}(P_{\mathfrak{w}}([x_{\mathfrak{v},v},z]^f),[x_{\mathfrak{w},w},z']^f)$ is uniformly bounded as the other one has a similar proof. We continue with the notations used in (1). From the above proof, we note that $d_{\mathfrak{v}\mathfrak{w}}(x_1',x_3')\leq C_1+\varepsilon+2K$. So $d_{\mathfrak{v}\mathfrak{w}}(x_1',x_2)\leq d_{\mathfrak{v}\mathfrak{w}}(x_1',x_3')+d_{\mathfrak{v}\mathfrak{w}}(x_3',x_3)+d_{\mathfrak{v}\mathfrak{w}}(x_3,x_2)\leq C_1+\varepsilon+2K+K+K=C_1+\varepsilon+4K$. Let $x'=P_{\mathfrak{w}}(x)$ for $x\in[x_1,z]^f\subseteq\mathscr{L}_{\mathfrak{v},v}$. If $x\in[x_3,z]^f$, then $\exists\, y\in[x_2,z']^f\subseteq\mathscr{L}_{\mathfrak{w},w}$ such that $h_{wv}(y)=x$ and $d_{\mathfrak{v}\mathfrak{w}}(y,x)\leq K$. So $d_{\mathfrak{v}\mathfrak{w}}(y,x')\leq 2K$. If $x\in[x_1,x_3]$, then $d_{\mathfrak{v}\mathfrak{w}}(x',x_2)\leq d_{\mathfrak{v}\mathfrak{w}}(x',x_1')+d_{\mathfrak{v}\mathfrak{w}}(x_1',x_2)\leq C_{5.1.17}+C_1+\varepsilon+4K=2(2C_1+\varepsilon+3K)$. So $P_{\mathfrak{w}}([x_{\mathfrak{v},v},z]^f)$ is contained in $2(2C_1+\varepsilon+3K)$-neighborhood of $[x_2,z']^f\subseteq\mathscr{L}_{\mathfrak{w},w}$ in the metric of $F_{\mathfrak{v}\mathfrak{w}}$. For the other inclusion, let $\xi'\in[x_2,z']^f\subseteq\mathscr{L}_{\mathfrak{w},w}$. Then there is $\xi\in$

$[x_3, z]^f \subseteq \mathscr{L}_{\mathfrak{v},v}$ such that $h_{wv}(\xi') = \xi$ and $d_{\mathfrak{v}\mathfrak{w}}(\xi, \xi') \leq K$. Then ξ' is contained in $2K$-neighborhood of $P_{\mathfrak{w}}([x_1, z]^f)$ in the path metric of $F_{\mathfrak{v}\mathfrak{w}}$. As ξ' is arbitrary in $[x_2, z']^f \subseteq \mathscr{L}_{\mathfrak{w},w}$. So, we take $\varepsilon_{5.1.17} = 2(2C_1 + \varepsilon + 3K)$. □

Definition 5.1.18 (Subladder). Suppose $\mathscr{L}$ is (K, C, ε)-ladder. A subladder $\mathscr{L}'$ in $\mathscr{L}$ is a (K', C', ε')-ladder whose fiber geodesics are subsegments of the corresponding fiber geodesics of the ladder $\mathscr{L}$ and the family of monotonic maps are restrictions of the given family $\{h_{wv}\}$. The constants K', C' and ε' depend on the given ones.

Definition 5.1.19 (Girth and Neck). Suppose $\mathscr{L}$ is a (K, C, ε)-ladder with a central base $\mathfrak{B}$. Let $B_1 \subseteq \mathfrak{B}$ and let $\ell(\alpha)$ denote the length of a fiber geodesic α. Girth of the ladder $\mathscr{L}$ over B_1 is denoted by $\mathscr{L}^g|_{B_1}$ and defined as $\inf\{\ell(\mathscr{L}_{b,v}) : v \in \pi_B(B_1), b \in B_v \cap B_1\}$. For a $A \geq 0$, A-neck of the ladder $\mathscr{L}$ inside B_1 is denoted by $\mathscr{L}^n(A)|_{B_1}$ and defined as $\{b \in B_1 : \ell(\mathscr{L}_{b,v}) \leq A,\ \pi_B(b) = v\}$.

Let $\mathscr{L}$ be a (K, C, ε)-ladder. Let $x, y \in \mathscr{L}$ and Σ_x, Σ_y be K-qi sections through x, y over B_x, B_y respectively. Suppose $B_{xy} = B_x \cap B_y$. Then the restriction of Σ_x and Σ_y over B_{xy} form a special $K_{2.4.14}(K)$-ladder over B_{xy}. We denote the restriction by $\mathscr{L}_{xy}$. With these notations, we have the following lemma.

Lemma 5.1.20. *Given $K \geq 1$, $C \geq 0, \varepsilon \geq 0$ and $A \geq M_K$ (coming from K-flaring condition) there exists $K_{5.1.20} = K_{5.1.20}(K, A)$ such that the following holds.*

Let $\mathscr{L}$ be a (K, C, ε)-ladder with a central base $\mathfrak{B}$. Then for $x, y \in \mathscr{L}$, $\mathscr{L}^n_{xy}(A)|_{B_{xy}}$ is $K_{5.1.20}$-quasiconvex in B_{xy}, and consequently, in B as well.

Proof. If $\mathscr{L}^n_{xy}(A)|_{B_{xy}} = \emptyset$, then there is nothing to prove. Suppose $\mathscr{L}^n_{xy}(A)|_{B_{xy}} \neq \emptyset$ and $a, b \in \mathscr{L}^n_{xy}(A)|_{B_{xy}}$. Without loss of generality, we assume that $d^f(\Sigma_x(s), \Sigma_y(s)) > A \geq M_K$, $\forall\, s \in [a, b] \setminus \{a, b\}$. So by Lemma 2.4.7 (1), $d_B(a, b) \leq \tau_{2.4.7}(K, A)$. Hence one can take $K_{5.1.20} := \tau_{2.4.7}(K, A)$. □

We finish this subsection by noting an interesting fact; which gives a criterion for a family of geodesic segments in the fibers to form a ladder.

Lemma 5.1.21. *Given $K' \geq 1, C' \geq 0, \varepsilon' \geq 0$, we have constants $k_{5.1.21} = k_{5.1.21}(K')$, $c_{5.1.21} = c_{5.1.21}(C')$ and $\varepsilon_{5.1.21} = \varepsilon_{5.1.21}(\varepsilon')$ such that the following holds.*

Suppose $\mathscr{L}$ is a collection of geodesic segments in the fibers such that:

(1) $T_{\mathscr{L}} := hull(\pi(\mathscr{L}))$. For all $v \in T_{\mathscr{L}}$, $\mathscr{L} \cap X_v$ form a special K'-ladder in X_v over B_v bounded by two K'-qi sections. We also have a subtree $\mathfrak{T}$ in $T_{\mathscr{L}}$ with the following. Suppose $v, w \in T_{\mathscr{L}}$ with $d_T(v, w) = 1$. Let $[\mathfrak{v}, \mathfrak{w}]$ be the edge joining $\mathfrak{v} \in B_v$ and $\mathfrak{w} \in B_w$. Let $\mathscr{L}_{\mathfrak{v},v} := \mathscr{L} \cap F_{\mathfrak{v},v} = [x_{\mathfrak{v},v}, y_{\mathfrak{v},v}]^f$ and $\mathscr{L}_{\mathfrak{w},w} := \mathscr{L} \cap F_{\mathfrak{w},w} =$

$[x_{\mathfrak{w},w}, y_{\mathfrak{w},w}]^f$. *If* $v,w \in \mathfrak{T}$, *then* $d_{\mathfrak{vw}}(x_{\mathfrak{v},v}, x_{\mathfrak{w},w}) \leq K'$ *and* $d_{\mathfrak{vw}}(y_{\mathfrak{v},v}, y_{\mathfrak{w},w}) \leq K'$. *Otherwise, if* $d_T(v,\mathfrak{T}) < d_T(w,\mathfrak{T})$, *then* $x_{\mathfrak{w},w}, y_{\mathfrak{w},w} \in N^{\mathfrak{vw}}_{K'}(\mathscr{L}_{\mathfrak{v},v})$.

(2) *In the second part of* (1), *where* $d_T(v,\mathfrak{T}) < d_T(w,\mathfrak{T})$, *we have,*

$$Hd_{\mathfrak{vw}}(P_{\mathfrak{w}}(\mathscr{L}_{\mathfrak{v},v}), \mathscr{L}_{\mathfrak{w},w}) \leq \varepsilon'.$$

(3) *Let* $v \in T_{\mathscr{L}}, w \notin T_{\mathscr{L}}$ *such that* $d_T(v,w) = 1$. *Let* $[\mathfrak{v},\mathfrak{w}]$ *be the edge joining* $\mathfrak{v} \in B_v$ *and* $\mathfrak{w} \in B_w$. *Then the pair* $(\mathscr{L}_{\mathfrak{v},v}, F_{\mathfrak{w},w})$ *is* C'*-cobounded in the path metric of* $F_{\mathfrak{vw}}$.

Then $\mathscr{L}$ *is a* $(k_{5.1.21}, c_{5.1.21}, \varepsilon_{5.1.21})$*-ladder with a central base* $\mathfrak{B} := \pi_B^{-1}(\mathfrak{T})$.

Proof. We only need to find $k_{5.1.21}$ and set an orientation on the fiber geodesics along with the family $\{h_{wv}\}$ of monotonic maps. Fix $u \in \mathfrak{T}$. Suppose $[v,w]$ is an edge in $T_{\mathscr{L}}$ with $d_T(u,v) < d_T(u,w)$. Let us fix once and for all $\bar{x}_{\mathfrak{v},v}, \bar{y}_{\mathfrak{v},v} \in \mathscr{L}_{\mathfrak{v},v}$ such that $d_{\mathfrak{vw}}(x_{\mathfrak{w},w}, \bar{x}_{\mathfrak{v},v}) \leq K'$, $d_{\mathfrak{vw}}(y_{\mathfrak{w},w}, \bar{y}_{\mathfrak{w},w}) \leq K'$ with $\bar{x}_{\mathfrak{v},v} \in \mathscr{L}_{\mathfrak{v},v} \cap [x_{\mathfrak{v},v}, \bar{y}_{\mathfrak{v},v}]^f$; and in the case if $v,w \in \mathfrak{T}$, then $\bar{x}_{\mathfrak{v},v} = x_{\mathfrak{v},v}, \bar{y}_{\mathfrak{v},v} = y_{\mathfrak{v},v}$. We inductively fix an orientation as discussed in Subsection 5.1.2.

Now we apply Lemma 2.2.4 on $F_{\mathfrak{vw}}$ and L'_0-quasi-geodesics $\mathscr{L}_{\mathfrak{v},v}$ and $\mathscr{L}_{\mathfrak{w},w}$. So we get a monotonic map, say, $h_{wv} : \mathscr{L}_{\mathfrak{w},w} \to \mathscr{L}_{\mathfrak{v},v}$ such that h_{wv} is $L_{2.2.4}(\delta'_0, L'_0, K')$-quasi-isometry. Also, we have $d_{\mathfrak{vw}}(x, h_{wv}(x)) \leq k_{2.2.4}(\delta'_0, L'_0, K')$ for all $x \in \mathscr{L}_{\mathfrak{w},w}$, and $h_{wv}(x_{\mathfrak{w},w}) = \bar{x}_{\mathfrak{v},v}$, $h_{wv}(y_{\mathfrak{w},w}) = \bar{y}_{\mathfrak{v},v}$. We fix once and for all such maps h_{wv}.

Therefore, one can take $k_{5.1.21} = max\{k_{2.2.4}(\delta'_0, L'_0, K'), K'\}$ and $c_{5.1.21} = C'$, $\varepsilon_{5.1.21} = \varepsilon'$. □

5.2 Hyperbolicity of ladder

In this section, we show that a uniform neighborhood of a ladder (see Definition 5.1.16) is uniformly hyperbolic with the induced path metric. We divide the proof into two cases. (1) Ladder with small girth (see Definition 5.1.19); here we construct paths for any pair of points in the ladder and prove that they satisfy the conditions of Proposition 2.2.6 (see Proposition 5.2.1). (2) For general ladder, we subdivide it into (uniformly) small (but not too small) girth ladders and show that they satisfy all conditions of Proposition 2.2.7 (see Theorem 5.2.11). Let $K \geq 1, C \geq 0, \varepsilon \geq 0$ and let $\mathscr{L}_K$ be a (K,C,ε)-ladder with a central base $\mathfrak{B}$. In this section, we fix **notation** $\boldsymbol{L_{Kr}} := \boldsymbol{N_r}(\mathscr{L}_{\boldsymbol{K}})$ for $r \geq 0$. Additionally, we will use the same notations as introduced in Definition 5.1.16 and in Notation 5.1.15 for ladders. In view of Remark 2.4.8, in this section, we require the tree of metric bundles (X,B,T) to satisfy $C^{(9)}_{2.4.12}(K)$-flaring condition (see below).

5.2.1 Hyperbolicity of ladders (small girth)

We refer to Remark 2.4.13 for the notation of $C^{(i)}_{2.4.12}(K)$, $i \in \mathbb{N} \cup \{0\}$. Fix $\kappa = C^{(3)}_{2.4.12}(K)$ for this Subsection 5.2.1. Given $A_0 \geq 0$, we let $A = max\{M_{C^{(i)}_{2.4.12}(K)}, A_0 : i = 0,1,2,3\}$, where $M_{C^{(i)}_{2.4.12}(K)}$ is coming from the $C^{(i)}_{2.4.12}(K)$-flaring condition. Note that $C^{(i+1)}_{2.4.12}(K) \geq C^{(i)}_{2.4.12}(K)$.

Proposition 5.2.1. *Suppose $R \geq 2\kappa$ and A as above. Then there exists $\delta_{5.2.1} = \delta_{5.2.1}(K, A_0, R)$ such that the following holds.*

Suppose $\mathscr{L}^g|_{\mathfrak{B}} \leq A_0$ (see Definition 5.1.19). Then $L_{KR} := N_R(\mathscr{L}_K)$ is $\delta_{5.2.1}$-hyperbolic with respect to the path metric induced from X.

Proof. **Idea of the proof:** The proof of this proposition is long, so we break it up into several cases. We first define a path, say, $c(x,y)$ for a pair of distinct points $x, y \in \mathscr{L}$; and we fix it once and for all. Then we show that this family of paths satisfies Proposition 2.2.6. Hence the hyperbolicity of L_{KR} follows.

Notation 5.2.2. Let $x,y,z \in \mathscr{L}$. We use the following notations for Proposition 5.2.1. For a fixed $u \in \mathfrak{T}$, $B_s = \mathfrak{B} \cup \pi_B^{-1}([u, \pi(s)])$, $s \in \{x,y,z\}$; $B_{xy} = B_x \cap B_y$, $B_{xyz} = B_x \cap B_y \cap B_z$. We use $\bar{x}$ to denote $\pi_X(x)$ (projection under π_X), and same way, $\bar{y} = \pi_X(y)$, $\bar{z} = \pi_X(z)$. We will denote the path metric on L_{KR} induced from X by d'.

Definition of family of paths: Let $x, y \in \mathscr{L}$. Suppose Σ_x, Σ_y are K-qi sections in $\mathscr{L}$ over B_x, B_y through x, y respectively. If $[\pi(x), \pi(y)] \cap \mathfrak{T} = \emptyset$ then we take u_{xy} as the center of the triangle $\triangle(u, \pi(x), \pi(y))$ for some $u \in \mathfrak{T}$. Otherwise, u_{xy} is the nearest point projection of $\pi(x)$ on $\mathfrak{T}$. Let $U_{xy} = \mathscr{L}^n_{xy}(A)|_{B_{xy}}$ be the A-neck of the special ladder $\mathscr{L}_{xy}$ bounded by Σ_x, Σ_y (see Definition 5.1.19) over the common base B_{xy}. Then U_{xy} is $K_{5.1.20}(K,A)$-quasiconvex (see Lemma 5.1.20). Let $\boldsymbol{t_{xy}}$ be a nearest point projection of $\bar{x}$ on U_{xy} and let $\boldsymbol{v_{xy}} := \pi_B(t_{xy})$. We take K-qi lifts $\tilde{\alpha}_{xy}$ and $\tilde{\gamma}_{xy}$ of geodesics $\boldsymbol{\alpha_{xy}} := [\bar{x}, t_{xy}]_B$ and $\boldsymbol{\gamma_{xy}} := [\bar{y}, t_{xy}]_B$ in Σ_x and Σ_y respectively. Denote $\mu_{xy} = [\Sigma_x(t_{xy}), \Sigma_y(t_{xy})]^f \subseteq \mathscr{L}_{t_{xy}, v_{xy}}$.

In general, $\tilde{\alpha}_{xy}$ and $\tilde{\gamma}_{xy}$ are not continuous. To make them continuous, we consider the following. Fix points $\bar{x} = a_1, a_2, \cdots, a_n = t_{xy}$ on $[\bar{x}, t_{xy}]_B$ such that $d_B(a_i, a_{i+1}) = 1$ for $1 \leq i \leq n-2$ and $d_B(a_{n-1}, a_n) \leq 1$. So we get a discrete path joining $\Sigma_x(a_1)$ and $\Sigma_y(a_n)$ with an order $\Sigma_x(a_1) < \Sigma_x(a_2) < \cdots < \Sigma_x(a_n)$. Consider the path $[\tilde{\alpha}_{xy}] = [\Sigma_x(a_1), \Sigma_x(a_2)] \cup [\Sigma_x(a_2), \Sigma_x(a_3)] \cup \cdots \cup [\Sigma_x(a_{n-1}), \Sigma_x(a_n)]$ based on this discrete path. Now $[\tilde{\alpha}_{xy}]$ is a continuous path. Similarly, we have the continuous path $[\tilde{\gamma}_{xy}]$ path corresponding to $\tilde{\gamma}_{xy}$.

We define $c(x,y) := \tilde{\alpha}_{xy} \cup \mu_{xy} \cup \tilde{\gamma}_{xy}$ and $[c(x,y)] = [\tilde{\alpha}_{xy}] \cup \mu_{xy} \cup [\tilde{\gamma}_{xy}]^{-}$, where $[\tilde{\gamma}_{xy}]^{-}$ denotes the path corresponding to $[\tilde{\gamma}_{xy}]$ with opposite orientation. We see that there is an asymmetry in the definition of $[c(x,y)]$ and the number of choices are involved. However, for each unordered pair $\{x,y\}$, we fix once and for all a choices and choose either $[c(x,y)]$ or $[c(y,x)]$ as the path joining x and y.

Note that $Hd_{L_{KR}}(c(x,y),[c(x,y)])$ is uniformly bounded. To prove condition (1) of Proposition 2.2.6 for our family of paths, we show that arc-length parametrization of $[c(x,y)]$ is properly embedded. For condition (2), we show the slimness of paths $c(x,y)$'s; and that is enough. Later on, we will use $c(x,y)$ for the notation of paths instead $[c(x,y)]$.

Different choices of geodesics, $[\bar{x},t_{xy}]$ and $[\bar{y},t_{xy}]$ give rise to path joining x,y, that are $2K\delta_0$ (uniformly) Hausdorff close to $c(x,y)$. However, we will have to think about other two natural questions as follows.

1. Are $c(x,y)$ and $c(y,x)$ uniformly Hausdorff close?

2. Suppose Σ'_x and Σ'_y are two different K-qi sections through x and y respectively lying inside $\mathscr{L}$. Let $c'(x,y)$ be a path joining x and y coming from the construction above for the qi sections Σ'_x, Σ'_y. Are $c(x,y)$ and $c'(x,y)$ uniformly Hausdorff close?

These two questions are proven in [10, Section 3] for the case, metric graph bundles (see [10, Definition 1.5]). However, we will establish that these are also true in our case (see Lemma 5.2.3 and Corollary 5.2.5). The proof idea involves playing with quasiconvex subsets U_{xy} and lifts in qi sections.

Lemma 5.2.3. *With the hypothesis as in Proposition 5.2.1, there exists $D_{5.2.3} = D_{5.2.3}(\kappa,A)$ such that $Hd'(c(x,y),c(y,x)) \leq D_{5.2.3}$.*

Proof. We can think of Σ_x,Σ_y as κ-qi sections in $\mathscr{L}$ through x,y respectively. By our notation, t_{yx} is a nearest point projection of $\bar{y}$ on $U_{yx}(=U_{xy})$. Let $\alpha = [t_{xy},t_{yx}]$. Also, $\alpha_{yx} = [\bar{y},t_{yx}], \gamma_{yx} = [t_{yx},\bar{x}]$; and $\tilde{\alpha}_{yx}, \tilde{\gamma}_{yx}$ are lifts of α_{yx},γ_{yx} in Σ_y,Σ_x respectively. Further, $v_{yx} := \pi_B(t_{yx})$ and $\mu_{yx} := [\Sigma_y(t_{yx}),\Sigma_x(t_{yx})]^f \subseteq \mathscr{L}_{t_{yx},v_{yx}}$. Finally, $c(y,x) = \tilde{\alpha}_{yx} \cup \mu_{yx} \cup \tilde{\gamma}_{yx}$. Since U_{xy} is $K_{5.1.20}(\kappa,A)$-quasiconvex, the arc-length parametrizations of $\alpha_{xy} \cup \alpha$ and $\alpha_{yx} \cup \alpha$ are $(3+2K_{5.1.20}(\kappa,A))$-quasi-geodesics (by [10, Lemma 1.31 (2)]). So by Lemma 2.2.2, there is D depending on δ_0, $3+2K_{5.1.20}(\kappa,A)$ such that $Hd_B(\gamma_{yx},\alpha_{xy} \cup \alpha) \leq D$ and $Hd_B(\gamma_{xy},\alpha_{yx} \cup \alpha) \leq D$. Again, $\alpha \subseteq B_{xy}$ and $t_{xy},t_{yx} \in U_{xy}$. So by Lemma 2.4.7 (2), $\forall\, s \in \alpha$, $d^f(\Sigma_x(s),\Sigma_y(s)) \leq R_{2.4.7}(\kappa,A)$. Below, we prove that $c(x,y)$ lies inside uniform neighborhood of $c(y,x)$. Then by the symmetry of proof we will be done.

Let $\xi \in c(x,y) \cap \tilde{\alpha}_{xy}$ and $\eta = \pi_X(\xi)$. Then $\exists\ \eta' \in \gamma_{yx}$ such that $d_B(\eta,\eta') = d_{B_{xy}}(\eta,\eta') \leq D$. So by taking κ-qi lift of $[\eta,\eta']$ in Σ_x (see Lemma 2.4.12 (3)), we get, $d'(\xi, c(y,x)) \leq d'(\xi, \tilde{\gamma}_{yx}) \leq 2\kappa D$.

Let $\xi \in c(x,y) \cap \mu_{xy}$. Then from above, $d'(\xi, c(y,x)) \leq 2\kappa D + A$.

Finally, let $\xi \in c(x,y) \cap \tilde{\gamma}_{xy}$ and $\eta = \pi_X(\xi)$. Then $\exists\ \eta' \in \alpha_{yx} \cup \alpha$ such that $d_B(\eta,\eta') \leq \delta_0$. If $\eta' \in \alpha_{yx}$, then by taking κ-qi lift of $[\eta,\eta']$ in Σ_y, we get, $d'(\xi, c(y,x)) \leq d'(\xi, \tilde{\alpha}_{yx}) \leq 2\kappa\delta_0$. Again if $\eta' \in \alpha$, then η' is further D-close to γ_{yx}, i.e. $\exists\ \eta'' \in \gamma_{yx}$ such that $d_B(\eta',\eta'') \leq D$. Therefore, by taking lifts of geodesics $[\eta,\eta']$ and $[\eta',\eta'']$ in Σ_y and Σ_x respectively, we get,

$$\begin{aligned} d'(\xi, \tilde{\gamma}_{yx}) &\leq d'(\Sigma_y(\eta), \Sigma_x(\eta'')) \\ &\leq d'(\Sigma_y(\eta), \Sigma_y(\eta')) + d'(\Sigma_y(\eta'), \Sigma_x(\eta')) + d'(\Sigma_x(\eta'), \Sigma_x(\eta'')) \\ &\leq 2\kappa\delta_0 + R_{2.4.7}(\kappa, A) + 2\kappa D \text{ (since } \eta' \in \alpha) \\ &= 2\kappa(\delta_0 + D) + R_{2.4.7}(\kappa, A) \end{aligned}$$

Therefore, we can take $D_{5.2.3} := 2\kappa(\delta_0 + D) + R_{2.4.7}(\kappa, A)$ so that $Hd'(c(x,y), c(y,x)) \leq D_{5.2.3}$. □

To prove (1), we first show that if we change one of the qi sections, then the path we get is uniformly Hausdorff close to the other one. In other words, suppose Σ'_x is another qi section through x. Let $c(x,y)$ and $c_1(x,y)$ be paths coming from the pairs (Σ_x, Σ_y) and (Σ'_x, Σ_y) respectively. Then $Hd'(c(x,y), c_1(x,y))$ is uniformly small, say, bounded by D. Hence we complete (1) by applying twice this process. Indeed, $Hd'(c(x,y), c'(x,y)) \leq Hd'(c(x,y), c_1(x,y)) + Hd'(c_1(x,y), c'(x,y)) \leq 2D$, where Σ'_y is another qi section through y and the path $c'(x,y)$ is coming from the pair (Σ'_x, Σ'_y).

Lemma 5.2.4. *With the hypothesis of Proposition 5.2.1, there is $D_{5.2.4} = D_{5.2.4}(\kappa, A)$ such that $Hd'(c(x,y), c_1(x,y)) \leq D_{5.2.4}$.*

Proof. Here also we consider Σ_x, Σ'_x and Σ_y as κ-qi sections. Let the special ladder formed by pair (Σ'_x, Σ_y) restricted over B_{xy} be $\mathscr{L}'_{xy}$ (see Lemma 5.1.20). Let V be the A-neck of the ladder $\mathscr{L} \cap X_{\mathfrak{B}}$ (restriction of $\mathscr{L}$ on $\mathfrak{B}$, see also Notation 5.1.15) and U'_{xy} be that of $\mathscr{L}'_{xy}$. Notice that $V \subseteq U_{xy} \cap U'_{xy}$. We assume that t'_{xy} is a nearest point projection of $\bar{x}$ on U'_{xy}. Let $\alpha'_{xy} = [\bar{x}, t'_{xy}]$ and $\gamma'_{xy} = [\bar{y}, t'_{xy}]$. First, we prove that $d_B(t_{xy}, t'_{xy})$ is uniformly small.

$d_B(t_{xy}, t'_{xy})$ is uniformly small: We fix a point $t \in V$ and a geodesic $\alpha = [\bar{x}, t]$. Now for $s \in \{t, \bar{x}\}$, $d^f(\Sigma_x(s), \Sigma'_x(s)) \leq A$. So by Lemma 2.4.7 (2), for all $s \in \alpha$, $d^f(\Sigma_x(s), \Sigma'_x(s)) \leq R_{2.4.7}(\kappa, A)$. Again U_{xy}, U'_{xy} are $K_{5.1.20}(\kappa, A)$-quasiconvex and so by [10, Lemma 1.31 (2)], the arc-length parametrizations of $\alpha_{xy} \cup [t_{xy}, t]$ and

$\alpha'_{xy} \cup [t'_{xy}, t]$ are $(3+2K_{5.1.20}(\kappa, A))$-quasi-geodesics. Thus by Lemma 2.2.2, there is D depending on δ_0, $3+2K_{5.1.20}(\kappa, A)$ such that $Hd_B(\alpha, \alpha_{xy} \cup [t_{xy}, t]) \leq D$ and $Hd_B(\alpha, \alpha'_{xy} \cup [t'_{xy}, t]) \leq D$. Then $Hd(\alpha_{xy} \cup [t_{xy}, t], \alpha'_{xy} \cup [t'_{xy}, t]) \leq 2D$. Hence $\exists\, t_0 \in \alpha_{xy}$ such that $d_B(t'_{xy}, t_0) \leq 3D + \delta_0$ or $\exists\, t_0 \in \alpha'_{xy}$ such that $d_B(t_{xy}, t_0) \leq 3D + \delta_0$. Without loss of generality, we assume that $d_B(t_{xy}, t_0) \leq 3D + \delta_0$ for $t_0 \in \alpha'_{xy}$. Since T is tree and B_v's are isometrically embedded in B, we can take $t_0 \in B_y$. In particular, $[t_{xy}, t_0] \subseteq B_{xy}$.

Again for $s \in \alpha'_{xy}$, $\exists\, s' \in \alpha$ such that $d_B(s, s') \leq D$. By taking lifts of geodesic $[s, s']$ in Σ_x and Σ'_x (see Lemma 2.4.12 (3)), $d_X(\Sigma_x(s), \Sigma'_x(s)) \leq d_X(\Sigma_x(s), \Sigma_x(s')) + d_X(\Sigma_x(s'), \Sigma'_x(s')) + d_X(\Sigma'_x(s'), \Sigma'_x(s)) \leq 2D\delta_0 + R_{2.4.7}(\kappa, A) + 2D\delta_0 = D_1$ (say). As fibers are ϕ-properly embedded, $d^f(\Sigma_x(s), \Sigma'_x(s)) \leq \phi(D_1)$. In particular,

$$d_X(\Sigma_x(t_0), \Sigma'_x(t_0)) \leq D_1 \text{ and } d^f(\Sigma_x(t_0), \Sigma'_x(t_0)) \leq \phi(D_1).$$

Note that $[t_{xy}, t_0] \subseteq B_{xy}$. Now by taking lifts of $[t_0, t_{xy}]$ in Σ_x and Σ_y, we have, $d_X(\Sigma_x(t_0), \Sigma_x(t_{xy})) \leq 2\kappa(3D + \delta_0)$ and $d_X(\Sigma_y(t_{xy}), \Sigma_y(t_0)) \leq 2\kappa(3D + \delta_0)$. Again, $d^f(\Sigma_y(t_{xy}), \Sigma_x(t_{xy})) \leq A$. Therefore, combining all these inequalities, we have, $d_X(\Sigma'_x(t_0), \Sigma_y(t_0)) \leq D_1 + 4\kappa(3D + \delta_0) + A = D_2$ (say). So $d^f(\Sigma'_x(t_0), \Sigma_y(t_0)) \leq \phi(D_2)$. Then by Lemma 2.4.7 (1), $d_B(t_0, t'_{xy}) \leq \tau_{2.4.7}(\kappa, \phi(D_2))$. Hence $d_B(t_{xy}, t'_{xy}) \leq d_B(t_{xy}, t_0) + d_B(t_0, t'_{xy}) \leq D_3$ where $D_3 = 3D + \delta_0 + \tau_{2.4.7}(\kappa, \phi(D_2))$.

Let us come back to the proof of Hausdorff closeness of $c(x,y)$ and $c_1(x,y)$. We only prove that $c(x,y)$ lies inside uniform neighborhood of $c_1(x,y)$. Then by the symmetry of the proof we will be done.

Let $\xi \in c(x,y) \cap \tilde{\alpha}_{xy}$ and $\eta = \pi_X(\xi)$. Then $d_B(\eta, \eta') \leq D_3 + \delta_0$ for some $\eta' \in \alpha'_{xy}$. Since $\eta' \in \alpha'_{xy}$, from the above paragraph, $d^f(\Sigma_x(\eta'), \Sigma'_x(\eta')) \leq \phi(D_1)$. Therefore, by taking lift of geodesic $[\eta, \eta']$ in Σ_x (see Lemma 2.4.12 (3)), we get, $d'(\xi, c_1(x,y)) \leq d'(\Sigma_x(\eta), \Sigma'_x(\eta')) \leq d'(\Sigma_x(\eta), \Sigma_x(\eta')) + d^f(\Sigma_x(\eta'), \Sigma'_x(\eta')) \leq 2\kappa(D_3 + \delta_0) + \phi(D_1)$.

Now let $\xi \in c(x,y) \cap \tilde{\gamma}_{xy}$ and $\eta = \pi_X(\xi)$. Then $\exists\, \eta' \in \gamma'_{xy}$ such that $d_B(\eta, \eta') \leq D_3 + \delta_0$. Taking lift of $[\eta, \eta']$ in Σ_y, we get, $d'(\xi, c_1(x,y)) \leq d'(\Sigma_y(\eta), \Sigma_y(\eta')) \leq 2\kappa(D_3 + \delta_0)$.

Finally, we assume that $\xi \in c(x,y) \cap \mu_{xy}$. Then $d'(\xi, c_1(x,y)) \leq 2\kappa(D_3 + \delta_0) + A$.

We note that $\phi(D_1) > A$. Therefore, $c(x,y) \subseteq N_{2\kappa(D_3+\delta_0)+\phi(D_1)}(c_1(x,y))$. Hence, we can take $D_{5.2.4} := 2\kappa(D_2 + \delta_0) + \phi(D_1)$. □

Corollary 5.2.5. *With the hypothesis of Proposition 5.2.1, there exists $D_{5.2.5} = D_{5.2.5}(\kappa, A)$ such that Hausdorff distance between any two paths joining $x, y \in \mathscr{L}$*

coming from the path-construction with κ-qi sections through x,y is bounded by $D_{5.2.5}$ in the path metric of L_{KR}.

Proof. We can take $D_{5.2.5} = D_{5.2.3}(\kappa,A) + 2D_{5.2.4}(\kappa,A)$. □

Now we show that this family of paths are fellow-travel ([9, Definition 1.60]). In other words, any two such paths whose starting points are same and ending points are at uniform distance are uniformly Hausdorff close.

Proposition 5.2.6 (Fellow-travelling property). *For all $r \geq 0$ there exists $D_{5.2.6} = D_{5.2.6}(\kappa,A,r)$ such that the following holds.*

With the hypothesis of Proposition 5.2.1, if $x,y,z \in \mathscr{L}$ such that $d_X(x,y) \leq r$, then

$$Hd'(c(x,z),c(y,z)) \leq D_{5.2.6}.$$

Proof. We will be working with $\Sigma_x, \Sigma_y, \Sigma_z$ as κ-qi sections respectively through x,y,z over B_x, B_y, B_z inside the ladder $\mathscr{L}$ (explained in Case 1 below). We consider the following three cases depending on the position of $\bar{x} = \pi_X(x)$ and $\bar{y} = \pi_X(y)$.

Case 1: Let $\bar{x} = \bar{y}$. Since the fibers are ϕ-properly embedded, without loss of generality, we assume that $d^f(x,y) \leq r$. Applying Lemma 2.4.12 (2), we may assume that these sections satisfy the following inclusion property when we restrict them to B_{xyz}. We have three possibilities $\Sigma_y|_{B_{xyz}} \subseteq \mathscr{L}_{xz}|_{B_{xyz}}$, $\Sigma_x|_{B_{xyz}} \subseteq \mathscr{L}_{yz}|_{B_{xyz}}$ or $\Sigma_z|_{B_{xyz}} \subseteq \mathscr{L}_{xy}|_{B_{xyz}}$. (To get this one has to consider $\Sigma_x, \Sigma_y, \Sigma_z$ as $\kappa = C^{(3)}_{2.4.12}(K)$-qi sections instead K-qi sections.) In the Subcase $(1A)$ below, we will see that the proof for the inclusions $\Sigma_y|_{B_{xyz}} \subseteq \mathscr{L}_{xz}|_{B_{xyz}}$ and $\Sigma_x|_{B_{xyz}} \subseteq \mathscr{L}_{yz}|_{B_{xyz}}$ are similar. So we proof this proposition when $\Sigma_y|_{B_{xyz}} \subseteq \mathscr{L}_{xz}|_{B_{xyz}}$ and $\Sigma_z|_{B_{xyz}} \subseteq \mathscr{L}_{xy}|_{B_{xyz}}$ in the following subcases.

Subcase (1A): Suppose $\Sigma_y|_{B_{xyz}} \subseteq \mathscr{L}_{xz}|_{B_{xyz}}$. Recall that $U_{xy} = \mathscr{L}^n_{xy}(A)|_{B_{xy}}$, $U_{xz} = \mathscr{L}^n_{xz}(A)|_{B_{xz}}$ and $U_{yz} = \mathscr{L}^n_{yz}(A)|_{B_{yz}}$ are $K_{5.1.20}(\kappa,A)$-quasiconvex (Lemma 5.1.20). Since $\Sigma_y|_{B_{xyz}} \subseteq \mathscr{L}_{xz}|_{B_{xyz}}$, so we have $U_{xz} \subseteq U_{xy} \cap U_{yz}$. By our notation, t_{xz} and t_{yz} are nearest point projections of $\bar{x}$ and $\bar{y}$ on U_{xz} and U_{yz} respectively. Hence for $M = max\{r,A\}$, $\forall\, s \in \alpha_{xz}$, $d^f(\Sigma_x(s),\Sigma_y(s)) \leq R_{2.4.7}(\kappa,M)$ (Lemma 2.4.7 (2)).

Claim: $d_B(t_{xz},t_{yz})$ is uniformly bounded.

Proof of the claim: Since $\alpha_{yz} \cup [t_{yz},t_{xz}]$ is $(3+2K_{5.1.20}(\kappa,A))$-quasi-geodesic, $\exists$ $t \in \alpha_{xz}$ such that $d_B(t_{yz},t) \leq D$ for some D depending on δ_0 and $3+2K_{5.1.20}(\kappa,A)$ (Lemma 2.2.2). Since T is a tree and B_v's are isometrically embedded in B, we can take $t \in \alpha_{xz} \cap B_{yz}$. Then by taking lifts of geodesic $[t_{yz},t]$ in Σ_y and Σ_z, we get, $d'(\Sigma_y(t),\Sigma_z(t)) \leq d'(\Sigma_y(t),\Sigma_y(t_{yz})) + d^f(\Sigma_y(t_{yz}),\Sigma_z(t_{yz})) + d'(\Sigma_z(t_{yz}),\Sigma_z(t)) \leq 2\kappa D + A + 2\kappa D = 4\kappa D + A \Rightarrow d^f(\Sigma_y(t),\Sigma_z(t)) \leq \phi(4\kappa D + A)$. So, $d^f(\Sigma_x(t),\Sigma_z(t)) \leq$

$d^f(\Sigma_x(t),\Sigma_y(t)) + d^f(\Sigma_y(t),\Sigma_z(t)) \leq R_{2.4.7}(\kappa,M) + \phi(4\kappa D + A) = R_1$ (say). Then by Lemma 2.4.7 (1), $d_B(t,t_{xz}) \leq \tau_{2.4.7}(\kappa,R_1)$. Hence by triangle inequality, $d_B(t_{yz},t_{xz}) \leq d_B(t_{yz},t) + d_B(t,t_{xz}) \leq D_1$, where $D_1 = D + \tau_{2.4.7}(\kappa,R_1)$.

Now we show the Hausdorff closeness of paths. We only prove that $c(x,z)$ lies in uniform neighborhood of $c(y,z)$. Then by the symmetry of the proof we will be done.

Let $\xi \in c(x,z) \cap \tilde{\alpha}_{xz}$ and $\eta = \pi_X(\xi)$. Then $\exists\ \eta' \in \alpha_{yz}$ such that $d_B(\eta,\eta') \leq D_1 + \delta_0$. Note that $[\eta,\eta'] \subseteq B_y$. Then by taking lift of geodesic $[\eta,\eta']$ in Σ_y, we get, $d'(\xi,c(y,z)) \leq d'(\Sigma_x(\eta),\Sigma_y(\eta')) \leq d'(\Sigma_x(\eta),\Sigma_y(\eta)) + d'(\Sigma_y(\eta),\Sigma_y(\eta')) \leq R_{2.4.7}(\kappa,M) + 2\kappa(D_1+\delta_0)$.

Now let $\xi \in c(x,z) \cap \tilde{\gamma}_{xz}$ and $\eta = \pi_X(\xi)$. Then $\exists\ \eta' \in \gamma_{yz}$ such that $d_B(\eta,\eta') \leq D_1 + \delta_0$. Note that $[\eta,\eta'] \subseteq B_z$. So, by taking lift of $[\eta,\eta']$ in Σ_z, $d'(\xi,c(y,z)) \leq d'(\Sigma_z(\eta),\Sigma_z(\eta')) \leq 2\kappa(D_1+\delta_0)$ (Lemma 2.4.12 (3)).

Finally, let $\xi \in c(x,z) \cap \mu_{xz}$. Then $d'(\xi,c(y,z)) \leq 2\kappa(D_1+\delta_0) + A$.

Let $R_2 := max\{R_{2.4.7}(\kappa,M) + 2\kappa(D_1+\delta_0), 2\kappa(D_1+\delta_0)+A\} = R_{2.4.7}(\kappa,M) + 2\kappa(D_1+\delta_0)$. Hence $c(x,z) \subseteq N_{R_2}(c(y,z))$. Therefore, $Hd'(c(x,z),c(y,z)) \leq R_2$.

Subcase (1B): Suppose $\Sigma_z|_{B_{xyz}} \subseteq \mathscr{L}_{xy}|_{B_{xyz}}$. Here also we will do the same as in Subcase (1*A*). Let a be the nearest point projection of $\bar{x} = \bar{y}$ on B_{xyz}. Since $\mathscr{L} \cap X_{\mathfrak{B}}$ has girth $\leq A_0 \leq A$, by Lemma 2.4.7 (2), $d^f(\Sigma_x(s),\Sigma_y(s)) \leq R_{2.4.7}(\kappa,M), \forall\, s \in [a,\bar{x}]$. (Note that $[a,\bar{x}]$ could be $\{\bar{x}\} = \{a\}$ if $\bar{x} \in B_{xyz}$.) In particular, $d^f(\Sigma_x(a),\Sigma_y(a)) \leq R_{2.4.7}(\kappa,M)$. Again, since $\Sigma_z \subseteq \mathscr{L}_{xy}|_{B_{xyz}}$, we have, $d^f(\Sigma_x(a),\Sigma_z(a)) \leq R_{2.4.7}(\kappa,M)$ and $d^f(\Sigma_y(a),\Sigma_z(a)) \leq R_{2.4.7}(\kappa,M)$. Note that t_{yz} and t_{xz} are also nearest point projections of a on U_{yz} and U_{xz} respectively. So by Lemma 2.4.7 (1), we have $D_2 = \tau_{2.4.7}(\kappa,R_{2.4.7}(\kappa,M))$ such that $d_B(a,t_{xz}) \leq D_2$ and $d_B(a,t_{yz}) \leq D_2$. Thus $d_B(t_{xz},t_{yz}) \leq 2D_2$. Now we only show that $c(x,z)$ lies in uniform neighborhood of $c(y,z)$. Then by symmetry of the proof we will be done.

Let $\xi \in c(x,z) \cap \tilde{\alpha}_{xz}$ and $\eta = \pi_X(\xi)$. Note that $\eta \in [t_{xz},a] \cup [a,\bar{x}]$. If $\eta \in [a,\bar{x}]$, then $d'(\xi,c(y,z)) \leq d^f(\Sigma_x(\eta),\Sigma_y(\eta)) \leq R_{2.4.7}(\kappa,M)$. If $\eta \in [t_{xz},a]$, then $d'(\xi,c(y,z)) \leq d'(\Sigma_x(\eta),\Sigma_x(a)) + d^f(\Sigma_x(a),\Sigma_y(a)) \leq 2\kappa D_2 + R_{2.4.7}(\kappa,M)$.

Now let $\xi \in c(x,z) \cap \tilde{\gamma}_{xz}$ and $\eta = \pi_X(\xi)$. Then $\exists\ \eta' \in \gamma_{yz}$ such that $d_B(\eta,\eta') \leq 2D_2 + \delta_0$. So taking lift of $[\eta,\eta']$ in Σ_z, we get, $d'(\xi,c(y,z)) \leq d'(\Sigma_z(\eta),\Sigma_z(\eta')) \leq 2\kappa(2D_2+\delta_0)$.

Finally, if $\xi \in c(x,z) \cap \mu_{xz}$, then $d'(\xi,c(y,z)) \leq 2\kappa(2D_2+\delta_0) + A$.

Therefore, $Hd'(c(x,z),c(y,z)) \leq max\{2\kappa(2D_2+\delta_0)+A, 2\kappa D_2 + R_{2.4.7}(\kappa,M)\} = R_3$ (say).

Let $\boldsymbol{R_4(\kappa,A,r)} := max\{R_2,R_3\}$.

Now for the rest of the proof for this proposition, we assume that all the paths $c(\zeta,\zeta')$ are constructed using the qi sections Σ_x, Σ_y and Σ_z, where $\zeta,\zeta' \in \Sigma_x \cup \Sigma_y \cup \Sigma_z$.

Case 2: Let $\pi(x) = \pi(y)$. Suppose $\Sigma_y(\bar{x}) = y_1$. We also assume that $\Sigma_{y_1} = \Sigma_y$. Since $d_X(x,y) \le r$, so $d_B(\bar{x},\bar{y}) \le r$. Now by taking lift of geodesic $[\bar{x},\bar{y}]$ in Σ_y, we get, $d'(y,y_1) \le 2\kappa r$. Thus $d_X(y_1,x) \le 2\kappa r + r$ and so $d^f(y_1,x) \le \phi(2\kappa r + r)$. Therefore, by Case 1, $Hd'(c(y_1,z),c(x,z)) \le R_4(\kappa,A,\phi(2\kappa r+r))$.

Now we investigate on $Hd'(c(y_1,z),c(y,z))$. For the consistency of notation, we let $\bar{y}_1 = \pi_X(y_1)$. Let t_{y_1z} be a nearest point projection of $\bar{y}_1$ on $U_{y_1z} = U_{yz}$ (since $\Sigma_y = \Sigma_{y_1}$). Again U_{y_1z} is $K_{5.1.20}(\kappa,A)$-quasiconvex and $d_B(\bar{y}_1,\bar{y}) \le 2\kappa r$, so by lemma 2.2.21 (1), we have, $d_B(t_{y_1z},t_{yz}) \le (2\kappa r+1)C_{2.2.21}(\delta_0,K_{5.1.20}(\kappa,A)) = D_3$ (say). Note that $\bar{z} = \pi_X(z)$ and $\alpha_{y_1z} = [\bar{y}_1,t_{y_1z}]$, $\gamma_{y_1z} = [\bar{z},t_{y_1z}]$. Then $Hd_B(\gamma_{y_1z},\gamma_{yz}) \le D_3+\delta_0$ and $Hd_B(\alpha_{y_1z},\alpha_{yz}) \le D_3+2\delta_0$ (note that $D_3 > 2\kappa r$). Thus $Hd'(\tilde{\alpha}_{yz},\tilde{\alpha}_{y_1z}) \le 2\kappa(D_3+2\delta_0)$ and $Hd'(\tilde{\gamma}_{yz},\tilde{\gamma}_{y_1z}) \le 2\kappa(D_3+\delta_0)$. Hence, $Hd'(c(y,z),c(y_1,z)) \le 2\kappa(D_3+2\delta_0)+A$.

Therefore, $Hd'(c(x,z),c(y,z)) \le R_4(\kappa,A,\phi(2\kappa r+r)) + 2\kappa(D_3+2\delta_0)+A =:$ $\boldsymbol{R_5(\kappa,A,r)}$ (say).

Case 3: Now we consider the general case. Let $\bar{x}_1$ and $\bar{y}_1$ be the nearest point projections of $\bar{x}$ and $\bar{y}$ on $B_{u_{xy}}$ respectively (see Definition of family of paths for u_{xy}). Let $\Sigma_x(\bar{x}_1) = x_1$ and $\Sigma_y(\bar{y}_1) = y_1$. Since $d_B(\bar{x},\bar{y}) \le r$, so $d_B(\bar{x},\bar{x}_1) \le r$ and $d_B(\bar{y}_1,\bar{y}) \le r$. Thus by taking lifts of geodesics $[\bar{x},\bar{x}_1]$ and $[\bar{y},\bar{y}_1]$ in Σ_x and Σ_y respectively, we get, $d'(x,x_1) \le 2\kappa r$ and $d'(y,y_1) \le 2\kappa r$. So by triangle inequality, $d_X(x_1,y_1) \le 4\kappa r + r$. Note that $\pi(x_1) = \pi(y_1)$. Hence by Case 2, $Hd'(c(x_1,z),c(y_1,z)) \le R_5(\kappa,A,4\kappa r+r)$. Since $d'(x,x_1) \le 2\kappa r$ and $d'(y,y_1) \le 2\kappa r$, we have, $Hd'(c(x,z),c(y,z)) \le R_5(\kappa,A,4\kappa r+r)+2\kappa r =:$ $\boldsymbol{R_6(\kappa,A,r)}$ (say).

Therefore, we can take $D_{5.2.6} = max\{R_i(\kappa,A,r) : i = 4,5,6\} = R_6(\kappa,A,r)$. □

The proof for slimness of triangle formed by three paths (as in path construction) inside special K-ladder (see Definition 2.4.11) was done in [10, Lemma 3.11 for small girth ladder] in case of metric graph bundles (see [10, Definition 1.5]). In their proof, without changing much, one can proof the same in case of metric bundles (Lemma 5.2.7); which we will see in Condition (2) below. So we omit the proof and state below for small girth ladder.

Lemma 5.2.7. ([10, Lemma 3.11]) *Given $k \ge 1, \mathscr{A} \ge 0$, there is $D_{5.2.7} = D_{5.2.7}(k,\mathscr{A})$ such that the following holds.*

Suppose $\mathscr{L}(\Sigma,\Sigma')$ is a special k-ladder (in a tree of metric bundles (X,B,T)) bounded by two k-qi sections Σ,Σ' over an isometrically embedded subspace $B_1 \subseteq B$ such that $inf\{d^f(\Sigma(a),\Sigma'(a)) : a \in B_1\} \le \mathscr{A}$. Let $x,y,z \in \mathscr{L}(\Sigma,\Sigma')$. Then the triangle *formed by paths $c(x,y), c(x,z)$ and $c(y,z)$, coming from the path construction, are $D_{5.2.7}$-slim in the induced path metric on $N_{2C^{(3)}_{2.4.12}(k)}(\mathscr{L}(\Sigma,\Sigma')) \subseteq X$.*

Proof of Proposition 5.2.1: We verify the condition (1) and (2) of Proposition 2.2.6 for our family of paths. Here $\mathscr{L}$ is R-dense in L_{KR}. We will be working with $\Sigma_x, \Sigma_y, \Sigma_z$ as κ-qi sections (explained in Condition (2), Case 1 below).

Condition (1): Let $x,y \in \mathscr{L}$ such that $d_X(x,y) = r$ for $r \in \mathbb{R}_{\geq 0}$. We want to show that the length of $c(x,y)$ in the path metric of (L_{KR}, d') is bounded in terms of r. Let $c \in [\bar{x},\bar{y}] \cap B_{u_{xy}}$ and $c_1 \in [\bar{x},t_{xy}] \cap B_{u_{xy}}, c_2 \in [\bar{y},t_{xy}] \cap B_{u_{xy}}$ such that $d_B(c,c_i) \leq \delta_0,\ i = 1,2$. (We refer to the 'definition of family of paths' for u_{xy}.) Since $d_B(\bar{x},\bar{y}) \leq d_X(x,y) \leq r$, so $d_B(\bar{x},c) \leq r$ and $d_B(\bar{y},c) \leq r$. Let $\Sigma_x(c) = \{x_1\}$ and $\Sigma_y(c) = \{y_1\}$. Now taking lifts of $[\bar{x},c]$ and $[\bar{y},c]$ in Σ_x and Σ_y respectively, we have, $d'(x,x_1) \leq 2n\kappa$ and $d'(y,y_1) \leq 2r\kappa$ (see Lemma 2.4.12 (3)). Then by triangle inequality, $d_X(x_1,y_1) \leq r(4\kappa+1)$. So $d_X(\Sigma_x(c_1),\Sigma_y(c_1)) \leq d'(\Sigma_x(c_1),\Sigma_x(c)) + d_X(\Sigma_x(c),\Sigma_y(c)) + d'(\Sigma_y(c),\Sigma_y(c_1)) \leq 2\kappa\delta_0 + r(4\kappa+1) + 2\kappa\delta_0 = 4\kappa\delta_0 + r(4\kappa+1)$. Thus $d^f(\Sigma_x(c_1),\Sigma_y(c_1)) \leq \phi(4\kappa\delta_0 + r(4\kappa+1))$. Since t_{xy} is a nearest point projection of $\bar{x}$ on U_{xy}, by Lemma 2.4.7 (1), $d_B(c_1,t_{xy}) \leq D$, where $D = \tau_{2.4.7}(\kappa, \phi(4\kappa\delta_0 + n(4\kappa+1)))$. So $d_B(c_2,t_{xy}) \leq d_B(c_2,c_1) + d_B(c_1,t_{xy}) \leq 2\delta_0 + D$. Again $d_B(\bar{x},c_1) \leq r + \delta_0$ and $d_B(\bar{y},c_2) \leq r + \delta_0$. Hence $d_B(\bar{x},t_{xy}) \leq d_B(\bar{x},c_1) + d_B(c_1,t_{xy}) \leq r + \delta_0 + D$ and $d_B(\bar{y},t_{xy}) \leq d_B(\bar{y},c_2) + d_B(c_2,t_{xy}) \leq r + 3\delta_0 + D$. Note that $\alpha_{xy} = [\bar{x},t_{xy}]$ and $\gamma_{xy} = [\bar{y},t_{xy}]$. Therefore, by taking lifts of α_{xy} and γ_{xy} in Σ_x and Σ_y respectively, we see that the length of $c(x,y)$ is bounded by $2\kappa(r+\delta_0+D) + A + 2\kappa(r+3\delta_0+D) = 4\kappa(r+D+2\delta_0)+A$. So the paths $c(x,y)$ are ψ-properly embedded, where $\psi : \mathbb{N} \to \mathbb{N}$ is a function such that

$$\psi(r) = 4\kappa(r+D+2\delta_0)+A \tag{5.2. 1}$$

Condition (2): Recall $U_{xy} = \mathscr{L}^n_{xy}(A)|_{B_{xy}}$, $U_{xz} = \mathscr{L}^n_{xz}(A)|_{B_{xz}}$ and $U_{yz} = \mathscr{L}^n_{yz}(A)|_{B_{yz}}$ are $K_{5.1.20}(\kappa,A)$-quasiconvex and so is in B. We show that paths $c(x,y), c(x,z), c(y,z)$, coming from the above qi sections, are uniformly slim in the path metric of (L_{KR}, d'). Depending on the position of t_{xy}, t_{xz}, t_{yz} with respect to B_{xyz}, we consider the following two cases. Note that by the definition of B_{xyz}, either all of t_{xy}, t_{xz}, t_{yz} are in B_{xyz} or at most one of them is outside of B_{xyz}.

Case 1: All of t_{xy}, t_{xz}, t_{yz} are in B_{xyz}.

In this case, without loss of generality, we assume that $\Sigma_y|_{B_{xyz}} \subseteq \mathscr{L}_{xz}|_{B_{xyz}}$, i.e. in the ladder $\mathscr{L}$, we have the order, $bot(\mathscr{L}_{a,v}) \leq \Sigma_x(a) \leq \Sigma_y(a) \leq \Sigma_z(a) \leq top(\mathscr{L}_{a,v})$, where $a \in B_{xyz}$ and $v = \pi_B(a)$. (To get this one has to consider $\Sigma_x, \Sigma_y, \Sigma_z$ as $\kappa = C^{(3)}_{2.4.12}(K)$-qi sections instead K-qi sections, see Lemma 2.4.12 (2)). Let $\bar{x}_1, \bar{y}_1, \bar{z}_1$ be (the) nearest point projections of $\bar{x}, \bar{y}, \bar{z}$ on B_{xyz} respectively. Let $\Sigma_x(\bar{x}_1) = x_1, \Sigma_y(\bar{y}_1) =$

y_1 and $\Sigma_z(\bar{z}_1) = z_1$. Further, we assume that the restriction of $c(x,y)$ from x_1 to y_1 is $c(x_1,y_1)$. Likewise, we have $c(x_1,z_1)$ and $c(y_1,z_1)$. Note that restriction of Σ_x and Σ_z over B_{xyz} form a special $C_{2.4.12}(\kappa) = C^{(4)}_{2.4.12}(K)$-ladder over B_{xyz} bounded by two qi sections $\Sigma_x|_{B_{xyz}}$ and $\Sigma_z|_{B_{xyz}}$ such that $inf\{d^f(\Sigma_x(s),\Sigma_z(s)) : s \in B_{xyz}\} \le A$. Since (X,B,T) satisfies $C^{(7)}_{2.4.12}(K) = C^{(3)}_{2.4.12}(C^{(4)}_{2.4.12}(K))$-flaring condition, so by Lemma 5.2.7, the triangle formed by the paths $c(x_1,y_1), c(x_1,z_1)$ and $c(y_1,z_1)$ are $D_{5.2.7}(C^{(4)}_{2.4.12}(K),A)$-slim in the path metric of L_{KR}. Let $D_1 = D_{5.2.7}(C^{(4)}_{2.4.12}(K),A)$.

For the point $\xi \in c(x,y)$ such that $\xi \notin c(x_1,y_1)$, ξ is $2\kappa\delta_0$-close to $c(x,z) \cup c(y,z)$ in the path metric of L_{KR}. Same for others.

Note that $2\kappa\delta_0 \le D_1$. Therefore, the triangle formed by the paths $c(x,y)$, $c(x,z)$ and $c(y,z)$ are D_1-slim in the path metric of L_{KR}.

Case 2: One of t_{xy}, t_{xz}, t_{yz} is not in B_{xyz} (see Figure 5.1). Without loss of generality, we assume that $t_{xy} \notin B_{xyz}$. In this case, we do not need to consider what exactly is happening to Σ_x, Σ_y and Σ_z over B_{xyz}.

Note that $\pi_B(t_{xy}) = v_{xy}$. Let $t \in B_{v_{xy}}$ such that $d_B(B_{xyz}, B_{v_{xy}}) = d_B(B_{xyz}, t)$. Let us fix $s \in \mathfrak{B} \cap U_{xy}$ such that $t \in [s, t_{xy}]$. (We can get such s as $\mathscr{L}^g|_{\mathfrak{B}} \le A_0 \le A$, see Definition 5.1.19 for notation.) Since $s, t_{xy} \in U_{xy}$, by Lemma 2.4.7 (2), for all $\zeta \in [s, t_{xy}]$, $d^f(\Sigma_x(\zeta), \Sigma_y(\zeta)) \le R_{2.4.7}(\kappa, A)$. In particular, $d^f(\Sigma_x(t), \Sigma_y(t)) \le R_{2.4.7}(\kappa, A)$. Then by the fellow-travelling property (see Proposition 5.2.6), we have,

$$Hd'(c(x,z)|_{[t,t_{xz}]\cup[t_{xz},\bar{z}]}, c(y,z)|_{[t,t_{yz}]\cup[t_{yz},\bar{z}]}) \le D_{5.2.6}(\kappa, A, R_{2.4.7}(\kappa, A)) = D_2 \text{ (say)}.$$

Again let t_x, t_y be the nearest point projections of $\bar{x}$ and $\bar{y}$ respectively on $B_{v_{xy}}$. Then $Hd'(c(x,y)|_{[t_x,\bar{x}]}, c(x,z)|_{[t_x,\bar{x}]}) \le 2\kappa\delta_0$ and $Hd'(c(x,y)|_{[t_y,\bar{y}]}, c(y,z)|_{[t_y,\bar{y}]}) \le 2\kappa\delta_0$.

Now we only need to analyse what is happening to the paths $c(x,y), c(x,z)$ and $c(y,z)$ over $B_{v_{xy}}$ to conclude the slimness, and here we go.

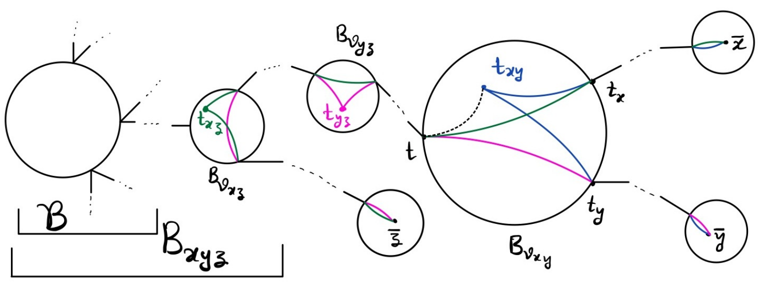


Figure 5.1: Case 2

A. The portion of the path $c(x,y)$ over $B_{v_{xy}}$ is uniformly close to $c(x,z) \cup c(y,z)$:

Let $\xi \in c(x,y) \cap (\tilde{\alpha}_{xy} \cup \tilde{\gamma}_{xy})$ such that $\eta = \pi_X(\xi)$. Then $\eta \in [t_{xy}, t_x] \cup [t_{xy}, t_y]$. First, we consider $\eta \in [t_{xy}, t_x]$. Since U_{xy} is $K_{5.1.20}(\kappa, A)$-quasiconvex, t_{xy} nearest point projection of $\bar{x}$ and $s \in U_{xy}$, so by [10, Lemma 1.31(2)], the arc-length parametrization of $[t_x, t_{xy}] \cup [t_{xy}, s]$ is a $(3 + 2K_{5.1.20}(\kappa, A))$-quasi-geodesic. In particular, $[t_x, t_{xy}] \cup [t_{xy}, t]$ is $(3 + 2K_{5.1.20}(\kappa, A))$-quasi-geodesic. Therefore, by Lemma 2.2.2, there is D_3 depending on δ_0 and $3 + 2K_{5.1.20}(\kappa, A)$ such that $d_B(\eta, \eta') \leq D_3$ for some $\exists\, \eta' \in [t_x, t]$. So by taking lift of $[\eta, \eta']$ in Σ_x (see Lemma 2.4.12 (3)), we get, $d'(\xi, c(x,z)) \leq d'(\Sigma_x(\eta), \Sigma_x(\eta')) \leq 2\kappa D_3$.

Now suppose $\eta \in [t_{xy}, t_y]$. Then the slimness of $\triangle(t_{xy}, t, t_y)$ says that $\eta \in N_{\delta_0}([t_{xy}, t] \cup [t, t_y])$. Let $\exists\, \eta' \in [t, t_y]$ such that $d_B(\eta, \eta') \leq \delta_0$. Then by taking lift of $[\eta, \eta']$ in Σ_y (see Lemma 2.4.12 (3)), we get, $d'(\xi, c(y,z)) \leq d'(\Sigma_y(\eta), \Sigma_y(\eta')) \leq 2\kappa\delta_0$. Now let $\eta' \in [t, t_{xy}]$ such that $d_B(\eta, \eta') \leq \delta_0$. Again, since $[t_x, t_{xy}] \cup [t_{xy}, t]$ is $(3 + 2K_{5.1.20}(\kappa))$-quasi-geodesic, $\exists\, \eta'' \in [t, t_x]$ such that $d_B(\eta', \eta'') \leq D_3$. Taking lift of the geodesic $[\eta, \eta']$ in Σ_y and that of the geodesic $[\eta', \eta'']$ in Σ_x, we get, $d'(\Sigma_y(\eta), \Sigma_y(\eta')) \leq 2\kappa\delta_0$ and $d'(\Sigma_x(\eta'), \Sigma_x(\eta'')) \leq 2\kappa D_3$. Recall that $\forall\, \zeta \in [s, t_{xy}]$, $d^f(\Sigma_x(\zeta), \Sigma_y(\zeta)) \leq R_{2.4.7}(\kappa, A)$; in particular, $d^f(\Sigma_x(\eta'), \Sigma_y(\eta')) \leq R_{2.4.7}(\kappa, A)$. So, by triangle inequality, $d'(\xi, c(x,z)) \leq d'(\Sigma_y(\eta), \Sigma_x(\eta'')) \leq 2\kappa\delta_0 + R_{2.4.7}(\kappa, A) + 2\kappa D_3 = D_4$ (say).

Again, if $\xi \in \mu_{xy}$, then $d'(\xi, c(x,z)) \leq 2\kappa D_3 + A \leq 2\kappa D_3 + R_{2.4.7}(\kappa, A) \leq D_4$.

B. The portion of the path $c(y,z)$ over $B_{v_{xy}}$ is uniformly close to $c(x,y) \cup c(x,z)$:

Note that the portion of $c(y,z)$ over $B_{v_{xy}}$ is $c(y,z) \cap \tilde{\alpha}_{yz}$. Let $\xi \in c(y,z) \cap \tilde{\alpha}_{yz}$ such that $\eta = \pi_X(\xi)$. Then $\eta \in [t_y, t]$, and the slimness of $\triangle(t_{xy}, t_y, t)$ says that $\eta \in N_{\delta_0}([t_y, t_{xy}] \cup [t_{xy}, t])$. First, we consider that $\exists\, \eta' \in [t_y, t_{xy}]$ such that $d_B(\eta, \eta') \leq \delta_0$. So by taking lift of $[\eta, \eta']$ in Σ_y (see Lemma 2.4.12 (3)), we get $d'(\xi, c(x,y)) \leq d'(\Sigma_y(\eta), \Sigma_y(\eta')) \leq 2\kappa\delta_0$. Now, suppose $\exists\, \eta' \in [t_{xy}, t]$ such that $d_B(\eta, \eta') \leq \delta_0$. Recall that $[t_x, t_{xy}] \cup [t_{xy}, t]$ is $(3 + 2K_{5.1.20}(\kappa))$-quasi-geodesic and so $\exists\, \eta'' \in [t_x, t]$ such that $d_B(\eta', \eta'') \leq D_3$ (defined above in **A**). Taking lifts of the geodesic $[\eta, \eta']$ in Σ_y and that of the geodesic $[\eta', \eta'']$ in Σ_x, we get, $d'(\Sigma_y(\eta), \Sigma_y(\eta')) \leq 2\kappa\delta_0$ and $d'(\Sigma_x(\eta'), \Sigma_x(\eta'')) \leq 2\kappa D_3$. Also, $\forall\, \zeta \in [t_{xy}, t], d^f(\Sigma_x(\zeta), \Sigma_y(\zeta)) \leq R_{2.4.7}(\kappa, A)$; in particular, $d^f(\Sigma_y(\eta'), \Sigma_x(\eta')) \leq R_{2.4.7}(\kappa, A)$. Therefore, by triangle inequality, $d'(\xi, c(x,z)) \leq d'(\Sigma_y(\eta), \Sigma_x(\eta'')) \leq 2\kappa\delta_0 + R_{2.4.7}(\kappa, A) + 2\kappa D_3 = D_4$ (defined above in **A**).

C. The portion of the path $c(x,z)$ over $B_{v_{xy}}$ is uniformly close to $c(x,y) \cup c(y,z)$:

Note that the portion of $c(x,z)$ over $B_{v_{xy}}$ is $c(x,z) \cap \tilde{\alpha}_{xz}$. Let $\xi \in c(x,z) \cap \tilde{\alpha}_{xz}$ such that $\eta = \pi_X(\xi)$. Then $\eta \in [t_x, t]$, and so $\eta \in N_{\delta_0}([t_x, t_{xy}] \cup [t_{xy}, t])$. If

$\exists\ \eta' \in [t_x, t_{xy}]$ such that $d_B(\eta,\eta') \leq \delta_0$, then by taking lift of the geodesic $[\eta,\eta']$ in Σ_x, we get, $d'(\xi, c(x,y)) \leq d'(\Sigma_x(\eta),\Sigma_x(\eta')) \leq 2\kappa\delta_0$. Now, let $\exists\ \eta' \in [t_{xy}, t]$ such that $d_B(\eta,\eta') \leq \delta_0$. Recall that $\forall\ \zeta \in [t, t_{xy}]$, $d^f(\Sigma_x(\zeta),\Sigma_y(\zeta)) \leq R_{2.4.7}(\kappa, A)$. Again, if we look at $\triangle(t_{xy}, t, t_y), \exists\ \eta'' \in [t_{xy}, t_y] \cup [t_y, t]$ such that $d_B(\eta', \eta'') \leq \delta_0$. If $\eta'' \in [t_{xy}, t_y]$, then by taking lifts of geodesics $[\eta,\eta']$ and $[\eta',\eta'']$ in Σ_x and Σ_y respectively, we get, $d'(\xi, c(x,y)) \leq d'(\Sigma_x(\eta),\Sigma_y(\eta'')) \leq d'(\Sigma_x(\eta),\Sigma_x(\eta')) + d^f(\Sigma_x(\eta'),\Sigma_y(\eta')) + d'(\Sigma_y(\eta'),\Sigma_y(\eta'')) \leq 2\kappa\delta_0 + R_{2.4.7}(\kappa, A) + 2\kappa\delta_0 = D_5$ (say). If $\eta'' \in [t_y, t]$, then the same inequality would imply that ξ is D_5-close to $c(y,z)$ in the path metric of L_{KR}.

Let $D' = max\{D_1, D_2, D_4, D_5\} + 2D_{5.2.5}(\kappa, A)$, representing the maximum of all constants obtained in Case 1, Case 2; additionally, considering Corollary 5.2.5, we add $2D_{5.2.5}(\kappa, A)$. Therefore, the triangle formed by the paths $c(x,y), c(x,z)$ and $c(y,z)$, which we started with to show the combing criterion, are D'-slim. Hence, by Proposition 2.2.6, L_{KR} is $\delta_{5.2.1} = \delta_{2.2.6}(\psi, D', R)$-hyperbolic, where ψ is defined in Condition (1), equation 5.2. 1. □

5.2.2 Hyperbolicity of ladders (general case)

Lemma 5.2.8 (Bisection of ladders). *There are constants $K_{5.2.8} = K_{5.2.8}(K) = C_{2.4.12}(K)$, $C_{5.2.8} = C_{5.2.8}(K, C, \varepsilon) \geq C, \varepsilon_{5.2.8} = \varepsilon_{5.2.8}(K, C, \varepsilon) \geq \varepsilon$ such that the following holds.*

Suppose $z \in \mathscr{L} \cap X_{\mathfrak{B}}$ and Σ_z is a maximal K-qi section in $\mathscr{L}$ through z. Then Σ_z divide the ladder $\mathscr{L}$ into two $(K_{5.2.8}, C_{5.2.8}, \varepsilon_{5.2.8})$-subladders, $\mathscr{L}^+$ and $\mathscr{L}^-$ with central base $\mathfrak{B}$ such that

$$top(\mathscr{L}^+) \subseteq top(\mathscr{L}),\ \Sigma_z \subseteq bot(\mathscr{L}^+) \text{ and}$$
$$bot(\mathscr{L}^-) \subseteq bot(\mathscr{L}),\ \Sigma_z \subseteq top(\mathscr{L}^-)$$

Proof. Since the proofs are similar, we prove only for, say, $\mathscr{L}^+$. Note that Σ_z is a maximal K-qi section over some base, say, $B_z \subseteq \pi_B^{-1}(T_{\mathscr{L}})$. Let $T_z = \pi_B(B_z)$. There are two kinds of segments in the fibers of $\mathscr{L}^+$ as follows.

First kind: For all $v \in T_z$ and for all $b \in B_v$, $\mathscr{L}^+_{b,v} = [top(\mathscr{L}_{b,v}), \Sigma_z(b)] \subseteq \mathscr{L}_{b,v}$.

Second kind: Let $w \in T_{\mathscr{L}} \setminus T_z$ and $v \in T_z$ such that $d_T(v,w) = 1$. Let S be the connected component of $T \setminus \{v\}$ containing $\{w\}$. If we have an order $h_{wv}(top(\mathscr{L}_{\mathfrak{w},w})) < \Sigma_z(\mathfrak{v}) \leq top(\mathscr{L}_{\mathfrak{v},v})$ (see Figure 5.2 left one), then $\mathscr{L}^+_{b,t} = \emptyset$ for $t \in S$ and $b \in B_t$. If the order is $bot(\mathscr{L}_{\mathfrak{v},v}) < \Sigma_z(\mathfrak{v}) \leq h_{wv}(bot(\mathscr{L}_{\mathfrak{w},w}))$ (see Figure 5.2 right one), then $\mathscr{L}^+_{b,t} = \mathscr{L}_{b,t}$ for $t \in T_{\mathscr{L}} \cap S$ and $b \in B_t$ with the same orientation as it was for $\mathscr{L}$. Also, the family of maps $\{h_{wv}\}$ for $\mathscr{L}^+$ are the restriction of that of $\mathscr{L}$.

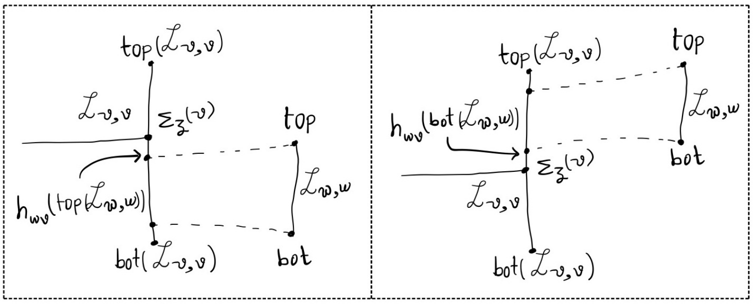


Figure 5.2

Now with the help of Lemma 5.1.21, we show that union of these fiber geodesics form a ladder. In view of Lemma 5.1.21, we have to find K', C', ε'. By Lemma 2.4.12 (2), one observes that $K' = C_{2.4.12}(K)$, $C' = C_{5.1.17}(K,C,\varepsilon)$ and $\varepsilon' = \varepsilon_{5.1.17}(K,C,\varepsilon)$ serve our purpose.

Therefore, by Lemma 5.1.21, $\mathscr{L}^+$ is a $(K_{5.2.8}, C_{5.2.8}, \varepsilon_{5.2.8})$-subladder in $\mathscr{L}$, where $K_{5.2.8} = k_{5.1.21}(K')$, $C_{5.2.8} = c_{5.1.21}(C')$ and $\varepsilon_{5.2.8} = \varepsilon_{5.1.21}(\varepsilon')$ for the above K', C', ε'. Since the family of maps $\{h_{wv}\}$ for $\mathscr{L}^+$ are restriction, so $k_{5.1.21}(K') = C_{2.4.12}(K)$. □

In the same line, we also have the following lemma. Since the proof is similar to that of Lemma 5.2.8, we omit the proof.

Lemma 5.2.9 (Trisection of ladders)**.** *There are constants $K_{5.2.9} = K_{5.2.9}(K) = C_{2.4.12}(K), C_{5.2.9} = C_{5.2.9}(K,C,\varepsilon), \varepsilon_{5.2.9} = \varepsilon_{5.2.9}(K,C,\varepsilon)$ such that the following holds.*

Suppose $x, y \in \mathscr{L} \cap X_{\mathfrak{B}}$. Let Σ_x, Σ_y be maximal K-qi sections through x, y over B_x, B_y respectively. We assume that $\forall\, v \in \pi_B(B_x \cap B_y)$ and $\forall\, b \in B_v$, we have an order $bot(\mathscr{L}_{b,v}) \leq \Sigma_x(b) \leq \Sigma_y(b) \leq top(\mathscr{L}_{b,v})$ in $\mathscr{L}_{b,v}$. Then we have $(K_{5.2.9}, C_{5.2.9}, \varepsilon_{5.2.9})$-subladder in $\mathscr{L}$ bounded by Σ_x, Σ_y with central base $B_x \cap B_y$.

Lemma 5.2.10. *For all $R \geq 2K_{5.2.8}(K)$ there exists $R_{5.2.10} = R_{5.2.10}(K,R)$ such that the following holds.*

Let $x \in \mathscr{L} \cap X_{\mathfrak{B}}$ and Σ_x be a maximal K-qi section in $\mathscr{L}$. Then we have two $(K_{5.2.8}, C_{5.2.8}, \varepsilon_{5.2.8})$-subladders, $\mathscr{L}^+$ and $\mathscr{L}^-$, coming from Lemma 5.2.8. Then $N_R(\mathscr{L}^+) \cap N_R(\mathscr{L}^-) \subseteq N_{R_{5.2.10}}(\Sigma_x)$ in both the path metric of $N_R(\mathscr{L}^+)$ and $N_R(\mathscr{L}^-)$.

Proof. For ease of notation, let $\mathscr{L}^{(1)} = \mathscr{L}^+$, $\mathscr{L}^{(2)} = \mathscr{L}^-$. Let d_i be the induced path metric on $N_R(\mathscr{L}^i)$, $i = 1,2$. Suppose $y \in N_R(\mathscr{L}^{(1)}) \cap N_R(\mathscr{L}^{(2)})$ and $y_i \in \mathscr{L}^{(i)}$ such that $d_i(y, y_i) \leq R$, $i = 1,2$. Then $d_X(y_1, y_2) \leq 2R$ and so $d_B(\bar{y}_1, \bar{y}_2) \leq 2R$, where $\pi_X(y_i) = \bar{y}_i$, $i = 1,2$. If $\mathfrak{T} \cap [\pi(y_1), \pi(y_2)] = \emptyset$, we let $u \in T$ such that

$d_T(\mathfrak{T},[\pi(y_1),\pi(y_2)]) = d_T(\mathfrak{T},u)$; otherwise, we take $u \in \mathfrak{T}\cap[\pi(y_1),\pi(y_2)]$ arbitrary. Fix $c \in B_u \cap [\bar{y}_1,\bar{y}_2]_B$. Then $d_B(\bar{y}_i,c) \leq 2R$. Let t_i be the nearest point projection of $\pi(y_i)$ on $\mathfrak{T}$ and $B_{y_i} = \mathfrak{B}\cup\pi_B^{-1}([t_i,\pi(y_i)])$ for $i=1,2$. Then there is $K_{5.2.8}(K)$-qi sections, say, Σ_{y_i} over B_{y_i} through y_i in $\mathscr{L}^{(i)}$, $i=1,2$. Let $K_{5.2.8}(K) = K_1$ Taking lifts of geodesic $[\bar{y}_i,c]_B$ in $\Sigma_{\bar{y}_i}$, we get, $d_i(y_i,\Sigma_{y_i}(c)) \leq 2K_1.2R = 4K_1R$, $i=1,2$. Then $d_X(\Sigma_{y_1}(c),\Sigma_{y_2}(c)) \leq d_X(\Sigma_{y_1}(c),y_1)+d_X(y_1,y_2)+d_X(y_2,\Sigma_{y_2}(c)) \leq 2R(4K_1+1)$. So $d^f(\Sigma_{y_1}(c),\Sigma_{y_2}(c)) \leq \phi(2R(4K_1+1))$. Since $\Sigma_x(c)\in[\Sigma_{y_1}(c),\Sigma_{y_2}(c)]^f \subseteq \mathscr{L}_{c,u}$, so $d^f(\Sigma_{y_i}(c),\Sigma_x(c)) \leq \phi(2R(4K_1+1))$. Hence $d_i(y,\Sigma_x) \leq d_i(y,\Sigma_x(c)) \leq d_i(y,y_i) + d_i(y_i,\Sigma_{y_i}(c)) + d_i(\Sigma_{y_i}(c),\Sigma_x(c)) \leq R+4K_1R+\phi(2R(4K_1+1))$, $i=1,2$. So, we can take $R_{5.2.10} := R(4K_1+1)+\phi(2R(4K_1+1))$. □

Now we are ready to state the main result of this Subsection 5.2.2.

Proposition 5.2.11. *For all $R \geq 2C^{(9)}_{2.4.12}(K)$, there exists $\delta_{5.2.11} = \delta_{5.2.11}(K,R)$ such that $L_{KR} := N_R(\mathscr{L}_K)$ is $\delta_{5.2.11}$-hyperbolic with respect to the path metric induced from X.*

Proof. **Subdivision of ladder**: We fix a fiber geodesic $\mathscr{L}_{a,u}$ for some $u\in\mathfrak{T}$ and $a \in B_u$. Let $K_1 = K_{5.2.9}(K)$. We also fix $A_0 > max\{\phi(2K+k_{2.2.4}(\delta_0',L_0',K)),\phi(4K(2R+1)+2R+1),\phi(8KR+2R),\phi(4KD_2+D_2)\}$ where D_2 is defined below in the verification of condition (4) of Proposition 2.2.7. For $x\in\mathscr{L}_{a,u}$, Σ_x denote a maximal K-qi section in $\mathscr{L}$ over, say, B_x. Let $\gamma:[0,l]\to\mathscr{L}_{a,u}$ be the arc length parametrization such that $\gamma(0) = bot(\mathscr{L}_{a,u})$ and $\gamma(l) = top(\mathscr{L}_{a,u})$. Now we inductively subdivide $\mathscr{L}$ into small girth ladders as follows. First, inductively we construct a finite sequence of points on $\mathscr{L}_{a,u}$ and K-qi sections through that, which will help in subdivision. Note that set map from $\mathfrak{B}$ to $bot(\mathscr{L})\cap\pi_X^{-1}(\mathfrak{B})$ is K-qi section in $\mathscr{L}$. Set $x_0 = \gamma(0)$ and $\Sigma_{x_0} = bot(\mathscr{L})\cap\pi_X^{-1}(\mathfrak{B})$. Suppose $x_i = \gamma(t_i)$ has been constructed. Let

$$\Omega_{i+1} = \{t\in(t_i,l] : \gamma(t) = x \text{ and } d^f(\Sigma_{x_i}(s),\Sigma_x(s)) > A_0, \forall\, s\in B_{x_i}\cap B_x\}.$$

If $\Omega_{i+1} = \emptyset$, then we define $x_{i+1} = \gamma(l)$ and stop the process. Otherwise, we take $x_{i+1} = \gamma(\min\{\inf\Omega_{i+1}+A_0/2, l\})$, and $x_{i+1}' = \gamma(\inf\Omega_{i+1}-A_0/2)$. The construction of these points and sections stop at n-th step if $x_n = \gamma(l)$.

Claim: Let $i > j$ and $d^f(\Sigma_{x_i}(t),\Sigma_{x_j}(t)) > A_0$, $\forall\, t\in B_{x_i}\cap B_{x_j}$. Then for $v\in\pi_B(B_{x_i}\cap B_{x_j})$ and $b\in B_v$, we have the order $bot(\mathscr{L}_{b,v}) \leq \Sigma_{x_i}(b) \leq \Sigma_{x_j}(b) \leq top(\mathscr{L}_{b,v})$ in the fiber geodesic $\mathscr{L}_{b,v}$.

Proof of the claim: Indeed, because we have the family of order preserving monotonic maps $\{h_{wv}\}$, if B_v is single vertex, then we are done. Otherwise, let $b,b'\in B_v$ such that $d_B(b,b') = 1$ and $bot(\mathscr{L}_{b',v}) \leq \Sigma_{x_i}(b') \leq \Sigma_{x_j}(b') \leq top(\mathscr{L}_{b',v})$ but

$bot(\mathscr{L}_{b,v}) \leq \Sigma_{x_j}(b) < \Sigma_{x_i}(b) \leq top(\mathscr{L}_{b,v})$. Let $\alpha = [bot(\mathscr{L}_{b',u}), \Sigma_{x_j}(b')]^f \subseteq \mathscr{L}_{b',u}$ and $\beta = [bot(\mathscr{L}_{b,u}), \Sigma_{x_j}(b)]^f \subseteq \mathscr{L}_{b,u}$. Consider the δ_0'-hyperbolic space $F_{b'b} := \pi_X^{-1}([b',b])$ (see Lemma 2.3.4). Then we apply Lemma 2.2.4, to L_0'-quasi-geodesic α, β in δ_0'-hyperbolic space $F_{b'b}$. So there is a point $z \in \beta$ such that $d_{X_u}(\Sigma_{x_i}(b'), z) \leq d_{F_{b'b}}(\Sigma_{x_i}(b'), z) \leq k_{2.2.4}(\delta_0', L_0', K)$. Thus by triangle inequality, $d_{X_u}(z, \Sigma_{x_i}(b)) \leq 2K + k_{2.2.4}(\delta_0', L_0', K)$. Hence $d^f(z, \Sigma_{x_i}(b)) \leq \phi(2K + k_{2.2.4}(\delta_0', L_0', K))$. Since $\Sigma_{x_j}(b) \in [z, \Sigma_{x_i}(b)]^f$, so $d^f(\Sigma_{x_j}(b), \Sigma_{x_i}(b)) \leq d^f(\Sigma_{x_i}(b), z) \leq \phi(2K + k_{2.2.4}(\delta_0', L_0', K)) < A_0$ – which contradicts to the fact that $d^f(\Sigma_{x_i}(t), \Sigma_{x_j}(t)) > A_0 \; \forall \; t \in B_{x_i} \cap B_{x_j}$.

Fact 1: Hence by Lemma 5.2.9, if $d^f(\Sigma_{x_i}(t), \Sigma_{x_j}(t)) > A_0 \; \forall \; t \in B_{x_i} \cap B_{x_j}$, the K-qi sections Σ_{x_i} and Σ_{x_j} bounds a $(K_1, C_1, \varepsilon_1)$-subladder in $\mathscr{L}_K$ over the central base $B_{x_i} \cap B_{x_j}$, where $K_1 = K_{5.2.9}(K), C_1 = C_{5.2.9}(K, C, \varepsilon)$ and $\varepsilon_1 = \varepsilon_{5.2.9}(K, C, \varepsilon)$. If $j = i+1$, we denote this subladder by $\mathscr{L}^{(i)} = \mathscr{L}(\Sigma_{x_i}, \Sigma_{x_{i+1}})$.

Again, if $d^f(\Sigma_{x_i}(t), \Sigma_{x_{i+1}}(t)) > A_0$, $\forall \; t \in B_{x_i} \cap B_{x_{i+1}}$, then $\Sigma_{x'_{i+1}}$ is a maximal K-qi section in $\mathscr{L}_K$ through x'_{i+1} over $B_{x'_{i+1}}$. Also, from the construction, $\exists \; a \in B_{x_i} \cap B_{x'_{i+1}}$ and $\exists \; b \in B_{x'_{i+1}} \cap B_{x_{i+1}}$ such that

$$d^f(\Sigma_{x_i}(a), \Sigma_{x'_{i+1}}(a)) \leq A_0 \text{ and } d^f(\Sigma_{x'_{i+1}}(b), \Sigma_{x_{i+1}}(b)) \leq A_0 \tag{5.2. 2}$$

It is very well possible that $\Sigma_{x'_{i+1}}$ does not lie fully in $\mathscr{L}^{(i)}$. In that case, considering Lemma 2.4.12 (2), we adjust $\Sigma_{x'_{i+1}}$ to lie inside $\mathscr{L}^{(i)}$, turning it into a $C_{2.4.12}(K)$-qi section over possibly a smaller base than $B_{x'_{i+1}}$. (We refer to the proof of Lemma 2.4.12 (2), i.e., [10, Lemma 3.1].) We still denote this modified qi section as $\Sigma_{x'_{i+1}}$ and its base as $B_{x'_{i+1}}$. We note that this modification will not effect to the girth condition 5.2. 2; and $K_1 = K_{5.2.9}(K) = C_{2.4.12}(K)$.

Therefore, by Lemma 5.2.8, the K_1-qi section $\Sigma_{x'_{i+1}}$ subdivides the ladder $\mathscr{L}^{(i)}$ into two $(K_2, C_2, \varepsilon_2)$-subladders, where $K_2 = K_{5.2.8}(K_1), C_2 = C_{5.2.8}(K_1, C_1, \varepsilon_1)$ and $\varepsilon_2 = \varepsilon_{5.2.8}(K_1, C_1, \varepsilon_1)$. Let us denote these subladders of $\mathscr{L}^{(i)}$ by $\mathscr{L}^{i1} = \mathscr{L}(\Sigma_{x_i}, \Sigma_{x'_{i+1}})$ and $\mathscr{L}^{i2} = \mathscr{L}(\Sigma_{x'_{i+1}}, \Sigma_{x_{i+1}})$. Note that $K_2 = C^{(2)}_{2.4.12}(K)$ and the ladders $\mathscr{L}^{i1}$ and $\mathscr{L}^{i2}$ satisfy the small girth condition 5.2. 2.

Therefore, the ladder $\mathscr{L}$ is subdivided into $(K_1, C_1, \varepsilon_1)$-subladders $\mathscr{L}^{(i)}$, $0 \leq i \leq n-1$. Also, $\mathscr{L}^{(i)}$'s are further subdivided into two $(K_2, C_2, \varepsilon_2)$-subladders $\mathscr{L}^{i1}, \mathscr{L}^{i2}$ in $\mathscr{L}^{(i)}$ except possibly for $i = n-1$.

Lemma 5.2.12. *Let $x \in \Sigma_{x_i}$ and $y \in \Sigma_{x_j}$ such that $d_X(x,y) \leq D$ and $i \neq j$. Then there is a point $c \in B_{x_i} \cap B_{x_j}$ such that $d^f(\Sigma_{X_i}(c), \Sigma_{x_j}(c)) \leq \phi(4KD + D)$.*

Proof. Let $\pi_X(x) = a$ and $\pi_X(y) = b$. Suppose $c \in [a,b]$ such that $a \in B_{x_i} \cap B_{x_j}$. Since B_{x_i}'s are isometrically embedded in B, $[a,c]_B \subseteq B_{x_i}$ and $[c,b]_B \subseteq B_{x_j}$. Now $d_B(a,b) \leq d_X(x,y) \leq D$ implies $d_B(a,c) \leq D$ and $d_B(c,b) \leq D$. By taking K-qi lift of $[a,c]_B$ and $[c,b]_B$ in Σ_{x_i} and Σ_{x_j} respectively, we have $d_X(x,\Sigma_{x_i}(c)) \leq 2KD$ and $d_X(y,\Sigma_{x_j}(c)) \leq 2KD$. Again by triangle inequality, $d_X(\Sigma_{x_i}(c),\Sigma_{x_j}(c)) \leq 4KD+D$. Hence $d^f(\Sigma_{X_i}(c),\Sigma_{x_j}(c)) \leq \phi(4KD+D)$. □

Proof of Theorem 5.2.11: We use the following notations for the proof.

$$X_i := N_R(\mathscr{L}^{(i)}),\ L^{i1} := N_R(\mathscr{L}^{i1}),\ L^{i2} := N_R(\mathscr{L}^{i2}),\ 0 \leq i \leq n-1$$

From the construction, it follows that $L_{KR} = \cup_{i=0}^{n-1} X_i$. We will verify the conditions of Proposition 2.2.7.

(1) *X_i's are uniformly hyperbolic,* $0 \leq i \leq n-1$.

Note that $(\mathscr{L}^{(i)})^g|_{B_{x_i}\cap B_{x_{i+1}}} > A_0$ except possibly for $i = n-1$ (see Definition 5.1.19 for notation). If $(\mathscr{L}^{(n-1)})^g|_{B_{x_{n-1}}\cap B_{x_n}} \leq A_0$ then by Proposition 5.2.1, X_{n-1} is $\delta_{5.2.1}(K_1,A_0,R)$-hyperbolic. Otherwise, the ladder $\mathscr{L}^{(i)}$ is subdivided by a K_1-qi section $\Sigma_{x'_{i+1}}$ into two (K_2,C_2,ε_2)-subladders, $\mathscr{L}^{i1}$ and $\mathscr{L}^{i2}$ such that their girth over central base $\leq A_0$ (see inequation 5.2. 2). Since (X,B,T) satisfies flaring condition, by Proposition 5.2.1, L^{i1} and L^{i2} are $\delta_{5.2.1}(K_2,A_0,R)$-hyperbolic. Note that $N_{2K_1}(\Sigma_{x'_{i+1}})$ is a connected subspace in $L^{i1}\cap L^{i2}$, and by Lemma 5.2.10, $L^{i1}\cap L^{i2} \subseteq N_{R_{5.2.10}(K_1,R)}(N_{2K_1}(\Sigma_{x'_{i+1}}))$. Again the inclusions $N_{2K_1}(\Sigma_{x'_{i+1}}) \hookrightarrow L^{i1}$ and $N_{2K_1}(\Sigma_{x'_{i+1}}) \hookrightarrow L^{i2}$ are $K_1(2K_1+1)$-qi embedding (see Lemma 2.4.12 (3)). So by Lemma 2.1.4, $L^{i1}\cap L^{i2}$ is L_1-qi embedded in both L^{i1} and L^{i2} for some L_1 depending on $K_1(2K_1+1)$ and $R_{5.2.10}(K_1,R)$. Therefore, by Remark 2.2.8, $L^{(i)}$ is $\delta' = \delta_{2.2.8}(\delta_{5.2.1}(K_2,A_0,R),L_1)$. Therefore, for $0 \leq i \leq n-1$, X_i is δ_1-hyperbolic metric space, where $\delta_1 = max\{\delta',\delta_{5.2.1}(K_1,A_0,R)\}$.

(2) Let $0 \leq i \leq n-2$. By Lemma 2.4.12 (3), $N_{2K}(\Sigma_{x_{i+1}})$ is $K(2K+1)$-qi embedded in both X_i and X_{i+1}. By *Fact* 1, Σ_{x_i} and $\Sigma_{x_{i+2}}$ bounds (K_1,C_1,ε_1)-ladder. So by Lemma 5.2.10, $X_i \cap X_{i+1} \subseteq N_{R_{5.2.10}(K_1,R)}(\Sigma_{x_{i+1}})$. So by Lemma 2.1.4, $X_i \cap X_{i+1}$ is L_2-qi embedded in both X_i and X_{i+1} for some L_2 depending on $K(2K+1)$ and $R_{5.2.10}(K_1,R)$.

(3) Let $x \in X_i$, $y \in X_{i+1}$ and α be a path in L_{KR} joining x and y.

Claim: There is a point in α which is R-close to $\mathscr{L}^{(i)}$ and $\mathscr{L}^{(i+1)}$.

Proof of the claim: Suppose this is not the case. Then there are points $z \in \alpha$, $z_i \in \mathscr{L}^{(i)}$ and $z_j \in \mathscr{L}^{(j)}$ such that $d_{X_i}(z,z_i) \leq R$, $d_{X_j}(z,z_j) \leq R$ and $j-i \geq 2$. So $d_X(z_i,z_j) \leq 2R$. Then by Lemma 5.2.12, $\exists\, c \in B_{x_i}\cap B_{x_j}$ such that $d^f(\Sigma_{x_i}(c),\Sigma_{x_j}(c)) \leq \phi(8KR+2R) < A_0$ – which contradicts to the construction of Σ_{x_i}'s.

(4) Now we want to prove that the pair (Y_i,Y_{i+1}) is uniformly cobonded for $1 \leq i \leq n-2$ where $Y_i = X_{i-1}\cap X_i$ and $Y_{i+1} = X_i \cap X_{i+1}$. Since X_i's are δ_1-hyperbolic and the inclusion $N_{2K}(\Sigma_{x_i}) \hookrightarrow X_i$ is $K(2K+1)$-qi embedding (see Lemma 2.4.12 (3)),

then Σ_{x_i}'s are K'-quasiconvex in X_i, where $K' = K_{2.2.22}(\delta_1, K(2K+1), 0) + 2K$ (see Lemma 2.2.22 (1)). By similar argument, we have that $\Sigma_{x_{i+1}}$ is also K'-quasiconvex in X_i.

We prove that the set of nearest point projections of Σ_{x_i} on $\Sigma_{x_{i+1}}$ in the metric of X_i is uniformly bounded; which will complete the proof. Indeed, let $\rho : \Sigma_{x_i} \to \Sigma_{x_{i+1}}$ be a nearest point projection map in X_i such that the diameter of $\rho(\Sigma_{x_i})$ is bonded by D in the metric of X_i. Then by Lemma 2.2.18 there is D_1 depending on δ_1, K' and D such that the pair $(\Sigma_{x_i}, \Sigma_{x_{i+1}})$ is D_1-cobounded in X_i. By *Fact* 1, $\Sigma_{x_{i-1}}$ and $\Sigma_{x_{i+1}}$ bounds a $(K_1, C_1, \varepsilon_1)$-ladder. So by Lemma 5.2.10, $Hd(Y_i, \Sigma_{x_i})$ and $Hd(Y_{i+1}, \Sigma_{x_{i+1}})$ are bounded by $R_{5.2.10}(K_1, R)$. Hence by Lemma 2.2.21 (2), the pair (Y_i, Y_{i+1}) is D'-cobounded where $D' = D_1 + 2E_{2.2.21}(\delta_1, K', R_{5.2.10}(K_1, R))$.

Let $\rho(y_j) = p_j$ for $y_j \in \Sigma_{x_i}$ and $p_j \in \Sigma_{x_{i+1}}$, $j = 1, 2$. We prove that $d_{X_i}(p_1, p_2)$ is bounded by D. By [10, Lemma 1.31(2)], the arc-length parametrizations of $[y_1, p_1]_{X_i} \cup [p_1, p_2]_{X_i}$ and $[y_2, p_2]_{X_i} \cup [p_2, p_1]_{X_i}$ are $(3+2K')$-quasi-geodesic in X_i.

Claim: $d_{X_i}(p_1, p_2) \leq L_{2.2.5}(\delta_1, 3+2K', 3+2K') =: D$.

Proof of claim: On contrary, suppose $d_{X_i}(p_1, p_2) > L_{2.2.5}(\delta_1, 3+2K', 3+2K')$. Then by Lemma 2.2.5, $[y_1, p_1]_{X_i} \cup [p_1, p_2]_{X_i} \cup [p_2, y_2]_{X_i}$ is λ-quasi-geodesic in X_i, where $\lambda = \lambda_{2.2.5}(\delta_1, 3+2K', 3+2K')$. Now by stability of quasi-geodesic (Lemma 2.2.2) in X_i and K'-quasiconvexity of Σ_{x_i} in X_i, $\exists\, z_1, z_2 \in \Sigma_{x_i}$ such that $d_{X_i}(p_j, z_j) \leq D_2$, where $D_2 = D_{2.2.2}(\delta_1, \lambda, \lambda) + K'$, $j = 1, 2$. In particular, $d_X(\Sigma_{x_i}, \Sigma_{x_{i+1}}) \leq D_2$. Then by Lemma 5.2.12, there is $c \in B_{x_i} \cap B_{x_{i+1}}$ such that $d^f(\Sigma_{x_i}(c), \Sigma_{x_{i+1}}(c)) \leq \phi(4KD_2 + D_2) < A_0$ which contradicts to our construction of Σ_{x_i}'s.

(5) On contrary, suppose $d_{X_i}(Y_i, Y_{i+1}) < 1$. Then $d_{X_i}(\Sigma_{x_i}, \Sigma_{x_{i+1}}) \leq 2R + 1$. Then by Lemma 5.2.12, $\exists\, c \in B_{x_i} \cap B_{x_{i+1}}$ such that $d^f(\Sigma_{x_i}(c), \Sigma_{x_{i+1}}(c)) \leq \phi(4K(2R+1) + 2R + 1) < A_0$ which contradicts to our construction of Σ_{x_i}'s.

Therefore, we have shown that the collection $\{X_i : 0 \leq i \leq n-1\}$ satisfies all conditions of Proposition 2.2.7. Hence, $N_R(\mathscr{L}_K) = L_{RK}$ is $\delta_{5.2.11}$-hyperbolic, where $\delta_{5.2.11} = \delta_{2.2.7}(\delta_1, K(2K+1), D')$. □

5.3 Hyperbolicity of flow spaces

Suppose $R = 6\delta_0 + \theta_{2.2.26}(\delta'_0, L'_0, \lambda'_0) + 4\lambda'_0 + 8\delta'_0 > R_{2.2.13}(\delta'_0, \lambda'_0) = 2\lambda'_0 + 5\delta'_0$ and $k = K_{2.4.12}$. Let $u \in T$ and $\mathscr{F}l_K(X_u)$ be the flow space of X_u obtained for the parameters R and k (see Definition 5.1.7). More precisely, $\mathscr{F}l_K(X_u)$ is (K, C, ε)-semicontinuous family, where $K = K_{5.1.9}(k, R)$, $C = C_{5.1.9}$, $\varepsilon = \varepsilon_{5.1.9}(R)$. This section is devoted to proving the (uniform) neighborhood of $\mathscr{F}l_K(X_u)$ is (uniformly) hyperbolic with the induced path metric. **In this section, we work with these flow**

spaces and these parameters. **So we reserve K, C and ε for the above values**. Sometimes we use the notations $\mathscr{U}_K := \mathscr{F}l_K(X_u)$ and $U_{KL} := Fl_{KL}(X_u)$. The idea is to apply Bowditch's criterion (see Proposition 2.2.6) to show that U_{KL} is hyperbolic (see Theorem 5.3.16). Given a pair of points, we first find a ladder inside $\mathscr{U}_K$ containing those points (see Corollary 5.3.8), and in which we take a fixed geodesic path joining them for the family of paths to apply Proposition 2.2.6. Then we show that this family of paths satisfies all conditions of Bowditch's criterion. This strategy is elaborated in [9, Chapter 5] when X is a tree of metric spaces. In the line of finding ladder, we prove something more in the following proposition. This proposition is kind of heart of this section. In view of Remark 2.4.8, for this section, we require the tree of metric bundles (X,B,T) to satisfy $max\{C^{(9)}_{2.4.12}(k_{5.3.1}), \mathscr{R}_0(2k_{5.3.8}+1)\} =: \boldsymbol{k_*}$ -flaring condition, where $\mathscr{R}_0 = L_{5.1.3}(k_{5.3.8}, c_{5.3.8}, \varepsilon_{5.3.8})$ is defined in the proof of Lemma 5.3.12, Case (1).

Proposition 5.3.1. *There are constants $k_{5.3.1} = k_{5.3.1}(K), c_{5.3.1} = c_{5.3.1}(K)$ and $\varepsilon_{5.3.1} = \varepsilon_{5.3.1}(K)$ such that the following hold.*

Suppose $x^i \in \mathscr{F}l_K(X_u)$ and Σ_i is a K-qi section through x^i over $B_{x^i} := B_{[u,\pi(x^i)]}$ lying inside $\mathscr{F}l_K(X_u)$, $i = 1,2,3$. Let $\mathfrak{B} = \bigcap_{i=1}^{3} B_{x^i}$ and $\mathfrak{T} = \pi_B(\mathfrak{B})$. Then we have the following.

1. *There is $(k_{5.3.1}, c_{5.3.1}, \varepsilon_{5.3.1})$-ladder $\mathscr{L}^i$, $i = 1,2,3$ containing Σ_i with a central base $\mathfrak{B}$ (possibly bigger) such that:*
   - *(a) Let $S_i = hull(\pi(\mathscr{L}^i))$ and $B_i = \pi_B^{-1}(S_i)$, $i = 1,2,3$, and $B_{123} = \cap_{i=1}^3 B_i$ and $S_{123} = \cap_{i=1}^3 S_i$. Then $\Xi = \{\bigcap_{i=1}^{3} \mathscr{L}^i_{b,v} : v \in S_{123},\ b \in B_v\}$ is a $k_{5.3.1}$-qi section over B_{123} and $\Xi \subseteq N^f_{5\delta_0}(\mathscr{U}_K)$.*
   - *(b) $\Sigma_i \subseteq bot(\mathscr{L}^i) \subseteq \mathscr{U}_K$ and $\Xi \subseteq top(\mathscr{L}^i)$, $i = 1,2,3$.*
   - *(c) $\mathscr{L}^i \subseteq N^f_{6\delta_0}(\mathscr{U}_K)$, $i = 1,2,3$.*
2. *There exist $(k_{5.3.1}, c_{5.3.1}, \varepsilon_{5.3.1})$-ladder $\mathscr{L}^{ij}$ with central base $\mathfrak{B}$ containing Σ_i and Σ_j such that $bot(\mathscr{L}^i) \subseteq top(\mathscr{L}^{ij})$, $bot(\mathscr{L}^j) \subseteq bot(\mathscr{L}^{ij})$. Also, $\mathscr{L}^{ij} \subseteq N^f_{2\delta_0}(\mathscr{U}_K)$.*

Although $k_{5.3.1}$, $c_{5.3.1}$ and $\varepsilon_{5.3.1}$ depend on the constants C, ε and the other structural constants, we keep those implicit.

Proof. The construction of $\mathscr{L}^i$, $\mathscr{L}^{ij}$ and Ξ are by induction on $d_T(u,v)$, where $v \in T$. As an initial step, first we explain how to get them in X_u. Note that $\Sigma_i \cap X_u$

is a K-qi section over B_u in the metric of X_u, $i=1,2,3$. Let $\Sigma_i \cap F_{b,u} = \{x^i_{b,u}\}$, $i=1,2,3$. Then by Lemma 2.4.15, $\forall\, b \in B_u$, δ_0-center, say, $z_{b,u}$ of geodesic triangle $\triangle_{b,u} = \triangle(x^1_{b,u}, x^2_{b,u}, x^3_{b,u})$ in the fiber $F_{b,u}$, forms a $k_{2.4.15}(K)$-qi section over B_u in the metric of X_u. Let $Y_{b,u} := \cup_{i=1}^3 [z_{b,u}, x^i_{b,u}]_{F_{b,u}}$, $b \in B_u$. We call $\cup_{b\in B_u} Y_{b,u}$ as tripod of ladder over B_u and $[z_{b,u}, x^i_{b,u}]_{F_{b,u}}$ as legs of the tripod $Y_{b,u}$ with vertices $\{x^i_{b,u} : i=1,2,3\}$. Recall that $Q_{b,u} = \mathscr{U}_K \cap F_{b,u}$. Now $Q_{b,u} (= F_{b,u})$ is $2\delta_0$-quasiconvex in $F_{b,u}$ implies $[x^i_{b,u}, x^j_{b,u}]_{F_{b,u}} \subseteq N^f_{2\delta_0}(\mathscr{U}_K)$ for all distinct $i,j \in \{1,2,3\}$. So δ_0-centers, $z_{b,u}$ of geodesic triangles $\triangle_{b,u}$ belong to $N^f_{5\delta_0}(\mathscr{U}_K)$ and $Y_{b,u} = \cup_{i=1}^3 [z_{b,u}, x^i_{b,u}]_{F_{b,u}} \subseteq N^f_{6\delta_0}(Q_{b,u})$. Here $\mathscr{L}^i_{b,u} = [z_{b,u}, x^i_{b,u}]_{F_{b,u}}$ with $top(\mathscr{L}^i_{b,u}) = z_{b,u}$, $bot(\mathscr{L}^i_{b,u}) = x^i_{b,u}$ for $i \in \{1,2,3\}$ and $\mathscr{L}^{ij}_{b,u} = [x^i_{b,u}, x^j_{b,u}]_{F_{b,u}}$ with $top(\mathscr{L}^{ij}_{b,u}) = bot(\mathscr{L}^i_{b,u}) = x^i_{b,u}, bot(\mathscr{L}^{ij}_{b,u}) = bot(\mathscr{L}^j_{b,u}) = x^j_{b,u}$ for all distinct $i,j \in \{1,2,3\}$. We note that $\{z_{b,u} : b \in B_u\} \subseteq \Xi$ (which we are constructing).

Now we assume the induction hypothesis. In other words, let $v, w \in \pi(\mathscr{U}_K)$ such that $d_T(u,v) = n$, $d_T(u,w) = n+1$ and $d_T(v,w) = 1$. Suppose we have constructed $\mathscr{L}^i$, L^{ij} and Ξ over B_t for all the vertices $t \in [u,v]$. Now we will explain how and when to extend $\mathscr{L}^i$, $\mathscr{L}^{ij}$ and Ξ inside X_w. Let $[\mathfrak{v}, \mathfrak{w}]$ be the edge joining $\mathfrak{v} \in B_v$ and $\mathfrak{w} \in B_w$.

We divide the construction into following cases and subcases. Before going into fabrication, let us fix some notations and collect some facts.

Notations: We use the following notations $\mathscr{L}^i_{a,t} := \mathscr{L}^i \cap F_{a,t}$, $\mathscr{L}^{ij}_{a,t} := \mathscr{L}^{ij} \cap F_{a,t}$ for $t \in T$ and $a \in B_t$. Consider the nearest point projection maps by $P_{\mathfrak{w}} : F_{\mathfrak{v}\mathfrak{w}} \to F_{\mathfrak{w},w}$, $P_Y : F_{\mathfrak{v}\mathfrak{w}} \to Y_{\mathfrak{v},v}$ and the modified projection (see Definition 2.2.25) map by $\bar{P}_Y : F_{\mathfrak{w},w} \to Y_{\mathfrak{v},v}$. In the construction, we will see that either $Y_{\mathfrak{v},v}$ is a genuine tripod or a degenerate tripod (i.e., a geodesic segment). We denote the tripod, in the former case, by $Y_{\mathfrak{v},v} := \cup_{i=1}^3 [x^i_{\mathfrak{v},v}, z_{\mathfrak{v},v}]^f$ (in Case 1 and Case 2 below); and in the later case, by $Y_{\mathfrak{v},v} := [x^i_{\mathfrak{v},v}, y^i_{\mathfrak{v},v}]^f$ (in Case 3 and Case 4 below) with $top(\mathscr{L}^i_{\mathfrak{v},v}) = y^i_{\mathfrak{v},v}$ and $bot(\mathscr{L}^i_{\mathfrak{v},v}) = x^i_{\mathfrak{v},v}$. Let T_{vw} be the connected component of $T \setminus \{v\}$ containing w and $B_{T_{vw}} = \pi_B^{-1}(T_{vw})$, $X_{T_{vw}} = \pi^{-1}(T_{vw})$.

Facts: Let $\bar{x}^i_{\mathfrak{v},v} \in \bar{Y}_{\mathfrak{v},v}$ be the point closest to $x^i_{\mathfrak{v},v} (= bot(\mathscr{L}^i_{\mathfrak{v},v}))$ in the induced path metric of $Y_{\mathfrak{v},v}$, $i=1,2,3$. Suppose $\bar{x}^i_{\mathfrak{w},w} := P_{\mathfrak{w}}(\bar{x}^i_{\mathfrak{v},v})$. Again if $\bar{Y}_{\mathfrak{v},v} \subseteq \mathscr{L}^i_{\mathfrak{v},v}$ then $\bar{x}^{i-1}_{\mathfrak{v},v} = \bar{x}^{i+1}_{\mathfrak{v},v}$ ($i \pm 1$ is considered in module 3). In this case, we set $\bar{y}^i_{\mathfrak{v},v} := \bar{x}^{i-1}_{\mathfrak{v},v}$. Note that $\bar{y}^i_{\mathfrak{v},v}$ is the point closest to $z_{\mathfrak{v},v}$ in the induced path metric of $Y_{\mathfrak{v},v}$. Suppose $\bar{y}^i_{\mathfrak{w},w} := P_{\mathfrak{w}}(\bar{y}^i_{\mathfrak{v},v})$.

Fact (1): Suppose $\bar{Y}_{\mathfrak{v},v} \subseteq \mathscr{L}^i_{\mathfrak{v},v}$. Then by Lemma 2.2.26 (2) (b),

$$d_{\mathfrak{v}\mathfrak{w}}(P_{\mathfrak{w}}(x^i_{\mathfrak{v},v}), \bar{x}^i_{\mathfrak{w},w}),\ d_{\mathfrak{v}\mathfrak{w}}(P_{\mathfrak{w}}(z_{\mathfrak{v},v}), \bar{y}^i_{\mathfrak{w},w}) \text{ and } d_{\mathfrak{v}\mathfrak{w}}(P_{\mathfrak{w}}(x^{i\pm1}_{\mathfrak{v},v}), \bar{y}^i_{\mathfrak{w},w})$$

are bounded by $D_{2.2.26}(\delta_0', L_0', \lambda_0') = D$ (say).

Fact (2): Suppose $\bar{Y}_{\mathfrak{v},v} \nsubseteq \mathscr{L}^i_{\mathfrak{v},v}$ for any $i \in \{1,2,3\}$, i.e., $z_{\mathfrak{v},v}$ is in the interior of $\bar{Y}_{\mathfrak{v},v}$ in the induced metric of $Y_{\mathfrak{v},v}$. By Lemma 2.2.26 (2) (a), $d_{\mathfrak{v}\mathfrak{w}}(P_{\mathfrak{w}}(x^i_{\mathfrak{v},v}), \tilde{x}^i_{\mathfrak{w},w}) \leq D$.

Now suppose the pair $(Y_{\mathfrak{v},v}, F_{\mathfrak{w},w})$ is not C-cobounded in $F_{\mathfrak{v}\mathfrak{w}}$. Then Lemma 2.2.13, $d_{\mathfrak{v}\mathfrak{w}}(Y_{\mathfrak{v},v}, F_{\mathfrak{w},w}) \leq 2\lambda_0' + 5\delta_0'$ and so $Hd_{\mathfrak{v}\mathfrak{w}}(P_Y(F_{\mathfrak{w},w}), P_{\mathfrak{w}}(Y_{\mathfrak{v},v})) \leq 4\lambda_0' + 8\delta_0'$.

Fact (3): Note that $\bar{x}^i_{\mathfrak{v},v} \in P_Y(F_{\mathfrak{w},w})$. So, $d_{\mathfrak{v}\mathfrak{w}}(\bar{x}^i_{\mathfrak{v},v}, \tilde{x}^i_{\mathfrak{w},w}) \leq 4\lambda_0' + 8\delta_0' \leq K$. Again $d^f(\bar{x}^i_{\mathfrak{v},v}, Q_{\mathfrak{v},v}) \leq 6\delta_0$ and since $R > 6\delta_0 + 4\lambda_0' + 8\delta_0'$, so $\tilde{x}^i_{\mathfrak{w},w} \in Q_{\mathfrak{w},w}$ (see construction of flow spaces 5.1.1).

Fact (4): Suppose $\bar{Y}_{\mathfrak{v},v} \subseteq \mathscr{L}^i_{\mathfrak{v},v}$ for some $i \in \{1,2,3\}$. By the same argument as in Fact (3), $\tilde{y}^i_{\mathfrak{w},w} \in Q_{\mathfrak{w},w}$ and $d_{\mathfrak{v}\mathfrak{w}}(\bar{y}^i_{\mathfrak{v},v}, \tilde{y}^i_{\mathfrak{w},w}) \leq 4\lambda_0' + 8\delta_0' \leq K$. Now suppose that $d_{\mathfrak{v}\mathfrak{w}}(x^{i+1}_{\mathfrak{v},v}, Q_{\mathfrak{w},w}) \leq K$. Then $d_{\mathfrak{v}\mathfrak{w}}(P_Y \circ P_{\mathfrak{w}}(x^{i+1}_{\mathfrak{v},v}), x^{i+1}_{\mathfrak{v},v}) \leq 2K$ and $P_Y \circ P_{\mathfrak{w}}(x^{i+1}_{\mathfrak{v},v}) \in \bar{Y}_{\mathfrak{v},v}$. So $d^f(P_Y \circ P_{\mathfrak{w}}(x^{i+1}_{\mathfrak{v},v}), x^{i+1}_{\mathfrak{v},v}) \leq \phi(2K)$. Again $z_{\mathfrak{v},v}$ is δ_0-center of geodesic triangle with vertices $\{x^i_{\mathfrak{v},v} : i = 1,2,3\}$ in the fiber $F_{\mathfrak{v},v}$ so $d^f(z_{\mathfrak{v},v}, [P_Y \circ P_{\mathfrak{w}}(x^{i+1}_{\mathfrak{v},v}), x^{i+1}_{\mathfrak{v},v}]^f) \leq 2\delta_0$. Thus $d^f(z_{\mathfrak{v},v}, x^{i+1}_{\mathfrak{v},v}) \leq \phi(2K) + 2\delta_0$. Hence $d_{\mathfrak{v}\mathfrak{w}}(z_{\mathfrak{v},v}, Q_{\mathfrak{w},w}) \leq K + \phi(2K) + 2\delta_0$

Fact (5): Fact (3) and Fact (4) also say that if $Y_{\mathfrak{v},v} = \mathscr{L}^i_{\mathfrak{v},v} = [x^i_{\mathfrak{v},v}, y^i_{\mathfrak{v},v}]$ (i.e., $Y_{\mathfrak{v},v}$ has one leg) then $d_{\mathfrak{v}\mathfrak{w}}(\bar{x}^i_{\mathfrak{v},v}, \tilde{x}^i_{\mathfrak{v},v}) \leq K$, $d_{\mathfrak{v}\mathfrak{w}}(\bar{y}^i_{\mathfrak{v},v}, \tilde{y}^i_{\mathfrak{v},v}) \leq K$. Also by Fact (1), $d_{\mathfrak{v}\mathfrak{w}}(P_{\mathfrak{w}}(x^i_{\mathfrak{v},v}), \tilde{x}^i_{\mathfrak{w},w}) \leq D$ and $d_{\mathfrak{v}\mathfrak{w}}(P_{\mathfrak{w}}(y^i_{\mathfrak{v},v}), \tilde{y}^i_{\mathfrak{w},w}) \leq D$.

Now we are ready to explain how and when to extend the tripod $Y_{\mathfrak{v},v}$, in particular, $\mathscr{L}^i, \mathscr{L}^{ij}$ and Ξ, first in $F_{\mathfrak{w},w}$ and then in the entire X_w. In the end of some cases and subcases, we make some note which will be used in Lemma 5.3.6 and Lemma 5.3.7. We recommend the reader first to read the construction and then look at those notes while reading Lemma 5.3.6 and Lemma 5.3.7. Also, all the time we refer to the Figure 5.3.

Case 1: Suppose $Y_{\mathfrak{v},v}$ has three legs and the pair $(Y_{\mathfrak{v},v}, F_{\mathfrak{w},w})$ is C-cobounded in $F_{\mathfrak{v}\mathfrak{w}}$. Depending on $\Sigma_i \cap F_{\mathfrak{w},w}$ is empty or nonempty, we consider the following subcases.

Subcase $(1A)$: If $\Sigma_i \cap F_{\mathfrak{w},w} \neq \emptyset, \forall\ i \in \{1,2,3\}$, then we have tripod of ladder inside X_w formed by qi sections $\Sigma_i \cap X_w,\ i = 1,2,3$ as described in initial step of the induction.

Subcase $(1B)$: Suppose $\Sigma_{i\pm1} \cap X_w \neq \emptyset$ and $\Sigma_i \cap X_w = \emptyset$. Now we prove that $d_{\mathfrak{v}\mathfrak{w}}(z_{\mathfrak{v},v}, \Sigma_{i-1}(\mathfrak{w})) \leq D_1$. Now the pair $(Y_{\mathfrak{v},v}, F_{\mathfrak{w},w})$ is C-cobounded in $F_{\mathfrak{v}\mathfrak{w}}$ and $d_{\mathfrak{v}\mathfrak{w}}(\Sigma_{i\pm1}(\mathfrak{w}), Y_{\mathfrak{v},v}) \leq K$ together imply $d_{\mathfrak{v}\mathfrak{w}}(\Sigma_{i+1}(\mathfrak{v}), \Sigma_{i-1}(\mathfrak{v})) \leq 4K + C$. Then $d^f(\Sigma_{i+1}(\mathfrak{v}), \Sigma_{i-1}(\mathfrak{v})) \leq \phi(4K+C)$ and so $d^f(\Sigma_{i-1}(\mathfrak{v}), z_{\mathfrak{v},v}) \leq \phi(4K+C) + \delta_0$ (as $z_{\mathfrak{v},v}$ is δ_0-center geodesic triangle with vertices $\{\Sigma_i(\mathfrak{v}) : i = 1,2,3\}$ in the fiber $F_{\mathfrak{v},v}$). Therefore, by triangle inequality, $d_{\mathfrak{v}\mathfrak{w}}(\Sigma_{i-1}(\mathfrak{w}), z_{\mathfrak{v},v}) \leq K + \phi(4K+C) + \delta_0 =: D_1$.

Let $\bar{x}^i_{\mathfrak{v},v} \in \mathscr{L}^i_{\mathfrak{v},v}$ be the point closest to $x^i_{\mathfrak{v},v}$ in the induced metric of $\mathscr{L}^i_{\mathfrak{v},v}$ such that $\bar{x}^i_{\mathfrak{v},v}$ is D_1 close (in $d_{\mathfrak{v}\mathfrak{w}}$-metric) to a point $\tilde{x}^i_{\mathfrak{v},v} \in Q_{\mathfrak{w},w}$. Let γ be a K-qi section through $\tilde{x}^i_{\mathfrak{w},w}$ lying inside $\mathscr{U}_K \cap X_w$ over B_w. Then we have tripod of ladder inside X_w (as described in the initial step of the induction) formed by qi sections $\Sigma_{i\pm 1} \cap X_w$ and γ.

Subcase (1*C*): Suppose $\Sigma_i \cap F_{\mathfrak{w},w} \neq \emptyset$ and $\Sigma_{i\pm 1} \cap F_{\mathfrak{w},w} = \emptyset$. We consider the point $\bar{y}^i_{\mathfrak{v},v} \in \mathscr{L}^i_{\mathfrak{v},v}$ closest to $z_{\mathfrak{v},v}$ (in the fiber metric) which is K-close (in the metric of $F_{\mathfrak{v}\mathfrak{w}}$) to a point in $Q_{\mathfrak{w},w}$. (Existence of such points are clear as $\Sigma_i \cap F_{\mathfrak{w},w} \neq \emptyset$.) We take $\tilde{y}^i_{\mathfrak{w},w} \in Q_{\mathfrak{w},w}$ a nearest point projection of $\bar{y}^i_{\mathfrak{w},w}$ on $Q_{\mathfrak{w},w}$ in $d_{\mathfrak{v}\mathfrak{w}}$-metric. Here we have a degenerate tripod of ladder (i.e., special ladder) inside X_w formed by the qi section $\Sigma_i \cap X_w$ and a K-qi section γ through $\tilde{y}^i_{\mathfrak{w},w}$ lying inside $\mathscr{U}_K \cap X_w$ over B_w. We set this as part of the ladder $\mathscr{L}^i$ with $top(\mathscr{L}^i_{a,w}) = \gamma(a)$ and $bot(\mathscr{L}^i_{a,w}) = \Sigma_i(a)$, $a \in B_w$. Also, we set $\mathscr{L}^{ii+1} \cap X_w = \mathscr{L}^i \cap X_w$ with same orientation as $\mathscr{L}^i$ has and the other $\mathscr{L}^{ij}$'s are empty over $B_{T_{vw}}$.

Note (1*C*): Since the pair $(Y_{\mathfrak{v},v}, F_{\mathfrak{w},w})$ is C-cobounded in $d_{\mathfrak{v}\mathfrak{w}}$-metric, so by the construction of $\tilde{y}^i_{\mathfrak{w},w}$, $d_{\mathfrak{v}\mathfrak{w}}(P_{\mathfrak{w}}(z_{\mathfrak{v},v}), \tilde{y}^i_{\mathfrak{w},w}) \leq C$ and $d_{\mathfrak{v}\mathfrak{w}}(P_{\mathfrak{w}}(x^{i\pm 1}_{\mathfrak{v},v}), \tilde{y}^i_{\mathfrak{w},w}) \leq C$.

Subcase (1*D*): Suppose $\Sigma_i \cap X_w = \emptyset, \forall\, i = 1,2,3$. Then $\mathscr{L}^i, \mathscr{L}^{ij}$ are empty over $B_{T_{vw}}$ for all distinct $i, j \in \{1,2,3\}$.

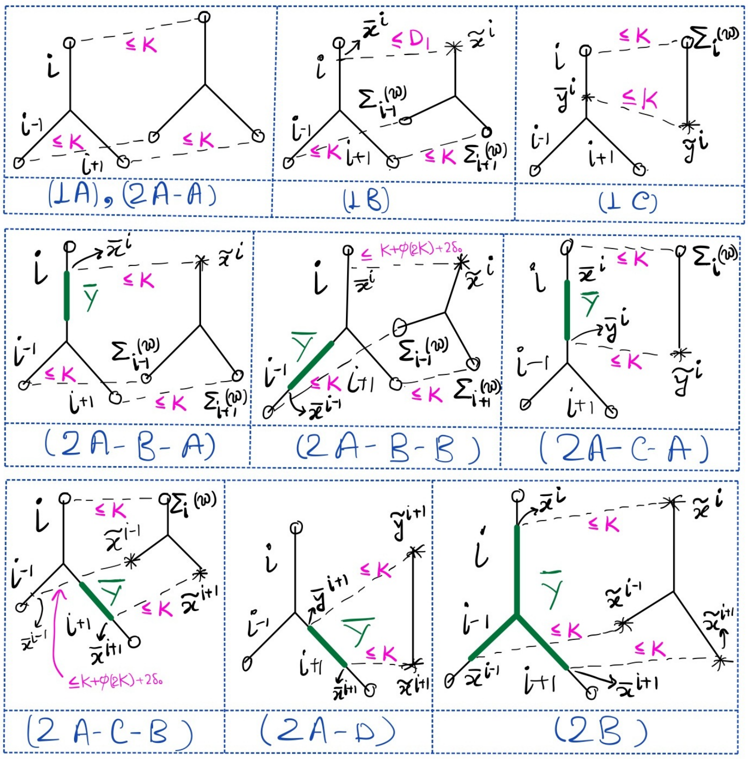


Figure 5.3: For ease of notation, we only use i, $\bar{x}^i$ to denote $\mathscr{L}^i_{\mathfrak{v},v}$, $\bar{x}^i_{\mathfrak{v},v}$ respectively in the figure and so on. We also omit some not to make it clumsy.

Case 2: Suppose $Y_{\mathfrak{v},v}$ has three legs and the pair $(Y_{\mathfrak{v},v}, F_{\mathfrak{w},w})$ is not C-cobounded in $F_{\mathfrak{v}\mathfrak{w}}$. We consider the following two subcases depending on $\bar{Y}_{\mathfrak{v},v} \subseteq \mathscr{L}^j_{\mathfrak{v},v}$ for some $j \in \{1,2,3\}$ or $\bar{Y}_{\mathfrak{v},v} \nsubseteq \mathscr{L}^j_{\mathfrak{v},v}$ for any $j \in \{1,2,3\}$.

Subcase $(2A)$: Suppose $\bar{Y}_{\mathfrak{v},v} \subseteq \mathscr{L}^j_{\mathfrak{v},v}$ for some $j \in \{1,2,3\}$. In this subcase, we consider further division $(2A\text{-}A)$, $(2A\text{-}B)$, $(2A\text{-}C)$, $(2A\text{-}D)$ as follows.

$(2A\text{-}A)$: Suppose $\Sigma_i \cap X_w \neq \emptyset, \forall\, i = 1,2,3$. Then we go back to Subcase $(1A)$.

$(2A\text{-}B)$: Suppose $\Sigma_{i\pm1} \cap X_w \neq \emptyset$ and $\Sigma_i \cap X_w = \emptyset$. Here in Subcase $(2A)$, depending on j, we have further following division $(2A\text{-}B\text{-}A)$ and $(2A\text{-}B\text{-}B)$.

$(2A\text{-}B\text{-}A)$: Let $\bar{Y}_{\mathfrak{v},v} \subseteq \mathscr{L}^i_{\mathfrak{v},v}$, i.e., $j = i$. Then by Fact (3), $d_{\mathfrak{v}\mathfrak{w}}(\bar{x}^i_{\mathfrak{v},v}, \bar{x}^i_{\mathfrak{w},w}) \leq K$. Here we will have a tripod of ladder inside X_w formed by K-qi sections $\Sigma_{i\pm1} \cap X_w$ and a K-qi section γ lying inside $\mathscr{U}_K \cap X_w$ through $\bar{x}^i_{\mathfrak{w},w}$ over B_w. To set $\mathscr{L}^i$ and $\mathscr{L}^{ij}$ over B_w, we go back to the initial step of induction with qi sections $\Sigma_{i\pm1} \cap X_w$ and γ.

Note $(2A\text{-}B\text{-}A)$: By Fact (1), $d_{\mathfrak{v}\mathfrak{w}}(P_{\mathfrak{w}}(x^i_{\mathfrak{v},v}), \bar{x}^i_{\mathfrak{w},w}) \leq D$.

$(2A\text{-}B\text{-}B)$: Suppose $\bar{Y}_{\mathfrak{v},v} \subseteq \mathscr{L}^j_{\mathfrak{v},v},\ j \in \{i \pm 1\}$ (i.e. $j \neq i$). By second part of Fact (4), $d_{\mathfrak{v}\mathfrak{w}}(z_{\mathfrak{v},v}, Q_{\mathfrak{w},w}) \leq K + \phi(2K) + 2\delta_0$. Take a point $\bar{x}^i_{\mathfrak{v},v} \in \mathscr{L}^i_{\mathfrak{v},v}$ closest to $x^i_{\mathfrak{v},v}$ (in the fiber metric) such that $\bar{x}^i_{\mathfrak{v},v}$ is $(K + \phi(2K) + 2\delta_0)$-close (in $d_{\mathfrak{v}\mathfrak{w}}$-metric) to a point $\bar{x}^i_{\mathfrak{w},w} \in Q_{\mathfrak{w},w}$. Let γ be a K-qi section through $\bar{x}^i_{\mathfrak{w},w}$ lying inside $\mathscr{U}_K \cap X_w$ over B_w. Then we have a tripod of ladder inside X_w (as discussed in the initial step of induction) formed by qi sections $\Sigma_{i\pm1} \cap X_w$ and γ.

$(2A\text{-}C)$: Suppose $\Sigma_i \cap X_w \neq \emptyset$ and $\Sigma_{i\pm1} \cap X_w = \emptyset$. Here in Subcase $(2A)$, depending on j, we have further following division $(2A\text{-}C\text{-}A)$ and $(2A\text{-}C\text{-}B)$.

$(2A\text{-}C\text{-}A)$: $\bar{Y}_{\mathfrak{v},v} \subseteq \mathscr{L}^i_{\mathfrak{v},v}$, i.e., $j = i$. We set $\bar{x}^i_{\mathfrak{w},w} = \Sigma_i(\mathfrak{w})$. Now by Fact (4), $d_{\mathfrak{v}\mathfrak{w}}(\bar{y}^i_{\mathfrak{v},v}, \bar{y}^i_{\mathfrak{w},w}) \leq K$. Here we will have a degenerate tripod of ladder inside X_w formed by the qi section $\Sigma_i \cap X_w$ and a K-qi section γ through $\bar{y}^i_{\mathfrak{w},w}$ lying inside $\mathscr{U}_K \cap X_w$ over B_w. We set this as part of the ladder $\mathscr{L}^i$ with $top(\mathscr{L}^i_{a,w}) = \gamma(a)$, $bot(\mathscr{L}^i_{a,w}) = \Sigma_i(a),\ a \in B_w$. Also, we set $\mathscr{L}^{ii\pm1} \cap X_w = \mathscr{L}^i \cap X_w$ with the same orientation as $\mathscr{L}^i \cap X_w$ has and $\mathscr{L}^{i+1i-1} \cap X_w$ is empty over $B_{T_{vw}}$.

Note $(2A\text{-}C\text{-}A)$: By Fact (1), $d_{\mathfrak{v}\mathfrak{w}}(P_{\mathfrak{w}}(x^{i\pm1}_{\mathfrak{v},v}), \bar{y}^i_{\mathfrak{w},w}) \leq D$ and $d_{\mathfrak{v}\mathfrak{w}}(P_{\mathfrak{w}}(z_{\mathfrak{v},v}), \bar{y}^i_{\mathfrak{w},w}) \leq D$. Also, $d_{\mathfrak{v}\mathfrak{w}}(P_{\mathfrak{w}}(x^i_{\mathfrak{v},v}), \bar{x}^i_{\mathfrak{w},w}) \leq 2K$.

$(2A\text{-}C\text{-}B)$: Let $\bar{Y}_{\mathfrak{v},v} \subseteq \mathscr{L}^j_{\mathfrak{v},v},\ j \in \{i \pm 1\}$, i.e., $j \neq i$. By second part of Fact (4), $d_{\mathfrak{v}\mathfrak{w}}(z_{\mathfrak{v},v}, Q_{\mathfrak{w},w}) \leq K + \phi(2K) + 2\delta_0$. We maintain the order of $+$ and $-$ depending on $j = i \pm 1$. Take a point $\bar{x}^{i\mp1}_{\mathfrak{v},v} \in \mathscr{L}^{i\mp1}_{\mathfrak{v},v}$ closest to $x^{i\mp1}_{\mathfrak{v},v}$ (in the fiber metric) such that $\bar{x}^{i\mp1}_{\mathfrak{v},v}$ is $(K + \phi(2K) + 2\delta_0)$-close (in $d_{\mathfrak{v}\mathfrak{w}}$-metric) to a point $\bar{x}^{i\mp1}_{\mathfrak{v},v} \in Q_{\mathfrak{w},w}$. Let γ and γ' be K-qi sections through $\bar{x}^{i-1}_{\mathfrak{w},w}$ and $\bar{x}^{i+1}_{\mathfrak{w},w}$ lying inside $\mathscr{U}_k \cap X_w$ over B_w. Then we will have a tripod of ladder inside X_w (as described in the initial step of induction) formed by the qi sections $\Sigma_i \cap X_w$, γ and γ'.

$(2A\text{-}D)$: Suppose $\Sigma_i \cap X_w = \emptyset, \forall\, i \in \{1,2,3\}$. Note that $\bar{Y}_{\mathfrak{v},v} \subseteq \mathscr{L}^j_{\mathfrak{v},v}$ (we are in Subcase $(2A)$). By Fact (3) and Fact (4), $d_{\mathfrak{v}\mathfrak{w}}(\bar{x}^j_{\mathfrak{v},v}, \tilde{x}^j_{\mathfrak{w},w}) \leq K$ and $d_{\mathfrak{v}\mathfrak{w}}(\bar{y}^j_{\mathfrak{v},v}, \tilde{y}^j_{\mathfrak{w},w}) \leq K$. Here we have a degenerate tripod of ladder inside X_w formed by K-qi sections γ and γ' through $\tilde{x}^j_{\mathfrak{w},w}$ and $\tilde{y}^j_{\mathfrak{w},w}$ lying inside $\mathscr{U}_K \cap X_w$ over B_w respectively. We consider this as a part of $\mathscr{L}^j$ with $top(\mathscr{L}^j_{a,w}) = \gamma'(a), bot(\mathscr{L}^j_{a,w}) = \gamma(a),\ a \in B_w$. Also, we set $\mathscr{L}^{jj\pm1} \cap X_w = \mathscr{L}^j \cap X_w$ with the same orientation as $\mathscr{L}^j \cap X_w$ has and $\mathscr{L}^{j+1 j-1} \cap X_w$ is empty over $B_{T_{vw}}$.

Note $(2A\text{-}D)$: By Fact (1), $d_{\mathfrak{v}\mathfrak{w}}(P_{\mathfrak{w}}(x^{j\pm1}_{\mathfrak{v},v}), \tilde{y}^j_{\mathfrak{w},w}) \leq D$, $d_{\mathfrak{v}\mathfrak{w}}(P_{\mathfrak{w}}(x^j_{\mathfrak{v},v}), \tilde{x}^j_{\mathfrak{w},w}) \leq D$ and $d_{\mathfrak{v}\mathfrak{w}}(z_{\mathfrak{v},v}, \tilde{y}^j_{\mathfrak{w},w}) \leq D$.

Subcase $(2B)$: Suppose $\bar{Y}_{\mathfrak{v},v} \nsubseteq \mathscr{L}^j_{\mathfrak{v},v}$ for any $j \in \{1,2,3\}$, i.e., $z_{\mathfrak{v},v}$ belong to the interior of $\bar{Y}_{\mathfrak{v},v}$ in the induced metric of $Y_{\mathfrak{v},v}$. We set $\tilde{x}^i_{\mathfrak{w},w}$ to be $\Sigma_i \cap F_{\mathfrak{w},w}$ provided $\Sigma_i \cap X_w \neq \emptyset$. We consider K-qi section through $\tilde{x}^i_{\mathfrak{w},w}$ lying inside $\mathscr{U}_K \cap X_w$ over B_w if $\Sigma_i \cap X_w = \emptyset$; otherwise, the qi section to be $\Sigma_i \cap X_w$. Then we have a tripod of ladder inside X_w as described in the initial step of the induction formed by these K-qi sections through $\tilde{x}^i_{\mathfrak{w},w}$.

Note $(2B)$: By Fact (2), $d_{\mathfrak{v}\mathfrak{w}}(P_{\mathfrak{w}}(\bar{x}^i_{\mathfrak{v},v}), \tilde{x}^i_{\mathfrak{w},w}) \leq max\{2K, D\}$ for $i = 1,2,3$.

Case 3: Suppose $Y_{\mathfrak{v},v}$ has only one leg, i.e., $Y_{\mathfrak{v},v} = \mathscr{L}^i_{\mathfrak{v},v}$ for some $i \in \{1,2,3\}$ and the pair $(Y_{\mathfrak{v},v}, F_{\mathfrak{w},w})$ is C-cobounded. So $\mathscr{L}^i \cap X_v$ is a special ladder bounded by two K-qi sections γ_1 and γ_2 lying inside $\mathscr{U}_K \cap X_v$ over B_v. Then by our construction in Case 1 and Case 2, either γ_1 (say) is restriction of some Σ_i or both γ_1 and γ_2 are not restriction of Σ_i's. In the later case, $\mathscr{L}^i$ is empty over $B_{T_{vw}}$. Now we assume that $\gamma_1 = \Sigma_i \cap X_v$. We consider the following subcases depending on whether $\Sigma_i \cap X_w$ is empty or non-empty.

Subcase $(3A)$: Suppose $\Sigma_i \cap X_w \neq \emptyset$. Note that $top(\mathscr{L}^i_{\mathfrak{v},v}) = y^i_{\mathfrak{v},v}$ and $bot(\mathscr{L}^i_{\mathfrak{v},v}) = x^i_{\mathfrak{v},v} = \Sigma_i \cap F_{\mathfrak{v},v}$. We set $\tilde{x}^i_{\mathfrak{w},w} = \Sigma_i(\mathfrak{w})$. Then take $\bar{y}^i_{\mathfrak{v},v} \in \mathscr{L}^i_{\mathfrak{v},v}$ is the closest to $y^i_{\mathfrak{v},v}$ (in the fiber metric) such that $\bar{y}^i_{\mathfrak{v},v}$ is K-close (in $d_{\mathfrak{v}\mathfrak{w}}$-metric) to a point $\tilde{y}^i_{\mathfrak{w},w} \in Q_{\mathfrak{w},w}$. (Existence of such points are clear as $\Sigma_i \cap F_{\mathfrak{w},w} \neq \emptyset$.) Let γ be a K-qi section through $\tilde{y}^i_{\mathfrak{w},w}$ lying inside $\mathscr{U}_K \cap X_w$ over B_w. Then we have a degenerate tripod of ladder inside X_w bounded by qi sections $\Sigma_i \cap X_w$ and γ. We set this as part of $\mathscr{L}^i$ with $top(\mathscr{L}^i_{a,w}) = \gamma(a)$ and $bot(\mathscr{L}^i_{a,w}) = \Sigma_i(a)$, $a \in B_w$. Also, note that this is the part of the same $\mathscr{L}^{ij}$ as it was over B_v with orientation same as $\mathscr{L}^i \cap X_w$.

Note $(3A)$: Since the pair $(Y_{\mathfrak{v},v}, F_{\mathfrak{w},w})$ is C cobounded in the metric of $F_{\mathfrak{v}\mathfrak{w}}$, so $d_{\mathfrak{v}\mathfrak{w}}(P_{\mathfrak{w}}(x^i_{\mathfrak{v},v}), \tilde{x}^i_{\mathfrak{w},w}) \leq 2K$ and $d_{\mathfrak{v}\mathfrak{w}}(P_{\mathfrak{w}}(y^i_{\mathfrak{v},v}), \tilde{y}^i_{\mathfrak{w},w}) \leq 2K + C$.

Subcase $(3B)$: Suppose $\Sigma_i \cap X_w = \emptyset$. Then $\mathscr{L}^i$ is empty over $B_{T_{vw}}$.

Case 4: Suppose $Y_{\mathfrak{v},v}$ has only one leg, i.e., $Y_{\mathfrak{v},v} = \mathscr{L}^i_{\mathfrak{v},v}$ for some $i \in \{1,2,3\}$ and the pair $(Y_{\mathfrak{v},v}, F_{\mathfrak{w},w})$ is not C-cobounded in $F_{\mathfrak{v}\mathfrak{w}}$. Then we have two extreme points $\bar{x}_{\mathfrak{v},v}$ and $\bar{y}_{\mathfrak{v},v}$ of $\bar{Y}_{\mathfrak{v},v} \subseteq \mathscr{L}^i_{\mathfrak{v},v}$, which are K-close to points $\tilde{x}^i_{\mathfrak{w},w}$ and $\tilde{y}^i_{\mathfrak{w},w}$ of $Q_{\mathfrak{w},w}$

respectively in $d_{\mathfrak{v}\mathfrak{w}}$-metric (see Fact (5)). If $\Sigma_i \cap X_w \neq \emptyset$, then we set $\tilde{x}^i_{\mathfrak{w},w} = \Sigma_i(\mathfrak{w})$. Suppose γ is a K-qi section through $\tilde{x}^i_{\mathfrak{w},w}$ lying inside $\mathscr{U}_K \cap X_w$ if $\Sigma_i \cap X_w = \emptyset$; otherwise, $\gamma = \Sigma_i \cap X_w$. Let γ' be K-qi sections through $\tilde{y}^i_{\mathfrak{w},w}$ lying inside $\mathscr{U}_K \cap X_w$ over B_w. Then we have a degenerate tripod of ladder inside X_w bounded by γ and γ' with orientation is same as described in Subcase $(3A)$. Further, this is the part of the same $\mathscr{L}^{ij}$ as it was over B_v with the orientation same as $\mathscr{L}^i \cap X_w$.

Note for Case 4: $d_{\mathfrak{v}\mathfrak{w}}(P_{\mathfrak{w}}(x^i_{\mathfrak{v},v}), \tilde{x}^i_{\mathfrak{w},w}) \leq \max\{2K+D\}$ and $d_{\mathfrak{v}\mathfrak{w}}(P_{\mathfrak{w}}(y^i_{\mathfrak{v},v}), \tilde{y}^i_{\mathfrak{w},w}) \leq D$ (see Fact (5)).

Lemma 5.3.2. *Suppose $\bar{Y}_{\mathfrak{v},v} \subseteq \mathscr{L}^i_{\mathfrak{v},v}$ for some $i \in \{1,2,3\}$. Then there is a constant $C_{5.3.2}$ satisfying the following.*

(1) ([9, Corollary 5.4]) *The pair $(\mathscr{L}^t_{\mathfrak{v},v}, F_{\mathfrak{w},w})$ is $C_{5.3.2}$-cobounded in the path metric of $F_{\mathfrak{v}\mathfrak{w}}$ for $t = i \pm 1$.*

(2) *The pair $(\mathscr{L}^{i+1 i-1}_{\mathfrak{v},v}, F_{\mathfrak{w},w})$ is $C_{5.3.2}$ cobounded in the path metric of $F_{\mathfrak{v}\mathfrak{w}}$.*

Proof. (1) Fix $t \in \{i \pm 1\}$. Then by Fact (1), $d_{\mathfrak{v}\mathfrak{w}}(P_{\mathfrak{w}}(x^t_{\mathfrak{v},v}), P_{\mathfrak{w}}(z_{\mathfrak{v},v})) \leq 2D$ (by triangle inequality). Again $d_{\mathfrak{v},\mathfrak{w}}(P_{\mathfrak{w}}(x^{i-1}_{\mathfrak{v},v}), P_{\mathfrak{w}}(x^{i+1}_{\mathfrak{v},v})) \leq 2D$. Then by Lemma 2.2.20, we can take $C_{5.3.2} := C_{2.2.20}(\delta'_0, L'_0, \lambda'_0, 2D)$. □

Lemma 5.3.3. *There is a constant $C_{5.3.3}$ satisfying the following.*

If the pair $(Y_{\mathfrak{v},v}, F_{\mathfrak{w},w})$ is C-cobounded in $F_{\mathfrak{v}\mathfrak{w}}$, then the pairs $(\mathscr{L}^{ij}_{\mathfrak{v},v}, F_{\mathfrak{w},w})$ and $(\mathscr{L}^i_{\mathfrak{v},v}, F_{\mathfrak{w},w})$ are $C_{5.3.3}$-cobounded in $F_{\mathfrak{v}\mathfrak{w}}$ for all distinct $i, j \in \{1,2,3\}$.

Proof. By Lemma 2.2.20, we can take $C_{5.3.3} = C_{2.2.20}(\delta'_0, L'_0, \lambda'_0, C)$. □

The above construction yields $\mathscr{L}^i$ and $\mathscr{L}^{ij}$ for all distinct $i, j \in \{1,2,3\}$ as a collection of geodesic segments in fibers. We still need to show that they form ladders (Lemma 5.3.6 and Lemma 5.3.7) using Lemma 5.1.21. Let $B_i := \pi_X(\mathscr{L}^i)$ and $S_i := \pi_B(B_i)$ for all $i \in \{1,2,3\}$. Let $v \in S_{123}$. Recall for $b \in B_v$, $z_{b,v}$ is δ_0-center of geodesic triangle in the fiber $F_{b,v}$ with vertices $\{bot(\mathscr{L}^i_{b,v}) : i = 1,2,3\}$. As we saw in the initial step of induction that $\{z_{b,v} : b \in B_v\}$ form $k_{2.4.15}(K)$-qi section in X_v over B_v. Therefore, to show $\Xi = \{z_{b,v} : v \in S_{123}, b \in B_v\}$ form a uniform qi section over B_{123} (Corollary 5.3.5), we only need to prove $d_{\mathfrak{v}\mathfrak{w}}(z_{\mathfrak{v},v}, z_{\mathfrak{w},w})$ is uniformly bounded, where $v, w \in S_{123}$, $d_T(v,w) = 1$ and $[\mathfrak{v}, \mathfrak{w}]$ is the edge joining $\mathfrak{v} \in B_v$ and $\mathfrak{w} \in B_w$.

Lemma 5.3.4. ([9, Lemma 5.8]) *There exists $k_{5.3.4}$ such that $d_{\mathfrak{v}\mathfrak{w}}(z_{\mathfrak{v},v}, z_{\mathfrak{w},w}) \leq k_{5.3.4}$, where $v, w \in S_{123}$ such that $d_T(v,w) = 1$ and $[\mathfrak{v}, \mathfrak{w}]$ is the edge joining $\mathfrak{v} \in B_v$ and $\mathfrak{w} \in B_w$.*

Proof. This situation happens in Subcase $(1A)$, Subcase $(1B)$, $(2A\text{-}A)$, $(2A\text{-}B\text{-}A)$, $(2A\text{-}B\text{-}B)$, $(2A\text{-}C\text{-}B)$ and Subcase $(2B)$. We denote $\triangle_v$ and $\hat{\triangle}_v$ for geodesic triangle with vertices $\bar{x}^i_{\mathfrak{v},v}, i = 1,2,3$ in $F_{\mathfrak{v},v}$ and $F_{\mathfrak{v}\mathfrak{w}}$ respectively. We also denote $\triangle_w$ and $\hat{\triangle}_w$ for geodesic triangle with vertices $\bar{x}^i_{\mathfrak{w},w}, i = 1,2,3$ in $F_{\mathfrak{w},w}$ and $F_{\mathfrak{v}\mathfrak{w}}$ respectively. In all these cases, $d_{\mathfrak{v}\mathfrak{w}}(\bar{x}^i_{\mathfrak{v},v}, \bar{x}^i_{\mathfrak{w},w}) \leq max\{D_1, K + \phi(2K) + 2\delta_0, K\} = D_2$ (say), $i \in \{1,2,3\}$. Since $z_{\mathfrak{v},v}$ is δ_0-center of $\triangle(x^1_{\mathfrak{v},v}, x^2_{\mathfrak{v},v}, x^3_{\mathfrak{v},v})$ in $F_{\mathfrak{v},v}$, so $z_{\mathfrak{v},v}$ is $3\delta_0$-center of $\triangle_v$ in $F_{\mathfrak{v},v}$. Thus $z_{\mathfrak{v},v}$ is $(3\delta_0 + D_{2.2.2}(\delta_0, L'_0, L'_0))$-center $\hat{\triangle}_v$ in $F_{\mathfrak{v}\mathfrak{w}}$ (as the inclusion $F_{\mathfrak{v},v} \hookrightarrow F_{\mathfrak{v}\mathfrak{w}}$ is L'_0-qi embedding, see Lemma 2.3.4). Since the corresponding end points of the triangles $\hat{\triangle}_v$ and $\hat{\triangle}_w$ are D_2-distance apart from each other in the path metric of $F_{\mathfrak{v}\mathfrak{w}}$, then by slimness of quadrilateral in $F_{\mathfrak{v}\mathfrak{w}}$, we get, $z_{\mathfrak{v},v}$ is $(3\delta_0 + D_{2.2.2}(\delta_0, L'_0, L'_0) + 2\delta'_0 + D_2)$-center of $\hat{\triangle}_w$. Again, as $z_{\mathfrak{w},w}$ is δ_0-center of $\triangle_w$ and so is $(\delta_0 + D_{2.2.2}(\delta_0, L'_0, L'_0))$-center of $\hat{\triangle}_w$. Thus $z_{\mathfrak{v},v}$ and $z_{\mathfrak{w},w}$ are two D_3-center of $\hat{\triangle}_w$ in the path metric of $F_{\mathfrak{v}\mathfrak{w}}$, where $D_3 = 3\delta_0 + D_{2.2.2}(\delta_0, L'_0, L'_0) + 2\delta'_0 + D_2$. Hence, by [9, Lemma 1.76], we have, $d_{\mathfrak{v}\mathfrak{w}}(z_{\mathfrak{v},v}, z_{\mathfrak{w},w}) \leq 2D_3 + 9\delta'_0 =: k_{5.3.4}$. □

Corollary 5.3.5. *We have a constant* $k_{5.3.5} = max\{k_{2.4.15}(K), k_{5.3.4}\}$ *such that* $\Xi = \bigcup_{b \in B_v,\ v \in S_{123}} \{z_{b,v}\}$ *forms a* $k_{5.3.5}$*-qi section over* B_{123} *with* $\Xi \subseteq N^f_{6\delta_0}(\mathscr{U}_K)$.

Here we show that $\mathscr{L}^i$ and $\mathscr{L}^{ij}$ form ladders.

Lemma 5.3.6. *There are constants* $k_{5.3.6}, c_{5.3.6}$ *and* $\varepsilon_{5.3.6}$ *such that* $\mathscr{L}^i$ *is a ladder with a central base* $\mathfrak{B}$ *containing* Σ_i *with constants* $k_{5.3.6}, c_{5.3.6}$ *and* $\varepsilon_{5.3.6}$, $i = 1,2,3$.

Proof. We check all conditions of Lemma 5.1.21.

Condition (1): Note that for all $v \in S_i$, $\mathscr{L}^i \cap X_v$ is a special $C_{2.4.12}(k_{2.4.15}(K))$-ladder over B_v. Let $v, w \in S_i$ such that $d_T(v,w) = 1$ and $[\mathfrak{v}, \mathfrak{w}]$ is the edge joining $\mathfrak{v} \in B_v$ and $\mathfrak{w} \in B_w$. If $v, w \in \mathfrak{T}$, then $d_{\mathfrak{v}\mathfrak{w}}(top(\mathscr{L}^i_{\mathfrak{v},v}), top(\mathscr{L}^i_{\mathfrak{w},w})) = d_{\mathfrak{v}\mathfrak{w}}(z_{\mathfrak{v},v}, z_{\mathfrak{w},w}) \leq k_{5.3.4}$ and $d_{\mathfrak{v}\mathfrak{w}}(bot(\mathscr{L}^i_{\mathfrak{v},v}), bot(\mathscr{L}^i_{\mathfrak{w},w})) = d_{\mathfrak{v}\mathfrak{w}}(\Sigma_i(\mathfrak{v}), \Sigma_i(\mathfrak{w})) \leq K$. Otherwise, suppose $d_T(v, \mathfrak{T}) < d_T(w, \mathfrak{T})$. Now let $v, w \in S_{123}$, i.e. both $Y_{\mathfrak{v},v}$ and $Y_{\mathfrak{w},w}$ have three legs (this happens in $(1A)$, $(1B)$, $(2A\text{-}A)$, $(2A\text{-}B\text{-}A)$, $(2A\text{-}B\text{-}B)$, $(2A\text{-}C\text{-}B)$ and $(2B)$). Note that $\bar{x}^i_{\mathfrak{w},w} = bot(\mathscr{L}^i_{\mathfrak{w},w}), z_{\mathfrak{w},w} = top(\mathscr{L}^i_{\mathfrak{w},w})$. Then $\bar{x}^i_{\mathfrak{w},w}$ and $z_{\mathfrak{w},w}$ are $\max\{k_{5.3.4}, D_2\} = k_{5.3.4}$-close to $\mathscr{L}^i_{\mathfrak{v},v}$, where D_2 is as in Lemma 5.3.4. Now let $v \in S_{123}$ and $w \notin S_{123}$, i.e. $Y_{\mathfrak{v},v}$ has three legs but $Y_{\mathfrak{w},w}$ has one (this happens in $(1C)$, $(2A\text{-}C\text{-}A)$, $(2A\text{-}D)$). Then (by our construction) $top(\mathscr{L}^i_{\mathfrak{w},w}), bot(\mathscr{L}^i_{\mathfrak{w},w})$ are K-close to $\mathscr{L}^i_{\mathfrak{v},v}$ in $d_{\mathfrak{v}\mathfrak{w}}$-metric. Finally, $v, w \notin S_{123}$, i.e., both $Y_{\mathfrak{v},v} = \mathscr{L}^i_{\mathfrak{v},v}$ and $Y_{\mathfrak{w},w} = \mathscr{L}^i_{\mathfrak{w},w}$ have one leg, then $top(\mathscr{L}^i_{\mathfrak{w},w}), bot(\mathscr{L}^i_{\mathfrak{w},w})$ are K-close to $\mathscr{L}^i_{\mathfrak{v},v}$. Therefore, for the condition (1) of Lemma 5.1.21, we can take $K' = max\{C_{2.4.12}(K_{2.4.15}(K)), k_{5.3.4}, K\}$.

Condition (2): Let $v, w \in S_i$ such that $d_T(v,w) = 1$ and $d_T(v, \mathfrak{T}) < d_T(w, \mathfrak{T})$. To show a uniform bound on $Hd_{\mathfrak{v}\mathfrak{w}}(P_{\mathfrak{w}}(\mathscr{L}^i_{\mathfrak{v},v}), \mathscr{L}^i_{\mathfrak{w},w})$, we first show that

$$d_{\mathfrak{v}\mathfrak{w}}(P_{\mathfrak{w}}(bot(\mathscr{L}^i_{\mathfrak{v},v})), bot(\mathscr{L}^i_{\mathfrak{w},w})) \text{ and } d_{\mathfrak{v}\mathfrak{w}}(P_{\mathfrak{w}}(top(\mathscr{L}^i_{\mathfrak{v},v})), top(\mathscr{L}^i_{\mathfrak{w},w}))$$

are uniformly bounded, and then we apply [9, Corollary 1.116].

Suppose both $Y_{\mathfrak{v},v}$ and $Y_{\mathfrak{w},w}$ have three legs (this happens in $(1A)$, $(1B)$, $(2A\text{-}A)$, $(2A\text{-}B\text{-}A)$, $(2A\text{-}B\text{-}B)$, $(2A\text{-}C\text{-}B)$ and $(2B)$). In all these cases, $d_{\mathfrak{vw}}(P_{\mathfrak{w}}(z_{\mathfrak{v},v}), z_{\mathfrak{w},w}) \leq 2k_{5.3.4}$. So we need bound only for $d_{\mathfrak{vw}}(P_{\mathfrak{w}}(x^i_{\mathfrak{v},v}), \tilde{x}^i_{\mathfrak{w},w})$. In $(1A)$ and $(2A\text{-}A)$, $d_{\mathfrak{vw}}(P_{\mathfrak{w}}(x^i_{\mathfrak{v},v}), \tilde{x}^i_{\mathfrak{w},w}) \leq 2K$. In $(1B)$, $d_{\mathfrak{vw}}(P_{\mathfrak{w}}(x^i_{\mathfrak{v},v}), \tilde{x}^i_{\mathfrak{w},w}) \leq max\{2K, 2D_1 + C\}$. In $(2A\text{-}B\text{-}A)$ and $(2B)$, $d_{\mathfrak{vw}}(P_{\mathfrak{w}}(x^i_{\mathfrak{v},v}), \tilde{x}^i_{\mathfrak{w},w}) \leq max\{2K, D\}$. In $(2A\text{-}B\text{-}B)$ and $(2A\text{-}C\text{-}B)$, $d_{\mathfrak{vw}}(P_{\mathfrak{w}}(x^i_{\mathfrak{v},v}), \tilde{x}^i_{\mathfrak{w},w}) \leq max\{2K, 2(K + \phi(2K) + \delta_0)\}$.

Suppose $Y_{\mathfrak{v},v}$ has three legs and $Y_{\mathfrak{w},w}$ has one (this happens in $(1C)$, $(2A\text{-}C\text{-}A)$, $(2A\text{-}D)$). Then $d_{\mathfrak{vw}}(P_{\mathfrak{w}}(x^i_{\mathfrak{v},v}), \tilde{x}^i_{\mathfrak{w},w})$ and $d_{\mathfrak{vw}}(P_{\mathfrak{w}}(z_{\mathfrak{v},v}), \tilde{y}^i_{\mathfrak{w},w})$ are bounded by $\max\{2K + C, 2K, D\}$ (see the Note in the end of each case).

Finally, we assume that both $Y_{\mathfrak{v},v}, Y_{\mathfrak{w},w}$ have one leg (Case 3 and Case 4). Then $d_{\mathfrak{vw}}(P_{\mathfrak{w}}(x^i_{\mathfrak{v},v}), \tilde{x}^i_{\mathfrak{w},w})$ and $d_{\mathfrak{vw}}(P_{\mathfrak{w}}(y_{\mathfrak{v},v}), \tilde{y}^i_{\mathfrak{w},w})$ are bounded by $max\{2K, 2K + C, 2K + D\}$ (see Note for Case 3 and Note for Case 4).

Let D_4 be the maximum of all the above constants. Now by [9, Corollary 1.116], we have C_1 depending on δ'_0, λ'_0 and L'_0 such that

$$Hd_{\mathfrak{vw}}(P_{\mathfrak{w}}(\mathscr{L}^i_{\mathfrak{v},v}), [P_{\mathfrak{w}}(bot(\mathscr{L}^i_{\mathfrak{v},v})), P_{\mathfrak{w}}(top(\mathscr{L}^i_{\mathfrak{v},v}))]^f) \leq C_1.$$

Since $\mathscr{L}^i_{\mathfrak{w},w}$ are L'_0-quasi-geodesic in $F_{\mathfrak{vw}}$, by slimness of quadrilateral in $F_{\mathfrak{vw}}$, there is ε' depending on δ'_0, D_4, C_1 and L'_0 such that $Hd_{\mathfrak{vw}}(P_{\mathfrak{w}}(\mathscr{L}^i_{\mathfrak{v},v}), \mathscr{L}^i_{\mathfrak{w},w}) \leq \varepsilon'$

Condition (3): Suppose $v \in S_i$ and $w \notin S_i$ such that $d_T(v,w) = 1$ and $[\mathfrak{v},\mathfrak{w}]$ is the edge joining $\mathfrak{v} \in B_v$ and $\mathfrak{w} \in B_w$. That is $\mathscr{L}^i_{\mathfrak{w},w} = \emptyset$. Then the pair $(\mathscr{L}^i_{\mathfrak{v},v}, F_{\mathfrak{w},w})$ is $C' = max\{C_{5.3.2}, C_{5.3.3}\}$-cobounded in $F_{\mathfrak{vw}}$ (see Lemma 5.3.2, Lemma 5.3.3).

Therefore, to conclude the lemma, we take, $k_{5.3.6} = k_{5.1.21}(K')$, $c_{5.3.6} = c_{5.1.21}(C')$ and $\varepsilon_{5.3.6} = \varepsilon_{5.1.21}(\varepsilon')$ for the above K', C', ε'. □

Lemma 5.3.7. *There are constants $k_{5.3.7}, c_{5.3.7}$ and $\varepsilon_{5.3.7}$ such that $\mathscr{L}^{ij}$ is a ladder with a central base $\mathfrak{B}$ containing Σ_i and Σ_j with constants $k_{5.3.7}, c_{5.3.7}$ and $\varepsilon_{5.3.7}$ for all distinct $i, j \in \{1,2,3\}$.*

Proof. Here also we verify all conditions of Lemma 5.1.21.

Condition (1): Note that for all $v \in \pi(\mathscr{L}^{ij})$, $\mathscr{L}^{ij} \cap X_v$ is a special $C_{2.4.12}(K)$-ladder over B_v. Suppose $v, w \in \pi(\mathscr{L}^{ij})$ such that $d_T(v,w) = 1$ and $[\mathfrak{v},\mathfrak{w}]$ is the edge joining $\mathfrak{v} \in B_v$ and $\mathfrak{w} \in B_w$. If $v, w \in \mathfrak{T}$, $d_{\mathfrak{vw}}(\Sigma_i(\mathfrak{v}), \Sigma_i(\mathfrak{w})) \leq K$ for all $i \in \{1,2,3\}$. Otherwise, we assume that $d_T(v,\mathfrak{T}) < d_T(w,\mathfrak{T})$. In the construction, we have seen that $\mathscr{L}^{ij}_{\mathfrak{w},w}$ matches with $\mathscr{L}^i_{\mathfrak{w},w}$ or $\mathscr{L}^j_{\mathfrak{w},w}$ unless both $Y_{\mathfrak{v},v}$ and $Y_{\mathfrak{w},w}$ have three legs. Note that $Hd^f(\mathscr{L}^i_{\mathfrak{v},v} \cup \mathscr{L}^j_{\mathfrak{v},v}, \mathscr{L}^{ij}_{\mathfrak{v},v}) \leq 2\delta_0$. Also, $\tilde{x}^i_{\mathfrak{w},w}$ and $\tilde{y}^i_{\mathfrak{w},w}$ are D_4-close (in $d_{\mathfrak{vw}}$-metric) to $\mathscr{L}^i_{\mathfrak{v},v} \cup \mathscr{L}^j_{\mathfrak{w},w}$, where D_4 is as in Lemma 5.3.6. Hence, in all the cases, there are points, $x_1, y_1 \in \mathscr{L}^{ij}_{\mathfrak{v},v}$ such that $d_{\mathfrak{v}\mathfrak{w}}(\tilde{x}^i_{\mathfrak{w},w}, x_1) \leq D_4 + 3\delta_0$ and $d_{\mathfrak{v}\mathfrak{w}}(\tilde{x}^j_{\mathfrak{w},w}, x_1) \leq D_4 + 3\delta_0$ with an order $bot(\mathscr{L}^j_{\mathfrak{v},v}) = bot(\mathscr{L}^{ij}_{\mathfrak{v},v}) \leq y_1 \leq x_1 \leq top(\mathscr{L}^{ij}_{\mathfrak{v},v}) = bot(\mathscr{L}^i_{\mathfrak{v},v})$.

Therefore, for the condition (1) of Lemma 5.1.21, we can take $K' = max\{D_4 + 3\delta_0, C_{2.4.12}(K)\}$.

Condition (2): Let $v, w \in \pi(\mathscr{L}^{ij})$ such that $d_T(v,w) = 1$ and $d_T(v,\mathfrak{T}) < d_T(w,\mathfrak{T})$. We prove that $Hd_{\mathfrak{v}\mathfrak{w}}(P_{\mathfrak{w}}(\mathscr{L}^{ij}_{\mathfrak{v},v}), \mathscr{L}^{ij}_{\mathfrak{w},w})$ is uniformly bounded.

If both $Y_{\mathfrak{v},v}$ and $Y_{\mathfrak{w},w}$ have one leg, then $Hd_{\mathfrak{v}\mathfrak{w}}(P_{\mathfrak{w}}(\mathscr{L}^{ij}_{\mathfrak{v},v}), \mathscr{L}^{ij}_{\mathfrak{w},w}) \leq \varepsilon_{5.3.6}$ (by Lemma 5.3.6). If $Y_{\mathfrak{v},v}$ has three legs but $Y_{\mathfrak{w},w}$ has one, then $\mathscr{L}^s_{\mathfrak{w},w} = \emptyset$ for s is equal to either i or j. So by Lemma 5.3.6, $\text{diam}(P_{\mathfrak{w}}(\mathscr{L}^s_{\mathfrak{v},v})) \leq c_{5.3.6}$ in $d_{\mathfrak{v}\mathfrak{w}}$-metric. Again if both $Y_{\mathfrak{v},v}$ and $Y_{\mathfrak{w},w}$ have three legs, $Hd_{\mathfrak{v}\mathfrak{w}}(P_{\mathfrak{w}}(\mathscr{L}^s_{\mathfrak{v},v}), \mathscr{L}^s_{\mathfrak{w},w}) \leq \varepsilon_{5.3.6}$ for $s \in \{i,j\}$. Now $Hd^f(\mathscr{L}^i_{\mathfrak{w},w} \cup \mathscr{L}^j_{\mathfrak{w},w}, \mathscr{L}^{ij}_{\mathfrak{w},w}) \leq 2\delta_0$ implies that in either case, $Hd_{\mathfrak{v}\mathfrak{w}}(P_{\mathfrak{w}}(\mathscr{L}^i_{\mathfrak{v},v} \cup \mathscr{L}^j_{\mathfrak{v},v}), \mathscr{L}^{ij}_{\mathfrak{w},w}) \leq \varepsilon_{5.3.6} + 2\delta_0 + c_{5.3.6}$. Again $Hd^f(\mathscr{L}^i_{\mathfrak{v},v} \cup \mathscr{L}^j_{\mathfrak{v},v}, \mathscr{L}^{ij}_{\mathfrak{v},v}) \leq 2\delta_0$ and so by Lemma 2.2.21 (1), $Hd_{\mathfrak{v}\mathfrak{w}}(P_{\mathfrak{w}}(\mathscr{L}^i_{\mathfrak{v},v} \cup \mathscr{L}^j_{\mathfrak{v},v}), P_{\mathfrak{w}}(\mathscr{L}^{ij}_{\mathfrak{v},v})) \leq C_{2.2.21}(\delta'_0, \lambda'_0).(2\delta_0 + 1)$. Therefore, combining these inequalities, we get, $Hd_{\mathfrak{v}\mathfrak{w}}(P_{\mathfrak{w}}(\mathscr{L}^{ij}_{\mathfrak{v},v}), \mathscr{L}^{ij}_{\mathfrak{w},w}) \leq \varepsilon$, where $\varepsilon = \varepsilon_{5.3.6} + 2\delta_0 + c_{5.3.6} + C_{2.2.21}(\delta'_0, \lambda'_0).(2\delta_0 + 1)$.

Therefore, for condition (2) of Lemma 5.1.21, we take $\varepsilon' = max\{\varepsilon, \varepsilon_{5.3.6}\} = \varepsilon$.

Condition (3): So for the condition (3) of Lemma 5.1.21, we can take $C' = max\{C_{5.3.2}, C_{5.3.3}\}$ (see Lemma 5.3.2, Lemma 5.3.3).

Hence, to conclude the lemma, we take, $k_{5.3.7} = k_{5.1.21}(K')$, $c_{5.3.7} = c_{5.1.21}(C')$ and $\varepsilon_{5.3.7} = \varepsilon_{5.1.21}(\varepsilon')$ for the above K', C', ε'. □

Therefore, to complete Proposition 5.3.1, we take $k_{5.3.1} = max\{k_{5.3.6}, k_{5.3.7}\}$, $c_{5.3.1} = max\{c_{5.3.6}, c_{5.3.7}\}$ and $\varepsilon_{5.3.1} = max\{\varepsilon_{5.3.6}, \varepsilon_{5.3.7}\}$. □

Therefore, Proposition 5.3.1 gives us the following Corollary 5.3.8. Below in Lemma 5.3.9, we will investigate the (uniform) hyperbolicity of a (uniform) neighborhood of $\cup_{i=1}^3 \mathscr{L}^i$ in a bit larger neighborhood of $\mathscr{F}l_K(X_u)$, where $\mathscr{L}^i$'s are the ladders obtained in Proposition 5.3.1.

Corollary 5.3.8. *Let $x, y \in \mathscr{F}l_K(X_u)$. Suppose Σ_x and Σ_y are K-qi sections through x and y lying inside $\mathscr{U}_K$ over $B_{[u,\pi(x)]}$ and $B_{[u,\pi(y)]}$ respectively. Then there is a $(k_{5.3.8}, c_{5.3.8}, \varepsilon_{5.3.8})$-ladder, $\mathscr{L}_{xy}$, with a central base (possibly bigger than) B_u containing the sections Σ_x, Σ_y such that $bot(\mathscr{L}_{xy}) \subseteq \mathscr{U}_K$, $top(\mathscr{L}_{xy}) \subseteq \mathscr{U}_K$ and $\mathscr{L}_{xy} \subseteq N^f_{2\delta_0}(\mathscr{U}_K)$, where $k_{5.3.8} = k_{5.3.1}$, $c_{5.3.8} = c_{5.3.1}$ and $\varepsilon_{5.3.8} = \varepsilon_{5.3.1}$.*

Lemma 5.3.9. *Given $R \geq 2C^{(9)}_{2.4.12}(k_{5.3.1})$, there exist $\delta_{5.3.9} = \delta_{5.3.9}(k_{5.3.1}, R)$ and $L_{5.3.9} = L_{5.3.9}(k_{5.3.1}, R)$ such that the following hold.*

1. *$Y := N_{R+2\delta_0}(\cup_{i=1}^3 \mathscr{L}^i)$ is a $\delta_{5.3.9}$-hyperbolic subspace (with the induced path metric) in $U_{K(R+8\delta_0)} := N_{(R+8\delta_0)}(\mathscr{F}l_K(X_u))$.*

2. *The inclusion $L^{ij} = N_R(\mathscr{L}^{ij}) \hookrightarrow Y$ is $L_{5.3.9}$-qi embedding with their induced path metric.*

Proof. (1) Note that $\mathscr{L}^i$ is a $(k_{5.3.1}, c_{5.3.1}, \varepsilon_{5.3.1})$-ladder such that $\mathscr{L}^i \subseteq N^f_{6\delta_0}(\mathscr{U}_K)$ for $i = 1,2,3$, and so $Y \subseteq U_{K(R+8\delta_0)}$. Let $L^i = N_{R+2\delta_0}(\mathscr{L}^i)$, $i = 1,2,3$. Since the tree of metric bundles (X,B,T) satisfies $C^{(9)}_{2.4.12}(k_{5.3.1})$-flaring condition, by Theorem 5.2.11, L^i is δ_1-hyperbolic, where $\delta_1 = \delta_{5.2.11}(k_{5.3.1}, R+2\delta_0)$. Now we apply Proposition 2.2.7 twice on L^i's, $i = 1,2,3$; first on $L^1 \cup L^2$ and then on $(L^1 \cup L^2) \cup L^3$ to show Y is hyperbolic.

$L^1 \cup L^2$ is Hyperbolic: We verify the conditions of Proposition 2.2.7 for $n = 2$ (see Remark 2.2.8).

(1) L^1 and L^2 are δ_1-hyperbolic.

(2) Note that $N_{2k_{5.3.1}}(\Xi) \subseteq L^1 \cap L^2$, and by Lemma 2.4.12 (3), $N_{2k_{5.3.1}}(\Xi)$ is $k_{5.3.1}(2k_{5.3.1}+1)$-qi embedded in both L^1 and L^2. Let $N^i_D(N_{2k_{5.3.1}}(\Xi))$ denote the D-neighborhood of $N_{2k_{5.3.1}}(\Xi)$ in L^i-metric for $i = 1,2$. If $L^1 \cap L^2 \subseteq N^i_D(N_{2k_{5.3.1}}(\Xi))$, then by Lemma 2.1.4, $L^1 \cap L^2$ is L_1-qi embedded in both L^1 and L^2 for some L_1 depending on D and $k_{5.3.1}(2k_{5.3.1}+1)$. Now we will find D.

Finding D: Let $x \in L^1 \cap L^2$. Then $\exists\, x_i \in \mathscr{L}^i$ such that $d_{L^i}(x,x_i) \le R+2\delta_0$, $i = 1,2$. So $d_X(x_1,x_2) \le 2(R+2\delta_0)$ and $d_B(a_1,a_2) \le 2(R+2\delta_0)$, where $\pi_X(x_i) = a_i, i = 1,2$. Let v be the center of the tripod with vertices $\pi(x_1), \pi(x_2), u$ in T. If any one of $\pi(x_1)$, $\pi(x_2)$ is u then we set v to be u. Let $c \in B_v \cap [a_1,a_2]$. Then $d_B(a_i,c) \le 2(R+2\delta_0), i = 1,2$. Suppose γ_i is $k_{5.3.1}$-lift through x_i of geodesic $[a_i,c]$ lying inside $\mathscr{L}^i$. Let $\gamma_i(a_i) = c_i, i = 1,2$. So $d_{L^i}(x_i,c_i) \le 4(R+2\delta_0)k_{5.3.1}, i = 1,2$. Thus $d_X(c_1,c_2) \le d_X(c_1,x_1) + d_X(x_1,x_2) + d_X(x_2,c_2) \le 2(R+2\delta_0)(4k_{5.3.1}+1)$, and so $d^f(c_1,c_2) \le \phi(2(R+2\delta_0)(4k_{5.3.1}+1)) = D_1$ (say). We also note that $v \in \mathfrak{B}$ where $\mathfrak{B}$ is a central base for all $\mathscr{L}^i$'s. Since $z_{c,v} \in \Xi$ is δ_0-center of geodesic triangle with vertices $\{bot(\mathscr{L}^j_{c,v}); j = 1,2,3\}$ in $F_{c,v}$, so $z_{c,v}$ is $3\delta_0$-close (in fiber distance) to a point $c_3 \in [c_1,c_2]^f$. Then, for $i = 1,2$, $d_{L^i}(c_i,z_{c,v}) \le d^f(c_i,z_{c,v}) \le d^f(c_i,c_3) + d^f(c_3,z_{c,v}) \le D_1 + 3\delta_0$. Hence, for $i = 1,2$, $d_{L^i}(x,z_{c,v}) \le d_{L^i}(x,x_i) + d_{L^i}(x_i,c_i) + d_{L^i}(c_i,z_{c,v}) \le R+2\delta_0 + 4(R+2\delta_0)k_{5.3.1} + D_1 + 3\delta_0 =: D$. Therefore, $L^1 \cap L^2 \subseteq N^i_D(\Xi) \subseteq N^i_D(N_{2k_{5.3.1}}(\Xi))$, where $N^i_D(\zeta)$ denotes D-neighborhood around $\zeta \in L^i$ in the path metric of L^i, $i = 1,2$.

Therefore, by Remark 2.2.8, we conclude that $L^1 \cup L^2$ is δ_2-hyperbolic, where $\delta_2 = \delta_{2.2.8}(\delta_1, L_1)$.

The exact proof works mutatis mutandis for $(L^1 \cup L^2) \cup L^3$, and we conclude that Y is uniformly hyperbolic with the induced path metric. We take that hyperbolic constant to be $\delta_{5.3.9}$.

(2) Note that $L^{ij} = N_R(\mathscr{L}^{ij}) \subseteq Y \subseteq U_{K(R+8\delta_0)}$ as $\mathscr{L}^{ij} \subseteq N^f_{\delta_0}(\mathscr{L}^i \cup \mathscr{L}^j)$. Since $\mathscr{L}^{ij}$ is a $(k_{5.3.1}, c_{5.3.1}, \varepsilon_{5.3.1})$-ladder (see Proposition 5.3.1), then by Corollary 5.1.5, the inclusion $L^{ij} \hookrightarrow X$ is $L_{5.1.5}(k_{5.3.1}, R)$-qi embedding and so is the inclusion $L^{ij} \hookrightarrow Y$. Therefore, we can take $L_{5.3.9} = L_{5.1.5}(k_{5.3.1}, R)$. $\square$

Given a pair of points in $\mathscr{F}l_K(X_u)$, we get a ladder according to Corollary 5.3.8. But this ladder is far from being canonical. In the following proposition, we show that different ladders for different choices of qi sections for the same pair of points give rise to uniform Hausdorff-close geodesic paths joining those points in the respective ladders. In fact, this is more generally true, but we prove it according to our requirements. For the proposition below, r_2 is defined in Lemma 5.3.14.

Proposition 5.3.10. *There is a constant $D_{5.3.10} = D_{5.3.10}(k_{5.3.8}, c_{5.3.8}, \varepsilon_{5.3.8})$ such that the following holds.*

Let $x, y \in \mathscr{F}l_K(X_u)$. Suppose Σ_x and Σ_y are K-qi sections through x and y lying inside $\mathscr{U}_K$ over $B_{[u,\pi(x)]}$ and $B_{[u,\pi(y)]}$ respectively. Let $\mathscr{L}^1_{xy}$ and $\mathscr{L}^2_{xy}$ be two $(k_{5.3.8}, c_{5.3.8}, \varepsilon_{5.3.8})$-ladders containing Σ_x, Σ_y (see Corollary 5.3.8). Further we assume that $c^1(x,y)$ and $c^2(x,y)$ are geodesics joining x, y inside $L^1_{xy} := N_{r_2}(\mathscr{L}^1_{xy})$ and $L^2_{xy} := N_{r_2}(\mathscr{L}^2 xy)$ respectively. Then $c^1(x,y)$ and $c^2(x,y)$ are $D_{5.3.10}$-Hausdorff-close in X.

Proof. In this proposition, we omit the subscript xy when denoting the ladders to avoid excessive notation. We denote $\mathscr{L}^i := \mathscr{L}^i_{xy}$, $L^i := N_{r_2}(\mathscr{L}^i)$ and the fibers of $\mathscr{L}^i$ by $\mathscr{L}^i_{b,v} \subseteq F_{b,v}, i = 1,2$. Let $\pi_X(\mathscr{L}^i) = B_i$, $\pi(\mathscr{L}^i) = S_i, i = 1,2$. Note that both the ladders $\mathscr{L}^1$, $\mathscr{L}^2$ contain Σ_x and Σ_y, so for $v \in \pi(\Sigma_x) \cup \pi(\Sigma_y)$ and $b \in B_v$, $d^f(\mathscr{L}^1_{b,v}, \mathscr{L}^2_{b,v}) \leq \delta_0$; in particular, the pair $(\mathscr{L}^1_{b,v}, \mathscr{L}^2_{b,v})$ is $5\delta_0$-close in $F_{b,v}$.

The proof is divided into two steps. In Step 1, we find a common base $\bar{B}$ and develop lemmas which are needed in Step 2. In Step 2, we show that there is a common subspace containing x, y which is qi embedded in both L^1, L^2. Finally, we conclude the proposition.

Step 1: ***Construction of common base $\bar{B}$***: Let $B' = \{b \in B_v : d^f(\mathscr{L}^1_{b,v}, \mathscr{L}^2_{b,v}) \leq 5\delta_0, v \in S_1 \cap S_2\}$. Note that $B_u \subseteq \pi_X(\Sigma_x) \cup \pi_X(\Sigma_y) \subseteq B'$. Let $B'_v = \text{hull}(B') \cap B_v$ and $\bar{B}_v = N_{\delta_0}(B'_v) \cap B_v$, where $v \in \pi_B(B')$. Suppose $\bar{B}_1 = \cup_{v \in \pi_B(B')} \bar{B}_v$ and so $\pi_B(\bar{B}_1)$ is a subtree of $S_1 \cap S_2$. Then by [9, Lemma 1.93] and the fact that B_v's are isometrically embedded in B, we note that $\bar{B}_1$ is $(1, 6\delta_0)$-qi embedded in B. Finally, we will add a few more vertices and edges to $\bar{B}_1$ to complete the construction of $\bar{B}$. Suppose $v \in \pi_B(\bar{B}_1), w \notin \pi_B(\bar{B}_1)$ and $w \in S_1 \cap S_2$ such that $d_T(v,w) = 1$. Let $[\mathfrak{v}, \mathfrak{w}]$ be the edge joining $\mathfrak{v} \in B_v$ and $\mathfrak{w} \in B_w$. Further, we assume that $\mathfrak{v} \in \bar{B}_v$ and the pair $(\mathscr{L}^1_{\mathfrak{v},v}, \mathscr{L}^2_{\mathfrak{v},v})$ is $5\delta_0$-close in the fiber $F_{\mathfrak{v},v}$. Then we include only the vertex $\mathfrak{w}$ and the edge $[\mathfrak{v},\mathfrak{w}]$ to $\bar{B}_1$. And, we will use the same notation for these extra vertices, i.e., here $\bar{B}_w = \bar{B}\cap B_w = \{\mathfrak{w}\}$. Notice that $\bar{B}$ is still $(1,6\delta_0)$-qi embedded in B. Let $\bar{S} = \pi_B(\bar{B})$.

Let $v \in S_1 \cap S_2$ and $b \in B_v$. Let $P^i_{b,v} : F_{b,v} \to \mathscr{L}^i_{b,v}$ be a nearest point projection map in $F_{b,v}$ (see Lemma 2.2.21 (1)) and $\bar{P}^i_{b,v} : F_{b,v} \to \mathscr{L}^i_{b,v}$ is modified projection (see Definition 2.2.25) corresponding to $P^i_{b,v}$, $i = 1,2$. We denote $\bar{\mathscr{L}}^1_{b,v} := \bar{P}^1_{b,v}(\mathscr{L}^2_{b,v}) \subseteq \mathscr{L}^1_{b,v}$ and $\bar{\mathscr{L}}^2_{b,v} := \bar{P}^2_{b,v}(\mathscr{L}^1_{b,v}) \subseteq \mathscr{L}^2_{b,v}$. We take $\bar{\mathscr{L}}^i := \bigcup_{b\in\bar{B}_v,\ v\in\bar{S}} \bar{\mathscr{L}}^i_{b,v}, i = 1,2$.

Note 5.3.11. (i) Let $v \in \pi_B(B')$ and $b \in B' \cap B_v$. Then by Remark 2.2.14 (2), $Hd^f(P^1_{b,v}(\mathscr{L}^2_{b,v}), P^2_{b,v}(\mathscr{L}^1_{b,v})) \le 2\delta_0 + 5\delta_0 = 7\delta_0$. So $Hd^f(\bar{\mathscr{L}}^1_{b,v}, \bar{\mathscr{L}}^2_{b,v}) \le 13\delta_0$ by Remark 2.2.27. However, we will prove below in Lemma 5.3.12 that $\forall\ v \in \bar{S}$ and $\forall\ b \in \bar{B}_v$, $Hd^f(\bar{\mathscr{L}}^1_{b,v}, \bar{\mathscr{L}}^2_{b,v})$ is uniformly bounded.

(ii) If $v \in S_1 \cap S_2$, then by Lemma 2.4.16, $\bar{\mathscr{L}}^i \cap X_v := \bigcup_{b\in B_v} \bar{\mathscr{L}}^i_{b,v}$ form a (uniformly) special $C_{2.4.12}(K_{2.4.16}(K))$-ladder bounded by $K_{2.4.16}(K)$-qi sections, $i = 1,2$.

Lemma 5.3.12. *With the above notations, we have $R_{5.3.12}$ such that $\forall\ v \in \bar{S}$ and $\forall\ b \in \bar{B}_v$,*

$$Hd^f(\bar{\mathscr{L}}^1_{b,v}, \bar{\mathscr{L}}^2_{b,v}) \le R_{5.3.12}.$$

Proof. Let $v \in \bar{S},\ c \in \bar{B}_v$. We divide the proof into following cases. First three cases deal with the vertices $v \in \pi_B(\bar{B}_1) = \pi_B(B')$ and Case (4) with the extra vertices.

Case 1: Suppose $c \in \text{hull}(B')$ such that $c \in [a,b]_B$ for some $a,b \in B'$ and $a \in B_u$. Let $w = \pi_B(b), v = \pi_B(c)$. We prove that $d^f(\mathscr{L}^1_{c,v}, \mathscr{L}^2_{c,v})$ is uniformly bounded and hence we are through by Remark 2.2.14 (2) and Remark 2.2.27. As the pair $(\mathscr{L}^1_{b,w}, \mathscr{L}^2_{b,w})$ is $5\delta_0$-close, we take $x \in \mathscr{L}^1_{b,w}$ such that $d^f(x, \mathscr{L}^2_{b,w}) \le 5\delta_0$. Consider $k_{5.3.8}$-qi section, say, γ_x through x over $B_{[u,w]}$ in the ladder $\mathscr{L}^1$ (since $\mathscr{L}^i$'s are $(k_{5.3.8}, c_{5.3.8}, \varepsilon_{5.3.8})$-ladders). Now we apply Mitra's retraction (see Theorem 5.1.3), $\rho_{\mathscr{L}^2}$ on γ_x and get a $\mathscr{R}_0(2k_{5.3.8}+1)$-qi section, say, γ'_x over $B_{[u,w]}$ in $\mathscr{L}^2$, where $\mathscr{R}_0 := L_{5.1.3}(k_{5.3.8}, c_{5.3.8}, \varepsilon_{5.3.8})$. Then we have two $\mathscr{R}_0(2k_{5.3.8}+1)$-qi sections γ_x, γ'_x over $B_{[u,w]}$ such that (as $a \in B_u$) $d^f(\gamma_x(s), \gamma'_x(s)) \le 5\delta_0,\ s \in \{a,b\}$. Again the tree of metric bundles (X,B,T) satisfies $\mathscr{R}_0(2k_{5.3.8}+1)$-flaring condition. Thus the restriction of γ_x, γ'_x on geodesic $[a,b]$ and Lemma 2.4.7 (2) imply $d^f(\gamma_x(c), \gamma'_x(c)) \le R_1$, where $R_1 = R_{2.4.7}(L_0(2k_{5.3.8}+1), 5\delta_0)$. Hence $d^f(\mathscr{L}^1_{c,v}, \mathscr{L}^2_{c,v}) \le R_1$ and so by Remark 2.2.14 (2) and Remark 2.2.27, $Hd^f(\bar{\mathscr{L}}^1_{c,v}, \bar{\mathscr{L}}^2_{c,v}) \le 8\delta_0 + R_1$.

Case 2: Suppose $c \in \text{hull}(B')$ such that $c \in [a,b]_B$ for some $a,b \in B'$ and none of a,b belong to B_u. Let $v = \pi_B(c)$. More precisely, we assume that v is the center of the geodesic triangle $\triangle(\pi_B(a), u, \pi_B(b))$, otherwise, it will land in Case 1. Let

$a' \in B_u$. Then by δ_0-slimness of the geodesic triangle $\triangle(a,b,a')$ and without loss of generality, we may assume that $d_B(c,c') \le \delta_0$ for some $c' \in [a',b]$. So by Case 1, we have $d^f(\mathscr{L}^1_{c',v}, \mathscr{L}^2_{c',v}) \le R_1$. Let $x \in \mathscr{L}^1_{c',v}, y \in \mathscr{L}^2_{c',v}$ such that $d^f(x,y) \le R_1$. Now we take $k_{5.3.8}$-qi lifts, say, γ_1 and γ_2 of geodesic $[c,c']$ in $\mathscr{L}^1$ and $\mathscr{L}^2$ through x and y respectively. Then $d_X(\gamma_1(c),\gamma_2(c)) \le d_X(\gamma_1(c),x) + d_X(x,y) + d_X(y,\gamma_2(c)) \le 2k_{5.3.8}\delta_0 + R_1 + 2k_{5.3.8}\delta_0 = 4k_{5.3.8}\delta_0 + R_1$. Thus $d^f(\mathscr{L}^1_{c,v}, \mathscr{L}^2_{c,v}) \le \phi(4k_{5.3.8}\delta_0 + R_1)$, where fibers are ϕ-properly embedded in X. Therefore, by Remark 2.2.14 (2) and Remark 2.2.27, $Hd^f(\bar{\mathscr{L}}^1_{c,v}, \bar{\mathscr{L}}^2_{c,v}) \le 8\delta_0 + \phi(4k_{5.3.8}\delta_0 + R_1)$.

Case 3: Suppose $c \in \bar{B}_1$ and $\pi_B(c) = v$. Then by construction of $\bar{B}_1$, there exists $c_1 \in B'_v \subseteq \text{hull}(B')$ such that $d_B(c,c_1) = d_{B_v}(c,c_1) \le \delta_0$. Since $c_1 \in \text{hull}(B')$, by Case 2, we know that $d^f(\mathscr{L}^1_{c_1,v}, \mathscr{L}^2_{c_1,v}) \le \phi(4k_{5.3.8}\delta_0 + R_1)$. Now we use the same argument used in the last part of Case 2. Let $x \in \mathscr{L}^1_{c_1,v}, y \in \mathscr{L}^2_{c_1,v}$ such that $d^f(x,y) \le \phi(4k_{5.3.8}\delta_0 + R_1)$. By taking $k_{5.3.8}$-qi lifts through points x and y of the geodesic $[c,c_1]$ in the ladders $\mathscr{L}^1$ and $\mathscr{L}^2$ respectively, one can conclude that $d^f(\mathscr{L}^1_{c,v}, \mathscr{L}^2_{c,v}) \le \phi(4k_{5.3.8}\delta_0 + \phi(4k_{5.3.8}\delta_0 + R_1))$. Hence, by Remark 2.2.14 (2) and Remark 2.2.27, $Hd^f(\bar{\mathscr{L}}^1_{c,v}, \bar{\mathscr{L}}^2_{c,v}) \le 8\delta_0 + \phi(4k_{5.3.8} + \phi(4k_{5.3.8}\delta_0 + R_1))$.

Case 4: Here we will check for extra vertices if any. Suppose $c \in \bar{B} \setminus \bar{B}_1$. Let $\pi_B(c) = w$ and $[v,w]$ be the edge such that $d_T(u,v) < d_T(u,w)$. Let $[\mathfrak{v},\mathfrak{w}]$ be the edge joining $\mathfrak{v} \in B_v$ and $\mathfrak{w} \in B_w$. Note that $c = \mathfrak{w}$. Then by the construction of $\mathfrak{w} \in \bar{B}$, $d^f(\mathscr{L}^1_{\mathfrak{v},v}, \mathscr{L}^2_{\mathfrak{v},v}) \le 5\delta_0$ and so $Hd^f(\bar{\mathscr{L}}^1_{\mathfrak{v},v}, \bar{\mathscr{L}}^2_{\mathfrak{v},v}) \le 13\delta_0$ (see Remark 2.2.14 (2)). Suppose $x \in \bar{\mathscr{L}}^1_{\mathfrak{v},v}$ and $y \in \bar{\mathscr{L}}^2_{\mathfrak{v},v}$ such that $d^f(x,y) \le 13\delta_0$. We consider $x' \in \mathscr{L}^1_{\mathfrak{w},w}$, $y' \in \mathscr{L}^2_{\mathfrak{w},w}$ such that $d_{\mathfrak{v}\mathfrak{w}}(P_{\mathfrak{w}}(x),x') \le \varepsilon_{5.3.8}$ and $d_{\mathfrak{v}\mathfrak{w}}(P_{\mathfrak{w}}(y),y') \le \varepsilon_{5.3.8}$ (since $\mathscr{L}^i$'s are $(k_{5.3.8}, c_{5.3.8}, \varepsilon_{5.3.8})$-ladders). Again $P_{\mathfrak{w}}$ is L'_1-coarsely Lipschitz retraction in the metric $F_{\mathfrak{v}\mathfrak{w}}$ (see Lemma 2.3.4 (2)) and so $d_{\mathfrak{v}\mathfrak{w}}(P_{\mathfrak{w}}(x),P_{\mathfrak{w}}(y)) \le L'_1 d_{\mathfrak{v}\mathfrak{w}}(x,y) + L'_1 \le L'_1(13\delta_0 + 1)$. Thus by triangle inequality, $d_{\mathfrak{v}\mathfrak{w}}(x',y') \le 2\varepsilon_{5.3.8} + L'_1(13\delta_0 + 1)$ $\Rightarrow d^f(x',y') \le \phi(2\varepsilon_{5.3.8} + L'_1(13\delta_0 + 1))$. Now $x' \in \mathscr{L}^1_{\mathfrak{w},w}$, $y' \in \mathscr{L}^2_{\mathfrak{w},w}$ and so by Remark 2.2.14 (2) and Remark 2.2.27, we have, $d^f(\bar{\mathscr{L}}^1_{\mathfrak{w},w}, \bar{\mathscr{L}}^2_{\mathfrak{w},w}) \le 8\delta_0 + \phi(2\varepsilon_{5.3.8} + L'_1(13\delta_0 + 1))$.

Therefore, we can take $R_{5.3.12}$ to be the maximum of all four constants we get in four cases, i.e., $R_{5.3.12} = max\{\phi(2k_{5.3.8} + \phi(2k_{5.3.8}\delta_0 + R_1)), 8\delta_0 + \phi(2\varepsilon_{5.3.1} + L'_1(13\delta_0 + 1))\}$. $\square$

Next we show that $\bar{\mathscr{L}}^i$'s are more general ladders. More precisely, they are semicontinuous families and they need not satisfy the condition (5) of Definition 5.1.1 trivially (i.e., in the notation of Definition 5.1.1, $B' \subsetneq \pi_B^{-1}(T_{\mathfrak{Y}})$), and they behave like ladders.

Lemma 5.3.13. *There are constants $k_{5.3.13}$, $c_{5.3.13}$ and $\varepsilon_{5.3.13}$ such that the following hold.*

Suppose $[v,w]$ is an edge in T such that $d_T(u,v) < d_T(u,w)$ and $[\mathfrak{v},\mathfrak{w}]$ is the edge joining $\mathfrak{v} \in B_v$ and $\mathfrak{w} \in B_w$. Let $\mathfrak{v},\mathfrak{w} \in \bar{B}$. Then:

1. $\bar{\mathscr{L}}^i_{\mathfrak{w},w} \subseteq N^{\mathfrak{v}\mathfrak{w}}_{k_{5.3.13}}(\bar{\mathscr{L}}^i_{\mathfrak{v},v})$.
2. $Hd_{\mathfrak{v}\mathfrak{w}}(P_{\mathfrak{w}}(\bar{\mathscr{L}}^i_{\mathfrak{v},v}), \bar{\mathscr{L}}^i_{\mathfrak{w},w}) \leq \varepsilon_{5.3.13}$.
3. *Suppose $a \in \bar{B}$, $b \notin \bar{B}$ with $d_B(a,b) = 1$ and $\pi_B(a) = s$, $\pi_B(b) = t$. Then $diam^f(\bar{\mathscr{L}}^i_{a,s}) \leq c_{5.3.13}$ if $s = t$ or the pair $(\bar{\mathscr{L}}^i_{a,s}, F_{b,t})$ is $c_{5.3.13}$-cobounded in the path metric of $F_{ab} := \pi_X^{-1}([a,b])$ if $s \neq t$.*

In particular, $\bar{\mathscr{L}}^i$ is $(k_{5.3.13}, c_{5.3.13}, \varepsilon_{5.3.13})$-semicontinuous family, $i = 1,2$.

Proof. (1) We will only prove for $i = 1$ as the proof for $i = 2$ involves a simple change of indices. Since $\mathscr{L}^i$'s are $(k_{5.3.8}, c_{5.3.8}, \varepsilon_{5.3.8})$-ladders, for $x \in \bar{\mathscr{L}}^1_{\mathfrak{w},w}$, $\exists\, x_1 \in \mathscr{L}^1_{\mathfrak{v},v}$ such that $d_{\mathfrak{v}\mathfrak{w}}(x,x_1) \leq k_{5.3.8}$. Let $y \in \bar{\mathscr{L}}^2_{\mathfrak{w},w}$, $y_1 \in \mathscr{L}^2_{\mathfrak{v},v}$ such that $d^f(x,y) \leq R_{5.3.12}$ (by Lemma 5.3.12) and $d_{\mathfrak{v}\mathfrak{w}}(y,y_1) \leq k_{5.3.8}$. Then $d_{\mathfrak{v}\mathfrak{w}}(x_1,y_1) \leq d_{\mathfrak{v}\mathfrak{w}}(x_1,x) + d^f(x,y) + d_{\mathfrak{v}\mathfrak{w}}(y,y_1) \leq 2k_{5.3.8} + R_{5.3.12}$. Hence $d^f(x_1,y_1) \leq R_2$, where $R_2 = \phi(2k_{5.3.8} + R_{5.3.12})$. Note that $P^1_{\mathfrak{v},v} : F_{\mathfrak{v},v} \to \mathscr{L}^1_{\mathfrak{v},v}$ is a nearest point projection map in the metric of $F_{\mathfrak{v},v}$. For simplicity, let $P = P^1_{\mathfrak{v},v}$. Then $d^f(P(x_1),P(y_1)) = d^f(x_1,P(y_1)) \leq C_{2.2.21}(\delta_0,\delta_0)(R_2 + 1)$ (see Lemma 2.2.21 (1)). So, $d^f(x_1,\bar{\mathscr{L}}^1_{\mathfrak{v},v}) \leq C_{2.2.21}(\delta_0,\delta_0)(R_2+1)$. Hence $d_{\mathfrak{v}\mathfrak{w}}(x,\bar{\mathscr{L}}^1_{\mathfrak{v},v}) \leq d_{\mathfrak{v}\mathfrak{w}}(x,x_1) + d^f(x_1,\bar{\mathscr{L}}^1_{\mathfrak{v},v}) \leq k_1$, where $k_1 = k_{5.3.8} + C_{2.2.21}(\delta_0,\delta_0)(R_2 + 1)$. Therefore, $\bar{\mathscr{L}}^i_{\mathfrak{w},w} \subseteq N^{\mathfrak{v}\mathfrak{w}}_{k_1}(\bar{\mathscr{L}}^i_{\mathfrak{v},v})$. Thus k_1 works for (1) but for the second part of this lemma, we have defined $k_{5.3.13} > k_1$ in the end.

(2) Here also we will only prove for $i = 1$. Let $x \in \bar{\mathscr{L}}^1_{\mathfrak{v},v}$ and we take $y \in \bar{\mathscr{L}}^2_{\mathfrak{v},v}$ such that $d^f(x,y) \leq R_{5.3.12}$ (by Lemma 5.3.12). We take $x' \in \mathscr{L}^1_{\mathfrak{w},w}$, $y' \in \mathscr{L}^2_{\mathfrak{w},w}$ such that $d_{\mathfrak{v}\mathfrak{w}}(P_{\mathfrak{w}}(x),x') \leq \varepsilon_{5.3.8}$ and $d_{\mathfrak{v}\mathfrak{w}}(P_{\mathfrak{w}}(y),y') \leq \varepsilon_{5.3.8}$ (since $\mathscr{L}^i$'s are $(k_{5.3.8}, c_{5.3.8}, \varepsilon_{5.3.8})$-ladders). Again $P_{\mathfrak{w}}$ is L'_1-coarsely Lipschitz in the metric of $F_{\mathfrak{v}\mathfrak{w}}$, so $d_{\mathfrak{v}\mathfrak{w}}(P_{\mathfrak{w}}(x),P_{\mathfrak{w}}(y)) \leq L'_1 d_{\mathfrak{v}\mathfrak{w}}(x,y) + L'_1 \leq L'_1(R_{5.3.12}+1)$. Therefore, (by triangle inequality) $d_{\mathfrak{v}\mathfrak{w}}(x',y') \leq 2\varepsilon_{5.3.8} + L'_1(R_{5.3.12}+1) \Rightarrow d^f(x',y') \leq \phi(2\varepsilon_{5.3.8} + L'_1(R_{5.3.12}+1)) = R_3$ (say). Note that $P^1_{\mathfrak{w},w} : F_{\mathfrak{w},w} \to \mathscr{L}^1_{\mathfrak{w},w}$ is a nearest point projection map in the metric of $F_{\mathfrak{w},w}$. For simplicity, let $P = P^1_{\mathfrak{w},w}$. Since $d^f(x',y') \leq R_3$, then by Lemma 2.2.21 (1), $d^f(x',P(y')) = d^f(P(x'),P(y')) \leq C_{2.2.21}(\delta_0,\delta_0)(R_3 + 1) = R_4$ (say). So $d^f(x',\bar{\mathscr{L}}^1_{\mathfrak{w},w}) \leq R_4$. Hence $d_{\mathfrak{v}\mathfrak{w}}(P_{\mathfrak{w}}(x),\bar{\mathscr{L}}^1_{\mathfrak{w},w}) \leq d_{\mathfrak{v}\mathfrak{w}}(P_{\mathfrak{w}}(x),x') + d_{\mathfrak{v}\mathfrak{w}}(x',\bar{\mathscr{L}}^1_{\mathfrak{w},w}) \leq \varepsilon_{5.3.8} + R_4 = \varepsilon_1$ (say), i.e., $P_{\mathfrak{w}}(\bar{\mathscr{L}}^1_{\mathfrak{v},v}) \subseteq N^{\mathfrak{v}\mathfrak{w}}_{\varepsilon_1}(\bar{\mathscr{L}}^1_{\mathfrak{w},w})$.

For the other inclusion, let $x \in \bar{\mathscr{L}}^1_{\mathfrak{w},w}$. Then by (1), $\exists\, x_1 \in \bar{\mathscr{L}}^1_{\mathfrak{v},v}$ such that $d_{\mathfrak{v}\mathfrak{w}}(x,x_1) \leq k_1$. Then $d_{\mathfrak{v}\mathfrak{w}}(P_{\mathfrak{w}}(x_1),x) \leq 2k_1$. Hence $\bar{\mathscr{L}}^1_{\mathfrak{w},w} \subseteq N^{\mathfrak{v}\mathfrak{w}}_{2k_1}(\bar{\mathscr{L}}^1_{\mathfrak{w},w})$.

Therefore, we can take $\varepsilon_{5.3.13} := max\{\varepsilon_1, 2k_1\}$.

(3) Suppose $s = t$. Then $a, b \in B_s$. Since $b \notin \bar{B}$, so $d^f(\mathscr{L}^1_{b,s}, \mathscr{L}^2_{b,s}) > 5\delta_0$. Then by Remark 2.2.14 (1), $\mathrm{diam}^f(\bar{\mathscr{L}}^i_{b,s}) \leq 8\delta_0$ for $i = 1,2$. Let $\bar{\mathscr{L}}^i_{a,s} = [\eta^i_{a,s}, \zeta^i_{a,s}]$ for $i = 1,2$. Since $d_B(a,b) = 1$, by Note 5.3.11 (ii), $d_X(\eta^i_{a,s}, \bar{\mathscr{L}}^i_{b,s}) \leq 2K_{2.4.16}(K)$ and $d_X(\zeta^i_{a,s}, \bar{\mathscr{L}}^i_{b,s}) \leq 2K_{2.4.16}(K)$ for $i = 1,2$. Then by triangle inequality, $d_X(\eta^i_{a,s}, \zeta^i_{a,s}) \leq 4K_{2.4.16}(K) + 8\delta_0$ for $i = 1,2$. Therefore, $\mathrm{diam}^f(\bar{\mathscr{L}}^i_{a,s}) \leq \phi(4K_{2.4.16}(K) + 8\delta_0)$ for $i = 1,2$.

Now suppose $s \neq t$. Note that since $a \in \bar{B},\ b \notin \bar{B}$, then $d_T(u,s) < d_T(u,t)$ and $[a,b]$ is the edge joining $a \in B_s$, $b \in B_t$. If $t \notin S_1 \cup S_2$, then the pair $(\mathscr{L}^i_{a,s}, F_{b,t})$ is $c_{5.3.8}$-cobounded in F_{ab}. So by Lemma 2.2.18, there is a constant C_1 depending on δ'_0, λ'_0 and $c_{5.3.8}$ such that the pair $(\bar{\mathscr{L}}^i_{a,s}, F_{b,t})$ is C_1-cobounded in F_{ab}.

Now let $t \in S_1 \cap S_2$. Since $b \notin \bar{B}$, then by the construction of $\bar{B}$, $d^f(\mathscr{L}^1_{a,s}, \mathscr{L}^2_{a,s}) > 5\delta_0$. Thus by Remark 2.2.14 (1), $\mathrm{diam}^f(\bar{\mathscr{L}}^i_{a,s}) \leq 8\delta_0$ for $i = 1,2$. Therefore, by Lemma 2.2.18, there is a constant C_2 depending on δ'_0, λ'_0 and $8\delta_0$ such that the pair $(\bar{\mathscr{L}}^i_{a,s}, F_{b,t})$ is C_2-cobounded in F_{ab} for $i = 1,2$.

Finally, we assume that t belong to only one of the S_1, S_2. Without of loss of generality, let $t \in S_1$ but $t \notin S_2$. Note that $s \in S_2$. Then the pair $(\mathscr{L}^2_{a,s}, F_{b,t})$ is $c_{5.3.8}$-cobounded in F_{ab}. So by Lemma 2.2.18, the pair $(\bar{\mathscr{L}}^2_{a,s}, F_{b,t})$ is C_1-cobounded in F_{ab} (where C_1 is defined above). Again, $Hd^f(\bar{\mathscr{L}}^1_{a,s}, \bar{\mathscr{L}}^2_{a,s}) \leq R_{5.3.12}$ and a nearest point projection map $P : F_{ab} \to F_{b,t}$ is L'_1-coarsely Lipschitz (see Lemma 2.3.4 (2)) together imply that the diameter (in the metric of F_{ab}) of $\{P(\bar{\mathscr{L}}^1_{a,s})\}$ is bounded by $2L'_1(R_{5.3.12} + 1) + C_1 = D$ (say). Then by Lemma 2.2.18, there is a constant C_3 depending on δ'_0, λ'_0 and D such that the pair $(\bar{\mathscr{L}}^1_{a,s}, F_{b,t})$ is C_3-cobounded in F_{ab}.

Therefore, we can take $c_{5.3.13} = max\{\phi(4K_{2.4.16}(K) + 8\delta_0), C_1, C_2, C_3\}$.

For second part, we note that $\bar{B}$ is $(1, 6\delta_0)$-qi embedded in B. Now by Note 5.3.11 (ii), $\bar{\mathscr{L}}^i \cap X_v$ is special $C_{2.4.12}(K_{2.4.16}(K))$-ladder in $X_v, v \in S_1 \cap S_2$. Therefore, by Lemma 5.1.21, $\bar{\mathscr{L}}^i$ is $(k_{5.3.13}, c_{5.3.13}, \varepsilon_{5.3.13})$-ladder, where $k_{5.3.13} = k_{5.1.21}(k') > k_1$ and $k' = max\{k_1, C_{2.4.12}(K_{2.4.16}(K))\}$, and $i = 1,2$. □

Lemma 5.3.14. *Let* $r_1 = max\{2k_{5.3.13}, 2\delta_0 + 1\}, r_2 = max\{2C^{(9)}_{2.4.12}(k_{5.3.8}), R_{5.3.12} + r_1 + \delta_0\}$ *and* $\bar{L}^i := N_{r_1}(\bar{\mathscr{L}}^i)$, $L^i := N_{r_2}(\mathscr{L}^i)$, $i = 1,2$. *Then there is a constant* $L_{5.3.14}$ *such that the inclusion* $\bar{L}^i \hookrightarrow L^i$ *is* $L_{5.3.14}$*-qi embedding.*

Proof. Note that $\bar{\mathscr{L}}^i$ is $(k_{5.3.13}, c_{5.3.13}, \varepsilon_{5.3.13})$-semicontinuous family (see Lemma 5.3.13) and $\bar{L}^i \subseteq L^i$. Then by Corollary 5.1.5, the inclusion $\bar{L}^i \hookrightarrow X$ is $L_{5.1.5}(k_{5.3.13}, r_1)$-qi embedding and so is the inclusion $\bar{L}^i \hookrightarrow L^i, i = 1,2$. Therefore, we can take $L_{5.3.14} = L_{5.1.5}(k_{5.3.13}, r_1)$. □

Step 2: **We fix r_1 and r_2 as in Lemma 5.3.14 for the rest of the proof**. Here we construct a common qi embedded subspace of L^1 and L^2 containing both x and y and which will show that $c^1(x,y)$ and $c^2(x,y)$ are uniformly Hausdorff-close.

Construction of common qi embedded subspace of L^1 and L^2: Let $v \in \bar{S}$ and $b \in \bar{B}_v$. Suppose $\mathscr{Z}_{b,v} := \text{hull}(\bar{\mathscr{L}}^1_{b,v} \cup \bar{\mathscr{L}}^2_{b,v}) \subseteq F_{b,v}$, $\mathscr{Z} := \bigcup \mathscr{Z}_{b,v}$ and $Z := N_{r_1}(\mathscr{Z})$, where the quasiconvex hull and its neighborhood is taken in the corresponding fiber. We also have $Hd^f(\bar{\mathscr{L}}^1_{b,v}, \bar{\mathscr{L}}^2_{b,v}) \leq R_{5.3.12}$ (by Lemma 5.3.12). Suppose L^i and $\bar{L}^i$ are as in Lemma 5.3.14. Then $\bar{L}^i \subseteq Z \subseteq N_{r_1+R_{5.3.12}+\delta_0}(\bar{\mathscr{L}}^i) = N_{R_{5.3.12}+\delta_0}(\bar{L}^i)$ in the metric L^i and so $Hd_{L^i}(\bar{L}^i, Z) \leq R_{5.3.12} + \delta_0$ for $i = 1,2$. The subspace Z is our required common subspace and in the below lemma we will see that it is uniformly qi embedded in L^i for $i = 1,2$.

Lemma 5.3.15. *With the above notations, there is a (uniform) constant $L_{5.3.15}$ such that the inclusion $Z \hookrightarrow L^i$ is $L_{5.3.15}$-qi embedding in the path metric of L^i, $i = 1,2$.*

Proof. By Lemma 5.3.14, the inclusion $\bar{L}^i \hookrightarrow L^i$ is $L_{5.3.14}$-qi embedding. Since $Hd_{L^i}(\bar{L}^i, Z) \leq R_{5.3.12} + \delta_0$, with the reference to [9, Lemma 1.19], our task is to show that Z is uniformly properly embedded in L^i.

Z is properly embedded in L^i: Let $x,y \in Z$ and $n \in \mathbb{N}$ such that $d_{L^i}(x,y) \leq n$. Then $\exists\, x_1, y_1 \in \bar{L}^i$ such that $d_Z(x,x_1) \leq R_{5.3.12} + \delta_0$, $d_Z(y,y_1) \leq R_{5.3.12} + \delta_0$. So, $d_{L^i}(x_1,y_1) \leq 2(R_{5.3.12} + \delta_0) + n$ and by Lemma 5.3.14, $d_Z(x_1,y_1) \leq d_{\bar{L}^i}(x_1,y_1) \leq (2(R_{5.3.12} + \delta_0) + n)L_{5.3.14} + L^2_{5.3.14} = D(n)$ (say). Thus,

$$\begin{aligned} d_Z(x,y) &\leq d_Z(x,x_1) + d_Z(x_1,y_1) + d_Z(y_1,y) \\ &\leq R_{5.3.12} + \delta_0 + D(n) + R_{5.3.12} + \delta_0 \\ &= 2(R_{5.3.12} + \delta_0) + D(n) =: g(n) \text{ (say)} \end{aligned}$$

So Z is g-properly embedded in L^i for the function $g : \mathbb{N} \to \mathbb{N}$ defined above. Therefore, for $i = 1,2$, the inclusion $Z \hookrightarrow L^i$ is $L_{5.3.15}$-qi embedding, where $L_{5.3.15}$ depends on $L_{5.3.14}$, $R_{5.3.12} + \delta_0$ and g (see [9, Lemma 1.19]). □

Conclusion: Let $c^*(x,y)$, $c^1(x,y)$ and $c^2(x,y)$ be geodesic paths in Z, L^1 and L^2 respectively joining x,y. Since (X,B,T) satisfies $C^{(9)}_{2.4.12}(k_{5.3.8})$-flaring condition. So by Theorem 5.2.11, L^i is $\delta_{5.2.11}(k_{5.3.8}, r_2)$-hyperbolic, $i = 1,2$. Therefore, by stability of quasi-geodesic (Lemma 2.2.2), $Hd_X(c^1(x,y), c^2(x,y)) \leq D_{5.3.10} := 2D_{2.2.2}(\delta_{5.2.11}(k_{5.3.8}, r_2), L_{5.3.15}, L_{5.3.15})$. □

Now, we are at a stage to show the uniform hyperbolicity of a uniform neighborhood of $\mathscr{F}l_K(X_u)$ with the induced path metric.

Theorem 5.3.16. *Suppose r_2 as in Lemma 5.3.14. Then for any $R \geq r_2 + 8\delta_0$ there is $\delta_{5.3.16} = \delta_{5.3.16}(K,R)$ such that $Fl_{KR}(X_u) := N_R(\mathscr{F}l_K(X_u))$ is $\delta_{5.3.16}$-hyperbolic metric space.*

Proof. We show that $Fl_{KR}(X_u)$ satisfies all the conditions of Proposition 2.2.6. Note that $\mathscr{F}l_K(X_u)$ is R-dense in $Fl_{KR}(X_u)$. For a point $x \in \mathscr{F}l_K(X_u)$, we fix once and for all a K-qi section Σ_x through x over $B_x := B_{[u,\Pi(x)]}$ lying inside $\mathscr{U}_K$. Now given a pair (x^1,x^2) of distinct points in $\mathscr{U}_K$, by Corollary 5.3.8, there is a $(k_{5.3.8}, c_{5.3.8}, \varepsilon_{5.3.8})$-ladder, say, $\tilde{\mathscr{L}}^{12}$ containing Σ_{x^1}, Σ_{x^2} such that $top(\tilde{\mathscr{L}}^{12}) \subseteq \mathscr{U}_K$, $bot(\tilde{\mathscr{L}}^{12}) \subseteq \mathscr{U}_K$ and $\tilde{\mathscr{L}}^{12} \subseteq N^f_{2\delta_0}(\mathscr{U}_K)$.

We take $\tilde{c}(x^1,x^2)$, a geodesic path joining x^1,x^2 in $\tilde{L}^{12} := N_{r_2}(\tilde{\mathscr{L}}^{12})$. For a given pair of points, once and for all, we fix this ladder and the geodesic path. These paths serve as family of paths for Proposition 2.2.6.

Let us start with three points $x^i \in \mathscr{F}l_K(X_u)$, $i = 1,2,3$ and geodesic paths $\tilde{c}(x^i,x^j)$ in the respective ladders $\tilde{L}^{ij} := N_{r_2}(\tilde{\mathscr{L}}^{ij})$ for all distinct $i,j \in \{1,2,3\}$. Note that $\tilde{L}^{ij} \subseteq U_{KR}$.

Condition (1): As $\tilde{L}^{ij}$ is $L_{5.1.5}(k_{5.3.8}, r_2)$-qi embedded in X and so is in U_{KR}. Then the path $\tilde{c}(x^i,x^j)$ is h-properly embedded in U_{KR}, where $h : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ sending $r \mapsto rL_{5.1.5}(k_{5.3.8}, r_2) + (L_{5.1.5}(k_{5.3.8}, r_2))^2$.

Condition (2): By Proposition 5.3.1, given any three points $x^i, i = 1,2,3$, we have, $(k_{5.3.1}, c_{5.3.1}, \varepsilon_{5.3.1})$-ladders, $\mathscr{L}^{ij}$ containing Σ_i, Σ_j such that $top(\mathscr{L}^{ij}) \subseteq \mathscr{U}_K$, $bot(\mathscr{L}^{ij}) \subseteq \mathscr{U}_K$ and $\mathscr{L}^{ij} \subseteq N^f_{2\delta_0}(\mathscr{U}_K)$. Let $c(x^i,x^j)$ be a geodesic path joining x^i,x^j in $L^{ij} := N_{r_2}(\mathscr{L}^{ij}) \subseteq U_{KR}$. Note that $k_{5.3.1} = k_{5.3.8}$, $c_{5.3.1} = c_{5.3.8}$ and $\varepsilon_{5.3.1} = \varepsilon_{5.3.8}$ (Lemma 5.3.8). So by Proposition 5.3.10, $Hd_X(\tilde{c}(x^i,x^j), c(x^i,x^j)) \leq D$, where $D = D_{5.3.10}(k_{5.3.8}, c_{5.3.8}, \varepsilon_{5.3.8})$. Thus by Proposition 5.1.11, their Hausdorff distance is bounded by $\eta_1(D)$ in the path metric of U_{KR}, where $\eta_1 := \eta_{5.1.11}(K,R)$.

Now by Lemma 5.3.9, there is a $\delta_{5.3.9}(k_{5.3.1}, r_2)$-hyperbolic subspace $Y (:= N_{r_2+2\delta_0}(\cup_{i=1}^3 \mathscr{L}^i))$ such that the inclusion $i : L^{ij} \hookrightarrow Y$ is $L_{5.3.9}(k_{5.3.1}, r_2)$-qi embedding. Also, note that $Y \subseteq U_{K(r_2+8\delta_0)} \subseteq U_{KR}$. Let $\delta_1 = \delta_{5.3.9}(k_{5.3.1}, r_2)$ and $L_1 = L_{5.3.9}(k_{5.3.1}, r_2)$. Then by Lemma 2.2.2, the triangle formed by the paths $c(x^i,x^j)$ for all distinct $i,j \in \{1,2,3\}$, are D_1-slim in the path metric of Y and so is in the path metric of U_{KR}, where $D_1 := 2D_{2.2.2}(\delta_1, L_1, L_1) + \delta_1$.

Hence the triangle formed by the paths $\tilde{c}(x^i,x^j)$ for all distinct $i,j \in \{1,2,3\}$, are D_2-slim in the path metric of U_{KR}, where $D_2 := 2\eta_1(D) + D_1$.

Therefore, by Proposition 2.2.6, $Fl_{KR}(X_u)$ is $\delta_{5.3.16}$-hyperbolic metric space with the induced path metric from X, where $\delta_{5.3.16} = \delta_{2.2.6}(h, D_2, R)$, where h and D_2 are defined above. ☐

5.4 Hyperbolicity of $N_D(\mathscr{F}l_K(X_u)\cup\mathscr{F}l_K(X_v))$

Let $u,v\in T$, and $\mathscr{F}l_K(X_u)$ and $\mathscr{F}l_K(X_v)$ are the flow spaces as described in the first paragraph of Section 5.3. We also assume that $\mathscr{F}l_K(X_u)\cap\mathscr{F}l_K(X_v)\neq\emptyset$. In this section, we will prove that $N_D(\mathscr{F}l_K(X_u)\cup\mathscr{F}l_K(X_v))$ is uniformly hyperbolic. We use the notations $Fl_{KD} := N_D(\mathscr{F}l_K(X_u))$, $Fl_{KD} := N_D(\mathscr{F}l_K(X_v))$ for $D\geq 0$. Here, we require (X,B,T) to satisfy $(2(L')^2(2K+1)+L')$-flaring condition where $L' = L_{5.1.10}(K)$.

So far we have the following $(\mathscr{H}0)-(\mathscr{H}6)$. We will use these properties in this section.

$(\mathscr{H}0)$ Suppose $w,w'\in T$ and e is the edge on $[w,w']$ incident on w'. Let T' be the maximal subtree of T containing w' not containing e. Then $\mathscr{F}l_K(X_w)\cap X_{T'}\subseteq \mathscr{F}l_K(X_{w'})\cap X_{T'}$.

$(\mathscr{H}1)$ For all $w\in T$, we have $L' := L_{5.1.10}(K)$-coarsely Lipschitz retraction $\rho_w : X\to\mathscr{F}l_K(X_w)$ such that $\forall\, x\in\mathscr{F}l_K(X_w)$, $\pi_X(\rho_w(x)) = \pi_X(x)$ (see Proposition 5.1.10).

$(\mathscr{H}2)$ Let $w\in T$. For all $x\in\mathscr{F}l_K(X_w)$ there is a K-qi section lying in $\mathscr{F}l_K(X_w)\cap \pi^{-1}([w,\pi(x)])$ through x over $B_{[w,\pi(x)]}$.

$(\mathscr{H}3)$ Let $w\in T$. For $L\geq 2K$, there is $\eta(L):\mathbb{R}_{\geq 0}\to\mathbb{R}_{\geq 0}$ such that the inclusion $Fl_{KL}(X_w)\hookrightarrow X$ is $\eta(L)$-proper embedding (see Proposition 5.1.11).

$(\mathscr{H}4)$ Let $\mathscr{G} = \{\gamma : \gamma \text{ is a } (2KL'+L')\text{-qi section over } B_{[u,v]}\}$. Note that $\mathfrak{T}\neq\emptyset$ as $\mathscr{F}l_K(X_u)\cap X_v\neq\emptyset$. For $w\in[u,v]$, $b\in B_w$ let $H_{b,w} = \text{hull}\{\gamma(b):\gamma\in\mathfrak{T}\}\subseteq F_{b,w}$ and $H = \bigcup_{w\in[u,v],\, b\in B_w} H_{b,w}$. (Here quasiconvex hull is considered in the corresponding fiber.) Then by Lemma 2.4.14, H is κ-metric bundle over $B_{[u,v]}$ where $\kappa = K_{2.4.14}(2KL'+L')\geq 2KL'+L'$. Now we consider flow of H with parameters κ,κ (see Definition 5.1.8). According to our notation (see 5.1.14), we have $\mathscr{F}l_{\kappa^{(1)}}(H)$ and it also satisfies the following. Let $w\in T$ and T_{uvw} be the tripod with vertices u,v,w. Since $\kappa\geq 2KL'+L'$, by Lemma 5.1.13, we have that for any $(2KL'+L')$-qi section γ over $B_{T_{uvw}}$, $\gamma\subseteq\mathscr{F}l_{\kappa^{(1)}}(H)$.

By Notation 5.1.14, we also have flow spaces $\mathscr{F}l_{\kappa^{(2)}}(X_u)$ containing $\mathscr{F}l_K(X_u)\cup \mathscr{F}l_{\kappa^{(1)}}(H)$ and $\mathscr{F}l_{\kappa^{(2)}}(X_v)$ containing $\mathscr{F}l_K(X_v)\cup\mathscr{F}l_{\kappa^{(1)}}(H)$.

$(\mathscr{H}5)$ Let R_0 be large enough so that $Fl_{\kappa^{(2)}R_0}(X_u)$ are δ-hyperbolic for some $\delta\geq 0$ (see Theorem 5.3.16).

$(\mathscr{H}6)$ Since flow spaces are semicontinuous family, for all $L\geq max\{2\kappa^{(1)}, 2\delta_0+1\}$ there is $\bar{L}(L)$ such that the inclusions $Fl_{KL}(X_u)\to X$, $Fl_{KL}(X_v)\to X$ and $Fl_{\kappa^{(1)}L}(H)\to X$ are $\bar{L}(L)$-qi embedding (see Proposition 5.1.5).

We know that uniform neighborhood of flow spaces are uniformly properly embedded in the total space (see ($\mathscr{H}3$)). In the following proposition, we prove the same for the union of two intersecting flow spaces.

Proposition 5.4.1. *Let $k_{5.4.1} = 2(L')^2(2K+1)+L'$. For all $L \geq M_{k_{5.4.1}}(\geq 2K)$ there exists $\eta_{5.4.1} = \eta_{5.4.1}(K,L) : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ such that the inclusion $N_L(\mathscr{F}l_K(X_u)\cup \mathscr{F}l_K(X_v)) \to X$ is $\eta_{5.4.1}$-proper embedding.*

Proof. Our proof goes in the same methodology as in the book [9] for trees of metric spaces (see [9, Subsection 6.1.1]). We denote the induced path metric on $Fl_{KL}(X_u)\cup F_{KL}(X_v)$ by d'. We divide the proof by reducing the tree T to intervals and the general tree in the following three cases. We first prove in all the cases that for $r\in\mathbb{R}_{\geq 0}$ and $x,y\in \mathscr{F}l_K(X_u)\cup\mathscr{F}l_K(X_v)$ with $d_X(x,y)=r$, $d'(x,y)$ is bounded in terms of r. In the end, we prove for the points in $Fl_{KL}(X_u)\cup F_{KL}(X_v)$.

Let $x,y\in\mathscr{F}l_K(X_u)\cup\mathscr{F}l_K(X_v)$ such that $d_X(x,y)=r$. Suppose $\pi(x)=u',\pi(y)=v',\pi_X(x)=\bar{x},\ \pi_X(y)=\bar{y}$. As $L\geq 2K$, we may assume that $x\in\mathscr{F}l_K(X_u)\setminus\mathscr{F}l_K(X_v)$ and $y\in\mathscr{F}l_K(X_v)\setminus\mathscr{F}l_K(X_u)$, otherwise, by ($\mathscr{H}3$), $d'(x,y)\leq\eta(L)(r)$.

Case 1: We first assume that $T=[u,v]$. Then $u'\neq v$, otherwise, $x\in X_v\subseteq \mathscr{F}l_K(X_v)$. Also $v'\neq u$, otherwise, $y\in X_u\subseteq\mathscr{F}l_K(X_u)$. Depending on positions of u',v',x and y, we consider following subcases.

Subcase (1A): Suppose $u'=v'$ and $x=\rho_u(y)$ (see ($\mathscr{H}1$)). Consider a K-qi section, γ_y over $B_{[v',v]}$ through y in $\mathscr{F}l_K(X_v)$ (see ($\mathscr{H}2$)). Since ρ_u is L'-coarsely Lipschitz retraction (see ($\mathscr{H}1$)) and $\mathscr{F}l_K(X_u)\cap X_v\neq\emptyset$, so applying ρ_u on γ_y, we get a $(2KL'+L')$-qi section, say, $\bar{\gamma}_y$ in $\mathscr{F}l_K(X_u)$ over $B_{[v',v]}$ such that $x=\rho_u(y)=\bar{\gamma}_y(\bar{y})$. Let b be the nearest point projection of $\bar{x}$ on B_v (note that such b exists as $v'=u'\neq v$). Applying ρ_v (see ($\mathscr{H}1$)) on $\bar{\gamma}_y$ and we get a $2.(2KL'+L')L'+L'=k_{5.4.1}$-qi section, say, $\bar{\bar{\gamma}}_y$ in $\mathscr{F}l_K(X_v)$ over $B_{[v',v]}$. Note that $\bar{\gamma}_y(b)=\bar{\bar{\gamma}}_y(b)$ (as $\bar{\gamma}_y(b)\in X_v$). Let $\bar{\bar{\gamma}}_y(\bar{x})=y'$. Then $\rho_v(x)=y'$ and since $\rho_v(y)=y$, $d_X(y',y)=d_X(\rho_v(x),\rho_v(y))\leq L'd_X(x,y)+L'\leq L'(r+1)$. So $d_X(x,y')\leq d_X(x,y)+d_X(y,y')\leq r(L'+1)+L'$ and $d^f(x,y')\leq\phi(r(L'+1)+L')$, where the fibers are ϕ-properly embedded in total space. Here we have two $k_{5.4.1}$-qi sections $\bar{\gamma}_y$ and $\bar{\bar{\gamma}}_y$ over $B_{[v',v]}$ such that $\bar{\gamma}_y(b)=\bar{\bar{\gamma}}_y(b)$ and $d^f(\bar{\gamma}_y(\bar{x}),\bar{\bar{\gamma}}_y(\bar{x}))=d^f(x,y')\leq\phi(r(L'+1)+L')$. Let a be the point on $[\bar{x},b]_B$ closest to $\bar{x}$ such that $d^f(\bar{\gamma}_y(a),\bar{\bar{\gamma}}_y(a))\leq M_{k_{5.4.1}}$. Since $L\geq M_{k_{5.4.1}}$, $d'(\bar{\gamma}_y(a),\bar{\bar{\gamma}}_y(a))\leq M_{k_{5.4.1}}$. Again, the tree of metric bundles (X,B,T) satisfies flaring condition, so by Lemma 2.4.7 (1), $d_B(\bar{x},a)\leq\tau_{2.4.7}(k_{5.4.1},C)$, where $C=max\{M_{k_{5.4.1}},\phi(r(L'+1)+L')\}$. Let $C_1=\tau_{2.4.7}(k_{5.4.1},C)$. Then by taking lifts of geodesic $[\bar{x},a]_B$ in $\bar{\gamma}_y$ and $\bar{\bar{\gamma}}_y$ (see Lemma 2.4.12 (3)), we get, $d'(x,\bar{\gamma}_y(a))\leq 2k_{5.4.1}C_1$ and $d'(y,\bar{\bar{\gamma}}_y(a))\leq 2k_{5.4.1}C_1$. Hence

$d'(x,y) \leq d'(x,\bar{\gamma}_y(a)) + d'(\bar{\gamma}_y(a),\bar{\bar{\gamma}}_y(a)) + d'(\bar{\bar{\gamma}}_y(a),y) \leq 4k_{5.4.1}C_1 + M_{k_{5.4.1}} =: \eta_1(r)$ for some $\eta_1 : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$.

Subcase (1B): Let $y' = \rho_u(y)$. In this subcase, y' need not be equal to x. Since $\rho_u(x) = x$, $d_X(x,y') = d_X(\rho_u(x),\rho_u(y)) \leq L'r + L' = L'(r+1)$. So $d_X(y',y) \leq d_X(y',x) + d_X(x,y) \leq L'(r+1) + r$. Since $L \geq 2K$, $Fl_{KL}(X_u)$ is $\eta(L)$-properly embedded in X (see $(\mathscr{H}3)$). So $d'(x,y') \leq \eta(L)(L'(r+1))$. Note that $\pi_X(y') = \pi_X(y)$ (as $\mathscr{F}l_K(X_u \cap X_v \neq \emptyset)$ and $y' \in \mathscr{F}l_K(X_u)$. Then by *Subcase* (1*A*), $d'(y,y') \leq \eta_1(L'(r+1)+r)$. Hence $d'(x,y) \leq d'(x,y') + d'(y',y) \leq \eta(L)(L'(r+1)) + \eta_1(L'(r+1)+r)$.

We assume $\boldsymbol{\zeta_1(r)} := max\{\eta_1(r), \eta(L)(L'(r+1)) + \eta_1(L'(r+1)+r)\}$, maximum distortion in this Case 1 for some $\zeta_1 : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$.

Case 2: We now assume that $T = [w,w'] \supsetneq [u,v]$ such that u is closest to w. Then $v' \notin [w,u]$, otherwise, by $(\mathscr{H}0)$, $y \in \mathscr{F}l_K(X_u)$. Also, $u' \notin [v,w']$, otherwise, by $(\mathscr{H}0)$, $x \in \mathscr{F}l_K(X_v)$. We consider the following subcases depending on the position of u' and v'.

Let $S = [u,v]$ and $X_S := \pi^{-1}(S)$. Let d'' denote the induced path metric on L-neighborhood (in X_S-metric) of $(\mathscr{F}l_K(X_u) \cup \mathscr{F}l_K(X_v)) \cap X_S$ inside X_S. We note that the restriction $\pi_X|_{X_S} : X_S \to B_S$ also satisfies flaring condition (Remark 2.4.8 (b)).

Subcase (2A): Suppose $u' \in [w,u]$ and $v' \in [v,w']$. Let b' be the nearest point projection of $\bar{x}$ on B_u and b'' be that of $\bar{y}$ on B_v. Then $d_B(\bar{x},\bar{y}) \leq d_X(x,y) \leq r$ implies $d_B(\bar{x},b') \leq r$, $d_B(b',b'') \leq r$ and $d_B(b'',\bar{y}) \leq r$. Let γ_x be K-qi lift through x of geodesic $[\bar{x},b']_B$ in $\mathscr{F}l_K(X_u)$ and γ_y be that through y of $[b'',\bar{y}]_B$ in $\mathscr{F}l_K(X_v)$ (see $(\mathscr{H}2)$). Let $\gamma_x(b') = x'$ and $\gamma_y(b'') = y'$. Then $d'(x,x') \leq 2Kr$ and $d'(y,y') \leq 2Kr$ (see Lemma 2.4.12 (3)). So by triangle inequality, $d_X(x',y') \leq 4Kr + r$, and that implies $d_{X_S}(x',y') \leq \eta_{2.4.3}(4Kr+r)$ (see Proposition 2.4.3). Then by Case 1, $d''(x',y') \leq \zeta_1(\eta_{2.4.3}(4Kr+r))$. Therefore, $d'(x,y) \leq d'(x,x') + d''(x',y') + d'(y',y) \leq 4Kr + \zeta_1(\eta_{2.4.3}(4Kr+r))$.

Subcase (2B): Suppose $u' \in (u,v)$ and $v' \in [v,w']$. Let b'' be the nearest point projection of $\bar{y}$ on B_v. Then $d_B(\bar{x},\bar{y}) \leq d_X(x,y) \leq r$ implies $d_B(\bar{y},b'') \leq r$ (as $u' \in (u,v)$). Let γ_y be a K-qi lift of the geodesic $[b'',\bar{y}]_B$ in $\mathscr{F}l_K(X_v)$ through y (see $(\mathscr{H}2)$) and let $\gamma_y(b'') = y'$. Then $d'(y,y') \leq 2Kr$ (see Lemma 2.4.12 (3)). Again, $d_X(y',x) \leq d_X(y',y) + d_X(y,x) \leq 2Kr + r$. So by Proposition 2.4.3, $d_{X_S}(y',x) \leq \eta_{2.4.3}(2Kr+r)$. Hence by *Subcase* (1*A*), $d''(x,y') \leq \zeta_1(\eta_{2.4.3}(2Kr+r))$. So $d'(x,y) \leq d'(x,y') + d'(y',y) \leq d''(x,y') + d'(y',y) \leq \zeta_1(\eta_{2.4.3}(2Kr+r)) + 2Kr$.

Subcase (2C): Suppose $u' \in [w,u]$ and $v' \in (u,v)$. Then this is a symmetry of *Subcase* (2*B*). So $d'(x,y) \leq \zeta_1(\eta_{2.4.3}(2Kr+r)) + 2Kr$.

Subcase (2D): Finally, we assume that $u', v' \in (u, v)$. Then by Proposition 2.4.3, $d_{X_S}(x,y) \leq \eta_{2.4.3}(r)$. So by Case 1, $d''(x,y) \leq \zeta_1(\eta_{2.4.3}(r))$. Hence $d'(x,y) \leq d''(x,y) \leq \zeta_1(\eta_{2.4.3}(r))$.

We assume $\boldsymbol{\zeta_2(r)} := max\{4Kr + \zeta_1(\eta_{2.4.3}(4Kr+r)), \zeta_1(\eta_{2.4.3}(2Kr+r)) + 2Kr, \zeta_1(\eta_{2.4.3}(r))\}$, maximum distortion in this Case 2 for some $\zeta_2 : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$.

Case 3: Here we consider the general case, where T is any tree. Depending on the position of u, v, u' and v', we consider the following subcases.

Let S be an interval in T containing u, v and $X_S := \pi^{-1}(X_S)$. We denote the induced path metric on L-neighborhood (in X_S-metric) of $(\mathscr{F}l_K(X_u) \cup \mathscr{F}l_K(X_v)) \cap X_S$ inside X_S by d''. We will use this notation below. We note that the restriction $\pi_X|_{X_S} : X_S \to B_S$ also satisfies flaring condition (see Remark 2.4.8 (b)).

Subcase (3A): Suppose u, v, u' and v' lie on an interval in T. We fix one such interval S in T containing u, v, u', v'. So $d_{X_S}(x,y) \leq \eta_{2.4.3}(r)$ (by Proposition 2.4.3). Now we restrict the flow spaces to X_S. Then by Case 2, $d''(x,y) \leq \zeta_2(\eta_{2.4.3}(r))$. So $d'(x,y) \leq d''(x,y) \leq \zeta_2(\eta_{2.4.3}(r))$.

Now we consider the subcases when all of u, v, u' and v' do not lie on an interval.

Subcase (3B): Suppose there is no interval containing u, v that contains both u', v'; but there is an interval containing u, v which contains one of u', v'. We give a proof when an interval containing u, v also contains u', and leave the other case because its involves only a change in indices. We fix one such interval S in T containing u, v and u'. Let t be the nearest point projection of v' on S in d_T-metric and b'' be that of $\bar{y}$ on B_t in d_B-metric. Since $d_X(x,y) \leq r$, then $d_B(\bar{y}, b'') \leq d_B(\bar{y}, \bar{x}) \leq r$. Let γ_y be a K-qi lift of the geodesic $[b'', \bar{y}]_B$ through y in $\mathscr{F}l_K(X_v)$ (see $(\mathscr{H}2)$). Suppose $\gamma_y(b'') = y'$. Then $d'(y,y') \leq 2Kr$ (see Lemma 2.4.12 (3)). Again $d_X(y',x) \leq d_X(y',y) + d_X(y,x) \leq 2Kr + r$, and so by Proposition 2.4.3, $d_{X_S}(y',x) \leq \eta_{2.4.3}(2Kr+r)$. Now we restrict the flow spaces to X_S. Hence by Case 2, $d''(y',x) \leq \zeta_2(\eta_{2.4.3}(2Kr+r))$. Therefore, $d'(x,y) \leq d'(x,y') + d'(y',y) \leq d''(x,y') + d'(y',y) \leq \zeta_2(\eta_{2.4.3}(2Kr+r)) + 2Kr$.

Subcase (3C): Suppose there is no interval containing u, v that contains either of u', v'. We fix $S = [u, v]$. Let t_1 and t_2 be the nearest point projections of u' and v' on S respectively. Then $t_1, t_2 \in (u, v)$, otherwise, it will land in Subcase $(3B)$. We divide the proof into two cases depending on whether t_1, t_2 are same or not.

(a) Suppose $t_1 \neq t_2$. Let b' be the nearest point projection of $\bar{x}$ on B_{t_1} and b'' be that of $\bar{y}$ on B_{t_2}. Since $d_B(\bar{x}, \bar{y}) \leq d_X(x,y) \leq r$, then $d_B(\bar{x}, b') \leq r$ and $d_B(\bar{y}, b'') \leq r$. Let γ_x be a K-qi lift of the geodesic $[\bar{x}, b']_B$ through x in $\mathscr{F}l_K(X_u)$ and γ_y be that of $[\bar{y}, b'']_B$ through y in $\mathscr{F}l_K(X_v)$ (see $(\mathscr{H}2)$). Let $x' = \gamma_x(b')$ and $y' = \gamma_y(b'')$. Then $d'(x,x') \leq 2Kr$ and $d'(y,y') \leq 2Kr$. So $d_X(x',y') \leq d_X(x',x) + d_X(x,y) + d_X(y,y') \leq 4Kr + r$. Then by Proposition 2.4.3, $d_{X_S}(x',y') \leq \eta_{2.4.3}(4Kr+r)$. Note that $x' \in \mathscr{F}l_K(X_u)$ and

$y' \in \mathscr{F}l_K(X_v)$. Now we restrict the flow spaces to $X_S = X_{[u,v]}$. Hence by Case 1, $d''(x',y') \leq \zeta_1(\eta_{2.4.3}(4Kr+r))$, and so $d'(x',y') \leq \zeta_1(\eta_{2.4.3}(4Kr+r))$. Therefore, $d'(x,y) \leq d'(x,x') + d'(x',y') + d'(y',y) \leq 4Kr + \zeta_1(\eta_{2.4.3}(4Kr+r))$.

(b) Suppose $t_1 = t_2 = t$ (say). Let s be the center of $\triangle(u',t,v')$ and $c \in [\bar{x},\bar{y}] \cap B_s$. Since $d_B(\bar{x},\bar{y}) \leq d_X(x,y) \leq r$, so $d_B(\bar{x},c) \leq r$ and $d_B(c,\bar{y}) \leq r$. Let γ_1 be a K-qi lift through x of the geodesic $[\bar{x},c]_B$ in $\mathscr{F}l_K(X_u)$ and γ_2 be that through y of the geodesic $[\bar{y},c]_B$ in $\mathscr{F}l_K(X_v)$ (see $(\mathscr{H}2)$). Let $x_1 = \gamma_1(c)$ and $y_1 = \gamma_2(c)$. Then $d'(x,x_1) \leq 2Kr$ and $d'(y,y_1) \leq 2Kr$.

Now we only need to show that $d'(x_1,y_1)$ is bounded in terms of r. We will apply the same trick as in Case 1. Let γ_y be a K-qi section over $B_{[s,v]}$ through y_1 in $\mathscr{F}l_K(X_v)$ (see $(\mathscr{H}2)$). Now we apply ρ_u (see $(\mathscr{H}1)$) on γ_y and get a $L'(2K+1)$-qi section, say, $\bar{\gamma}_y$ over $B_{[s,v]}$ in $\mathscr{F}l_K(X_u)$ (see Figure 5.4). (This is possible as $[s,v] \subseteq \pi(\mathscr{F}l_K(X_u))$.) By triangle inequality, $d_X(x_1,y_1) \leq 4Kr + r$. Let $\rho_u(y_1) = x_2$. Since $\rho_u(x_1) = x_1$, then $d_X(x_1,x_2) = d_X(\rho_u(x_1),\rho_u(y_1)) \leq L'd_X(x_1,y_1) + L' \leq L'(4Kr+r+1)$. Then $d_X(y_1,x_2) \leq d_X(y_1,x_1) + d_X(x_1,x_2) \leq 4Kr + r + L'(4Kr + r + 1) = (4Kr+r)(L'+1) + L'$. Again we apply ρ_v (see $(\mathscr{H}1)$) on $\bar{\gamma}_y$ and get a $k_{5.4.1}$-qi section, say, $\bar{\bar{\gamma}}_y$ over $B_{[s,v]}$ in $\mathscr{F}l_K(X_v)$ (see Figure 5.4). Let $\rho_v(x_2) = y_2$. Since $\rho_v(y_1) = y_1$, $d_X(y_1,y_2) \leq d_X(\rho_v(y_1),\rho_v(x_2)) \leq L'd_X(y_1,x_2) + L' \leq L_1$, where $L_1 = L'((4Kr+r)(L'+1)+L') + L'$. Then $d_X(x_2,y_2) \leq d_X(x_2,y_1) + d_X(y_1,y_2) \leq L_2$, where $L_2 = (4Kr + r(L'+1) + L' + L_1$. So $d^f(x_2,y_2) \leq \phi(L_2)$.

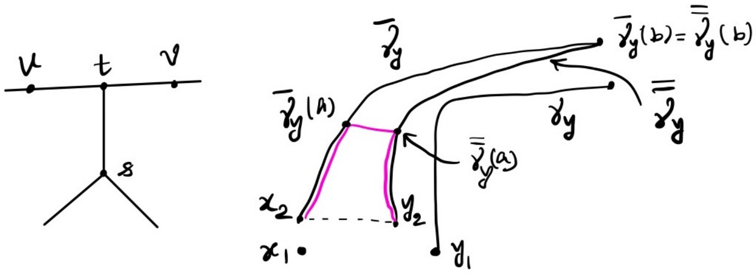


Figure 5.4

Note that $d_X(x_1,x_2) \leq L'(4Kr+r+1)$ and $d_X(y_1,y_2) \leq L_1$. Again $x_1,x_2 \in \mathscr{F}l_K(X_u)$ and $y_1,y_2 \in \mathscr{F}l_K(X_v)$, so by $(\mathscr{H}3)$, $d'(x_1,x_2) \leq \eta(L)(L'(4Kr+r+1))$ and $d'(y_1,y_2) \leq \eta(L)(L_1)$. So to get a bound on $d'(x_1,y_1)$, we need to get a bound on $d'(x_2,y_2)$; which we will show now.

Let b be the nearest point projection of c on B_v. Then $\bar{\gamma}_y(b) \in X_v$ and so $\bar{\gamma}_y(b) = \bar{\bar{\gamma}}_y(b)$. Note that $\bar{\gamma}_y$ and $\bar{\bar{\gamma}}_y$ are two $k_{5.4.1}$-qi sections over $B_{[s,v]}$ such that $d^f(\bar{\gamma}_y(c),\bar{\bar{\gamma}}_y(c)) = d^f(x_2,y_2) \leq \phi(L_2)$ and $\bar{\gamma}_y(b) = \bar{\bar{\gamma}}_y(b)$. Now we restrict the qi sections $\bar{\gamma}_y$ and $\bar{\bar{\gamma}}_y$ on the geodesic $[c,b]_B \subseteq B$. Let a be the point on $[c,b]$ closest to c such that $d^f(\bar{\gamma}_y(a),\bar{\bar{\gamma}}_y(a))\leq M_{k_{5.4.1}}$. Since the tree of metric bundles (X,B,T) satisfies flaring condition, by Lemma 2.4.7 (1), $d_B(c,a)\leq\tau_{2.4.7}(k_{5.4.1},D)$, where $D=max\{M_{k_{5.4.1}},\phi(L_2)\}$. Let $D_1=\tau_{2.4.7}(k_{5.4.1},D)$. Then by taking $k_{5.4.1}$-qi lifts of the geodesic $[c,a]_B$ in $\bar{\gamma}_y$ and $\bar{\bar{\gamma}}_y$, we get, $d'(x_2,\bar{\gamma}_y(a))\leq 2k_{5.4.1}D_1$ and $d'(\bar{\bar{\gamma}}_y(a),y_2)\leq 2k_{5.4.1}D_1$ (note that $\bar{\gamma}_y(c)=x_2$, $\bar{\bar{\gamma}}_y(c)=y_2$). Again $L\geq M_{k_{5.4.1}}$ implies $d'(\bar{\gamma}_y(a),\bar{\bar{\gamma}}_y(a))\leq M_{k_{5.4.1}}$. Hence $d'(x_2,y_2)\leq 4k_{5.4.1}D_1+M_{k_{5.4.1}}$ (by triangle inequality).

Again by triangle inequality, $d'(x_1,y_1)\leq d'(x_1,x_2)+d'(x_2,y_2)+d'(y_2,y_1)\leq L_3$, where $L_3=\eta(L)(L'(4Kr+r+1))+4k_{5.4.1}D_1+M_{k_{5.4.1}}+\eta(L)(L_1)$. So $d'(x,y)\leq d'(x,x_1)+d'(x_1,y_1)+d'(y_1,y)\leq 4Kr+L_3$.

Let $\boldsymbol{\zeta_3(r)}:=max\{\zeta_2(\eta_{2.4.3}(r)),\ \zeta_2(\eta_{2.4.3}(2Kr+r))+2Kr,\ \zeta_1(\eta_{2.4.3}(4Kr+r))+4Kr,\ 4Kr+L_3\}$, maximum distortion in this Case 3 for some $\zeta_3:\mathbb{R}_{\geq 0}\to\mathbb{R}_{\geq 0}$.

Let $\zeta:\mathbb{R}_{\geq 0}\to\mathbb{R}_{\geq 0}$ such that $\zeta(r):=max\{\eta(L)(r),\zeta_1(r),\zeta_2(r),\zeta_3(r)\}$ for $r\in\mathbb{R}_{\geq 0}$. We have proved that if $x,y\in\mathscr{F}l_K(X_u)\cup\mathscr{F}l_K(X_v)$ are at most r-distance apart in the metric of X, then they are at most $\zeta(r)$-distance apart in the induced metric on $Fl_{KL}(X_u)\cup F_{KL}(X_v)$. Now we take points $x,y\in Fl_{KL}(X_u)\cup F_{KL}(X_v)$ such that $d_X(x,y)\leq r$ for $r\in\mathbb{R}_{\geq 0}$. Let $x_1,y_1\in\mathscr{F}l_K(X_u)\cup\mathscr{F}l_K(X_v)$ such that $d'(x,x_1)\leq L$ and $d'(y,y_1)\leq L$. Then $d_X(x,y)\leq d_X(x,x_1)+d_X(x_1,y_1)+d_X(y_1,y)\leq r+2L$. So $d'(x_1,y_1)\leq\zeta(r+2L)$. Hence $d'(x,y)\leq d'(x,x_1)+d'(x_1,y_1)+d'(y_1,y)\leq 2L+\zeta(r+2L)$.

Therefore, we can take $\eta_{5.4.1}:\mathbb{R}_{\geq 0}\to\mathbb{R}_{\geq 0}$ sending $r\mapsto\zeta(r+2L)+2L$. □

To show the hyperbolicity of $N_D(\mathscr{F}l_K(X_u)\cup\mathscr{F}l_K(X_v))$, we construct a bigger uniformly hyperbolic space $Y=Y_1\cup Y_2$ containing both $\mathscr{F}l_K(X_u)$ and $\mathscr{F}l_K(X_v)$ as uniformly quasiconvex subsets. As $\mathscr{F}l_K(X_u)\cap X_v\neq\emptyset$, so a uniform neighborhood of $\mathscr{F}l_K(X_u)\cup\mathscr{F}l_K(X_v)$ in Y is uniformly hyperbolic. Let $N'_D(y)$ denote D-neighborhood at $y\in Y$ in the path metric of Y. Next, we show that $N'_D(\mathscr{F}l_K(X_u)\cup\mathscr{F}l_K(X_v))\subseteq Y$ and $N_D(\mathscr{F}l_K(X_u)\cup\mathscr{F}l_K(X_v))\subseteq X$ are (uniformly) quasi-isometric, and that completes the proof.

Construction of the space Y: By $(\mathscr{H}5)$, $Fl_{\kappa^{(2)}R_0}(X_u)$ is δ-hyperbolic metric space. Also by $(\mathscr{H}6)$, $Fl_{KR_0}(X_u)$ is $\bar{L}(R_0)$-qi embedded in X and so is in $Fl_{\kappa^{(2)}R_0}(X_u)$. Then by Lemma 2.2.22 (1), there is K_1 depending on δ and $\bar{L}(R_0)$ such that $Fl_{KR_0}(X_u)$ is K_1-quasiconvex in $Fl_{\kappa^{(2)}R_0}(X_u)$. So $\mathscr{F}l_K(X_u)$ is (K_1+R_0)-quasiconvex in $Fl_{\kappa^{(2)}R_0}(X_u)$. Also by $(\mathscr{H}6)$, $Fl_{\kappa^{(1)}R_0}(H)$ is $\bar{L}(R_0)$-qi embedded in X and so is in $Fl_{\kappa^{(2)}R_0}(X_u)$. Then by Lemma 2.2.22 (1), there is K_2 depending on δ and $\bar{L}(R_0)$ such that $Fl_{\kappa^{(1)}R_0}(H)$ is K_2-quasiconvex in $Fl_{\kappa^{(2)}R_0}(X_u)$. So $\mathscr{F}l_{\kappa^{(1)}}(H)$ is (K_2+R_0)-quasiconvex in $Fl_{\kappa^{(2)}R_0}(X_u)$. Let $K_3=max\{K_1+R_0,K_2+R_0\}$. As $\mathscr{F}l_K(X_u)\cap\mathscr{F}l_{\kappa^{(1)}}(H)\neq\emptyset$, so $\mathscr{F}l_K(X_u)\cup\mathscr{F}l_{\kappa^{(1)}}(H)$ is $(K_3+\delta)$-quasiconvex in $Fl_{\kappa^{(2)}R_0}(X_u)$. Let $Y'_{1R}:=N'_R(\mathscr{F}l_K(X_u)\cup\mathscr{F}l_{\kappa^{(1)}}(H))$ be R-neighborhood of $\mathscr{F}l_K(X_u)\cup\mathscr{F}l_{\kappa^{(1)}}(H)\subseteq$

$Fl_{\kappa^{(2)}R_0}(X_u)$ in the induced path metric on $Fl_{\kappa^{(2)}R_0}(X_u)$ where

$$\boldsymbol{R} \;=\; max\{K_3+\delta+1,\; M_{k_{5.4.1}}\} \tag{5.4. 1}$$

Hence, by Lemma 2.2.23 (1), there is L_1 depending on δ, K_3 and R such that the inclusion $Y'_{1R} \hookrightarrow Fl_{\kappa^{(2)}R_0}(X_u)$ is L_1-qi embedding.

We fix this *R* for the rest of this section. Thus there is δ_1 depending on δ and L_1 such that Y'_{1R} is δ_1-hyperbolic with the induced path metric. Again, the inclusion $Fl_{\kappa^{(2)}R_0}(X_u) \hookrightarrow X$ is $\bar{L}(R_0)$-qi embedding (see ($\mathscr{H}6$)). Thus the inclusion $Y'_{1R} \hookrightarrow X$ is L_2-qi embedding for some L_2 depending on L_1 and $\bar{L}(R_0)$.

Let $Y'_{2R} := N'_R(\mathscr{F}l_K(X_v) \cup \mathscr{F}l_{\kappa^{(1)}}(H))$ be R-neighborhood of $\mathscr{F}l_K(X_v) \cup \mathscr{F}l_{\kappa^{(1)}}(H)$ $\subseteq Fl_{\kappa^{(2)}R_0}(X_v)$ in the induced path metric on $Fl_{\kappa^{(2)}R_0}(X_v)$. Then by similar argument, we can show that Y'_{2R} is δ_1-hyperbolic and the inclusion $Y'_{2R} \hookrightarrow Fl_{\kappa^{(2)}R_0}(X_v)$ is L_2-qi embedding.

We take $Y := Y_{1R} \cup Y_{2R}$ where $Y_{1R} := N_R(\mathscr{F}l_K(X_u) \cup \mathscr{F}l_{\kappa^{(1)}}(H)) \subseteq X$ and $Y_{2R} := N_R(\mathscr{F}l_K(X_v) \cup \mathscr{F}l_{\kappa^{(1)}}(H)) \subseteq X$. Note that these neighborhoods are considered in X.

Hyperbolicity of Y:

Lemma 5.4.2. *There is a uniform constant $\delta_{5.4.2}$ such that Y_{iR} is $\delta_{5.4.2}$-hyperbolic metric space with the induced path metric for $i = 1,2$.*

Proof. Since $Y'_{iR} \hookrightarrow X$ is L_2-qi embedding, so is the inclusion $Y'_{iR} \hookrightarrow Y_{iR}$. Also $Y'_{iR} \hookrightarrow Y_{iR}$ is R-coarsely surjective. Hence the inclusion $Y'_{iR} \hookrightarrow Y_{iR}$ is (L_2, L_2, R)-quasi-isometry for $i = 1,2$ (see Subsection 2.1). Since the hyperbolicity is quasi-isometry invariant and Y'_{iR} is δ_1-hyperbolic, Y_{iR} is $\delta_{5.4.2}$-hyperbolic for $i = 1,2$, where $\delta_{5.4.2}$ depends on δ_1, L_2, R. □

Lemma 5.4.3. *There exists a uniform function $\eta_{5.4.3} : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ such that the inclusion $Y_{iR} \hookrightarrow X$ is $\eta_{5.4.3}$-proper embedding for $i = 1,2$.*

Proof. We prove it only for $i = 1$ as the proof for $i = 2$ is similar. We denote the induced path metric on Y_{1R} and Y'_{1R} by d_1 and d'_1 respectively. Let $x, y \in Y_{1R}$ such that $d_X(x,y) = r$ for $r \in \mathbb{R}_{\geq 0}$. We take points $x_1, y_1 \in \mathscr{F}l_K(X_u) \cup \mathscr{F}l_{\kappa^{(1)}}(H) \subseteq Y'_{1R}$ such that $d_1(x,x_1) \leq R,\ d_1(y,y_1) \leq R$. So $d_X(x_1,y_1) \leq r+2R$. Since Y'_{1R} is L_2-qi embedded in X, then $d'_1(x_1,y_1) \leq (r+2R)L_2 + L_2^2$. Since $Y'_{1R} \subseteq Y_{1R}$ and so $d_1(x,y) \leq d_1(x,x_1) + d'_1(x_1,y_1) + d_1(y_1,y) \leq (r+2R)L_2 + L_2^2 + 2R =: \eta_{5.4.3}(r)$. □

Lemma 5.4.4. *Let d_i denote the induced path metric on Y_{iR} for $i = 1,2$. There is a uniform constant $D_{5.4.4}$ such that $Y_{1R} \cap Y_{2R} \subseteq N^i_{D_{5.4.4}}(Y_0)$, where $N^i_{D_{5.4.4}}(Y_0)$ denotes the $D_{5.4.4}$-neighborhood of Y_0 in d_i-metric.*

Proof. Let $x \in Y_{1R}\cap Y_{2R}$. Then there exist $x_1 \in \mathscr{F}l_K(X_u)\cup\mathscr{F}l_{\kappa^{(1)}}(H)$ and $x_2 \in \mathscr{F}l_K(X_v)\cup\mathscr{F}l_{\kappa^{(1)}}(H)$ such that $d_i(x,x_i)\le R,\ i=1,2$. So $d_X(x_1,x_2)\le 2R$. Without loss of generality, we assume that $x_1\in\mathscr{F}l_K(X_u)\setminus\mathscr{F}l_{\kappa^{(1)}}(H),\ x_2\in\mathscr{F}l_K(X_v)\setminus\mathscr{F}l_{\kappa^{(1)}}(H)$, otherwise, $x\in Y_0:=Fl_{\kappa^{(1)}R}(H)$. Let $\pi_X(x_i)=\bar{x}_i,\ \pi(x_i)=t_i$ for $i=1,2$. Let w_i be the nearest point projection of t_i on $[u,v], i=1,2$. We consider the following two cases, depending on whether $w_1=w_2$ or $w_1\neq w_2$.

Case 1: Suppose $w_1\neq w_2$. Let $\bar{y}_i$ be the nearest point projection of $\bar{x}_i$ on B_{w_i} for $i=1,2$. Then $d_B(\bar{x}_1,\bar{x}_2)\le d_X(x_1,x_2)\le 2R$ implies $d_B(\bar{x}_i,\bar{y}_i)\le 2R$ for $i=1,2$. Let γ_{x_1} be a K-qi section through x_1 inside $\mathscr{F}l_K(X_u)$ over $B_{[t_1,u]}$ and γ_{x_2} be that through x_2 inside $\mathscr{F}l_K(X_v)$ over $B_{[t_2,v]}$ (see $(\mathscr{H}2)$). Suppose $\gamma_{x_i}(\bar{y}_i)=y_i,\ i=1,2$. Then by taking lift of the geodesic $[\bar{x}_i,\bar{y}_i]_B$ in γ_{x_i}, we get, $d_i(x_i,y_i)\le 2K.2R=4KR$ (see Lemma 2.4.12 (3)) for $i=1,2$. Now we restrict the K-qi section γ_{x_2} over $B_{[w_2,v]}$ and apply ρ_u (see $(\mathscr{H}1)$) on this restriction of γ_{x_2} over $B_{[w_2,v]}$. We set this projection as $\bar{\gamma}_2$. Note that $B_{[w_2,v]}\subseteq\pi(\mathscr{F}l_K(X_u))$, then $\bar{\gamma}_2$ is a $(2KL'+L')$-qi section over $B_{[w_2,v]}$ inside $\mathscr{F}l_K(X_u)$. As $\mathscr{F}l_K(X_u\cap X_v\neq\emptyset$, then we can extend $\bar{\gamma}_2$ to a $(2KL'+L')$-qi section over $B_{[u,v]}$. Then in particular, we have, $\bar{\gamma}_2\subseteq H$ (see $(\mathscr{H}4)$). Again, $d_X(y_1,y_2)\le d_X(y_1,x_1)+d_X(x_1,x_2)+d_X(x_2,y_2)\le 8KR+2R$. Note that $\rho_u(y_1)=y_1$ and $\rho_u(y_2)=\bar{\gamma}_2(\bar{y}_2)$. Since ρ_u is L'-coaresly Lipschitz retraction (see $(\mathscr{H}1)$), $d_X(y_1,\bar{\gamma}_2(\bar{y}_2))\le L'd_X(y_1,y_2)+L'\le L'(8KR+2R)+L'$. Since $y_1,\bar{\gamma}_2(\bar{y}_2)\in\mathscr{F}l_K(X_u)\subseteq Y_{1R}$, by Lemma 5.4.3, $d_1(y_1,\bar{\gamma}_2(\bar{y}_2))\le\eta_{5.4.3}(L'(8KR+2R)+L')$. Now $\bar{\gamma}_2\subseteq H$ implies $d_1(x,Y_0)\le d_1(x,\bar{\gamma}_2(\bar{y}_2))\le d_1(x,x_1)+d_1(x_1,y_1)+d_1(y_1,\bar{\gamma}_2(\bar{y}_2))\le R+4KR+\eta_{5.4.3}(L'(8KR+2R)+L')$.

Again, $d_X(x_2,\bar{\gamma}_2(\bar{y}_2))\le d_X(x_2,x_1)+d_X(x_1,y_1)+d_X(y_1,\bar{\gamma}_2(\bar{y}_2))\le 2R+4KR+L'(8KR+2R)+L'$. Since $x_2\in\mathscr{F}l_K(X_v)\subseteq Y_{2r}$ and $\bar{\gamma}_2\subseteq H\subseteq Y_{2r}$, so by Lemma 5.4.3, $d_2(x_2,\bar{\gamma}_2(\bar{y}_2))\le\eta_{5.4.3}(2R+4KR+L'(8KR+2R)+L')$. Thus $d_2(x,Y_0)\le d_2(x,x_2)+d_2(x_2,\bar{\gamma}_2(\bar{y}_2))\le R+\eta_{5.4.3}(2R+4KR+L'(8KR+2R)+L')$.

Case 2: Suppose $w_1=w_2$. Let w be the center of the tripod $\triangle(t_1,t_2,w_1)$. Suppose $\bar{y}_i$ is the nearest point projection of $\bar{x}_i$ on B_w for $i=1,2$. Let γ_{x_1} be a K-qi section through x_1 inside $\mathscr{F}l_K(X_u)$ over $B_{[t_1,u]}$ and γ_{x_2} be that through x_2 inside $\mathscr{F}l_K(X_v)$ over $B_{[t_2,v]}$ (see $(\mathscr{H}2)$). Again $d_B(\bar{x}_1,\bar{x}_2)\le 2R$ implies $d_B(\bar{x}_i,\bar{y}_i)\le 2R$ for $i=1,2$. Let $\gamma_{x_i}(\bar{y}_i)=y_i,\ i=1,2$. Then by taking lift of the geodesic $[\bar{x}_i,\bar{y}_i]_B$ in γ_{x_i}, we have $d_i(x_i,y_i)\le 2K.2R=4KR$ (see Lemma 2.4.12 (3)). Now let us restrict the K-qi section, γ_{x_2}, over $B_{[w,v]}$ and apply ρ_u (see $(\mathscr{H}1)$) on this restriction of γ_{x_2} over $B_{[w,v]}$. We denote the image under ρ_u by γ_2. Since $B_{[w,v]}\subseteq\pi(\mathscr{F}l_K(X_u))$, γ_2 is a $(2KL'+L')$-qi section over $B_{[w,v]}$. Let T_{uvw} is the tripod in T with vertices u,v,w and $B_{T_{uvw}}:=\pi_B^{-1}(T_{uvw})$. As $\mathscr{F}l_K(X_u)\cap X_v\neq\emptyset$, we can extend γ_2 to a $(2KL'+L')$-qi section over $B_{T_{uvw}}$. Then in particular, we have $\gamma_2\subseteq\mathscr{F}l_{\kappa^{(1)}}(H)$ (see $(\mathscr{H}4)$). Now we apply line by line argument as in Case 1. Note that $d_X(y_1,y_2) \leq d_X(y_1,x_1)+d_X(x_1,x_2)+d_X(x_2,y_2) \leq 8KR+2R$ and $\rho_u(y_1)=y_1$, $\rho_u(y_2)=\gamma_2(\bar{y}_2)$. So $d_X(y_1,\gamma_2(\bar{y}_2)) \leq L'd_X(y_1,y_2)+L' \leq L'(8KR+2R)+L'$. Since $y_1,\gamma_2(\bar{y}_2) \in \mathscr{F}l_K(X_u \subseteq Y_{1R}$, by Lemma 5.4.3, $d_1(y_1,\gamma_2(\bar{y}_2)) \leq \eta_{5.4.3}(L'(8KR+2R)+L')$. Again, we have, $\gamma_2 \subseteq \mathscr{F}l_{\kappa^{(1)}}(H)$, so $d_1(x,Y_0) \leq d_1(x,x_1)+d_1(x_1,y_1)+d_1(y_1,\gamma_2(\bar{y}_2)) \leq R+4KR+\eta_{5.4.3}(L'(8KR+2R)+L')$.

Again, $d_X(x_2,\gamma_2(\bar{y}_2)) \leq d_X(x_2,x_1)+d_X(x_1,y_1)+d_X(y_1,\gamma_2(\bar{y}_2)) \leq 2R+4KR+L'(8KR+2R)+L'$. Since $x_2 \in \mathscr{F}l_K(X_v) \subseteq Y_{2R}$ and $\gamma_2 \subseteq \mathscr{F}l_{\kappa^{(1)}}(H) \subseteq Y_{2R}$, so by Lemma 5.4.3, $d_2(x_2,\gamma_2(\bar{y}_2)) \leq \eta_{5.4.3}(2R+4KR+L'(8KR+2R)+L')$. Thus $d_2(x,Y_0) \leq d_2(x,x_2)+d_2(x_2,\gamma_2(\bar{y}_2)) \leq R+\eta_{5.4.3}(2R+4KR+L'(8KR+2R)+L')$.

Therefore, we can take $D_{5.4.4}=max\{R+4KR+\eta_{5.4.3}(L'(8KR+2R)+L'), R+\eta_{5.4.3}(2R+4KR+L'(8KR+2R)+L')\}$. □

Lemma 5.4.5. *There is $\delta_{5.4.5}=\delta_{5.4.5}(R)$ such that Y is $\delta_{5.4.5}$-hyperbolic metric space.*

Proof. We verify all the conditions of Proposition 2.2.7 for $n=2$ (see Remark 2.2.8). Note that $Y=Y_{1R}\cup Y_{2R}$.

(1) Y_{iR}, $i=1,2$ are $\delta_{5.4.2}$-hyperbolic.

(2) Y_0 is $\bar{L}(R)$-qi embedded in X (see ($\mathscr{H}6$)), so is in both Y_{1R} and Y_{1R}. Again $Y_{1R}\cap Y_{1R} \subseteq N_{D_{5.4.4}}(Y_0)$ (see Lemma 5.4.4), so by Lemma 2.1.4, the inclusion $Y_{1R}\cap Y_{1R} \hookrightarrow Y_{iR}$ is $L_{2.1.4}(\bar{L}(R),D_{5.4.4})$-qi embedding for $i=1,2$.

Therefore, Y is $\delta_{5.4.5}:=\delta_{2.2.8}(\delta_{5.4.2},L_{2.1.4}(\bar{L}(R),D_{5.4.4}),1)$-hyperbolic. □

Lemma 5.4.6. *The inclusion $Y\hookrightarrow X$ is $\eta_{5.4.6}=\eta_{5.4.6}(R)$-proper embedding for some uniform function $\eta_{5.4.6}:\mathbb{R}_{\geq 0}\to\mathbb{R}_{\geq 0}$.*

Proof. Let $r\in\mathbb{R}_{\geq 0}$ and $x,y\in Y$ such that $d_X(x,y)\leq r$. Then $\exists\, x_1,y_1 \in \mathscr{F}l_K(X_u)\cup\mathscr{F}l_{\kappa^{(1)}}(H)\cup\mathscr{F}l_K(X_v)$ such that $d_Y(x,x_1)\leq R$ and $d_Y(y,y_1)\leq R$. So by triangle inequality, $d_X(x_1,y_1)\leq r+2R$. Without loss of generality, we may assume that $x_1\in\mathscr{F}l_K(X_u)$ and $y_1\in\mathscr{F}l_K(X_v)$. Otherwise, by Lemma 5.4.3, $d_Y(x_1,y_1)\leq\eta_{5.4.3}(r+2R)$. Again $R\geq M_{k_{5.4.1}}$, and thus by Proposition 5.4.1, $d_Y(x_1,y_1)\leq\eta_{5.4.1}(K,R)(r+2R)$. Therefore, in either case, $d_Y(x,y)\leq 2R+max\{\eta_{5.4.3}(r+2R),\eta_{5.4.1}(K,R)(r+2R)\}=:\eta_{5.4.6}(r)$. □

Proposition 5.4.7. *There exists a constant $D_{5.4.7}$ such that for all $D\geq D_{5.4.7}$ we have $\delta_{5.4.7}=\delta_{5.4.7}(D)$ for which $N_D(\mathscr{F}l_K(X_u)\cup\mathscr{F}l_K(X_v))$ is $\delta_{5.4.7}$-hyperbolic metric space with the induced path metric from X.*

Proof. In the proof, we define $D_{5.4.7}$. For $D \geq D_{5.4.7}$, we denote the induced path metric on $N_D(\mathscr{F}l_K(X_u) \cup \mathscr{F}l_K(X_v))$ by $\bar{d}$. By $(\mathscr{H}6)$, $Fl_{KR}(X_u)$ is $\bar{L}(R)$-qi embedded in X and so is in Y. Hence $Fl_{KR}(X_u)$ is K_1-quasiconvex in Y for some K_1 depending on $\delta_{5.4.5}(R)$ and $\bar{L}(R)$. So $\mathscr{F}l_K(X_u$ is K_2-quasiconvex in Y, where $K_2 = K_1 + R$. Also, by the similar argument $\mathscr{F}l_K(X_v)$ is K_2-quasiconvex in Y. Since Y is $\delta_{5.4.5}(R)$-hyperbolic and $\mathscr{F}l_K(X_u) \cap \mathscr{F}l_K(X_v) \neq \emptyset$, so $\mathscr{F}l_K(X_u) \cup \mathscr{F}l_K(X_v)$ is $(K_2 + \delta_{5.4.5}(R))$-quasiconvex in Y. Let $N'_D(\mathscr{F}l_K(X_u) \cup \mathscr{F}l_K(X_v))$ be the D-neighborhood of $\mathscr{F}l_K(X_u) \cup \mathscr{F}l_K(X_v)$ (inside Y) in the path metric on Y. We set $D_{5.4.7} > K_2 + \delta_{5.4.5}(R) + 1$. Thus for $D \geq D_{5.4.7}$, (by Lemma 2.2.23 (1)) the inclusion $N'_D(\mathscr{F}l_K(X_u) \cup \mathscr{F}l_K(X_v)) \hookrightarrow Y$ is L_1-qi embedding, where $L_1 = L_{2.2.23}(\delta_{5.4.5}(R), K_2 + \delta_{5.4.5}(R), D)$. Therefore, $N'_D(\mathscr{F}l_K(X_u) \cup \mathscr{F}l_K(X_v))$ is δ_1-hyperbolic, where δ_1 depends on $\delta_{5.4.5}(R)$ and L_1. Now we show that the subset $N'_D(\mathscr{F}l_K(X_u) \cup \mathscr{F}l_K(X_v)) \subseteq N_D(\mathscr{F}l_K(X_u) \cup \mathscr{F}l_K(X_v))$ satisfies all the conditions of Proposition 2.2.6. Note that $N'_D(\mathscr{F}l_K(X_u) \cup \mathscr{F}l_K(X_v))$ is a D-dense in $N_D(\mathscr{F}l_K(X_u) \cup \mathscr{F}l_K(X_v))$ in the $\bar{d}$-metric. For any pair (x,y) of distinct points in $N'_D(\mathscr{F}l_K(X_u) \cup \mathscr{F}l_K(X_v))$, we fix once and for all a geodesic path, say, $c(x,y)$ joining x and y in the δ_1-hyperbolic space $N'_D(\mathscr{F}l_K(X_u) \cup \mathscr{F}l_K(X_v))$. These paths serve as family of paths for Proposition 2.2.6. Then any triangle formed by these paths are δ_1-slim in the induced path metric of $N'_D(\mathscr{F}l_K(X_u) \cup \mathscr{F}l_K(X_v))$ and so is in $N_D(\mathscr{F}l_K(X_u) \cup \mathscr{F}l_K(X_v))$. Hence we are left to show the proper embedding of these paths in $N_D(\mathscr{F}l_K(X_u) \cup \mathscr{F}l_K(X_v))$. Indeed, suppose $x, y \in N'_D(\mathscr{F}l_K(X_u) \cup \mathscr{F}l_K(X_v))$ such that $\bar{d}(x,y) \leq r$ for some $r \in \mathbb{R}_{\geq 0}$. Then $d_X(x,y) \leq r$ and by Lemma 5.4.6, $d_Y(x,y) \leq \eta_{5.4.6}(r)$. Since $N'_D(\mathscr{F}l_K(X_u) \cup \mathscr{F}l_K(X_v))$ is L_1-qi embedded in Y, the path $c(x,y)$ is η_1-properly embedded, where $\eta)1 : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ sending $r \mapsto \eta_{5.4.6}(r)L_1 + L_1^2$.

Therefore, $N_D(\mathscr{F}l_K(X_u) \cup \mathscr{F}l_K(X_v))$ is $\delta_{5.4.7}$-hyperbolic metric space with the induced path metric from X, where $\delta_{5.4.7} = \delta_{2.2.6}(\eta_1, \delta_1, D)$. □

5.5 Proof of Theorem 1.2.4

We think of the tree of metric bundles (X,B,T) as a tree of metric spaces $\pi : X \to T$ as explained in Remark 1.2.5. For a vertex $u \in T$, we take $\mathscr{M}(X_u) = \mathscr{F}l_K(X_u)$ as in Lemma 5.1.9. Now we show that it satisfies property $(\mathscr{P}0) - (\mathscr{P}4)$ of Chapter 4. Note that $(\mathscr{P}0)$ follows from the definition of $\mathscr{F}l_K(X_u)$. Again by Proposition 5.1.10, $L' = L_{5.1.10}(K)$ and we have $C = D_{2.2.13}(\delta'_0, L'_0)$ for $(\mathscr{P}1)$. Taking into account of Proposition 5.4.7, we set L_0 large enough so that for $L \geq L_0$, we can take $\eta'(L) = \eta_{5.4.1}(K,L)$ (by Proposition 5.4.1) for $(\mathscr{P}3)$ and $\delta(L) = \delta_{5.4.7}(L)$ (by

Proposition 5.4.7) for $(\mathscr{P}4)$. Finally, by Proposition 5.1.11, for $L \geq L_0$, we can take $\eta(L) = \eta_{5.1.11}(K,L)$ for $(\mathscr{P}2)$. □

5.6 Applications to complexes of groups

We refer to [23, Chapter III.C], [21] and [22] or Section 2.5 for basic notions of developable complexes of groups. All the groups we consider here are finitely generated.

The construction of a tree of metric bundles for a given complex of groups in the setup $\mathscr{C}$ explained in Introduction 1.2 follows from the idea of [22], [21]. We briefly discuss the same below.

Suppose $\mathscr{Y}$ is a finite connected simplicial complex and $\mathscr{G}(\mathscr{Y})$ is a developable complex of groups over $\mathscr{Y}$. Let G be the fundamental group of $\mathscr{G}(\mathscr{Y})$. As in [22] and more generally, [21, Theorem 3.4.1], we consider a cellular aspherical realization (see [21, Definition 3.3.4]) $\mathscr{X}$ of the complex of groups $\mathscr{G}(\mathscr{Y})$ with cellular map $p : \mathscr{X} \to \mathscr{Y}$. Note that $\mathscr{X}$ is constructed by gluing along the Eilenberg-Mac Lane complexes of the local groups of the complex of groups $\mathscr{G}(\mathscr{Y})$; where for each local group G_σ corresponding to a face σ of $\mathscr{Y}$, the 0-skeleton of $K(G_\sigma, 1)$ is a point x_σ and the 1-skeletons form wedge of circles coming from a finite generating set of G_σ. Now we consider the universal covering $\pi : \tilde{\mathscr{X}} \to \mathscr{X}$ with the standard CW-complex structure on $\tilde{\mathscr{X}}$ coming from $\mathscr{X}$. We identify G with the group of deck transformation of the covering map $p : \tilde{\mathscr{X}} \to \mathscr{X}$. Let $y \in \mathscr{Y}$ and σ be the face containing y in its interior, and a be the barycenter of σ. Now we collapse each connected component of $\{(p \circ \pi)^{-1}(y)\}$ to a point. Note that since $\mathscr{G}(\mathscr{Y})$ is developable, the inclusion $p^{-1}(a) \hookrightarrow \mathscr{X}$ is π_1-injective and hence $\{(p \circ \pi)^{-1}(y)\}$ are copies of universal cover of $\mathscr{X}_a := p^{-1}(a)$. We do this for all $y \in \mathscr{Y}$. Let $\mathscr{B}$ be the space we get after collapsing and $q : \tilde{\mathscr{X}} \to \mathscr{B}$ be the quotient map. There is a natural G-equivariant isomorphism between $\mathscr{B}$ and the universal cover of $\mathscr{G}(\mathscr{Y})$ as in Definition 2.5.16. Thus we get the quotient map $\bar{\pi} : \mathscr{B} \to \mathscr{B}/G = \mathscr{Y}$ and the following commutative diagram.

$$\begin{array}{ccc} \tilde{\mathscr{X}} & \xrightarrow{q} & \mathscr{B} \\ {\scriptstyle\pi}\downarrow & \curvearrowright & \downarrow{\scriptstyle\bar{\pi}} \\ \mathscr{X} & \xrightarrow{p} & \mathscr{Y} \end{array}$$

Figure 5.5

Let $X := \tilde{\mathscr{X}}^{(1)}$ and $B := \mathscr{B}^{(1)}$, where $Z^{(1)}$ denotes the 1-skeleton of a CW-complex Z. Assume that each edge in X and B has length 1. Put the length metric on X and B. Since the groups are finitely generated, by covering space argument, we have the following facts.

1. The restriction of the map q on X, $q|_X : X \to B$ is G-equivariant, surjective and 1-Lipschitz.

2. The action of G on X is proper and cocompact; and, G-action on B is cocompact (but not necessarily proper unless local groups are all finite).

3. There is an isomorphism of graphs $r : B/G \to \mathscr{Y}^{(1)}$ such that for all $\sigma_0 \in \mathscr{Y}^{(0)}$ and $a \in \{r^{-1}(\sigma_0)\}$, G_a (the stabilizer subgroup of $a \in B^{(0)}$) is conjugate to G_{σ_0} in G.

4. Let $a \in B^{(0)}$ and $F_a := (q|_X)^{-1}(a)$. Then the action of G_a (the stabilizer subgroup of a) on $F_a^{(0)}$ is transitive and on $F_a^{(1)}$ is uniformly cofinite. In particular, G_a is uniformly quasi-isometric to F_a (with the induced path metric from X).

5. Since a finitely generated subgroup of a finitely generated group is properly embedded (with respect to their finite generating sets), for all $a \in B^{(0)}$, the inclusion $F_a \hookrightarrow X$ is uniformly properly embedding, where $F_a := (q|_X)^{-1}(a)$.

Complexes of groups as explained in setup $\mathscr{C}$: Now suppose $\mathscr{G}(\mathscr{Y}, Y)$ is a complex of groups over $\mathscr{Y}$ as explained in Introduction 1.2. Note that for this discussion, we do not require the hypotheses of Problem 1.2.1 but those of Theorem 1.2.2. Then we have a natural graph of groups, say, $(\mathscr{G}, Y)$ over Y such that the vertex groups are $G_s := \pi_1(\mathscr{G}_s(\mathscr{Y}_s)), \forall\, s \in Y^{(0)}$ and the edge groups are G_e for $e = [u, v]$ joining two vertices $u, v \in \mathscr{Y}$ so that restriction of p_Y on e is injective. Note that the monomorphisms from edge groups to the corresponding vertex groups are the restriction of $\mathscr{G}(\mathscr{Y}, Y)$.

As a corollary of [23, Proposition 3.9, III.C], we have the following lemma.

Lemma 5.6.1. *The complex of groups $\mathscr{G}(\mathscr{Y}, Y)$ is developable.*

5.6.1 Trees of metric bundles coming from complexes of groups

Consider the above discussion on complexes of groups for $\mathscr{G}(\mathscr{Y}, Y)$ (setup $\mathscr{C}$). For now onward, we denote the restriction map $q|_X$ by π_X. Let $s \in Y^{(0)}$ and $G_s =$

$\pi_1(\mathcal{G}_s(\mathcal{Y}_s))$. Consider the corresponding graph of groups $(\mathcal{G}, Y)$ as explained above. Then note that X is the corresponding tree of metric spaces over the Base-Serre tree of the graph of groups $(\mathcal{G}, Y)$. Let T be the Base-Serre tree and $\pi : X \to T$ be the projection map. Again vertex spaces of X are acted (properly and cocompactly) upon by the conjugates of G_s in G, $s \in Y^{(0)}$. Now by condition (2) of setup $\mathcal{C}$ (see Introduction 1.2), it follows from [33, Section 3.3] that the vertex spaces of $\pi : X \to T$ are metric graph bundles with uniform parameters. For instance, if $s \in Y^{(0)}$, $g \in G$, $u = gG_s \in T^{(0)}$ and B_s is the 1-skeleton of the universal cover of $G_s(\mathcal{Y}_s)$, then $X_u := \pi^{-1}(u)$ is the metric graph bundle over $B_u = gB_s \subseteq B$ and the subgroup gG_sg^{-1} acts on X_u properly and cocompactly. Also, note that T is obtained by collapsing the universal cover of $\mathcal{G}_s(\mathcal{Y}_s)$ (for $s \in \mathcal{Y}^{(0)}$) in B and its G-translates to points. Let $\pi_B : B \to T$ be the projection map. The maps $\pi_X : X \to B$ and $\pi_B : B \to T$ are G-equivariant. Therefore, we have the following proposition.

Proposition 5.6.2. *Suppose G is the fundamental group of $\mathcal{G}(\mathcal{Y})$. Then there is a natural tree of metric bundles (X, B, T) and an action of G by isometries on both X and B such that the following hold.*

1. *The map π_X is G-equivariant.*
2. *The action of G on X is proper and cocompact; and, G-action on B is cocompact (but not necessarily proper unless all local groups are finite).*
3. *There is an isomorphism of graphs $r : B/G \to \mathcal{Y}^{(1)}$ such that for all $\sigma_0 \in \mathcal{Y}^{(0)}$ and $a \in \{r^{-1}(\sigma_0)\}$, G_a (the stabilizer subgroup of $a \in B^{(0)}$) is conjugate to G_{σ_0} in G.*
4. *Let $a \in B^{(0)}$ and $u = \pi_B(a)$, $F_{a,u} := \pi_X^{-1}(a) = q^{-1}(a)^{(1)}$. Then the action of G_a on $F_{a,u}^{(0)}$ is transitive and on $F_{a,u}^{(1)}$ is uniformly cofinite. In particular, if $\sigma_0 \in \mathcal{Y}^{(0)}$ and G_{σ_0} is hyperbolic, then for all $a \in \{r^{-1}(\sigma_0)\}$, $F_{a,u}$ is uniformly hyperbolic, where $u = \pi_B(a)$.*

Note that condition (2) of setup $\mathcal{C}$ (see Introduction 1.2) in Proposition 5.6.2 is necessary to get trees of metric bundles.

Now we are ready to prove the main application of Theorem 1.2.4.

Proof of Theorem 1.2.2 : It follows from Proposition 5.6.2 and Theorem 1.2.4. □

Proof of Corollary 1.2.7: It follows from Proposition 5.6.2 and Remark 1.2.6. □

Chapter 6

Further Questions

6.1 On Cannon-Thurston maps

In [9, Chapter 9], Kapovich and Sardar generalize the theorem of Mj-Pal ([40]) to a subtree of relatively hyperbolic spaces. This motivates to the following question.

Question 6.1.1. *Prove a result analogous to Theorem 1.1.6 in relatively hyperbolic setup.*

Suppose $\pi' : Y \to B$ is a hyperbolic metric bundle and A is a qi embedded subspace of B. In [33], Krishna and Sardar showed that the inclusion $(\pi')^{-1}(A) \hookrightarrow Y$ admits the CT map. Keeping these in mind, we have Question 6.1.2 below. This question put both the theorems of Kapvich-Sardar [9, Theorem 8.11] and Krishna-Sardar in a single frame.

Question 6.1.2. *Let $\pi_X : X \to B$ be a tree of metric bundles as in Theorem 1.2.4. Suppose A is a qi embedded subspace of B. Prove that the inclusion $\pi_X^{-1}(A) \hookrightarrow X$ admits the CT map.*

6.2 On combination theorems

Motivated by the combination theorem of Bestvina-Feighn ([6]), Mj and Reeves proved an analogous combination theorem for trees of relatively hyperbolic spaces

([41]). In [42], Krishna proved a combination theorem for relatively hyperbolic metric bundle. Subsuming these two we have the following question.

Question 6.2.1. *Prove a combination theorem analogous to Theorem 1.2.4 for relatively hyperbolic spaces.*

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